# THE NONSTATIONARY IDEAL ON $P_\kappa(\lambda)$ FOR $\lambda$ SINGULAR

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**Abstract**

We give a new characterization of the nonstationary ideal on $P_\kappa(\lambda)$ in the case when $\kappa$ is a regular uncountable cardinal and $\lambda$ a singular strong limit cardinal of cofinality at least $\kappa$.

## 0 Introduction

**Let $\kappa$ be a regular uncountable cardinal and $\lambda \geq \kappa$ be a cardinal.**
As [10] and [11] of which it is a continuation, this paper investigates ideals on $P_\kappa(\lambda)$ with some degree of normality. For $\delta \leq \lambda$, let $NS^\delta_{\kappa,\lambda}$ denotes the least $\delta$-normal ideal on $P_\kappa(\lambda)$. Thus $NS^\delta_{\kappa,\lambda}$ = the noncofinal ideal $I_{\kappa,\lambda}$ for any $\delta < \kappa$, and $NS^\lambda_{\kappa,\lambda}$ = the nonstationary ideal $NS_{\kappa,\lambda}$. $NSS_{\kappa,\lambda}$ denotes the least seminormal ideal on $P_\kappa(\lambda)$. It is simple to see that $NSS_{\kappa,\lambda} = NS_{\kappa,\lambda}$ in case $\mathrm{cf}(\lambda) < \kappa$. If $\lambda$ is regular, then by a result of Abe [1], $NSS_{\kappa,\lambda} = \bigcup_{\delta<\lambda} NS^\delta_{\kappa,\lambda}$.

One problem we address in the paper is whether for $\lambda > \kappa$ $NS_{\kappa,\lambda}$ is the restriction of a smaller ideal, i.e. whether $NS_{\kappa,\lambda} = J \mid A$ for some ideal $J \subset NS_{\kappa,\lambda}$ and some $A \in NS^*_{\kappa,\lambda}$. The question as stated has a positive answer (see [2]) with $J = \nabla^\lambda I_{\kappa,\lambda}$. By a result of Abe [1] we can also take $J = NSS_{\kappa,\lambda}$ in case $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. We investigate the possibility of taking $J = \bigcup_{\delta<\xi} NS^\delta_{\kappa,\lambda}$ for some $\xi \leq \lambda$. If $\lambda$ is regular, no such $J$ will work since then, by an argument of [7],

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there is no $A$ such that $NS_{\kappa,\lambda} = NSS_{\kappa,\lambda} \mid A$.

Let $\mathcal{H}_{\kappa,\lambda}$ assert that $\overline{\text{cof}}(NS_{\kappa,\tau}) \leq \lambda$ for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$, where $\overline{\text{cof}}(NS^{\tau}_{\kappa,\lambda})$ denotes the reduced cofinality of $NS^{\tau}_{\kappa,\lambda}$. Clearly, $\mathcal{H}_{\kappa,\lambda}$ follows from $2^{<\lambda} = \lambda$. But there are other situations in which $\mathcal{H}_{\kappa,\lambda}$ holds. For instance, if in $V$, GCH holds and $\lambda$ is a limit cardinal, and $\mathbb{P}$ is the forcing notion to add $\lambda^+$ Cohen reals, then in $V^{\mathbb{P}}, 2^{\aleph_0} > \lambda$ but, by results of [7], for every cardinal $\tau$ with $\kappa \leq \tau < \lambda, \text{cof}(NS_{\kappa,\tau}) = \tau^+$ and hence $\overline{\text{cof}}(NS_{\kappa,\tau}) \leq \lambda$.

It is known ([16], [11]) that if $\text{cf}(\lambda) < \kappa$, then $\mathcal{H}_{\kappa,\lambda}$ holds just in case $NS_{\kappa,\lambda} = I_{\kappa,\lambda} \mid A$ for some $A$. We will prove the following.

**Proposition 0.1.** *Suppose $\kappa \leq \text{cf}(\lambda) < \lambda$ and $\mathcal{H}_{\kappa,\lambda}$ holds. Then (a) $NS_{\kappa,\lambda} = NS^{\text{cf}(\lambda)}_{\kappa,\lambda} \mid A$ for some $A$, but (b) there is no $B$ such that $NS_{\kappa,\lambda} = (\bigcup\limits_{\delta<\text{cf}(\lambda)} NS^{\delta}_{\kappa,\lambda}) \mid B$.*

It is not known whether the converse holds :

**Question**. Suppose $\kappa \leq \text{cf}(\lambda) < \lambda$ and $NS_{\kappa,\lambda} = NS^{\text{cf}(\lambda)}_{\kappa,\lambda} \mid A$ for some $A$. Does it follow that $\mathcal{H}_{\kappa,\lambda}$ holds ?

If $\lambda$ is singular and $\mathcal{H}_{\kappa,\lambda}$ holds, then by the results above $NS_{\kappa,\lambda} = NS^{\text{cf}(\lambda)}_{\kappa,\lambda} \mid A$ for some $A$. The following problem is open.

**Question**. Is it consistent that "$\lambda$ is singular but $NS_{\kappa,\lambda} \neq NS^{\delta}_{\kappa,\lambda} \mid A$ for every $\delta < \lambda$ and every $A \in NS^{*}_{\kappa,\lambda}$" ?

For any infinite cardinal $\tau < \lambda$, let $u(\tau,\lambda) =$ the least size of any cofinal subset of $(P_\tau(\lambda), \subset)$.
Now suppose $\kappa \leq \text{cf}(\lambda) < \lambda$. Then by results of [11], there is no $A$ such that $NS_{\kappa,\lambda} = I_{\kappa,\lambda} \mid A$. And it is shown in [10] that for any $\delta$ such that $\kappa \leq \delta < \text{cf}(\lambda)$ and $u(\mid \delta \mid^+, \lambda) = \lambda$, there is no $A$ such that $NS_{\kappa,\lambda} = NS^{\delta}_{\kappa,\lambda} \mid A$. Thus assuming Shelah's Strong Hypothesis (SSH), $NS_{\kappa,\lambda} \neq NS^{\delta}_{\kappa,\lambda} \mid A$ for every $\delta < \text{cf}(\lambda)$ and every $A \in NS^{*}_{\kappa,\lambda}$.

**Question**. Is it consistent relative to some large cardinal that "$\kappa < \text{cf}(\lambda) < \lambda$ and $NS_{\kappa,\lambda} = NS^{\delta}_{\kappa,\lambda} \mid A$ for some $\delta < \text{cf}(\lambda)$ and some $A \in NS^{*}_{\kappa,\lambda}$"?

Another problem we consider is whether $NS^{\delta}_{\kappa,\lambda}$ is nowhere precipitous, where $\delta \leq \lambda$. As shown by Matsubara and Shioya [14], $I_{\kappa,\lambda}$ is nowhere precipitous, and in fact so is any ideal $J$ on $P_\kappa(\lambda)$ of cofinality $u(\kappa,\lambda)$. Thus for every ideal $J$ on $P_\kappa(\lambda)$,

$$\overline{\text{cof}}(J) \leq \lambda \Rightarrow \text{cof}(J) = u(\kappa,\lambda) \Rightarrow J \text{ is nowhere precipitous.}$$

We establish the following.

**Proposition 0.2.** *Suppose $\mathcal{H}_{\kappa,\lambda}$ holds, and let $\xi > \kappa$ be such that (a) $\xi$ is either a successor ordinal, or a limit ordinal of cofinality at least $\kappa$, and (b) $\xi \leq \eta$, where $\eta$ equals $\lambda + 1$ if $\mathrm{cf}(\lambda) < \kappa$, and $\mathrm{cf}(\lambda)$ otherwise. Then $\overline{\mathrm{cof}}\Big(\bigcup_{\delta<\xi} NS^{\delta}_{\kappa,\lambda}\Big) \leq \lambda$.*

It follows from Propositions 0.1 and 0.2 that if $\mathcal{H}_{\kappa,\lambda}$ holds, then $NSS_{\kappa,\lambda} \mid A = NS^{\delta}_{\kappa,\lambda} \mid A$ for some $A \in NS^{*}_{\kappa,\lambda}$, where $\delta$ equals $\mathrm{cf}(\lambda)$ if $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, and 0 otherwise.

Let us next consider cases when $\kappa \leq \delta \leq \lambda$ and $\mathrm{cof}\big(NS^{\delta}_{\kappa,\lambda}\big) > u(\kappa,\lambda)$. Goldring [7] and the second author proved that if $\lambda$ is regular and $\mu > \lambda$ is Woodin, then in $V^{\mathrm{Col}(\lambda,<\mu)}$ $NS_{\kappa,\lambda}$ is precipitous. On the other hand Matsubara and the second author [13] showed [(1)] that if $\lambda$ is a strong limit cardinal with $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, then $NS_{\kappa,\lambda}$ is nowhere precipitous. We establish the following.

**Proposition 0.3.** *Suppose $\kappa \leq \mathrm{cf}(\lambda) \leq \delta < \lambda$.*

(i) *If $\mid \delta \mid= \mathrm{cf}(\lambda)$ and $\tau^{\mathrm{cf}(\lambda)} < \lambda$ for every cardinal $\tau < \lambda$, then $NS^{\delta}_{\kappa,\lambda}$ is nowhere precipitous.*

(ii) *If $\mathrm{cf}(\lambda) <\mid \delta \mid$ and $\tau^{c(\kappa,\mid\delta\mid)} < \lambda$ for every cardinal $\tau < \lambda$, where $c(\kappa, \mid \delta \mid)$ denotes the least size of any closed unbounded subset of $P_\kappa(\mid \delta \mid)$, then $NS^{\delta}_{\kappa,\lambda}$ is nowhere precipitous.*

Note that if $\kappa \leq \mathrm{cf}(\lambda) \leq \delta < \lambda$ and the hypothesis of (i) (respectively, (ii)) of Proposition 0.3. holds, then $\lambda^{<\mathrm{cf}(\lambda)} = \lambda$, so by results of [11],

$$\mathrm{cof}(NS^{\delta}_{\kappa,\lambda}) \geq \overline{\mathrm{cof}}(NS^{\delta}_{\kappa,\lambda}) > \lambda = u(\kappa,\lambda).$$

The paper is organized as follows. Section 1 collects basic definitions and facts concerning ideals on $P_\kappa(\lambda)$. It is shown in Section 2 that $\overline{\mathrm{cof}}(NS^{\pi}_{\kappa,\lambda})$ is a nondecreasing function of $\pi$. In Section 3 we establish that if $\lambda$ is regular, then $\overline{\mathrm{cof}}(NSS_{\kappa,\lambda}) = \lambda$ just in case $\mathcal{H}_{\kappa,\lambda}$ holds. In Section 4, Proposition 0.2 is proved. In Section 5 we show that it is consistent relative to a large cardinal that "$\lambda$ is regular and $\overline{\mathrm{cof}}(NS_{\kappa,\lambda} \mid A) < \lambda$ for some $A$". It is shown in Section 6 that if $\lambda$ is singular and $\mathcal{H}_{\kappa,\lambda}$ holds, then $NS_{\kappa,\lambda} = NS^{\mathrm{cf}(\lambda)}_{\kappa,\lambda} \mid A$ for some $A$. Finally in Section 7 we prove Proposition 0.3 and note that it is consistent relative to a large cardinal that "there is an ideal $J$ on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) > \lambda$ but $\mathrm{cof}(J) = u(\kappa,\lambda)$."

1 At some point the first author claimed to have found an error in the proof but it turned out that the mistake was his.

# 1 Ideals on $P_\kappa(\lambda)$

In this section we collect basic material concerning ideals on $P_\kappa(\lambda)$.
$NS_\kappa$ denotes the nonstationary ideal on $\kappa$.
For a set $A$ and a cardinal $\rho$, let $P_\rho(A) = \{a \subseteq A :\mid a \mid < \rho\}$.
Given four cardinals $\tau, \rho, \chi$ and $\sigma$, we define $\mathrm{cov}(\tau, \rho, \chi, \sigma)$ as follows. If there is $X \subseteq P_\rho(\tau)$ with the property that for any $a \in P_\chi(\tau)$, we may find $Q \in P_\sigma(X)$ with $a \subseteq \cup Q$, we let $\mathrm{cov}(\tau, \rho, \chi, \sigma)$ = the least cardinality of any such $X$. Otherwise we let $\mathrm{cov}(\tau, \rho, \chi, \sigma) = 0$.
We let $\mathrm{cov}(\tau, \rho, \chi, \sigma) = u(\tau, \chi)$ in case $\rho = \chi$ and $\sigma = 2$.

**LEMMA 1.1.** ([15], pp. 85-86)) *Let $\tau, \rho, \chi$ and $\sigma$ be four cardinals such that $\tau \geq \rho \geq \chi \geq \omega$ and $\chi \geq \sigma \geq 2$. Then the following hold :*

(i) *If $\tau > \rho$, then $\mathrm{cov}(\tau, \rho, \chi, \sigma) \geq \tau$.*

(ii) $\mathrm{cov}(\tau, \rho, \chi, \sigma) = \mathrm{cov}(\tau, \rho, \chi, \omega \cdot \sigma)$.

(iii) $\mathrm{cov}(\tau^+, \rho, \chi, \sigma) = \tau^+ \cdot \mathrm{cov}(\tau, \rho, \chi, \sigma)$

(iv) *If $\tau > \rho$ and $\mathrm{cf}(\tau) < \sigma = \mathrm{cf}(\sigma)$, then* $\mathrm{cov}(\tau, \rho, \chi, \sigma) = \bigcup_{\rho \leq \tau' < \tau} \mathrm{cov}(\tau', \rho, \chi, \sigma)$.

(v) *If $\tau$ is a limit cardinal such that $\tau > \rho$ and $\mathrm{cf}(\tau) \geq \chi$, then* $\mathrm{cov}(\tau, \rho, \chi, \sigma) = \bigcup_{\rho \leq \tau' < \tau} \mathrm{cov}(\tau', \rho, \chi, \sigma)$.

$I_{\kappa,\lambda}$ denotes the set of all $A \subseteq P_\kappa(\lambda)$ such that $\{a \in A : b \subseteq a\} = \emptyset$ for some $a \in P_\kappa(\lambda)$.
By an *ideal* on $P_\kappa(\lambda)$, we mean a collection $J$ of subsets of $P_\kappa(\lambda)$ that is closed under subsets (i.e. $P(A) \subseteq J$ for all $A \in J$), $\kappa$-complete (i.e. $\cup X \in J$ for every $X \in P_\kappa(J)$), fine (i.e. $I_{\kappa,\lambda} \subseteq J$) and proper (i.e. $P_\kappa(\lambda) \notin J$ ).
Given an ideal $J$ on $P_\kappa(\lambda)$, let $J^+ = \{A \subseteq P_\kappa(\lambda) : A \notin J\}$ and $J^* = \{A \subseteq P_\kappa(\lambda) : P_\kappa(\lambda) \setminus A \in J\}$. For $A \in J^+$, let $J \mid A = \{B \subseteq P_\kappa(\lambda) : B \cap A \in J\}$.
Given a cardinal $\chi > \lambda$ and $f : P_\kappa(\lambda) \to P_\kappa(\chi)$, we let

$$f(J) = \{X \subseteq P_\kappa(\chi) : f^{-1}(X) \in J\}.$$

$\mathcal{M}_J$ denotes the collection of all maximal antichains in the partially ordered set $(J^+, \subseteq)$, i.e. of all $Q \subseteq J^+$ such that (i) $A \cap B \in J$ for any distinct $A, B \in Q$, and (ii) for every $C \in J^+$, there is $A \in Q$ with $A \cap C \in J^+$.
For a cardinal $\rho$, $J$ is *$\rho$-saturated* if $\mid Q \mid < \rho$ for every $Q \in \mathcal{M}_J$.
$\mathrm{cof}(J)$ denotes the least cardinality of any $X \subseteq J$ such that $J = \bigcup_{A \in X} P(A)$.
$\overline{\mathrm{cof}}(J)$ denotes the least size of any $Y \subseteq J$ with the property that for every $A \in J$, there is $y \in P_\kappa(Y)$ with $A \subseteq \cup y$.
$\mathrm{non}(J)$ denotes the least cardinality of any $A \in J^+$.
Note that $\mathrm{cof}(J) \geq \mathrm{non}(J) \geq \mathrm{non}(I_{\kappa,\lambda}) = u(\kappa, \lambda)$.

The following is well-known (see e.g. [10] and [11]).

**LEMMA 1.2.**

(i) $\lambda^{<\kappa} = 2^{<\kappa} \cdot u(\kappa, \lambda)$.

(ii) $\overline{\mathrm{cof}}(I_{\kappa,\lambda}) = \lambda$.

(iii) *Let $J$ be an ideal on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) \leq \lambda$. Then $\mathrm{cof}(J) = u(\kappa, \lambda)$.*

Shelah's Strong Hypothesis (SSH) asserts that for any two uncountable cardinals $\chi$ and $\rho$ with $\chi \geq \rho = \mathrm{cf}(\rho)$, $u(\rho, \chi)$ equals $\chi$ if $\mathrm{cf}(\chi) \geq \rho$, and $\chi^+$ otherwise.

**LEMMA 1.3.** ([8])

(i) *Suppose there is a $\pi$-saturated ideal on $P_\nu(\lambda)$, where $\pi$ and $\nu$ are two cardinals such that $\omega < \nu = \mathrm{cf}(\nu) \leq \lambda$ and $\pi < \nu \cap \kappa^+$. Then $u(\kappa, \lambda)$ equals $\lambda$ if $\mathrm{cf}(\lambda) \geq \kappa$, and $\lambda^+$ otherwise.*

(ii) *Suppose there is a regular uncountable cardinal $\nu < \lambda$ that is mildly $\pi^+$-ineffable for every cardinal $\pi$ with $\nu \leq \pi < \lambda$. Then (a) $u(\kappa, \lambda)$ equals $\lambda$ if $\mathrm{cf}(\lambda) \geq \kappa$, and $\lambda^+$ if $\omega < \mathrm{cf}(\lambda) < \kappa$, and (b) $\mathrm{cov}(\lambda, \kappa, \kappa, \omega_1) = \lambda$ if $\mathrm{cf}(\lambda) = \omega$.*

Numerous variations on the original notion of ideal normality have been considered over the years. One such variant is the concept of $\delta$-normality which has been studied by Abe [1].

Let $\delta \leq \lambda$. An ideal $J$ on $P_\kappa(\lambda)$ is *$\delta$-normal* if given $A \in J^+$ and $f : A \to \delta$ with the property that $f(a) \in a$ for all $a \in A$, there exists $B \in J^+ \cap P(A)$ such that $f$ is constant on $B$.

$NS^\delta_{\kappa,\lambda}$ denotes the smallest $\delta$-normal ideal on $P_\kappa(\lambda)$.
Note that $\lambda$-normality is the same as normality, so $NS^\lambda_{\kappa,\lambda} = NS_{\kappa,\lambda}$.
$c(\kappa, \lambda)$ denotes the least size of any closed unbounded subset of $P_\kappa(\lambda)$.

**LEMMA 1.4.**

(i) ([1]) *Let $\delta$ be an ordinal such that $\delta + \kappa \leq \lambda$. Then $NS^{\delta+\kappa}_{\kappa,\lambda} \setminus NS^\delta_{\kappa,\lambda} \neq \emptyset$.*

(ii) ([10]) *Suppose $\kappa \leq \delta < \lambda$. Then $NS^\delta_{\kappa,\lambda} = NS^{|\delta|}_{\kappa,\lambda} \mid A$ for some $A$.*

(iii) ([10]) *Let $\delta$ and $\eta$ be two ordinals such that $\mid \delta \mid < \mid \eta \mid < \lambda$ and $\kappa \leq \eta$. Then there is no $A$ such that $NS^\eta_{\kappa,\lambda} = NS^\delta_{\kappa,\lambda} \mid A$.*

**LEMMA 1.5.**

(i) ([11]) *$\overline{\mathrm{cof}}(NS^\delta_{\kappa,\lambda}) \geq \lambda$ for every $\delta \leq \lambda$.*

(ii) ([8], [11]) *Let $\delta \leq \lambda$. Then $\overline{\mathrm{cof}}(NS^\delta_{\kappa,\lambda} \mid A) = \overline{\mathrm{cof}}(NS^\delta_{\kappa,\lambda})$ for every $A \in NS^*_{\kappa,\lambda}$.*

(iii) ([11]) $\overline{\text{cof}}(NS_{\kappa,\lambda}) \geq \overline{\text{cof}}(NS_{\kappa,\rho})$ *for every cardinal* $\rho$ *with* $\kappa \leq \rho < \lambda$.

(iv) ([11]) *Suppose* $\text{cf}(\lambda) \geq \kappa$. *Then* $\overline{\text{cof}}(NS_{\kappa,\lambda}) > \lambda$.

The concept of $[\delta]^{<\theta}$-normality generalizes that of $\delta$-normality.

Let $\delta \leq \lambda$, and let $\theta$ be a cardinal with $\theta \leq \kappa$. An ideal $J$ on $P_\kappa(\lambda)$ is $[\delta]^{<\theta}$*-normal* if given $A \in J^+$ and $f : A \to P_\theta(\delta)$ with the property that $f(a) \in P_{|a\cap\theta|}(a \cap \delta)$ for all $a \in A$, there exists $B \in J^+ \cap P(A)$ such that $f$ is constant on $B$.
Note that for $\theta = \kappa$, $[\lambda]^{<\theta}$-normality is the same as strong normality.
**We set $\overline{\theta} = \theta$ if $\theta < \kappa$, or $\theta = \kappa$ and $\kappa$ is a limit cardinal, and $\overline{\theta} = \nu$ if $\theta = \kappa = \nu^+$.**

**LEMMA 1.6.** ([10])

(i) *Suppose that* $\delta < \kappa$, *or* $\theta < \kappa$, *or* $\kappa$ *is not a limit cardinal. Then there exists a* $[\delta]^{<\theta}$*-normal ideal on* $P_\kappa(\lambda)$ *if and only if* $|P_{\overline{\theta}}(\rho)| < \kappa$ *for every cardinal* $\rho < \kappa \cap (\delta + 1)$.

(ii) *Suppose that* $\delta \geq \kappa, \theta = \kappa$ *and* $\kappa$ *is a limit cardinal. Then there exists a* $[\delta]^{<\theta}$*-normal ideal on* $P_\kappa(\lambda)$ *if and only if* $\kappa$ *is a Mahlo cardinal.*

(iii) *Suppose there exists a* $[\kappa]^{<\theta}$*- normal ideal on* $P_\kappa(\lambda)$. *Then* $\kappa^{<\overline{\theta}} = \kappa$, *and* $\left(\pi^{<\overline{\theta}}\right)^{<\overline{\theta}} = \pi^{<\overline{\theta}}$ for every cardinal $\pi > \kappa$.

Assuming there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ denotes the smallest such ideal.

**LEMMA 1.7.** ([10])

(i) *Suppose* $\theta < 2$ *or* $\delta < \kappa$. *Then* $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda}$.

(ii) *Suppose* $2 \leq \theta \leq \omega$. *Then* $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{\delta}$.

(iii) *Suppose* $\mid \delta \mid^{<\overline{\theta}} = \mid \eta \mid^{<\overline{\pi}}$, *where* $\kappa \leq \eta \leq \lambda$ *and* $\pi$ *is a cardinal with* $2 \leq \pi \leq \kappa$. *Then* $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A = NS_{\kappa,\lambda}^{[\eta]^{<\pi}} \mid A$ *for some* $A \in \left(NS_{\kappa,\lambda}^{[\gamma]^{<\rho}}\right)^*$, *where* $\gamma = \delta \cup \eta$ *and* $\rho = \theta \cup \pi$.

Given an ordinal $\eta$, a cardinal $\pi$ and $f : P_\pi(\eta) \to P_\kappa(\lambda)$, let $C_f^{\kappa,\lambda}$ be the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \pi \neq \emptyset$ and $f(e) \subseteq a$ for every $e \in P_{|a\cap\pi|}(a \cap \eta)$.

**LEMMA 1.8.** ([10]) *Suppose* $A \subseteq P_\kappa(\lambda)$, $\kappa \leq \delta \leq \lambda$, *and* $\theta$ *is a cardinal with* $2 \leq \theta \leq \kappa$. *Then the following are equivalent :*

(i) $A \in NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

(ii) $A \cap C_f^{\kappa,\lambda} = \emptyset$ *for some* $f : P_{\overline{\theta}\cdot 3}(\delta) \to P_\kappa(\lambda)$.

(iii) $A \cap \{a \in C_g^{\kappa,\lambda} : a \cap \kappa \in \kappa\} = \emptyset$ for some $g : P_{\overline{\theta},3}(\delta) \to P_3(\lambda)$.

**LEMMA 1.9.** ([11]) *Let $\chi$ and $\theta$ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. Then the following hold :*

(i) *Let $J$ be a $[\chi]^{<\theta}$- normal ideal on $P_\kappa(\lambda)$. Then either* $\mathrm{cf}\big(\overline{\mathrm{cof}}(J)\big) < \kappa$, *or* $\mathrm{cf}\big(\overline{\mathrm{cof}}(J)\big) > \chi^{<\overline{\theta}}$.

(ii) *If $\chi^{<\overline{\theta}} < \lambda$, then* $\overline{\mathrm{cof}}\big(NS_{\kappa,\lambda}^{[\chi]^{<\theta}}\big) \geq \lambda$.

**LEMMA 1.10.** ([10], [11]) *Suppose $\kappa \leq \delta < \lambda$, and $\theta$ is a cardinal with $2 \leq \theta \leq \kappa$. Then the following hold :*

(i) $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \overline{\mathrm{cof}}(NS_{\kappa,|\delta|}^{|\delta|^{<\theta}}) \cdot \mathrm{cov}\big(\lambda, (|\,\delta\,|^{<\overline{\theta}})^+, (|\,\delta\,|^{<\overline{\theta}})^+, \kappa\big)$ *and*
$\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \mathrm{cof}(NS_{\kappa,|\delta|}^{|\delta|^{<\theta}}) \cdot \mathrm{cov}\big(\lambda, (|\,\delta\,|^{<\overline{\theta}})^+, (|\,\delta\,|^{<\overline{\theta}})^+, 2\big)$.

(ii) *If $\lambda$ is a limit cardinal and either* $\mathrm{cf}(\lambda) < \kappa$ *or* $\mathrm{cf}(\lambda) > |\,\delta\,|^{<\overline{\theta}}$, *then* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \bigcup_{\delta<\tau<\lambda} \overline{\mathrm{cof}}(NS_{\kappa,\tau}^{[\delta]^{<\theta}})$.

For a cardinal $\tau, \mathfrak{d}_{\kappa,\lambda}^{\tau}$ denotes the smallest cardinality of any family $F$ of functions from $\tau$ to $P_\kappa(\lambda)$ with the property that for any $g : \tau \to P_\kappa(\lambda)$, there is $f \in F$ such that $g(\alpha) \subseteq f(\alpha)$ for every $\alpha < \tau$.

**LEMMA 1.11.** ([10])

(i) *For any cardinal $\tau > 0$,* $\mathrm{cf}(\mathfrak{d}_{\kappa,\lambda}^{\tau}) > \tau$.

(ii) *Suppose $0 < \delta \leq \lambda$, and $\theta$ is a cardinal with $0 < \theta \leq \kappa$. Then* $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A) = \mathfrak{d}_{\kappa,\lambda}^{|P_{\overline{\theta}}(\delta)|}$ *for every* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$.

Next let us recall a few facts concerning the notion of precipitousness.

An ideal $J$ on $P_\kappa(\lambda)$ is *precipitous* if whenever $A \in J^+$ and $< Q_n : n < \omega >$ is a sequence of members of $\mathcal{M}_{J|A}$ such that $Q_{n+1} \subseteq \bigcup_{B \in Q_n} P(B)$ for all $n < \omega$, there exists $f \in \prod_{n\in\omega} Q_n$ such that $f(0) \supseteq f(1) \supseteq \ldots$ and $\bigcap_{n<\omega} f(n) \neq \emptyset$. $J$ is *nowhere precipitous* if for each $A \in J^+, J \mid A$ is not precipitous.
Let $G(J)$ denote the following two-player game lasting $\omega$ moves, with player I making the first move : I and II alternately pick members of $J^+$, thus building a sequence $< X_n : n < \omega >$, subject to the condition that $X_0 \supseteq X_1 \supseteq \ldots$ II wins $G(J)$ just in case $\bigcap_{n<\omega} X_n = \emptyset$.

**LEMMA 1.12.** ([5]) *An ideal $J$ on $P_\kappa(\lambda)$ is nowhere precipitous if and only if* II *has a winning strategy in the game $G(J)$.*

The following is a straightforward generalization of a result of Foreman [4] :

**PROPOSITION 1.13.** *Let $\chi$ and $\theta$ be two cardinals such that $\chi \leq \lambda$ and $\theta \leq \kappa$. Then every $[\chi]^{<\theta}$-normal, $(\chi^{<\overline{\theta}})^+$-saturated ideal on $P_\kappa(\lambda)$ is precipitous.*

**LEMMA 1.14.** ([14]) *Suppose $J$ is an ideal on $P_\kappa(\lambda)$ such that $\mathrm{cof}(J) = non(J)$. Then $J$ is nowhere precipitous.*

Thus for an ideal $J$ on $P_\kappa(\lambda)$,

$$\overline{\mathrm{cof}}(J) \leq \lambda \Rightarrow \mathrm{cof}(J) = u(\kappa, \lambda) \Rightarrow J \text{ is nowhere precipitous.}$$

Let $\tau$ be a cardinal with $\kappa \leq \tau \leq \lambda$. It is simple to see that if GCH holds and either $\mathrm{cf}(\lambda) < \kappa$ or $\tau < \mathrm{cf}(\lambda)$, then $\mathrm{cof}(NS^{\tau}_{\kappa,\lambda}) = u(\kappa, \lambda)$. Note that if SSH holds and $\kappa \leq \mathrm{cf}(\lambda) \leq \tau$, then by Lemmas 1.5 (i) and 1.9, $\mathrm{cof}(NS^{\tau}_{\kappa,\lambda}) > u(\kappa, \lambda)$.

**PROPOSITION 1.15.** *Suppose $\sigma$ is a strong limit cardinal with $\mathrm{cf}(\sigma) < \kappa < \sigma \leq \lambda \leq 2^\sigma$. Then the following hold :*

(i) $\mathrm{cof}(NS^{\tau}_{\kappa,\lambda}) = u(\kappa, \lambda)$ *for every cardinal $\tau$ with $\kappa \leq \tau \leq \sigma$.*

(ii) *Suppose $2^\lambda = 2^\sigma$. Then $\mathrm{cof}(NS^{\tau}_{\kappa,\lambda}) = u(\kappa, \lambda)$ for every cardinal $\tau$ with $\sigma < \tau \leq \lambda$.*

**Proof.**

(i) : Let $\tau$ be a cardinal with $\kappa \leq \tau \leq \sigma$. If $\tau = \lambda$, then

$$\mathrm{cof}(NS^{\tau}_{\kappa,\lambda}) \leq 2^\lambda = \lambda^{<\kappa} = u(\kappa, \lambda).$$

Otherwise by Lemma 1.10

$$\mathrm{cof}(NS^{\tau}_{\kappa,\lambda}) = \mathrm{cof}(NS_{\kappa,\tau}) \cdot u(\tau^+, \lambda) \leq \lambda^\tau = \sigma^\tau = \sigma^{\mathrm{cf}(\sigma)} \leq \lambda^{<\kappa} = u(\kappa, \lambda).$$

(ii) : Given a cardinal $\tau$ with $\sigma < \tau \leq \lambda$,

$$\mathrm{cof}(NS^{\tau}_{\kappa,\lambda}) \leq 2^\lambda = 2^\sigma = \sigma^{\mathrm{cf}(\sigma)} = u(\kappa, \lambda). \qquad \square$$

## 2 $\overline{\mathbf{cof}}(NS^{\chi}_{\kappa,\lambda})$

By Lemma 1.11 (ii), $\mathrm{cof}(NS^{\chi}_{\kappa,\lambda}) = \partial^{\chi}_{\kappa,\lambda}$ for any cardinal $\chi$ with $\kappa \leq \chi \leq \lambda$. We now derive a similar formula for $\overline{\mathrm{cof}}(NS^{\chi}_{\kappa,\lambda})$.

For a cardinal $\tau$, $\overline{\mathfrak{d}}^{\tau}_{\kappa,\lambda}$ denotes the smallest cardinality of any family $F$ of functions from $\tau$ to $P_\kappa(\lambda)$ with the property that for any $g : \tau \to P_\kappa(\lambda)$, there is $Z \in P_\kappa(F)$ such that $g(\alpha) \subseteq \bigcup_{f \in Z} f(\alpha)$ for every $\alpha < \tau$.

**LEMMA 2.1.** *Let $\theta$ and $\chi$ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. Then* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\chi]^{<\theta}}) \leq \overline{\mathfrak{d}}_{\kappa,\lambda}^{\chi^{<\overline{\theta}}}$.

**Proof.** Select a collection $G$ of functions from $P_{\overline{\theta}\cdot 3}(\chi)$ to $P_\kappa(\lambda)$ so that $\mid G \mid = \overline{\mathfrak{d}}_{\kappa,\lambda}^{\chi^{<\overline{\theta}}}$ and for any $k : P_{\overline{\theta}\cdot 3}(\chi) \to P_\kappa(\lambda)$, there is $Z_k \in P_\kappa(G)$ such that $k(e) \subseteq \bigcup_{g \in Z_k} g(e)$ for all $e \in P_{\overline{\theta}\cdot 3}(\chi)$. Then clearly for each $k : P_{\overline{\theta}\cdot 3}(\chi) \to P_\kappa(\lambda)$, $\bigcap_{g \in Z_k} C_g^{\kappa,\lambda} \subseteq C_k^{\kappa,\lambda}$. Hence $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\chi]^{<\theta}}) \leq \mid G \mid$. $\square$

**LEMMA 2.2.** *Let $\theta$ and $\chi$ be two cardinals such that $\omega \leq \theta = \mathrm{cf}(\theta) < \kappa \leq \chi \leq \lambda$. Then* $\overline{\mathfrak{d}}_{\kappa,\lambda}^{\chi^{<\theta}} \leq u\big(\theta, \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\chi]^{<\theta}})\big)$.

**Proof.** Pick a collection $H$ of functions from $P_\theta(\chi) \to P_3(\lambda)$ so that $\mid H \mid = \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\chi]^{<\theta}})$ and for any $A \in (NS_{\kappa,\lambda}^{[\chi]^{<\theta}})^*$, there is $Q \in P_\kappa(H) \setminus \{\emptyset\}$ with $\big\{b \in \bigcap_{h \in Q} C_h^{\kappa,\lambda} : b \cap \kappa \in \kappa\big\} \subseteq A$. Select $\mathfrak{X} \subseteq P_\theta(H) \setminus \{\emptyset\}$ so that $\mid \mathfrak{X} \mid = u(\theta, \mid H \mid)$ and for any $Z \in P_\theta(H)$, there is $X \in \mathfrak{X}$ with $Z \subseteq X$. For $X \in \mathfrak{X}$, define $t_X : P_\theta(\chi) \to P_\kappa(\lambda)$ by $t_x(e) = \cap T_{X,e}$, where

$$T_{X,e} = \Big\{b \in \bigcap_{h \in X} C_h^{\kappa,\lambda} : e \cup \theta \subseteq b \text{ and } b \cap \kappa \in \kappa\Big\}.$$

Note that $t_X(e) \in T_{X,e}$, and $t_Y(e) \subseteq t_X(e)$ for all $Y \in \mathfrak{X} \cap P(X)$.
Now fix $f : P_\theta(\chi) \to P_\kappa(\lambda)$. We may find $W \in P_\kappa(\mathfrak{X})$ such that

$$\Big\{b \in \bigcap_{h \in \cup W} C_h^{\kappa,\lambda} : b \cap \kappa \in \kappa\Big\} \subseteq C_f^{\kappa,\lambda},$$

$\theta \leq \mid W \mid$ and for any $K \in P_\theta(W)$, there is $Z \in W$ with $\cup K \subseteq Z$. For $e \in P_\theta(\chi)$, put $b_e = \bigcup_{X \in W} t_X(e)$. Note that $b_e \cap \kappa \in \kappa$.

**Claim.** *Let $k \in \cup W$. Then $b_e \in C_k^{\kappa,\lambda}$.*

**Proof of Claim.** Fix $d \in P_\theta(b_e \cap \chi)$. Pick $\varphi : d \to W$ so that $\beta \in t_{\varphi(\beta)}(e)$ for every $\beta \in d$. Select $Y \in W$ with $k \in Y$. There must be $Z \in W$ such that $Y \bigcup \big(\bigcup_{\beta \in d} \varphi(\beta)\big) \subseteq Z$. Then $d \in P_\theta(t_Z(e))$ and $t_Z(e) \in C_k^{\kappa,\lambda}$, so $k(d) \subseteq t_Z(e) \subseteq b_e$. This completes the proof of the claim.

Thus $b_e \in \bigcap_{h \in \cup W} C_h^{\kappa,\lambda}$. Hence $b_e \in C_f^{\kappa,\lambda}$, and consequently $f(e) \subseteq b_e$. $\square$

**PROPOSITION 2.3.** *Let $\chi$ be a cardinal with $\kappa \leq \chi \leq \lambda$. Then $\overline{\mathrm{cof}}(NS^{\chi}_{\kappa,\lambda}) = \overline{\mathfrak{d}}^{\chi}_{\kappa,\lambda}$.*

**Proof.** By Lemmas 2.1 and 2.2. □

**COROLLARY 2.4.** *Let $\pi$ and $\chi$ be two cardinals such that $\kappa \leq \pi < \chi \leq \lambda$.*

*Then $\overline{\mathrm{cof}}(NS^{\pi}_{\kappa,\lambda}) \leq \overline{\mathrm{cof}}(NS^{\chi}_{\kappa,\lambda})$.*

## 3 $NSS_{\kappa,\lambda}$

An ideal $J$ on $P_\kappa(\lambda)$ is *seminormal* if it is $\delta$-normal for every $\delta < \lambda$.
$NSS_{\kappa,\lambda}$ denotes the smallest seminormal ideal on $P_\kappa(\lambda)$.

**LEMMA 3.1.**

(i) (Folklore) *Suppose* $\mathrm{cf}(\lambda) < \kappa$. *Then $NS_{\kappa,\lambda} = NSS_{\kappa,\lambda}$.*

(ii) ([1]) *Suppose $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. Then $NS_{\kappa,\lambda} = NSS_{\kappa,\lambda} \mid A$ for some $A$.*

**PROPOSITION 3.2.** *Suppose $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. Then $\overline{\mathrm{cof}}(NSS_{\kappa,\lambda}) > \lambda$.*

**Proof.** By Lemmas 1.5 (iv) and 3.1. □

We will see that "$\overline{\mathrm{cof}}(NSS_{\kappa,\lambda}) > \lambda$" needs not hold in case $\lambda$ is regular. Note that if $\lambda$ is regular, then by Lemma 1.5 (iv), $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) > \lambda$.

**LEMMA 3.3.** ([1]) *Suppose $\lambda$ is regular. Then $NSS_{\kappa,\lambda} = \bigcup_{\delta<\lambda} NS^{\delta}_{\kappa,\lambda}$.*

**Proof.** It is immediate that $\bigcup_{\delta<\lambda} NS^{\delta}_{\kappa,\lambda} \subseteq NSS_{\kappa,\lambda}$. To show the reverse inclusion, fix $A \in \left(\bigcup_{\delta<\lambda} NS^{\delta}_{\kappa,\lambda}\right)^+$, $\eta$ with $\kappa \leq \eta < \lambda$, and $f : A \longrightarrow \eta$ with the property that $f(a) \in a$ for all $a \in A$. For $\xi$ with $\eta \leq \xi < \lambda$, we may find $B_\xi \in (NS^{\xi}_{\kappa,\lambda})^+ \cap P(A)$ and $\gamma_\xi < \eta$ such that $f$ takes the constant value $\gamma_\xi$ on $B_\xi$. There must be $\beta < \eta$ and $Z \subseteq \{\xi : \eta \leq \xi < \lambda\}$ such that $\mid Z \mid \; = \lambda$ and $\gamma_\xi = \beta$ for all $\xi \in Z$. Now set $C = \bigcup_{\xi\in Z} B_\xi$. Then clearly $C \in \left(\bigcup_{\delta<\lambda} NS^{\delta}_{\kappa,\lambda}\right)^+$. Moreover $f$ is identically $\beta$ on $C$. □

**LEMMA 3.4.** ([11]) *Suppose $\theta$ is a cardinal with $2 \leq \theta \leq \kappa$, and $J$ is an ideal on $P_\kappa(\lambda)$ such that $J \subseteq NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}$ and $\overline{\mathrm{cof}}(J) \leq \lambda^{<\overline{\theta}}$. Then $J \mid A = I_{\kappa,\lambda} \mid A$ for* *some* $A \in \left(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$.

In particular, if $J \subseteq NS_{\kappa,\lambda}$ and $\overline{\mathrm{cof}}(J) \leq \lambda$, then $J \mid D = I_{\kappa,\lambda} \mid D$ for some $D \in NS_{\kappa,\lambda}^*$.

**LEMMA 3.5.** ([11]) *Suppose $\theta$ is a cardinal with $2 \leq \theta \leq \kappa$, and let $\sigma$ be the least cardinal $\tau$ such that $\tau^{<\overline{\theta}} \geq \lambda$. Then $\overline{\mathrm{cof}}(I_{\kappa,\lambda} \mid A) \geq \sigma$ for every $A \in \left(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$.*

**PROPOSITION 3.6.** *Suppose $\theta$ is a cardinal with $2 \leq \theta \leq \kappa$, and $J$ is an ideal on $P_\kappa(\lambda)$ with $J \subseteq NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}$. Let $\sigma$ be the least cardinal $\tau$ such that $\tau^{<\overline{\theta}} \geq \lambda$. Then $\overline{\mathrm{cof}}(J) \geq \sigma$.*

**Proof.** If $\overline{\mathrm{cof}}(J) > \lambda^{<\overline{\theta}}$, there is nothing to prove. Otherwise, there is by Lemma 3.4 $A \in \left(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$ such that $J \mid A = I_{\kappa,\lambda} \mid A$. Then by Lemma 3.5, $\sigma \leq \overline{\mathrm{cof}}(I_{\kappa,\lambda} \mid A) \leq \overline{\mathrm{cof}}(J)$. □

In particular, $\overline{\mathrm{cof}}(J) \geq \lambda$ for any ideal $J \subseteq NS_{\kappa,\lambda}$.

**LEMMA 3.7.** ([8])

(i) *Suppose $\lambda$ is a successor cardinal, say $\lambda = \nu^+$. Then $NSS_{\kappa,\lambda} \mid C = I_{\kappa,\lambda} \mid C$ for some $C \in NS_{\kappa,\lambda}^*$ if and only if $\overline{\mathrm{cof}}(NS_{\kappa,\nu}) \leq \lambda$.*

(ii) *Suppose $\lambda$ is a regular limit cardinal. Then $NSS_{\kappa,\lambda} \mid C = I_{\kappa,\lambda} \mid C$ for some $C \in NS_{\kappa,\lambda}^*$ if and only if $\overline{\mathrm{cof}}(NS_{\kappa,\tau}) \leq \lambda = \mathrm{cov}(\lambda, \tau^+, \tau^+, \kappa)$ for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$.*

Recall from the introduction that $\mathcal{H}_{\kappa,\lambda}$ is said to hold if $\overline{\mathrm{cof}}(NS_{\kappa,\tau}) \leq \lambda$ for every cardinal $\tau$ with $\kappa \leq \tau < \lambda$.

**PROPOSITION 3.8.** *Suppose $\lambda$ is a regular cardinal. Then the following are equivalent :*

(i) *$\mathcal{H}_{\kappa,\lambda}$ holds.*

(ii) *$\overline{\mathrm{cof}}(NSS_{\kappa,\lambda}) = \lambda$.*

(iii) *$NSS_{\kappa,\lambda} \mid C = I_{\kappa,\lambda} \mid C$ for some $C \in NS_{\kappa,\lambda}^*$.*

**Proof.**
(i)$\longrightarrow$ (ii) : By Proposition 3.6, $\overline{\mathrm{cof}}(NSS_{\kappa,\lambda}) \geq \lambda$. For the reverse inequality, we consider two cases. First suppose $\lambda$ is a successor cardinal, say $\kappa = \nu^+$. Then by Lemma 3.3 $NSS_{\kappa,\lambda} = \bigcup_{\nu \leq \delta < \lambda} NS_{\kappa,\lambda}^\delta$. Now for $\nu \leq \delta < \lambda$,

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^\delta) \leq \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^\nu) = \overline{\mathrm{cof}}(NS_{\kappa,\nu}) \cdot \mathrm{cov}(\lambda, \lambda, \lambda, \kappa) \leq \lambda \cdot \lambda = \lambda$$

by Lemmas 1.4 (ii) and 1.10. Hence $\overline{\mathrm{cof}}(\bigcup_{\nu\le\delta<\lambda} NS^{\delta}_{\kappa,\lambda}) \le \lambda$.

Next suppose $\lambda$ is a limit cardinal. Given a cardinal $\chi$ with $\kappa \le \chi < \lambda$, by Corollary 2.4 $\overline{\mathrm{cof}}(NS^{\chi}_{\kappa,\tau}) \le \lambda$ for every cardinal $\tau$ with $\chi \le \tau < \lambda$, so by Lemma 1.10 $\overline{\mathrm{cof}}(NS^{\chi}_{\kappa,\lambda}) \le \lambda$. It follows that $\overline{\mathrm{cof}}(NSS_{\kappa,\lambda}) \le \lambda$ since by Lemma 3.3 $NSS_{\kappa,\lambda} = \bigcup_{\kappa\le\chi<\lambda} NS^{\chi}_{\kappa,\lambda}$.

(ii)$\longrightarrow$ (iii) : By Lemma 3.4.
(iii) $\longrightarrow$ (i) : By Lemmas 1.5 (iii) and 3.7. □

# 4 Ideals $J$ on $P_\kappa(\lambda)$ with $\overline{\mathbf{cof}}(J) = \lambda$

In this section we look for cases when $\overline{\mathrm{cof}}(\bigcup_{\delta<\xi} NS^{\delta}_{\kappa,\lambda}) = \lambda$, where $\kappa < \xi \le \lambda+1$. We start with the following observation.

**LEMMA 4.1.** *Suppose $K \subseteq NS_{\kappa,\lambda}$ is an ideal on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(K) \le \lambda$, and $\xi$ is an ordinal such that (a) $\kappa < \xi \le \lambda + 1$, (b) $\xi$ is either a successor ordinal, or a limit ordinal of cofinality at least $\kappa$, and (c) $\bigcup_{\delta<\xi} NS^{\delta}_{\kappa,\lambda} \subseteq K$. Then $\overline{\mathrm{cof}}(\bigcup_{\delta<\xi} NS^{\delta}_{\kappa,\lambda}) = \lambda$.*

**Proof.** By Lemma 3.5 we may find $A \in NS^{*}_{\kappa,\lambda}$ such that $K \mid A = I_{\kappa,\lambda} \mid A$. For any cardinal $\chi$ with $\kappa \le \chi < \xi$, $NS^{\chi}_{\kappa,\lambda} \mid A = I_{\kappa,\lambda} \mid A$, so by Lemma 1.5 (ii) $\overline{\mathrm{cof}}(NS^{\chi}_{\kappa,\lambda}) \le \lambda$. Hence by Lemma 1.4 (ii) $\overline{\mathrm{cof}}(NS^{\delta}_{\kappa,\lambda}) \le \lambda$ for every $\delta$ with $\kappa \le \delta < \xi$. It easily follows that $\overline{\mathrm{cof}}(\bigcup_{\delta<\xi} NS^{\delta}_{\kappa,\lambda}) \le \lambda$. the reverse inequality holds by Proposition 3.6. □

So we are looking for a large $K \subseteq NS_{\kappa,\lambda}$ with $\overline{\mathrm{cof}}(K) \le \lambda$. Assuming $\mathcal{H}_{\kappa,\lambda}$ holds, we can take $K = \bigcup_{\delta<\mathrm{cf}(\lambda)} NS^{\delta}_{\kappa,\lambda}$ if $\lambda$ is a singular cardinal of cofinality at least $\kappa$, and $K = NSS_{\kappa,\lambda}$ otherwise.

**LEMMA 4.2.** ([11]) *Let $\theta$ be a cardinal with $2 \le \theta \le \kappa$. Suppose $\overline{\theta} \le \mathrm{cf}(\lambda) < \kappa$. Then for any cardinal $\nu$ with $\kappa \le \nu < \lambda$, $\overline{\mathrm{cof}}\big(NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}\big) \le \bigcup_{\nu\le\tau<\lambda} \overline{\mathrm{cof}}(NS^{[\tau]^{<\theta}}_{\kappa,\tau})$.*

**PROPOSITION 4.3.** *Let $\theta$ be a cardinal with $2 \le \theta \le \kappa$. Suppose $\overline{\theta} \le \mathrm{cf}(\lambda) < \kappa$ and there is a cardinal $\nu$ with $\kappa \le \nu < \lambda$ such that for any cardinal $\tau$ with $\nu \le \tau < \lambda$, $\overline{\mathrm{cof}}\big(NS^{[\tau]^{<\theta}}_{\kappa,\tau}\big) \le \lambda$ and $\tau^{<\overline{\theta}} < \lambda$. Then $\overline{\mathrm{cof}}\big(NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}\big) = \lambda$.*

**Proof.** By Proposition 3.6 and Lemma 4.2. $\square$

In particular, if $\mathrm{cf}(\lambda) < \kappa$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) = \lambda$.

Note that if $\overline{\mathrm{cof}}\big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big) = \lambda$, then by Lemma 3.4 $NS_{\kappa,\lambda}^{[\lambda]^{<\theta}} = I_{\kappa,\lambda} \mid C$ for some $C$.

**LEMMA 4.4.** ([10]) *Let $A \in I_{\kappa,\lambda}^+$ be such that $|\ \{a \in A : b \subseteq a\}| = |A|$ for every $b \in P_\kappa(\lambda)$. Then $A$ can be decomposed into $|A|$ pairwise disjoint members of $I_{\kappa,\lambda}^+$.*

**PROPOSITION 4.5.** *Let $\theta$ be a cardinal with $2 \leq \theta \leq \kappa$. Suppose there is $C$ such that $NS_{\kappa,\lambda}^{[\lambda]^{<\theta}} = I_{\kappa,\lambda} \mid C$. Then $P_\kappa(\lambda)$ can be split into $\pi$ members of $\big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^+$, where $\pi$ is the least size of any member of $\big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^*$.*

**Proof.** Pick $D \in \big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^*$. Then by Lemma 4.4, $C \cap D$ can be decomposed into $\pi$ pairwise disjoints members of $\big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^+$. $\square$

In particular, if $NS_{\kappa,\lambda} = I_{\kappa,\lambda} \mid C$ for some $C$, then $P_\kappa(\lambda)$ can be split into $c(\kappa, \lambda)$ disjoint stationary sets.

**PROPOSITION 4.6.** *Suppose $\theta$ and $\rho$ are two cardinals such that $\omega \leq \theta = \mathrm{cf}(\theta) < \kappa \leq \rho \leq \lambda$, $u(\theta, \lambda) = \lambda$, and either $\mathrm{cf}(\lambda) < \kappa$ or $\mathrm{cf}(\lambda) > \rho^{<\theta}$. Suppose further that for every cardinal $\tau$ with $\rho \leq \tau < \lambda$, $\overline{\mathrm{cof}}\big(NS_{\kappa,\tau}^{[\tau]^{<\theta}}\big) \leq \lambda$. Then $\overline{\mathrm{cof}}\big(NS_{\kappa,\lambda}^{[\rho]^{<\theta}}\big) \leq \lambda$.*

**Proof.** It suffices to show that $\overline{\mathrm{cof}}\big(NS_{\kappa,\tau}^{[\rho]^{<\theta}}\big) \leq \lambda$ for any cardinal $\tau$ with $\rho \leq \tau < \lambda$ since by Lemmas 1.1 and 1.10 $\overline{\mathrm{cof}}\big(NS_{\kappa,\lambda}^{[\rho]^{<\theta}}\big) = \bigcup\limits_{\rho<\tau<\lambda} \overline{\mathrm{cof}}\big(NS_{\kappa,\tau}^{[\rho]^{<\theta}}\big)$ if $\lambda$ is a limit cardinal, and $\overline{\mathrm{cof}}\big(NS_{\kappa,\lambda}^{[\rho]^{<\theta}}\big) = \lambda \cdot \overline{\mathrm{cof}}\big(NS_{\kappa,\nu}^{[\rho]^{<\theta}}\big)$ if $\lambda = \nu^+$. Now for any cardinal $\tau$ with $\rho \leq \tau < \lambda$,

$$\overline{\mathrm{cof}}\big(NS_{\kappa,\tau}^{[\rho]^{<\theta}}\big) \leq \overline{\partial}_{\kappa,\tau}^{\rho^{<\overline{\theta}}} \leq \overline{\partial}_{\kappa,\tau}^{\tau^{<\overline{\theta}}} \leq u\big(\theta, \overline{\mathrm{cof}}\big(NS_{\kappa,\tau}^{[\tau]^{<\theta}}\big)\big) \leq u(\theta, \lambda) = \lambda$$

by Lemmas 2.1 and 2.2. $\square$

**PROPOSITION 4.7.** *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, and $\xi$ is an ordinal such that (a) $\kappa < \xi \leq \eta$, where $\eta$ equals $\lambda + 1$ if $\mathrm{cf}(\lambda) < \kappa$, and $\mathrm{cf}(\lambda)$ otherwise, and (b) $\xi$ is either a successor ordinal, or a limit ordinal of cofinality at least $\kappa$. Then $\overline{\mathrm{cof}}\Big(\bigcup\limits_{\delta<\xi} NS_{\kappa,\lambda}^{\delta}\Big) = \lambda$.*

**Proof.** By Lemmas 1.4 (ii) and 4.1 and Propositions 3.8, 4.3 and 4.6. $\square$

In particular if $\mathcal{H}_{\kappa,\lambda}$ holds and $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, then $\overline{\mathrm{cof}}\Big(\bigcup_{\delta<\mathrm{cf}(\lambda)} NS^\delta_{\kappa,\lambda}\Big) = \lambda$ (and hence by Lemma 1.5 (iv) there is no $A$ such that $NS_{\kappa,\lambda} = \Big(\bigcup_{\delta<\mathrm{cf}(\lambda)} NS^\delta_{\kappa,\lambda}\Big)|A)$.

# 5 Ideals $J$ on $P_\kappa(\lambda)$ with $\overline{\mathbf{cof}}(J) < \lambda$

There may exist ideals $J$ on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) < \lambda$. Some examples were presented in [11]. We now give some more.

Given two cardinals $\pi \leq \kappa$ and $\chi \geq \lambda$, $\mathcal{A}_{\kappa,\lambda}(\pi,\chi)$ asserts the existence of $Z \leq P_\pi(\lambda)$ with $\mid Z \mid = \chi$ such that $\mid Z \cap P(a) \mid < \kappa$ for every $a \in P_\kappa(\lambda)$.

**LEMMA 5.1.** ([11]) *Let $\theta$ and $\chi$ be two cardinals such that (a) $2 \leq \theta \leq \kappa$, $\lambda < \chi$ and there is a $[\chi]^{<\theta}$-normal ideal on $P_\kappa(\chi)$, and (b) $\mathcal{A}_{\kappa,\lambda}(\pi,\chi)$ holds for some regular uncountable cardinal $\pi < \kappa$. Then $\overline{\mathrm{cof}}(I_{\kappa,\chi} \mid A) \leq \lambda$ for some $A \in \big(NS^{[\chi]^{<\theta}}_{\kappa,\chi}\big)^+$.*

**LEMMA 5.2.** ([9]) *Let $\tau$ be the largest limit cardinal less than or equal to $\kappa$. Assume $\mathrm{cf}(\lambda) < \kappa$ and one of the following conditions is satisfied :*

(a) $\tau = \kappa$.

(b) $\tau > \mathrm{cf}(\lambda)$ *and* $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\tau)$.

(c) $\tau > \mathrm{cf}(\lambda) = \mathrm{cf}(\tau)$ *and* $\min\{\mathrm{pp}(\tau), \tau^{+3}\} < \kappa$.

(d) $\tau \leq \mathrm{cf}(\lambda)$ *and* $\min\{2^{\mathrm{cf}(\lambda)}, \big(\mathrm{cf}(\lambda)\big)^{+3}\} < \kappa$.

*Then $\mathcal{A}_{\kappa,\lambda}\big(\big(\mathrm{cf}(\lambda)\big)^+, \lambda^+\big)$ holds.*

Suppose for instance that $\kappa$ is a limit cardinal and $\mathrm{cf}(\lambda) < \kappa$. Then by Lemmas 5.1 and 5.2, $\overline{\mathrm{cof}}(I_{\kappa,\lambda^+} \mid B) \leq \lambda$ for some $B \in NS^+_{\kappa,\lambda^+}$.

Note that in case $\kappa$ is the successor of a cardinal of cofinality $\mathrm{cf}(\lambda)$, Lemma 5.2 does not apply, as none of the conditions (a) - (d) is satisfied. To handle this case, we introduce the following principle.

Given a cardinal $\chi \geq \lambda$, $\mathcal{B}_{\kappa,\lambda}(\chi)$ asserts the existence of $Z \subseteq P_\kappa(\lambda)$ with $\mid Z \mid = \chi$ such that for every $e \subseteq Z$ with $\mid e \mid = \kappa$, there is a $< \kappa$ -to-one function in $\prod_{z \in e} z$.

**LEMMA 5.3.** ([11]) *Let $\theta$ and $\chi$ be two cardinals such that (a) $2 \leq \theta \leq \kappa$, $\lambda < \chi$ and there is a $[\chi]^{<\theta}$-normal ideal on $P_\kappa(\chi)$, and (b) $\mathcal{B}_{\kappa,\lambda}(\chi)$ holds. Then*

$\overline{\mathrm{cof}}(I_{\kappa,\chi} \mid A) \leq \lambda$ *for some* $A \in \big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big)^+$.

Note that in case cf$(\lambda < \kappa$, $\mathcal{B}_{\kappa,\lambda}(\lambda^+)$ follows from $ADS_\lambda$, where $ADS_\lambda$ asserts the existence of $y_\alpha \subseteq \lambda$ for $\alpha < \lambda^+$ such that (a) for any $\alpha < \lambda^+$, $\cup y_\alpha = \lambda$ and o.t.$(y_\alpha) = \mathrm{cf}(\lambda)$, and (b) given $\beta < \lambda^+$, there is $g : \beta \longrightarrow \lambda$ such that

$$\big(y_\alpha \setminus g(\alpha)\big) \cap \big(y_{\alpha'} \setminus g(\alpha')\big) = \emptyset$$

for any $\alpha, \alpha' \in \beta$ with $\alpha \neq \alpha'$.

For more on the existence of $A \in \big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big)^+$ such that $\overline{\mathrm{cof}}(I_{\kappa,\chi} \mid A) < \chi$, see [9] and [11].

**PROPOSITION 5.4.** *Suppose $\theta$ and $\chi$ are two cardinals such that $2 \leq \theta \leq \kappa$ and $\lambda < \chi$, and $A \in \big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big)^+$ is such that $\overline{\mathrm{cof}}(I_{\kappa,\chi} \mid A) \leq \lambda$. Then there is $B \in \big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big)^+$ and a function $f$ such that (a) $f$ is an isomorphism from $(P_\kappa(\lambda), \subset)$ onto $(B, \subset)$, and (b) for any $\delta \leq \lambda$, $f\big(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}\big) = NS_{\kappa,\chi}^{[\delta]^{<\theta}} \mid B$ (and hence $\overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\delta]^{<\theta}} \mid B\big) \leq \overline{\mathrm{cof}}\big(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}\big)$ and $\mathrm{cof}\big(NS_{\kappa,\chi}^{[\delta]^{<\theta}}\big) \leq \mathrm{cof}\big(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}\big)$.*

**Proof.** Select $x_\beta \in P_\kappa(\chi)$ for $\beta < \lambda$ so that for each $X \in I_{\kappa,\chi}$, there is $z \in P_\kappa(\lambda)$ with $X \cap \{y \in A : \bigcup_{\beta \in z} x_\beta \subseteq y\} = \emptyset$. For $\lambda \leq \alpha < \chi$, pick $z_\alpha \in P_\kappa(\lambda)$ with

$$\{y \in A : \bigcup_{\beta \in z_\alpha} x_\beta \subseteq y\} \subseteq \{t \in P_\kappa(\chi) : \alpha \in t\}.$$

Let $C$ be the set of all $x \in P_\kappa(\chi)$ such that $\big(\bigcup_{\beta \in x \cap \lambda} x_\beta\big) \cup \big(\bigcup_{\alpha \in x \setminus \lambda} z_\alpha\big) \subseteq x$. Note that $C \in NS_{\kappa,\chi}^*$.

**Claim 1.** *Let $x \in A \cap C$. Then $x \setminus \lambda = \{\alpha \in \chi \setminus \lambda : z_\alpha = x \cap \lambda\}$.*

**Proof of Claim 1.** Since $x \in C$, $x \setminus \lambda \subseteq \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq x \cap \lambda\}$. To show the reverse inclusion, fix $\alpha \in \chi \setminus \lambda$ with $z_\alpha \subseteq x \cap \lambda$. Then $\bigcup_{\beta \in z_\alpha} x_\beta \subseteq x$, and hence $\alpha \in x$, which completes the proof of Claim 1.

**Claim 2.** Let $a \in P_\kappa(\lambda)$. Then $\mid \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq a\} \mid < \kappa$.

**Proof of Claim 2.** Pick $x \in A \cap C$ with $a \subseteq x$. Then by Claim 1,

$$\{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq a\} \subseteq \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq x \cap \lambda\} \subseteq X,$$

which completes the proof of Claim 2.

Using Claim 2, define $f : P_\kappa(\lambda) \longrightarrow P_\kappa(\chi)$ by $f(a) = a \cup \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq a\}$. Put $B = \mathrm{ran}(f)$. By Claim 1, $x = f(x \cap \lambda)$ for any $x \in A \cap C$, so $A \cap C \subseteq B$.

It follows that $B \in \big(NS^{[\chi]^{<\theta}}_{\kappa,\chi}\big)^+$.

As is easily seen, $f$ is an isomorphism from $\big(P_\kappa(\lambda), \subset\big)$ onto $(B, \subset)$, and moreover $f^{-1}(X) \in I_{\kappa,\lambda}$ for any $X \in I_{\kappa,\chi}$. Now fix $\delta \leq \lambda$. Set $J = NS^{[\delta]^{<\theta}}_{\kappa,\lambda}$. It is simple to see that $f(J)$ is an ideal on $P_\kappa(\chi)$ with the property that $B \in \big(f(J)\big)^*$.

**Claim 3.** $f(J)$ is $[\delta]^{<\theta}$-normal.

**Proof of Claim 3.** Fix $X \in \big(f(J)\big)^+ \cap P(B)$ and $h : X \longrightarrow P_\theta(\delta)$ such that $h(x) \in P_{|x\cap\theta|}(x)$ for every $x \in X$. Define $k : f^{-1}(X) \longrightarrow P_\theta(\delta)$ by $k(a) = h\big(f(a)\big)$. There must be $A \in J^+ \cap P\big(f^{-1}(X)\big)$ such that $k$ is constant on $A$. Then clearly $f''A \in \big(f(J)\big)^+ \cap P(X)$, and moreover $h$ is constant on $f''A$, which completes the proof of the claim.

It immediately follows from Claim 3 that $NS^{[\delta]^{<\theta}}_{\kappa,\chi} \mid B \subseteq f(J)$.

To establish the reverse inclusion fix $Y \in f(J)$. Since $f^{-1}(Y \cap B) \in J$, we may find $g : P_\theta(\delta) \longrightarrow P_\kappa(\lambda)$ such that $f^{-1}(Y \cap B) \cap C^{\kappa,\lambda}_g = \emptyset$. Then clearly $(Y \cap B) \cap C^{\kappa,\chi}_g = \emptyset$ and hence $Y \cap B \in NS^{[\delta]^{<\theta}}_{\kappa,\chi}$. $\square$

Let $\kappa = (2^\rho)^+$, where $\rho$ is an infinite cardinal, and suppose $\lambda$ is a strong limit cardinal with $\mathrm{cf}(\lambda) \leq \rho$. Then $\mathcal{A}_{\kappa,\lambda}(\rho^+, 2^\lambda)$ holds, since $| P_{\rho^+}(\lambda) \cap P(a) | \leq 2^\rho$ for any $a \in P_\kappa(\lambda)$. Hence by Lemmas 4.2 and 5.1 and Proposition 5.4, $\overline{\mathrm{cof}}\big(NS^\lambda_{\kappa,2^\lambda} \mid B\big) \leq \lambda$ for some $B \in NS^+_{\kappa,2^\lambda}$.

**PROPOSITION 5.5.** *Suppose that* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) \leq \lambda^+$*, and there is* $A \in NS^+_{\kappa,\lambda^+}$ *such that* $\overline{\mathrm{cof}}(I_{\kappa,\lambda^+} \mid A) \leq \lambda$*. Then* $\overline{\mathrm{cof}}(NSS_{\kappa,\lambda^+} \mid B) < \lambda^+$ *for some* $B \in NS^+_{\kappa,\lambda^+}$.

**Proof.** By Lemma 3.7 (i), there is $C \in NS^*_{\kappa,\lambda^+}$ such that $NSS_{\kappa,\lambda^+} \mid C = I_{\kappa,\lambda^+} \mid C$. Then $B = A \cap C$ is as desired. $\square$

For example, suppose $\kappa = \omega_4$ and $\lambda = \beth_\alpha$ for some infinite limit ordinal $\alpha$ of cofinality $\omega$. Then by Lemmas 4.2, 5.1 and 5.2 and Proposition 5.5, $\overline{\mathrm{cof}}\big(NSS_{\kappa,\lambda^+} \mid B\big) \leq \lambda$ for some $B \in NS^+_{\kappa,\lambda^+}$.

If $\lambda$ is singular, then by Lemma 3.1 $NS_{\kappa,\lambda} = NSS_{\kappa,\lambda} \mid B$ for some $B$, so $\overline{\mathrm{cof}}(NSS_{\kappa,\lambda} \mid A) < \lambda$ for some $A \in NS^+_{\kappa,\lambda}$ just in case $\overline{\mathrm{cof}}(NS_{\kappa,\lambda} \mid D) < \lambda$ for some $D \in NS^+_{\kappa,\lambda}$.

Suppose $\overline{\mathrm{cof}}(NS_{\kappa,\lambda} \mid D) < \lambda$ for some $D \in NS^+_{\kappa,\lambda}$. Then setting $\sigma = \overline{\mathrm{cof}}(NS_{\kappa,\lambda} \mid D)$,

$$\operatorname{cof}(NS_{\kappa,\lambda}) \le u(\kappa,\sigma) \le u(\kappa,\lambda) \le \operatorname{cof}(NS_{\kappa,\lambda})$$

by Lemma 1.11 (ii), so $\operatorname{cof}(NS_{\kappa,\lambda}) = u(\kappa,\sigma) = u(\kappa,\lambda)$. Hence by Lemma 1.5 (iv), SSH does not hold.

**PROPOSITION 5.6.** *Let $\theta$ and $\chi$ be two cardinals such that $2 \le \theta \le \kappa$ and $\lambda < \chi$. Suppose that $\overline{\operatorname{cof}}(NS_{\kappa,\chi}) \le \chi^{<\overline{\theta}}$, and there is $A \in \big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big)^+$ such that $\overline{\operatorname{cof}}(I_{\kappa,\chi} \mid A \mid \le \lambda$. Then $\overline{\operatorname{cof}}(NS_{\kappa,\chi} \mid B) \le \lambda$ for some $B \in \big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big)^+$.*

**Proof.** By Lemma 3.4 there is $C \in \big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big)^*$ such that $NS_{\kappa,\chi} \mid C = I_{\kappa,\chi} \mid C$. Then $B = A \cap C$ is as desired. □

Here is an example of a situation where Proposition 5.6 applies. Starting from a $\mathcal{P}^3(\nu)$-hypermeasurable, Cummings [3] constructs a generic extension $W$ of $V$ in which for any infinite cardinal $\rho$, $2^\rho$ equals $\rho^+$ if $\rho$ is a successor cardinal, and $\rho^{++}$ otherwise. In $W$, let $\sigma$ be a regular uncountable cardinal, and $\mu > \sigma$ be a cardinal of cofinality less than $\sigma$. Suppose that (a) $\sigma$ is not the successor of a cardinal $\tau$ with $\operatorname{cf}(\tau) \le \operatorname{cf}(\mu)$, and (b) $\sigma$ is not the successor of the successor of a limit cardinal $\pi$ with $\operatorname{cf}(\pi) \le \operatorname{cf}(\mu)$. Then by Lemmas 5.1 and 5.2 and Proposition 5.6, $\overline{\operatorname{cof}}\big(NS_{\sigma,\mu^+} \mid B\big) \le \mu$ for some $B \in \big(NS_{\sigma,\mu^+}^{[\mu^+]^{<(\operatorname{cf}(\mu))^+}}\big)^+$.

## 6 Cases when $NS_{\kappa,\lambda} = NS_{\kappa,\lambda}^{\operatorname{cf}(\lambda)} \mid A$ for some $A$

In this section we establish that if $\kappa \le \operatorname{cf}(\lambda) < \lambda$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then $NS_{\kappa,\lambda} = NS_{\kappa,\lambda}^{\operatorname{cf}(\lambda)} \mid A$ for some $A$. Note that if $\operatorname{cf}(\lambda) < \kappa$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then by Lemmas 3.5 and 4.1, $NS_{\kappa,\lambda} = I_{\kappa,\lambda} \mid A$ for some $A$. Note further that if $\lambda$ is regular, then trivially $NS_{\kappa,\lambda} = NS_{\kappa,\lambda}^{\lambda} \mid P_\kappa(\lambda)$. By combining the three cases, we obtain that if $\overline{\operatorname{cof}}(NS_{\kappa,\tau}) \le \lambda$ for every cardinal $\tau$ with $\kappa \cdot \operatorname{cf}(\lambda) \le \tau < \lambda$, then $NS_{\kappa,\lambda} = NS_{\kappa,\lambda}^{\operatorname{cf}(\lambda)} \mid A$ for some $A$.

**PROPOSITION 6.1.** *Let $\pi, \theta$ and $\chi$ be three cardinals with $\kappa \le \pi < \lambda$ and $2 \le \theta \le \kappa \le \chi \le \lambda$. Suppose that (a) $\lambda$ is singular, (b) $\overline{\theta} \le \operatorname{cf}(\lambda)$ in case $\chi = \lambda$, and (c) $\overline{\operatorname{cof}}\big(NS_{\kappa,\tau}^{[\chi\cap\tau]^{<\theta}}\big) \le \lambda^{<\overline{\theta}}$ for every cardinal $\tau$ with $\pi \le \tau < \lambda$. Then there is $A \in \big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^*$ such that $NS_{\kappa,\lambda}^{[\chi]^{<\theta}} \subseteq NS_{\kappa,\lambda}^{\operatorname{cf}(\lambda)} \mid A$.*

**Proof.** Set $\mu = \operatorname{cf}(\lambda)$ and select an increasing sequence of cardinals $< \lambda_\eta : \eta < \mu >$ so that (i) $\bigcup_{\eta<\mu} \lambda_\eta = \lambda$, (ii) $\lambda_0 > \pi \cdot \mu$, and (iii) $\lambda_0 \ge \chi$ in case $\chi < \lambda$. For $\eta < \mu$, pick a family $G_\eta$ of functions from $P_{\overline{\theta}\cdot 3}(\chi \cap \lambda_\eta)$ to $P_3(\lambda_\eta)$ so that $\mid G_\eta \mid \le \overline{\operatorname{cof}}\big(NS_{\kappa,\lambda_\eta}^{[\chi\cap\lambda_\eta]^{<\theta}}\big)$ and for every $H \in (NS_{\kappa,\lambda_\eta}^{[\chi\cap\lambda_\eta]^{<\theta}})^*$, there is $y \in P_\kappa(G_\eta) \setminus \{\emptyset\}$

such that $\{b \in \bigcap_{g \in y} C_g^{\kappa,\lambda_\eta} : b \cap \kappa \in \kappa\} \subseteq H$. Let $\bigcup_{\eta<\mu} G_\eta = \{g_e : e \in P_{\overline{\theta}}(\lambda)\}$.
Let $A$ be the set of all $a \in P_\kappa(\lambda)$ such that

- $\overline{\theta} \subseteq a$ in case $\overline{\theta} < \kappa$ ;
- $\omega \subseteq a$ ;
- $a \cap \kappa \in \kappa$ ;
- $k(\alpha) \in a$ for every $\alpha \in a$, where $k : \lambda \to \mu$ is defined by $k(\alpha) =$ the least $\eta < \mu$ such that $\alpha \in \lambda_\eta$ ;
- If $\chi = \lambda$, then $i(v) \in a$ for every $v \in P_{|a \cap (\overline{\theta} \cdot 3)|}(a)$, where $i : P_{\overline{\theta} \cdot 3}(\lambda) \to \mu$ is defined by $i(v) =$ the least $\eta < \mu$ such that $v \subseteq \lambda_\eta$ ;
- $g_e(u) \subseteq a$ whenever $e \in P_{|a \cap \overline{\theta}|}(a)$ and $u \in P_{|a \cap (\overline{\theta} \cdot 3)|}(a) \cap \mathrm{dom}(g_e)$.

It is immediate that $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$. Let us check that $A$ is as desired. Thus fix $f : P_{\overline{\theta} \cdot 3}(\chi) \to P_3(\lambda)$. Given $\eta < \mu$, define $p_\eta : P_{\overline{\theta} \cdot 3}(\chi \cap \lambda_\eta) \to P_2(\lambda_\eta)$ by $p_\eta(v) = \{\zeta\}$, where $\zeta =$ the least $\sigma$ such that $\eta \leq \sigma < \mu$ and $f(v) \subseteq \lambda_\sigma$. Also define $q_\eta : P_{\overline{\theta} \cdot 3}(\chi \cap \lambda_\eta) \to P_3(\lambda_\eta)$ by $q_\eta(v) = \lambda_\eta \cap f(v)$. Select $x_\eta, y_\eta \in P_\kappa(P_{\overline{\theta}}(\lambda)) \setminus \{\emptyset\}$ so that $\{g_e : e \in x_\eta \cup y_\eta\} \subseteq G_\eta$, $\{b \in \bigcap_{e \in x_\eta} C_{g_e}^{\kappa,\lambda_\eta} : b \cap \kappa \in \kappa\} \subseteq C_{p_\eta}^{\kappa,\lambda_\eta}$ and $\{b \in \bigcap_{e \in y_\eta} C_{g_e}^{\kappa,\lambda_\eta} : b \cap \kappa \in \kappa\} \subseteq C_{q_\eta}^{\kappa,\lambda_n}$. Finally define $u : \mu \longrightarrow P_\kappa(\lambda)$ by $u(\eta) = \cup(x_\eta \cup y_\eta)$, and $t : P_2(\mu) \longrightarrow P_\kappa(\lambda)$ so that for any $\eta \in \mu$, $t(\{\eta\})$ equals $u(\eta)$ if $\overline{\theta} < \kappa$, and $u(\eta) \cup \mid u(\eta) \mid^+$ otherwise.
We claim that $A \cap C_t^{\kappa,\lambda} \subseteq C_f^{\kappa,\lambda}$. Thus let $a \in A \cap C_t^{\kappa,\lambda}$ and $v \in P_{|a \cap (\overline{\theta} \cdot 3)|}(a \cap \chi)$. There must be $\eta \in a \cap \mu$ such that $v \subseteq \lambda_\eta$. Then $a \cap \lambda_\eta \in C_{p_\eta}^{\kappa,\lambda_\eta}$ since $x_\eta \subseteq P_{|a \cap \overline{\theta}|}(a)$. It follows that $v \cup f(v) \subseteq \lambda_\sigma$ for some $\sigma \in a \cap \mu$. Now $a \cap \lambda_\sigma \in C_{q_\sigma}^{\kappa,\lambda_\sigma}$, since $y_\sigma \subseteq P_{|a \cap \overline{\theta}|}(a)$, so $f(v) \subseteq a$. □

In Proposition 6.1 we assumed that $\overline{\theta} \leq \mathrm{cf}(\lambda)$ in case $\chi = \lambda$. Some condition of this kind is necessary. In fact if $\mathrm{cf}(\lambda) < \kappa$ and $u(\kappa, \lambda^{<\overline{\theta}}) = \lambda^{<\overline{\theta}}$, then for each $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$, $NS_{\kappa,\lambda}^{[\lambda]^{<\theta}} \neq NS_{\kappa,\lambda}^{\mathrm{cf}(\lambda)} \mid A$ since by Lemma 1.11,

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{[\lambda]^{<\overline{\theta}}}) > \lambda^{<\overline{\theta}} \geq \lambda \geq \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{\mathrm{cf}(\lambda)} \mid A).$$

**COROLLARY 6.2.** *Suppose that (a)* SSH *holds, or (b) there exists a $\sigma$-saturated ideal on $P_\nu(\lambda)$, where $\sigma$ and $\nu$ are two cardinals such that $\omega < \nu = \mathrm{cf}(\nu) < \lambda$ and $\sigma < \nu$, or (c) there is a regular uncountable cardinal $\tau < \lambda$ that is mildly $\pi$-ineffable for every cardinal $\pi$ with $\tau \leq \pi < \lambda$. Let $\theta$ and $\chi$ be two cardinals such that $2 \leq \theta \leq \kappa$, $\kappa \cdot \mathrm{cf}(\lambda) \leq \chi < \lambda$ and $\overline{\mathrm{cof}}(NS_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \lambda^{<\overline{\theta}}$. Then*

$NS^{[\chi]^{<\theta}}_{\kappa,\lambda} \mid A = NS^{\mathrm{cf}(\lambda)}_{\kappa,\lambda} \mid A$ *for some* $A \in \big(NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}\big)^*$.

**Proof.** Use Lemmas 1.3 and 1.10. □

**COROLLARY 6.3.** *Suppose* $\mathrm{cf}(\lambda) < \kappa$, *and* $\theta$ *and* $\chi$ *are two cardinals such that* $2 \leq \theta \leq \kappa \leq \chi < \lambda$ *and* $\overline{\mathrm{cof}}\big(NS^{[\chi]^{<\theta}}_{\kappa,\tau}\big) \leq \lambda^{<\overline{\theta}}$ *for every cardinal* $\tau$ *with* $\chi \leq \tau < \lambda$. *Then* $\mathrm{cof}\big(NS^{[\chi]^{<\theta}}_{\kappa,\lambda}\big) = u(\kappa,\lambda)$.

**Proof.** By Proposition 6.1 there is $A \in \big(NS^{[\lambda]^{<\theta}}_{\kappa,\lambda}\big)^*$ such that $NS^{[\chi]^{<\theta}}_{\kappa,\lambda} \mid A = I_{\kappa,\lambda} \mid A$. Then by Lemma 1.11 (ii), $\mathrm{cof}\big(NS^{[\chi]^{<\theta}}_{\kappa,\lambda}\big) = \mathrm{cof}\big(NS^{[\chi]^{<\theta}}_{\kappa,\lambda} \mid A\big) = \mathrm{cof}\big(I_{\kappa,\lambda} \mid A\big) = \mathrm{cof}(I_{\kappa,\lambda}) = u(\kappa,\lambda)$. □

**COROLLARY 6.4.**

(i) *Suppose* $\lambda$ *is singular and* $\mathcal{H}_{\kappa,\lambda}$ *holds. Then* $NS_{\kappa,\lambda} = NS^{\mathrm{cf}(\lambda)}_{\kappa,\lambda} \mid A$ *for some* $A$.

(ii) *Let* $\chi$ *be a cardinal such that* $\kappa \cdot \mathrm{cf}(\lambda) \leq \chi < \lambda$ *and* $\overline{\mathrm{cof}}(NS^{\chi}_{\kappa,\tau}) \leq \lambda$ *for every cardinal* $\tau$ *with* $\chi \leq \tau < \lambda$. *Then* $NS^{\chi}_{\kappa,\lambda} \mid A = NS^{\mathrm{cf}(\lambda)}_{\kappa,\lambda} \mid A$ *for some* $A \in NS^*_{\kappa,\lambda}$.

**COROLLARY 6.5.**

(i) *Let* $\chi \geq \kappa$ *be a cardinal, and* $\alpha < \kappa$ *be a limit ordinal such that* $\overline{\mathrm{cof}}(NS_{\kappa,\chi}) \leq \chi^{+\alpha}$. *Then* $NS^{\chi}_{\kappa,\chi^{+\alpha}}|A = I_{\kappa,\chi^{+\alpha}}|A$ *for some* $A \in NS^*_{\kappa,\chi^{+\alpha}}$.

(ii) *Let* $\chi > \kappa$ *be a cardinal such that* $\overline{\mathrm{cof}}(NS_{\kappa,\chi}) < \chi^{+\kappa}$. *Then* $NS^{\chi}_{\kappa,\chi^{+\kappa}}|A = NS^{\kappa}_{\kappa,\chi^{+\kappa}}|A$ *for some* $A \in NS^*_{\kappa,\chi^{+\kappa}}$.

**Proof.** Use Lemmas 1.1 and 1.10. □

Note that we do get a better result by considering the reduced cofinality ($\overline{\mathrm{cof}}$) instead of the usual one (cof). For example, suppose that GCH holds in $V$. By a result of [12], there is a $< \kappa-$closed, $\kappa^+$-cc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}, \overline{\mathrm{cof}}(NS_{\kappa,\kappa}) = \kappa^{+\omega}$ and $\mathrm{cof}(NS_{\kappa,\kappa}) = \kappa^{+(\omega+1)}$. Then in $V^{\mathbb{P}}$, there is by Corollary 6.5 (i) $A \in NS^*_{\kappa,\kappa^{+\omega}}$ such that $NS^{\kappa}_{\kappa,\kappa^{+\omega}}|A = I_{\kappa,\kappa^{+\omega}}|A$.

Let us next discuss the condition in Proposition 6.1 that $\overline{\mathrm{cof}}\big(NS^{[\tau]^{<\theta}}_{\kappa,\tau}\big) \leq \lambda^{<\overline{\theta}}$ for almost all cardinals $\tau < \lambda$.

**PROPOSITION 6.6.** *Let* $\theta$ *and* $\chi$ *be two cardinals such that* $2 \leq \theta \leq \kappa \leq \chi < \lambda$. *Suppose* $\overline{\mathrm{cof}}\big(NS^{[\chi]^{<\theta}}_{\kappa,\chi}\big) \leq \lambda^{<\overline{\theta}}$ *and* $\chi^{<\overline{\theta}} \geq \lambda$. *Then* $\overline{\mathrm{cof}}\big(NS^{[\chi]^{<\theta}}_{\kappa,\lambda}\big) =$

$\overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \le \chi$.

**Proof.** By Lemma 1.6 (iii) $\chi^{<\overline{\theta}} = \lambda^{<\overline{\theta}}$, so by Lemma 3.5 $NS_{\kappa,\chi}^{[\chi]^{<\theta}} = I_{\kappa,\chi} \mid A$ for some $A$. It follows that $\overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \le \chi$. Moreover by Lemma 1.10

$$\overline{\mathrm{cof}}\big(NS_{\kappa,\lambda}^{[\chi]^{<\theta}}\big) = \overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \cdot \mathrm{cov}\big(\lambda, (\lambda^{<\overline{\theta}})^+, (\lambda^{<\overline{\theta}})^+, \kappa\big) = \overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big).$$

□

**COROLLARY 6.7.** *Let $\theta$ and $\chi$ be two cardinals such that $2 \le \theta \le \kappa \le \chi < \chi^{<\overline{\theta}} = \lambda$. Suppose $\overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \le \chi^{<\overline{\theta}}$. Then there is $A \in \big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^*$ such that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda} \mid A) \le \chi$.*

**Proof.** By Lemma 1.7 we may find $A \in \big(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^*$ such that $NS_{\kappa,\lambda} \mid A = NS_{\kappa,\lambda}^{[\chi]^{<\theta}} \mid A$. Then by Proposition 6.6, $\overline{\mathrm{cof}}(NS_{\kappa,\lambda} \mid A) \le \overline{\mathrm{cof}}\big(NS_{\kappa,\lambda}^{[\chi]^{<\theta}}\big) \le \chi$. □

**Question**. Suppose $\theta$ and $\chi$ are two cardinals such that $2 \le \theta \le \kappa \le \chi$ and $\overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \le \chi^{<\overline{\theta}}$. Does then $\chi^{<\overline{\theta}} = \chi$ hold ?

**PROPOSITION 6.8.**

(i) *Suppose $\theta$ and $\sigma$ are two cardinals such that $2 \le \theta \le \kappa \le \sigma < \lambda$, $\overline{\theta} \le \mathrm{cf}(\lambda)$ and $\overline{\mathrm{cof}}\big(NS_{\kappa,\tau}^{[\tau]^{<\theta}}\big) \le \lambda^{<\overline{\theta}}$ for every cardinal $\tau$ with $\sigma \le \tau < \lambda$. Then there is a cardinal $\pi$ with $\sigma \le \pi < \lambda$ such that $\overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \le \lambda$ for every cardinal $\chi$ with $\pi \le \chi < \lambda$.*

(ii) *Let $\theta$ and $\pi$ be two cardinals with $2 \le \theta \le \kappa \le \pi < \lambda$. Suppose $\kappa \le \mathrm{cf}(\lambda) < \lambda$, and $\overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \le \lambda$ for every cardinal $\chi$ with $\pi \le \chi < \lambda$. Then $\overline{\mathrm{cof}}\big(NS_{\kappa,\rho}^{[\rho]^{<\theta}}\big) < \lambda$ for every cardinal $\rho$ with $\pi \cdot \mathrm{cf}(\lambda) \le \rho < \lambda$.*

**Proof.**

(i) : If $\nu^\rho < \lambda$ for every cardinal $\nu < \lambda$ and every cardinal $\rho < \overline{\theta}$, then $\lambda^{<\overline{\theta}} = \lambda$, and $\pi = \sigma$ is as desired. Now suppose there are two cardinals $\nu < \lambda$ and $\rho < \overline{\theta}$ such that $\nu^\rho \ge \lambda$. Set $\pi = \nu \cdot \sigma$. Let $\chi$ be a cardinal with $\pi \le \chi < \lambda$. Then $\chi^{<\overline{\theta}} = \lambda^{<\overline{\theta}}$, so by Proposition 6.6 $\overline{\mathrm{cof}}\big(NS_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \le \chi$.

(ii) : By Lemma 1.9. □

In particular, if $\kappa \le \mathrm{cf}(\lambda) < \lambda$, then $\mathcal{H}_{\kappa,\lambda}$ holds just in case $\overline{\mathrm{cof}}\big(NS_{\kappa,\tau}\big) < \lambda$ for every cardinal $\tau$ with $\kappa \le \tau < \lambda$.

Suppose $\lambda$ is a limit cardinal and $\chi$ is a cardinal with $\kappa \leq \chi \leq \lambda$. If either $\mathrm{cf}(\lambda) < \kappa$ or $\mathrm{cf}(\lambda) > \chi$, then by Lemmas 1.10 and 4.1

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{\chi}) \leq \bigcup_{\pi \leq \tau < \lambda} \overline{\mathrm{cof}}(NS_{\kappa,\tau}^{\chi \cap \tau}),$$

where $\pi$ equals $\kappa$ if $\chi = \lambda$, and $\chi$ otherwise. We will now deal with the case when $\kappa \leq \mathrm{cf}(\lambda) \leq \chi$. The proof of the following is a modification of that of Proposition 6.1.

**PROPOSITION 6.9** *Let $\chi$ be a cardinal such that $\kappa \cdot \mathrm{cf}(\lambda) \leq \chi \leq \lambda$. Set $\pi = \kappa$ if $\chi = \lambda$, and $\pi = \chi$ otherwise. Then $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{\chi}) \leq \overline{\mathrm{cof}}(NS_{\kappa,\rho}^{\mathrm{cf}(\lambda)})$ and $\mathrm{cof}(NS_{\kappa,\lambda}^{\chi}) \leq \mathrm{cof}(NS_{\kappa,\rho}^{\mathrm{cf}(\lambda)})$ where $\rho = \bigcup_{\pi \leq \tau < \lambda} \overline{\mathrm{cof}}(NS_{\kappa,\tau}^{\chi \cap \tau})$.*

**Proof.** We can assume that $\mathrm{cf}(\lambda) < \chi$ since otherwise the result is trivial. We show that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{\chi}) \leq \overline{\mathrm{cof}}(NS_{\kappa,\rho}^{\mathrm{cf}(\lambda)})$ and leave the proof of the other assertion to the reader. Put $\mu = \mathrm{cf}(\lambda)$ and pick an increasing sequence $< \lambda_\eta : \eta < \mu >$ of cardinals cofinal in $\lambda$ so that $\lambda_0 > \kappa \cdot \mu$, and $\lambda_0 \geq \chi$ in case $\chi < \lambda$. For $\eta < \mu$, select a family $G_\eta$ of functions from $P_3(\chi \cap \lambda_\eta)$ to $P_3(\lambda_\eta)$ so that $\mid G_\eta \mid \leq \overline{\mathrm{cof}}(NS_{\kappa,\lambda_\eta}^{\chi \cap \lambda_\eta})$ and for any $H \in (NS_{\kappa,\lambda_\eta}^{\chi \cap \lambda_\eta})^*$, there is $y \in P_\kappa(G_\eta) \setminus \{\emptyset\}$ with $\{b \in \bigcap_{g \in y} C_g^{\kappa,\lambda_\eta} : b \cap \kappa \in \kappa\} \subseteq H$. Let $\bigcup_{\eta < \mu} G_\eta = \{g_\xi : \xi < \rho\}$. For $\xi < \rho$, let $g_\xi \in G_{\eta_\xi}$. Let $A$ be the set of all $a \in P_\kappa(\lambda)$ such that $\omega \subseteq a$, $a \cap \kappa \in \kappa$ and $k(\alpha) \in a$ for all $\alpha \in a$, where $k : \lambda \to \mu$ is defined by $k(\alpha) =$ the least $\eta < \mu$ such that $\alpha \in \lambda_\eta$. Clearly $A \in NS_{\kappa,\lambda}^*$, so by Lemma 1.5 (ii) $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{\chi} \mid A) = \overline{\mathrm{cof}}(NS_{\kappa,\lambda}^{\chi})$.

By Proposition 2.3 we may find a collection $T$ of functions from $\mu$ to $P_\kappa(\rho)$ such that $\mid T \mid = \overline{\mathrm{cof}}(NS_{\kappa,\rho}^{\mu})$ and for any $u : \mu \to P_\kappa(\rho)$, there is $z \in P_\kappa(T)$ with the property that $u(\eta) \subseteq \bigcup_{t \in z} t(\eta)$ for every $\eta \in \mu$. For $t \in T$, let $D_t$ be the set of all $a \in P_\kappa(\lambda)$ such that for any $\eta \in a \cap \mu$ and any $\xi \in t(\eta)$, $a \cap \lambda_{n_\xi} \in C_{g_\xi}^{\kappa,\lambda_{\eta_\xi}}$. Note that $D_t \in (NS_{\kappa,\lambda}^{\chi})^*$.

Now fix $f : P_3(\chi) \to P_3(\lambda)$. Given $\eta < \mu$, define $p_\eta : P_3(\chi \cap \lambda_\eta) \to P_2(\lambda_\eta)$ by $p_n(v) = \{\zeta\}$, where $\zeta =$ the least $\sigma$ such that $\eta \leq \sigma < \mu$ and $f(v) \leq \lambda_\sigma$, and $q_n : P_3(\chi \cap \lambda_\eta) \to P_3(\lambda_\eta)$ by $q_\eta(v) = \lambda_\eta \cap f(v)$. Select $x_\eta, y_\eta \in P_\kappa(\rho) \setminus \{\emptyset\}$ so that $\{g_\xi : \xi \in x_\eta \cup y_\eta\} \subseteq G_\eta$, $\{b \in \bigcap_{\xi \in x_\eta} C_{g_\xi}^{\kappa,\lambda_\eta} : b \cap \kappa \in \kappa\} \subseteq C_{p_\eta}^{\kappa,\lambda_\eta}$ and $\{b \in \bigcap_{\xi \in y_\eta} C_{g_\xi}^{\kappa,\lambda_\eta} : b \cap \kappa \in \kappa\} \subseteq C_{q_\eta}^{\kappa,\lambda_\eta}$. We may find $z \in P_\kappa(T)$ such that $x_\eta \cup y_\eta \subseteq \bigcup_{t \in z} t(\eta)$ for every $\eta \in \mu$.

Let us show that $A \cap \big(\bigcap_{t \in z} D_t\big) \subseteq C_f^{\kappa,\lambda}$. Thus let $a \in A \cap \big(\bigcap_{t \in z} D_t\big)$ and $v \in P_3(a \cap \chi)$. There must be $\eta \in a \cap \mu$ such that $v \subseteq \lambda_\eta$. Then $a \cap \lambda_\eta \in \bigcap_{\xi \in x_\eta} C_{g_\xi}^{\kappa,\lambda_\eta}$, so $v \cup f(v) \subseteq \lambda_\sigma$ for some $\sigma \in a \cap \mu$. Now $a \cap \lambda_\sigma \in \bigcap_{\xi \in y_\sigma} C_{g_\xi}^{\kappa,\lambda_\sigma}$, and therefore $f(v) \subseteq a$. $\square$

## 7 Nowhere precipitousness of $NS_{\kappa,\lambda}^{\nu}$

**Throughout this section it is assumed that** $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. **Let** $\nu$ **be a cardinal with** $\mathrm{cf}(\lambda) \leq \nu < \lambda$. We will show that under certain conditions, $NS_{\kappa,\lambda}^{\nu}$ is nowhere precipitous. Our proof will follow that of Theorem 2.1 in [13], except that we do not appeal to pcf theory.
**Set** $\mu = \mathrm{cf}(\lambda)$. **We assume that** $c(\kappa, \nu) < \lambda$ **in case** $\nu > \mu$. **Let** $\rho < \lambda$ **be a regular cardinal such that** $\rho > \mu$ **if** $\nu = \mu$, **and** $\rho > c(\kappa, \nu)$ **otherwise**.
Select a continuous, increasing sequence $< \lambda_\beta : \beta < \mu >$ of cardinals so that $\bigcup_{\beta < \mu} \lambda_\beta = \lambda$ and $\lambda_0 > \rho$. Let $E$ be the set of all limit ordinals $\alpha < \mu$ with $\mathrm{cf}(\alpha) < \kappa$. We define $D$ as follows. If $\nu = \mu$, we set $D = E$. Otherwise we pick $D$ in $NS_{\kappa,\nu}^{*}$ so that $\mid D \mid = c(\kappa, \nu)$. For $d \in D$, put $\alpha(d) = \cup(d \cap \mu)$. Note that $\alpha(d) \in E$. Moreover $\alpha(d) = d$ in case $\nu = \mu$.

Let $B$ be the set of all $a \in P_\kappa(\lambda)$ such that (i) $0 \in a$, (ii) $\gamma + 1 \in a$ for every $\gamma \in a \cap \nu$, (iii) $a \cap \kappa \in \kappa$, (iv) $a \cap \mu \subseteq \{\beta \in \mu : \lambda_\beta \in a\}$, and (v) $a \cap \nu \in D$ in case $\nu > \mu$. Then clearly, $B \in \big(NS_{\kappa,\lambda}^{\nu}\big)^{*}$. For $d \in D$, define $W_d$ by letting $W_d = \{a \in B : \cup(a \cap \mu) = d\}$ if $\nu = \mu$, and $W_d = \{a \in B : a \cap \nu = d\}$ otherwise. Note that $\cup(a \cap \lambda_{\alpha(d)}) = \lambda_{\alpha(d)}$ for every $a \in W_d$.

**LEMMA 7.1.** *Suppose there is $T \subseteq P_\kappa(\lambda)$ such that (a) $|T \cap P(a)| < \rho$ for any $a \in P_\kappa(\lambda)$, and (b) $u(\rho, \tau) \leq |T|$ for every cardinal $\tau$ with $\rho \leq \tau < \lambda$. Then for every $R \in (NS_{\kappa,\lambda}^{\nu})^{+}$*

$$\{d \in D : |\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\} \geq u(\rho, \lambda_{\alpha(d)})\}$$

*lies in $NS_\mu^{+}$ if $\nu = \mu$, and in $NS_{\kappa,\nu}^{+}$ otherwise.*

**Proof.** For $\eta \in \mu$, select $Z_\eta \in I_{\rho,\lambda_\eta}^{+}$ with $|Z_\eta| \leq |T|$. Then clearly there is $Q \subseteq T$ with $|\bigcup_{\eta < \mu} Z_\eta| = |Q|$. Pick a bijection $i : \bigcup_{\eta < \mu} Z_\eta \to Q$ and let $j$ denote the inverse of $i$. For $\alpha \in E$, define $k_\alpha : P_\kappa(\lambda_\alpha) \to P_\rho(\lambda_\alpha)$ by $k_\alpha(b) = \bigcup_{e \in Q \cap P(b)} (\lambda_\alpha \cap j(e))$.

**Claim.** *Let $S \in \left(NS^{\nu}_{\kappa,\lambda}\right)^+$. Then there is $d \in D$ such that*

$$k''_{\alpha(d)}(\{a \cap \lambda_{\alpha(d)} : a \in S \cap W_d\}) \in I^+_{\rho,\lambda_{\alpha(d)}}.$$

**Proof of the claim.** Assume otherwise. For $d \in D$, select $y_d \in P_\rho(\lambda_{\alpha(d)})$ so that $y_d \setminus k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) \neq \emptyset$ for every $a \in S \cap W_d$. Set $y = \bigcup_{d \in D} y_d$. Note that $y \in P_\rho(\lambda)$. For $\eta \in \mu$, pick $z_\eta \in Z_\eta$ so that $y \cap \lambda_\eta \subseteq z_\eta$. Now let $H$ be the set of all $a \in P_\kappa(\lambda)$ such that $i(z_\eta) \in \bigcup_{\zeta \in a \cap \mu} P(a \cap \lambda_\zeta)$ for every $\eta \in a \cap \mu$. Since $H \in (NS^{\mu}_{\kappa,\lambda})^*$, we can find $a \in S \cap B \cap H$. Set $d = \cup(a \cap \mu)$ if $\nu = \mu$, and $d = a \cap \nu$ otherwise. Then $a \in W_d$ and

$$y_d \subseteq y \cap \lambda_{\alpha(d)} = \bigcup_{\eta \in a \cap \mu} (y \cap \lambda_\eta) \subseteq \bigcup_{\eta \in a \cap \mu} z_\eta = \bigcup_{\eta \in a \cap \mu} j(i(z_\eta)) \subseteq k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}).$$

This contradiction completes the proof of the claim.

It is now easy to show that the conclusion of the lemma holds: Fix $R \in (NS^{\nu}_{\kappa,\lambda})^+$, and $A$ such that $A \in NS^*_\mu$ if $\nu = \mu$, and $A \in NS^*_{\kappa,\nu}$ otherwise. Set $Y = \bigcup_{d \in D \cap A} W_d$. Since $Y \in (NS^{\nu}_{\kappa,\lambda})^*$, there must be some $d \in D$ such that

$$k''_{\alpha(d)}\big(\{a \cap \lambda_{\alpha(d)} : a \in (R \cap Y) \cap W_d\}\big) \in I^+_{\rho,\lambda_{\alpha(d)}}.$$

Then clearly, $d \in A$ and $|\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})$. □

Concerning condition (a) in Lemma 7.1 let us observe the following :

**LEMMA 7.2.** *Let $T \subseteq P_\kappa(\lambda)$ be such that $\mid T \cap P(a) \mid \leq u(\kappa,\lambda)$ for any $a \in P_\kappa(\lambda)$. Then $\mid T \mid \leq u(\kappa,\lambda)$.*

**Proof.** Suppose otherwise. Select $A \in I^+_{\kappa,\lambda}$ with $\mid A \mid = u(\kappa,\lambda)$. Define $g : T \to A$ so that $t \subseteq g(t)$ for all $t \in T$. Then $g$ must be constant on some $D \subseteq T$ with $\mid D \mid = \mid A \mid^+$. Contradiction. □

Consider for instance the following situation : In $V$, GCH holds, $\sigma$ is a strong cardinal with $\rho < \sigma < \lambda$, and $\pi$ a cardinal greater than $\lambda$. Then by a result of Gitik and Magidor [6], there is a cardinal preserving, $\sigma^+$-cc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$, (a) no new bounded subsets of $\sigma$ are added, (b) $\sigma$ changes its cofinality to $\omega$, and (c) $2^\sigma \geq \pi$. Now working in $V^{\mathbb{P}}$, let $T = P_{\omega_1}(\sigma)$. Then clearly $|T \cap P(a)| \leq 2^{|a|} \leq \kappa$ for any $a \in P_\kappa(\lambda)$. Moreover for any two uncountable cardinals $\chi$ and $\tau$ with $\mathrm{cf}(\chi) = \chi < \sigma \leq \tau \leq \pi$,

$$u(\chi,\tau) = 2^{<\chi} \cdot u(\chi,\tau) = \tau^{<\chi} = \sigma^{<\chi} = \sigma^{\aleph_0} = \mid T \mid .$$

Hence by Lemma 7.1, for any $R \in (NS^{\nu}_{\kappa,\lambda})^+$,

$$\{d \in D :| \{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\} |\geq u(\rho, \lambda_{\alpha(d)})\}$$

lies in $NS^+_{\mu}$ if $\nu = \mu$, and in $NS^+_{\kappa,\nu}$ otherwise. Note that for any cardinal $\chi$ with $\kappa \leq \chi \leq \sigma$, $\mathrm{cof}(NS^{\chi}_{\kappa,\lambda}) = u(\kappa, \lambda)$ since $\mathrm{cof}(NS^{\chi}_{\kappa,\lambda}) \leq (\lambda^{<\kappa})^{\chi} = (2^{\sigma})^{\chi} = 2^{\sigma}$, and moreover, by Lemma 1.9 and Proposition 3.6, $\overline{\mathrm{cof}}(NS^{\chi}_{\kappa,\lambda}) > \lambda$ in case $\mu \leq \chi$.

**PROPOSITION 7.3.** *Suppose there is $T \subseteq P_{\kappa}(\lambda)$ and a cardinal $\pi$ with $\rho \leq \pi < \lambda$ such that (a) $|T \cap P(a)| < \rho$ for any $a \in P_{\kappa}(\lambda)$, and (b) $\tau^{\nu} \leq u(\rho, \tau) \leq |T|$ for every cardinal $\tau$ with $\pi < \tau < \lambda$. Then $NS^{\nu}_{\kappa,\lambda}$ is nowhere precipitous.*

**Proof.** By Lemma 1.12 it suffices to show that II has a winning strategy in the game $G(NS^{\nu}_{\kappa,\lambda})$. We can assume without loss of generality that $\lambda_0 > \pi$. For $g : P_3(\nu) \to P_3(\lambda)$ and $\alpha < \mu$, define $g_{\alpha} : P_3(\nu) \to P_3(\lambda_{\alpha})$ by $g_{\alpha}(e) = \lambda_{\alpha} \cap g(e)$.

**Claim 1.** *Let $g : P_3(\nu) \to P_3(\lambda)$. Then*

$$\Big\{d \in D : \{a \in W_d : a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_{g_{\alpha(d)}}\} \subseteq C^{\kappa,\lambda}_g\Big\}$$

lies in $(NS_{\mu} \mid E)^*$ if $\nu = \mu$, and in $NS^*_{\kappa,\nu}$ otherwise.

**Proof of Claim 1.** We prove the claim in the case when $\nu > \mu$, and leave the proof in the case when $\nu = \mu$ to the reader. Define $h : P_3(\nu) \to \mu$ by $h(e) =$ the least $\beta < \mu$ such that $g(e) \subseteq \lambda_{\beta}$. Let $Q$ be the set of all $d \in D$ such that $h(e) \in d$ for every $e \in P_3(d)$. Then clearly $Q \in NS^*_{\kappa,\nu}$. Now fix $d \in Q$ and $a \in W_d$ such that $a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_{g_{\alpha(d)}}$. Let $e \in P_3(a \cap \nu)$. Then $h(e) \in d$, so $g(e) \subseteq \lambda_{\alpha(d)}$. It follows that $g(e) \subseteq a$, since $\lambda_{\alpha(d)} \cap g(e) \subseteq a$. Thus $a \in C^{\kappa,\lambda}_g$. This completes the proof of Claim 1.

**Claim 2.** *Let $X \in (NS^{\nu}_{\kappa,\lambda})^+$ and $Y \subseteq B$. Suppose that*

$$\{a \in Y \cap W_d : a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_k\} \neq \emptyset$$

*whenever $d \in D$ and $k : P_3(\nu) \to P_3(\lambda_{\alpha(d)})$ are such that*

$$|\{a \cap \lambda_{\alpha(d)} : a \in A \cap W_d and a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_k\}| = \lambda^{\nu}_{\alpha(d)}.$$

*Then $Y \in (NS^{\nu}_{\kappa,\lambda})^+$.*

**Proof of Claim 2.** Fix $g : P_3(\nu) \to P_3(\lambda)$. By Lemma 7.1 and Claim 1, there must be $d \in D$ such that

$$| \{a \cap \lambda_{\alpha(d)} : a \in (X \cap C^{\kappa,\lambda}_g) \cap W_d\} |= \lambda^{\nu}_{\alpha(d)}$$

and

$$\{a \in W_d : a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_{g_{\alpha(d)}}\} \subseteq C^{\kappa,\lambda}_g.$$

Then

$$\{a \in Y \cap W_d : a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_{g_{\alpha(d)}}\} \neq \emptyset$$

since $a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_{g_{\alpha(d)}}$ for every $a \in X \cap C^{\kappa,\lambda}_g \cap W_d$. Hence $Y \cap C^{\kappa,\lambda}_g \neq \emptyset$. This completes the proof of the claim.

For $d \in D$, consider the following two-person game $G_d$ consisting of $\omega$ moves, with player I making the first move : I and II alternately pick subsets of $W_d$, thus building a sequence $< X_n : n < \omega >$ subject to the following two conditions : (1) $X_0 \supseteq X_1 \supseteq \ldots$, and (2) $\{a \in X_{2n+1} : a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_k\} \neq \emptyset$ for every $k : P_3(\nu) \to P_3(\lambda_{\alpha(d)})$ such that

$$|\{a \cap \lambda_{\alpha(d)} : a \in X_{2n} \text{ and } a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_k\}| = \lambda^{\nu}_{\alpha(d)}.$$

II wins the game if and only if $\bigcap_{n<\omega} X_n = \emptyset$.

**Claim 3.** *Let $d \in D$. Then II has a winning strategy $\tau_d$ in the game $G_d$.*

**Proof of Claim 3.** Let $X_0, X_1, X_2, \ldots$ be the successive moves of player I. For $n \in \omega$, let $K_n$ be the set of all $k : P_3(\nu) \to P_3(\lambda_{\alpha(d)})$ such that

$$|\{a \cap \lambda_{\alpha(d)} : a \in X_n \text{ and } a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_k\}| = \lambda^{\nu}_{\alpha(d)}.$$

**Case 1 :** $|K_n| = \lambda^{\nu}_{\alpha(d)}$ for every $n < \omega$.
Given $n < \omega$, set $K_n = \{k_{n,\xi} : \xi < \lambda^{\nu}_{\alpha(d)}\}$ and let

$$\tau_d(X_0, \ldots, X_n) = \{y_{n,\xi} : \xi < \lambda^{\nu}_{\alpha(d)}\},$$

where $y_{n,\xi} \in X_n$ and

$$y_{n,\xi} \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_{k_{n,\xi}} \setminus \{y_{q,\zeta} : q < n \text{ and } \zeta \leq \xi\}.$$

**Case 2 :** There is $n \in \omega$ such that $|K_n| < \lambda^{\nu}_{\alpha(d)}$.
Let $m$ be the least such $n$. Pick $A \subseteq X_m$ with $|A| < \lambda^{\nu}_{\alpha(d)}$ so that

$$\{a \in A : a \cap \lambda_{\alpha(d)} \in C^{\kappa,\lambda_{\alpha(d)}}_k\} \neq \emptyset$$

for every $k \in K_m$. Then set $\tau_d(X_0, \ldots, X_m) = A$ and $\tau_d(X_0, \ldots, X_m, X_{m+1}) = \emptyset$.

The proof of Claim 3 is now complete.

Finally, consider the strategy $\tau$ for player II in $G(NS^{\nu}_{\kappa,\lambda})$ defined by

$$\tau(X_0, \ldots, X_n) = \bigcup_{d \in D} \tau_d(X_0 \cap W_d, X_1 \cap W_d, \ldots, X_n \cap W_d).$$

Using Claims 2 and 3, it is easy to check that the strategy $\tau$ is a winning one. $\square$

**COROLLARY 7.4.**

(i) *Suppose* $\tau^\mu < \lambda$ *for every cardinal* $\tau < \lambda$. *Then* $NS^\mu_{\kappa,\lambda}$ *is nowhere precipitous.*

(ii) *Suppose* $\nu > \mu$, *and* $\tau^{c(\kappa,\nu)} < \lambda$ *for every cardinal* $\tau < \lambda$. *Then* $NS^\nu_{\kappa,\lambda}$ *is nowhere precipitous.*

(iii) *Suppose* $\mathcal{H}_{\kappa,\lambda}$ *holds, and* $\tau^\mu < \lambda$ *for every cardinal* $\tau < \lambda$. *Then* $NS_{\kappa,\lambda}$ *is nowhere precipitous.*

**Proof.** By Proposition 7.3 and Corollary 6.4 (i). □

**Question**. Suppose $\mathcal{H}_{\kappa,\lambda}$ holds and $\tau^\mu < \lambda$ for every cardinal $\tau < \lambda$. Does it then follow that $\lambda$ is a strong limit cardinal ?

We conclude the section with the following remark. Suppose there exist $T$ and $\pi$ as in the statement of Proposition 7.3. Then either $\mathrm{cof}(NS^\nu_{\kappa,\lambda}) = u(\kappa,\lambda)$, or $\lambda^{<\mu} = \lambda$. To establish this, note that $u(\kappa,\lambda) \leq \lambda^{<\kappa} \leq \lambda^{<\mu} \leq \mid T \mid$, so by Proposition 7.2 $\mid T \mid = u(\kappa,\lambda) = \lambda^{<\mu}$. It is now simple to see that $\mid T \mid = \lambda$ if $\tau^\nu < \lambda$ for every cardinal $\tau < \lambda$, and $\mid T \mid = \lambda^\nu$ otherwise.

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