WHEN FIRST ORDER $T$ HAS LIMIT MODELS

SH868

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Abstract. We to a large extent sort out when does a (first order complete theory) $T$ have a superlimit model in a cardinal $\lambda$. Also we deal with related notions of being limit.

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§0 Introduction, pg.3
[We give background and the basic definitions. We then present existence results for stable $T$ which have models which are saturated or closed to being saturated.]

§1 On countable superstable not $\aleph_0$-stable, pg.8
[Consistently $2^{\aleph_1} \geq \aleph_2$ and some such (complete first order) $T$ has a superlimit (non-saturated) model of cardinality $\aleph_1$. This shows that we cannot prove a non-existence result fully complementary to Lemma 0.9.]

§2 A strictly stable consistent example, pg.10
[Consistently $\aleph_1 < 2^{\aleph_0}$ and some countable stable not superstable $T$, has a (non-saturated) model of cardinality $\aleph_1$ which satisfies some relatives of being superlimit.]

§3 On the non-existence of limit models, pg.14
[The proofs here are in ZFC. If $T$ is unstable it has no superlimit models of cardinality $\lambda$ when $\lambda \geq \aleph_1 + |T|$. For unsuperstable $T$ we have similar results but with “few” exceptional cardinals $\lambda$ on which we do not know: $\lambda < \lambda^{\aleph_0}$ which are $< \beth_\omega$. Lastly, if $T$ is superstable and $\lambda \geq |T| + 2^{|T|}$ then $T$ has a superlimit model of cardinality $\lambda$ iff $|D(T)| \leq \lambda$ iff $T$ has a saturated model. Lastly, we get weaker results on weaker relatives of superlimit.]
§0A Background and Content

Recall that ([Sh:c, Ch.III]). If $T$ is (first order complete and) superstable then for $\lambda \geq 2^{|T|}$, $T$ has a saturated model $M$ of cardinality $\lambda$ and moreover

(*) if $\{M_\alpha : \alpha < \delta\}$ is $\prec$-increasing, $\delta$ a limit ordinal $< \lambda^+$ and $\alpha < \delta \Rightarrow M_\alpha \cong M$ then $\cup\{M_\alpha : \alpha < \delta\}$ is isomorphic to $M$.

When investigating categoricity of an a.e.c. (abstract elementary classes) $t = (K_t, \leq_t)$, the following property turns out to be central: $M$ is $\leq_t$-universal model of cardinality $\lambda$ with the property (*) above (called superlimit) - possibly with addition parameter $\kappa = \text{cf}(\kappa) \leq \lambda$ (or stationary $S \subseteq \lambda^+$): we also consider some relatives, mainly limit, weakly limit and strongly limit. Those notions were suggested for a.e.c. in [Sh:88, 3.1] or see the revised version [Sh:88r, 3.3] and see [Sh:h] or here in 0.7. But though coming from investigating non-elementary classes, they are meaningful for elementary classes and here we try to investigate them for elementary classes.

Recall that for a first order complete $T$, we know $\{\lambda : T$ has a saturated model of $T$ of cardinality $\lambda\}$, that is, it is $\{\lambda : \lambda^{<\lambda} \geq |D(T)|$ or $T$ is stable in $\lambda\}$, on the definitions of $D(T)$ and other notions see §0B below. What if we replace saturated by superlimit (or some relative)? Let $EC_\lambda(T)$ be the class of models $M$ of $T$ of cardinality $\lambda$.

If there is a saturated $M \in EC_\lambda(T)$ we have considerable knowledge on the existence of limit model for cardinal $\lambda$, this was as mentioned in [Sh:88r, 3.6] by [Sh:c], see 0.9(1),(2). E.g. for superstable $T$ in $\lambda \geq 2^{|T|}$ there is a superlimit model (the saturated one). It seems a natural question on [Sh:88r, 3.6] whether it exhausts the possibilities of $(\lambda, *)$-superlimit and $(\lambda, \kappa)$-superlimit models for elementary classes.

Clearly the cases of the existence of such models of a (first order complete) theory $T$ where there are no saturated (or special) models are rare, because even the weakest version of Definition [Sh:88, 3.1] $= [Sh:88r, 3.3]$ or here Definition 0.7 for $\lambda$ implies that $T$ has a universal model of cardinality $\lambda$, which is rare (see Kojman Shelah [KjSh:409] which includes earlier history and recently Djamonza [Mirar]).

So the main question seems to be whether there are such cases at all. We naturally look at some of the previous cases of consistency of the existence of a universal model (for $\lambda < \lambda^{<\lambda}$), i.e., those for $\lambda = \aleph_1$.

E.g. a sufficient condition for some versions is the existence of $T' \supseteq T$ of cardinality $\lambda$ such that $PC(T', T)$ is categorical in $\lambda$, see 0.4(3). By [Sh:100] we have consistency results for such $T_1$ so naturally we first deal with the consistency results from [Sh:100]. In §1 we deal with the case of the countable superstable $T_0$ from [Sh:100] which is not $\aleph_0$-stable. By [Sh:100] consistently $\aleph_1 < 2^{\aleph_0}$ and for some $T_0' \supseteq T_0$ of cardinality $\aleph_1$, $PC(T_0', T_0)$ is categorical in $\aleph_1$. We use this to get the consistency of $T_0$ has a superlimit model of cardinality $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$.

In §2 we prove that for some stable not superstable countable $T_1$ we have a parallel but weaker result. We relook at the old consistency results of “some $PC(T_1', T_1), [T_1'] = \aleph_1 > |T_1|$, is categorical in $\aleph_1$” from [Sh:100]. From this we deduce that in this universe, $T_1$ has a strongly $(\aleph_1, \aleph_0)$-limit model.
It is a reasonable thought that we can similarly have a consistency result on the
to theory of linear order, but this is still not clear.

In §3 we show that if $T$ has a superlimit model in $\lambda \geq |T| + \aleph_1$ then $T$ is stable and $T$ is superstable except possibly under some severe restrictions on the cardinal $\lambda$ (i.e., $\lambda < \beth_\omega$ and $\lambda < \lambda^{\aleph_0}$). We then prove some restrictions on the existence of some (weaker) relatives.

Summing up our results on the strongest notion, superlimit, by 1.1 + 3.1 we have:

**Conclusion 0.1.** Assume $\lambda \geq |T| + \beth_\omega$. Then $T$ has a superlimit model of cardinality $\lambda$ iff $T$ is superstable and $\lambda \geq |D(T)|$.

In subsequent work we shall show that for some unstable $T$ (e.g. the theory of linear orders), if $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)$, then $T$ has a medium $(\lambda, \kappa)$-limit model, whereas if $T$ has the independence property even weak $(\lambda, \kappa)$-limit models do not exist; see [Sh:877] and more in [Sh:900], [Sh:906], [Sh:950], [Sh:F1054].

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**§0.1 Basic Definitions**

Notation 0.2. 1) Let $T$ denote a complete first order theory which has infinite models but $T_1, T'$, etc. are not necessarily complete.

2) Let $M, N$ denote models, $|M|$ the universe of $M$ and $\|M\|$ its cardinality and $M < N$ means $M$ is an elementary submodel of $N$.

3) Let $\tau_T = \tau(T), \tau_M = \tau(M)$ be the vocabulary of $T, M$ respectively.

4) Let $M \models \"\varphi[\bar{a}]\"$ means that the model $M$ satisfies $\varphi[\bar{a}]$ iff the statement stat is true (or is 1 rather than 0)).

**Definition 0.3.** 1) For $\bar{a} \in \omega^{|M|}$ and $B \subseteq M$ let $\text{tp}(\bar{a}, B, M) = \{\varphi(x, \bar{b}) : \varphi = \varphi(x, \bar{b}) \in L(\tau_M), \bar{b} \in g(\bar{a})B \text{ and } M \models \varphi[\bar{a}, \bar{b}]\}$.

2) Let $D(T) = \{\text{tp}(\bar{a}, \emptyset, M) : M \text{ a model of } T \text{ and } \bar{a} \text{ a finite sequence from } M\}$.

3) If $A \subseteq M$ then $S^m(A, M) = \{\text{tp}(\bar{a}, A, N) : M < N \text{ and } a \in \ell_m N\}$, if $m = 1$ we may omit it.

4) A model $M$ is $\lambda$-saturated when: if $A \subseteq M, |A| < \lambda$ and $p \in S(A, M)$ then $p$ is realized by some $a \in M$, i.e. $p \subseteq \text{tp}(a, A, M)$; if $\lambda = \|M\|$ we may omit it.

5) A model $M$ is special when letting $\lambda = \|M\|$, there is an increasing sequence $\langle \lambda_i : i < \text{cf}(\lambda) \rangle$ of cardinals with limit $\lambda$ and a $\prec$-increasing sequence $\langle M_i : i < \text{cf}(\lambda) \rangle$ of models with union $M$ such that $M_{i+1}$ is $\lambda_i$-saturated of cardinality $\lambda_{i+1}$ for $i < \text{cf}(\lambda)$.

**Definition 0.4.** 1) For any $T$ let $\text{EC}(T) = \{M : M \text{ is a } \tau_T \text{-model of } T\}$.

2) $\text{EC}_\lambda(T) = \{M \in \text{EC}(T) : M \text{ is of cardinality } \lambda\}$.

3) For $T \subseteq T'$ let $PC(T', T) = \{M \upharpoonright \tau_T : M \text{ is model of } T'\}$

$PC_\lambda(T', T) = \{M \in PC(T', T) : M \text{ is of cardinality } \lambda\}$.

4) We say $M$ is $\lambda$-universal for $T_1$ when it is a model of $T_1$ and every $N \in \text{EC}(T)$ can be elementarily embedded into $M$; if $T_1 = \text{Th}(M)$ we may omit it.

5) We say $M \in \text{EC}(T)$ is universal when it is $\lambda$-universal for $\lambda = \|M\|$.
We are here mainly interested in

**Definition 0.5.** Given $T$ and $M \in \text{EC}_\lambda(T)$ we say that $M$ is a superlimit or $\lambda$-superlimit model when: $M$ is universal and if $\delta < \lambda^+$ is a limit ordinal, $(M_\alpha : \alpha \leq \delta)$ is $\prec$-increasing continuous, and $M_\alpha$ is isomorphic to $M$ for every $\alpha < \delta$ then $M_\delta$ is isomorphic to $M$.

**Remark 0.6.** Concerning the following definition we shall use strongly limit in 2.14(1), medium limit in 2.14(2).

**Definition 0.7.** Let $\lambda$ be a cardinal $\geq |T|$. For parts 3) - 7) but not 8), for simplifying the presentation we assume the axiom of global choice and $F$ is a class function; alternatively restrict yourself to models with universe an ordinal $\in [\lambda, \lambda^+)$.

1) For non-empty $\Theta \subseteq \{ \mu : \aleph_0 \leq \mu < \lambda \text{ and } \mu \text{ is regular \} }$ and $M \in \text{EC}_\lambda(T)$ we say that $M$ is a $(\lambda, \Theta)$-superlimit when: $M$ is universal and if $(M_i : i \leq \mu)$ is $\prec$-increasing, $M_i \cong M$ for $i < \mu$ and $\mu \in \Theta$ then $\cup\{M_i : i < \mu\} \cong M$.

2) If $\Theta$ is a singleton, say $\Theta = \{ \theta \}$, we may say that $M$ is $(\lambda, \theta)$-superlimit.

3) Let $S \subseteq \lambda^+$ be stationary. A model $M \in \text{EC}_\lambda(T)$ is called $S$-strongly limit or $(\lambda, S)$-strongly limit when for some function: $F : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T)$ we have:

(a) for $N \in \text{EC}_\lambda(T)$ we have $N \prec F(N)$

(b) if $\delta \in S$ is a limit ordinal and $(M_i : i < \delta)$ is a $\prec$-increasing continuous sequence \(^1\) in $\text{EC}_\lambda(T)$ and $i < \delta \Rightarrow F(M_{i+1}) \prec M_{i+2}$, then $M \cong \cup\{M_i : i < \delta\}$.

4) Let $S \subseteq \lambda^+$ be stationary. $M \in \text{EC}_\lambda(T)$ is called $S$-limit or $(\lambda, S)$-limit if for some function $F : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T)$ we have:

(a) for every $N \in \text{EC}_\lambda(T)$ we have $N \prec F(N)$

(b) if $(M_i : i < \lambda^+)$ is a $\prec$-increasing continuous sequence of members of $\text{EC}_\lambda(T)$ such that $F(M_{i+1}) \prec M_{i+2}$ for $i < \lambda^+$ then for some closed unbounded \(^2\) subset $C$ of $\lambda^+$,

$$\delta \in S \cap C \Rightarrow M_\delta \cong M.$$  

5) We define \(^3\) “$S$-weakly limit”, “$S$-medium limit” like “$S$-limit”, “$S$-strongly limit” respectively by demanding that the domain of $F$ is the family of $\prec$-increasing continuous sequence of members of $\text{EC}_\lambda(T)$ of length $< \lambda^+$ and replacing “$F(M_{i+1}) \prec M_{i+2}$” by “$M_{i+1} \cong F(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$”.

6) If $S = \lambda^+$ then we may omit $S$ (in parts (3), (4), (5)).

7) For non-empty $\Theta \subseteq \{ \mu : \mu \leq \lambda \text{ and } \mu \text{ is regular \} }$, $M$ is $(\lambda, \Theta)$-strongly limit \(^4\) if $M$ is $\{ \delta < \lambda^+: cf(\delta) \in \Theta \}$-strongly limit. Similarly for the other notions. If we do not write $\lambda$ we mean $\lambda = \|M\|$.

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\(^1\)no loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $F$, i.e., only $F(M')$ for $M' \cong M$ is required

\(^2\)alternatively, we can use as a parameter a filter on $\lambda^+$ extending the co-bounded filter

\(^3\)Note that $M$ is $(\lambda, S)$-strongly limit iff $M$ is $(\langle \lambda, cf(\delta) : \delta \in S \rangle)$-strongly limit.

\(^4\)in [Sh:88t] we consider: we replace “limit” by “limit” if $F(M_{i+1}) \prec M_{i+2}$; “$M_{i+1} \prec F(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$” are replaced by “$F(M_i) \prec M_{i+1}$”, “$M_i \prec F(\langle M_j : j \leq i \rangle) \prec M_{i+1}$” respectively. But $(\text{EC}(T), \prec)$ has amalgamation.
8) We say that $M \in K_\lambda$ is invariantly strong limit when in part (3), $F$ is just a subset of $\{(M,N)/ \equiv : M \prec N \text{ are from } EC_\lambda(T)\}$ and in clause (b) of part (3) we replace “$F(M_{i+1}) \prec M_{i+2}$” by “$(\exists N)(M_{i+1} \prec N \prec M_{i+2} \land ((M,N)/ \equiv) \in F)$”. But abusing notation we still write $N = F(M)$ instead $((M,N)/ \equiv) \in F$. Similarly with the other notions, so we use the isomorphism type of $M^{-}\langle N \rangle$ for “weakly limit” and “medium limit”.

9) In the definitions above we may say “$F$ witness $M$ is ...”

\{y.5c\} Observation 0.8. 1) Assume $F_1, F_2$ are as above and $F_1(N) \prec F_2(N)$ (or $F_1(\bar{N}) \prec F_2(\bar{N})$) whenever defined. If $F_1$ is a witness then so is $F_2$.

2) All versions of limit models implies being a universal model in $EC_\lambda(T)$.

\{y.6\} 3) The Obvious implications diagram: For non-empty $\Theta \subseteq \{\theta : \theta \text{ is regular } \leq \lambda\}$ and stationary $S_1 \subseteq \{\theta \prec \lambda^+ : cf(\theta) \in \Theta\}$:

\[
superlimit = (\lambda, \{\mu : \mu \leq \lambda \text{ regular}\})\text{-superlimit} \\
\downarrow \\
(\lambda, \Theta)\text{-superlimit} \\
\downarrow \\
S_1\text{-strongly limit} \\
\downarrow \\
S_1\text{-medium limit,} \\
\downarrow \\
S_1\text{-limit} \\
\downarrow \\
S_1\text{-weakly limit.}
\]

\{y.7\} Lemma 0.9. Let $T$ be a first order complete theory.

1) If $\lambda$ is regular, $M$ a saturated model of $T$ of cardinality $\lambda$, then $M$ is $(\lambda, \lambda)$-superlimit.

2) If $T$ is stable, and $M$ is a saturated model of $T$ of cardinality $\lambda \geq \aleph_1 + |T|$ and $\Theta = \{\mu : \kappa(T) \leq \mu \leq \lambda \text{ and } \mu \text{ is regular}\}$, then $M$ is $(\lambda, \Theta)$-superlimit (on $\kappa(T)$)-see [Sh:c, III, §3].

3) If $T$ is stable in $\lambda$ and $\kappa = cf(\kappa) \leq \lambda$ then $T$ has an invariantly strongly $(\lambda, \kappa)$-limit model.

\{y.8\} Remark 0.10. Concerning 0.9(2), note that by [Sh:c] if $\lambda$ is singular or just $\lambda < \lambda^\lambda$ and $T$ has a saturated model of cardinality $\lambda$ then $T$ is stable (even stable in $\lambda$) and $cf(\lambda) \geq \kappa(T)$.

Proof. 1) Let $M_i$ be a $\lambda$-saturated model of $T$ of cardinality $\lambda$ for $i < \lambda$ and $\langle M_i : i < \lambda \rangle$ is $\prec$-increasing and $M_\lambda = \bigcup_{i < \lambda} M_i$. Now for every $A \subseteq M_\lambda$ of cardinality $< \lambda$ there is $i < \lambda$ such that $A \subseteq M_i$ hence every $p \in S(A, M_\lambda)$ is realized in $M_i$ hence in $M_\lambda$ hence clearly $M_\lambda$ is $\lambda$-saturated. Remembering the uniqueness of a $\lambda$-saturated model of $T$ of cardinality $\lambda$ we finish.

2) Use [Sh:c, III, 3.11]: if $M_i$ is a $\lambda$-saturated model of $T$, $\langle M_i : i < \delta \rangle$ increasing $cf(\delta) \geq \kappa(T)$ then $\bigcup_{i < \delta} M_i$ is $\lambda$-saturated.

3) Let $K_{\lambda, \kappa} = \{\bar{M} : \bar{M} = \langle M_i : i \leq \kappa \rangle \text{ is } (\prec, \subseteq)\text{-increasing continuous, } M_i \in EC_\lambda(T) \text{ and } (M_{i+2}, e)_{i \in M_{i+1}} \text{ is saturated for every } i < \kappa\}$. Clearly $\bar{M}, \bar{N} \in K_{\lambda, \kappa} \Rightarrow M_\kappa \equiv
$N_{\kappa}$. Also for every $M \in EC_{\lambda}(T)$ there is $N$ such that $M \prec N$ and $(N,c) \in E_M$ is saturated, as also $Th((M,c) \in E_M)$ is stable in $\lambda$; so there is an invariant $F : EC_{\lambda}(T) \to EC_{\lambda}(T)$ such that $M \prec F(M)$ and $(F(M),c) \in E_M$ is saturated; such $F$ witness the desired conclusion. $\square_{0.9}$

**Definition 0.11.**

0) For regular $\kappa < \lambda$ let $S_{\theta}^\lambda = \{ \delta < \lambda : cf(\delta) = \lambda \}$.

1) For a regular uncountable cardinal $\lambda$ let $I[\lambda] = \{ S \subseteq \lambda : \text{some pair } (E,\vec{a}) \}

2) We say that $(E,\vec{u})$ is a witness for $S \in I[\lambda]$ iff:

   (a) $E$ is a club of the regular cardinal $\lambda$

   (b) $\vec{u} = \langle u_\alpha : \alpha < \lambda \rangle$, $u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$

   (c) for every $\delta \in E \cap S$, $u_\delta$ is an unbounded subset of $\delta$ of order-type $cf(\delta)$ (and $\delta$ is a limit ordinal).

By [Sh:420, §1]

**Claim 0.12.** If $\kappa^+ < \lambda$ and $\kappa,\lambda$ are regular then some stationary $S \subseteq \{ \delta < \lambda : cf(\delta) = \kappa \}$ belongs to $I[\lambda]$.

By [Sh:108]

**Claim 0.13.** If $\lambda = \mu^+, \theta = cf(\theta) \leq cf(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\theta} \leq \mu$ then $S_{\theta}^\lambda \in I[\lambda]$. 
1. ON SUPERSTABLE NOT $\aleph_0$-STABLE $T$

We first note that superstable $T$ tend to have superlimit models.

**Claim 1.1.** Assume $T$ is superstable and $\lambda \geq |T| + 2^{\aleph_0}$. Then $T$ has a superlimit model of cardinality $\lambda$ iff $T$ has a saturated model of cardinality $\lambda$. Moreover, $T$ has a universal model of cardinality $\lambda$ iff $\lambda \geq |D(T)|$.

**Proof.** By [Sh:c, III, §5] we know that $T$ is stable in $\lambda$ iff $\lambda \geq |D(T)|$. Now if $|T| \leq \lambda < |D(T)|$ trivially there is no universal model of $T$ of cardinality $\lambda$ hence no saturated model and no superlimit model, etc., recalling 0.8(2). If $\lambda \geq |D(T)|$, then $T$ is stable in $\lambda$ hence has a saturated model of cardinality $\lambda$ by [Sh:c, III] (hence universal) and the class of $\lambda$-saturated models of $T$ is closed under increasing elementary chains by [Sh:c, III] so we are done. \( \square_{1.1} \)

The following are the prototypical theories which we shall consider.

**Definition 1.2.**
1) $T_0 = \text{Th}(\mathbb{N})$ when $\eta E_0 \nu \iff \eta | n = \nu | n$.
2) $T_1 = \text{Th}(\omega_1, E_0)_{n<\omega}$ where $\eta E_0 \nu \iff \eta | n = \nu | n$.
3) $T_2 = \text{Th}(\mathbb{R}, <)$.

Recall

**Observation 1.3.**
0) $T_0$ is a countable complete first order theory for $\ell = 0, 1, 2$.
1) $T_0$ is superstable not $\aleph_0$-stable.
2) $T_1$ is strictly stable, that is, stable but not superstable.
3) $T_2$ is unstable.
4) $T_2$ has elimination of quantifiers for $\ell = 0, 1, 2$.

**Claim 1.4.** It is consistent with ZFC that $\aleph_1 < 2^{\aleph_0}$ and some $M \in EC_{\aleph_1}(T_0)$ is a superlimit model.

**Proof.** By [Sh:100], for notational simplicity we start with $V = L$.

So $T_0$ is defined in 1.2(1) and it is the $T$ from Theorem [Sh:100, 1.1] and let $S$ be the set of $\eta \in (\omega)^{\omega_1}$ as the following theory:

@1 (i) $T_0$, or just for each $n$ the sentence saying $E_n$ is an equivalence relation with $2^n$ equivalence classes, each $E_n$ equivalence class divided to two by $E_n+1$, $E_n+1$ refine $E_n$, $E_0$ is trivial

(ii) the sentences saying that

(a) for every $x$, the function $z \mapsto F(x, z)$ is one-to-one and

(b) $xE_n(F(x, z))$ for each $n < \omega$

(iii) $E_n(\eta, \nu)\mathcal{H}(\eta[n = \nu]|n)$ for $\eta, \nu \in S$.

In [Sh:100] it is proved that in some forcing\(^5\) extension $L^P$ of $L$, $P$ an $\aleph_2$-c.c. proper forcing of cardinality $\aleph_2$, in $V = L^P$, the class $PC(T_0, T_0) = \{M | \pi_0 : M$ is a $\tau$-model of $T'\}$ is categorical in $\aleph_1$.

However, letting $M^*$ be any model from $PC(T_0, T_0)$ of cardinality $\aleph_1$, it is easy to see that (in $V = L^P$):

@2 the following conditions on $M$ are equivalent

(a) $M$ is isomorphic to $M^*$

\(^5\) We can replace $L$ by any $V_0$ which satisfies $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$.\n
(b) $M \in \text{PC}(T', T_0)$
(c) (α) $M$ is a model of $T_0$ of cardinality $\aleph_1$
(β) $M^*$ can be elementarily embedded into $M$
(γ) for every $a \in M$ the set $\cap\{a/E^M_n : n < \omega\}$ has cardinality $\aleph_1$.

But
⊕₃ every model $M_1$ of $T$ of cardinality $\leq \aleph_1$ has a proper elementary extension to a model satisfying (c), i.e., (α), (β), (γ) of ⊕₂ above
⊕₄ if $\langle M_\alpha : \alpha < \delta \rangle$ is an increasing chain of models satisfying (c) of ⊕₂ and $\delta < \omega_2$ then also $\cup\{M_\alpha : \alpha < \delta\}$ does.

Together we are done. □₁₄

Naturally we ask

Question 1.5. What occurs to $T_0$ for $\lambda > \aleph_1$ but $\lambda < 2^{\aleph_0}$? {0.1.3}

Question 1.6. Does the theory $T_2$ of linear order consistently have an $(\aleph_1, \aleph_0)$-superlimit? (or only strongly limit?) but see §3. {0.1}

Question 1.7. What is the answer for $T$ when $T$ is countable superstable not $\aleph_0$-stable and $D(T)$ countable for $\aleph_1 < 2^{\aleph_0}$ for $\aleph_2 < 2^{\aleph_0}$?

So by the above for some such $T$, in some universe, for $\aleph_1$ the answer is yes, there is a superlimit.
2. A Strictly Stable Consistent Example

We now look at models of $T_1$ (redefined below) in cardinality $\aleph_1$; recall

Definition 2.1. $T_1 = \text{Th}(\omega(\omega_1), E_n)_{n<\omega}$ where $E_n = \{ (\eta, \nu) : \eta, \nu \in \omega(\omega_1) \text{ and } \eta \upharpoonright n = \nu \upharpoonright n \}$. 

Remark 2.2.

(a) Note that $T_1$ has elimination of quantifiers.

(b) If $\lambda = \Sigma \{ \lambda_n : n < \omega \}$ and $\lambda_n = \lambda^{\aleph_n}_n$, then $T_1$ has a $(\lambda, \aleph_0)$-superlimit model in $\lambda$ (see 2.15).

Definition/Claim 2.3. 1) Any model of $T_1$ of cardinality $\lambda$ is isomorphic to $M_{A,h} := \langle \{ (\eta, \epsilon) : \eta \in A, \epsilon < h(\eta) \} ; E_n \rangle_{n<\omega}$ for some $A \subseteq \omega\lambda$ and $h : \omega\lambda \rightarrow \text{Car} \cap \lambda^+$ \{0\} where $(\eta_1, \epsilon_1)E_n(\eta_2, \epsilon_2) \iff \eta_1 \upharpoonright n = \eta_2 \upharpoonright n$, pedantically we should write $E^{M_{A,h}}_n = E_n\lvert M_{A,h} \rangle$.

2) We write $M_A$ for $M_{A,h}$ when $A$ is as above and $h : A \rightarrow \{ |A| \}$, so constantly $|A|$ when $A$ is infinite.

3) For $A \subseteq \omega\lambda$ and $h$ as above the model $M_{A,h}$ is a model of $T_1$ if $A$ is non-empty and $(\forall \eta \in A)(\forall n < \omega)(\exists \nu \in A) (|\nu| = n \wedge \nu(n) \neq \eta(n))$.

4) Above $M_{A,h}$ has cardinality $\lambda$ iff $\Sigma \{ h(\eta) : \eta \in A \} = \lambda$.

Definition 2.4. 1) We say that $A$ is a $(T_1, \lambda)$-witness when

(a) $A \subseteq \omega\lambda$ has cardinality $\lambda$

(b) if $B_1, B_2 \subseteq \omega\lambda$ are $(T_1, A)$-big (see below) of cardinality $\lambda$ then $(B_1 \uplus \omega\lambda, A)$ is isomorphic to $(B_2 \uplus \omega\lambda, A)$.

2) A set $B \subseteq \omega\lambda$ is called $(T_1, A)$-big when it is $(\lambda, \lambda) - (T_1, A)$-big; see below.

3) $B$ is $(\mu, \lambda) - (T_1, A)$-big means: $B \subseteq \omega\lambda, |B| = |A| = \mu$ and for every $\eta \in \omega\lambda$ there is an isomorphism $f$ from $(\omega\lambda, \prec)$ onto $(\{ \nu : \nu \in \omega\lambda \}, \prec)$ mapping $A$ into $\{ \nu : \eta \prec \nu \in B \}$.

4) $A \subseteq \omega(\omega_1)$ is $\aleph_1$-suitable when

(a) $|A| = \aleph_1$

(b) for a club of $\delta < \omega_1, A \cap \omega\delta$ is everywhere not meagre in the space $\omega\delta$, i.e., for every $\eta \in \omega\delta$ the set $\{ \nu \in A \cap \omega\delta : \eta \prec \nu \}$ is a non-meagre subset of $\omega\delta$ (that is what really is used in [Sh:100]).

Claim 2.5. It is consistent with ZFC that $2^{\aleph_0} > \aleph_1 + \text{there is a } (T_1, \aleph_1)-\text{witness;}\text{ moreover every } \aleph_1\text{-suitable set is a } (T_1, \aleph_1)-\text{witness.}$

Proof. By [Sh:100, §2].

Remark 2.6. The witness does not give rise to an $(\aleph_1, \aleph_0)$-limit model, as for the union of any “fast enough” $\prec$-increasing $\omega$-chain of members of EC$\aleph_1(T_1)$, the relevant sets are meagre.

Definition 2.7. Let $A$ be a $(T_1, \lambda)$-witness. We define $K_{T_1,A}^1$ as the family of $M = (|M|, <^M, P^M_\alpha)_{\alpha \leq \omega}$ such that:

(a) $(|M|, <^M)$ is a tree with $(\omega + 1)$ levels

(b) $P^M_\alpha$ is the $\alpha$-th level; let $P^M_{<\omega} = \bigcup \{ P^M_n : n < \omega \}$
\( M \) is isomorphic to \( M^1_B \) for some \( B \subseteq \omega \lambda \) of cardinality \( \lambda \) where \( M^1_B \) is defined by \( |M^1_B| = (\omega^\omega \lambda) \cup B, T^n_{M^1_B} = \omega \lambda, P_\omega^{M^1_B} = B \) and \( <M^1_B = <]|M^1_B| \), i.e., being an initial segment

(\( \delta \)) moreover \( B \) is such that some \( f \) satisfies:

(\( a \)) \( f : \omega^\omega \lambda \to \omega \) and \( f(\langle \rangle) = 0 \) for simplicity

(\( b \)) \( \eta \leq \nu \in \omega^\omega \lambda \Rightarrow f(\eta) \leq f(\nu) \)

(\( c \)) if \( \eta \in B \) then \( \langle f(\eta \downharpoonright n) : n < \omega \rangle \) is eventually constant

(\( d \)) if \( \eta \in \omega^\omega \lambda \) then \( \{ \nu \in \omega^\omega \lambda : \eta \leq \nu \in B \) and \( m < \omega \Rightarrow f(\eta \downharpoonright (\nu \upharpoonright m)) = f(\eta) \} \) is \( (T_1, A) \)-big

(\( e \)) for \( \eta \in \omega^\omega \lambda \) and \( n \in \{ f(\eta), \omega \} \) for \( \lambda \) ordinals \( \alpha < \lambda \), we have \( f(\eta \downharpoonright \langle \alpha \rangle) = n \).

Claim 2.8. [The Global Axiom of Choice] If \( A \) is a \((T_1, N_1)\)-witness then

(\( a \)) \( K_{T_1, A}^1 \neq \emptyset \)

(\( b \)) any two members of \( K_{T_1, A}^1 \) are isomorphic

(\( c \)) there is a function \( F \) from \( K_{T_1, A}^1 \) to itself (up to isomorphism, i.e., \((M, F(M)) \) is defined only up to isomorphism) satisfying \( M \subseteq F(M) \) such that \( K_{T_1, A}^1 \) is closed under increasing unions of sequence \( \langle M_n : n < \omega \rangle \) such that \( F(M_n) \subseteq M_{n+1} \).

Proof. Clause (a): Trivial.

Clause (b): By the definition of “\( A \) is a \((T_1, N_1)\)-witness” and of \( K_{T_1, A}^1 \).

Clause (c):

We choose \( F \) such that

(\( \circ \)) if \( M \in K_{T_1, A}^1 \) then \( M \subseteq F(M) \in K_{T_1, A}^1 \) and for every \( k < \omega \) and \( a \in P^M_k \), the set \( \{ b \in F^M(a) : a < F(M), b \) and \( b \notin M \} \) has cardinality \( N_1 \).

Assume \( M = \cup\{ M_n : n < \omega \} \) where \( \langle M_n : n < \omega \rangle \) is \( \subseteq \)-increasing, \( M_n \in K_{T_1, A}^1, F(M_n) \subseteq M_{n+1} \). Clearly \( M \) is as required in the beginning of Definition 2.7, that is, satisfies clauses (\( a \)), (\( \beta \)), (\( \gamma \)) there. To prove clause (\( \delta \)), we define \( f : P^M_{\omega} \to \omega \) by \( f(a) = \text{Min}\{ n : a \in M_n \} \). Pendentically, \( F \) is defined only up to isomorphism.

So we are done.  \( \square_{2.8} \)

Claim 2.9. [The Global Axiom of Choice]

If \( A \) is a \((T_1, \lambda)\)-witness then

(\( a \)) \( K_{T_1, A}^1 \neq \emptyset \)

(\( b \)) any two members of \( K_{T_1, A}^1 \) are isomorphic

(\( c \)) if \( M_n \in K_{T_1, A}^1 \) and \( n < \omega \Rightarrow M_n \subseteq M_{n+1} \) then \( M := \cup\{ M_n : n < \omega \} \in K_{T_1, A}^1 \).

Remark 2.10. If we omit clause (\( b \)), we can weaken the demand on the set \( A \).
Proof. Assume $M = \cup\{M_n : n < \omega\}, M_n \subseteq M_{n+1}, M_n \in K^1_{T_1,A}$ and $f_n$ witnesses $M_n \in K^1_{T_1,A}$. Clearly $M$ satisfies clauses (a), (β), (γ) from Definition 2.7, we just have to find a witness $f$ as in clause (δ) there.

For each $a \in M$ let $n(a) = \operatorname{Min}\{n : a \in M_n\}$, clearly if $M \models "a < b < c"$ then $n(a) \leq n(b)$ and $n(a) = n(c) \Rightarrow n(a) = n(b)$. Let $g_n : M \rightarrow M$ be defined by:

$$g_n(a) = b \text{ if } b \leq^M a, b \in M_n \text{ and } b \leq^M \text{-maximal under those restrictions; clearly it is well defined. Now we define } f'_n : M \rightarrow \omega \text{ by induction on } n < \omega \text{ such that } m < n \Rightarrow f'_m \subseteq f'_n, \text{ as follows. }$$

If $n = 0$ let $f'_n = f_n$.

If $n = m + 1$ and $a \in M_n$ we let $f'_{n}(a)$ be $f'_{m}(a)$ if $a \in M_m$ and be $(f_n(a) - f_n(g_m(a))) + f'_m(g_m(a)) + 1$ if $a \in M_n \setminus M_m$. Clearly $f := \cup\{f'_n : n < \omega\}$ is a function from $M$ to $\omega$, $a \leq^M b \Rightarrow f(a) \leq f(b)$, and for any $a \in M$ the set $\{b \in M : a \leq^M b \text{ and } f(b) = f(a)\}$ is equal to $\{b \in M(a) : f_n(a)(a) = f_n(a)(b) \text{ and } a \leq^M b\}$.

So we are done. \(\square\)

**Definition 2.11.** Let $A$ be a $(T_1, \lambda)$-witness. We define $K^2_{T_1,A}$ as in Definition 2.7 but $f$ is constant zero.

**Claim 2.12.** [The Global Axiom of Choice] If $A$ is a $(T_1, \mathfrak{N}_1)$-witness then

(a) $K^2_{T_1,A} \neq \emptyset$

(b) any two members of $K^2_{T_1,A}$ are isomorphic

(c) there is a function $F$ from $\bigcup\{\alpha + 2 : K^{\alpha + 2}_{T_1,A} \} : \alpha < \omega_1\}$ to $K^2_{T_1,A}$ which satisfies:

- (α) $M = \langle M_i : i \leq \alpha + 1\rangle$ is an $\prec$-increasing sequence of models of $T$ then $M_{\alpha + 1} \subseteq F(M) \in K^2_{T_1,A}$

- (β) the union of any increasing $\omega_1$-sequence $M = \langle M_\alpha : \alpha < \omega_1\rangle$ of members of $K^2_{T_1,A}$ belongs to $K^2_{T_1,A}$ when $\omega_1 = \sup\{\alpha : F(M + (\alpha + 2)) \subseteq M_{\alpha + 2}\}$ and is a well defined embedding of $M_\alpha$ into $M_{\alpha + 2}\}$.

**Remark 2.13.** Instead of the global axiom of choice, we can restrict the models to have universe a subset of $\lambda^+$ (or just a set of ordinals).

**Proof.** Clause (a): Easy.

Clause (b): By the definition.

Clause (c): Let $\langle \mathcal{U}_\varepsilon : \varepsilon < \omega_1\rangle$ be an increasing sequence of subsets of $\omega_1$ with union $\omega_1$ such that $\varepsilon < \omega_1 \Rightarrow |\mathcal{U}_\varepsilon| \cup \bigcup_{\varepsilon < \varepsilon} |\mathcal{U}_\varepsilon| = \aleph_1$. Let $M^* \in K^2_{T_1,A}$ be such that $\omega^*(\omega_1) \subseteq |M^*| \subseteq \omega^*(\omega_1)$ and $M^*_\varepsilon = M^* \upharpoonright \omega^*(\mathcal{U}_\varepsilon)$ belongs to $K^2_{T_1,A}$ for every $\varepsilon < \omega_1$.

We choose a pair $(F, f)$ of functions with domain $\{M : \bar{M} \text{ an increasing sequence of members of } K^2_{T_1,A} \text{ of length } < \omega_1\}$ such that:

(a) $F(M)$ is an extension of $\cup\{M_i : i < \ell g(M)\}$ from $K^2_{T_1,A}$

(b) $f(M)$ is an embedding from $M^*_{\ell g(M)}$ into $F(M)$

(γ) if $M^\ell = \langle M_\alpha : \alpha < \alpha_1\rangle$ for $\ell = 1, 2$ and $\alpha_1 < \alpha_2, M^1 = M^2 \upharpoonright \alpha_1$ and $F(M^1) \subseteq M_{\alpha_1}$ then $f(M^1) \subseteq f(M^2)$

(δ) if $a \in F(M)$ and $n < \omega$ then for some $b \in M^*_{\ell g(M)}$ we have $F(M) \models aE_n(f(M)(b))$. 
Now check. □_{2.12} \{s1.7\}

**Conclusion 2.14.** Assume there is a \((T_1, \aleph_1)\)-witness (see Definition 2.4) for the first-order complete theory \(T_1\) from 2.1:

1) \(T_1\) has an \((\aleph_1, \aleph_1)\)-strongly limit model.
2) \(T_1\) has an \((\aleph_1, \aleph_1)\)-medium limit model.
3) \(T_1\) has a \((\aleph_1, \aleph_0)\)-superlimit model.

**Proof.**
1) By 2.8 the reduction of problems on \((EC(T_1), \preceq)\) to \(K_{T_1,A}\) (which is easy) is exactly as in [Sh:100]. □_{2.12} \{s1.5\}
2) By 2.12. □_{2.14} \{s1.5.1\} \{s1.11\}
3) Like part (1) using claim 2.9. □_{2.14}

**Claim 2.15.** If \(\lambda = \sum\{\lambda_n : n < \omega\}\) and \(\lambda_n = \lambda_{n+1}\), then \(T_1\) has a \((\lambda, \aleph_0)\)-superlimit model in \(\lambda\).

**Proof.** Let \(M_n\) be the model \(M_{A_n, h_n}\) where \(A_n = \omega(\lambda_n)\) and \(h_n : A_n \to \lambda_n^+\) is constantly \(\lambda_n\).

Clearly

\((*)_1\) \(M_n\) is a saturated model of \(T_1\) of cardinality \(\lambda_n\)
\((*)_2\) \(M_n \preceq M_{n+1}\)
\((*)_3\) \(M_\omega = \bigcup\{M_n : n < \omega\}\) is a special model of \(T_1\) of cardinality \(\lambda\).

The main point:

\((*)_4\) \(M_\omega\) is \((\lambda, \aleph_0)\)-superlimit model of \(T_1\).

Why? Toward this assume

\((a)\) \(N_n\) is isomorphic to \(M_n\) say \(f_n : M_\omega \to N_n\) is such isomorphic
\((b)\) \(N_n \preceq N_{n+1}\) for \(n < \omega\).

Let \(N_\omega = \bigcup\{N_n : n < \omega\}\) and we should prove \(N_\omega \cong M_\omega\), so just \(N_\omega\) is a special model of \(T_1\) of cardinality \(\lambda\) suffice.

Let \(N'_n = N_\omega(\bigcup\{f_n(M_k) : k \leq n\})\). Easily \(N'_n \preceq N'_{n+1} \preceq N_\omega\) and \(\bigcup\{N'_n : n < \omega\} = N_\omega\) and \(\|N'_n\| = \lambda_n\). So it suffices to prove that \(N'_n\) is saturated and by direct inspection shows this. □_{2.15}
ON NON-EXISTENCE OF LIMIT MODELS

Naturally we assume that non-existence of superlimit models for unstable $T$ is easier to prove. For other versions we need to look more. We first show that for $\lambda \geq |T|+\aleph_1$, if $T$ is unstable then it does not have a superlimit model of cardinality $\lambda$ and if $T$ is unsuperstable, we show this for “most” cardinals $\lambda$. On “$\Phi$ proper for $K_\omega$ or $K_\omega^*$”, see [Shc, VII] or [Sh:E50] or hopefully some day [Sh:III]. We assume some knowledge on stability.

CLAIM 3.1. 1) If $T$ is unstable, $\lambda \geq |T|+\aleph_1$, then $T$ has no superlimit model of cardinality $\lambda$.

2) If $T$ is stable not superstable and $\lambda \geq |T|+\beth_\omega$ or $\lambda = \lambda^{\aleph_0} \geq |T|$ then $T$ has no superlimit model of cardinality $\lambda$.

Remark 3.2. 1) We assume some knowledge on EM models for linear orders $I$ and members of $K_\omega$ as index models, see, e.g. [Sh:c, VII].

2) We use the following definition in the proof, as well as a result from [Sh:c, VII].

DEFINITION 3.3. For cardinals $\lambda > \kappa$ let $\lambda^{[\kappa]}$ be the minimal $\mu$ such that for some, equivalently for every set $A$ of cardinality $\lambda$ there is $\mathcal{P}_A \subseteq [A]^{\leq \kappa} = \{B \subseteq A : |B| \leq \kappa\}$ of cardinality $\lambda$ such that any $B \in [\lambda]^{\leq \kappa}$ is the union of $< \kappa$ members of $\mathcal{P}_A$.

Proof. 1) Towards a contradiction assume $M^*$ is a superlimit model of $T$ of cardinality $\lambda$. As $T$ is unstable we can find $m, \varphi(x, \hat{y})$ such that

$$\varphi(x, \hat{y}) \in L_{\tau(T)}$$

linearly orders some infinite $I \subseteq \mathcal{M}$, $\mathcal{M} \models T$ so $\ell g(x) = \ell g(\hat{y}) = m$.

We can find a $\Phi$ which is proper for linear orders (see [Shc, VII]) and $F_\ell (\ell < m)$ such that $F_\ell \in \tau \setminus \tau_T$ is a unary function symbol for $\ell < m$, $\tau_T \subseteq \tau(\Phi)$ and for every linear order $I$, $EM(I, \Phi)$ has Skolem functions and its $\tau_T$-reduct $EM_{\tau(T)}(I, \Phi)$ is a model of $T$ of cardinality $|T|+|I|$ and $\tau(\Phi)$ is of cardinality $|T|+\aleph_1$ and $\langle a_s : s \in I \rangle$ is the Skeleton of $EM(I, \Phi)$, that is, it is an indiscernible sequence in $EM(I, \Phi)$ and $EM(I, \Phi)$ is the Skolem hull of $\{a_s : s \in I\}$, and letting $a_s = (F_\ell(a_s) : \ell < m)$ in $EM(I, \Phi)$ we have $EM_{\tau(T)}(I, \Phi) \models \varphi(a_s, \bar{a}_s^{[\ell < \tau]}$ for $s, t \in I$.

Next we can find $\Phi_n$ (for $n < \omega$) such that:

- $(a)$ $\Phi_n$ is proper for linear order and $\Phi_0 = \Phi$
- $(b)$ $EM_{\tau(T)}(I, \Phi_n) \prec EM_{\tau(T)}(I, \Phi_{n+1})$ for every linear order $I$ and $n < \omega$; moreover
- $(b)^+$ $\tau(\Phi_n) \subseteq \tau(\Phi_{n+1})$ and $EM(I, \Phi_n) \prec EM_{\tau(T)}(I, \Phi_{n+1})$ for every $n < \omega$ and linear order $I$
- $(c)$ if $|I| \leq n$ then $EM_{\tau(T)}(I, \Phi_n) = EM_{\tau(T)}(I, \Phi_{n+1})$ and $EM_{\tau(T)}(I, \Phi_n) \cong M^*$
- $(d)$ $|\tau(\Phi_n)| = \lambda$.

This is easy. Let $\Phi_\omega$ be the limit of $\langle \Phi_n : n < \omega \rangle$, i.e. $\tau(\Phi_\omega) = \cup \{\tau(\Phi_n) : n < \omega\}$ and if $k < \omega$ then $EM_{\tau(T)}(I, \Phi_\omega) = \cup \{EM_{\tau(T)}(I, \Phi_n) : n \in [k, \omega)\}$. So as $M^*$ is a superlimit model, for any linear order $I$ of cardinality $\lambda$, $EM_{\tau(T)}(I, \Phi_\omega)$ is the direct limit of $(EM_{\tau(T)}(J, \Phi_n) : J \subseteq I$ finite), each isomorphic to $M^*$, so as we have assumed that $M^*$ is a superlimit model it follows that $EM_{\tau(T)}(I, \Phi_\omega)$ is isomorphic
to $M^*$. But by [Sh:300, III] or [Sh:E59] which may eventually be [Sh:c, III] there are $2^\lambda$ many pairwise non-isomorphic models of this form varying $I$ on the linear orders of cardinality $\lambda$, contradiction.

2) First assume $\lambda = \lambda^{\aleph_0}$. Let $\tau \subseteq \tau^T$ be countable such that $T'' = T \cap L(\tau)$ is not superstable. Clearly if $M^*$ is $(\lambda, \aleph_0)$-limit model then $M^* \upharpoonright \tau'$ is not $\aleph_1$-saturated. [Why? As in [Sh:a, Ch.VI, 2] \{\lambda\} orders of cardinality $\lambda$.

Also not fork over $\aleph_0$ is some $\bar{a}$ but by [Sh:c, VIII], or see [Sh:E59] assuming $\alpha < \beta < \mu$:

Hence by [Sh:c, VIII], or see [Sh:E59] assuming $M^*$ is a universal model of $T$ of cardinality $\lambda$:

@1 there is $\Phi$ such that

(a) $\Phi$ is proper for $K^{\tau^T}_I$, $\tau^T \subseteq \tau(\Phi)$, $|\tau(\Phi)| = \lambda \geq |T| + \aleph_0$

(b) for $I \subseteq \tau^T$, $EM(\tau, I, \Phi)$ is a model of $T$ and $I \subseteq J \Rightarrow EM(I, \Phi) \prec EM(J, \Phi)$

(c) for some two-place function symbol $F$ if for $I \in K^{\tau^T}_I$ and $\eta \in P^I$, $I$ a subtruct of $\tau^*\lambda$ for transparency we let $I_{\eta, n} = \{F(a_n, a_{\eta}) : \nu \in I\}$ then $EM(\tau, I_{\eta, n}) = EM(\tau, I_{\eta, n})$ if $\eta \in \eta_n \in \nu_n$.

Also

@2 if $\Phi_1$ satisfies (a),(b),(c) of @1 and $M$ is a universal model of $T$ then there is $\Phi_2$ satisfying (a),(b),(c) of @2 and $\Phi_1 \prec \Phi_2$ see @3(a) and for every finitely generated $J \in K^{\tau^T}_I$, see @3(b) below, there is $M' \equiv M$ such that $EM(\tau, (\phi_1) \prec M' \prec EM(\tau, (\phi_2)$

@3 (a) we say $\Phi_1 \preceq \Phi_2$ when $\tau(\phi_1) \subseteq \tau(\phi_2)$ and $J \in K^{\tau^T}_I \Rightarrow EM(J, \phi_1) \prec EM(J, \phi_2)$

(b) we say $J \subseteq I$ is finitely generated if it has the form $\{\eta_\ell : \ell < n\} \cup \{\rho : \text{for some } n, \ell \text{ we have } \rho \in P^I_n \text{ and } \rho < \eta_\ell\}$ for some $\eta_0, \ldots, \eta_{n-1} \in P^I_n$

@4 if $M \in EC_3(T)$ is superlimit (or just weakly S-limit, $S \subseteq \lambda^+$ stationary) then there is $\Phi$ as in @1 above such that $EM(\tau, (\phi_1) \equiv M$ for every finitely generated $J \in K^{\tau^T}_I$

@5 we fix $\Phi$ as in @4 for $M \in EC_3(T)$ superlimit.
Hence (mainly by clause (b) of $\oplus_{2,1}$ and $\oplus_{2,4}$ as in the proof of part (1))

$\oplus_3$ if $I \in K^{\omega}_{\kappa}$ has cardinality $\leq \lambda$ then $\text{EM}_{\tau}(\Phi)(I, \Phi)$ is isomorphic to $M^*$.

Now by [Sh:460], we can find regular uncountable $\kappa < \beth_\omega$ such that $\lambda = \lambda[\kappa]$, see Definition 3.3.

Let $S = \{ \delta < \kappa : \text{cf}(\delta) = \aleph_0 \}$ and $\eta = \langle \eta_\delta : \delta \in S \rangle$ be such that $\eta_\delta$ an increasing sequence of length $\omega$ with limit $\delta$.

For a model $M$ of $T$ let $\text{OB}_\eta(M) = \{ a : a = \langle a_\eta, \alpha : \delta \in W \text{ and } \alpha < \kappa \rangle, W \subseteq S \}
\text{ and in }$ $M$ they are as in $\oplus_1(b), (d)$.

For $a \in \text{OB}_\eta(M)$ let $W[a]$ be $W$ as above and let
\[
\Xi(a, M) = \{ \eta \in \omega_\kappa : \text{ there is an indiscernible set}
\]
\[
I = \{ a, \alpha : \alpha < \kappa \} \text{ in } M \text{ such that for every } n
\]
\[
\text{ for some } \delta \in W[a], \eta \upharpoonright n = \eta_\delta \upharpoonright n \text{ and}
\]
\[
\text{Av}_{\Delta_\alpha}(M, I) = \text{Av}_{\Delta_\alpha}(M, \{ a_\eta, \alpha : \alpha < \kappa \}).
\]

Clearly

$\oplus_4$

(a) if $M \prec N$ then $\text{OB}_\eta(M) \subseteq \text{OB}_\eta(N)$

(b) if $M \prec N$ and $a \in \text{OB}_\eta(M)$ then $\Xi(a, M) \subseteq \Xi(a, N)$.

Now by the choice of $\kappa$ it should be clear that

$\oplus_5$ if $M \models T$ is of cardinality $\lambda$ then we can find an elementary extension $N$ of $M$ of cardinality $\lambda$ such that for every $a \in \text{OB}_\eta(M)$ with $W[a]$ a stationary subset of $\kappa$, for some stationary $W'' \subseteq W[a]$ the set $\Xi[a, N]$ includes $\{ \eta \in \omega_\kappa : (\forall n)(\exists \delta \in W'')(\eta \upharpoonright n = \eta_\delta \upharpoonright n) \}$, (moreover we can even find $\varepsilon^* < \kappa$ and $W_\varepsilon \subseteq W$ for $\varepsilon < \varepsilon^*$ satisfying $W[a] = \cup \{ W_\varepsilon : \varepsilon < \varepsilon^* \}$

$\oplus_6$ we can find $M \in \text{EC}_\lambda(T)$ isomorphic to $M^*$ such that for every $a \in \text{OB}_\eta(M)$ with $W[a]$ a stationary subset of $\kappa$, we can find a stationary subset $W''$ of $W[a]$ such that the set $\Xi[a, M]$ includes $\{ \eta \in \omega_\mu : (\forall n)(\exists \delta \in W''')(\eta \upharpoonright n = \eta_\delta \upharpoonright n) \}.$

[Why? We choose $(M_i, N_i)$ for $i < \kappa^+$ such that

(a) $M_i \in \text{EC}_\lambda(T)$ is $\prec$-increasing continuous

(a) $M_{i+1}$ is isomorphic to $M^*$

(a) $M_i \prec N_i \prec M_{i+1}$

(a) $(M_i, N_i)$ are like $(M, N)$ in $\oplus_5$.

Now $M = \cup \{ M_i : i < \kappa^+ \}$ is as required.

Now the model $M$ is isomorphic to $M^*$ as $M^*$ is superlimit.]

Now the model from $\oplus_6$ is not isomorphic to $M' = \text{EM}_{\tau}(\Phi)(\omega^\kappa \cup \{ \eta_\delta : \delta \in S \}, \Phi)$ where $\Phi$ is from $\oplus_{2.1}$. But $M' \cong M^*$ by $\oplus_3$.

Together we are done.

$\square_{3.1}$

The following claim says in particular that if some not unreasonable pcf conjectures holds, the conclusion holds for every $\lambda \geq 2^{\aleph_0}$.
Claim 3.4. Assume $T$ is stable not superstable, $\lambda \geq |T|$ and $\lambda \geq \kappa = \text{cf}(\kappa ) > \aleph_0$. 
1) $T$ has no $(\lambda, \kappa )$-superlimit model provided that $\kappa = \text{cf}(\kappa ) > \aleph_0, \lambda \geq \kappa^{\aleph_0}$ and $\lambda = \bigcup D(\lambda ) := \text{Min}\{ |\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\kappa} \}$ and for every $f : \kappa \rightarrow \lambda$ for some $u \in \mathcal{P}$ we have $\{ \alpha < \kappa : f(\alpha ) \in u \} \in D^+$, where $D$ is a normal filter on $\kappa$ to which $\{ \delta < \kappa : \text{cf}(\delta ) = \aleph_0 \}$ belongs.

2) Similarly if $\lambda \geq 2^{\aleph_0}$ and letting $J_0 = \{ u \subseteq \kappa : |u| \leq \aleph_0 \}$, $J_1 = \{ u \subseteq \kappa : u \cap S_\kappa^{\aleph_0} \text{ non-stationary} \}$ we have $\lambda = \bigcup_{J_0,J_1}(\lambda ) := \text{Min}\{ |\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\aleph_0} \}$, if $u \in J_1$, $f : (\kappa \setminus u) \rightarrow \lambda$ then for some countable infinite $w \subseteq \kappa(u)$ and $v \in \mathcal{P}$, $\text{Rang}(f|w) \subseteq v$.

Proof. Like 3.1(2).

Claim 3.5. 1) Assume $T$ is unstable and $\lambda \geq |T| + \beth_\omega$. Then for at most one regular $\kappa \leq \lambda$ does $T$ have a weakly $(\lambda, \kappa)$-limit model and even a weakly $(\lambda, S)$-limit model for some stationary $S \subseteq S_\kappa^\kappa$.

2) Assume $T$ is un-stable and $\lambda \geq |T| + \beth_\omega(\kappa_2)$ and $\kappa_1 = \aleph_0 < \kappa_2 = \text{cf}(\kappa_2)$. Then $T$ has no model which is a weak $(\lambda, S)$-limit where $S \subseteq \kappa$ and $S \cap S_\kappa^{\aleph_0}$ is stationary for $\ell = 1, 2$.

Proof. 1) Assume $\kappa_1 \neq \kappa_2$ form a counterexample. Let $\kappa < \beth_\omega$ be regular large enough such that $\lambda = \lambda^{[\kappa]}$, see Definition 3.3 and $\kappa \notin \{ \kappa_1, \kappa_2 \}$. Let $m, \varphi(\bar{x}, \bar{y})$ be as in the proof of 3.1

\[ (+) \text{ if } M \in E\mathcal{C}_\lambda(\ell) \text{ then there is } N \text{ such that} \]
\[ (a) N \in E\mathcal{C}_\lambda(\ell) \]
\[ (b) M \prec N \]
\[ (c) \text{ if } \bar{a} = \langle a_i : i < \kappa \rangle \in \kappa^\kappa M \text{ for } \alpha < \kappa \text{ then for some } \mathcal{U} \subseteq [\kappa]^\kappa \text{ for every uniform ultrafilter } D \text{ on } \kappa \text{ to which } \mathcal{U} \text{ belongs there is } \bar{a}_D \in \kappa^N \text{ such that } \text{tp}(\bar{a}_D, N, N) = \text{Av}(\bar{a}/D, M) = \{ \psi(\bar{x}, \bar{c}) : \psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_\ell), \bar{c} \in \ell^\varphi(\bar{x})M \text{ and } \{ \alpha < \kappa : N \models \psi[\bar{a}_\alpha, \bar{c}] \in D \} \}. \]

Similarly

\[ \mathbb{B}_1 \text{ for every function } F \text{ with domain } \{ M : M \text{ an } \prec \text{-increasing sequence of models of } T \text{ of length } < \lambda^+ \text{ each with universe } \in \lambda^+ \} \text{ such that } M_i \prec F(M) \text{ for } i < \ell \mathcal{G}(\bar{M}) \text{ and } F(M) \text{ has universe } \in \lambda^+ \text{ there is a sequence } \langle M_\varepsilon : \varepsilon < \lambda^+ \rangle \text{ obeying } F \text{ such that: for every } \varepsilon < \lambda^+ \text{ and } \bar{a} \in \kappa^\kappa(M_\varepsilon) \text{ for } \alpha < \kappa, \text{ there is } \mathcal{U} \subseteq [\kappa]^\kappa \text{ such that for every ultrafilter } D \text{ on } \kappa \text{ to which } \mathcal{U} \text{ belongs, for every } \zeta \in (\varepsilon, \lambda^+) \text{ there is } \bar{a}_{D,\varepsilon} \in \kappa^{M_{\zeta+1}} \text{ realizing } \text{Av}(\bar{a}/D, M_\zeta) \text{ in } M_{\zeta+1}. \]

Hence

\[ \mathbb{B}_2 \text{ for } (M_\alpha : \alpha < \lambda^+) \text{ as in } \mathbb{B}_1 \text{ for every limit } \delta < \lambda^+ \text{ of cofinality } \neq \kappa \text{ for every } \bar{a} = \langle a_i : i < \kappa \rangle \in \kappa^\kappa(M_\delta), \text{ there is } \mathcal{U} \subseteq [\kappa]^\kappa \text{ such that for every ultrafilter } D \text{ on } \kappa \text{ to which } \mathcal{U} \text{ belongs, there is a sequence } \langle b_\varepsilon : \varepsilon < \text{cf}(\delta) \rangle \in \mathcal{E}(\delta)(\kappa^\kappa(M_\delta)) \text{ such that for every } \psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_\gamma) \text{ and } \bar{c} \in \ell^\varphi(\bar{x})M_\delta \text{ for every } \varepsilon < \text{cf}(\delta) \text{ large enough, } M_\delta \models \psi[b_\varepsilon, \bar{c}] \text{ iff } \psi(\bar{x}, \bar{c}) \in \text{Av}(\bar{a}/D, M_\delta). \]

The rest should be clear.

2) Combine the above and the proof of 3.1(2).
REVIEWS


[Sh:88r] Saharon Shelah, Abstract elementary classes near $\aleph_1$, Chapter I. 0705.4137. arxiv:0705.4137.


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