THE NONSTATIONARY IDEAL ON $P_\kappa(\lambda)$
FOR $\lambda$ SINGULAR

Pierre MATET * and Saharon SHELAH †

Abstract

We give a new characterization of the nonstationary ideal on $P_\kappa(\lambda)$ in the case when $\kappa$ is a regular uncountable cardinal and $\lambda$ a singular strong limit cardinal of cofinality at least $\kappa$.

1 Introduction

Let $\kappa$ be a regular uncountable cardinal and $\lambda \geq \kappa$ be a cardinal.

As [10] and [11] of which it is a continuation, this paper investigates ideals on $P_\kappa(\lambda)$ with some degree of normality. For $\delta \leq \lambda$, let $NS^\delta_{\kappa,\lambda}$ denotes the least $\delta$-normal ideal on $P_\kappa(\lambda)$. Thus $NS^\delta_{\kappa,\lambda}$ = the noncofinal ideal $I_{\kappa,\lambda}$ for any $\delta < \kappa$, and $NS^{\lambda}_{\kappa,\lambda}$ = the nonstationary ideal $NS_{\kappa,\lambda}$. $NSS_{\kappa,\lambda}$ denotes the least seminormal ideal on $P_\kappa(\lambda)$. It is simple to see that $NSS_{\kappa,\lambda} = NS_{\kappa,\lambda}$ in case $\operatorname{cf}(\lambda) < \kappa$.

If $\lambda$ is regular, then by a result of Abe [1], $NSS_{\kappa,\lambda} = \bigcup_{\delta < \lambda} NS^\delta_{\kappa,\lambda}$.

One problem we address in the paper is whether for $\lambda > \kappa$ $NS_{\kappa,\lambda}$ is the restriction of a smaller ideal, i.e. whether $NS_{\kappa,\lambda} = J|A$ for some ideal $J \subset NS_{\kappa,\lambda}$ and some $A \in NSS_{\kappa,\lambda}$. The question as stated has a positive answer (see [2]) with $J = \nabla^\lambda I_{\kappa,\lambda}$. By a result of Abe [1] we can also take $J = NSS_{\kappa,\lambda}$ in case $\kappa \leq \operatorname{cf}(\lambda) < \lambda$. We investigate the possibility of taking $J = \bigcup_{\delta < \xi} NS^\delta_{\kappa,\lambda}$ for some $\xi \leq \lambda$. If $\lambda$ is regular, no such $J$ will work since then, by an argument of [11], there is no $A$ such that $NS_{\kappa,\lambda} = NSS_{\kappa,\lambda} | A$.

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Let $H_{\kappa,\lambda}$ assert that $\text{cof}(\text{NS}_{\kappa,\tau}) \leq \lambda$ for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$, where $\text{cof}(\text{NS}_{\kappa,\tau}^{\delta})$ denotes the reduced cofinality of $\text{NS}_{\kappa,\tau}^{\delta}$. Clearly, $H_{\kappa,\lambda}$ follows from $2^{<\lambda} = \lambda$. But there are other situations in which $H_{\kappa,\lambda}$ holds. For instance, if in $V$, GCH holds, $\lambda$ is a limit cardinal, $\chi$ is a regular uncountable cardinal less than $\kappa$, and $P$ is the forcing notion to add $\lambda^+$ Cohen subsets of $\chi$, then in $V^P, 2^\chi > \lambda$ but, by results of [11], for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$, $\text{cof}(\text{NS}_{\kappa,\tau}) = \tau^+$ and hence $\text{cof}(\text{NS}_{\kappa,\tau}) \leq \lambda$.

It is known ([16], [10]) that if $\text{cf}(\lambda) < \kappa$, then $H_{\kappa,\lambda}$ holds just in case $\text{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$ for some $A$. We will prove the following.

**Theorem 1.1.** Suppose that $\kappa \leq \text{cf}(\lambda) < \lambda$ and $H_{\kappa,\lambda}$ holds. Then (a) $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\text{cf}(\lambda)}|A$ for some $A$, but (b) there is no $B$ such that $\text{NS}_{\kappa,\lambda} = \left( \bigcup_{\delta < \text{cf}(\lambda)} \text{NS}_{\kappa,\lambda}^{\delta} \right)|B$.

It is not known whether the converse holds:

**Question.** Suppose that $\kappa \leq \text{cf}(\lambda) < \lambda$ and $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\text{cf}(\lambda)}|A$ for some $A$. Does it follow that $H_{\kappa,\lambda}$ holds?

If $\lambda$ is singular and $H_{\kappa,\lambda}$ holds, then by the results above $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\text{cf}(\lambda)}|A$ for some $A$. The following problem is open.

**Question.** Is it consistent that “$\lambda$ is singular but $\text{NS}_{\kappa,\lambda} \neq \text{NS}_{\kappa,\lambda}^{\delta}|A$ for every $\delta < \lambda$ and every $A \in \text{NS}_{\kappa,\lambda}$”?

For any infinite cardinal $\tau < \lambda$, let $u(\tau, \lambda)$ be the least size of any cofinal subset of $(\mathcal{P}_\tau(\lambda), \subseteq)$.

Now suppose $\kappa \leq \text{cf}(\lambda) < \lambda$. Then by results of [10], there is no $A$ such that $\text{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$. And it is shown in [11] that for any $\delta$ such that $\kappa \leq \delta < \text{cf}(\lambda)$ and $u(\delta^+, \lambda) = \lambda$, there is no $A$ such that $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\delta}|A$. Thus assuming Shelah’s Strong Hypothesis (SSH), $\text{NS}_{\kappa,\lambda} \neq \text{NS}_{\kappa,\lambda}^{\delta}|A$ for every $\delta < \text{cf}(\lambda)$ and every $A \in \text{NS}_{\kappa,\lambda}$.

**Question.** Is it consistent relative to some large cardinal that “$\kappa < \text{cf}(\lambda) < \lambda$ and $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\delta}|A$ for some $\delta < \text{cf}(\lambda)$ and some $A \in \text{NS}_{\kappa,\lambda}$”?

Another problem we consider is whether $\text{NS}_{\kappa,\lambda}^\delta$ is nowhere precipitous, where $\delta \leq \lambda$. As shown by Matsubara and Shioya [14], $I_{\kappa,\lambda}$ is nowhere precipitous, and in fact so is any ideal $J$ on $\mathcal{P}_\kappa(\lambda)$ of cofinality $u(\kappa, \lambda)$. Thus for every ideal $J$ on $\mathcal{P}_\kappa(\lambda)$,

$$\text{cof}(J) \leq \lambda \Rightarrow \text{cof}(J) = u(\kappa, \lambda) \Rightarrow J \text{ is nowhere precipitous}.$$  

We establish the following.
Proposition 1.2. Suppose that $\mathcal{H}_{\kappa, \lambda}$ holds, and let $\xi > \kappa$ be such that
\begin{itemize}
  \item $\xi$ is either a successor ordinal, or a limit ordinal of cofinality at least $\kappa$ ;
  \item $\xi \leq \eta$, where $\eta$ equals $\lambda + 1$ if $\text{cf}(\lambda) < \kappa$, and $\text{cf}(\lambda)$ otherwise.
\end{itemize}
Then $\text{cof}(\bigcup_{\delta < \xi} \text{NS}^\delta_{\kappa, \lambda}) \leq \lambda$.

It follows from Theorem 1.1 and Proposition 1.2 that if $\mathcal{H}_{\kappa, \lambda}$ holds, then $\text{NSS}_{\kappa, \lambda}|_A = \text{NS}_{\kappa, \lambda}^\delta|_A$ for some $A \in \text{NS}_{\kappa, \lambda}^*$, where $\delta$ equals $\text{cf}(\lambda)$ if $\kappa \leq \text{cf}(\lambda) < \lambda$, and 0 otherwise.

Let us next consider cases when $\kappa \leq \delta \leq \lambda$ and $\text{cof}(\text{NS}_{\kappa, \lambda}^\delta) > u(\kappa, \lambda)$. Goldring [7] and the second author proved that if $\lambda$ is regular and $\mu > \lambda$ is Woodin, then in $V^{\text{Col}(\lambda, < \mu)}$, $\text{NS}_{\kappa, \lambda}$ is precipitous. On the other hand Matsubara and the second author [13] showed (1) that if $\lambda$ is a strong limit cardinal with $\kappa \leq \text{cf}(\lambda) < \lambda$, then $\text{NS}_{\kappa, \lambda}$ is nowhere precipitous. We establish the following.

Theorem 1.3. Let $\sigma$ be a cardinal such that $\kappa \leq \text{cf}(\lambda) \leq \sigma < \lambda$ Then the following hold:

(i) If $\sigma = \text{cf}(\lambda)$ and $\tau \text{cf}(\lambda) < \lambda$ for every cardinal $\tau < \lambda$, then $\text{NS}_{\kappa, \lambda}^\sigma$ is nowhere precipitous.

(ii) If $\text{cf}(\lambda) < \sigma$ and $\tau \text{cf}(\kappa, \sigma) < \lambda$ for every cardinal $\tau < \lambda$, where $c(\kappa, \sigma)$ denotes the least size of any closed unbounded subset of $P_\kappa(\sigma)$, then $\text{NS}_{\kappa, \lambda}^\sigma$ is nowhere precipitous.

Note that if $\kappa \leq \text{cf}(\lambda) \leq \sigma < \lambda$ and the hypothesis of (i) (respectively, (ii)) of Theorem 1.3 holds, then $\lambda^{< \text{cf}(\lambda)} = \lambda$, so by results of [10],
\[
\text{cof}(\text{NS}_{\kappa, \lambda}^\sigma) \geq \text{cof}(\text{NS}_{\kappa, \lambda}^\sigma) > \lambda = u(\kappa, \lambda).
\]
By combining Theorems 1.1 and 1.3, we obtain the following.

Theorem 1.4. Suppose that $\mathcal{H}_{\kappa, \lambda}$ holds, $\kappa \leq \text{cf}(\lambda) < \lambda$, and $\tau \text{cf}(\lambda) < \lambda$ for every cardinal $\tau < \lambda$. Then $\text{NS}_{\kappa, \lambda}$ is nowhere precipitous.

It is not clear whether Theorem 1.4 constitutes a real improvement in comparison to the result of Matsubara and the second author quoted above.

Question. Suppose that $\mathcal{H}_{\kappa, \lambda}$ holds, $\kappa \leq \text{cf}(\lambda) < \lambda$, and $\tau \text{cf}(\lambda) < \lambda$ for every cardinal $\tau < \lambda$. Does it then follow that $\lambda$ is a strong limit cardinal?

\footnote{1 At some point the first author claimed to have found an error in the proof but it turned out that the mistake was his.}

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The paper is organized as follows. Section 2 collects basic definitions and facts concerning ideals on $P_\kappa(\lambda)$. It is shown in Section 3 that $\text{cof}(\text{NSS}_\kappa^\lambda)\) is a non-decreasing function of $\pi$. In Section 4 we establish that if $\lambda$ is regular, then $\text{cof}(\text{NSS}_\kappa^\lambda) = \lambda$ just in case $\mathcal{H}_{\kappa, \lambda}$ holds. In Section 5, Proposition 1.2 is proved. In Section 6 we show that it is consistent relative to a large cardinal that “$\lambda$ is regular and $\text{cof}(\text{NSS}_\kappa^\lambda[A]) < \lambda$ for some $A$”. It is shown in Section 7 that if $\lambda$ is singular and $\mathcal{H}_{\kappa, \lambda}$ holds, then $\text{NSS}_\kappa^\lambda = \text{NSS}^{\text{cf}(\lambda)}_\kappa[A]$ for some $A$. Finally in Section 8 we prove Theorem 1.3 and note that it is consistent relative to a large cardinal that “there is an ideal $J$ on $P_\kappa(\lambda)$ such that $\text{cof}(J) > \lambda$ but $\text{cof}(J) = u(\kappa, \lambda)$.”

2 Ideals on $P_\kappa(\lambda)$

In this section we collect basic material concerning ideals on $P_\kappa(\lambda)$.

$\text{NS}_\kappa$ denotes the nonstationary ideal on $\kappa$.

For a set $A$ and a cardinal $\rho$, let $P_\rho(A) = \{a \subseteq A : |a| < \rho\}$.

Given four cardinals $\tau, \rho, \chi$ and $\sigma$, we define $\text{cov}(\tau, \rho, \chi, \sigma)$ as follows. If there is $X \subseteq P_\rho(\tau)$ with the property that for any $a \in P_\rho(\tau)$, we may find $Q \in P_\rho(X)$ with $a \subseteq \bigcup Q$, we let $\text{cov}(\tau, \rho, \chi, \sigma) = \text{the least cardinality of any such } X$.

Otherwise we let $\text{cov}(\tau, \rho, \chi, \sigma) = 0$.

We let $\text{cov}(\tau, \rho, \chi, \sigma) = \text{u}(\tau, \chi)$ in case $\rho = \chi$ and $\sigma = 2$.

**FACT 2.1.** ([15, pp. 85-86]) Let $\tau, \rho, \chi$ and $\sigma$ be four cardinals such that $\tau \geq \rho \geq \chi \geq \omega$ and $\chi \geq \sigma \geq 2$. Then the following hold:

(i) If $\tau > \rho$, then $\text{cov}(\tau, \rho, \chi, \sigma) \geq \tau$.

(ii) $\text{cov}(\tau, \rho, \chi, \sigma) = \text{cov}(\tau, \rho, \chi, \text{max}\{\omega, \sigma\})$.

(iii) $\text{cov}(\tau^+, \rho, \chi, \sigma) = \text{max}\{\tau^+, \text{cov}(\tau, \rho, \chi, \sigma)\}$.

(iv) If $\tau > \rho$ and $\text{cf}(\tau) < \sigma = \text{cf}(\sigma)$, then $\text{cov}(\tau, \rho, \chi, \sigma) = \text{sup}\{\text{cov}(\tau', \rho, \chi, \sigma) : \rho \leq \tau' < \tau\}$.

(v) If $\tau$ is a limit cardinal such that $\tau > \rho$ and $\text{cf}(\tau) \geq \chi$, then $\text{cov}(\tau, \rho, \chi, \sigma) = \text{sup}\{\text{cov}(\tau', \rho, \chi, \sigma) : \rho \leq \tau' < \tau\}$.

$I_{\kappa, \lambda}$ denotes the set of all $A \subseteq P_\kappa(\lambda)$ such that $\{a \in A : b \subseteq a\} = \emptyset$ for some $a \in P_\kappa(\lambda)$.

By an ideal on $P_\kappa(\lambda)$, we mean a collection $J$ of subsets of $P_\kappa(\lambda)$ that is closed under subsets (i.e. $P(A) \subseteq J$ for all $A \in J$), $\kappa$-complete (i.e. $\bigcup X \in J$ for every $X \in P_\kappa(J)$), fine (i.e. $I_{\kappa, \lambda} \subseteq J$) and proper (i.e. $P_\kappa(\lambda) \not\in J$).

Given an ideal $J$ on $P_\kappa(\lambda)$, let $J^+ = \{A \subseteq P_\kappa(\lambda) : A \not\in J\}$ and $J^* = \{A \subseteq P_\kappa(\lambda) : P_\kappa(\lambda) \setminus A \in J\}$. For $A \in J^+$, let $J|A = \{B \subseteq P_\kappa(\lambda) : B \cap A \in J\}$. Given a cardinal $\chi > \lambda$ and $f : P_\kappa(\chi) \to P_\kappa(\lambda)$, we let

$f(J) = \{X \subseteq P_\kappa(\lambda) : f^{-1}(X) \in J\}$. 

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\( M_J \) denotes the collection of all maximal antichains in the partially ordered set 
\((J^+, \subseteq)\), i.e. of all \( Q \subseteq J^+ \) such that

- \( A \cap B \in J \) for any distinct \( A, B \in Q \);
- for every \( C \in J^+ \), there is \( A \in Q \) with \( A \cap C \in J^+ \).

For a cardinal \( \rho \), \( J \) is \( \rho \)-saturated if \( |Q| < \rho \) for every \( Q \in M_J \).

\( \text{cof}(J) \) denotes the least cardinality of any \( X \subseteq J \) such that \( J = \bigcup_{A \in X} P(A) \).

\( \text{cof}(J) \) denotes the least size of any \( Y \subseteq J \) with the property that for every \( A \in J \), there is \( y \in P_\kappa(Y) \) with \( A \subseteq \bigcup y \).

\( \text{non}(J) \) denotes the least cardinality of any \( A \in J^+ \).

Note that \( \text{cof}(J) \geq \text{non}(J) \geq \text{non}(I_{\kappa, \lambda}) = u(\kappa, \lambda) \).

The following is well-known (see e.g. [10] and [11]).

**FACT 2.2.**

(i) \( \lambda^\kappa = \max\{2^{<\kappa}, u(\kappa, \lambda)\} \).

(ii) \( \text{cof}(I_{\kappa, \lambda}) = \lambda \).

(iii) Let \( J \) be an ideal on \( P_\kappa(\lambda) \) such that \( \text{cof}(J) \leq \lambda \). Then \( \text{cof}(J) = u(\kappa, \lambda) \).

Shelah's Strong Hypothesis (SSH) asserts that for any two uncountable cardinals \( \chi \) and \( \rho \) with \( \chi \geq \rho = \text{cf}(\rho) \), \( u(\rho, \chi) \) equals \( \chi \) if \( \text{cf}(\chi) \geq \rho \), and \( \chi^+ \) otherwise.

**FACT 2.3.** ([8])

(i) Suppose that there is a \( \pi \)-saturated ideal on \( P_\nu(\lambda) \), where \( \pi \) and \( \nu \) are two cardinals such that \( \omega < \nu = \text{cf}(\nu) \leq \lambda \) and \( \pi < \nu \cap \kappa^+ \). Then \( u(\kappa, \lambda) \) equals \( \lambda \) if \( \text{cf}(\lambda) \geq \kappa \), and \( \lambda^+ \) otherwise.

(ii) Suppose that there is a regular uncountable cardinal \( \nu < \lambda \) that is mildly \( \pi^+ \)-ineffable for every cardinal \( \pi \) with \( \nu \leq \pi < \lambda \). Then the following hold:

- \( u(\kappa, \lambda) \) equals \( \lambda \) if \( \text{cf}(\lambda) \geq \kappa \), and \( \lambda^+ \) if \( \omega < \text{cf}(\lambda) < \kappa \).

- \( \text{cov}(\lambda, \kappa, \kappa, \omega_1) = \lambda \) if \( \text{cf}(\lambda) = \omega \).

Numerous variations on the original notion of ideal normality have been considered over the years. One such variant is the concept of \( \delta \)-normality which has been studied by Abe [1].

Let \( \delta \leq \lambda \). An ideal \( J \) on \( P_\kappa(\lambda) \) is \( \delta \)-normal if given \( A \in J^+ \) and \( f : A \to \delta \) with the property that \( f(a) \in a \) for all \( a \in A \), there exists \( B \in J^+ \cap P(A) \) such that \( f \) is constant on \( B \).

\( \text{NS}_{\kappa, \lambda}^\delta \) denotes the smallest \( \delta \)-normal ideal on \( P_\kappa(\lambda) \).

Note that \( \lambda \)-normality is the same as normality, so \( \text{NS}_{\kappa, \lambda}^\lambda = \text{NS}_{\kappa, \lambda} \).
\( c(\kappa, \lambda) \) denotes the least size of any closed unbounded subset of \( P_\kappa(\lambda) \).

**FACT 2.4.**

(i) ([1]) Let \( \delta \) be an ordinal such that \( \delta + \kappa \leq \lambda \). Then \( \text{NS}^{\delta + \kappa}_{\kappa, \lambda} \setminus \text{NS}^\delta_{\kappa, \lambda} \neq \emptyset \).

(ii) ([11]) Suppose \( \kappa \leq \delta < \lambda \). Then \( \text{NS}^\delta_{\kappa, \lambda} = \text{NS}^{[\delta]}_{\kappa, \lambda} A \) for some \( A \).

(iii) ([11]) Let \( \delta \) and \( \eta \) be two ordinals such that \( |\delta| < |\eta| < \lambda \) and \( \kappa \leq \eta \). Then there is no \( A \) such that \( \text{NS}^\eta_{\kappa, \lambda} = \text{NS}^\delta_{\kappa, \lambda} A \).

**FACT 2.5.**

(i) ([10]) \( \text{cof}(\text{NS}^\delta_{\kappa, \lambda}) \geq \lambda \) for every \( \delta \leq \lambda \).

(ii) ([8], [10]) Let \( \delta \leq \lambda \). Then \( \text{cof}(\text{NS}^\delta_{\kappa, \lambda}|A) = \text{cof}(\text{NS}^\delta_{\kappa, \lambda}) \) for every \( A \in \text{NS}^*_\kappa, \lambda \).

(iii) ([10]) \( \text{cof}(\text{NS}^\delta_{\kappa, \lambda}) \geq \text{cof}(\text{NS}^\rho_{\kappa, \lambda}) \) for every cardinal \( \rho \) with \( \kappa \leq \rho < \lambda \).

(iv) ([10]) Suppose \( \text{cf}(\lambda) \geq \kappa \). Then \( \text{cof}(\text{NS}^\delta_{\kappa, \lambda}) > \lambda \).

The concept of \( [\delta]^{<\theta} \)-normality generalizes that of \( \delta \)-normality.

Let \( \delta \leq \lambda \), and let \( \theta \) be a cardinal with \( \theta \leq \kappa \). An ideal \( J \) on \( P_\kappa(\lambda) \) is \( [\delta]^{<\theta} \)-normal if given \( A \in J^+ \) and \( f : A \rightarrow P_\theta(\delta) \) with the property that \( f(a) \in P_{|\theta \cap a \cap \delta|} \) for all \( a \in A \), there exists \( B \in J^+ \cap P(A) \) such that \( f \) is constant on \( B \).

We set \( \overline{\theta} = \theta \) if \( \theta < \kappa \), or \( \overline{\theta} = \kappa \) and \( \kappa \) is a limit cardinal, and \( \overline{\theta} = \nu \) if \( \theta = \kappa = \nu^+ \).

**FACT 2.6.** ([11])

(i) Suppose that \( \delta < \kappa \), or \( \theta < \kappa \), or \( \kappa \) is not a limit cardinal. Then there exists a \( [\delta]^{<\theta} \)-normal ideal on \( P_\kappa(\lambda) \) if and only if \( |P_\theta(\rho)| < \kappa \) for every cardinal \( \rho < \kappa \cap (\delta + 1) \).

(ii) Suppose that \( \delta \geq \kappa, \theta = \kappa \) and \( \kappa \) is a limit cardinal. Then there exists a \( [\delta]^{<\theta} \)-normal ideal on \( P_\kappa(\lambda) \) if and only if \( \kappa \) is a Mahlo cardinal.

(iii) Suppose that there exists a \( [\kappa]^{<\theta} \)-normal ideal on \( P_\kappa(\lambda) \). Then \( \kappa^{<\theta} = \kappa \), and \( (\pi^{<\theta})^{<\theta} = \pi^{<\theta} \) for every cardinal \( \pi > \kappa \).

Assuming that there exists a \( [\delta]^{<\theta} \)-normal ideal on \( P_\kappa(\lambda) \), \( \text{NS}^{[\delta]^{<\theta}}_{\kappa, \lambda} \) denotes the smallest such ideal.

**FACT 2.7.** ([11])

(i) Suppose that \( \theta < 2 \) or \( \delta < \kappa \). Then \( \text{NS}^{[\delta]^{<\theta}}_{\kappa, \lambda} = I_{\kappa, \lambda} \).
(ii) Suppose that \( 2 \leq \theta \leq \omega \). Then \( \text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta} = \text{NS}_{\kappa, \lambda}^{\delta} \).

(iii) Suppose that \( [\delta]^{\vartheta} = [\eta]^{\tau} \), where \( \kappa \leq \eta \leq \lambda \) and \( \pi \) is a cardinal with \( 2 \leq \pi \leq \kappa \). Then \( \text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta} | A = \text{NS}_{\kappa, \lambda}^{[\eta]<\tau} | A \) for some \( A \in \langle \text{NS}_{\kappa, \lambda}^{[\gamma]<\rho} \rangle^* \), where \( \gamma = \max\{\delta, \eta\} \) and \( \rho = \max\{\theta, \pi\} \).

Given an ordinal \( \eta \), a cardinal \( \pi \) and \( f : P_\pi(\eta) \to P_\kappa(\lambda) \), let \( C(f, \kappa, \lambda) \) be the set of all \( a \in P_\kappa(\lambda) \) such that \( a \cap \pi \neq \emptyset \) and \( f(e) \subseteq a \) for every \( e \in P_{|a \cap \pi|}(a \cap \eta) \).

**FACT 2.8.** ([11]) Suppose that \( A \subseteq P_\kappa(\lambda) \), \( \kappa \leq \delta \leq \lambda \), and \( \theta \) is a cardinal with \( 2 \leq \theta \leq \kappa \). Then the following are equivalent:

(i) \( A \in \text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta} \).

(ii) \( A \cap C(f, \kappa, \lambda) = \emptyset \) for some \( f : P_{\max(\bar{\gamma}, 3)}(\delta) \to P_\kappa(\lambda) \).

(iii) \( A \cap \{ a \in C(g, \kappa, \lambda) : a \cap \kappa \in \kappa \} = \emptyset \) for some \( g : P_{\max(\bar{\gamma}, 3)}(\delta) \to \mathcal{P}_\lambda(\delta) \).

**FACT 2.9.** ([10]) Let \( \chi \) and \( \theta \) be two cardinals such that \( 2 \leq \theta \leq \kappa \leq \chi \leq \lambda \). Then the following hold:

(i) Let \( J \) be a \( [\chi]^{<\vartheta} \)-normal ideal on \( P_\kappa(\lambda) \). Then either \( \text{cf}(\overline{\text{col}}(J)) < \kappa \), or \( \text{cf}(\overline{\text{col}}(J)) > \chi^{<\vartheta} \).

(ii) If \( \chi^{<\vartheta} < \lambda \), then \( \overline{\text{col}}(\text{NS}_{\kappa, \lambda}^{[\chi]<\vartheta}) \geq \lambda \).

**FACT 2.10.** ([10], [11]) Suppose that \( \kappa \leq \delta < \lambda \), and \( \theta \) is a cardinal with \( 2 \leq \theta \leq \kappa \). Then the following hold:

(i) \( \overline{\text{cf}}(\text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta}) = \max(\overline{\text{col}}(\text{NS}_{\kappa, \delta}^{[\delta]<\vartheta}), \text{cov}(\kappa, (\delta)^{<\vartheta} + , (\delta)^{<\vartheta} + , \kappa)) \) and \( \overline{\text{cf}}(\text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta}) = \max(\overline{\text{col}}(\text{NS}_{\kappa, \delta}^{[\delta]<\vartheta}), \text{cov}(\kappa, (\delta)^{<\vartheta} + , (\delta)^{<\vartheta} + , 2)) \).

(ii) If \( \lambda \) is a limit cardinal and either \( \text{cf}(\lambda) < \kappa \) or \( \text{cf}(\lambda) > |\delta|^{<\vartheta} \), then \( \overline{\text{col}}(\text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta}) = \sup(\overline{\text{col}}(\text{NS}_{\kappa, \delta}^{[\delta]<\vartheta}) : \delta < \tau < \lambda) \).

For a cardinal \( \tau, \overline{\text{d}}_{\kappa, \lambda}^\tau \) denotes the smallest cardinality of any family \( F \) of functions from \( \tau \) to \( P_\kappa(\lambda) \) with the property that for any \( g : \tau \to P_\kappa(\lambda) \), there is \( f \in F \) such that \( g(\alpha) \subseteq f(\alpha) \) for every \( \alpha < \tau \).

**FACT 2.11.** ([11])

(i) For any cardinal \( \tau > 0 \), \( \text{cf}(\overline{\text{d}}_{\kappa, \lambda}^\tau) > \tau \).

(ii) Suppose that \( 0 < \delta \leq \lambda \), and \( \theta \) is a cardinal with \( 0 < \theta \leq \kappa \). Then \( \overline{\text{cf}}(\text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta} | A) = \overline{\text{d}}_{\kappa, \lambda}^{[\delta]}(\text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta}+) \) for every \( A \in \langle \text{NS}_{\kappa, \lambda}^{[\delta]<\vartheta} \rangle^* \).
Next let us recall a few facts concerning the notion of precipitousness.

An ideal \( J \) on \( P_\kappa(\lambda) \) is precipitous if whenever \( A \in J^+ \) and \( \langle Q_n : n < \omega \rangle \) is a sequence of members of \( \mathcal{M}_{J,A} \) such that \( Q_{n+1} \subseteq \bigcup_{B \in Q_n} P(B) \) for all \( n < \omega \), there exists \( f \in \prod_{n \in \omega} Q_n \) such that \( f(0) \supseteq f(1) \supseteq \ldots \) and \( \bigcap_{n<\omega} f(n) \neq \emptyset \). \( J \) is nowhere precipitous if for each \( A \in J^+ \), \( J|A \) is not precipitous.

Let \( G(\mathcal{M}) \) denote the following two-player game lasting \( \omega \) moves, with player I making the first move: I and II alternately pick members of \( J^+ \), thus building a sequence \( \langle X_n : n < \omega \rangle \), subject to the condition that \( X_0 \supseteq X_1 \supseteq \ldots \). II wins \( G(\mathcal{M}) \) just in case \( \bigcap_{n<\omega} X_n = \emptyset \).

**FACT 2.12.** ([5]) An ideal \( J \) on \( P_\kappa(\lambda) \) is nowhere precipitous if and only if II has a winning strategy in the game \( G(\mathcal{M}) \).

The following is a straightforward generalization of a result of Foreman [4]:

**PROPOSITION 2.13.** Let \( \chi \) and \( \theta \) be two cardinals such that \( \chi \leq \lambda \) and \( \theta \leq \kappa \). Then every \( [\chi]^{<\theta} \)-normal, \( (\chi^{<\theta})^+ \)-saturated ideal on \( P_\kappa(\lambda) \) is precipitous.

**FACT 2.14.** ([14]) Suppose that \( J \) is an ideal on \( P_\kappa(\lambda) \) such that \( \text{cof}(J) = \text{non}(J) \). Then \( J \) is nowhere precipitous.

Thus for an ideal \( J \) on \( P_\kappa(\lambda) \),

\[
\overline{\text{cof}(J)} \leq \lambda \Rightarrow \text{cof}(J) = u(\kappa, \lambda) \Rightarrow J \text{ is nowhere precipitous.}
\]

Let \( \tau \) be a cardinal with \( \kappa \leq \tau \leq \lambda \). It is simple to see that if GCH holds and either \( \text{cf}(\lambda) < \kappa \) or \( \tau < \text{cf}(\lambda) \), then \( \text{cof}(\text{NS}_{\kappa,\lambda}^\tau) = u(\kappa, \lambda) \). Note that if SSH holds and \( \kappa \leq \text{cf}(\lambda) \leq \tau \), then by Facts 2.5 (i) and 2.9, \( \text{cof}(\text{NS}_{\kappa,\lambda}^\tau) > u(\kappa, \lambda) \).

**PROPOSITION 2.15.** Suppose that \( \sigma \) is a strong limit cardinal with \( \text{cf}(\sigma) < \kappa < \sigma \leq \lambda \leq 2^\sigma \). Then the following hold:

(i) \( \text{cof}(\text{NS}_{\kappa,\lambda}^\tau) = u(\kappa, \lambda) \) for every cardinal \( \tau \) with \( \kappa \leq \tau \leq \sigma \).

(ii) Suppose \( 2^\lambda = 2^\sigma \). Then \( \text{cof}(\text{NS}_{\kappa,\lambda}^\tau) = u(\kappa, \lambda) \) for every cardinal \( \tau \) with \( \sigma < \tau \leq \lambda \).

**Proof.**

(i) : Let \( \tau \) be a cardinal with \( \kappa \leq \tau \leq \sigma \). If \( \tau = \lambda \), then

\[
\text{cof}(\text{NS}_{\kappa,\lambda}^\tau) \leq 2^\lambda = \lambda^{<\kappa} = u(\kappa, \lambda).
\]

Otherwise by Fact 2.10, \( \text{cof}(\text{NS}_{\kappa,\lambda}^\tau) = \max\{\text{cof}(\text{NS}_{\kappa,\tau}), u(\tau^+, \lambda)\} \leq \lambda^\tau = \sigma^\tau = \sigma^{\text{cf}(\sigma)} \leq \lambda^{<\kappa} = u(\kappa, \lambda) \).
(ii): Given a cardinal $\tau$ with $\sigma < \tau \leq \lambda$,
\[
\text{cof}(\text{NS}_{\kappa,\lambda}^\tau) \leq 2^\lambda = 2^\sigma = \sigma^{\text{cf}(\sigma)} = u(\kappa, \lambda).
\]

3 \ \text{cof}(\text{NS}_{\kappa,\lambda}^\chi)

By Fact 2.11 (ii), $\text{cof}(\text{NS}_{\kappa,\lambda}^\chi) = \text{cof}(\chi)$ for any cardinal $\chi$ with $\kappa \leq \chi \leq \lambda$. We now derive a similar formula for $\text{cof}(\text{NS}_{\kappa,\lambda}^\chi)$.

For a cardinal $\tau$, $\text{cof}(\chi)$ denotes the smallest cardinality of any family $F$ of functions from $\tau$ to $P_{\alpha}(\lambda)$ with the property that for any $g : \tau \to P_{\alpha}(\lambda)$, there is $Z \in P_{\alpha}(F)$ such that $g(\alpha) \subseteq \bigcup_{f \in Z} f(\alpha)$ for every $\alpha < \tau$.

**Lemma 3.1.** Let $\theta$ and $\chi$ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. Then $\text{cof}(\text{NS}_{\kappa,\lambda}^{\chi < \theta}) \leq \chi^\theta$.

**Proof.** Select a collection $G$ of functions from $P_{\max[\theta, 3]}(\chi)$ to $P_{\kappa}(\lambda)$ so that $|G| = \chi^\theta$ and for any $k : P_{\max[\theta, 3]}(\chi) \to P_{\kappa}(\lambda)$, there is $Z_k \in P_{\kappa}(G)$ such that $k(e) \subseteq \bigcup_{g \in Z_k} g(e)$ for all $e \in P_{\max[\theta, 3]}(\chi)$. Then clearly for each $k$,
\[
P_{\max[\theta, 3]}(\chi) \to P_{\kappa}(\lambda), \bigcap_{g \in Z_k} C(g, \kappa, \lambda) \subseteq C(k, \kappa, \lambda).
\]
Hence $\text{cof}(\text{NS}_{\kappa,\lambda}^{\chi < \theta}) \leq |G|$.

**Lemma 3.2.** Let $\theta$ and $\chi$ be two cardinals such that $\omega \leq \theta = \text{cf}(\theta) < \kappa \leq \chi \leq \lambda$. Then $\text{cof}(\text{NS}_{\kappa,\lambda}^{\chi < \theta}) = u(\theta, \text{cof}(\text{NS}_{\kappa,\lambda}^{\chi < \theta}))$.

**Proof.** Pick a collection $H$ of functions from $P_{\theta}(\chi) \to P_{3}(\lambda)$ so that $|H| = \text{cof}(\text{NS}_{\kappa,\lambda}^{\chi < \theta})$ and for any $A \in (\text{NS}_{\kappa,\lambda}^{\chi < \theta})^*$, there is $Q \in P_{\alpha}(H) \setminus \{\theta\}$ with $\{b \in \bigcap_{h \in Q} C(h, \kappa, \lambda) : b \cap \kappa \in \kappa\} \subseteq A$. Select $X \subseteq P_{\theta}(H) \setminus \{\theta\}$ so that $|X| = u(\theta, |H|)$ and for any $Z \in P_{\theta}(H)$, there is $X \in X$ with $Z \subseteq X$. For $X \in X$, define $t_X : P_{\theta}(\chi) \to P_{\kappa}(\lambda)$ by $t_X(e) = \bigcap T_{X,e}$, where
\[
T_{X,e} = \left\{b \in \bigcap_{h \in X} C(h, \kappa, \lambda) : e \cup \theta \subseteq b \text{ and } b \cap \kappa \in \kappa\right\}.
\]
Note that $t_X(e) \in T_{X,e}$, and $t_Y(e) \subseteq t_X(e)$ for all $Y \in X \cap P(X)$. Now fix $f : P_{\theta}(\chi) \to P_{\kappa}(\lambda)$. We may find $W \in P_{\theta}(X)$ such that
\[
\left\{b \in \bigcap_{h \in W} C(h, \kappa, \lambda) : b \cap \kappa \in \kappa\right\} \subseteq C(f, \kappa, \lambda),
\]
$\theta \leq |W|$ and for any $K \in P_{\theta}(W)$, there is $Z \in W$ with $\bigcup K \subseteq Z$. For $e \in P_{\theta}(\chi)$, put $b_e = \bigcup_{X \in W} t_X(e)$. Note that $b_e \cap \kappa \in \kappa$.

**Claim.** Let $k \in \bigcup W$. Then $b_e \in C(k, \kappa, \lambda)$.
Proof of Claim. Fix $d \in P(b_c \cap \chi)$. Pick $\varphi : d \rightarrow W$ so that $\beta \in t_{\varphi(\beta)}(e)$ for every $\beta \in d$. Select $Y \in W$ with $k \in Y$. There must be $Z \in W$ such that $Y \cup \{ \bigcup_{\beta \in d} \varphi(\beta) \} \subseteq Z$. Then $d \in P(t_Z(e))$ and $t_Z(e) \in C(k, \kappa, \lambda)$, so $k(d) \subseteq t_Z(e) \subseteq b_e$. This completes the proof of the claim.

Thus $b_e \in \bigcap_{b \in \cup W} C(b, \kappa, \lambda)$. Hence $b_e \in C(f, \kappa, \lambda)$, and consequently $f(e) \subseteq b_e$. □

PROPOSITION 3.3. Let $\chi$ be a cardinal with $\kappa \leq \chi \leq \lambda$. Then $\text{col}(\text{NS}_{\kappa, \lambda}^\chi) = \delta_{\kappa, \lambda}^\kappa$.

Proof. By Lemmas 3.1 and 3.2. □

COROLLARY 3.4. Let $\pi$ and $\chi$ be two cardinals such that $\kappa \leq \pi < \chi \leq \lambda$. Then $\text{col}(\text{NS}_{\pi, \kappa, \lambda}) \leq \text{col}(\text{NS}_{\kappa, \lambda}^\chi)$.

4 NSS$_{\kappa, \lambda}$

An ideal $J$ on $P_\kappa(\lambda)$ is seminormal if it is $\delta$-normal for every $\delta < \lambda$. NSS$_{\kappa, \lambda}$ denotes the smallest seminormal ideal on $P_\kappa(\lambda)$.

FACT 4.1.

(i) (Folklore) Suppose $\text{cf}(\lambda) < \kappa$. Then $\text{NS}_{\kappa, \lambda} = \text{NSS}_{\kappa, \lambda}$.

(ii) ([1]) Suppose $\kappa \leq \text{cf}(\lambda) < \lambda$. Then $\text{NS}_{\kappa, \lambda} = \text{NSS}_{\kappa, \lambda}|A$ for some $A$.

PROPOSITION 4.2. Suppose $\kappa \leq \text{cf}(\lambda) < \lambda$. Then $\text{col}(\text{NSS}_{\kappa, \lambda}) > \lambda$.

Proof. By Facts 2.5 (iv) and 4.1. □

We will see that $\text{col}(\text{NSS}_{\kappa, \lambda}) > \lambda$ needs not hold in case $\lambda$ is regular. Note that if $\lambda$ is regular, then by Fact 2.5 (iv), $\text{col}(\text{NS}_{\kappa, \lambda}) > \lambda$.

FACT 4.3. ([1]) Suppose that $\lambda$ is regular. Then $\text{NSS}_{\kappa, \lambda} = \bigcup_{\delta < \lambda} \text{NSS}_{\kappa, \lambda}^\delta$.

Proof. It is immediate that $\bigcup_{\delta < \lambda} \text{NSS}_{\kappa, \lambda}^\delta \subseteq \text{NSS}_{\kappa, \lambda}$. To show the reverse inclusion, fix $A \in (\bigcup_{\delta < \lambda} \text{NSS}_{\kappa, \lambda}^\delta)^+$. For every $a \in A$, we may find $B_\xi \in (\text{NS}_{\kappa, \lambda}^\delta)^+ \cap \text{P}(A)$ and $\gamma_\xi < \eta$ such that $f$ takes the constant value $\gamma_\xi$ on $B_\xi$. There must be $\beta < \eta$ and $Z \subseteq \{ \xi : \eta \leq \xi < \lambda \}$ such that $|Z| = \lambda$ and
\[ \gamma_\xi = \beta \text{ for all } \xi \in Z. \] Now set \( C = \bigcup_{\xi \in Z} B_\xi \). Then clearly \( C \in \left( \bigcup_{\xi \in \kappa} \text{NS}_{\kappa, \lambda}^{\xi} \right)^+ \). Moreover \( f \) is identically \( \beta \) on \( C \).

**FACT 4.4.** ([10]) Suppose that \( \theta \) is a cardinal with \( 2 \leq \theta \leq \kappa \), and \( J \) is an ideal on \( P_\kappa(\lambda) \) such that \( J \subseteq \text{NS}_{\kappa, \lambda}^{\langle \theta, \lambda \rangle} \) and \( \text{cof}(J) \leq \lambda^{<\theta} \). Then \( J|A = I_{\kappa, \lambda}|A \) for some \( A \in \left( \text{NS}_{\kappa, \lambda}^{\langle \theta, \lambda \rangle} \right)^+ \).

In particular, if \( J \subseteq \text{NS}_{\kappa, \lambda} \) and \( \text{cof}(J) \leq \lambda \), then \( J|D = I_{\kappa, \lambda}|D \) for some \( D \in \text{NS}_{\kappa, \lambda}^* \).

**FACT 4.5.** ([10]) Suppose that \( \theta \) is a cardinal with \( 2 \leq \theta \leq \kappa \), and let \( \sigma \) be the least cardinal \( \tau \) such that \( \tau < \theta \geq \lambda \). Then \( \text{cof}(I_{\kappa, \lambda}|A) \geq \sigma \) for every \( A \in \left( \text{NS}_{\kappa, \lambda}^{\langle \theta, \lambda \rangle} \right)^+ \).

**PROPOSITION 4.6.** Suppose that \( \theta \) is a cardinal with \( 2 \leq \theta \leq \kappa \), and \( J \) is an ideal on \( P_\kappa(\lambda) \) with \( J \subseteq \text{NS}_{\kappa, \lambda}^{\langle \theta, \lambda \rangle} \). Let \( \sigma \) be the least cardinal \( \tau \) such that \( \tau < \theta \geq \lambda \). Then \( \text{cof}(J) \geq \sigma \).

**Proof.** If \( \text{cof}(J) > \lambda \), there is nothing to prove. Otherwise, there is by Fact 4.4 \( A \in \left( \text{NS}_{\kappa, \lambda}^{\langle \theta, \lambda \rangle} \right)^+ \) such that \( J|A = I_{\kappa, \lambda}|A \). Then by Fact 4.5, \( \sigma \leq \text{cof}(I_{\kappa, \lambda}|A) \leq \text{cof}(J) \).

In particular, \( \text{cof}(J) \geq \lambda \) for any ideal \( J \subseteq \text{NS}_{\kappa, \lambda} \).

**FACT 4.7.** ([8])

(i) Suppose that \( \lambda \) is a successor cardinal, say \( \lambda = \nu^+ \). Then \( \text{NS}_{\kappa, \lambda}|C = I_{\kappa, \lambda}|C \) for some \( C \in \text{NS}_{\kappa, \lambda}^* \) if and only if \( \text{cof}(\text{NS}_{\kappa, \nu}) \leq \lambda \).

(ii) Suppose that \( \lambda \) is a regular limit cardinal. Then \( \text{NS}_{\kappa, \lambda}|C = I_{\kappa, \lambda}|C \) for some \( C \in \text{NS}_{\kappa, \lambda}^* \) if and only if \( \text{cof}(\text{NS}_{\kappa, \tau}) \leq \lambda = \text{cov}(\lambda, \tau^+, \tau^+, \kappa) \) for every cardinal \( \tau \) with \( \kappa \leq \tau < \lambda \).

Recall from the introduction that \( \mathcal{H}_{\kappa, \lambda} \) is said to hold if \( \text{cof}(\text{NS}_{\kappa, \tau}) \leq \lambda \) for every cardinal \( \tau \) with \( \kappa \leq \tau < \lambda \).

**PROPOSITION 4.8.** Suppose that \( \lambda \) is a regular cardinal. Then the following are equivalent:

(i) \( \mathcal{H}_{\kappa, \lambda} \) holds.

(ii) \( \text{cof}(\text{NSS}_{\kappa, \lambda}) = \lambda \).

(iii) \( \text{NSS}_{\kappa, \lambda}|C = I_{\kappa, \lambda} | C \) for some \( C \in \text{NS}_{\kappa, \lambda}^* \).
Proof. 
(i) → (ii) : By Proposition 4.6, \( \text{cof}(\text{NSS}_{\kappa,\lambda}) \geq \lambda \). For the reverse inequality, we consider two cases. First suppose that \( \lambda \) is a successor cardinal, say \( \lambda = \nu^+ \). Then by Fact 4.3 \( \text{NSS}_{\kappa,\lambda} = \bigcup_{\nu \leq \delta < \lambda} \text{NS}_{\delta}^{\kappa,\lambda} \). Now for \( \nu \leq \delta < \lambda \), \( \text{cof}(\text{NSS}_{\kappa,\lambda}) = \text{max}\{\text{cof}((\text{NSS}_{\kappa,\nu})), \text{cov}(\lambda, \lambda, \lambda, \kappa)\} \leq \max\{\lambda, \lambda\} = \lambda \) by Facts 2.4 (ii) and 2.10. Hence \( \text{cof}(\bigcup_{\nu \leq \delta < \lambda} \text{NS}_{\delta}^{\kappa,\lambda}) \leq \lambda \).

Next suppose that \( \lambda \) is a limit cardinal. Given a cardinal \( \chi \) with \( \kappa \leq \chi < \lambda \), by Corollary 3.4 \( \text{cof}(\text{NS}_{\chi}^{\kappa,\lambda}) \leq \lambda \) for every cardinal \( \tau \) with \( \chi \leq \tau < \lambda \), so by Fact 2.10 \( \text{cof}(\text{NSS}_{\kappa,\lambda}) \leq \lambda \). It follows that \( \text{cof}(\text{NSS}_{\kappa,\lambda}) \leq \lambda \) since by Fact 4.3 \( \text{NSS}_{\kappa,\lambda} = \bigcup_{\kappa \leq \chi < \lambda} \text{NS}_{\chi}^{\kappa,\lambda} \).

(ii) → (iii) : By Fact 4.4.
(iii) → (i) : By Facts 2.5 (iii) and 4.7. \( \square \)

5 Ideals \( J \) on \( P_{\kappa}(\lambda) \) with \( \text{cof}(J) = \lambda \)

In this section we look for cases when \( \text{cof}(\bigcup_{\delta < \xi} \text{NS}_{\delta}^{\kappa,\lambda}) = \lambda \), where \( \kappa < \xi \leq \lambda + 1 \).

We start with the following observation.

LEMMA 5.1. Suppose that \( K \subseteq \text{NS}_{\kappa,\lambda} \) is an ideal on \( P_{\kappa}(\lambda) \) with \( \text{cof}(K) \leq \lambda \), and \( \xi \) is an ordinal such that

- \( \kappa < \xi \leq \lambda + 1 \);
- \( \xi \) is either a successor ordinal, or a limit ordinal of cofinality at least \( \kappa \);
- \( \bigcup_{\delta < \xi} \text{NS}_{\delta}^{\kappa,\lambda} \subseteq K \).

Then \( \text{cof}(\bigcup_{\delta < \xi} \text{NS}_{\delta}^{\kappa,\lambda}) = \lambda \).

Proof. By Fact 4.5 we may find \( A \in \text{NS}_{\kappa,\lambda}^* \) such that \( K|A = I_{\kappa,\lambda}|A \). For any cardinal \( \chi \) with \( \kappa \leq \chi < \xi \), \( \text{NS}_{\chi}^{\kappa,\lambda}|A = I_{\kappa,\lambda}|A \), so by Lemma 2.5 (ii) \( \text{cof}(\text{NS}_{\chi}^{\kappa,\lambda}) \leq \lambda \). Hence by Fact 2.4 (ii) \( \text{cof}(\text{NS}_{\delta}^{\kappa,\lambda}) \leq \lambda \) for every \( \delta \) with \( \kappa \leq \delta < \xi \). It easily follows that \( \text{cof}(\bigcup_{\delta < \xi} \text{NS}_{\delta}^{\kappa,\lambda}) \leq \lambda \). The reverse inequality holds by Proposition 4.6. \( \square \)

So we are looking for a large \( K \subseteq \text{NS}_{\kappa,\lambda} \) with \( \text{cof}(K) \leq \lambda \). Assuming that \( \mathcal{H}_{\kappa,\lambda} \) holds, we can take \( K = \bigcup_{\delta < \text{cf}(\lambda)} \text{NS}_{\delta}^{\kappa,\lambda} \) if \( \lambda \) is a singular cardinal of cofinality at least \( \kappa \), and \( K = \text{NSS}_{\kappa,\lambda} \) otherwise.

FACT 5.2. ([10]) Let \( \theta \) be a cardinal with \( 2 \leq \theta \leq \kappa \). Suppose \( \theta \leq \text{cf}(\lambda) < \kappa \). Then for any cardinal \( \nu \) with \( \kappa \leq \nu < \lambda \), \( \text{cof}(\text{NS}_{\nu}^{\kappa,\lambda}) \leq \bigcup_{\nu \leq \tau < \lambda} \text{cof}(\text{NS}_{\nu}^{\kappa,\tau}) \).
PROPOSITION 5.3. Let $\theta$ be a cardinal with $2 \leq \theta \leq \kappa$. Suppose that $\theta \leq \text{cf}(\lambda) < \kappa$ and there is a cardinal $\nu$ with $\kappa \leq \nu < \lambda$ such that for any cardinal $\tau$ with $\nu \leq \tau < \lambda$, $\text{cf}(\text{NS}_{\kappa,\tau}^{[\kappa,\nu]}) \leq \lambda$ and $\tau^{<\kappa} < \lambda$. Then $\text{cf}(\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]}) = \lambda$.

Proof. By Proposition 4.6 and Fact 5.2. □

In particular, if $\text{cf}(\lambda) < \kappa$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\text{cf}(\text{NS}_{\kappa,\lambda}) = \lambda$.

Note that if $\text{cf}(\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]}) = \lambda$, then by Fact 4.4 $\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]} = I_{\kappa,\lambda}|C$ for some $C$.

FACT 5.4. (11) Let $A \in I_{\kappa,\lambda}^+$ be such that $|\{a \in A : b \subseteq a\}| = |A|$ for every $b \in P_\kappa(\lambda)$. Then $A$ can be decomposed into $|A|$ pairwise disjoint members of $I_{\kappa,\lambda}^+$.

Proof. Pick $D \in (\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]})^*$. Then by Fact 5.4, $C \cap D$ can be decomposed into $\pi$ pairwise disjoint members of $(\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]})^+$. □

In particular, if $\text{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|C$ for some $C$, then $P_\kappa(\lambda)$ can be split into $c(\kappa,\lambda)$ disjoint stationary sets.

PROPOSITION 5.5. Let $\theta$ be a cardinal with $2 \leq \theta \leq \kappa$. Suppose that there is $C$ such that $\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]} = I_{\kappa,\lambda}|C$. Then $P_\kappa(\lambda)$ can be split into $\pi$ members of $(\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]})^+$, where $\pi$ is the least size of any member of $(\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]})^*$.

Proof. Pick $D \in (\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]})^*$. Then by Fact 5.4, $C \cap D$ can be decomposed into $\pi$ pairwise disjoint members of $(\text{NS}_{\kappa,\lambda}^{[\kappa,\nu]})^+$. □

In particular, if $\text{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|C$ for some $C$, then $P_\kappa(\lambda)$ can be split into $c(\kappa,\lambda)$ disjoint stationary sets.

PROPOSITION 5.6. Suppose that $\theta$ and $\rho$ are two cardinals such that $\omega \leq \theta = \text{cf}(\theta) < \kappa \leq \rho \leq \lambda$, $u(\theta,\lambda) = \lambda$, and either $\text{cf}(\lambda) < \kappa$ or $\text{cf}(\lambda) > \rho^{<\theta}$. Suppose further that for every cardinal $\tau$ with $\rho \leq \tau < \lambda$, $\text{cf}(\text{NS}_{\kappa,\tau}^{[\kappa,\lambda]}) \leq \lambda$. Then $\text{cf}(\text{NS}_{\kappa,\lambda}^{[\kappa,\rho]}) \leq \lambda$.

Proof. It suffices to show that $\text{cf}(\text{NS}_{\kappa,\tau}^{[\kappa,\rho]}) \leq \lambda$ for any cardinal $\tau$ with $\rho \leq \tau < \lambda$ since by Facts 2.1 and 2.10 $\text{cf}(\text{NS}_{\kappa,\tau}^{[\kappa,\rho]}) = \bigcup_{\rho < \tau < \lambda} \text{cf}(\text{NS}_{\kappa,\tau}^{[\kappa,\rho]})$ if $\lambda$ is a limit cardinal, and $\text{cf}(\text{NS}_{\kappa,\lambda}^{[\kappa,\rho]}) = \max\{\lambda, \text{cf}(\text{NS}_{\kappa,\rho}^{[\kappa,\rho]})\}$ if $\lambda = \nu^+$. Now for any cardinal $\tau$ with $\rho \leq \tau < \lambda$,

$$\text{cf}(\text{NS}_{\kappa,\tau}^{[\kappa,\rho]}) \leq \omega^{\kappa,\rho} \leq \omega^{\kappa,\tau} \leq \omega(\theta, \text{cf}(\text{NS}_{\kappa,\tau}^{[\kappa,\rho]})) \leq \omega(\theta, \lambda) = \lambda$$

by Lemmas 3.1 and 3.2. □

PROPOSITION 5.7. Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, and $\xi$ is an ordinal such that

- $\kappa < \xi \leq \eta$, where $\eta$ equals $\lambda + 1$ if $\text{cf}(\lambda) < \kappa$, and $\text{cf}(\lambda)$ otherwise;
- $\xi$ is either a successor ordinal, or a limit ordinal of cofinality at least $\kappa$. 


Then $\text{cof}(\bigcup_{\delta < \xi} \text{NS}^d_{\kappa, \lambda}) = \lambda$.

**Proof.** By Facts 2.4 (ii) and 5.1 and Propositions 4.8, 5.3 and 5.6. \hfill \Box

In particular if $\mathcal{H}_{\kappa, \lambda}$ holds and $\kappa \leq \text{cf}(\lambda) < \lambda$, then $\text{cof}(\bigcup_{\delta < \text{cf}(\lambda)} \text{NS}^d_{\kappa, \lambda}) = \lambda$

(and hence by Fact 1.5 (iv) there is no $A$ such that $\text{NS}_{\kappa, \lambda} = (\bigcup_{\delta < \text{cf}(\lambda)} \text{NS}^d_{\kappa, \lambda})\upharpoonright A$).

## 6 Ideals $J$ on $P_\kappa(\lambda)$ with $\text{cof}(J) < \lambda$

There may exist ideals $J$ on $P_\kappa(\lambda)$ such that $\text{cof}(J) < \lambda$. Some examples were presented in [10]. We now give some more.

Given two cardinals $\pi \leq \kappa$ and $\chi \geq \lambda$, $A_{\kappa, \lambda}(\pi, \chi)$ asserts the existence of $Z \subseteq P_\pi(\lambda)$ with $|Z| = \chi$ such that $|Z \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$.

**FACT 6.1.** ([10]) Let $\theta$ and $\chi$ be two cardinals such that

1. $2 \leq \theta \leq \kappa$, $\lambda < \chi$ and there is a $[\chi]^{<\theta}$-normal ideal on $P_\pi(\chi)$;
2. $A_{\kappa, \lambda}(\pi, \chi)$ holds for some regular uncountable cardinal $\pi < \kappa$.

Then $\text{cof}(I_{\kappa, \chi} \upharpoonright A) \leq \lambda$ for some $A \in (\text{NS}^\kappa_{\kappa, \chi})^+$.

**FACT 6.2.** ([9]) Let $\tau$ be the largest limit cardinal less than or equal to $\kappa$. Assume $\text{cf}(\lambda) < \kappa$ and one of the following conditions is satisfied:

1. $\tau = \kappa$.
2. $\tau > \text{cf}(\lambda)$ and $\text{cf}(\lambda) \neq \text{cf}(\tau)$.
3. $\tau > \text{cf}(\lambda) = \text{cf}(\tau)$ and $\min\{\text{pp}(\tau), \tau^{+3}\} < \kappa$.
4. $\tau \leq \text{cf}(\lambda)$ and $\min\{2^{\text{cf}(\lambda)}, (\text{cf}(\lambda))^{+3}\} < \kappa$.

Then $A_{\kappa, \lambda}\big((\text{cf}(\lambda))^{+}, \lambda^+\big)$ holds.

Suppose for instance that $\kappa$ is a limit cardinal and $\text{cf}(\lambda) < \kappa$. Then by Facts 6.1 and 6.2, $\text{cof}(I_{\kappa, \lambda} \upharpoonright B) \leq \lambda$ for some $B \in \text{NS}^{\kappa, \lambda+}$.

Note that in case $\kappa$ is the successor of a cardinal of cofinality $\text{cf}(\lambda)$, Fact 6.2 does not apply, as none of the conditions (a) - (d) is satisfied. To handle this case, we introduce the following principle.

Given a cardinal $\chi \geq \lambda$, $B_{\kappa, \lambda}(\chi)$ asserts the existence of $Z \subseteq P_\kappa(\lambda)$ with $|Z| = \chi$ such that for every $e \subseteq Z$ with $|e| = \kappa$, there is a $< \kappa$-to-one function in $\prod_{z \in e} z$. 

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FACT 6.3. ([10]) Let $\theta$ and $\chi$ be two cardinals such that

- $2 \leq \theta \leq \kappa$, $\lambda < \chi$ and there is a $[\chi]^{<\theta}$-normal ideal on $P_\kappa(\chi)$;
- $B_{\kappa,\lambda}(\chi)$ holds.

Then $\text{cof}(I_{\kappa,\chi}|A) \leq \lambda$ for some $A \in (\text{NS}_{\kappa,\chi}[^{<\theta}])^+$. 

Note that in case $\text{cf}(\lambda < \kappa)$, $B_{\kappa,\lambda}(\lambda^+)$ follows from ADS$_\kappa$, where ADS$_\kappa$ asserts the existence of $y_\alpha \subseteq \lambda$ for $\alpha < \lambda^+$ such that

- for any $\alpha < \lambda^+$, $\sup y_\alpha = \lambda$ and $o.t.(y_\alpha) = \text{cf}(\lambda)$;
- given $\beta < \lambda^+$, there is $g : \beta \rightarrow \lambda$ such that $(y_\alpha \setminus g(\alpha)) \cap (y_{\alpha'} \setminus g(\alpha')) = \emptyset$

for any $\alpha, \alpha' \in \beta$ with $\alpha \neq \alpha'$.

For more on the existence of $A \in (\text{NS}_{\kappa,\chi}[^{<\theta}])^+$ such that $\text{cof}(I_{\kappa,\chi}|A) < \chi$, see [9] and [10].

PROPOSITION 6.4. Suppose that $\theta$ and $\chi$ are two cardinals such that

$2 \leq \theta \leq \kappa$ and $\lambda < \chi$, and $A \in (\text{NS}_{\kappa,\chi}[^{<\theta}])^+$ is such that $\text{cof}(I_{\kappa,\chi}|A) \leq \lambda$. Then there is $B \in (\text{NS}_{\kappa,\chi}[^{<\theta}])^+$ and a function $f$ such that

- $f$ is an isomorphism from $(P_\kappa(\lambda), \subset)$ onto $(B, \subset)$;
- for any $\delta < \lambda$, $f(\text{NS}_{\kappa,\chi}[^{<\theta}]) = \text{NS}_{\kappa,\chi}[^{<\theta}]B$ (and hence $\text{cof}(\text{NS}_{\kappa,\chi}[^{<\theta}]B) \leq \text{cof}(\text{NS}_{\kappa,\chi}[^{<\theta}])$).

Proof. Select $x_\beta \in P_\kappa(\chi)$ for $\beta < \lambda$ so that for each $X \in I_{\kappa,\chi}$, there is $z \in P_\kappa(\lambda)$ with $X \cap \{y \in A : \bigcup_{\beta \in z} x_\beta \subseteq y\} = \emptyset$. For $\lambda \leq \alpha < \chi$, pick $z_\alpha \in P_\kappa(\lambda)$ with \[ \{y \in A : \bigcup_{\beta \in z_\alpha} x_\beta \subseteq y\} \subseteq \{t \in P_\kappa(\chi) : \alpha \in t\}. \]

Let $C$ be the set of all $x \in P_\kappa(\chi)$ such that $\bigcup_{\beta \in x \cap \lambda} x_\beta \cup \bigcup_{\alpha \in x \cap \lambda} z_\alpha \subseteq x$. Note that $C \in \text{NS}_{\kappa,\chi}$. 

Claim 1. Let $x \in A \cap C$. Then $x \setminus \lambda = \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq x \cap \lambda\}$.

Proof of Claim 1. Since $x \in C$, $x \setminus \lambda \subseteq \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq x \cap \lambda\}$. To show the reverse inclusion, fix $\alpha \in \chi \setminus \lambda$ with $z_\alpha \subseteq x \cap \lambda$. Then $\bigcup_{\beta \in z_\alpha} x_\beta \subseteq x$, and hence $\alpha \in x$, which completes the proof of Claim 1.

Claim 2. Let $a \in P_\kappa(\lambda)$. Then $|\{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq a\}| < \kappa$.

Proof of Claim 2. Pick $x \in A \cap C$ with $a \subseteq x$. Then by Claim 1,
\{\alpha \in \chi \setminus \lambda : z_{\alpha} \subseteq a\} \subseteq \{\alpha \in \chi \setminus \lambda : z_{\alpha} \subseteq x \cap \lambda\} \subseteq x,

which completes the proof of Claim 2.

Using Claim 2, define \( f : P_\kappa(\chi) \to P_\kappa(\chi) \) by \( f(a) = a \cup \{\alpha \in \chi \setminus \lambda : z_{\alpha} \subseteq a\} \). Put \( B = \text{ran}(f) \). By Claim 1, \( x = f(x \cap \lambda) \) for any \( x \in A \cap C \), so \( A \cap C \subseteq B \).

It follows that \( B \in (\text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}})^+ \).

As is easily seen, \( f \) is an isomorphism from \( (P_\kappa(\chi), \subseteq) \) onto \((B, \subseteq)\), and moreover \( f^{-1}(X) \in I_{\kappa,\lambda} \) for any \( X \in I_{\kappa,\lambda} \). Now fix \( \delta \leq \lambda \). Set \( J = \text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} \). It is simple to see that \( f(J) \) is an ideal on \( P_\kappa(\chi) \) with the property that \( B \in (f(J))^+ \).

\textbf{Claim 3.} \( f(J) \) is \([\delta]^{<\theta}\)-normal.

\textbf{Proof of Claim 3.} Fix \( X \in (f(J))^+ \cap P(B) \) and \( h : X \to P_\delta(\delta) \) such that \( h(x) \in P_{[\delta]^{<\theta}}(x) \) for every \( x \in X \). Define \( k : f^{-1}(X) \to P_\delta(\delta) \) by \( k(a) = h(f(a)) \). There must be \( A \in J^+ \cap P(f^{-1}(X)) \) such that \( k \) is constant on \( A \). Then clearly \( f^{-1}A \in (f(J))^+ \cap P(X) \), and moreover \( h \) is constant on \( f^{-1}A \), which completes the proof of the claim.

It immediately follows from Claim 3 that \( \text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} |B \subseteq f(J)\).

To establish the reverse inclusion fix \( Y \in f(J) \). Since \( f^{-1}(Y \cap B) \in J \), we may find \( g : P_\delta(\delta) \to P_\kappa(\lambda) \) such that \( f^{-1}(Y \cap B) \cap C(g, \kappa, \lambda) = \emptyset \). Then clearly \( (Y \cap B) \cap C(g, \kappa, \chi) = \emptyset \) and hence \( Y \cap B \in \text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} \). \( \square \)

Let \( \kappa = (2^\theta)^+ \), where \( \rho \) is an infinite cardinal, and suppose that \( \lambda \) is a strong limit cardinal with \( \text{cf}(\lambda) \leq \rho \). Then \( A_{\kappa,\lambda}(\rho^+, 2^\lambda) \) holds, since \( |P_{[\rho^+]^+}(\lambda) \cap P(a)| \leq 2^\rho \) for any \( a \in P_\kappa(\lambda) \). Hence by Facts 5.2 and 6.1 and Proposition 6.4, \( \text{col}(\text{NS}_{\kappa,2^\lambda}^\lambda | B) \leq \lambda \) for some \( B \in \text{NS}_{\kappa,2^\lambda}^\lambda \).

\textbf{Proposition 6.5.} Suppose that \( \text{col}(\text{NS}_{\kappa,\lambda}) \leq \lambda^+ \), and there is \( A \in \text{NS}_{\kappa,\lambda}^+ \) such that \( \text{col}(I_{\kappa,\lambda}^+ | A) \leq \lambda \). Then \( \text{col}(\text{NS}_{\kappa,\lambda}^+ | B) < \lambda^+ \) for some \( B \in \text{NS}_{\kappa,\lambda}^\lambda \).

\textbf{Proof.} By Fact 4.7 (i), there is \( C \in \text{NS}_{\kappa,\lambda}^\lambda \) such that \( \text{NSS}_{\kappa,\lambda} | C = I_{\kappa,\lambda}^+ | C \). Then \( B = A \cap C \) is as desired. \( \square \)

For example, suppose that \( \kappa = \omega_4 \) and \( \lambda = 3_\alpha \) for some infinite limit ordinal \( \alpha \) of cofinality \( \omega \). Then by Facts 5.2, 6.1 and 6.2 and Proposition 6.5, \( \text{col}(\text{NSS}_{\kappa,\lambda} | B) \leq \lambda \) for some \( B \in \text{NS}_{\kappa,\lambda}^\lambda \).

If \( \lambda \) is singular, then by Fact 4.1 \( \text{NS}_{\kappa,\lambda} = \text{NSS}_{\kappa,\lambda} | B \) for some \( B \), so \( \text{col}(\text{NSS}_{\kappa,\lambda} | A) < \lambda \) for some \( A \in \text{NS}_{\kappa,\lambda}^+ \) just in case \( \text{col}(\text{NSS}_{\kappa,\lambda} | D) < \lambda \) for some \( D \in \text{NS}_{\kappa,\lambda}^\lambda \).
Suppose that $\text{cof}(\text{NS}_{\kappa,\lambda}|D) < \lambda$ for some $D \in \text{NS}^+_{\kappa,\lambda}$. Then setting $\sigma = \text{cof}(\text{NS}_{\kappa,\lambda}|D)$, 

$$\text{cof}(\text{NS}_{\kappa,\lambda}) \leq u(\kappa, \sigma) \leq u(\kappa, \lambda) \leq \text{cof}(\text{NS}_{\kappa,\lambda})$$

by Fact 2.11 (ii), so $\text{cof}(\text{NS}_{\kappa,\lambda}) = u(\kappa, \sigma) = u(\kappa, \lambda)$. Hence by Fact 2.5 (iv), SSH does not hold.

**PROPOSITION 6.6.** Let $\theta$ and $\chi$ be two cardinals such that $2 \leq \theta \leq \kappa$ and $\lambda < \chi$. Suppose that $\text{cof}(\text{NS}_{\kappa,\chi}) \leq \chi < \theta$, and there is $A \in (\text{NS}_{\kappa,\chi}[^{<\theta}\kappa,\chi])^+$ such that $\text{cof}(I_{\kappa,\chi}|A) \leq \lambda$. Then $\text{cof}(\text{NS}_{\kappa,\chi}|B) \leq \lambda$ for some $B \in (\text{NS}_{\kappa,\chi}[^{<\theta}\kappa,\chi])^+$.

**Proof.** By Fact 4.4 there is $C \in (\text{NS}_{\kappa,\chi}[^{<\theta}\kappa,\chi])^+$ such that $\text{NS}_{\kappa,\chi}|C = I_{\kappa,\chi}|C$. Then $B = A \cap C$ is as desired. $\square$

Here is an example of a situation where Proposition 6.6 applies. Starting from a $\mathcal{P}^3(\nu)$-hypermeasurable, Cummings [3] constructs a generic extension $W$ of $V$ in which for any infinite cardinal $\rho$, $2^\rho$ equals $\rho^+$ if $\rho$ is a successor cardinal, and $\rho^{++}$ otherwise. In $W$, let $\sigma$ be a regular uncountable cardinal, and $\mu > \sigma$ be a cardinal of cofinality less than $\sigma$. Suppose that

1. $\sigma$ is not the successor of a cardinal $\tau$ with $\text{cf}(\tau) \leq \text{cf}(\mu)$;
2. $\sigma$ is not the successor of the successor of a limit cardinal $\pi$ with $\text{cf}(\pi) \leq \text{cf}(\mu)$.

Then by Facts 6.1 and 6.2 and Proposition 6.6, $\text{cof}(\text{NS}_{\sigma,\mu}|B) \leq \mu$ for some $B \in (\text{NS}_{\sigma,\mu}[^{<\text{cf}(\mu)}\sigma])^+$.

### 7 Cases when $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\text{cf}(\lambda)}|A$ for some $A$

In this section we establish that if $\kappa \leq \text{cf}(\lambda) < \lambda$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\text{cf}(\lambda)}|A$ for some $A$. Note that if $\text{cf}(\lambda) < \kappa$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then by Facts 4.5 and 5.1, $\text{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$ for some $A$. Note further that if $\lambda$ is regular, then trivially $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^\kappa|P_\kappa(\lambda)$. By combining the three cases, we obtain that if $\text{cof}(\text{NS}_{\kappa,\tau}) \leq \lambda$ for every cardinal $\tau$ with $\max\{\kappa, \text{cf}(\lambda)\} \leq \tau < \lambda$, then $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\text{cf}(\lambda)}|A$ for some $A$.

**THEOREM 7.1.** Let $\pi, \theta$ and $\chi$ be three cardinals with $\kappa \leq \pi < \lambda$ and $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. Suppose that

1. $\lambda$ is singular ;
2. $\overline{\theta} \leq \text{cf}(\lambda)$ in case $\chi = \lambda$ ;
3. $\text{cof}(\text{NS}_{\kappa,\tau}) \leq \lambda^{<\overline{\theta}}$ for every cardinal $\tau$ with $\pi \leq \tau < \lambda$.

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Then there is \( A \in (\text{NS}_{\kappa, \lambda}^{[\lambda]^{<\sigma}})^* \) such that \( \text{NS}_{\kappa, \lambda}^{[\lambda]^{<\sigma}} \subseteq \text{NS}_{\kappa, \lambda}^{\text{cf}(\lambda)} \mid A \).

**Proof.** Set \( \mu = \text{cf}(\lambda) \) and select an increasing sequence of cardinals \( \langle \eta_\alpha : \eta < \mu \rangle \) so that

- \( \sup \{ \eta_\alpha : \eta < \mu \} = \lambda ; \)
- \( \lambda_0 > \max \{ \pi, \mu \} ; \)
- \( \lambda_0 \geq \chi \) in case \( \chi < \lambda. \)

For \( \eta < \mu \), pick a family \( G_\eta \) of functions from \( P_{\max (\eta)} \min (\chi, \lambda_\eta)) \) to \( P_3(\eta) \) so that \( |G_\eta| \leq \text{cof}(\text{NS}_{\kappa, \lambda}^{[\lambda]^{<\sigma}}) \) and for every \( H \in (\text{NS}_{\kappa, \lambda}^{[\lambda]^{<\sigma}})^* \), there is \( y \in P_\kappa(G_\eta) \setminus \{ \emptyset \} \) such that \( \{ b \in \bigcap_{\eta \in y} C(g, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa \} \subseteq H. \) Let \( \bigcup_{\eta < \mu} G_\eta = \{ g_\eta : e \in P_\kappa(\lambda) \}. \) Let \( A \) be the set of all \( a \in P_\kappa(\lambda) \) such that

- \( \emptyset \subseteq a \) in case \( \emptyset < \kappa ; \)
- \( \omega \subseteq a ; \)
- \( a \cap \kappa \in \kappa ; \)
- \( k(a) \in a \) for every \( a \in a, \) where \( k : \lambda \to \mu \) is defined by \( k(\alpha) = \) the least \( \eta < \mu \) such that \( a \cap \lambda_\eta ; \)
- \( \text{if } \chi = \lambda, \text{ then } i(v) \subseteq a \) for every \( v \in P_{[a \cap \min (\eta)]} (a), \) where \( i : P_{\max (\eta)} (\lambda) \to \mu \) is defined by \( i(v) = \) the least \( \eta < \mu \) such that \( v \subseteq \lambda_\eta ; \)
- \( g_\eta (v) \subseteq a \) whenever \( e \in P_{[a \cap \eta]} (a) \) and \( u \in P_{[a \cap \max (\eta)]} (a) \cap \text{dom}(g_\eta). \)

It is immediate that \( A \in (\text{NS}_{\kappa, \lambda}^{[\lambda]^{<\sigma}})^*. \) Let us check that \( A \) is as desired. Thus fix \( f : P_{\max (\eta)} (\lambda) \to P_3(\lambda). \) Given \( \eta < \mu, \) define \( p_\eta : P_{\max (\eta)} (\min (\chi, \lambda_\eta)) \to P_2(\lambda_\eta) \) by \( p_\eta (v) = \{ \zeta \}, \) where \( \zeta = \text{the least } \sigma \text{ such that } \eta \leq \sigma < \mu \text{ and } f(v) \subseteq \lambda_\sigma. \) Also define \( q_\eta : P_{\max (\eta)} (\min (\chi, \lambda_\eta)) \to P_3(\lambda_\eta) \) by \( q_\eta (v) = \lambda_\eta \cap f(v). \) Select \( x_\eta, y_\eta \in P_\kappa(P_\kappa(\lambda)) \setminus \{ \emptyset \} \) so that

- \( \{ g_\eta : e \in x_\eta \cup y_\eta \} \subseteq G_\eta ; \)
- \( \{ b \in \bigcap_{\eta \in x_\eta} C(g_\eta, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa \} \subseteq C(p_\eta, \kappa, \lambda_\eta) ; \)
- \( \{ b \in \bigcap_{\eta \in y_\eta} C(g_\eta, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa \} \subseteq C(q_\eta, \kappa, \lambda_\eta) . \)

Finally define \( u : \mu \to P_3(\lambda) \) by \( u(\eta) = \bigcup(x_\eta \cup y_\eta), \) and \( t : P_2(\mu) \to P_3(\lambda) \) so that for any \( \eta \in \mu, t(\eta) \) equals \( u(\eta) \) if \( \emptyset < \kappa, \) and \( u(\eta) \cup u(\eta) \) otherwise. We claim that \( A \cap C_{\sigma}^{\kappa, \lambda} \subseteq C_{\sigma}^{\kappa, \lambda}. \) Thus let \( a \in A \cap C_{\sigma}^{\kappa, \lambda} \) and \( v \in P_{[a \cap \min (\eta)]} (a \cap \chi). \) There must be \( \eta \in a \cap \mu \) such that \( v \subseteq \lambda_\eta. \) Then \( a \cap \lambda_\eta \in C(p_\eta, \kappa, \lambda_\eta) \) since \( x_\eta \subseteq P_{[a \cap \eta]} (a). \) It follows that \( v \cup f(v) \subseteq \lambda_\sigma \) for some \( \sigma \in a \cap \mu. \) Now
\( a \cap \lambda \sigma \in C(q \sigma, \kappa, \lambda \sigma) \), since \( y \sigma \subseteq P_{\text{cf}(\tau)}(a) \), so \( f(v) \subseteq a \). \( \square \)

In Theorem 7.1 we assumed that \( \theta \leq \text{cf}(\lambda) \) in case \( \chi = \lambda \). Some condition of this kind is necessary. In fact if \( \text{cf}(\lambda) < \kappa \) and \( u(\kappa, \lambda, \theta) = \chi < \theta \), then for each
\[ A \in (\text{NS}_\kappa^{(\lambda)} \backslash^{< \theta} \sigma)^*, \text{NS}_\kappa^{(\lambda)} | A \neq \text{NS}_{\kappa, \lambda}^{\text{cf}(\lambda)} \] since by Fact 2.11,
\[ \overline{\text{cof}}(\text{NS}_\kappa^{(\lambda)} \backslash^{< \theta} \sigma) > \lambda \leq \chi < \theta \geq \overline{\text{cof}}(\text{NS}_\kappa^{\text{cf}(\lambda)} | A). \]

**COROLLARY 7.2.** Suppose that one of the following holds:

(i) SSH holds.

(ii) There exists a \( \sigma \)-saturated ideal on \( P_\nu(\lambda) \), where \( \sigma \) and \( \nu \) are two cardinals such that \( \omega < \nu = \text{cf}(\nu) < \lambda \) and \( \sigma < \nu \).

(iii) There is a regular uncountable cardinal \( \tau < \lambda \) that is mildly \( \pi \)-ineffable for every cardinal \( \pi \) with \( \tau \leq \pi < \lambda \).

Let \( \theta \) and \( \chi \) be two cardinals such that \( 2 \leq \theta \leq \kappa \), \( \max\{\kappa, \text{cf}(\lambda)\} \leq \chi < \lambda \) and \( \overline{\text{cof}}(\text{NS}_\kappa^{(\lambda)} \backslash^{< \theta} \sigma) \leq \lambda \). Then \( \text{NS}_\kappa^{(\lambda)} | A = \text{NS}_{\kappa, \lambda}^{\text{cf}(\lambda)} | A \) for some \( A \in (\text{NS}_\kappa^{(\lambda)} \backslash^{< \theta} \sigma)^* \).

**Proof.** Use Facts 2.3 and 2.10. \( \square \)

**COROLLARY 7.3.** Suppose that \( \text{cf}(\lambda) < \kappa \), and \( \theta \) and \( \chi \) are two cardinals such that \( 2 \leq \theta \leq \kappa \leq \chi < \lambda \) and \( \overline{\text{cof}}(\text{NS}_\kappa^{\chi} \backslash^{< \theta} \sigma) \leq \lambda \). Then \( \text{NS}_\kappa^{\chi} | A = \text{NS}_{\kappa, \lambda}^{\text{cf}(\lambda)} | A \) for some \( A \in (\text{NS}_\kappa^{\chi} \backslash^{< \theta} \sigma)^* \).

**Proof.** By Theorem 7.1 there is \( A \in (\text{NS}_\kappa^{(\lambda)} \backslash^{< \theta} \sigma)^* \) such that \( \text{NS}_\kappa^{(\lambda)} | A = I_{\kappa, \lambda} | A \). Then by Fact 2.11 (ii), \( \overline{\text{cof}}(\text{NS}_\kappa^{(\lambda)} \backslash^{< \theta} \sigma) = \overline{\text{cof}}(\text{NS}_\kappa^{(\lambda)} | A) = \overline{\text{cof}}(I_{\kappa, \lambda}) = u(\kappa, \lambda). \) \( \square \)

**COROLLARY 7.4.**

(i) Suppose that \( \lambda \) is singular and \( H_{\kappa, \lambda} \) holds. Then \( \text{NS}_{\kappa, \lambda} = \text{NS}_{\kappa, \lambda}^{\text{cf}(\lambda)} | A \) for some \( A \).

(ii) Let \( \chi \) be a cardinal such that \( \max\{\kappa, \text{cf}(\lambda)\} \leq \chi < \lambda \) and \( \overline{\text{cof}}(\text{NS}_\kappa^{\chi} \backslash^{< \theta} \sigma) \leq \lambda \) for every cardinal \( \tau \) with \( \chi \leq \tau < \lambda \). Then \( \text{NS}_\kappa^{\chi} | A = \text{NS}_{\kappa, \lambda}^{\text{cf}(\lambda)} | A \) for some \( A \in (\text{NS}_\kappa^{\chi} \backslash^{< \theta} \sigma)^* \).

**COROLLARY 7.5.**
(i) Let $\chi \geq \kappa$ be a cardinal, and $\alpha < \kappa$ be a limit ordinal such that $\text{cof}(\text{NS}_{\kappa,\chi}) \leq \chi^{+\alpha}$. Then $\text{NS}_{\kappa,\chi+\alpha}^\chi|A = I_{\kappa,\chi+\alpha}|A$ for some $A \in \text{NS}_{\kappa,\chi+\alpha}^\chi$.

(ii) Let $\chi > \kappa$ be a cardinal such that $\text{cof}(\text{NS}_{\kappa,\chi}) < \chi^+\kappa$. Then $\text{NS}_{\kappa,\chi+\alpha}^\chi|A = \text{NS}_{\kappa,\chi+\alpha}^\chi|A$ for some $A \in \text{NS}_{\kappa,\chi+\alpha}^\chi$.

Proof. Use Facts 2.1 and 2.10.

Note that we do get a better result by considering the reduced cofinality ($\text{cof}$) instead of the usual one (cf). For example, suppose that GCH holds in $V$. By a result of [12], there is a $<\kappa$-closed, $\kappa^+$-cc forcing notion $\mathbb{P}$ such that in $V^\mathbb{P}$, $\text{cf}(\text{NS}_{\kappa,\lambda}) = \kappa^+$ and $\text{cf}(\text{NS}_{\kappa,\lambda}) = \kappa^+(\omega+1)$. Then in $V^\mathbb{P}$, there is by Corollary 7.5 (i) $A \in \text{NS}_{\kappa,\kappa^+\omega}^\chi$ such that $\text{NS}_{\kappa,\kappa^+\omega}^\chi|A = I_{\kappa,\kappa^+\omega}|A$.

Let us next discuss the condition in Theorem 7.1 that $\text{cof}(\text{NS}_{\kappa,\chi}) \leq \lambda^{<\theta}$ for almost all cardinals $\tau < \lambda$.

PROPOSITION 7.6. Let $\theta$ and $\chi$ be two cardinals such that $2 \leq \theta < \kappa < \chi < \lambda$. Suppose that $\text{cof}(\text{NS}_{\kappa,\chi}) \leq \lambda^{<\theta}$ and $\chi^{<\theta} \geq \lambda$. Then $\text{cof}(\text{NS}_{\kappa,\chi}^{(\theta)}) = \text{cof}(\text{NS}_{\kappa,\chi}) \leq \chi$.

Proof. By Fact 2.6 (iii) $\chi^{<\theta} = \lambda^{<\theta}$, so by Fact 4.5 $\text{NS}_{\kappa,\chi}^{(\theta)} = I_{\kappa,\chi}|A$ for some $A$. It follows that $\text{cof}(\text{NS}_{\kappa,\chi}^{(\theta)}) \leq \chi$. Moreover by Fact 2.10

$$\text{cof}(\text{NS}_{\kappa,\chi}^{(\theta)}) = \max(\text{cof}(\text{NS}_{\kappa,\chi})^{(\theta)}, \text{cov}(\lambda, \lambda^{<\theta}^{+}, \lambda^{<\theta}^{+}, \kappa)) = \text{cof}(\text{NS}_{\kappa,\chi})$$.

\[ \square \]

COROLLARY 7.7. Let $\theta$ and $\chi$ be two cardinals such that $2 \leq \theta < \kappa < \chi < \chi^{<\theta} = \lambda$. Suppose $\text{cof}(\text{NS}_{\kappa,\chi}) \leq \chi^{<\theta}$. Then there is $A \in (\text{NS}_{\kappa,\chi}^{(\theta)})^*$ such that $\text{cof}(\text{NS}_{\kappa,\lambda} | A) \leq \chi$.

Proof. By Fact 2.7 we may find $A \in (\text{NS}_{\kappa,\lambda}^{(\theta)})^*$ such that $\text{NS}_{\kappa,\lambda}|A = \text{NS}_{\kappa,\lambda}^{(\theta)}|A$. Then by Proposition 7.6, $\text{cof}(\text{NS}_{\kappa,\lambda}|A) \leq \text{cof}(\text{NS}_{\kappa,\lambda}^{(\theta)}) \leq \chi$.

\[ \square \]

Question. Suppose that $\theta$ and $\chi$ are two cardinals such that $2 \leq \theta < \kappa < \chi$ and $\text{cof}(\text{NS}_{\kappa,\chi}) \leq \chi^{<\theta}$. Does then $\chi^{<\theta} = \chi$ hold?

PROPOSITION 7.8.

(i) Suppose that $\theta$ and $\sigma$ are two cardinals such that $2 \leq \theta < \kappa < \sigma < \lambda$, $\theta \leq \text{cf}(\lambda)$ and $\text{cof}(\text{NS}_{\kappa,\sigma}^{(\theta)}) \leq \lambda^{<\theta}$ for every cardinal $\tau$ with $\sigma \leq \tau < \lambda$.
Then there is a cardinal \( \pi \) with \( \sigma < \pi < \lambda \) such that \( \text{cof}(\text{NS}_\chi^{[< \sigma]}) \leq \lambda \) for every cardinal \( \chi \) with \( \pi < \chi < \lambda \).

(ii) Let \( \theta \) and \( \pi \) be two cardinals with \( 2 \leq \theta \leq \kappa \leq \pi < \lambda \). Suppose that \( \kappa < \text{cf}(\lambda) < \lambda \), and \( \text{cof}(\text{NS}_\chi^{[< \sigma]}) \leq \lambda \) for every cardinal \( \chi \) with \( \pi < \chi < \lambda \). Then \( \text{cof}(\text{NS}_\chi^{[< \sigma]}) \leq \lambda \) for every cardinal \( \rho \) with \( \max \{ \pi, \text{cf}(\lambda) \} \leq \rho < \lambda \).

Proof.

(i) : If \( \nu^\theta \) for every cardinal \( \nu < \lambda \) and every cardinal \( \rho < \theta \), then \( \lambda^{< \theta} = \lambda \), and \( \rho = \sigma \) is as desired. Now suppose there are two cardinals \( \nu < \lambda \) and \( \rho < \theta \) such that \( \nu^\theta \geq \lambda \). Set \( \pi = \max \{ \nu, \sigma \} \). Let \( \chi \) be a cardinal with \( \pi \leq \chi < \lambda \). Then \( \chi^{< \theta} = \lambda^{< \theta} \), so by Proposition 7.6 \( \text{cof}(\text{NS}_\chi^{[< \sigma]}) \leq \chi \).

(ii) : By Fact 2.9. \( \square \)

In particular, if \( \kappa \leq \text{cf}(\lambda) < \lambda \), then \( \mathcal{H}_{\kappa, \lambda} \) holds just in case \( \text{cof}(\text{NS}_{\kappa, \pi}) < \lambda \) for every cardinal \( \tau \) with \( \kappa \leq \tau < \lambda \).

Suppose that \( \lambda \) is a limit cardinal and \( \chi \) is a cardinal with \( \kappa \leq \chi \leq \lambda \). If either \( \text{cf}(\lambda) < \kappa \) or \( \text{cf}(\lambda) > \chi \), then by Fact 2.10 and Lemma 5.1,

\[
\text{cof}(\text{NS}_{\kappa, \lambda}) \leq \sup \{ \text{cof}(\text{NS}_{\kappa, \pi}) : \pi < \tau < \lambda \},
\]

where \( \pi \) equals \( \kappa \) if \( \chi = \lambda \), and \( \chi \) otherwise. We will now deal with the case when \( \kappa < \text{cf}(\lambda) \leq \chi \). The proof of the following is a modification of that of Theorem 7.1.

**PROPOSITION 7.9** Let \( \chi \) be a cardinal such that \( \max \{ \kappa, \text{cf}(\lambda) \} \leq \chi < \lambda \). Let \( \pi = \kappa \) if \( \chi = \lambda \), and \( \pi = \chi \) otherwise. Then \( \text{cof}(\text{NS}_{\kappa, \pi}) \leq \text{cof}(\text{NS}_{\kappa, \pi}^{[< \sigma]}) \) and \( \text{cof}(\text{NS}_{\kappa, \lambda}) \leq \text{cof}(\text{NS}_{\kappa, \pi}^{[< \sigma]}) \) where \( \rho = \sup \{ \text{cof}(\text{NS}_{\kappa, \tau}^{[< \sigma]}) : \pi < \tau < \lambda \} \).

**Proof.** We can assume that \( \text{cf}(\lambda) < \chi \) since otherwise the result is trivial. We show that \( \text{cof}(\text{NS}_{\kappa, \lambda}) \leq \text{cof}(\text{NS}_{\kappa, \pi}^{[< \sigma]}) \) and leave the proof of the other assertion to the reader. Put \( \mu = \text{cf}(\lambda) \) and pick an increasing sequence \( \{ \lambda_\eta : \eta < \mu \} \) of cardinals cofinal in \( \lambda \) so that \( \lambda_0 > \max \{ \kappa, \mu \} \), and \( \lambda_0 > \chi \) in case \( \chi < \lambda \). For every \( \eta < \mu \), select a family \( G_\eta \) of functions from \( P_{\eta}(\min \{ \chi, \lambda_\eta \}) \) to \( P_{\eta}(\lambda_\eta) \) so that \( |G_\eta| \leq \text{cof}(\text{NS}_{\kappa, \lambda_\eta}) \) and for any \( H \in (\text{NS}_{\kappa, \lambda_\eta})^{[< \sigma]} \), there is \( y \in P_{\eta}(G_\eta) \setminus \{ \emptyset \} \) with \( \{ b \in \bigcap_{\eta \in \xi} G(y, \kappa, \lambda_\eta) : b \cap \kappa \in \xi \} \subseteq H \). Let \( \bigcup_{\eta < \rho} G_\eta = \{ g_\xi : \xi < \rho \} \). For \( \xi < \rho \), let \( g_\xi \in G_\xi \). Let \( A \) be the set of all \( a \in P_{\kappa}(\lambda) \) such that \( \omega \subseteq a \), \( a \cap \kappa \in \kappa \) and \( k(a) \in a \) for all \( a \in a \), where \( k : \lambda \to \mu \) is defined by \( k(a) = \) the least \( \eta < \mu \) such that \( a \in \lambda_\eta \). Clearly \( A \in \text{NS}_{\kappa, \lambda} \), so by Fact 2.5 (ii) \( \text{cof}(\text{NS}_{\kappa, \pi}^{[< \sigma]}) = \text{cof}(\text{NS}_{\kappa, \lambda}) \).

By Proposition 2.3 we may find a collection \( T \) of functions from \( \mu \) to \( P_{\kappa}(\rho) \) such that \( |T| = \text{cof}(\text{NS}_{\kappa, \rho}^{[< \sigma]}) \) and for any \( u : \mu \to P_{\kappa}(\rho) \), there is \( z \in P_{\kappa}(\mu) \) with the
property that \( u(\eta) \subseteq \bigcup_{\xi \in z} t(\eta) \) for every \( \eta \in \mu \). For \( t \in T \), let \( D_t \) be the set of all \( a \in P_\kappa(\lambda) \) such that for any \( \eta \in a \cap \mu \) and any \( \xi \in t(\eta) \), \( a \cap \lambda_\xi \subseteq C(g_\xi, \kappa, \lambda_\eta) \). Note that \( D_t \in (\text{NS}_\kappa^\nu)^* \).

Now fix \( f : P_3(\chi) \rightarrow P_3(\lambda) \). Given \( \eta < \mu \), define \( p_\eta : P_3(\min\{\chi, \lambda_\eta\}) \rightarrow P_2(\lambda_\eta) \) by \( p_\eta(v) = \{\zeta\} \), where \( \zeta \) is the least \( \sigma \) such that \( \eta \leq \sigma < \mu \) and \( f(v) \leq \lambda_\sigma \), and \( q_\eta : P_3(\min\{\chi, \lambda_\eta\}) \rightarrow P_3(\lambda_\eta) \) by \( q_\eta(v) = \lambda_\eta \cap f(v) \). Select \( x_\eta, y_\eta \in P_\kappa(\rho) \setminus \{\emptyset\} \) so that

- \( \{g_\xi : \xi \in x_\eta \cup y_\eta\} \subseteq G_\eta \);
- \( \{b \in \bigcap_{\xi \in x_\eta} C(g_\xi, \kappa, \lambda_\eta) : b \cap \kappa \subseteq C(p_\eta, \kappa, \lambda_\eta)\} \);
- \( \{b \in \bigcap_{\xi \in y_\eta} C(g_\xi, \kappa, \lambda_\eta) : b \cap \kappa \subseteq C(q_\eta, \kappa, \lambda_\eta)\} \).

We may find \( z \in P_\kappa(T) \) such that \( x_\eta \cup y_\eta \subseteq \bigcup_{\xi \in z} t(\eta) \) for every \( \eta \in \mu \).

Let us show that \( A \cap (\bigcap_{\xi \in z} D_\xi) \subseteq C(f, \kappa, \lambda) \). Thus let \( a \in A \cap (\bigcap_{\xi \in z} D_\xi) \) and \( v \in P_3(a \cap \chi) \). There must be \( \eta \in a \cap \mu \) such that \( v \subseteq \lambda_\eta \). Then \( a \cap \lambda_\eta \in \bigcap_{\xi \in z} C(g_\xi, \kappa, \lambda_\eta) \), so \( v \cup f(v) \subseteq \lambda_\sigma \) for some \( \sigma \in a \cap \mu \). Now \( a \cap \lambda_\sigma \subseteq \bigcap_{\xi \in y_\eta} C(g_\xi, \kappa, \lambda_\sigma) \), and therefore \( f(v) \subseteq a \).

\[ \square \]

### 8 Nowhere precipitousness of \( \text{NS}_{\kappa,\lambda}^\nu \)

Throughout this section it is assumed that \( \kappa \leq \text{cf}(\lambda) < \lambda \). Let \( \nu \) be a cardinal with \( \text{cf}(\lambda) \leq \nu < \lambda \). We will show that under certain conditions, \( \text{NS}_{\kappa,\lambda}^\nu \) is nowhere precipitous. Our proof will follow that of Theorem 2.1 in [13], except that we do not appeal to pcf theory.

Set \( \mu = \text{cf}(\lambda) \). We assume that \( \text{cf}(\kappa, \nu) < \lambda \) in case \( \nu > \mu \). Let \( \rho < \lambda \) be a regular cardinal such that \( \rho > \mu \) if \( \nu = \mu \), and \( \rho > \text{cf}(\kappa, \nu) \) otherwise. Select a continuous, increasing sequence \( \langle \lambda_\beta : \beta < \mu \rangle \) of cardinals so that \( \sup\{\lambda_\beta : \beta < \mu\} = \lambda \) and \( \lambda_0 > \rho \). Let \( E \) be the set of all infinite limit ordinals \( \alpha < \mu \) with \( \text{cf}(\alpha) < \kappa \). We define \( D \) as follows. If \( \nu = \mu \), we set \( D = E \). Otherwise we pick \( D \) in \( \text{NS}_{\kappa,\nu}^\nu \) so that

- for any \( d \in D \), \( \text{sup}(d \cap \mu) \) is an infinite limit ordinal;
- \( |D| = \text{cf}(\kappa, \nu) \).

We will show that if \( \text{cf}|D| < \lambda \) for every cardinal \( \tau < \lambda \), then \( \text{NS}_{\kappa,\lambda}^\nu \) is nowhere precipitous.

For \( d \in D \), put \( \alpha(d) = \text{sup}(d \cap \mu) \). Note that \( \alpha(d) \in E \). Moreover \( \alpha(d) = d \) in case \( \nu = \mu \).

Let \( W \) be the set of all \( a \in P_\kappa(\lambda) \) such that
Then clearly, \( W \in (\text{NS}^+_{\kappa, \lambda})^* \). For \( d \in D \), define \( W_d = \{ a \in W : \text{sup}(a \cap \mu) = d \} \) if \( \nu = \mu \), and \( W_d = \{ a \in W : a \cap \nu = d \} \) otherwise. Note that \( W \) is the disjoint union of the \( W_d \)'s. Moreover, \( \text{sup}(a \cap \lambda_{\alpha(d)}) = \lambda_{\alpha(d)} \) for every \( a \in W_d \).

**Lemma 8.1.** Suppose that there is \( T \subseteq P_\kappa(\lambda) \) such that

(a) \( |T \cap P(a)| < \rho \) for any \( a \in P_\kappa(\lambda) \);

(b) \( u(\rho, \tau) \leq |T| \) for every cardinal \( \tau \) with \( \rho \leq \tau < \lambda \).

Then for every \( R \in (\text{NS}^+_{\kappa, \lambda})^* \),

\[
\{ d \in D : |\{ a \cap \lambda_{\alpha(d)} : a \in R \cap W_d \}| \geq u(\rho, \lambda_{\alpha(d)}) \}
\]

lies in \( \text{NS}^+_{\mu} \) if \( \nu = \mu \), and in \( \text{NS}^+_{\kappa, \nu} \) otherwise.

**Proof.** For \( \beta \in \mu \), select \( Z_\beta \in I^+_{\rho, \lambda_{\beta}} \) with \( |Z_\beta| \leq |T| \). Then clearly there is \( Q \subseteq T \) with \( |\bigcup_{\beta<\mu} Z_\beta| = |Q| \). Pick a bijection \( i : \bigcup_{\beta<\mu} Z_\beta \to Q \) and let \( j \) denote the inverse of \( i \). For \( \alpha \in E \), define \( k_\alpha : P_\kappa(\lambda_{\alpha}) \to P_\rho(\lambda_{\alpha}) \) by \( k_\alpha(b) = \bigcup_{\beta \in Q \cap P(b)} (j(e) \cap \lambda_{\alpha}) \).

**Claim.** Let \( S \in (\text{NS}^+_{\kappa, \lambda})^* \). Then there is \( d \in D \) such that

\[
\{ k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) : a \in S \cap W_d \} \in I^+_{\rho, \lambda_{\alpha(d)}}.
\]

**Proof of the claim.** Assume otherwise. For \( d \in D \), select \( y_d \in P_\rho(\lambda_{\alpha(d)}) \) so that \( y_d \setminus k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) \neq \emptyset \) for every \( a \in S \cap W_d \). Set \( y = \bigcup_{d \in D} y_d \). Note that \( y \in P_\rho(\lambda) \). For \( \beta \in \mu \), pick \( z_\beta \in Z_\beta \) so that \( y \cap \lambda_{\beta} \subseteq z_\beta \). Now let \( H \) be the set of all \( a \in P_\kappa(\lambda) \) such that \( i(z_\beta) \in \bigcup_{\xi \in \epsilon^\alpha_\mu} P(a \cap \lambda_{\xi}) \) for every \( \beta \in a \cap \mu \).

Since \( H \in (\text{NS}^+_{\kappa, \lambda})^* \), we can find \( d \) in \( S \cap B \cap H \). Set \( d = \text{sup}(a \cap \mu) \) if \( \nu = \mu \), and \( d = a \cap \nu \) otherwise. Then \( a \in W_d \) and \( y_d \subseteq y \cap \lambda_{\alpha(d)} = \bigcup_{\beta \in \epsilon^\alpha_\mu} (y \cap \lambda_{\beta}) \subseteq \bigcup_{\beta \in \epsilon^\alpha_\mu} z_\beta = \bigcup_{\beta \in \epsilon^\alpha_\mu} j(i(z_\beta)) \subseteq k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) \). This contradiction completes the proof of the claim.

It is now easy to show that the conclusion of the lemma holds: Fix \( R \in (\text{NS}^+_{\kappa, \lambda})^* \), and \( A \) such that \( A \in \text{NS}^+_{\mu} \) if \( \nu = \mu \), and \( A \in \text{NS}^+_{\kappa, \nu} \) otherwise. Set \( Y = \bigcup_{d \in D \cap A} W_d \). Since \( Y \in (\text{NS}^+_{\kappa, \lambda})^* \), there must be some \( d \in D \) such that
Consider for instance the following situation: In $V$, GCH holds, $\sigma$ is a strong cardinal with $\rho < \sigma < \lambda$, and $\eta$ a cardinal greater than $\lambda$. Then by a result of Gitik and Magidor [6], there is a cardinal preserving, $\sigma^+$-cc forcing notion $P$ such that in $V^P$,

- no new bounded subsets of $\sigma$ are added;
- $\sigma$ changes its cofinality to $\omega$;
- $2^\sigma \geq \eta$.

Now working in $V^P$, let $T = P_{\omega_1}(\sigma)$. Then clearly $|T \cap P(a)| \leq 2^{\omega_1} \leq \kappa < \rho$ for any $a \in P_\kappa(\lambda)$. Moreover for any two uncountable cardinals $\chi$ and $\theta$ with $\text{cf}(\chi) = \chi < \sigma \leq \theta < \eta$,

$$u(\chi, \theta) = \max\{2^{\chi}, u(\chi, \theta)\} = \theta^{<\chi} = \sigma^{<\chi} = \sigma^{<\theta} = |T|.$$ 

Hence $u(\rho, \tau) \leq |T|$ for every cardinal $\tau$ with $\rho \leq \tau < \lambda$, so by Lemma 8.1 for any $R \in (\text{NS}^\nu_{\kappa, \lambda})^+$,

$$\begin{align*}
\{d \in D : |\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})\}
\end{align*}$$

lies in $\text{NS}^\nu_{\mu}$ if $\nu = \mu$, and in $\text{NS}^\nu_{\kappa, \nu}$ otherwise.

Note that for any cardinal $\chi$ with $\kappa \leq \chi \leq \sigma$, $\text{cf}(\text{NS}^\chi_{\kappa, \lambda}) = u(\kappa, \lambda)$ since $\text{cf}(\text{NS}^\chi_{\kappa, \lambda}) \leq (\lambda^{<\kappa})^{\chi} = (2^\kappa)^{\chi} = 2^\sigma$, and moreover, by Fact 2.9 and Proposition 4.6, $\text{cof}^{\nu}(\text{NS}^\chi_{\kappa, \lambda}) > \lambda$ in case $\mu \leq \chi$.

Let us observe that if $T \subseteq P_\kappa(\lambda)$ is, as in condition (a) of Lemma 8.1, such that $|T \cap P(a)| \leq u(\kappa, \lambda)$ for any $a \in P_\kappa(\lambda)$, then it is easy to see that $|T| \leq u(\kappa, \lambda)$.

**PROPOSITION 8.2.** Suppose that there is $T \subseteq P_\kappa(\lambda)$ and a cardinal $\pi$ with $\rho \leq \pi < \lambda$ such that

- $|T \cap P(a)| < \rho$ for any $a \in P_\kappa(\lambda)$;
- $\tau^{<\pi} \leq u(\rho, \tau) \leq |T|$ for every cardinal $\tau$ with $\pi < \tau < \lambda$.

Then $\text{NS}^{\nu}_{\kappa, \lambda}$ is nowhere precipitous.

**Proof.** By Fact 2.12 it suffices to show that $\Pi$ has a winning strategy in the game $G(\text{NS}^{\nu}_{\kappa, \lambda})$. We can assume without loss of generality that $\lambda_0 > \pi$. For $g : P_\kappa(\nu) \to P_\lambda(\kappa) \cap (\alpha < \mu, \text{define } g_\alpha : P_\kappa(\nu) \to P_\lambda(\alpha) \text{ by } g_\alpha(e) = g(e) \cap \alpha$.

**Claim 1.** Let $g : P_\kappa(\nu) \to P_\lambda(\kappa)$. Then

$$\{d \in D : \{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(g_\alpha(\lambda_{\alpha(d)}), \kappa, \lambda_{\alpha(d)})\} \subseteq C(g, \kappa, \lambda)\}$$
lies in $(\text{NS}_\mu|E)^*$ if $\nu = \mu$, and in $\text{NS}_{\kappa, \nu}^*$ otherwise.

**Proof of Claim 1.** We prove the claim in the case when $\nu > \mu$, and leave the proof in the case when $\nu = \mu$ to the reader. Define $h : P_3(\nu) \to \mu$ by $h(e) =$ the least $\beta < \mu$ such that $g(e) \subseteq \lambda_\beta$. Let $Q$ be the set of all $d \in D$ such that $h(e) \in d$ for every $e \in P_3(d)$. Then clearly $Q \in \text{NS}_{\kappa, \nu}^*$. Now fix $d \in Q$ and $a \in W_d$ such that $a \cap \lambda_\alpha(d) \subseteq g(a_\alpha(d), \kappa, \lambda_\alpha(d))$. Let $e \in P_3(a \cap \nu)$. Then $h(e) \in d$, so $g(e) \subseteq \lambda_\alpha(d)$. It follows that $g(e) \subseteq a$, since $g(e) \cap \lambda_\alpha(d) \subseteq a$. Thus $a \in C(g, \kappa, \lambda)$. This completes the proof of Claim 1.

**Claim 2.** Let $X \in (\text{NS}_{\kappa, \lambda}^*)^+$ and $Y \subseteq W$. Suppose that

$$Y \cap \{a \in W_d : a \cap \lambda_\alpha(d) \cap C(k, \kappa, \lambda_\alpha(d)) \neq \emptyset\} \neq \emptyset$$

whenever $d \in D$ and $k : P_3(\nu) \to P_3(\lambda_\alpha(d))$ are such that

$$\{|a \cap \lambda_\alpha(d) : a \in X \cap W_d \cap a \cap \lambda_\alpha(d) \subseteq C(k, \kappa, \lambda_\alpha(d))\} \subseteq u(\rho, \lambda_\alpha(d)).$$

Then $Y \cap (\text{NS}_{\kappa, \lambda}^*)^+$.

**Proof of Claim 2.** Fix $g : P_3(\nu) \to P_3(\lambda)$. By Lemma 8.1 and Claim 1, there must be $d \in D$ such that

$$\{|a \cap \lambda_\alpha(d) : a \in \{X \cap C(g, \kappa, \lambda) \cap W_d\} \subseteq u(\rho, \lambda_\alpha(d))\}$$

and

$$\{a \in W_d : a \cap \lambda_\alpha(d) \subseteq C(g_\alpha(d), \kappa, \lambda_\alpha(d))\} \subseteq C(g, \kappa, \lambda).$$

Then

$$Y \cap \{a \in W_d : a \cap \lambda_\alpha(d) \subseteq C(g_\alpha(d), \kappa, \lambda_\alpha(d))\} \neq \emptyset$$

since $a \cap \lambda_\alpha(d) \subseteq C(g_\alpha(d), \kappa, \lambda_\alpha(d))$ for every $a \in X \cap C(g, \kappa, \lambda) \cap W_d$. Hence $Y \cap C(g, \kappa, \lambda) \neq \emptyset$. This completes the proof of the claim.

Now to describe a strategy $\tau$ for player II in the game $G(\text{NS}_{\kappa, \lambda}^*)$, let

$$X_0, Y_0, X_1, \ldots, Y_{n-1}, X_n$$

be a partial play of the game. We may assume $X_0 \subseteq W$. We define a subset of $X_n, Y_n \in (\text{NS}_{\kappa, \lambda}^*)^+$ and its 1-1 enumeration $(y^n_{d, \xi} : d \in D \text{ and } \xi < |K^n_d|)$. Here $K^n_d$ is the set of all $k : P_3(\nu) \to P_3(\lambda_\alpha(d))$ such that

$$|X_n \cap \{a \in W_d : a \cap \lambda_\alpha(d) \subseteq C(k, \kappa, \lambda_\alpha(d))\}| \geq u(\rho, \lambda_\alpha(d)).$$

Fix $d \in D$ with $K^n_d$ nonempty. Enumerate $K^n_d$ as $(k^n_{d, \xi} : \xi < |K^n_d|)$. Note that $|k^n_d| \leq \lambda_\alpha(d) \leq u(\rho, \lambda_\alpha(d))$ (and $K^n_d \subseteq K^{n-1}_d$ by $X_n \subseteq X_{n-1}$). So by induction on $\xi < |k^n_d|$ we can choose $y^n_{d, \xi}$ from

$$X_n \cap \{a \in W_d : a \cap \lambda_\alpha(d) \subseteq C(k^n_{d, \xi}, \kappa, \lambda_\alpha(d))\} \setminus \{|y^n_{d, \xi} : \xi < \xi| \cup \{|y^n_{d, \xi} : \xi \leq \xi|\}.$$

Define $Y_n = \{y^n_{d, \xi} : d \in D \text{ and } \xi < |k^n_d|\}$. Then $Y_n$ is a subset of $X_n$ by construction, and is an element of $(\text{NS}_{\kappa, \lambda}^*)^+$ by Claim 2. Moreover the enumeration is 1-1 by construction and the definition of $W_d$.

To see that $\tau$ is a winning strategy, suppose that $X_0, Y_0, X_1, \ldots$ is a play during which player II obeyed the strategy $\tau$. We claim that $\bigcap_{n<\omega} Y_n = \emptyset$. Suppose to the contrary that $x \in \bigcap_{n<\omega} Y_n$. Let $d$ be sup$(x \cap \mu)$ if $\nu = \mu$, and $x \cap \nu$ otherwise. Then $d \in D$ and for each $n < \omega$, there is $\xi(n)$ such that $x = y^n_{d, \xi(n)}$. By
the choice of \( y^m_i \), we have \( \xi(n) < \xi(n-1) \) for each \( 0 < n < \omega \), a contradiction. \( \square \)

Let us observe the following. Suppose that there exist \( T \) and \( \pi \) as in the statement of Proposition 8.2. Then either \( \text{cof}(\text{NS}_{\kappa, \lambda}^\mu) = u(\kappa, \lambda) \), or \( \lambda^{< \mu} = \lambda \). To establish this, note that \( u(\kappa, \lambda) \leq \lambda^{< \kappa} \leq \lambda^{< \mu} \leq |T| \leq u(\kappa, \lambda) \), so \( |T| = u(\kappa, \lambda) = \lambda^{< \mu} \). It is now simple to see that \( |T| = \lambda \) if \( \tau^\nu < \lambda \) for every cardinal \( \tau < \lambda \), and \( |T| = \lambda^n \) otherwise.

**THEOREM 8.3.**

(i) Suppose that \( \tau^\mu < \lambda \) for every cardinal \( \tau < \lambda \). Then \( \text{NS}_{\kappa, \lambda}^\mu \) is nowhere precipitous.

(ii) Suppose that \( \nu > \mu \), and \( \tau^{\kappa(\nu)} < \lambda \) for every cardinal \( \tau < \lambda \). Then \( \text{NS}_{\kappa, \lambda}^\nu \) is nowhere precipitous.

(iii) Suppose that \( \mathcal{H}_{\kappa, \lambda} \) holds, and \( \tau^\mu < \lambda \) for every cardinal \( \tau < \lambda \). Then \( \text{NS}_{\kappa, \lambda} \) is nowhere precipitous.

**Proof.**

(i) : Put \( \nu = \mu \), \( \rho = \mu^+ \), \( \pi = 2^\mu = 2^{< \rho} \) and \( T = P_2(\lambda) \). Then clearly, \( |T \cap P(a)| \leq |a| < \kappa < \rho \) for any \( a \in P_\kappa(\lambda) \). Moreover, \( \tau^\nu = \tau^{< \rho} = u(\mu, \tau) \lambda = |T| \) for any cardinal \( \tau \) with \( \pi < \tau < \lambda \). Hence by Proposition 8.2, \( \text{NS}_{\kappa, \lambda}^\mu \) is nowhere precipitous.

(ii) : Put \( \rho = c(\kappa, \nu)^+ \), \( \pi = 2^{< \rho} \) and \( T = P_2(\lambda) \). Then clearly, \( |T \cap P(a)| \leq |a| < \kappa < \rho \) for all \( a \in P_\kappa(\lambda) \). Moreover, \( \tau^\nu \leq \tau^{< \rho} = u(\rho, \tau) < \lambda = |T| \) for every cardinal \( \tau \) with \( \pi < \tau < \lambda \). Now apply Proposition 8.2.

(iii) : Use (i) and Corollary 7.4 (i).

\( \square \)

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**References**


Université de Caen - CNRS
Laboratoire de Mathématiques
BP 5186
14032 Caen Cedex
France

27
pierre.matet@unicaen.fr

Institute of Mathematics
The Hebrew University of Jerusalem
91904 Jerusalem
Israel
shelah@math.huji.ac.il

and

Department of Mathematics
Rutgers University
New Brunswick, NJ 08854
USA
shelah@math.huji.ac.il