NON-FORKING FRAMES IN ABSTRACT ELEMENTARY CLASSES

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Abstract. The stability theory of first order theories was initiated by Saharon Shelah in 1969. The classification of abstract elementary classes was initiated by Shelah, too. In several papers, he introduced non-forking relations. Later, in [Sh:h].II, he introduced the good non-forking frame, an axiomatization of the non-forking notion.

We improve results of Shelah on good non-forking frames, mainly by weakening the stability hypothesis in several important theorems, replacing it by the almost λ-stability hypothesis: The number of types over a model of cardinality λ is at most λ^+.

We present conditions on K_λ, that imply the existence of a model in $K_{\lambda^{+n}}$ for all n. We do this by providing sufficiently strong conditions on K_λ, that they are inherited by a properly chosen subclass of K_{λ^+}. What are these conditions? We assume that there is a 'non-forking' relation which satisfies the properties of the non-forking relation on superstable first order theories. Note that here we deal with models of a fixed cardinality, λ.

While in [Sh:h].II we assume stability in λ, so we can use brimmed (=limit) models, here we assume almost stability only, but we add an assumption: The conjugation property.

In the context of elementary classes, the superstability assumption gives the existence of types with well-defined dimension and the ω-stability assumption gives the existence and uniqueness of models prime over sets. In our context, the local character assumption is an analog to superstability and the density of the class of uniqueness triples with respect to the relation $\preceq_{bs}$ is the analog to ω-stability.

Introduction

The book [Sh:c], on elementary classes, i.e., classes of first order theories, presents properties of theories, which are so called 'dividing lines' and investigates them. When such a property is satisfied, the theory is low, i.e., we can prove structure theorems, such as:

(1) The fundamental theorem of finitely generated Abelian groups.
(2) ArtinWedderburn Theorem on semi-simple rings.
(3) If V is a vector space, then it has a basis B, and V is the direct sum of the subspaces $span\{b\}$ where $b \in B$.

(We do not assert that these results follow from the model theoretic analysis, but they merely illustrate the meaning of 'structure'.) But when such a property is not satisfied, we have *non-structure*, namely, there is a witness that the theory is complicated, and there are no structure theorems. This witness can be the existence of many models in the same power.

There has been much work on classification of elementary classes, and some work on other classes of models.

The main topic in the recently published book, ([Sh:h]), is *abstract elementary classes* (in short AEC). There are two additional books which deal with AEC's ([Ba] and [Gr]).

From the viewpoint of the algebraist, model theory of first order theories is somewhat close to universal algebra. But he prefers focusing on the structures, rather than on sentences and formulas. Our context, abstract elementary classes, is closer to universal algebra, as our definitions do not mention sentences or formulas.

As superstability is one of the better dividing lines for first order theories, it is natural to generalize this notion to AEC's. A reasonable generalization is that of the existence of a good λ-frame, (see Definition 2.1.1), introduced in [Sh:h].II. In [Sh:h].II we assume existence of a good λ-frame and either get a non-structure property (in λ^{++}, at least where $2^{\lambda} < 2^{\lambda^+} < 2^{\lambda^{++}}$) or derive a good λ^+-frame from it.

The main tool in studying superstability is the independence relation, so called 'non-forking'. So let us discuss the issue of independence.

"In the 1930's, van der Waerden [van der Waerden 1949] and Whitney [Whitney 1935] abstracted the following properties of linear independence in vector spaces and algebraic independence in fields and used them to define the general notion of an independence relation" [Bal 88]. Let us describe van der Waerden's notion in terms of an element a depending on a set X:

(1) (Reflexivity) a depends on $\{a\}$.
(2) (Monotonicity) If a depends on X and $X \subseteq Y$ then a depends on Y.
(3) (Transitivity) If a depends on X and each $x \in X$ depends on Y then a depends on Y.
(4) (Exchange axiom) If a depends on $X \bigcup \{b\}$ but a does not depend on X then b depends on $X \bigcup \{a\}$.
(5) (Finite character) If a depends on X then a depends on a finite subset of X.

The notion of forking (in the context of first order theories) also specializes to linear independence and algebraic independence. It is not, strictly speaking, a generalization of the usual notion, since it is stronger in some respects, weaker in others. However, it retains the most important consequence of the theory, the ability to assign a dimension to each member of certain classes of models (see [JrSi3]).

In stability theory of first order theories we deal with a ternary relation, 'non-forking', which intuitively means 'A is free from B relative to C'. Baldwin [Bal 88] presents three differences between this notion and the standard one:

(1) In stability theory of first order theories the transitivity of dependence fails, but we have transitivity of independence: If 'A is free from B relative to C' and '$A \bigcup B$ is free from D relative to B', then 'A is free from D relative to C'.

(2) The element a is replaced by a set A. Since a singleton is a set, in this sense we generalize the independence relation.

(3) In stability theory we define a is independent from X over A instead of only over empty set and study what happens when A changes.

Here we deal with a much more general case: Abstract elementary classes (in short AEC's). If we consider the study of first order theory T as the study of the class of models $\{M : M \models T\}$, then the context of abstract elementary classes is a generalization of that of first order theories. There are well-known theorems on first order theories, that are wrong or very hard to prove in the context of AEC's. The main reason is that the Compactness Theorem fails. Concerning AEC's see Section 1.

Shelah defines in [Sh:h].II a set of axioms, which a non-forking relation should satisfy, in the context of AEC. An AEC with a non-forking relation that satisfies this set of axioms is called 'a good frame'. This non-forking relation deals essentially with an element and a model. [Actually it is a relation on quadruples (M_0, M_1, a, M_3) which intuitively means 'a is free from M_1 relative to M_0' (M_3 is an ambient model, which is needed in the AEC context, because we cannot use a monster model as in the stability theory for first order theories).]

Until this point we have spoken about the following independence notions:

(1) The standard: between an element and a set.
(2) Non-forking in the context of first order theories: essentially between sets.
(3) Axioms for a non-forking relation on AEC's: essentially between an element and a model.

The current work is a generalization of [Sh:h].II. We replace the stability assumption by the almost stability assumption, categoricity in λ and the conjugation property. We define a semi-good λ-frame as a good λ-frame minus stability in λ with almost stability in λ.

A note about the hypotheses: When we write a hypothesis, we assume it until we write another hypothesis, but usually we recall the hypothesis at the beginning of the following section. Sometimes we write 'but we do not use local character'. It is important to write this because we want to apply theorems we prove here, in papers, in which local character is not assumed (for example [JrSh 940]). For the same reason, in Hypothesis 3.0.9 we assume weak assumptions.

Notations: We use the letters k, l, m, n for natural numbers or integer numbers, $\alpha, \beta, \gamma, i, j, \varepsilon, \zeta$ for ordinal numbers, δ for a limit ordinal number, κ, λ, μ for cardinal numbers. We use p, q for types and P for a set of types. We use K for a class of models ,$\preceq, \prec$ for relations on K, $\prec^*_\lambda$ for a relation on models of cardinality λ, while we use $\preceq^{NF}_{\lambda^+}$, $\prec^+_{\lambda^+}$, $\preceq^{\otimes}_{\lambda^+}$ for relations on models of cardinality λ^+. $\subseteq$ denotes the relation of being submodel. We use NF,$\bigcup$,$\widehat{NF}$ for relations on quadruples of models. We use x for an invariant (an element or a symbol of the meta-language), R, P, E for relations or for predicates and f, g, h for functions or for function symbols. So sometimes we use P for a set of types and sometimes for a relation or a predicate.

Contents

1. Introduction

We prove two main theorems. We present them now, but the technical terms will be defined as the paper progresses (most of them in the first 4 sections, but the hypothesis of the second theorem will be explained in Section 11). In the first theorem we provide conditions mainly on K_λ, that imply the existence of a model in $K_{\lambda^+}, K_{\lambda^{+2}}, K_{\lambda^{+3}}$.

Theorem 1.0.1. *Suppose:*

(1) $\mathfrak{s} = (K, \preceq, S^{bs}, \bigcup)$ *is a semi-good* λ*-frame with conjugation.*
(2) $K^{3,uq}$ *is dense with respect to* $\preceq_{bs}$.
(3) $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$.

Then

(1) *There is a good* λ^+*-frame* $\mathfrak{s}^+ = ((K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})^{up}, S^{bs,+}, \overset{+}{\bigcup})$, *such that* $K^{sat} \subseteq K_{\lambda^+}$ *and the relation* $\preceq^{NF}_{\lambda^+} \restriction K^{sat}$ *is included in the relation* $\preceq \restriction K^{sat}$.
(2) $\mathfrak{s}^+$ *satisfies the conjugation property.*
(3) *There is a model in* K *of cardinality* λ^{+2}.
(4) *There is a model in* K *of cardinality* λ^{+3}.

In the second theorem we provide conditions mainly on K_λ, that imply the existence of a model in $K_{\lambda^{+n}}$ for all n. We do this by providing sufficiently strong conditions on K_λ, that they are inherited by a properly chosen subclass of K_{λ^+}.

Theorem 1.0.2. *Suppose:*

(1) $\mathfrak{s} = (K, \preceq, S^{bs}, \bigcup)$ *is a semi-good* λ*-frame with conjugation.*
(2) $m < \omega \Rightarrow I(\lambda^{+(2+m)}, K) < \mu_{unif}(\lambda^{+(2+m)}, 2^{\lambda^{+(1+m)}})$.
(3) $2^{\lambda^{+m}} < 2^{\lambda^{+m+1}}$ *and for every* $m < \omega$, $WDmId(\lambda^{+1+m})$ *is not saturated in* $\lambda^{+(2+m)}$.
(4) *Conjecture 11.1.4.*

Then there is a model in K^n *of cardinality* λ^{+n} *for every* $n < \omega$.

The main idea is that a class of models is posited to have 'good' properties on models of size λ. By induction, a decreasing sequence of abstract elementary classes $(K^n, \preceq^n)$ are defined such that $(K^n, \preceq^n)$ satisfies the 'good' properties on models of size λ^{+n} (where λ^{+n} is the n-th successor of λ). Condition 2 (of Theorem 1.0.2) is a precise way of saying there are fewer than the maximal number of models in each λ^{+n} to carry out an essential part of the inductive step, provided a rather weak set theoretic hypotheses. Rather, the main part of the argument given here moves through several approximations to transfer a dependence relation which behaves abstractly like first-order superstability on the models of K of cardinality λ to a similar relation on a subclass of K of cardinality λ^+.

Definition 1.0.3 (Abstract Elementary Classes).

(1) Let K be any class of models for a fixed vocabulary and let $\preceq$ be a 2-place relation on K. The pair $(K, \preceq)$ is an *AEC* if the following axioms are satisfied:
   (a) $K, \preceq$ are closed under isomorphisms. In other words, if $M_1 \in K$, $M_0 \preceq M_1$ and $f : M_1 \to N_1$ is an isomorphism, then $N_1 \in K$ and $f[M_0] \preceq f[M_1] = N_1$.
   (b) $\preceq$ is a partial order on K and it is included in the inclusion relation.
   (c) If $\langle M_\alpha : \alpha < \delta \rangle$ is a $\preceq$-increasing continuous sequence, then
   $$M_0 \preceq \bigcup\{M_\alpha : \alpha < \delta\} \in K.$$
   (d) Smoothness: If $\langle M_\alpha : \alpha < \delta \rangle$ is a $\preceq$-increasing continuous sequence, and for every $\alpha < \delta$, $M_\alpha \preceq N$, then
   $$\bigcup\{M_\alpha : \alpha < \delta\} \preceq N.$$
   (e) If $M_0 \subseteq M_1 \subseteq M_2$ and $M_0 \preceq M_2 \wedge M_1 \preceq M_2$, then $M_0 \preceq M_1$.
   (f) There is a Lowenheim Skolem Tarski number, $LST(K, \preceq)$, which is the first cardinal λ, such that for every model $N \in K$ and a subset A of it, there is a model $M \in K$ such that $A \subseteq M \preceq N$ and the cardinality of M is $\leq \lambda + |A|$.

(2) $(K, \preceq)$ is an *AEC in* λ if: The cardinality of every model in K is λ, and it satisfies Axioms a,b,d,e, and for sequences $\langle M_\alpha : \alpha < \delta \rangle$ with $\delta < \lambda^+$ it satisfies Axiom c too.

Remark 1.0.4. Considering a natural Class of models, usually we can check if it is an AEC, by the following rules:
(1) If K is any class of models for a fixed vocabulary, then $(K, \subseteq)$ satisfies Axioms b,d,e of AEC (Definition 1.0.3.1).
(2) Suppose $(K, \preceq)$ is an AEC. If $(K, \subseteq)$ satisfies Axiom 1.0.3.1.c, then $(K, \subseteq)$ is an AEC.
(3) If $(K, \preceq)$ is an AEC and $K' \subseteq K$ then $(K', \preceq \restriction K')$ satisfies Axioms b,d,e of AEC (Definition 1.0.3.1).

We give some simple examples of AEC's. One can see more examples in [Gr 21].

Example 1.0.5. Let T be a first order theory. Denote $K =: \{M : M \models T\}$. Define $M \preceq N$ if M is an elementary submodel of N. $(K, \preceq)$ is an AEC.

Example 1.0.6. Let T be a first order theory with Π_2 axioms, namely, axioms of the form $\forall x \exists y \varphi(x, y)$ [it is allowed to use dummy variables]. Denote $K =: \{M : M \models T\}$. Then $(K, \subseteq)$ is an AEC.

Example 1.0.7. The class of *locally-finite groups* (the subgroup generated by every finite subset of the group is finite) with the relation $\subseteq$ is an AEC.

Example 1.0.8. Let K be the class of groups. Let $\preceq =: \{(M, N) : M, N$ are groups, and M is a *pure* subgroup of $N\}$ (M is a pure subgroup of N if and only if $N \models (\exists y)ry = m$ implies $M \models (\exists y)ry = m$ for every integer r and every $m \in M$). $(K, \preceq)$ is an AEC.

Example 1.0.9. The class of models that are isomorphic to $(\mathbb{N}, <)$ with the relation $\subseteq$ is *not* an AEC, because it does not satisfy Axiom 1.0.3.1.c: $\bigcup\{\{-n, -n+1, -n+2..0, 1, 2...\} : 0 \leq n\}$ is isomorphic to $(\mathbb{Z}, <)$ although for every n $\{-n, -n+1, -n+2..0, 1, 2...\}$ is isomorphic to $(\mathbb{N}, <)$.

But the class of models that are isomorphic to $(\mathbb{N}, 0, <)$ with the relation $\subseteq$ is an AEC, (the relation $\subseteq$ in this case is actually the equality, and this AEC has just one model).

Example 1.0.10. Let K be the class of *well-ordered sets*. Let $\preceq$ be the relation of being an *edge extension* ($(M, \leq) \preceq (N, \leq)$ if $M \subseteq N$ and for each $a \in M$ and $b \in N - M$ $N \models a < b$). Then the pair $(K, \preceq)$ satisfies Axioms a,b,c,d,e of Definition 1.0.3.1, but $(K, \preceq)$ does not satisfy Axiom f.

Example 1.0.11. The class of *Banach spaces* with the relation $\subseteq$ is *not* an AEC, because it does not satisfy Axiom 1.0.3.1.c.

Example 1.0.12. The class of *sets* (i.e. models without relations or functions) of cardinality less than κ, where $\aleph_0 \leq \kappa$ and the relation is $\subseteq$, is *not* an AEC, because it does not satisfy Axiom 1.0.3.1.c.

The class of sets with the relation $\preceq = \{(M, N) : M \subseteq N \text{ and } ||N - M|| > \kappa\}$ where $\aleph_0 \leq \kappa$, is not an AEC, because it does *not* satisfy smoothness (Axiom 1.0.3.1.d).

Definition 1.0.13. $K_\lambda =: \{M \in K : ||M|| = \lambda\}$, $K_{<\lambda} = \{M \in K : ||M|| < \lambda\}$, etc.

Definition 1.0.14. We say $M \prec N$ when $M \preceq N$ and $M \neq N$.

Definition 1.0.15. Let K be a class of models which is closed under isomorphisms and let λ be a cardinal. $I(\lambda, K)$ is the number of models in K_λ up to isomorphism.

Definition 1.0.16. $(K, \preceq)^{up} := (K^{up}, \preceq^{up})$ where we define:

(1) K^{up} is the class of models with the vocabulary of K, such that there are a directed order I, and a set of models $\{M_s : s \in I\}$ such that: $M = \bigcup\{M_s : s \in I\}$ and $s \leq_I t \Rightarrow M_s \preceq M_t$.
(2) For $M, N \in K^{up}$, $M \preceq^{up} N$ iff there are directed orders I, J and sets of models $\{M_s : s \in I\}$, $\{N_t : t \in J\}$, respectively, such that: $M = \bigcup\{M_s : s \in I\}$, $N = \bigcup\{N_t : t \in J\}$, $I \subseteq J$, $s \leq_J t \Rightarrow N_s \preceq N_t$, $s \leq_I t \Rightarrow M_s \preceq M_t \preceq N_t$.

Proposition 1.0.17. *If*

(1) $(K_1, \preceq_1), (K_2, \preceq_2)$ *are AEC's in* λ.
(2) $K_1 \subseteq K_2$.
(3) $\preceq_2 \restriction K_1$ *is* $\preceq_1$.

Then $K_1^{up} \subseteq K_2^{up}$ *and* $(\preceq_2)^{up} \restriction K_1^{up}$ *is* $(\preceq_1)^{up}$.

Proof. Easy. $\dashv$

Fact 1.0.18 (Lemma 1.23 in [Sh:h].II)**.** *Let $(K, \preceq)$ be an AEC in λ. Then*

(1) $(K, \preceq)^{up}$ *is an AEC.*
(2) $(K^{up})_\lambda = K$.
(3) $\preceq^{up}\restriction K$ *is* $\preceq$.
(4) $LST(K, \preceq)^{up} = \lambda$.

Definition 1.0.19.

(1) Let M, N be models in K, f is an injection of M to N. We say that f is a $\preceq$-*embedding* and write $f : M \to N$, or shortly f is an *embedding* (if $\preceq$ is clear from the context), when f is an injection with domain M and $Im(f) \preceq N$.
(2) A function $f : B \to C$ is *over* A, if $A \subseteq B \bigcap C$ and $x \in A \Rightarrow f(x) = x$.

Definition 1.0.20.

(1) We say that $(K_\lambda, \preceq\restriction K_\lambda)$ satisfies *the amalgamation property* when: For every M_0, M_1, M_2 in K_λ, such that $n < 3 \Rightarrow M_0 \preceq M_n$, there are f_1, f_2, M_3 such that: $f_n : M_n \to M_3$ is an embedding over M_0, i.e., the diagram below commutes. In such a case, we say that (f_1, f_2, M_3) is an amalgamation of M_1 and M_2 over M_0 or that M_3 is an amalgam of M_1, M_2 over M_0.

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] & M_3 \\ M_0 \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & M_2 \arrow[u, "f_2"'] \end{tikzcd}$$

(2) We say that K_λ satisfies *the joint embedding property* when: If $M_1, M_2 \in K_\lambda$, then there are f_1, f_2, M_3 such that for $n = 1, 2$ $f_n : M_n \to M_3$ is an embedding and $M_3 \in K_\lambda$.
(3) $M \in K$ is $\preceq$-*maximal* if there is no $N \in K$ with $M \prec N$.

Now we want to define Galois-types ('types' in short). First we define classes of triples. Then we define when two triples are 'of the same type'. Then we define a Galois-type as an equivalence class of triples (under being 'of the same Galois-type').

Definition 1.0.21.

(1) $K^3_{K,\preceq} =: \{(M, N, a) : M, N \in K,\ M \preceq N,\ a \in N\}$. When the class $(K, \preceq)$ is clear from the context we omit it and write K^3.
(2) $K^3_\lambda := \{(M, N, a) : M, N \in K_\lambda,\ M \preceq N,\ a \in N\}$.

Now we define the equivalence relation E, the relation of being 'of the same Galois-type'.

Definition 1.0.22.

(1) $E^*_{K,\preceq}$ is the following relation on $K^3_{K,\preceq}$: $(M_0, N_0, a_0)E^*(M_1, N_1, a_1)$ iff $M_1 = M_0$ and for some $N_2 \in K_\lambda$ with $N_1 \preceq N_2$ there is an embedding $f : N_0 \to N_2$ over M_0 with $f(a_0) = a_1$.
(2) $E_{K,\preceq}$ is the closure of $E^*_{K,\preceq}$ under transitivity, i.e., the closure to an equivalence relation.

When $(K, \preceq)$ is clear from the context we omit it writing E^*, E.

Proposition 1.0.23.

(1) For every $M, N_0, N_1 \in K_\lambda$, $a \in N_0 - M$ and $b \in N_1 - M$, $(M, N_0, a)E^*$ (M, N_1, b) iff there is an amalgamation (f_0, f_1, N) of N_0, N_1 over M such that $f_0(a) = f_1(b)$.
(2) E^* is a reflexive, symmetric relation.
(3) If $(K_\lambda, \preceq\restriction K_\lambda)$ satisfies the amalgamation property, then E^*_λ is an equivalence relation.

Proof. Easy. $\dashv$

Definition 1.0.24.

(1) For every $(M, N, a) \in K^3$ let $tp_{K,\preceq}(a, M, N)$, the *Galois-type* of a in N over M, be the equivalence class of (M, N, a) under $E_{K,\preceq}$. When the class $(K, \preceq)$ is clear from the context we omit it, writing $tp(a, M, N)$ (in other texts, it is called '$ga - tp(a/M, N)$').
(2) For every $M \in K$, $S(M) := \{tp(a, M, N) : (M, N, a) \in K^3\}$ and $S^{na}(M) := \{tp(a, M, N) : (M, N, a) \in K^3$ and $a \in N - M\}$. A Galois-type in $S^{na}(M)$ is called non-algebraic Galois-type.
(3) If $p = tp(a, M_1, N)$ and $M_0 \preceq M_1$, then we define $p \restriction M_0 = tp(a, M_0, N)$,

Definition 1.0.25. Let $M, N \in K$, $N \preceq M$. M is said to be *full* over N when M realizes $S(N)$. M is said to be *saturated* in λ^+ over λ, when $M \in K_{\lambda^+}$ and for every model $N \in K_\lambda$ with $N \preceq M$, M is full over N.

Remark 1.0.26. This is the reasonable sense of saturated model we can use in our context, since we do not want to assume anything about $K_{<\lambda}$, especially not stability and not the amalgamation property, (so a saturated model in λ^+ over λ may not be full over a model $N \in K_{<\lambda}$, $N \preceq M$), see the following example from [BKS].

Example 1.0.27. Let τ contain infinitely many unary predicates P_n and one binary predicate E. Define a first order theory T such that $P_{n+1}(x) \Rightarrow P_n(x)$, E is an equivalence relation with two classes, which are each represented be exactly one point in $P_n - P_{n+1}$, for each n. Now let K be the class of models in T, that omit the type of two inequivalent points that satisfy all the P_n. Then a model $M \in K$ is determined up to isomorphism by $\mu(M) := |\{x \in M : (\forall n)P_n(x)\}|$. So K is categorical in every uncountable powers, but has $\aleph_0$ countable models (none of them is finite). Now let $\preceq$ be the relation of being submodel. Then $(K, \preceq)$

is an AEC with $L.S.T.(K, \preceq) = \aleph_0$. Let $M_0, M_1, M_2 \in K$ be such that $\mu(M_0) = 0, \mu(M_1) = \mu(M_2) = 1$ and M_1, M_2 are not isomorphic over M_0. Then there is no amalgamation of M_1, M_2 over M_0. Now if $\lambda > \aleph_0$ then every model $M \in K_{\lambda^+}$ is saturated (over λ). But it is not saturated over $\aleph_0$, since it realizes $tp(a_1, M_0, M_1)$ if and only if it does not realize $tp(a_2, M_0, M_2)$, (where a_n is the unique element of $M_n - M_0$ of course).

Definition 1.0.28. Let M be a model in K_{λ^+}. M is said to be *homogenous* in λ^+ over λ if for every $N_1, N_2 \in K_\lambda$ with $N_1 \preceq M \wedge N_1 \preceq N_2$, there is a $\preceq$-embedding $f : N_2 \to M$ over N_1.

Definition 1.0.29. A *representation* of a model M is an $\preceq$-increasing continuous sequence $\langle M_\alpha : \alpha < ||M|| \rangle$ of models with union M, such that $||M_\alpha|| < ||M||$ for each α and if $||M|| = \lambda^+$ then $||M_\alpha|| = \lambda$ for each α.

The following proposition is a version of Fodor's Lemma (there is no mathematical reason to choose this version, but we think that it is comfortable).

Proposition 1.0.30. *There are no $\langle M_\alpha : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha : \alpha \in \lambda^+ \rangle$, $\langle f_\alpha : \alpha \in \lambda^+ \rangle, S$ such that the following conditions are satisfied:*

(1) The sequences $\langle M_\alpha : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha : \alpha \in \lambda^+ \rangle$ are $\preceq$-increasing continuous sequences of models in K_λ.
(2) For every $\alpha < \lambda^+$, $f_\alpha : M_\alpha \to N_\alpha$ is a $\preceq$-embedding.
(3) $\langle f_\alpha : \alpha \in \lambda^+ \rangle$ is an increasing continuous sequence.
(4) S is a stationary subset of λ^+.
(5) For every $\alpha \in S$, there is $a \in M_{\alpha+1} - M_\alpha$ such that $f_{\alpha+1}(a) \in N_\alpha$.

Proof. Suppose there are such sequences. Denote $M = \bigcup \{f_\alpha[M_\alpha] : \alpha \in \lambda^+\}$. By clauses 4,5 $||M|| = K_{\lambda^+}$. $\langle f_\alpha[M_\alpha] : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha \bigcap M : \alpha \in \lambda^+ \rangle$ are representations of M. So they are equal on a club of λ^+. Hence there is $\alpha \in S$ such that $f_\alpha[M_\alpha] = N_\alpha \bigcap M$. Hence $f_\alpha[M_\alpha] \subseteq N_\alpha \bigcap f_{\alpha+1}[M_{\alpha+1}] \subseteq N_\alpha \bigcap M = f_\alpha[M_\alpha]$ and so all are equal. Especially $f_{\alpha+1}[M_{\alpha+1}] \bigcap N_\alpha = f_\alpha[M_\alpha]$, in contradiction to condition 5. $\dashv$

By Lemma 1.14 in [Sh:h].II:

Proposition 1.0.31 (saturativity = model homogeneity)**.** *Let $(K, \preceq)$ be an AEC such that K_λ satisfies the amalgamation property, and $LST(K, \preceq) \leq \lambda$. Let M be a model in K_{λ^+}. Then M is saturated in λ^+ over λ iff M is a homogenous model in λ^+ over λ.*

Now we discuss the uniqueness of the saturated model, although we do not know its existence. The proof idea for homogenous models is due to Jonsson from 1960. It is proved as Lemma 1.14 in [Sh:h].II.

Theorem 1.0.32 (the uniqueness of the saturated model)**.** *Suppose $(K_\lambda, \preceq \restriction K_\lambda)$ satisfies the amalgamation property and $LST(K, \preceq) \leq \lambda$.*

(1) Suppose $N \in K_\lambda$ and for $n = 1, 2$, $N \preceq M_n$, and M_n is saturated in λ^+ over λ. Then M_1, M_2 are isomorphic over N.

(2) If M_1, M_2 are saturated in λ^+ over λ and $(K_\lambda, \preceq\restriction K_\lambda)$ satisfies the joint embedding property, then M_1, M_2 are isomorphic.

We can prove now that if $(K_\lambda, \preceq\restriction K_\lambda)$ is stable in λ, then there is a saturated model in λ^+ over λ. But we prefer to define semi-good frames and then to prove a stronger theorem (Theorem 2.5.8).

2. Non-forking frames

2.1. **The plan.** Suppose we know something about K_λ, especially that there is no $\preceq$-maximal model. Can we say something about $K_{\lambda^{+n}}$? At least we want to prove that $K_{\lambda^{+n}} \neq \emptyset$. It is easy to prove that $K_{\lambda^+} \neq \emptyset$ [How? We choose M_α by induction on $\alpha < \lambda^+$ such that $M_\alpha \prec M_{\alpha+1}$ and if α is limit we define $M_\alpha := \bigcup\{M_\beta : \beta < \alpha\}$ (by Definition 1.0.3.1.c $M_\alpha \in K$). At the end $M_{\lambda^+} \in K_{\lambda^+}$]. What about $K_{\lambda^{+2}}$? The main topic in this paper is semi-good frames. If there is a semi-good λ-frame, then by Proposition 3.1.8.2 there is no $\preceq$-maximal model in K_{λ^+}. So $K_{\lambda^{++}} \neq \emptyset$. Moreover, Theorem 11.1.5.1 says that if $\mathfrak{s}$ is a semi-good λ-frame with some additional assumptions and λ satisfies specific set-theoretic assumptions, then there is a good λ^+-frame $\mathfrak{s}^+ = (K^+, \preceq^+, S^{bs,+}, \overset{+}{\bigcup})$, such that $K^+ \subseteq K$ and the relation $\preceq^+\restriction K^+$ is included in the relation $\preceq\restriction K^+$ (so $K_{\lambda^{+3}} \neq \emptyset$).

If we want to use Theorem 11.1.5.1 ω times, then we have to assume set-theoretic assumptions on λ^{+n} for each $n \in \omega$. In this way we obtain semi-good λ^{+n}-frame for each $n \in \omega$, assuming the existence of a semi-good λ-frame. In particular, we conclude that $K_{\lambda^{+n}}$ is not empty for each $n \in \omega$.

Definition 2.1.1 is an axiomatization of the non-forking relation in a superstable first order theory. If we omit the local character (see Definition 2.1.1(3)(c)) from the definition of semi-good frame then we get the basic properties of the non-forking relation in $(K_\lambda, \preceq\restriction K_\lambda)$ where $(K, \preceq)$ is stable in λ.

Sometimes we do not find a natural independence relation on all the types. So first we extend the notion of an AEC in λ by adding a new function S^{bs} which assigns a collection of basic (because they are basic for our construction) types to each model in K_λ, and then we add an independence relation $\bigcup$ on basic types.

It is reasonable to assume *categoricity* in some cardinality λ for some reasons:

1. If K is not categorical in any cardinality, then we know $\{\lambda : K$ is categorical in $\lambda\}$, it is the empty set.
2. If there is a superlimit model in K_λ, then we can reduce $(K_\lambda, \preceq\restriction K_\lambda)$ to the models which are isomorphic to it, and therefore obtain categoricity in λ (see Section 1 in [Sh:h].II). However this case requires stability.

We do not assume *the amalgamation property*, but we assume the amalgamation property in $(K_\lambda, \preceq\restriction K_\lambda)$. This is a reasonable assumption because it is proved in [Sh:h].I that if an AEC is categorical in λ and the amalgamation property fails in λ then under a plausible set theoretic assumption there are 2^{λ^+} models in K_{λ^+}.

Definition 2.1.1. $\mathfrak{s} = (K, \preceq, S^{bs}, \bigcup\!\!\!\!\bigcup)$ is a *good* λ*-frame* if:
(0)

(a) $(K, \preceq)$ is an AEC.
(b) $LST(K, \preceq) \leq \lambda$.

(1)

(a) $(K_\lambda, \preceq\restriction K_\lambda)$ satisfies the joint embedding property.
(b) $(K_\lambda, \preceq\restriction K_\lambda)$ satisfies the amalgamation property.
(c) There is no $\preceq$-maximal model in K_λ.

(2) S^{bs} is a function with domain K_λ, which satisfies the following axioms:

(a) $S^{bs}(M) \subseteq S^{na}(M) = \{tp(a, M, N) : M \prec N \in K_\lambda,\ a \in N - M\}$.
(b) It respects isomorphisms: If $tp(a, M, N) \in S^{bs}(M)$ and $f : N \to N'$ is an isomorphism, then $tp(f(a), f[M], N') \in S^{bs}(f[M])$.
(c) Density of the basic types: If $M, N \in K_\lambda$ and $M \prec N$, then there is $a \in N - M$ such that $tp(a, M, N) \in S^{bs}(M)$.
(d) Basic stability: For every $M \in K_\lambda$, the cardinality of $S^{bs}(M)$ is $\leq \lambda$.

(3) the relation $\bigcup\!\!\!\!\bigcup$ satisfies the following axioms:

(a) $\bigcup\!\!\!\!\bigcup$ is a set of quadruples (M_0, M_1, a, M_3) where $M_0, M_1, M_3 \in K_\lambda$, $a \in M_3 - M_1$ and for $n = 0, 1$ $tp(a, M_n, M_3) \in S^{bs}(M_n)$ and it respects isomorphisms: If $\bigcup\!\!\!\!\bigcup(M_0, M_1, a, M_3)$ and $f : M_3 \to M_3'$ is an isomorphism, then $\bigcup\!\!\!\!\bigcup(f[M_0], f[M_1], f(a), M_3')$.
(b) Monotonicity: If $M_0 \preceq M_0^* \preceq M_1^* \preceq M_1 \preceq M_3 \preceq M_3^*$, $M_1^* \bigcup\{a\} \subseteq M_3^{**} \preceq M_3^*$, then $\bigcup\!\!\!\!\bigcup(M_0, M_1, a, M_3) \Rightarrow \bigcup\!\!\!\!\bigcup(M_0^*, M_1^*, a, M_3^{**})$. From now on, '$p \in S^{bs}(N)$ does not fork over M' will be interpreted as 'for some a, N^+ we have $p = tp(a, N, N^+)$ and $\bigcup\!\!\!\!\bigcup(M, N, a, N^+)$'. See Proposition 2.1.2.
(c) Local character: For every limit ordinal $\delta < \lambda^+$ if $\langle M_\alpha : \alpha \leq \delta + 1\rangle$ is an increasing continuous sequence of models in K_λ, and $tp(a, M_\delta, M_{\delta+1}) \in S^{bs}(M_\delta)$, then there is $\alpha < \delta$ such that $tp(a, M_\delta, M_{\delta+1})$ does not fork over M_α.
(d) Uniqueness of the non-forking extension: If $M, N \in K_\lambda$, $M \preceq N$, $p, q \in S^{bs}(N)$ do not fork over M, and $p \restriction M = q \restriction M$, then $p = q$.
(e) Symmetry: If $M_0, M_1, M_3 \in K_\lambda$, $M_0 \preceq M_1 \preceq M_3$, $a_1 \in M_1$, $tp(a_1, M_0, M_3) \in S^{bs}(M_0)$, and $tp(a_2, M_1, M_3)$ does not fork over M_0, *then* there are $M_2, M_3^* \in K_\lambda$ such that $a_2 \in M_2$, $M_0 \preceq M_2 \preceq M_3^*$, $M_3 \preceq M_3^*$, and $tp(a_1, M_2, M_3^*)$ does not fork over M_0.
(f) Existence of non-forking extension: If $M, N \in K_\lambda$, $p \in S^{bs}(M)$ and $M \prec N$, then there is a type $q \in S^{bs}(N)$ such that q does not fork over M and $q \restriction M = p$.

(g) Continuity: Let $\delta < \lambda^+$ and $\langle M_\alpha : \alpha \leq \delta \rangle$ an increasing continuous sequence of models in K_λ and let $p \in S(M_\delta)$. If for every $\alpha \in \delta$, $p \upharpoonright M_\alpha$ does not fork over M_0, then $p \in S^{bs}(M_\delta)$ and does not fork over M_0.

Proposition 2.1.2. *If* $\bigcup(M_0, M_1, a, M_3)$ *and* $tp(b, M_1, M_3^*) = tp(a, M_1, M_3)$*, then by Definition 2.1.1.3.b (the monotonicity axiom)* $\bigcup(M_0, M_1, b, M_3^*)$.

Proof. By Definition 1.0.24, there is an amalgamation (id_{M_3}, f, M_3^{**}) of M_3 and M_3^* over M_1 with $f(b) = a$. By Definition 2.1.1.3.b, $\bigcup(M_0, M_1, a, M_3^{**})$. Using again Definition 2.1.1.3.b, we get $\bigcup(M_0, M_1, a, f[M_3^*])$. Hence, since $f(b) = a$, we have $\bigcup(M_0, M_1, f(b), f[M_3^*])$. Therefore by Definition 2.1.1.3.a $\bigcup(M_0, M_1, b, M_3^*)$. $\dashv$

While in [Sh:h].II we study good frames, so basic stability is assumed; here we assume basic *almost* stability so the following definition is central:

Definition 2.1.3. $\mathfrak{s} = (K^{\mathfrak{s}}, \preceq^{\mathfrak{s}}, S^{bs,\mathfrak{s}}, \overset{\mathfrak{s}}{\bigcup}) = (K, \preceq, S^{bs}, \bigcup)$ is a *semi*-good λ-frame, if $\mathfrak{s}$ satisfies the axioms of a good λ-frame except that instead of assuming basic stability, we assume that $\mathfrak{s}$ satisfies *basic almost stability*, namely, for every $M \in K_\lambda$ $S^{bs}(M)$ is of cardinality at most λ^+.

$\mathfrak{s}$ is said to be a *semi-good frame* if it is a semi-good λ-frame for some λ.

Remark 2.1.4. If for each $M \in K_\lambda$ $S^{bs}(M) = S^{na}(M)$, then the continuity axiom is an easy consequence of the local character.

Can we define in our context independence, orthogonality and more things like in superstable theories? The answer is: See [Sh:h].III (mainly Sections 5,6) and [JrSi3].

2.2. **Examples.** We give examples of good frames and examples of semi-good frames . The propositions and definitions that appear in this subsection are important for this subsection only.

Example 2.2.1. Let T be a superstable first order theory and let λ be a cardinal $\geq |T| + \aleph_0$ such that T is stable in λ. Let $K_{T,\lambda}$ be the class of models of T of cardinality at least λ. Let $\preceq$ denote the relation of being an elementary submodel. Let $S^{bs}(M)$ be $S^{na}(M)$. Let $\bigcup$ be as usual. Then by Claim 3.1 on page 283 in [Sh:h].II (or see [Sh 91]) $(K_{T,\lambda}, \preceq, S^{bs}, \bigcup)$ is a good λ-frame.

Definition 2.2.2. Let $(K, \preceq)$ be an AEC. We say that $\mathfrak{s} := (K, \preceq, S^{bs}, \bigcup)$ is *the trivial λ-frame of* $(K, \preceq)$ if S^{bs} is S^{na} and the relation $\bigcup$ is $\{(M_0, M_1, a, M_3) : M_0, M_1, M_3 \in K_\lambda,\ a \in M_3 - M_1\}$.

Proposition 2.2.3. *Suppose:*

(1) $(K, \preceq)$ *is an AEC.*
(2) $LST(K, \preceq) \leq \lambda$.
(3) $(K_\lambda, \preceq \upharpoonright K_\lambda)$ *satisfies the joint embedding property, the amalgamation property and has no maximal model.*

(4) *For each* $M \in K_\lambda$ $1 \leq |S(M)| \leq \lambda^+$.
(5) *For each* $M, N \in K_\lambda$ *with* $M \preceq N$ *and each* $p \in S^{na}(M)$, *there is exactly one type* $q \in S^{na}(N)$ *with* $p \subset q$.

Then the trivial λ*-frame of* $(K, \preceq)$ *satisfies the axioms of a semi-good* λ*-frame except maybe the symmetry axiom.*

Proof. Check the axioms. $\dashv$

Example 2.2.4. Let λ be a cardinal. Let P a family of λ^+ subsets of λ. Let $\tau := \{R_\alpha : \alpha < \lambda\}$ where each R_α is an unary predicate. Let K be the class of models M for τ such that for each $a \in M$ $\{\alpha \in \lambda : M \models R_\alpha(a)\} \in P$. Let $\preceq$ be the inclusion relation on K. Then $(K, \subseteq)$ is an AEC, $LST(K, \subseteq) = \aleph_0$ and $(K_\lambda, \subseteq \restriction K_\lambda)$ satisfies the joint embedding property, the amalgamation property and has no maximal model. Moreover, for every $M, N_1, N_2 \in K_\lambda$ with $M \subseteq N_1 \wedge M \subseteq N_2$ and every $a_1 \in N_1 - M$ and $a_2 \in N_2 - M$ $tp(a_1, M, N_1) = tp(a_2, M, N_2)$ iff $\{\alpha \in \lambda : N_1 \models R_\alpha[a_1]\} = \{\alpha \in \lambda : N_2 \models R_\alpha[a_2]\}$. So by Proposition 2.2.3, the trivial λ-frame of $(K, \subseteq)$ satisfies the axioms of a semi-good λ-frame except maybe the symmetry axiom (Definition 2.1.1.3.e). But it satisfies the symmetry axiom, too. On the other hand, it is not a good λ-frame.

The following proposition presents a simple way to create a semi-good frame from a class of models.

But first we have to present a way to create an AEC from a class of models. Roughly, if K is a class of models, then we define a class K' by: each model $M \in K'$ is a disjoint union of models of K (up to isomorphism) enriched by an equivalence relation, whose classes are the models in K. The partial order $\preceq'$ is defined naturally.

Definition 2.2.5. Let τ be a relational vocabulary and let λ be a cardinal. Let K be a class of λ^+ (up to isomorphism) τ-models each of cardinality at most λ. Let E be a binary predicate not in τ.

Then $(K', \preceq')$ is defined as follows:

K' be the class of models M for $\tau \bigcup \{E\}$ such that:

(1) E^M is an equivalence relation.
(2) For every $a \in M$ aE^M is isomorphic to some model in K.
(3) For every predicate $R \in \tau$ if $R^M(a_1 ... a_n)$, then the elements $a_1 ... a_n$ are in the same class under E^M.

$\preceq'$ is the relation on K' which is defined by: $M \preceq' N$ if $M \subseteq N$ and for every $a \in N - M$ and $b \in M$ $\neg a E^N b$.

Proposition 2.2.6. *Let* τ *be a relational vocabulary and let* λ *be a cardinal. Let* K *be a class of* λ^+ *(up to isomorphism)* τ*-models each of cardinality exactly* λ*. Let* E *be a binary predicate not in* τ*.*

Then $(K', \preceq')$ *is an AEC and the trivial frame of it is a semi-good* λ*-frame which is not a good-frame. Moreover,* $(K, \preceq)$ *satisfies the following properties:*

(1) $(K, \preceq)$ *is* λ*-tame (see Definition 1.11 on page 8 in* [GrVa 24]*).*
(2) $(K, \preceq)$ *is stable in all cardinalities greater than* λ*.*
(3) $I(\mu, K) = \mu$ *for each* μ *with* $\mu > \lambda$*.*

Proof. It is easy to prove that it is an AEC. For example, we prove that it has a LST-number and actually its LST-number is λ. Let $N \in K'$ and let $A \subset N$. Let M be the sub-model of N with universe $\{b \in N : aE^N b$ for some $a \in A\}$. Now $|M| = |A| \times \lambda$ and $M \preceq' N$.

We have to prove that the trivial frame of $(K', \preceq')$ is a semi-good λ-frame. So we have to check conditions 3-5 of Proposition 2.2.3 and the symmetry axiom (Definition 2.1.1.3.e).

3. Easy.

4. Let $M, N_1, N_2 \in K'_\lambda$ with $M \preceq' N_1$ and $M \preceq' N_2$ and let $a_1 \in N_1$, $a_2 \in N_2$. Then $tp(a_1, M, N_1) = tp(a_2, M, N_2)$ iff there is an isomorphism $f : a_1 E^{N_1} \to a_2 E^{N_2}$ with $f(a_1) = a_2$. But for every $(M, N, a) \in S(M)$, aE^N is isomorphic to some model in K and $|K/\cong| = \lambda^+$. So $\lambda^+ \leq |S(M)| \leq \lambda \times \lambda^+ = \lambda^+$.

5. Easy.

By Proposition 2.2.3 it is enough to prove the symmetry axiom. We leave it to the reader.

It remains to prove that the trivial frame is not a good frame, namely, that for some model in $M \in K'$ we have $|S(M)| \succeq \lambda^+$ (so $= \lambda^+$). Take an $M \in K'$ of cardinality λ. For each model $N \in K$, we define a model $M_N \in K'$ such that its universe is a disjoint union of M and N, $E^{M_N} := E^M \bigcup \{(a, b) : a, b \in N\}$, for each predicate $R \in \tau$ $R^{M_N}(a_0, a_1, ...a_n)$ iff $R^M(a_0, a_1, ...a_n)$ or $R^N(a_0, a_1, ...a_n)$.

Let $N_1, N_2 \in K$ and let $a_1 \in M_{N_1}, a_2 \in M_{N_2}$. If $(M, M_{N_1}, a), (M, M_{N_2}, b)$ realize the same galois type, then the embedding witnessing it must map the equivalence class of a onto the equivalence class of b and so N_1 must be isomorphic to N_2.

$\dashv$

Example 2.2.7. Let λ be a cardinal. Let K be the class of well orderings of cardinality λ at most. So $|K/\cong| = \lambda^+$. Let $(K', \preceq')$ be as in Proposition 2.2.6. Then the trivial frame of $(K', \preceq')$ is a semi-good λ-frame, but is not a good λ-frame.

2.3. **A family of examples.** The following assertions of Shelah yield examples of semi-good frames, which in general do not have to be stable; understanding the argument requires a careful examination of the first chapter of [Sh:h].

In [Sh:h].II, Shelah presents a way to derive a good-frame, using results from [Sh:h].I. Here, Proposition 2.3.4 presents a way to derive a semi-good $\aleph_0$-frame, using [Sh:h].II and [Sh:h].I.

First we give some definitions.

Definition 2.3.1. Let $(K,\preceq)$ be an AEC and let $M\in K_{\aleph_0}$. We define $K_M=\{N\in K : N\equiv_{L_{\infty,\omega}} M\}$ and $\preceq_M=\{(N_1,N_2) : N_1,N_2\in K_M,\ N_1\preceq N_2,\ and\ N_1\preceq_{L_{\infty,\omega}} N_2\}$.

Definition 2.3.2. $(K,\preceq)$ is said to be $PC_{\aleph_0}$ when: K is the class of reductions to a smaller language, of some countable elementary class, which omit a countable set of types, and the relation $\preceq$ is defined similarly.

Let $(K,\preceq)$ be an AEC, M_1,M_2 be models in K and A a subset of $M_1\bigcap M_2$. Shelah defines (Definitions 5.5,5.7 of [Sh:h].I) when $tp(a,M_1,M_2)$ *is definable over* A. We should note, that while in [Sh:h].I, we deal with the syntactic types that are materialized (see 4.3 of [Sh:h].I), in [Sh:h].II, the types are galois. Shelah discusses this issue in the proof of Theorem 3.4 of [Sh:h].II and shows that in the context of Theorem 3.4 galois types are, actually, those types which are materialized.

Definition 2.3.3. Let $(K,\preceq)$ be an AEC. The *finitely definable* λ*-frame* of $(K,\preceq)$ is $(K,\preceq,S^{na},\bigcup)$ where we define $\bigcup:=\{(M_0,M_1,a,M_2) : M_0,M_1,M_2\in K,\ ||M_0||=||M_1||=\lambda,\ M_0\preceq M_1\prec M_2$ and $ga-tp(a,M_1,M_2)$ is definable (in the sense of Definitions 5.5,5.7 of [Sh:h].I) over some finite subset A of $M_0\}$.

Proposition 2.3.4. *Let $(K,\preceq)$ be an AEC with a countable vocabulary, $LST(K,\preceq)=\aleph_0$, $(K,\preceq)$ is $PC_{\aleph_0}$, $0<I(\aleph_1,K)<2^{\aleph_1}$ and $2^{\aleph_0}<2^{\aleph_1}$. Then:*

(1) There is a model M in $K_{\aleph_0}$ such that $(K_M)_{\aleph_1}\neq\emptyset$,
(2) the finitely definable $\aleph_0$-frame of $(K_M,\prec_M)$ is a semi-good $\aleph_0$-frame.

Proof. (1) By Proposition 2.3.5,
(2) by Proposition 2.3.10.

$\dashv$

Proposition 2.3.5. *Let $(K,\preceq)$ be an AEC with a countable vocabulary, $LST(K,\preceq)=\aleph_0$, $(K,\preceq)$ is $PC_{\aleph_0}$ $(0<I(\aleph_1,K)<2^{\aleph_1}$ and $2^{\aleph_0}<2^{\aleph_1}$. Then there is a model $M\in K_{\aleph_0}$ such that $(K_M)_{\aleph_1}\neq\emptyset$.*

In order to prove Proposition 2.3.5, we use theorems from [BLS].

Definition 2.3.6. Let L^* be a fragment of $L_{\omega_1,\omega}$. A model is L^*-small if it realizes only countably many $L^*(\tau)$-types over $\emptyset$.

The following fundamental result is du to Keisler (see [Ke] or Theorem 2.4 of [BLS] or Theorem 5.2.5 of [Ba]).

Theorem 2.3.7 (Keisler)**.** *If a PC_δ over $L_{\omega_1,\omega}$ class K has an uncountable model but less than 2^{ω_1} models of power $\aleph_1$ then for any countable fragment L^* of $L_{\omega_1,\omega}$ every member M of K is L^*-small. That is, each $M\in K$ realizes only countably many L^*-types over $\emptyset$.*

The following theorem is a translation of Theorem 2.7 from [BLS] (this theorem was certainly known to Shelah when Sh88 was proved).

Theorem 2.3.8. *If the class K is $PC_{\aleph_0}$ and every model of cardinality $\aleph_1$ is L^*-small for every countable fragment L^* of $L_{\omega_1,\omega}$ then K has a $L_{\omega_1,\omega}$-small models M' of cardinality $\aleph_1$.*

Now we can prove Proposition 2.3.5:

Proof. By Theorem 2.3.7, the assumptions of Theorem 2.3.8 are satisfied. So by Theorem 2.3.8, K has a $L_{\omega_1,\omega}$-small model of cardinality $\aleph_1$. Now it is enough to prove Proposition 2.3.9. $\dashv$

The following proposition is a version of the Lowenheim-Skolem-Tarski downward Theorem. It combines the logic $L_{\omega_1,\omega}$ with the concept of AEC.

Proposition 2.3.9. *Let $(K, \preceq)$ be an AEC with $LST(K, \preceq) = \aleph_0$ and let M be a $L_{\omega_1,\omega}$-small model of K of cardinality $\aleph_1$. Then we can find a countable submodel N of M such that:*

(1) $N \in K$.
(2) $N \preceq M$.
(3) $N \preceq_{L_{\omega_1,\omega}} M$.

Proof. If we can choose an increasing sequence of countable submodels of M, $\langle N_n : n < \omega \rangle$ such that $N_{2n} \preceq M$ and $N_{2n+1} \preceq_{L_{\omega_1,\omega}} M$ for each $n < \omega$, then the union of this sequence satisfies the needed conditions. Since $LST(K, \preceq) = \aleph_0$, we can choose N_n for an even number. Since M is $L_{\omega_1,\omega}$-small, by a Lowenheim-Skolem-Tarski argument, we can choose N_n for an odd number. $\dashv$

Proposition 2.3.10. *Let $(K, \preceq)$ be an AEC with a countable vocabulary, $LST(K, \preceq) = \aleph_0$, $(K, \preceq)$ is $PC_{\aleph_0}$, $0 < I(\aleph_1, K) < 2^{\aleph_1}$ and $2^{\aleph_0} < 2^{\aleph_1}$. Let M be a model in $K_{\aleph_0}$ with $(K_M)_{\aleph_1} \neq \emptyset$. Then the finitely definable $\aleph_0$-frame of $(K_M, \prec_M)$ is a semi-good $\aleph_0$-frame.*

Proof. By the proof of Theorem 3.4 on page 285 in [Sh:h].II: We assumed here assumptions $(\alpha), (\beta), (\gamma)$ of Theorem 3.4. So by Theorem 3.4.1 for some $M \in K_{\aleph_0}$ we have $(\delta^-), (\varepsilon)$ too. So if (δ) (namely stability) holds then by Theorem 3.4.2, $\mathfrak{s}$ is a good $\aleph_0$-frame.

We have two problems concerning (δ): the first problem is that we know (δ^-) (namely almost stability) only. But at the beginning of the proof of item 2 (the last line on page 287), it is written 'we assume (δ^-) instead of (δ)'.

The second problem is that the proof of almost stability uses [Sh:h].I, where the types are not galois. But shelah shows (in the proof of Theorem 3.4) that galois types are in this case a certain kind of syntactic type, those which are materialized.

By the continuation of the proof, we see that $\mathfrak{s}$ is a semi-good $\aleph_0$-frame $\dashv$

2.4. **Specific Examples.** Example 2.4.6 is a specific semi-good frame. Note that Example 2.4.6 is not of the same kind as the family in Subsection 2.3, because in 2.4.6 there are $2^{\aleph_1}$ models of cardinality $\aleph_1$.

Definition 2.4.1. *A transitive linear order* is a linear order, M, such that for every two elements $a, b \in M$ there is an automorphism f of M with $f(a) = b$.

The following lemma is implied by Corollary 8.6(2) on page 123 in [Rosenstein]
.

Lemma 2.4.2. *Let K be the class of transitive linear orders. Then $I(\aleph_0, K) = \aleph_1$.*

Definition 2.4.3. *A transitive partial order* is a partial order, M, such that for each element $a \in M$, the connectedness component of a, namely, $\{b \in M : b < a \vee a < b\}$ is a transitive linear order.

The following AEC can be called 'The AEC of transitive partial orders with countable connectedness components'. But we prefer a shorter name.

Definition 2.4.4. *The AEC of transitive partial orders*, $(K, \preceq)$ is defined by: K is the class of transitive partial orders, whose each connectedness component is countable. $M \preceq N$ means $M \subseteq N$ and for each $a \in M$ and $b \in N - M$, neither $a <^N b$ nor $b <^N a$ (new elements belong to new connectedness components).

Proposition 2.4.5. *The AEC of transitive partial orders is an AEC which is $PC_{\aleph_0}$ and it has LST-number $\aleph_0$.*

Proof. We prove that K is $PC_{\aleph_0}$ only. Define a vocabulary $\tau^+ := \{<\} \bigcup \{f_n : n < \omega\}$, where $<$ is a binary relation, and f_n is a unary function for each n. Define $\tau := \{<\}$. Let φ be the sentence '$<$ is a partial order' and let φ_n be the sentence 'f_n is a τ-automorphism'. Let $\mathbb{T}$ be the theory $\{\varphi\} \cup \{\varphi_n : n < \omega\}$. We define a type $p(x, y) := \{f_n(x) \neq y : n < \omega\}$. Now K is the class of reductions to τ of τ^+-models of $\mathbb{T}$ which omit $p(x, y)$. ⊣

Example 2.4.6. Let $(K, \preceq)$ be the AEC of transitive partial orders. Let S^{bs} be S^{na}. Let $\bigcup\!\!\!\!|$ be the trivial non-forking relation ('always' the type does not fork).

Remark 2.4.7. Let $M_0, M_1, M_2 \in K_\lambda$, $M_0 \preceq M_1, M_2$ and let $a_1 \in M_1 - M_0$ and $a_2 \in M_2 - M_0$. Then $ga - tp(a_1, M_0, M_1) = ga - tp(a_2, M_0, M_2)$ if and only if there is an isomorphism $f : a_1 E^{M_1} \to a_2 E^{M_2}$ with $f(a_1) = a_2$).

Claim 2.4.8. *$(K, \preceq, S^{bs}, \bigcup\!\!\!\!|)$ is a semi-good $\aleph_0$-frame.*

Proof. It is easy to prove the existence and uniqueness of the non-forking extension, using Remark 2.4.7. In order to prove almost stability, we have to use Lemma 2.4.2. It is easy to prove the remain axioms. ⊣

2.5. **Additional properties of a frame.** The following definition appears in [Sh E46].

Definition 2.5.1. Let $p_0 \in S(M_0), p_1 \in S(M_1)$. We say that p_0, p_1 are *conjugate* if for some $a_0, M_0^+, a_1, M_1^+, f$, the following hold:

(1) For $n = 0, 1$, $tp(a_n, M_n, M_n^+) = p_n$.
(2) $f : M_0^+ \to M_1^+$ is an isomorphism.
(3) $f \restriction M_0 : M_0 \to M_1$ is an isomorphism.
(4) $f(a_0) = a_1$.

Proposition 2.5.2. *Assume that p_0, p_1 are conjugate and the types p_1, p_2 are conjugate. Then the types p_0, p_2 are conjugate.*

Proof. Compose the isomorphisms. $\dashv$

Definition 2.5.3. Let $p = tp(a, M, N)$. Let f be a bijection with domain M. Define *f(p)* $:= tp(f(a), f[M], f^+[N])$, where f^+ is an extension of f (and the relations and functions on $f^+[N]$ are defined such that $f^+ : N \to f^+[N]$ is an isomorphism).

Remark 2.5.4. The definition of $f(p)$ in Definition 2.5.3 does not depend on the representative $(M, N, a) \in p$.

Definition 2.5.5. Let $\mathfrak{s}$ be a semi-good λ-frame. We say that $\mathfrak{s}$ *satisfies the conjugation property* when: K_λ is categorical and if $M_1, M_2 \in K_\lambda$, $M_1 \preceq M_2$ and $p_2 \in S^{bs}(M_2)$ is the non-forking extension of $p_1 \in S^{bs}(M_1)$, then the types p_1, p_2 are conjugate.

By Claim 2.18 in [Sh:h].II:

Proposition 2.5.6 (The transitivity proposition)**.** *Suppose $\mathfrak{s}$ is a semi-good λ-frame. Then: If $M_0 \preceq M_1 \preceq M_2$, $p \in S^{bs}(M_2)$ does not fork over M_1 and $p \restriction M_1$ does not fork over M_0, then p does not fork over M_0.*

By Claim 2.16 in [Sh:h].II:

Proposition 2.5.7. *Suppose*
(1) $\mathfrak{s}$ satisfies the axioms of a semi-good λ-frame.
(2) $n < 3 \Rightarrow M_0 \preceq M_n$.
(3) For $n = 1, 2$, $a_n \in M_n - M_0$ and $tp(a_n, M_0, M_n) \in S^{bs}(M_0)$.

Then there is an amalgamation (f_1, f_2, M_3) of M_1, M_2 over M_0 such that for $n = 1, 2$ $tp(f_n(a_n), f_{3-n}[M_{3-n}], M_3)$ does not fork over M_0.

Now we prove almost stability (and more). Note that while in Claim 4.2 of [Sh:h].II, Shelah uses local character in the proof of stability, here we do not use local character.

Theorem 2.5.8. *Suppose $\mathfrak{s}$ satisfies conditions 1 and 2 of a semi-good λ-frame (so actually the relation $\bigcup$ is irrelevant).*
(1) Suppose:
  (a) $\langle M_\alpha : \alpha \leq \lambda^+ \rangle$ is an increasing continuous sequence of models in K_λ.
  (b) There is a stationary set $S \subseteq \lambda^+$ such that for every $\alpha \in S$ and every model N, with $M_\alpha \prec N$, there is a type $p \in S^{bs}(M_\alpha)$ which is realized in M_{λ^+} and in N.

Then M_{λ^+} is full over M_0 and is saturated in λ^+ over λ.

(2) *Suppose:*
   (a) *$\langle M_\alpha : \alpha \leq \lambda^+\rangle$ is an increasing continuous sequence of models in K_λ.*
   (b) *For every $\alpha \in \lambda^+$ and every $p \in S^{bs}(M_\alpha)$, there is $\beta \in (\alpha, \lambda^+)$ such that p is realized in M_β.*

   Then M_{λ^+} is full over M_0 and M_{λ^+} is saturated in λ^+ over λ.
(3) *There is a model in K_{λ^+} which is saturated in λ^+ over λ.*
(4) *$M \in K_\lambda \Rightarrow |S(M)| \leq \lambda^+$ (we know that $|S^{bs}(M)| \leq \lambda^+$, but the point is that $|S(M)| \leq \lambda^+$).*

Proof. We will show 1 implies the rest and then prove 1. Obviously $1 \Rightarrow 2$.

$3 \Rightarrow 4$: Let $M \in K_\lambda$ and let $M^+ \in \lambda^+$ be a saturated model in λ^+ over λ. Since $LST(K, \preceq) \leq \lambda$ we can find $M_1 \in K_\lambda$ with $M_1 \preceq M^+$. Since $(K_\lambda, \preceq\restriction K_\lambda)$ satisfies the joint embedding property, we can find a joint embedding (f_1, id_{M_1}, M_2) of M and M_1. By Proposition 1.0.31 (the saturativity = model homogeneity Proposition) we can find an embedding $f_2 : M_2 \to M^+$ over M_1. Now $|S(M)| = |S(f_2 \circ f_1[M])| \leq ||M^+|| = \lambda^+$.

To show $2 \Rightarrow 3$, we construct a chain satisfying the hypotheses of 2. Let cd be an injection from $\lambda^+ \times \lambda^+$ onto λ^+. Define by induction on $\alpha < \lambda^+$ M_α and $\langle p_{\alpha,\beta} : \beta < \lambda^+\rangle$ such that:

(1) $\langle M_\alpha : \alpha < \lambda^+\rangle$ is an increasing continuous sequence of models in K_λ.
(2) $\{p_{\alpha,\beta} : \beta < \lambda^+\} = S^{bs}(M_\alpha)$.
(3) $M_{\alpha+1}$ realizes $p_{\gamma,\beta}$, where we denote: $A_\alpha := \{cd(\gamma, \beta) : \gamma \leq \alpha,\ p_{\gamma,\beta}$ is not realized in $M_\alpha\}$, $\varepsilon_\alpha = \mathrm{Min}(A_\alpha)$ and $(\gamma, \beta) = cd^{-1}(\varepsilon_\alpha)$.

We argue that $M_{\lambda^+} := \bigcup\{M_\alpha : \alpha < \lambda^+\}$ is saturated in λ^+ over λ. By 2 it is sufficient to prove that for every $\alpha \in \lambda^+$ and every $p \in S^{bs}(M_\alpha)$ there is $\beta \in (\alpha, \lambda^+)$ such that p is realized in M_β. Towards a contradiction, choose α^* so that $p \in S^{bs}(M_{\alpha^*})$ is not realized in M_{λ^+}. There is $\beta < \lambda^+$ such that $p = p_{\alpha^*,\beta}$. Denote $\varepsilon := cd(\alpha^*, \beta)$. For every $\alpha \geq \alpha^*$ $\varepsilon \in A_\alpha$, so A_α is nonempty and ε_α is defined. But $\varepsilon_\alpha \neq \varepsilon$, (because otherwise p is realized in $M_{\alpha+1}$), so $\varepsilon_\alpha < \varepsilon$. The function $f : [\alpha^*, \lambda^+) \to \varepsilon$, $f(\alpha) = \varepsilon_\alpha$ is an injection which is impossible.

It remains to prove item 1. Fix N, with $M_0 \prec N$. It is sufficient to prove that there is an embedding of N to M_{λ^+} over M_0. We choose $(\alpha_\varepsilon, N_\varepsilon, f_\varepsilon)$ by induction on $\varepsilon < \lambda^+$ such that:

$$\begin{tikzcd}
N \arrow[r, "id"] & N_\varepsilon \arrow[r, "id"] & N_{\varepsilon+1} \\
M_0 \arrow[u, "f_0"] \arrow[r, "id"] & M_{\alpha_\varepsilon} \arrow[u, "f_\varepsilon"] \arrow[r, "id"] & M_{\alpha_{\varepsilon+1}} \arrow[u, "f_{\varepsilon+1}"]
\end{tikzcd}$$

(1) $\langle \alpha_\varepsilon : \varepsilon < \lambda^+\rangle$ is an increasing continuous sequence of ordinals in λ^+.
(2) The sequence $\langle N_\varepsilon : \varepsilon < \lambda^+\rangle$ is increasing and continuous.

(3) $\langle f_\varepsilon : \varepsilon < \lambda^+\rangle$ is increasing continuous.
(4) $N_0 := N$, $\alpha_0 := 0$ and $f_0 = id_{M_0}$.
(5) $f_\varepsilon : M_{\alpha_\varepsilon} \to N_\varepsilon$ is an embedding.
(6) For every $\alpha \in S$ there is $a \in M_{\alpha_{\varepsilon+1}} - M_{\alpha_\varepsilon}$ such that $f_{\varepsilon+1}(a) \in N_\varepsilon$.

By Proposition 1.0.30 we cannot carry out this construction. Where will we get stuck? For $\varepsilon = 0$ or limit, we will not get stuck. Suppose we have defined $(\alpha_\zeta, N_\zeta, f_\zeta)$ for $\zeta \leq \varepsilon$. If $f_\varepsilon[M_{\alpha_\varepsilon}] = N_\varepsilon$, then $f_\varepsilon^{-1} \restriction N$ is an embedding of N into M_{λ^+} over M_0, hence we are finished. So, without loss of generality, $f_\varepsilon[M_{\alpha_\varepsilon}] \neq N_\varepsilon$. If $\alpha_\varepsilon \notin S$, then we define $\alpha_{\varepsilon+1} := \alpha_\varepsilon + 1$ and use the amalgamation property in $(K_\lambda, \preceq\restriction K_\lambda)$ to find $N_{\varepsilon+1}, f_{\varepsilon+1}$ as needed.

Suppose $\alpha_\varepsilon \in S$. By the theorem's assumption, there is a type $p \in S(M_{\alpha_\varepsilon})$ such that p is realized in M_{λ^+} and $f_\varepsilon(p)$ is realized in N_ε. Define $\alpha_{\varepsilon+1} := \text{Min}\{\alpha \in \lambda^+ : p \text{ is realized in } M_\alpha \}$. Take $a \in M_{\alpha_{\varepsilon+1}}$ such that $tp(a, M_{\alpha_\varepsilon}, M_{\alpha_{\varepsilon+1}}) = p$ and take $b \in N_\varepsilon$ such that $tp(b, f_\varepsilon(M_{\alpha_\varepsilon}), N_\varepsilon) = f_\varepsilon(p)$. Then $f_\varepsilon(tp(a, M_{\alpha_\varepsilon}, M_{\lambda^+})) = tp(b, M_{\alpha_\varepsilon}, N_\varepsilon)$. By the definition of type (Definition 1.0.24.1), there are $N_{\varepsilon+1}, f_{\varepsilon+1}$ with $N_\varepsilon \preceq N_{\varepsilon+1}$, $f_{\varepsilon+1}$ is an embedding of $M_{\alpha_{\varepsilon+1}}$ into $N_{\varepsilon+1}$, $f_\varepsilon \subseteq f_{\varepsilon+1}$ and $f_{\varepsilon+1}(a) = b$.

Since the hypotheses of 5 applies to any cofinal segment of the sequence $\langle M_\alpha : \alpha < \lambda^+\rangle$ and any submodel of size λ lies in some M_α, we conclude that M_{λ^+} is saturated in λ^+ over λ. $\dashv$

2.6. **Non-forking with larger models.** Now we extend our non-forking notion to include models of cardinality greater than λ.

Definition 2.6.1. $\overset{\geq\lambda}{\bigcup\!\!\!\!\bigcup}$ is the class of quadruples (M_0, a, M_1, M_2) such that:

(1) $\lambda \leq ||M_i||$ for each $i < 3$.
(2) $M_0 \preceq M_1 \preceq M_2$ and $a \in M_2 - M_1$.
(3) For some model $N_0 \in K_\lambda$ with $N_0 \preceq M_0$ for each model $N \in K_\lambda$, $N_0 \bigcup\{a\} \subseteq N \preceq M_1 \Rightarrow \bigcup\!\!\!\!\bigcup(N_0, a, N, M_2)$.

Definition 2.6.2. Let M_0, M_1 be models in $K_{\geq\lambda}$ with $M_0 \preceq M_1$ and $p \in S(M_1)$. We say that p does not fork over M_0, when for some triple $(M_1, M_2, a) \in p$ we have $\overset{\leq\lambda}{\bigcup\!\!\!\!\bigcup}(M_0, a, M_1, M_2)$.

Remark 2.6.3. We can replace the quantification 'for some' (M_1, M_2, a) in Definition 2.6.2 by 'for each'.

Definition 2.6.4. Let $M \in K_{>\lambda}$, $p \in S(M)$. p is said to be *basic* when there is $N \in K_\lambda$ such that $N \preceq M$ and p does not fork over N. For every $M \in K_{>\lambda}$, $S^{bs}_{>\lambda}(M)$ is the set of basic types over M. Sometimes we write $S^{bs}_{\geq\lambda}(M)$, meaning $S^{bs}(M)$ or $S^{bs}_{>\lambda}(M)$ (the unique difference is the cardinality of M).

Now we present a weak version of local character which is needed for a later paper.

Definition 2.6.5. Let $\mathfrak{s}$ be a semi-good λ-frame except local character. $\mathfrak{s}$ is said to satisfy *weak local character for $\prec^*_\lambda$-increasing sequences* when: If $\alpha^* < \lambda^+$ and $\langle M_\alpha : \alpha \leq \alpha^* + 1\rangle$ is an $\prec^*_\lambda$ $-$increasing continuous sequence of models, then for some element $a \in M_{\alpha^*+1} - M_{\alpha^*}$ and ordinal $\alpha < \alpha^*$, $tp(a, M_{\alpha^*}, M_{\alpha^*+1})$ does not fork over M_α.

Definition 2.6.6. Let $\mathfrak{s}$ be a semi-good λ-frame except local character. $\mathfrak{s}$ is said to satisfy *weak local character* when for some relation $\prec^*_\lambda$ the following hold:

(1) $\prec^*_\lambda$ is a relation on K_λ.
(2) If $M_0 \prec^*_\lambda M_1$ then $M_0 \prec M_1$ (so $M_0 \neq M_1$).
(3) If $M_0 \prec^*_\lambda M_1 \preceq M_2 \in K_\lambda$ then $M_0 \prec^*_\lambda M_2$.
(4) $\mathfrak{s}$ satisfies weak local character for $\prec^*_\lambda$-increasing sequences.
(5) If $M_0 \in K_\lambda$, $M_0 \prec M_2 \in K_{\lambda^+}$, then there is a model $M_1 \in K_\lambda$ such that: $M_0 \prec^*_\lambda M_1 \preceq M_2$.

Remark 2.6.7. If $\mathfrak{s}$ is a semi-good λ-frame (i.e. satisfies local character) and $\prec^*_\lambda$ is a relation on K_λ such that $M \prec^*_\lambda N \Rightarrow M \preceq N$, then $\mathfrak{s}$ satisfies weak local character for $\prec^*_\lambda$-increasing sequences.

The following theorem asserts that a non-forking relation in $(K_\lambda, \preceq\restriction K_\lambda)$ can be lifted to $K_{\geq\lambda}$ with many properties preserved. Assuming local character, we can prove that density, monotonicity, transitivity, local character and continuity are preserved. Without assuming local character, we can prove that monotonicity, transitivity and continuity are preserved.

Theorem 2.6.8. *Let $\mathfrak{s}$ be a semi-good λ-frame, except local character.*

(1) Density: If $\mathfrak{s}$ satisfies weak local character and $M \prec N$, $M \in K_{\geq\lambda}$, then there is $a \in N - M$ such that $tp(a, M, N) \in S^{bs}_{\geq\lambda}(M)$.
(2) Monotonicity: Suppose $M_0 \preceq M_1 \preceq M_2$, $n < 3 \Rightarrow M_n \in K_{\geq\lambda}$, $||M_2|| > \lambda$. If $p \in S^{bs}_{\geq\lambda}(M_2)$ does not fork over M_0, then
  (a) p does not fork over M_1.
  (b) $p \restriction M_1$ does not fork over M_0.
(3) Transitivity: Suppose $M_0, M_1, M_2 \in K_{\geq\lambda}$ and $M_0 \preceq M_1 \preceq M_2$. If $p \in S^{bs}_{\geq\lambda}(M_2)$ does not fork over M_1, and $p \restriction M_1$ does not fork over M_0, then p does not fork over M_0.
(4) About local character: Let δ be a limit ordinal. Suppose $\mathfrak{s}$ satisfies local character or $\lambda^+ \leq cf(\delta)$. If $\langle M_\alpha : \alpha \leq \delta\rangle$ is an increasing continuous sequence of models in $K_{>\lambda}$, and $p \in S^{bs}_{>\lambda}(M_\delta)$ then there is $\alpha < \delta$ such that p does not fork over M_α.
(5) Continuity: Suppose $\langle M_\alpha : \alpha \leq \delta + 1\rangle$ is an increasing continuous sequence of models in $K_{\geq\lambda}$. Let $c \in M_{\delta+1} - M_\delta$. Denote $p_\alpha = tp(c, M_\alpha, M_{\delta+1})$. If for every $\alpha < \delta$, p_α does not fork over M_0, then p_δ does not fork over M_0.

Proof. (1) Density: Suppose $M \prec N$.
Case 1: $||M|| = \lambda$. Choose $a \in N - M$. $LST(K, \preceq) \leq \lambda$ and so there is

$N^* \prec N$ such that: $||N^*|| = \lambda$ and $M \bigcup \{a\} \subseteq N^*$. By Axiom e of AEC $M \preceq N^*$ But $a \in N^* - M$ and so $M \prec N^*$. By the existence axiom in $\mathfrak{s}$, there is $c \in N^* - M$ such that $tp(c, M, N^*)$ is basic. So $tp(c, M, N) \in S^{bs}(M)$.
Case 2: $||M|| > \lambda$. We choose M_n, N_n by induction on $n < \omega$ such that:

$$\begin{tikzcd}
c \in N_n \arrow[rr, "id"] & & N_n \arrow[r, "id"] & N_\omega \arrow[r, "id"] & N \\
M_n \arrow[u, "id"] \arrow[r, "id"] & M_{n,c} \arrow[r, "id"] & M_{n+1} \arrow[u, "id"] \arrow[r, "id"] & N_\omega \arrow[u, "id"] \arrow[r, "id"] & M \arrow[u, "id"]
\end{tikzcd}$$

(a) $\langle N_n : n \leq \omega \rangle$ is an $\prec$ $-$increasing continuous sequence of models in K_λ.
(b) $\langle M_n : n \leq \omega \rangle$ is an $\prec^*_\lambda$ $-$increasing continuous sequence of models in K_λ.
(c) $M_n \prec M$ (see the end of Definition 2.1.1).
(d) $N_n \prec N$.
(e) $N_0 \nsubseteq M$.
(f) For every $c \in N_n$, $M_{n,c} \subseteq M_{n+1}$ where we choose $M_{n,c} \in K_\lambda$ such that: If $tp(c, M_n, N_n) \in S^{bs}(M_n)$ but does fork over M_n then $M_{n,c}$ is a witness for this, namely, $M_n \prec M_{n,c} \prec M$ and $tp(c, M_{n,c}, N)$ forks over M_n. Otherwise $M_{n,c} = M_n$.

The construction is, of course, possible.

Now we define $M_\omega := \bigcup \{M_n : n < \omega\}$ and $N_\omega := \bigcup \{N_n : n < \omega\}$. By Definition 1.0.3.1.d (smoothness), $M_\omega \preceq N_\omega$. By local character for $\prec^*_\lambda$-increasing sequences, for some element $c \in N_\omega - M_\omega$ and there is $n < \omega$ such that $tp(c, M_\omega, N_\omega) \in S^{bs}(M_\omega)$ does not fork over M_n. By monotonicity without loss of generality $c \in N_n$. We will prove that $tp(c, M, N)$ does not fork over M_ω. Take M^* with $M_\omega \prec M^* \prec M$. By way of contradiction suppose $tp(c, M^*, N)$ forks over M_ω. By the monotonicity in $\mathfrak{s}$ (Axiom b), $tp(c, M^*, N)$ forks over M_n. So by the definition of $M_{n,c}$, $tp(c, M_{n,c}, N)$ forks over M_n. Hence by Axiom b (monotonicity) $tp(c, M_\omega, N)$ forks over M_n, a contradiction.

(2) Monotonicity: We use the same witness.

(3) Transitivity:

$$\begin{tikzcd}
 & N \arrow[r, "id"] & N^{**} \arrow[r, "id"] & M_2 & p \\
N_1 \arrow[rr, "id"] & & N^* \arrow[u, "id"] \arrow[r, "id"] & M_1 \arrow[u, "id"] & \\
 & N_0 \arrow[uu, "id"] \arrow[ur, "id"] \arrow[rr, "id"] & & M_0 \arrow[u, "id"] &
\end{tikzcd}$$

Suppose $M_0 \prec M_1 \prec M_2$, $p \in S^{bs}(M_2)$ does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0. We can find $N_0 \prec M_0$ such that N_0 witnesses that $p \upharpoonright M_1$ does not fork over M_0. We will prove that N_0 witnesses that p

does not fork over M_0. Let $N \in K_\lambda$ be such that $N_0 \prec N \prec M_2$. We have to prove that $p \restriction N$ does not fork over N_0. First we find a model N_1 that witnesses that p does not fork over M_1. As $LST(K, \preceq) \leq \lambda$, there is $N^* \in K_\lambda$ such that $N_0 \bigcup N_1 \subseteq N^* \preceq M_1$ and there is $N^{**} \in K_\lambda$ such that $N^* \bigcup N \subseteq N^{**} \preceq M_2$. As N_1 witnesses that p does not fork over M_1, $p \restriction N^{**}$ does not fork over N_1. By the Definition 2.1.1.3.b (monotonicity), $p \restriction N^{**}$ does not fork over N^*. N_0 witnesses that $p \restriction M_1$ does not fork over M_0, so $p \restriction N^*$ does not fork over N_0. By the transitivity proposition (Proposition 2.5.6), $p \restriction N^{**}$ does not fork over N_0. So by Definition 2.1.1.3.b (monotonicity), $p \restriction N$ does not fork over N_0.

(4) About local character: Let $\langle M_\alpha : \alpha < \delta \rangle$ be an increasing continuous sequence of models in $K_{>\lambda}$. Let $p \in S^{bs}_{>\lambda}(M_\delta)$ and N^* be a witness for this, i.e., p does not fork over $N^* \in K_\lambda$. Let $\langle \alpha(\varepsilon) : \varepsilon \leq cf(\delta) \rangle$ be an increasing continuous sequence of ordinals with $\alpha(cf(\delta)) = \delta$.

Case a: $\lambda^+ \leq cf(\delta)$. By cardinality considerations, there is $\varepsilon < cf(\delta)$ such that: $N^* \subseteq M_{\alpha(\varepsilon)}$. By Definition 1.0.3.1.e $N^* \preceq M_{\alpha(\varepsilon)}$. As N^* witnesses that the type p is basic, by Definition 2.6.1, N^* witnesses that p does not fork over $M_{\alpha(\varepsilon)}$.

Case b: $\mathfrak{s}$ satisfies local character and $cf(\delta) \leq \lambda$. Using $LST(K, \preceq) \leq \lambda$ and smoothness, we can choose $N_{\alpha(\varepsilon)}$ by induction on $\varepsilon \leq cf(\delta)$ such that:

$$\begin{tikzcd}
N^* \arrow[r, "id"] & N_\delta \arrow[r, "id"] & M_\delta & p \\
 & N_{\alpha(\varepsilon)} \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha(\varepsilon)} \arrow[u, "id"] &
\end{tikzcd}$$

(a) $N_{\alpha(\varepsilon)} \in K_\lambda$.
(b) $\langle N_{\alpha(\varepsilon)} : \varepsilon \leq cf(\delta) \rangle$ is an increasing continuous sequence.
(c) $M_{\alpha(\varepsilon)} \bigcap N^* \subseteq N_{\alpha(\varepsilon)} \preceq M_{\alpha(\varepsilon)}$.

By Definition 1.0.3.1.e, $N^* \preceq N_\delta \preceq M_\delta$. Since p does not fork over N^*, by monotonicity (Theorem 2.6.8.2) p does not fork over N_δ. By local character, for some $\varepsilon < cf(\delta)$, $p \restriction N_\delta$ does not fork over $N_{\alpha(\varepsilon)}$. By transitivity (Theorem 2.6.8.3), p does not fork over $N_{\alpha(\varepsilon)}$. By monotonicity (Theorem 2.6.8.2), p does not fork over $M_{\alpha(\varepsilon)}$.

(5) Continuity: For every $\alpha \in \delta$ denote $p_\alpha := p \restriction M_\alpha$. p_7 does not fork over M_0. So for some $N_0 \in K_\lambda$, $N_0 \preceq M_0$ and p_7 does not fork over N_0. By monotonicity (Theorem 2.6.8.2) and transitivity (Theorem 2.6.8.2) for every $\alpha < \delta$, p_α does not fork over N_0. We will prove that p does not fork over N_0. Take $N^* \in K_\lambda$ with $N_0 \preceq N^* \preceq M_\delta$. We have to prove that $p \restriction N^*$ does not fork over N_0. Let $\langle \alpha(\varepsilon) : \varepsilon \leq cf(\delta) \rangle$ be an increasing continuous sequence of ordinals with $\alpha(cf(\delta)) = \delta$.

Case a: $\lambda^+ \leq cf(\delta)$. By cardinality considerations, there is $\varepsilon < cf(\delta)$ such that $N^* \subseteq M_{\alpha(\varepsilon)}$. But $M_{\alpha(\varepsilon)} \preceq M_\delta$ and $N^* \preceq M_\delta$, so by Axiom

1.0.3.1.e $N^* \preceq M_{\alpha(\varepsilon)}$. Since $p_{\alpha(\varepsilon)}$ does not fork over N_0, by monotonicity (Theorem 2.6.8.2) $p \upharpoonright N^*$ does not fork over N_0.

Case b: $cf(\delta) \leq \lambda^+$. We choose $N_{\alpha(\varepsilon)}$ by induction on $\varepsilon \in (0, cf(\delta)]$ such that:

(a) The sequence $\langle N_{\alpha(\varepsilon)} : \varepsilon \leq cf(\delta)\rangle$ is increasing continuous.
(b) $\varepsilon \leq cf(\delta) \Rightarrow N^* \bigcap M_{\alpha(\varepsilon)} \subseteq N_{\alpha(\varepsilon)} \preceq M_{\alpha(\varepsilon)}$.
(c) $N_{\alpha(\varepsilon)} \in K_\lambda$.

For every $\varepsilon < \mathrm{cf}(\delta)$, $p_{\alpha(\varepsilon)}$ does not fork over N_0, so $p \upharpoonright N_{\alpha(\varepsilon)}$ does not fork over N_0. So by Definition 2.1.1.3.g (continuity) (in $\mathfrak{s}$), $p \upharpoonright N_\delta$ does not fork over N_0. $N^* \subseteq N_\delta$, hence by Axiom 1.0.3.1.e $N^* \preceq N_\delta$. Therefore by Definition 2.1.1.3.b (monotonicity), $p \upharpoonright N^*$ does not fork over N_0. $\dashv$

3. The decomposition and amalgamation method

In this section, there is no reason to assume any version of stability or local character.

Hypothesis 3.0.9. $\mathfrak{s}$ is a semi-good λ-frame, except basic almost stability and local character.

Discussion. In Section 2 (Definition 2.6.2) we defined an extension of the non-forking notion to cardinals bigger than λ. But we did not prove all the good frame axioms (we proved only Theorem 2.6.8). The purpose from here until the end of the paper is to construct a good λ^+-frame, which is derived from the semi-good λ-frame. In a sense, the main problem is that the amalgamation property in $(K_\lambda, \preceq \upharpoonright K_\lambda)$ may not imply the amalgamation property in $(K_{\lambda^+}, \preceq \upharpoonright K_{\lambda^+})$. The solution is to define a special notion of a submodel, $\preceq^{NF}_{\lambda^+}$ (see Definition 6.1.4).

Suppose for $n < 3$ $M_n \in K_{\lambda^+}$, $M_0 \preceq M_n$ and we want to amalgamate M_1, M_2 over M_0. We take representations of M_0, M_1, M_2 as unions of models of size λ. We want to amalgamate M_1, M_2 by amalgamating their representations. For this goal, we will find in Section 5, a relation of 'a non-forking amalgamation'. Sections 3,4 are preparations for Section 5. If the reader wants to know the plan of the other sections now, he may see the discussion at the beginning of Section 10.

The decomposition and amalgamation method. Suppose for $n = 1, 2$, $M_0 \preceq M_n$ and we want to prove that there is an amalgamation of M_1, M_2 over M_0 which satisfies specific properties (for example disjointness or uniqueness, see below). We will define various subclasses of K^3 and study them in general under the name $K^{3,*}$. We will decompose a model into a chain such that each extension is in $K^{3,*}$ and draw conclusions from such a decomposition.

Theorem 3.2.3 says, under some assumptions, that we can decompose an extension of M_1 over M_0 by triples in $K^{3,*}$. By Proposition 3.1.8.2 we can amalgamate M_2 with the decomposition we have obtained.

Applications of the decomposition and amalgamation method.

(1) By Proposition 3.1.8(2) there is no $\preceq$-maximal model in K_{λ^+}.
(2) By Proposition 3.3.4 the reduced triples are dense with respect to $\preceq_{bs}$ (see Definition 3.1.1.2). It enables to prove Theorem 3.3.5 (the disjoint amalgamation existence), by the decomposition and disjoint method.
(3) By Hypothesis 5.1.1, the uniqueness triples are dense with respect to $\preceq_{bs}$. The density enables to prove Theorem 5.3.7 (the existence theorem for NF).
(4) Using again Hypothesis 5.1.1, we prove Proposition 5.4.6. But for this, we have to prove Proposition 3.1.10, a generalization of 3.1.8, which says that we can amalgamate two sequences of models, not just a model and a sequence.

3.1. $(K^{3,bs},\preceq_{bs})$ **and amalgamations.** We define $K^{3,bs}$ as the class of those triples which represent basic types. The reader may feel that this definition is not new, because we have defined basic types. But while we studied triples modulo an equivalence relation, now we want to study the triples themselves. We define a partial ordering, $\preceq_{bs}$ on $K^{3,bs}$.

Definition 3.1.1.

(1) $K^{3,bs} =: \{(M,N,a) : M,N \in K_\lambda,\ a \in N-M$ and $tp(a,M,N) \in S^{bs}(M)\}$.
(2) $\preceq_{bs}$ is the relation on $K^{3,bs}$ defined by: $(M,N,a) \preceq_{bs} (M^*,N^*,a^*)$ iff $M \preceq M^*,\ N \preceq N^*,\ a^* = a$ and $tp(a,M^*,N^*)$ does not fork over M. In particular, $tp(a,M^*,N^*)$ extends $tp(a,M,N)$.

The pair $(K^{3,bs},\preceq_{bs})$ satisfies most of the axioms of AEC. The comparison between the properties of $(K^{3,bs},\preceq_{bs})$ and the axioms of AEC helps to remember the properties of $(K^{3,bs},\preceq_{bs})$. For this comparison we have to define a new vocabulary.

Definition 3.1.2. Let $(K,\preceq)$ be an AEC with vocabulary τ. *The vocabulary of triples* means $\tau\bigcup\{P,c\}$, where P is an unary predicate not in τ, c is a 0-ary function not in τ, and we interpret (M,N,a) by: N is a τ-model, M is the interpretation of P and a the interpretation of c.

$(K^{3,bs},\preceq_{bs})$ should not be an AEC. If $(K^{3,bs},\preceq_{bs})$ is an AEC, then for each $(M_0,N_0,a),(M_1,N_1,a) \in K^{3,bs}$, $(M_0,N_0,a) \preceq_{bs} (M_1,N_1,a)$ implies $(M_0,N_0,a) \subset (M_1,N_1,a)$ and it implies $M_1\bigcap N_0 = M_0$. But why does it imply that $M_1\bigcap N_0 = M_0$? If (M_0,N_0,a) is reduced (see Definition 3.3.2), then it implies that $M_1\bigcap N_0 = M_0$.

We can replace the relation $\preceq_{bs}$ by the following relation:

Definition 3.1.3. $\preceq_{bs}^{disjoint}$ is the binary relation on K^3_{bs} defined by: $(M_0,N_0,a) \preceq_{bs}^{disjoint} (M_1,N_1,a)$ iff $(M_0,N_0,a) \preceq_{bs} (M_1,N_1,a)$ and $N_0\bigcap M_1 = M_0$.

The unique use of the Definition 3.1.3 is in the following remark. This remark is not used later.

Remark 3.1.4. $(K^{3,bs}, \preceq_{bs}^{disjoint})$ is an AEC in λ (for the vocabulary of triples). See also Remark 3.3.3.

Proof. Easy, using Proposition 3.1.6. $\dashv$

As we said above, the relation $\preceq_{bs}$ should not be included in the submodel relation. So in order to compare the properties of $(K^{3,bs}, \preceq_{bs})$ with the axioms of AEC, we have to define the notion of increasing continuous sequence in this context.

Definition 3.1.5. The sequence $\langle (M_\alpha, N_\alpha, a) : \alpha < \theta \rangle$ is said to be $\preceq_{bs}$-increasing continuous if $\alpha < \theta \Rightarrow (M_\alpha, N_\alpha, a) \preceq_{bs} (M_{\alpha+1}, N_{\alpha+1}, a)$ and the sequences $\langle M_\alpha : \alpha < \theta \rangle$, $\langle N_\alpha : \alpha < \theta \rangle$ are continuous (and increasing).

Proposition 3.1.6. $(K^{3,bs}, \preceq_{bs})$ *satisfies the axioms of AEC in* λ *except one: the relation* $\preceq_{bs}$ *should not be included in the submodel relation.*

Proof. First we note that $K^{3,bs}$ is not the empty set [there is $M \in K_\lambda$, and as K_λ has no $\preceq$-maximal model, there is $N \in K_\lambda$ with $M \prec N$. Now by Definition 2.1.1.3.f, there is $a \in N - M$ such that $tp(M, N, a) \in S^{bs}(M)$. So $(M, N, a) \in K^{3,bs}$]. Now we check the axioms of Definition 1.0.3.1.

a. Trivial.

b. $\preceq_{bs}$ is transitive by Proposition 2.5.6. It should not be included in the submodel relation.

c. Suppose $\delta < \lambda^+$ and $\langle (M_\alpha, N_\alpha, a) : \alpha < \delta \rangle$ is increasing and continuous. Denote $M = \bigcup \{M_\alpha : \alpha < \delta\}$, $N = \bigcup \{N_\alpha : \alpha < \delta\}$. By Axiom c of AEC, $M, N \in K_\lambda$ and for each $\alpha < \delta$ $M_\alpha \preceq M$, $N_\alpha \preceq N$. By the definition of $\preceq_{bs}$ for every $\alpha < \delta$, $tp(a, M_\alpha, N_\alpha)$ does not fork over M_0. So by Definition 2.1.1.3.g (continuity), $tp(a, M, N)$ is basic and does not fork over M_0. By smoothness, $M \preceq N$. By Axiom c of AEC $M_0 \preceq M$ and $N_0 \preceq N$. So $(M_0, N_0, a) \preceq_{bs} (M, N, a) \in K^{3,bs}$.

d. Why is smoothness satisfied? Suppose $\langle (M_\alpha, N_\alpha, a) : \alpha \leq \delta + 1 \rangle$ is continuous and for every α, β with $\alpha < \beta \leq \delta + 1$, we have $\alpha \neq \delta \Rightarrow (M_\alpha, N_\alpha, a) \preceq_{bs} (M_\beta, N_\beta, a)$. We should prove that $(M_\delta, N_\delta, a) \preceq_{bs} (M_{\delta+1}, N_{\delta+1}, a)$. $\delta \neq \alpha < \beta \leq \delta + 1 \Rightarrow M_\alpha \preceq M_\beta$. But by the continuity of the sequence $\langle (M_\alpha, N_\alpha, a) : \alpha \leq \delta + 1 \rangle$, we have $M_\delta = \bigcup \{M_\alpha : \alpha < \delta\}$. So by smoothness of $(K, \preceq)$, $M_\delta \preceq M_{\delta+1}$. In a similar way $N_\delta \preceq N_{\delta+1}$. $(M_0, N_0, a) \preceq_{bs} (M_{\delta+1}, N_{\delta+1}, a)$, so by the definition, $tp(a, M_{\delta+1}, N_{\delta+1})$ does not fork over M_0. Therefore by Definition 2.1.1.3.b (monotonicity), $tp(a, M_{\delta+1}, N_{\delta+1})$ does not fork over M_δ.

e. Suppose $(M_0, N_0, a) \subseteq (M_1, N_1, a) \preceq (M_2, N_2, a)$, $(M_0, N_0, a) \preceq_{bs} (M_2, N_2, a)$. By the definition of $\preceq_{bs}$, we have $M_0 \subseteq M_1 \preceq M_2$ and $M_0 \preceq M_2$. Hence by Axiom 1.0.3.1.e we have $M_0 \preceq M_1$. In a similar way $N_0 \preceq N_1$. By the definition of $\preceq_{bs}$, $tp(a, M_2, N_2)$ does not fork over M_0. By 2.1.1.3.b (monotonicity), $tp(a, M_1, N_1)$ does not fork over M_0. So $(M_0, N_0, a) \preceq_{bs} (M_1, N_1, a)$. $\dashv$

Proposition 3.1.7. *$K^{3,bs}$ has no $\preceq_{bs}$-maximal model.*

Proof. Let $(M_0, N_0, a) \in K^{3,bs}$. In K_λ there is no $\preceq$-maximal element, and so there is $M_1^* \in K_\lambda$ with $M_0 \prec M_1^*$. By Proposition 2.5.7 there is $N_1 \in K_\lambda$ with $N_0 \preceq N_1$ and there is an embedding $f : M_1^* \to N_1$ such that $tp(a, M_1, N_1)$ does not fork over M_0 where $M_1 := f[M_1^*]$. Hence $(M_0, N_0, a) \preceq_{bs} (M_1, N_1, a)$. $\dashv$

Roughly, the following proposition says that we can amalgamate the union of increasing continuous sequence of models $\langle M_\alpha : \alpha < \theta\rangle$ and a model N extending M_0 over M_0 such that many types do not fork.

Proposition 3.1.8.

$$\begin{tikzcd}
N_0 \arrow[r, "id"] & N_1 \arrow[r, "id"] & N_2 \arrow[r, "id"] & N_\alpha \arrow[r, "id"] & N_{\alpha+1} \arrow[r, "id"] & N_\theta \\
M_0 \arrow[r, "id"] \arrow[u, "id"] & M_1 \arrow[r, "id"] \arrow[u, "id"] & M_2 \arrow[r, "id"] \arrow[u, "id"] & M_\alpha \arrow[r, "id"] \arrow[u, "id"] & M_{\alpha+1} \arrow[r, "id"] \arrow[u, "id"] & M_\theta \arrow[u, "id"]
\end{tikzcd}$$

Let $\langle M_\alpha : \alpha \leq \theta\rangle$ be an increasing continuous sequence of models in K_λ. Let $N \in K_\lambda$ with $M_0 \prec N$, and for $\alpha < \theta$, let $a_\alpha \in M_{\alpha+1} - M_\alpha$, $(M_\alpha, M_{\alpha+1}, a_\alpha) \in K^{3,bs}$ and $b \in N - M_0$, $(M_0, N, b) \in K^{3,bs}$. Then there are $f, \langle N_\alpha : \alpha \leq \theta\rangle$ such that:

(1) f is an isomorphism of N to N_0 over M_0.
(2) $\langle N_\alpha : \alpha \leq \theta\rangle$ is an increasing continuous sequence.
(3) $M_\alpha \preceq N_\alpha$.
(4) $tp(a_\alpha, N_\alpha, N_{\alpha+1})$ does not fork over M_α.
(5) $tp(f(b), M_\alpha, N_\alpha)$ does not fork over M_0.

Note that N_θ is an amalgam of M_θ and N over M_0.

Proof. First note that the argument uses the symmetry axiom. Now we explain the idea of the proof. If we 'fix' the models in the sequence $\langle M_\alpha : \alpha \leq \theta\rangle$, then we will 'change' N θ times. So in limit steps, we will encounter a problem. The solution is to fix N, and 'change' the sequence $\langle M_\alpha : \alpha \leq \theta\rangle$. At the end of the proof, we 'return the sequence to its place'.

The proof itself: We choose (N^*_α, f_α) by induction on α such that $(*)_\alpha$ holds where $(*)_\alpha$ is:

(i) $\alpha \leq \theta \Rightarrow N^*_\alpha \in K_\lambda$.
(ii) $(N^*_0, f_0) = (N, id_{M_0})$.
(iii) The sequence $\langle N^*_\alpha : \alpha \leq \theta\rangle$ is increasing and continuous.
(iv) For every $\alpha \leq \theta$, the function f_α is an embedding of M_α to N^*_α.
(v) The sequence $\langle f_\alpha : \alpha \leq \theta\rangle$ is increasing and continuous.
(vi) For every $\alpha < \theta$ $tp(f_{\alpha+1}(a_\alpha), N^*_\alpha, N^*_{\alpha+1})$ does not fork over $f_\alpha[M_\alpha]$.
(vii) For every $\alpha \leq \theta$ $tp(b, f_\alpha[M_\alpha], N^*_\alpha)$ does not fork over M_0.

Note that in limit steps we do not choose any element and by smoothness, $f_\alpha[M_\alpha] \leq N^*_\alpha$.

Now $f_\theta : M_\theta \to N^*_\theta$ is an embedding. Extend f_θ^{-1} to a function with domain N^*_θ and define $f := g \restriction N$. Define $N_\alpha := g[N^*_\alpha]$.

$\dashv$

Proposition 3.1.9.

1. *$K_{\lambda^+} \neq \emptyset$, and it has no $\preceq$-maximal model.*
2. *There is a model in K of cardinality λ^{+2}.*

Proof. (1) $K_{\lambda^+} \neq \emptyset$, as we can choose an increasing continuous sequence of models in K_λ, $\langle M_\alpha : \alpha < \lambda^+ \rangle$, and so its union is a model in K_{λ^+}. [As there is no $\preceq$-maximal model in K_λ and in limit step, use Axiom 1.0.3.1.c.]

Why is there no maximal model in K_{λ^+}? Let $M \in K_{\lambda^+}$. Let $\langle M_\alpha : \alpha < \lambda^+ \rangle$ be a representation of M. By the Definition 2.1.1.3.f (existence), for every $\alpha \in \lambda^+$ there is an element $a_\alpha \in M_{\alpha+1} - M_\alpha$ (we do not use a_α, but as we have written it in 1, for shortness, we have to write it here). As in K_λ there is no maximal model, there is a model N such that $M_0 \prec N \in K_\lambda$ and, without loss of generality, $N \bigcap M = M_0$. By Definition 2.1.1.2.c (the density of basic types), there is $b \in N - M_0$ such that $tp(b, M_0, N)$ is basic. Now by Proposition 3.1.8.1, there is an increasing continuous sequence $\langle N_\alpha : \alpha < \lambda^+ \rangle$ and f such that $f : N \to N_0$ is an isomorphism over M_0 and for $\alpha \in \lambda^+$ we have $M_\alpha \preceq N_\alpha$ and $tp(f(b), M_\alpha, N_\alpha)$ does not fork over M_0. So by Definition 2.1.1, $f(b)$ does not belong to M_α for $\alpha \in \lambda^+$. So $f(b)$ does not belong to M. But it belongs to N_{λ^+}, so $M \neq N_{\lambda^+}$, and for this we defined b. But it is easy to see that $M \subseteq N_{\lambda^+}$ and $N_{\lambda^+} \in K_{\lambda^+}$. By smoothness (i.e. Definition 1.0.3.1.d) $M \preceq N_{\lambda^+}$. So M is not a maximal model.

(2) We construct a strictly increasing continuous sequence of models in K_{λ^+}, $\langle M_\alpha : \alpha < \lambda^{+2} \rangle$. So its union is a model in $K_{\lambda^{+2}}$. As by 2 there is no maximal model in K_{λ^+}, there is no problem to choose this sequence. $\dashv$

The following proposition will be used in the proof of Proposition 5.4.6.

Proposition 3.1.10 (a rectangle which amalgamates two sequences)**.** *For $x = a, b$ let $\langle M_{x,\alpha} : \alpha < \theta^x \rangle$ be an increasing continuous sequence of models in K_λ such that $M_{a,0} = M_{b,0}$ and let $\langle d_{x,\alpha} : \alpha < \theta^x \rangle$ be a sequence such that $d_{x,\alpha} \in M_{x,\alpha+1} - M_{x,\alpha}$, and the type $tp(d_{x,\alpha}, M_{x,\alpha}, M_{x,\alpha+1})$ is basic. Denote $\alpha^* = \theta^a$, $\beta^* = \theta^b$. Then there are a "rectangle of models" $\{M_{\alpha,\beta} : \alpha < \alpha^*,\ \beta < \beta^*\}$ and a sequence $\langle f_\beta : \beta < \beta^* \rangle$ such that:*

1. $(\alpha < \alpha^* \wedge \beta < \beta^*) \Rightarrow M_{\alpha,\beta} \in K_\lambda$.
2. *$f_\beta : M_{b,\beta} \to M_{0,\beta}$ is an isomorphism.*
3. $M_{\alpha,0} = M_{a,\alpha}$.
4. *f_0 is the identity on $M_{a,0} = M_{b,0}$.*
5. *$\langle f_\beta : \beta < \beta^* \rangle$ is increasing and continuous.*
6. *For every α, β which satisfies $\alpha + 1 < \alpha^*$ and $\beta < \beta^*$, the type $tp(d_{a,\alpha}, M_{\alpha,\beta}, M_{\alpha+1,\beta})$ does not fork over $M_{\alpha,0}$.*
7. *For every α, β which satisfies $\alpha < \alpha^*$ and $\beta + 1 < \beta^*$, the type $tp(d_{b,\beta}, M_{\alpha,\beta}, M_{\alpha,\beta+1})$ does not fork over $M_{0,\beta}$.*

(8) *If* $\bigcup\{Im(f_\beta) : \beta < \beta^*\}\bigcap\bigcup\{M_{a,\alpha} : \alpha < \alpha^*\} = \bigcup\{M_{b,\beta} : \beta < \beta^*\}\bigcap\bigcup\{M_{a,\alpha} : \alpha < \alpha^*\} = M_{a,0}$, *then* $(\forall\beta \in \beta^*) f_\beta = id \restriction M_{b,\beta}$.

(9) *For all* $\alpha(1) < \alpha^*$ *the sequence* $\langle M_{\alpha(1),\beta} : \beta < \beta^*\rangle$ *is increasing and continuous.*

(10) *For all* $\beta(1) < \beta^*$ *the sequence* $\langle M_{\alpha,\beta(1)} : \alpha < \alpha^*\rangle$ *is increasing and continuous.*

$$\begin{tikzcd}
d_{a,\alpha} \in M_{\alpha+1,0} = M_{a,\alpha+1} \arrow[r, "id"] & M_{\alpha+1,\beta} \arrow[r, "id"] & M_{\alpha+1,\beta+1} \\
M_{\alpha,0} = M_{a,\alpha} \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha,\beta} \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha,\beta+1} \arrow[u, "id"] \\
M_{0,0} = M_{a,0} = M_{b,0} \arrow[u, "id"] \arrow[r, "id"] & M_{0,\beta} = f_\beta[M_{b,\beta}] \arrow[u, "id"] \arrow[r, "id"] & M_{0,\beta+1} = f_{\beta+1}[M_{b,\beta+1}] \arrow[u, "id"]
\end{tikzcd}$$

Proof. We define by induction on $\beta < \beta^*$ $f_\beta, \{M_{\alpha,\beta} : \alpha < \alpha^*\}$ such that the conditions 1-6 and 8,9 are satisfied. For $\beta = 0$ see 3,4. For β a limit ordinal, we define $f_\beta = \bigcup\{f_\gamma : \gamma < \beta\}$, $M_{\alpha,\beta} = \bigcup\{M_{\alpha,\gamma} : \gamma < \beta\}$. Why does 6 satisfy, i.e., why for every α, does $tp(d_{a,\alpha}, M_{\alpha,\beta}, M_{\alpha+1,\beta})$ not fork over $M_{\alpha,0}$? By the induction hypothesis, 6 is satisfied for every $\gamma < \beta$, i.e., $tp(d_{a,\alpha}, M_{\alpha,\gamma}, M_{\alpha+1,\gamma}) = tp(d_{a,\alpha}, M_{\alpha,\gamma}, M_{\alpha+1,\gamma})$ does not fork over $M_{0,\gamma}$. By Definition 2.1.1.3.b (monotonicity) and Definition 2.1.1.3.g (continuity), $tp(d_{a,\alpha}, M_{\alpha,\beta}, M_{\alpha+1,\beta})$ does not fork over $M_{\alpha,0}$. So condition 6 is satisfied. For $\beta = \gamma + 1$ use Proposition 3.1.8.1. So we can carry out the induction. Now, without loss of generality, condition 7 is satisfied, too. $\dashv$

3.2. **Decomposition.** When we speak about $tp(a, M, N)$, the N is rather peripheral; any larger model will do. Now we consider classes $K^{3,*}$ of triples (M, N, a) where the role of N is very important. For example, N is the algebraic closure of $M\bigcup\{a\}$, where $(K, \preceq)$ is the class of fields with the partial order of being sub-field.

Definition 3.2.1. Let $K^{3,*} \subseteq K^{3,bs}$, such that $K^{3,*}$ is closed under isomorphisms (i.e., if $(M, N, a) \in K^{3,*}$ and $f : N \to N^*$ is an isomorphism, then $(f[M], f[N], f(a)) \in K^{3,*}$).

(1) $K^{3,*}$ is *dense with respect to* $\preceq_{bs}$ if for every $(M, N, a) \in K^{3,bs}$, there is $(M^*, N^*, a^*) \in K^{3,*}$ such that $(M, N, a) \preceq_{bs} (M^*, N^*, a^*)$.

(2) $K^{3,*}$ *satisfies the existence property* if for every $(M, N, a) \in K^{3,bs}$, there are N^*, a^* such that $tp(a^*, M, N^*) = tp(a, M, N)$ and $(M, N^*, a^*) \in K^{3,*}$. In other words, if $p \in S^{bs}(M)$ then $p\bigcap K^{3,*} \neq \emptyset$.

Definition 3.2.2. Let $K^{3,*} \subseteq K^{3,bs}$, $K^{3,*}$ be closed under isomorphisms. Let $M^* \in K_\lambda$. We say that M^* is *decomposable by* $K^{3,*}$ over M, if there is a sequence $\langle d_\varepsilon, N_\varepsilon : \varepsilon < \alpha^*\rangle^\frown\langle N_{\alpha^*}\rangle$ with $N_{\alpha^*} = \bigcup\{N_\varepsilon : \varepsilon < \alpha\}$ such that:

(1) $\alpha^* < \lambda^+$ and for each $\varepsilon < \alpha^*$ $N_\varepsilon \in K_\lambda$.

(2) $\langle N_\varepsilon : \varepsilon \preceq \alpha^* \rangle$ is increasing and continuous.
(3) $N_0 = M$.
(4) $(N_\varepsilon, N_{\varepsilon+1}, d_\varepsilon) \in K^{3,*}$.

In such a case, we say that the sequence $\langle d_\varepsilon, N_\varepsilon : \varepsilon < \alpha^* \rangle^\frown \langle N_{\alpha^*} \rangle$ is a decomposition of M^* over M by $K^{3,*}$.

Theorem 3.2.3 (the extensions decomposition theorem)**.** *Let $K^{3,*} \subseteq K^{3,bs}$ be closed under isomorphisms.*

(1) Suppose $\mathfrak{s}$ satisfies the conjugation property. If $K^{3,*}$ is dense with respect to $\preceq_{bs}$, then it satisfies the existence property.
(2) Suppose $K^{3,*}$ satisfies the existence property. If $N \in K_\lambda$ and $p = tp(a, M, N) \in S^{bs}(M)$, then there are N^*, N^+ such that $(M, N^*, a) \in K^{3,*} \bigcap p$, $N \preceq N^+$, $N^* \preceq N^+$.
(3) Suppose $K^{3,*}$ satisfies the existence property, $M, N \in K_\lambda$ and $M \prec N$. Then there is $M^* \in K_\lambda$ such that $M^* \succeq N$ and M^* is decomposable over M by $K^{3,*}$. Moreover, letting $a \in N - M$, $tp(a, M, N)$ is basic, we can choose $d_0 = a$, where d_0 is the element which appears in Definition 3.2.2.

Proof. (1) Suppose $p = tp(M, N, a) \in S^{bs}(M)$. As $K^{3,*}$ is dense with respect to $\preceq_{bs}$, there are M^*, N^*, b with $(M, N, a) \preceq_{bs} (M^*, N^*, b)$. As $\mathfrak{s}$ satisfies the conjugation property, $p^* =: tp(M^*, N^*, b)$ and p are conjugate. $K^{3,*}$ is closed under isomorphisms and so $p \bigcap K^{3,*} \neq \emptyset$.
(2) $K^{3,*}$ satisfies the existence property and so there are b, N^* such that: $tp(b, M, N^*) = p$, $(M, N^*, b) \in K^{3,*}$. By the definition of a type (i.e., the definition of equivalence between triples in a type), there are a model N^+, $N \preceq N^+$ and an embedding $f : N^* \to N^+$ over M such that $f(b) = a$. Denote $N^{**} = f[N^*]$. Now as $K^{3,*}$ respects isomorphisms, $(M, N^{**}, a) \in K^{3,*}$. $M \preceq N^{**} \preceq N^+$.
(3) Assume toward a contradiction that $M \prec N$ and there is no M^* as required. We try to construct $M_\alpha, a_\alpha, N_\alpha$ by induction on $\alpha \in \lambda^+$ such that (see the diagram below):
(a) $M_0 = M$, $N_0 = N$.
(b) $(M_\alpha, M_{\alpha+1}, d_\alpha) \in K^{3,*}$.
(c) $M_\alpha \preceq N_\alpha$.
(d) For every $\alpha \in \lambda^+$, $d_\alpha \in M_{\alpha+1} \bigcap N_\alpha - M_\alpha$.
(e) The sequence $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is increasing and continuous.
(f) The sequence $\langle N_\alpha : \alpha < \lambda^+ \rangle$ is increasing and continuous.

$$\begin{tikzcd}
N_0 \arrow[r, "id"] & N_1 \arrow[r, "id"] & N_\alpha & \\
M_0 \arrow[u, "id"] \arrow[r, "id"] & M_1 \arrow[u, "id"] \arrow[r, "id"] & M_\alpha \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha+1} \ni a_\alpha
\end{tikzcd}$$

We cannot succeed because otherwise substituting the sequences $\langle M_\alpha : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha : \alpha \in \lambda^+ \rangle$, $\langle id_{M_\alpha} : \alpha \in \lambda^+ \rangle$ in Proposition 1.0.30, we get a contradiction. So where will we get stuck? For $\alpha = 0$ there is no problem. For α limit take unions. 3 is satisfied by (smoothness) (Definition 1.0.3.1.d). What will we do for $\alpha + 1$, (assuming we have defined $(M_\alpha, N_\alpha, d_\alpha)$? If $N_\alpha = M_\alpha$, then N_α is decomposable over M by $K^{3,*}$ and the proof is complete. Otherwise by the existence of the basic types (2.1.1), there is $d_\alpha \in N_\alpha - M_\alpha$ such that $(M_\alpha, N_\alpha, d_\alpha) \in K^{3,bs}$ (and for the "moreover" take $d_0 = a$ if $\alpha = 0$). By assumption, $K^{3,*}$ satisfies the existence property, so there are $d^*_\alpha, M^*_{\alpha+1}$ such that: $(M_\alpha, M^*_{\alpha+1}, d^*_\alpha) \in K^{3,*}$, $tp(d^*_\alpha, M_\alpha, M^*_{\alpha+1}) = tp(d_\alpha, M_\alpha, N_\alpha)$. By the definition of a type, there are $N_{\alpha+1}$, $N_\alpha \preceq N_{\alpha+1}$ and an embedding $f : M^*_{\alpha+1} \to N_{\alpha+1}$ over M_α such that $f(d^*_\alpha) = d_\alpha$. Denote $M_{\alpha+1} = Im(f)$. We have $N_\alpha \preceq N_{\alpha+1}$, $M_{\alpha+1} \preceq N_{\alpha+1}$ and $(M_\alpha, M_{\alpha+1}, d_\alpha) \in K^{3,*}$. So 2,3,4 are guaranteed. $\dashv$

The following proposition will be used twice: once in the proof of Theorem 5.4.7 and once in the proof of Proposition 5.5.3.

Proposition 3.2.4 (existence of decomposition over two models)**.** *If M_0, $M_1, N \in K_\lambda$ and $n < 2 \Rightarrow M_n \preceq N$, then there is $M^* \in K_\lambda$ such that: $N \preceq M^*$ and M^* is decomposable over M_0 and over M_1.*

Proof. Choose an increasing continuous sequence $\langle M_n : 2 \preceq n \leq \omega\rangle$ such that:

(1) $N \preceq M_2$.
(2) For every $n \in \omega$, M_{n+2} is decomposable over M_n.

The construction is possible by Theorem 3.2.3. Now by the following proposition, M_ω is decomposable over M_0 and M_1. $\dashv$

Proposition 3.2.5 (the decomposable extensions transitivity)**.** *Let $\langle M_\varepsilon : \varepsilon \leq \alpha^*\rangle$ be an increasing continuous sequence of models, such that for every $\varepsilon < \alpha^*$, $M_{\varepsilon+1}$ is decomposable over M_ε. Then M_{α^*} is decomposable over M_0.*

Proof. Easy. $\dashv$

3.3. **A disjoint amalgamation.** The next goal is to prove the existence of a disjoint amalgamation. For this we are going to prove the density of the reduced triples. (M, N, a) is reduced means that a dominates N in a weak way. We will use the decomposition method where the class of reduced triples stands for $K^{3,*}$.

Definition 3.3.1. The amalgamation f_1, f_2, M_3 of M_1, M_2 over M_0 is said to be *disjoint* when $f_1[M_1] \bigcap f_2[M_2] = M_0$.

Definition 3.3.2. The triple $(M, N, a) \in K^{3,bs}_\lambda$ is *reduced* if $(M, N, a) \preceq_{bs} (M^*, N^*, a) \Rightarrow M^* \bigcap N = M$. We define

$$K^{3,r} := \{(M, N, a) \in K^{3,bs} : (M, N, a) \text{ is reduced}\}.$$

Remark 3.3.3. $(K^{3,r}, \subseteq)$ is an AEC in λ (see Proposition 3.1.6).

Proposition 3.3.4. *The reduced triples are dense with respect to $\preceq_{bs}$: For every $(M,N,a) \in K^{3,bs}_\lambda$, there is a reduced triple (M^*,N^*,a) which is $\preceq_{bs}$-bigger than it.*

Proof. Suppose towards contradiction that over (M,N,a) there is no reduced triple. We will construct models M_α, N_α by induction on $\alpha < \lambda^+$ such that:

(i) $(M_0,N_0,a) = (M,N,a)$.
(ii) For every $\alpha \in \lambda^+$, $(M_\alpha,N_\alpha,a) \preceq_{bs} (M_{\alpha+1},N_{\alpha+1},a)$.
(iii) For every $\alpha \in \lambda^+$, $M_{\alpha+1} \bigcap N_\alpha \neq M_\alpha$.
(iv) The sequence $\langle (M_\alpha,N_\alpha,a) : \alpha < \lambda^+ \rangle$ is continuous, (see Definition 3.1.1).

Why can we carry out the construction? For $\alpha = 0$ see clause (i) of the construction. For limit α see clause (iv). Suppose we have defined $\langle M_\beta, N_\beta, a) : \beta \leq \alpha \rangle$. By Proposition 3.1.6 $(K^{3,bs}, \preceq_{bs})$ is closed under increasing union. So by clauses (i),(ii),(iv) $(M,N,a) \preceq_{bs} (M_\alpha,N_\alpha,a)$. So by the assumption (M_α,N_α,a) is not a reduced triple, i.e., there are $M_{\alpha+1}, N_{\alpha+1}$ which satisfies clauses (ii),(iii). Hence we can carry out this construction.

Now we have:

(1) The sequences $\langle M_\alpha : \alpha < \lambda^+ \rangle$, $\langle N_\alpha : \alpha < \lambda^+ \rangle$ are increasing (by clause (ii) and the definition of $\preceq_{bs}$).
(2) These sequences are continuous (by clause (iv)).
(3) For $\alpha \in \lambda^+$, $M_\alpha \subseteq N_\alpha$ (by the definition of $K^{3,bs}$).
(4) For every $\alpha \in \lambda^+$, $M_{\alpha+1} \bigcap N_\alpha \neq M_\alpha$ (by clause (iii)).

We got a contradiction to Proposition 1.0.30. ⊣

The existence of non-forking extension implies that if M_1 and M_2 are extensions of M_0 then we can find an amalgamation (f_1,f_2,M_3) of M_1 and M_2 over M_0 such that $f_1[M_1] \neq f_2[M_2]$, namely, there is $a \in M_1 - M_0$ with $f_1(a) \notin f_2[M_2]$. By the following theorem, we can find an amalgamation (f_1,f_2,M_3) of M_1 and M_2 over M_0 such that for *each* $a \in M_1 - M_2$ $f_1(a) \notin f_2[M_2]$.

Theorem 3.3.5 (The disjoint amalgamation existence theorem)**.** *Assume that $\mathfrak{s}$ satisfies the conjugation property. Let M_0, M_1, M_2 be models in K_λ such that $M_0 \preceq M_1$ and $M_0 \preceq M_2$.*

Then there are M_3, f such that $f : M_2 \to M_3$ is an embedding over M_0, $M_1 \preceq M_3$, and $f[M_2] \bigcap M_1 = M_0$. Moreover, if $a \in M_1 - M_0$ and $tp(a,M_0,M_1) \in S^{bs}(M_0)$, then we can add that $tp(a,f[M_2],M_3)$ does not fork over M_0.

Proof. If $M_1 = M_0$ then the theorem is trivial. Otherwise by the density of basic types (see Definition 2.1.1), there is an element $a \in M_1 - M_0$ such that $tp(a,M_0,M_1) \in S^{bs}(M_0)$. So it is sufficient to prove the "moreover". By Proposition 3.3.4 the reduced triples are dense with respect to $\preceq_{bs}$. So by Theorem 3.2.3 (the extensions decomposition theorem), as $\mathfrak{s}$ satisfies the conjugation property, there is a model M_1^* such that $M_1 \preceq M_1^*$ and M_1^* is decomposable over M_1 by reduced triples, i.e., there is an increasing continuous sequence $\langle N_{0,\alpha} : \alpha \leq \delta \rangle$ of models in K_λ such that: $N_{0,0} = M_0,\ M_{0,\delta} = M_1^*$ and there is a sequence $\langle d_\alpha : \alpha < \delta \rangle$ such that $(N_{0,\alpha}, N_{0,\alpha+1}, d_\alpha)$ is a reduced triple and $d_0 = a$. By Proposition 3.1.8.1, there is an isomorphism f of M_2 over M_0 and there is an increasing continuous sequence $\langle N_{1,\alpha} : \alpha \leq \delta \rangle$ such that: $N_{0,\alpha} \preceq N_{1,\alpha},\ f[M_2] = N_{1,0}$ and $tp(d_\alpha, N_{1,\alpha}, N_{1,\alpha+1})$ does not fork over $N_{0,\alpha}$. So for $\alpha < \delta$, $(N_{0,\alpha}, N_{0,\alpha+1}, d_\alpha) \preceq_{bs} (N_{1,\alpha}, N_{1,\alpha+1}, d_\alpha)$. But the triple $(N_{0,\alpha}, N_{0,\alpha+1}, d_\alpha)$ is reduced, so $N_{1,\alpha} \bigcap N_{0,\alpha+1} = N_{0,\alpha}$. Hence $N_{1,0} \bigcap N_{0,\delta} = N_{0,0}$ [Why? let $x \in N_{1,0} \bigcap N_{0,\delta}$. Let α be the first ordinal such that $x \in N_{0,\alpha}$. α cannot be a limit ordinal as the sequence is continuous. If $\alpha = \beta + 1$ then $x \in N_{0,\alpha} \bigcap N_{1,\beta} = N_{0,\beta}$, in contradiction to the definition of α as the first such an ordinal. So we must have $\alpha = 0$, i.e., $x \in N_{0,0}$]. Hence $M_1 \bigcap f[M_2] = N_{0,0} = N_0$. Denote $M_3 = N_{0,\delta}$. $\dashv$

4. Uniqueness triples

4.1. **Introduction.** In Section 7 we amalgamate models in K_{λ^+} by amalgamating their approximations in K_λ. In Sections 4,5 we study amalgamations of models in K_λ. Now we define equivalence relation on amalgamations in K_λ.

Hypothesis 4.1.1. $\mathfrak{s}$ is a semi-good λ-frame.

Definition 4.1.2. Suppose

(1) $M_0, M_1, M_2 \in K_\lambda$, $M_0 \preceq M_1 \wedge M_0 \preceq M_2$.
(2) For $x = a, b$, (f_1^x, f_2^x, M_3^x) is an amalgamation of M_1, M_2 over M_0.

$(f_1^a, f_2^a, M_3^a), (f_1^b, f_2^b, M_3^b)$ are said to be *equivalent* over M_0 if there are f^a, f^b, M_3^{ab} such that $f^a : M_3^a \to M_3^{ab}$, $f^b : M_3^b \to M_3^{ab}$, $f^b \circ f_1^b = f^a \circ f_1^a$ and $f^b \circ f_2^b = f^a \circ f_2^a$, namely, the following diagram commutes:

$$\begin{tikzcd}
& M_3^b \arrow[r, "f^b"] & M_3^{ab} \\
M_1 \arrow[ur, "f_1^b"] \arrow[rr, "f_1^a"] & & M_3^a \arrow[u, "f^a"'] \\
M_0 \arrow[u, "id"] \arrow[r, "id"'] & M_2 \arrow[uu, "f_2^b"] \arrow[ur, "f_2^a"'] &
\end{tikzcd}$$

We denote the relation 'to be equivalent over M_0' between amalgamations over M_0, by E_{M_0}.

Proposition 4.1.3. *The relation E_{M_0} is an equivalence relation.*

Proof. Assume $(f_1^a, f_2^a, M_3^a) E_{M_0} (f_1^b, f_2^b, M_3^b)$ and $(f_1^b, f_2^b, M_3^b) E_{M_0} (f_1^c, f_2^c, M_3^c)$. We have to prove that $(f_1^a, f_2^a, M_3^a) E_{M_0} (f_1^c, f_2^c, M_3^c)$. Take witnesses g_1, g_2,

$M_3^{a,b}$ for $(f_1^a, f_2^a, M_3^a)E_{M_0}(f_1^b, f_2^b, M_3^b)$, and witnesses $g_3, g_4, M_3^{b,c}$ for $(f_1^b, f_2^b, M_3^b)E_{M_0}(f_1^c, f_2^c, M_3^c)$. Amalgamate $M_3^{a,b}$ and $M_3^{b,c}$ over M_3^b. $\dashv$

Example 4.1.4. Let K be the class of graphs, namely, $K := \{G = (|G|, E^G) : E_G$ is a binary relation on $|G|$ and for every $x_1, x_2 \in |G|$, $x_1 E^G x_2$ implies $x_2 E^G x_1$. The pair $(K, \subseteq)$ ($\subseteq$ is the relation of being subgraph), is an AEC.

Define three graphs by: $G_0 := \{0\}, E^{G_0} := \emptyset$, $G_1 := \{0,1\}, E^{G_1} := \emptyset$, $G_2 := \{0,2\}, E^{G_2} := \emptyset$. Now $G_0 \subseteq G_1$ and $G_0 \subseteq G_2$.

Up to E_{G_0} (equivalence over M_0, see Definition 4.1.2) there are exactly three non–equivalent amalgamations of G_1, G_2 over G_0:

(1) (f_1^a, f_2^a, G_3^a) is the non-disjoint amalgamation of G_1 and G_2 over G_0, namely: $G_3^a := G_1^a$, $f_1^a : G_1 \to G_3^a$, $(\forall x \in G_1) f_1^a(x) = x$, $f_2^a : G_2 \to G_3^a$, $f_2^a(0) = 0$, $f_2^a(2) = 1$.
(2) (f_1^b, f_2^b, G_3^b) is a disjoint amalgamation, where $|G_3^b| := \{0,1,2\}$, $E^{G_3^b} := \emptyset$, $f_2^b : G_1 \to G_3^b$ $(\forall x \in G_2) f_2^b(x) = x$.
(3) (f_1^c, f_2^c, G_3^c) is a disjoint amalgamation, where $|G_3^c|, f_1^c, f_2^c$ are $|G_3^b|$, f_1^b, f_2^b, respectively, but in contrast to the previous amalgamation $E^{G_3^c} := \{(1,2), (2,1)\}$.

We use the equivalence relation E_M to define a class of triples (M, N, a) such that the element a represents the extension N over M:

Definition 4.1.5. $K^{3,uq} = K_{\mathfrak{s}}^{3,uq}$ is the class of triples $(M, N, a) \in K^{3,bs}$ such that if $M \preceq M_1 \in K_\lambda$, then up to E_M there is a unique amalgamation (f_1, f_2, N_1) of N and M_1 over M such that $tp(f_1(a), f_2[M_1], N_1)$ does not fork over M. A *uniqueness triple* is a triple in $K^{3,uq}$.

Along the paper we use uniqueness implicitly, via the weak uniqueness of NF (see Theorem 5.4.7).

We define a variant of domination in order to compare it with the notion of a uniqueness triple. The main difference between our definition and the definition in the context of stable first order theories (as defined in Definition 3.2 on page 153 in [Bal 88]), is that in our variant, $a \downarrow_M b$ is replaced by '$tp(a, M_1, N_1)$ does not fork over M for some models M_1, N_1 with $M \preceq M_1 \preceq N_1$, $b \in M_1$, $tp(b, M, M_1) \in S^{bs}(M)$ and $N \preceq N_1$'. By symmetry, we can replace the assumption '$tp(a, M_1, N_1)$ does not fork over M' by 'for some N_2 with $M \preceq N_2 \preceq N_1$ and $a \in N_2$, the (Galois) type $tp(b, N_2, N_1)$ does not fork over M', so it is more similar to the first order case.

Definition 4.1.6. Let M, N be models in K_λ with $M \preceq N$ and let a be an element in $N - M$. We say that *a dominates* N over M when: For every models $M_1, N_1 \in K_\lambda$ with $M \preceq M_1 \preceq N_1$ and $N \preceq N_1$ and every element $b \in M_1$ with $tp(b, M, M_1) \in S^{bs}(M)$, if $tp(a, M_1, N_1)$ does not fork M, then $tp(b, N, N_1)$ does not fork over M.

Proposition 4.1.7. *If (M, N, a) is a uniqueness triple, then a dominates N over M.*

Proof. See in Section 6 of [JrSi3]. ⊣

Since the definition of $K^{3,uq}$ is confusing, we clarify it by the following proposition:

Proposition 4.1.8. *$(M,N,a) \in K^{3,uq}$ iff the following holds: If for $n = 1,2$ $(M,N,a) \preceq_{bs} (M^*_n, N^*_n, a)$ and $f : M^*_1 \to M^*_2$ is an isomorphism over $M \bigcup \{a\}$, then for some f_1, f_2, N^* the following hold: $f_n : N^*_n \to N^*$ is an embedding over N, and $f_1 \restriction M^*_1 = f_2 \restriction M^*_2 \circ f$.*

Proof. $\Rightarrow$: Suppose $(M,N,a) \in K^{3,uq}$ and for $n = 1,2$ $(M,N,a) \preceq_{bs} (M^*_n, N^*_n, a)$ and $f : M^*_1 \to M^*_2$ is an isomorphism over M over $M \bigcup \{a\}$. We have to prove that for some f_1, f_2, N^* the following hold: $f_n : N^*_n \to N^*$ is an embedding over N, and $f_1 \restriction M^*_1 = f_2 \restriction M^*_2 \circ f$. $(id_{M^*_1}, id_N, N^*_1), (f, id_N, N^*_2)$ are two amalgamations of M^*_1 and N over M. By the definition of the relation $\preceq_{bs}$ (Definition 3.1.1.2), $tp(a, M^*_1, N^*_1)$ does not fork over M and $tp(f(a), f[M^*_1], N^*_2) = tp(a, M^*_2, N^*_2)$ does not fork over M. Hence by Definition 4.1.5 $(id_{M^*_1}, id_N, N^*_1) E_M (f, id_N, N^*_2)$. So by Definition 4.1.2 there are f_1, f_2, N^* as needed.

$\Leftarrow$: We leave to the reader. ⊣

We give an example of a trivial frame such that it is very easy to compute $K^{3,uq}$.

Example 4.1.9. Let $\tau := (E,P)$ where E is a binary predicate and P is an unary predicate. Let K be the class of τ-models (G,E,P) such that:

(1) $(|G|, E)$ is a graph.
(2) For each $a,b \in G$, $aEb \Rightarrow [P(a) \wedge P(b)]$.

$(K, \subseteq)$ is an AEC with LST-number $\aleph_0$. Let λ be a cardinal. The trivial λ-frame (see Definition 2.2.2) of $(K, \subseteq)$ is of course not a semi-good λ-frame. But if we ignore this fact, and define $K^{3,uq}$ as in Definition 4.1.5 then $K^{3,uq} = \{(M,N,a) : (\forall x \in N - M) \neg P(x)\}$.

We will not use the following proposition later.

Proposition 4.1.10. *If for every $M,N \in K_\lambda$ with $M \preceq N$ and for every $a \in N - M$, the type $tp(a,M,N)$ is basic then every uniqueness triple is reduced.*

Proof. Let (M,N,a) be a uniqueness triple. By Proposition 4.1.7, a dominates a N over M. Suppose $(M,N,a) \preceq_{bs} (M',N',a)$. We have to prove that $M' \bigcap N = M$. Take $b \in M' - M$. We have to prove that $b \notin N$. Now by assumption, $tp(b,M,M')$ is basic. By the definition of $\preceq_{bs}$, $tp(a,M',N')$ does not fork over M. So since a dominates N over M, $tp(b,N,N')$ does not fork over M. Hence $b \notin N$. ⊣

Since we do not want to assume that every type is basic, Item 2 of the following proposition is important.

Proposition 4.1.11.

(1) If p_0, p_1 are conjugate types and in p_0 there is a uniqueness triple, then also in p_1 there is such a triple.
(2) If $\mathfrak{s}$ satisfies the conjugation property, then every uniqueness triple is reduced.

Proof.
(1) Suppose $p_0 = tp(a, M, N)$, $(M, N, a) \in K^{3,uq}$. Let f be an isomorphism with domain M, such that $f(p_0) = p_1$. $K, \preceq$ are closed under isomorphisms, so it is easy to prove that $(f[M], f^+[N], f^+(a)) \in K^{3,uq}$, where $f \subseteq f^+$, $dom(f^+) = N$. But $(f[M], f^+[N], f^+(a)) \in p_1$.
(2) First note that we do not use item 1. Suppose $(M_0, N_0, a) \in K^{3,uq}$ and $(M_0, N_0, a) \preceq_{bs} (M_1, N_1, a)$. Since $\mathfrak{s}$ satisfies the conjugation property, by Theorem 3.3.5 (the existence of a disjoint amalgamation), there are f, N_2 such that $f : M_1 \to N_2$ is an embedding over M_0, $N_0 \preceq N_2$, $f[M_1] \bigcap N_0 = M_0$ and $tp(a, f[M_1], N_2)$ does not fork over M_0. By Definition 4.1.5, there are f_1, f_2, N^* such that: $f_n : N_n \to N^*$ and embedding over N_0 and $f_1 \restriction M_1 = f_2 \circ f$. For the sake of contradiction assume that $x \in M_1 \bigcap N_0 - M_0$. On one hand, since $x \in N_0$, we have $f_1(x) \in f_2[N_0]$. But on the other hand, since $x \in M_1 - M_0$ we have $f_1(x) \notin f_2[N_0]$ $[f(x) \notin N_0$ because $f[M_1] \bigcap N_0 = M_0$. So $f_2(f(x)) \notin f_2[N_0]$. But $f_1(x) = f_2(f(x))]$. A contradiction .

$\dashv$

Proposition 4.1.12.
(1) If $K^{3,uq}$ is dense with respect to $\preceq_{bs}$ and $\mathfrak{s}$ satisfies the conjugation property then $K^{3,uq}$ satisfies the existence property.
(2) Suppose that $K^{3,uq}$ satisfies the existence property. If $p = tp(a, M, N) \in S^{bs}(M)$, then there is a model N^* such that $(M, N^*, a) \in K^{3,uq} \bigcap p$.

Proof.
(1) Substitute $K^{3,*} := K^{3,uq}$ in Theorem 3.2.3.1.
(2) By Theorem 3.2.3(2).

$\dashv$

5. Non-forking amalgamation

5.1. **The hypotheses.**

Hypothesis 5.1.1.
(1) $\mathfrak{s}$ is a semi-good λ-frame.
(2) $\mathfrak{s}$ satisfies the conjugation property.
(3) $K^{3,uq}$ satisfies the existence property.

Remark 5.1.2. Actually we do not use the local character in this section (we assume it implicitly, see Definition 2.1.1.3.c). So in [JrSh 940] we can use the results in this section, although we do not have local character.

5.2. **The axioms of non-forking amalgamation.**

Introduction: We want to find a relation of a non-forking amalgamation (see the discussion at the beginning of Section 3). In Definition 5.2.1 we define the properties this relation has to satisfy.

Definition 5.2.1. Let $R \subseteq {}^4(K_\lambda)$ be a relation. We say $\bigotimes_R$ when the following axioms are satisfied (where $M_0, M_1, M_2, M_3, N_0, N_1, N_2, N_3, M_{a,i}, M_{b,i}$ are models of cardinality λ):

(a) If $R(M_0, M_1, M_2, M_3)$, then $n \in \{1, 2\} \Rightarrow M_0 \preceq M_n \preceq M_3$ *and* $M_1 \cap M_2 = M_0$.
(b) Monotonicity: If $R(M_0, M_1, M_2, M_3)$ *and* $N_0 = M_0, n < 3 \Rightarrow N_n \preceq M_n \wedge N_0 \preceq N_n \preceq N_3, (\exists N^*)[M_3 \preceq N^* \wedge N_3 \preceq N^*]$, then $R(N_0, N_1, N_2, N_3)$. [Proposition 5.2.2 clarifies this axiom]
(c) Existence: For every $N_0, N_1, N_2 \in K_\lambda$ if $l \in \{1, 2\} \Rightarrow N_0 \preceq N_l$ and $N_1 \bigcap N_2 = N_0$, then there is N_3 such that $R(N_0, N_1, N_2, N_3)$.
(d) Weak uniqueness: Suppose for $x = a, b$ $R(N_0, N_1, N_2, N_3^x)$. Then there is a joint embedding of N_3^a, N_3^b over $N_1 \bigcup N_2$. In other words, if $R(N_0, N_1, N_2, N_3)$ then $N_1 \bigcup N_2$ is an amalgamation base.
(e) Symmetry: $R(N_0, N_1, N_2, N_3) \Leftrightarrow R(N_0, N_2, N_1, N_3)$.
(f) Long transitivity: For $x = a, b$ let $\langle M_{x,i} : i \leq \alpha^* \rangle$ an increasing continuous sequence of models in K_λ. Suppose $i < \alpha^* \Rightarrow R(M_{a,i}, M_{a,i+1}, M_{b,i}, M_{b,i+1})$. Then $R(M_{a,0}, M_{a,\alpha^*}, M_{b,0}, M_{b,\alpha^*})$.

Proposition 5.2.2. *We can replace item b from Definition 5.2.1 by the conjunction of the following two assumptions:*

(1) If $R(M_0, M_1, M_2, M_3)$ and $M_0 \preceq N_1 \preceq M_1$, then $R(M_0, N_1, M_2, M_3)$.
(2) If $M_1 \bigcup M_2 \subseteq N_3 \preceq M_3$, then $R(M_0, M_1, M_2, M_3) \Leftrightarrow R(M_0, M_1, M_2, N_3)$.

Proof. Suppose $\bigotimes_R$.

(1) If $R(M_0, M_1, M_2, M_3)$ and $M_0 \preceq N_1 \preceq M_1$, then by Definition 5.2.1.b (where $N^* := M_3$, $N_3 := M_3$ and $N_2 := M_2$) $R(M_0, N_1, M_2, M_3)$.

(2) Easy, too.

Conversely, suppose R satisfies items a,c,d,e,f from Definition 5.2.1 and items 1,2 from our proposition. By item 1, without loss or generality, $N_1 = M_1$. Using again item 1, by Definition 5.2.1.e (symmetry) without loss of generality $N_2 = M_2$. By item 2, $R(M_0, M_1, M_2, N^*)$. Using again item 2, we get $R(M_0, M_1, M_2, N_3)$, namely, $R(N_0, N_1, N_2, N_3)$. $\dashv$

Example 5.2.3. Let K be the class of graphs. Let $\preceq$ be the relation on K of being subgraph. Let λ be any cardinal. Define $R_1 := \{(M_0, M_1, M_2, M_3) \in {}^4K_\lambda : M_0 \preceq M_1 \preceq M_3,\ M_0 \preceq M_2 \preceq M_3,\ M_1 \bigcap M_2 = M_0$ and for every $a_1 \in M_1 - M_0$ and $a_2 \in M_2 - M_0$ $\neg(a_1 E^{M_3} a_2)$. Define R_2 like R_1 but at the end: $(a_1 E^{M_3} a_2)$. Now $\bigotimes_{R_1}$ and $\bigotimes_{R_2}$.

We give another version of weak uniqueness:

Proposition 5.2.4. *Suppose*

(1) $\bigotimes_R$.
(2) $R(M_0, M_1, M_2, M_3)$ *and* $R(M_0, M_1^*, M_2^*, M_3^*)$.
(3) *For* $n = 1, 2$ *there is an isomorphism* $f_n : M_n \to M_n^*$ *over* M_0.

Then there are M, f *such that:*

(1) *For* $n < 3$ $f \restriction M_n = f_n$.
(2) $M_3^* \preceq M$.
(3) $f[M_3] \preceq M$.

Proof. $M_1 \bigcap M_2 = M_0$, so there is a function g with domain M_3 such that $f_1 \bigcup f_2 \subseteq g$. So $g[M_1] = M_1^*$ and $g[M_2] = M_2^*$. Hence $R(M_0, M_1^*, M_2^*, g[M_3])$ and $R(M_0, M_1^*, M_2^*, M_3^*)$. Therefore we can use the weak uniqueness from Definition 5.2.1.d. $\dashv$

Roughly, the following proposition says that finding a relation R that satisfies clauses a,c,d of Definition 5.2.1 is equivalent to assigning to each triple $(M_0, M_1, M_2) \in D := \{(M_0, M_1, M_2) : M_0, M_1, M_2 \in K_\lambda,\ M_0 \preceq M_1, M_0 \preceq M_2\}$ a disjoint amalgamation (see Definition 3.3.1) (f_1, f_2, M_3) of M_1, M_2 over M_0 up to E_{M_0} (see Definition 4.1.2).

Proposition 5.2.5. *Let* R *be a relation that satisfies clauses a,c,d of Definition 5.2.1. Denote* $D := \{(M_0, M_1, M_2) : M_0, M_1, M_2$ *are models in* K_λ *and* $M_0 \preceq M_1, M_0 \preceq M_2\}$. *Then:*

(1) *There is a function* G *with domain* D *which assigns to each triple* (M_0, M_1, M_2) *an amalgamation* (f_1, f_2, M_3) *of* M_1, M_2 *over* M_0*, such that* $R(M_0, f_1[M_1], f_2[M_2], M_3)$ *(in proving this item we do not use clause d).*
(2) *If* G_1, G_2 *are two functions as in item 1 (with respect to* R*), then for every* $(M_0, M_1, M_2) \in D$, $G_1((M_0, M_1, M_2)) E_{M_0} G_2((M_0, M_1, M_2))$.
(3) *If* G *is a function with domain* D *which assigns to each triple* (M_0, M_1, M_2) *a disjoint amalgamation, then the relation* $R := \{(M_0, M_1, M_2, M_3) : M_1 \bigcap M_2 = M_0,\ G((M_0, M_1, M_2)) E_{M_0} (id_{M_1}, id_{M_2}, M_3)\}$ *satisfies clauses a,c,d of Definition 5.2.1.*

Proof. We leave to the reader. $\dashv$

Definition 5.2.6. Suppose $\bigotimes_R$. R is said to *respect the frame* $\mathfrak{s}$ when: if $R(M_0, M_1, M_2, M_3)$ and $tp(a, M_0, M_1) \in S^{bs}(M_0)$, then $tp(a, M_2, M_3)$ does not fork over M_0.

5.3. **The relation** NF. First we define a relation NF^* and then we define a relation NF as its monotonicity closure, see Definition 5.3.2. Theorem 5.5.4 asserts that the relation NF is the unique relation R which satisfies $\bigotimes_R$ and respects the frame $\mathfrak{s}$.

Definition 5.3.1. Define a relation $NF^* = NF^*_\lambda$ on ${}^4(K_\lambda)$ by: $NF^*(N_0, N_1, N_2, N_3)$ if there is $\alpha^* < \lambda^+$ and for l=1,2 there are an increasing continuous sequence $\langle N_{l,i} : i \leq \alpha^* \rangle$ and a sequence $\langle d_i : i < \alpha^* \rangle$ such that:

$$\begin{tikzcd}
N_2 = N_{2,0} \arrow[r, "id"] & N_{2,i} \arrow[r, "id"] & N_{2,i+1} \arrow[r, "id"] & N_{2,\alpha^*} = N_3 \\
N_0 = N_{1,0} \arrow[r, "id"] \arrow[u, "id"] & N_{1,i} \arrow[r, "id"] \arrow[u, "id"] & N_{1,i+1} \arrow[r, "id"] \arrow[u, "id"] & N_{1,\alpha^*} = N_1 \arrow[u, "id"]
\end{tikzcd}$$

(a) $n < 3 \Rightarrow N_0 \preceq N_n \preceq N_3$.
(b) $N_{1,0} = N_0, N_{1,\alpha^*} = N_1, N_{2,0} = N_2, N_{2,\alpha^*} = N_3$.
(c) $i \leq \alpha^* \Rightarrow N_{1,i} \preceq N_{2,i}$.
(d) $d_i \in N_{1,i+1} - N_{1,i}$.
(e) $(N_{1,i}, N_{1,i+1}, d_i) \in K^{3,uq}$.
(f) $tp(d_i, N_{2,i}, N_{2,i+1})$ does not fork over $N_{1,i}$.

In this case, $\langle N_{1,i}, d_i : i < \alpha^* \rangle^\frown \langle N_{1,\alpha^*} \rangle$ is said to be the first witness for $NF^*(N_0, N_1, N_2, N_3)$, d_i is said to be the i-th element in the first witness for NF^* and $\langle N_{2,i} : i \leq \alpha^* \rangle$ is said to be the second witness for $NF^*(N_0, N_1, N_2, N_3)$.

Definition 5.3.2. $NF = NF_\lambda$ is the class of quadruples (M_0, M_1, M_2, M_3) of models in K_λ such that $M_0 \preceq M_1 \preceq M_3$, $M_0 \preceq M_2 \preceq M_3$ and there are models N_0, N_1, N_2, N_3 such that: $N_0 = M_0, l < 4 \Rightarrow M_l \preceq N_l$ and $NF^*(N_0, N_1, N_2, N_3)$.

Proposition 5.3.3. *The relations NF^*, NF are closed under isomorphisms.*

Proof. Trivial. $\dashv$

Proposition 5.3.4. *Suppose for $x = a, b$ $(f_{x,1}, f_{x,2}, M_{x,3})$ is an amalgamation of M_1, M_2 over M_0. If $(f_{a,1}, f_{a,2}, M_{a,3}) E_{M_0} (f_{b,1}, f_{b,2}, M_{b,3})$, then*

$$NF(M_0, f_{a,1}[M_1], f_{a,2}[M_2], M_{a,3}) \Leftrightarrow NF(M_0, f_{b,1}[M_1], f_{b,2}[M_2], M_{b,3})$$

Proof. Easy. $\dashv$

Recall that by Definition 3.3.2 a triple $(M, N, a) \in K^{3,bs}_\lambda$ is *reduced* if $(M, N, a) \preceq_{bs} (M^*, N^*, a) \Rightarrow M^* \bigcap N = M$.

Proposition 5.3.5. *Every triple in $K^{3,uq}$ is reduced.*

Proof. Suppose $(N_0, N_1, d) \preceq_{bs} (N_2, N_3, d), (N_0, N_1, d) \in K^{3,uq}$. By Hypothesis 5.1.1 and Proposition 3.3.5, there is a disjoint amalgamation of N_1, N_2 over N_0, such that the type of d does not fork over N_0, so by the definition of uniqueness triple (Definition 4.1.5), N_3 is a disjoint amalgamation of N_1, N_2 over N_0. $\dashv$

Proposition 5.3.6.

(1) If $NF^*(N_0, N_1, N_2, N_3)$ then $N_1 \bigcap N_2 = N_0$.
(2) If $NF(N_0, N_1, N_2, N_3)$ then $N_1 \bigcap N_2 = N_0$.

Proof. (1) Let $x \in N_1 \bigcap N_2$. We will prove that $x \in N_0$. Let $\langle N_{1,\alpha}, d_\alpha : \alpha < \alpha^* \rangle^\frown \langle N_{1,\alpha^*} \rangle, \langle N_{2,\alpha} : \alpha \leq \alpha^* \rangle$ be witnesses for $NF^*(N_0, N_1, N_2, N_3)$. Let α be the first ordinal such that $x \in N_{1,\alpha}$. α is not a limit ordinal, because a first witness for NF^* is especially a continuous sequence. We prove that α is not a successor ordinal, so we conclude that $\alpha = 0$. Suppose $\alpha = \beta + 1$. By Definition 5.3.1.e $(N_{1,\beta}, N_{1,\beta+1}, d_\beta) \in K^{3,uq}$. By Definition 5.3.1.f $tp(d_\beta, N_{1,\beta}, N_{1,\beta+1})$ does not fork over $N_{0,\beta}$. So by Proposition 5.3.5 $N_{1,\beta+1} \bigcap N_{2,\beta} = N_{1,\beta}$. But $x \in N_{1,\beta+1} \bigcap N_2 \subseteq N_{1,\beta+1} \bigcap N_{2,\beta}$, so $x \in N_{1,\beta}$ in contradiction to the assumption that α is the minimal ordinal with $x \in N_{1,\alpha}$.

(2) By item 1. $\dashv$

Theorem 5.3.7 (the existence theorem for NF)**.** *Suppose that for $n = 1, 2$ $N_0 \preceq N_n$ and $N_1 \bigcap N_2 = N_0$.*

(a) For some model $N_3 \in K_\lambda$, $NF(N_0, N_1, N_2, N_3)$.
(b) If N_1 is decomposable over N_0 by $K^{3,uq}$, then for some $N_3 \in K_\lambda$, $NF^*(N_0, N_1, N_2, N_3)$.
(c) If N_1 is decomposable over N_0 by $K^{3,uq}$ and $a \in N_1 - N_0$, then for some $N_3 \in K_\lambda$, $NF^*(N_0, N_1, N_2, N_3)$. Moreover, we can choose a as the first element in the first witness for NF^*.

Proof.

(a) By Theorem 3.2.3.3 (the extensions decomposition theorem), (and assumption 5.1.1), there is a model N_1^* with $N_1 \preceq N_1^*$ which is decomposable over N_0, i.e., there is a sequence $\langle N_{1,\alpha}, d_\alpha : \alpha < \alpha^* \rangle^\frown \langle N_{1,\alpha^*} \rangle$, such that: $N_0 = N_{1,0}, (N_{1,\alpha}, N_{1,\alpha+1}, d_\alpha) \in K^{3,uq}, N_1 \preceq N_{1,\alpha^*} = N_1^*$. Therefore we can use item b.
(b) Let $\langle N_{1,\alpha}, d_\alpha : \alpha < \alpha^* \rangle^\frown \langle N_{1,\alpha^*} \rangle$ be an increasing continuous sequence with $N_{1,0} = N_0$ and $N_{1,\alpha^*} = N_1$. By Proposition 3.1.8.1 there is a sequence $\langle N_{2,\alpha} : \alpha \leq \alpha^* \rangle$ which is a corresponding second witness for $NF^*(N_0, N_{1,\alpha^*}, N_2, N_{2,\alpha^*})$.
(c) By the 'moreover' in Theorem 3.2.3.3 (the decomposition extensions theorem).

$\dashv$

The following theorem is a preparatory version for NF^* of Theorem 5.5.1, i.e., the long transitivity theorem for NF.

Proposition 5.3.8 (long transitivity theorem for NF^*)**.** *For $x = a, b$ let $\langle M_{x,\alpha} : \alpha \leq \alpha^* \rangle$ be an increasing continuous sequence of models. Suppose $\alpha < \alpha^* \Rightarrow NF^*(M_{a,\alpha}, M_{a,\alpha+1}, M_{b,\alpha}, M_{b,\alpha+1})$. Then $NF^*(M_{a,0}, M_{a,\alpha^*}, M_{b,0}, M_{b,\alpha^*})$.*

Proof. Concatenate all the sequences together. $\dashv$

Proposition 5.3.9 (the monotonicity theorem)**.**

(1) If $NF^*(N_0, N_1, N_2, N_3)$ and $N_0 \preceq M_2 \preceq N_2$, then $NF^*(N_0, N_1, M_2, N_3)$.

(2) If $NF(M_0, M_1, M_2, M_3)$, then we can find N_1, N_3 such that $NF^*(M_1, N_1, M_2, N_3)$ and $M_1 \preceq N_1 \preceq N_3 \wedge M_3 \preceq N_3$.
(3) $NF^*(M_0, M_1, M_2, M_3) \wedge M_3 \preceq M_3^* \Rightarrow NF(M_0, M_1, M_2, M_3^*)$.
(4) The relation NF satisfies monotonicity (in the sense of Definition 5.2.1.b).

Proof.
(1) Let $\langle N_{1,\alpha}, d_\alpha : \alpha < \alpha^* \rangle$, $\langle N_{2,\alpha} : \alpha < \alpha^* \rangle$ be witnesses for $NF^*(N_0, N_1, N_2, N_3)$. Then $\langle N_{1,\alpha} : \alpha < \alpha^* \rangle$, $\langle M_2 \rangle^\frown \langle N_{2,\alpha} : 0 < \alpha < \alpha^* \rangle$ are witnesses for $NF^*(N_0, N_1, N_2, N_3)$ (notice that by Definition 2.1.1.3.b (monotonicity) $tp(d_0, M_2, N_{2,1})$ does not fork over N_0).
(2) By the definition of NF (Definition 5.3.2) and item 1.
(3)

$$\begin{tikzcd}
a \in M_1^* \arrow[rr, "f"] & & M_3^{**} \\
M_1 \arrow[u, "id"] \arrow[r, "id"] & M_3 \arrow[r, "id"] & M_3^* \arrow[u, "id"'] \\
M_0 \arrow[u, "id"] \arrow[r, "id"] & M_2 \arrow[u, "id"] &
\end{tikzcd}$$

Take $p \in S^{bs}(M_1)$, and take M_1^*, a such that $(M_1, M_1^*, a) \in p \bigcap K^{3,uq}$. By Definition 2.1.1.1.3.f, there is an amalgamation $(f, id_{M_3^*}, M_3^{**})$ of M_1^*, M_3^* over M_1 such that $tp(a, f[M_3^*], M_3^{**})$ does not fork over M_1. So $NF^*(M_1, f[M_1^*], M_3^*, M_3^{**})$. Hence by item 1, $NF^*(M_1, f[M_1^*], M_3, M_3^{**})$. Now by Proposition 5.3.8 $NF^*(M_0, M_1^*, M_2, M_3^{**})$. So the definition of NF (Definition 5.3.2), $NF(M_0, M_1, M_2, M_3^*)$.
(4) Suppose $M_0^* = M_0$, $0 < n < 3 \Rightarrow M_0^* \preceq M_n^* \preceq M_3^*$, $M_n^* \preceq M_n$, $M_3^* \preceq M_3^{**}$, $M_3 \preceq M_3^{**}$, $NF(M_0, M_1, M_2, M_3)$.

$$\begin{tikzcd}
 & & M_3^{**} \arrow[r, "f"] & M_3^{***} \\
N_1 \arrow[rrr, "\cancel{id}"] & & & N_3 \arrow[u, "id"] \\
M_1 \arrow[u, "id"] \arrow[rr, "id"] & & M_3 \arrow[uu, "id"] \arrow[ur, "id"] & \\
M_1^* \arrow[u, "id"] \arrow[r, "id"] & M_3^* \arrow[uuur, "id"] & & \\
M_0 \arrow[u, "id"] \arrow[r, "id"] & M_2^* \arrow[u, "id"] \arrow[r, "id"] & M_2 = N_2 \arrow[uu, "id"] &
\end{tikzcd}$$

By item 2, for some N_1, N_3, $NF^*(M_0, N_1, M_2, N_3)$, $M_1 \preceq N_1 \preceq N_3$ and $M_3 \preceq N_3$. Take an amalgamation (f, id_{N_3}, M_3^{***}) of M_3^{**} and N_3 over M_3

(so over $M_1^* \bigcup M_2^*$). By item 3, $NF(M_0, N_1, M_2, M_3^{***})$. So by the definition of NF (Definition 5.3.2), $NF(M_0, M_1^*, M_2^*, f[M_3^*])$. But the relation NF is closed under isomorphisms, so $NF(M_0, M_1^*, M_2^*, M_3^*)$. $\dashv$

5.4. **Weak Uniqueness.** We want to show that NF satisfies weak uniqueness and long transitivity. Proposition 5.4.4 is a key point. To emphasize the exact hypotheses involved in the proof, we focus on a small set of consequences $\bigotimes_R$.

Item (3) of the following definition follows from $\otimes_R$ by existence and long transitivity.

Definition 5.4.1. Let $R \subseteq {}^4(K_\lambda)$ be a relation. We say $\bigotimes_R^-$ when:

(1) If $R(M_0, M_1, M_2, M_3)$ *then* $n \in \{1, 2\} \Rightarrow M_0 \preceq M_n \preceq M_3$.
(2) Weak Uniqueness: Suppose for $x = a, b$ (f_1^x, f_2^x, N_3^x) is an amalgamation of N_1 and N_2 over N_0 and $R(N_0, f_1^x[N_1], f_2^x[N_2], N_3^x)$. Then $(f_1^a, f_2^a, N_3^a) E_{N_0} (f_1^b, f_2^b, N_3^b)$.
(3) If $R(M_0, M_1, M_2, M_3)$ and $f : M_2 \to M_4$ is an embedding, then there is an amalgamation (g, id_{M_4}, M_5) of M_3, M_4 over M_2 such that $R(f[M_0], g[M_1], M_4, M_5)$.

$$\begin{tikzcd}
M_1 \arrow[r, "id"] & M_3 \arrow[r, "g"] & M_5 \\
M_0 \arrow[u, "id"] \arrow[r, "id"] & M_2 \arrow[u, "id"] \arrow[r, "f"] & M_4 \arrow[u, "id"]
\end{tikzcd}$$

Definition 5.4.2. $NF^{uq} := \{(M_0, M_1, M_2, M_3)$:there is $a \in M_1 - M_0$ such that $(M_0, M_1, a) \in K^{3,uq}$ and $tp(a, M_2, M_3)$ does not fork over $M_0\}$.

Proposition 5.4.3.

(1) $\bigotimes_{NF^{uq}}^-$.
(2) For every relation R, $\bigotimes_R \Rightarrow \bigotimes_R^-$.

Proof.

(1) By the definition of $K^{3,uq}$ (Definition 4.1.5), Definition 2.1.1.3.f and Definition 2.1.1.1.d (to get M_5).
(2) By Axioms d,f in Definition 5.2.1 and by Proposition 2.5.6.

$\dashv$

We show that weak transitivity is preserved by unions of chains.

Proposition 5.4.4 (the transitivity of weak uniqueness)**.** *Suppose*

(1) $\bigotimes_R^-$.
(2) $\alpha^* \leq \lambda^+$.
(3) For every $\alpha < \alpha^*$, $N_{1,\alpha}, N_{2,\alpha}^a, N_{2,\alpha}^b \in K_\lambda$.
(4) $\langle N_{1,\alpha} : \alpha \leq \alpha^* \rangle$, $\langle N_{2,\alpha}^a : \alpha \leq \alpha^* \rangle$, $\langle N_{2,\alpha}^b : \alpha \leq \alpha^* \rangle$ *are increasing continuous sequences.*
(5) $N_{2,0}^a = N_{2,0}^b$.

(6) For every $\alpha \leq \alpha^*$, $f^a_\alpha : N_{1,\alpha} \to N^a_{2,\alpha}$ and $f^b_\alpha : N_{1,\alpha} \to N^b_{2,\alpha}$.
(7) $(\alpha < \alpha^* \wedge x \in \{a,b\}) \Rightarrow R(f^x_\alpha[N_{1,\alpha}], f^x_{\alpha+1}[N_{1,\alpha+1}], N^x_{2,\alpha}, N^x_{2,\alpha+1})$.

Then $(f^a_{\alpha^*}, id_{N^a_{2,0}}, N^a_{2,\alpha^*}) E_{N_{1,0}} (f^a_{\alpha^*}, id_{N^a_{2,0}}, N^b_{2,\alpha^*})$.

Proof. We choose $N_{2,\alpha}, g_{a,\alpha}, g_{b,\alpha}$ by induction on $\alpha \leq \alpha^*$, such that for $x = a, b$ and $\alpha \leq \alpha^*$ the following hold:

(i) $g_{x,\alpha} : N^x_{2,\alpha} \to N_{2,\alpha}$ is an embedding such that $g_{a,\alpha} \circ f^a_\alpha = g_{b,\alpha} \circ f_{b,\alpha}$.
(ii) $N_{2,0} = N^x_{2,0}, g_{x,0} = identity$.
(iii) $\langle N_{2,\alpha} : \alpha \leq \alpha^* \rangle$ is an increasing continuous sequence.
(iv) $\langle g_{x,\alpha} : \alpha \leq \alpha^* \rangle$ is an increasing continuous sequence.

If we can construct this, then the following diagram commutes:

$$\begin{tikzcd}
 & N^a_{2,\alpha^*} \arrow[rr, "g_{a,\alpha^*}"] & & N_{2,\alpha^*} \\
N_{1,\alpha} \arrow[ur, "f^a_{\alpha^*}"] \arrow[rr, "f^b_{\alpha^*}"] & & N^b_{2,\alpha^*} \arrow[ur, "g_{b,\alpha^*}"] & \\
N_{1,0} \arrow[u, "id"] \arrow[r, "id"] & N_{2,0} \arrow[uu, "id"] \arrow[ur, "id"] & &
\end{tikzcd}$$

[By clause (i) $g_{a,\alpha^*} \circ f^a_{\alpha^*} = g_{b,\alpha^*} \circ f^b_{\alpha^*}$, and by clauses $(ii), (iv)$ $g_{x,\alpha^*} \supseteq g_{x,0} = id_{N_{2,0}}$].

Therefore $(g_{a,\alpha^*}, g_{b,\alpha^*}, N_{2,\alpha^*})$ witnesses that $(f^a_{\alpha^*}, id_{N_{2,0}}, N^a_{2,\alpha^*}) E_{N_{1,0}} (f^b_{\alpha^*}, id_{N_{2,0}}, N^b_{2,\alpha^*})$.

Why can we construct this? For $\alpha = 0$, only clause (ii) is relevant. For α limit ordinal, take unions, and by smoothness, $g_{x,\alpha}$ is $\preceq$-embedding. What will we do for $\alpha + 1$? By clause 7 for $x = a, b$ $R(f^x_\alpha[N_{1,\alpha}], f^x_{\alpha+1}[N_{1,\alpha+1}], N^x_{2,\alpha}, N^x_{2,\alpha+1})$. By clause (i), $g_{x,\alpha}[N^x_{2,\alpha}] \preceq N_{2,\alpha}$ and by clause 1, $\bigotimes^-_R$. So by Definition 5.4.1.3, we can find g_x, N^x such that the following hold:

(1) $g_x : N^x_{2,\alpha+1} \to N^x$ is an embedding.
(2) $g_{x,\alpha} \subset g_x$.
(3) $R(g_x \circ f^a_\alpha[N_{1,\alpha}], g_x \circ f^a_{\alpha+1}[N_{1,\alpha+1}], N_{2,\alpha}, N^x)$.

$$\begin{tikzcd}
 & & & N^a \arrow[r, "h^a"] & N_{2,\alpha+1} \\
 & N^a_{2,\alpha+1} \arrow[urr, "g_a"] & & & \\
 & & & & N^b \arrow[uu, "h^b"'] \\
N_{1,\alpha+1} \arrow[uur, "f^a_{\alpha+1}"] \arrow[rr, "f^b_{\alpha+1}", "id"'] & & N^b_{2,\alpha+1} \arrow[urr, "g_b"] & & \\
 & N^a_{2,\alpha} \arrow[uuu] \arrow[rr, "g_{a,\alpha}", "id"'] & & N_{2,\alpha} \arrow[uuuu, "id"] \arrow[uur, "id"'] & \\
N_{1,\alpha} \arrow[uu, "id"] \arrow[ur, "f^a_\alpha"] \arrow[rr, "f^b_\alpha"] & & N^b_{2,\alpha} \arrow[uu] \arrow[ur, "g_{b,\alpha}"'] & &
\end{tikzcd}$$

Hence by Definition 5.4.1.2 $(g_a \restriction f^a_{\alpha+1}[N_{1,\alpha+1}], id_{N_{2,\alpha}}, N^a) E_{f^a_\alpha[N_{1,\alpha}]} (g_b \restriction f^b_{\alpha+1}[N_{1,\alpha+1}], id_{N_{2,\alpha}}, N^b)$. So there is a joint embedding $(h^a, h^b, N_{2,\alpha+1})$ of N^a, N^b such that for $x = a, b$ $id_{N_{2,\alpha}} \subseteq h^x$ and $h^a \circ g_\alpha \circ f^a_{\alpha+1} = h^b \circ g_b \circ f^b_{\alpha+1}$. Now we define $g_{x,\alpha+1} := h^x \circ g_x$. $\dashv$

The following proposition asserts that we have weak uniqueness over the first witness for NF^*.

Proposition 5.4.5. *If for $x = a, b$ $NF^*(N_0, N_1, N_2, N^x_3)$ and they have the same first witness, then there is a joint embedding of N^a_3, N^b_3 over $N_1 \bigcup N_2$.*

Proof. By Proposition 5.4.3.1, $\bigotimes^-_{NF^{uq}}$. Hence it follows by Proposition 5.4.4. $\dashv$

The following proposition is similar to weak uniqueness for NF^*, but note the order of N_1, N_2 in the two quadruples.

Proposition 5.4.6 (the opposite uniqueness proposition)**.** *Suppose $NF^*(N_0, N_1, N_2, N^a_3)$ and $NF^*(N_0, N_2, N_1, N^b_3)$. Then there is a joint embedding of N^a_3 and N^b_3 over $N_1 \bigcup N_2$.*

Proof. Suppose that $\langle N^a_\alpha, d^a_\alpha : \alpha < \alpha^* \rangle^\frown \langle N^a_{\alpha^*} \rangle$ is a first witness for $NF^*(N_0, N_1, N_2, N^a_3)$ and $\langle N^b_\beta, d^b_\beta : \beta < \beta^* \rangle^\frown \langle N^b_{\beta^*} \rangle$ is a first witness for $NF^*(N_0, N_2, N_1, N^b_3)$. By Proposition 3.1.10, there is a rectangle $\{M_{\alpha,\beta} : \alpha \leq \alpha^*, \beta \leq \beta^*\}$ such that:

(1) $M_{\alpha,0} = N^a_\alpha$.
(2) $M_{0,\beta} = N^b_\beta$.
(3) $tp(d^a_\alpha, M_{\alpha,\beta}, M_{\alpha+1,\beta})$ does not fork over $M_{\alpha,0}$.
(4) $tp(d^b_\beta, M_{\alpha,\beta}, M_{\alpha,\beta+1})$ does not fork over $M_{0,\beta}$.

$$
\begin{tikzcd}
 & & & N_3^a \arrow[r, "f^a"] & N_3^{a,*} \arrow[r, "g^a"] & N^* \\
N_1 = N_{1,\alpha^*} \arrow[urrr, "id"] \arrow[rrrr, "id"] \arrow[drrrrr, "id"] & & & & M_{\alpha^*,\beta^*} \arrow[u, "id"] \arrow[r, "id"] & N_3^{b,*} \arrow[u, "g^b"'] \\
d_\alpha^a \in N_{\alpha+1}^a \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha+1,\beta} \arrow[r, "id"] & M_{\alpha+1,\beta+1} \arrow[urr, "id"] & & & N_3^b \arrow[u, "f^b"'] \\
N_\alpha^a = M_{\alpha,0} \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha,\beta} \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha,\beta+1} \arrow[u, "id"] & & & \\
N_1^a = M_{1,0} \arrow[u, "id"] \arrow[r, "id"] & M_{1,\beta} \arrow[u, "id"] \arrow[r, "id"] & M_{1,\beta+1} \arrow[u, "id"] & & & \\
N_0 = M_{0,0} \arrow[u, "id"] \arrow[r, "id"] & N_\beta^b = M_{0,\beta} \arrow[u, "id"] \arrow[r, "id"] & N_{\beta+1}^b \arrow[u, "id"] \arrow[rr, "id"] & & N_2 = M_{0,\beta^*} \arrow[uuuuul, "id"] \arrow[uuuu, "id"] \arrow[uuur, "id"'] &
\end{tikzcd}
$$

By clauses 1,3, $\langle d_\alpha^a, N_\alpha^a : \alpha < \alpha^a \rangle$ is a first witness for $NF^*(N_0, N_1, N_2, M_{\alpha^*,\beta^*})$. But by definition this is also a first witness for $NF^*(N_0, N_1, N_2, N_3^a)$. So by Proposition 5.4.5, there is a joint embedding $(id_{M_{\alpha^*,\beta^*}}, f^a, N_3^{a,*})$ of $M_{\alpha^*,\beta^*}, N_3^a$ over $N_1 \bigcup N_2$. Similarly by clauses 2,4, there is a joint embedding $(id_{M_{\alpha^*,\beta^*}}, f^b, N_3^{b,*})$ of $M_{\alpha^*,\beta^*}, N_3^b$ over $N_1 \bigcup N_2$. Since $(K_\lambda, \preceq \restriction K_\lambda)$ satisfies the amalgamation property, there is an amalgamation (g^a, g^b, N_3) of $N_3^{a,*}, N_3^{b,*}$ over M_{α^*,β^*}. N_3 is an amalgam of N_3^a, N_3^b over $N_1 \bigcup N_2$. $\dashv$

Theorem 5.4.7 (weak uniqueness for NF)**.** *Suppose for $x = a, b$ $NF(M_0, M_1, M_2, M^x)$. Then there is a joint embedding of M^a, M^b over $M_1 \bigcup M_2$.*

Proof. First note that since $M_1 \bigcap M_2 = M_0$, the conclusion of the theorem is equivalent to $(id_{M_1}, id_{M_2}, M^a) E_{M_0} (id_{M_1}, id_{M_2}, M^b)$.

Case a: $NF^*(M_0, M_1, M_2, M^x)$ and M_2 is decomposable over M_0. In this case, by Theorem 5.3.7.b (the existence theorem for NF) there is M^c such that $NF^*(M_0, M_2, M_1, M^c)$. By Proposition 5.4.6 for $x = a, b$ $(id_{M_1}, id_{M_2}, M^x) E_{M_0} (id_{M_1}, id_{M_2}, M^c)$. But the relation E_{M_0} is an equivalence relation, so it is transitive.

The general case: Since $NF(M_0, M_1, M_2, M^a)$ by Proposition 5.3.9.5, there are $N_1^a, N^{a,-}$ such that $NF^*(M_0, N_1^a, M_2, N^{a,-})$ and $M_1 \preceq N_1^a \preceq N^{a,-} \wedge M^a \preceq N^{a,-}$. Similarly there are $N_1^b, N^{b,-}$ such that $NF^*(M_0, N_1^b, M_2, N_3^b)$ and $M_1 \preceq N_1^b \preceq N^{b,-} \wedge M^b \preceq N^{b,-}$. By Theorem 3.2.3 (the extensions decomposition theorem), there is a model $M_2^+ \succeq M_2$ which is decomposable over M_0. Without loss of generality for $x = a, b$, $M_2^+ \bigcap N^{x,-} = M_2$.

So by Theorem 3.2.3.3 (the extensions decomposition theorem), there is $N^x \succeq N^{x,-}$ such that $NF^*(M_0, N_1^x, M_2^+, N^x)$.

$$\begin{tikzcd}
 & & N^{a,+} & \\
 & N_1 \arrow[ur, "g_1^a"] \arrow[rr, "g_1^b"] & & N^{b,+} \\
N_1^a \arrow[ur, "f_1^a"] \arrow[rr, "id"] & & N^a \arrow[uu, "g^a"] & \\
 & N_1^b \arrow[uu, "f_1^b"] \arrow[rr, "id"] & & N^b \arrow[uu, "g^b"] \\
 & M^a \arrow[ur] & & \\
M_1 \arrow[uuu, "id"] \arrow[uur, "id"] \arrow[ur, "id"] \arrow[rrr, "id"] & & & M^b \arrow[uu, "id"] \\
M_0 \arrow[u, "id"] \arrow[rr, "id"] & & M_2^+ \arrow[uuuu, "id"] \arrow[uuur, "id"] &
\end{tikzcd}$$

By Proposition 3.2.4 there is an amalgamation (f_1^a, f_1^b, N_1) of N_1^a, N_1^b over M_1 such that N_1 is decomposable over N_1^a and over N_1^b. Hence for $x = a, b$ there is an amalgamation $(g_1^x, g^x, N^{x,+})$ of N_1, N^x over N_1^x such that $NF^*(g_1^x \circ f_1^x[N_1^x], g_1^x[N_1], g^x[N^x], N^{x,+})$. So for $x = a, b$ by Proposition 5.3.9.8 (a private case of transitivity), since $NF^*(M_0, N_1^x, M_2^+, N^x)$ and $NF^*(N_1^x, N^x, N_1, N^{x,+})$ it follows that $NF^*(M_0, N_1, M_2^+, N^{x,+})$. So by case a $(g_1^a, g^a \restriction M_2^+, N^{a,+})E_{M_0}(g_1^b, g^b \restriction M_2^+, N^{b,+}$. Therefore $(g_1^a \restriction M_1, g^a \restriction M_2, N^{a,+})E_{M_0}(g_1^b \restriction M_1, g^b \restriction M_2, N^{b,+})$. $\dashv$

Proposition 5.4.8. $\bigotimes_{NF}^-$.

Proof. We have to check clauses 1,2,3 of Definition 5.4.1.

1. Trivial.
2. By Theorem 5.4.7.
3. Suppose $NF(M_0, M_1, M_2, M_3)$ and $f : M_2 \to M_4$ is an embedding. We have to find a model M_5 and an embedding $g : M_3 \to M_5$ over M_2 such that $NF(f[M_0], g[M_1], M_4, M_5)$. By Theorem 5.3.9.2, we can find N_1, N_3 such that $NF^*(M_1, N_1, M_2, N_3)$ and $M_1 \preceq N_1 \preceq N_3 \wedge M_3 \preceq N_3$. By Theorem 5.3.7.b (the existence theorem for NF, we can find a model M_5 with $M_4 \preceq M_5$ and an embedding $h : N_3 \to M_5$ such that $NF^*(M_0, M_4, N_1, M_5)$. Hence $NF(M_0, M_1, M_4, M_5)$. Now we define $g := h \restriction M_3$. $\dashv$

Theorem 5.4.9 (the symmetry theorem)**.** $NF(N_0, N_1, N_2, N_3) \Leftrightarrow NF(N_0, N_2, N_1, N_3)$.

Proof. By monotonicity of NF, i.e., Proposition 5.3.9.3, it is sufficient to prove $NF^*(N_0, N_1, N_2, N_3) \Rightarrow NF(N_0, N_2, N_1, N_3)$. Suppose $NF^*(N_0, N_1, N_2, N_3)$. By Theorem 3.2.3 (the extensions decomposition theorem), there is $N_2^+ \succeq N_2$ which is decomposable over N_0. By Theorem 5.3.7.b, there is an amalgamation (id_{N_1}, f, N_3^+) of N_1, N_2^+ over N_2 such that $NF^*(N_0, N_1, f[N_2^+], N_3^+)$. So $N_1 \bigcap f[N_2^+] = N_0$. Hence by Theorem 5.3.7.b, there is a model N^* such that $NF^*(N_0, f[N_2^+], N_1, N^*)$. By Proposition 5.4.6 (the opposite uniqueness proposition) there is a joint embedding $id_{N_3^+}, g, N^{**}$ of N_3^+ and N^* over $N_1 \bigcup f[N_2^+]$. Since NF^* is closed under isomorphisms, $NF^*(N_0, f[N_2^+], N_1, g[N^*])$. Now we have to use the monotonicity of NF twice. Since $N_0 \preceq N_2 \preceq f[N_2^*]$, it follows that $NF^*(N_0, N_2, N_1, g[N^*])$. Since $N_3 \preceq N_3^* \preceq N^{**}$ and $g[N^*] \preceq N^{**}$, it follows that $NF(N_0, N_2, N_1, N_3)$. $\dashv$

Theorem 5.4.10. *NF respects $\mathfrak{s}$ (see Definition 5.2.6).*

Proof. Suppose $NF(M_0, M_1, M_2, M_3)$, $tp(a, M_0, M_1) \in S^{bs}(M_0)$. We must prove that $tp(a, M_2, M_3)$ does not fork over M_0. Without loss of generality, $NF^*(M_0, M_1, M_2, M_3)$ [Why? see Definition 2.1.1.3.b (monotonicity)]. By the definition of NF^*, M_1 is decomposable over M_0. By Theorem 5.3.7.c (the existence theorem for NF), there is M_3^* such that $NF^*(M_0, M_1, M_2, M_3^*)$ and the first element in the first witness is a.

$$\begin{tikzcd}
 & M_3^* & \\
a \in M_1 \arrow[ur, "id"] \arrow[rr, "id"] & & M_3 \\
M_0 \arrow[u, "id"] \arrow[r, "id"] & M_2 \arrow[uu, "id"] \arrow[ur, "id"] &
\end{tikzcd}$$

By the definition of a first witness, $tp(a, M_2, M_3^*)$ does not fork over M_0. By the weak uniqueness theorem (Theorem 5.4.7), there are f, M_3^{**} such that $M_3 \preceq M_3^{**}$, and $f : M_3^* \to M_3^{**}$ is an embedding over $M_1 \bigcup M_2$. So $tp(a, M_2, M_3) = tp(a, M_2, f[M_3^*]) = tp(a, M_2, M_3^*)$ does not fork over M_0. $\dashv$

5.5. **Long transitivity.** Now we prove the long transitivity for NF. We are going to use decompositions of models of cardinality λ^+ in the definition of a new relation, $\preceq_{\lambda^+}^{NF}$ on K_{λ^+}. Long transitivity is applied on beginnings of the decompositions, $\langle a_{x,\varepsilon} : \varepsilon \leq \alpha^* \rangle$,$\langle b_{x,\varepsilon} : \varepsilon \leq \alpha^* \rangle$. In particular, long transitivity is used in the proofs of the properties of the relations $\widehat{NF}$ (see Proposition 6.1.3) and $\preceq_{\lambda^+}^{NF}$ (see Proposition 6.1.6) and in the proofs of Propositions ,7.1.7,8.1.4.

Theorem 5.5.1 (long transitivity for NF)**.** *For $x = a, b$, let $\langle M_{x,\varepsilon} : \varepsilon \leq \alpha^* \rangle$ be a $\prec$-increasing continuous sequence of models in K_λ. Suppose $\varepsilon < \alpha^* \Rightarrow NF(M_{a,\varepsilon}, M_{a,\varepsilon+1}, M_{b,\varepsilon}, M_{b,\varepsilon+1})$. Then $NF(M_{a,0}, M_{a,\alpha^*}, M_{b,0}, M_{b,\alpha^*})$.*

Similarly to the proof of Proposition 2.5.6 (the transitivity proposition), we use the existence and weak uniqueness theorems to prove the long transitivity. But here the proof is more complicated, and it is divided into four cases, each one is based on its predecessor and generalizes it.

Proof. Case a: $\varepsilon < \alpha^* \Rightarrow NF^*(M_{a,\varepsilon}, M_{a,\varepsilon+1}, M_{b,\varepsilon}, M_{b,\varepsilon+1})$. Concatenate all the sequences together.

In the other cases we are going to use the following claim:

Claim 5.5.2. *It is enough to find $(N_{b,\varepsilon}, f_\varepsilon)$ for $\varepsilon \leq \alpha^*$ such that:*

(1) $M_{b,0} \preceq N_{b,0}$.
(2) $\langle N_{b,\varepsilon} : \varepsilon \leq \alpha^* \rangle$ *is an increasing continuous sequence of models in* K_λ.
(3) f_ε *is an embedding of* $M_{a,\varepsilon}$ *to* $N_{b,\varepsilon}$.
(4) $f_0 = id_{M_{a,0}}$.
(5) $\langle f_\varepsilon : \varepsilon \leq \alpha^* \rangle$ *is an increasing continuous sequence.*
(6) For $\varepsilon < \alpha^*$, $NF(f_\varepsilon[M_{a,\varepsilon}], f_{\varepsilon+1}[M_{a,\varepsilon+1}], N_{b,\varepsilon}, N_{b,\varepsilon+1})$.
(7) $NF(M_{a,0}, f_{\alpha^*}[M_{a,\alpha^*}], N_{b,0}, N_{b,\alpha^*})$.

Proof. Suppose we found $(N_{b,\varepsilon}, f_\varepsilon)$ for $\varepsilon \leq \alpha^*$ such that clauses 1-7 are satisfied. By Proposition 5.4.8, $\bigotimes_{NF}^-$. Therefore by Proposition 5.4.4 (the transitivity of the uniqueness), $(id_{M_{a,\alpha^*}}, id_{M_{b,0}}, M_{b,\alpha^*}) E_{M_{a,0}} (f^a_{\alpha^*}, id_{M_{b,0}}, N_{b,\alpha^*})$ [Substitute $\langle M_{a,\varepsilon} : \varepsilon \leq \alpha^* \rangle$, $\langle M_{b,\varepsilon} : \varepsilon \leq \alpha^* \rangle$, $\langle N_{b,\varepsilon} : \varepsilon \leq \alpha^* \rangle$, $\langle id_{M_{a,\varepsilon}} : \varepsilon \leq \alpha^* \rangle$, $\langle f_\varepsilon : \varepsilon \leq \alpha^* \rangle$ in place of $\langle N_{1,\alpha} : \alpha \leq \alpha^* \rangle$, $\langle N^a_{2,\alpha} : \alpha \leq \alpha^* \rangle$, $\langle N^b_{2,\alpha} : \alpha \leq \alpha^* \rangle$, $\langle f^a_\alpha : \alpha \leq \alpha^* \rangle$, $\langle f^b_\alpha : \alpha \leq \alpha^* \rangle$] . By clause 7, $NF(M_{a,0}, M_{a,\alpha^*}, N_{b,0}, N_{b,\alpha^*})$. So by Proposition 5.3.4 $NF(M_{a,0}, M_{a,\alpha^*}, M_{b,0}, M_{b,\alpha^*})$. $\dashv$

Case b: For every ε, $M_{a,\varepsilon+1}$ is *decomposable* over $M_{a,\varepsilon}$. In this case, we choose $(N_{b,\varepsilon}, f_\varepsilon)$ such that clauses 1-6 of Claim 5.5.2 are satisfied: For $\varepsilon = 0$, we define $N_{b,0} := M_{b,0}$. In the successor step, we use Theorem 5.3.7.a. For ε limit, we define $N_{b,\varepsilon} := \bigcup \{N_{b,\zeta} : \zeta < \varepsilon\}$, $f_\varepsilon := \bigcup \{f_\zeta : \zeta < \varepsilon\}$. Now clause 7 is satisfied by case a of the proof.

Case c: $\alpha^* \leq \omega$. In this case we apply Claim 5.5.2 with $f_\varepsilon = id_{M_{a,\varepsilon}}$.

$$\begin{tikzcd}
N_{b,0} \arrow[r, "id"] & N_{b,1} \arrow[r, "id"] & N_{b,2} \arrow[r, "id"] & N_{b,\varepsilon} \arrow[r, "id"] & N_{b,\varepsilon+1} \arrow[r, "id"] & N_{b,\alpha^*} \\
M_{b,0} \arrow[u, "id"] & & & & & \\
M_{a,0} \arrow[u, "id"] \arrow[r, "id"] & N_{a,1} \arrow[uu, "id"] \arrow[r, "id"] & N_{a,2} \arrow[uu, "id"] \arrow[r, "id"] & N_{a,\varepsilon} \arrow[uu, "id"] \arrow[r, "id"] & N_{a,\varepsilon+1} \arrow[uu, "id"] \arrow[r, "id"] & N_{a,\alpha^*} \arrow[uu, "id"] \\
M_{a,0} \arrow[u, "id"] \arrow[r, "id"] & M_{a,1} \arrow[u, "id"] \arrow[r, "id"] & M_{a,2} \arrow[u, "id"] \arrow[r, "id"] & M_{a,\varepsilon} \arrow[u, "id"] \arrow[r, "id"] & M_{a,\varepsilon+1} \arrow[u, "id"] \arrow[r, "id"] & M_{a,\alpha^*} \arrow[u, "id"]
\end{tikzcd}$$

By Proposition 5.5.3.a (see below), there is an increasing continuous sequence
$\langle N_{a,\varepsilon} : \varepsilon \leq \alpha^* \rangle$ such that: $N_{a,0} = M_{a,0}$, $M_{a,\varepsilon} \preceq N_{a,\varepsilon}$, $N_{a,\varepsilon+1}$ is decomposable over $N_{a,\varepsilon}$ and over $M_{a,\varepsilon+1}$ and $\varepsilon < \alpha^* \Rightarrow NF(M_{a,\varepsilon}, M_{a,\varepsilon+1}, N_{a,\varepsilon}, N_{a,\varepsilon+1})$. Since $\alpha^* \leq \omega$, by Proposition 5.5.3.b (see below), there is an increasing continuous sequence $\langle N_{b,\varepsilon} : \varepsilon \leq \alpha^* \rangle$ such that $N_{b,0} \succ M_{b,0}$, for $\varepsilon \leq \alpha^*$ $N_{b,\varepsilon}$ is decomposable over $N_{a,\varepsilon}$ and $NF^*(N_{a,\varepsilon}, N_{a,\varepsilon+1}, N_{b,\varepsilon}, N_{b,\varepsilon+1})$.

Now it is enough to prove that $\langle (N_{b,\varepsilon}, id_{M_{a,\varepsilon}}) : \varepsilon \leq \alpha^* \rangle$ satisfies clauses 1-7 of Claim 5.5.2. Clauses 1-5 are satisfied trivially. We check clauses 6,7.

6. First assume $\varepsilon > 0$. As $NF(M_{a,\varepsilon}, M_{a,\varepsilon+1}, N_{a,\varepsilon}, N_{a,\varepsilon+1})$, $NF(N_{a,\varepsilon}, N_{a,\varepsilon+1}, N_{b,\varepsilon}, N_{b,\varepsilon+1})$, $N_{a,\varepsilon}$ is decomposable over $M_{a,\varepsilon}$, and $N_{b,\varepsilon}$ is decomposable over $N_{a,\varepsilon}$, by case b (for $\alpha^* = 2$), $NF(M_{a,\varepsilon}, M_{a,\varepsilon+1}, N_{b,\varepsilon}, N_{b,\varepsilon+1})$. Secondly assume $\varepsilon = 0$. As $NF(N_{a,0}, N_{a,1}, N_{b,0}, N_{b,1})$, $N_{a,0} = M_{a,0}$ and $M_{a,1} \preceq N_{a,1}$, by the monotonicity of NF, $NF(M_{a,0}, M_{a,1}, N_{b,0}, N_{b,1})$

7. By case b, we have $NF(N_{a,0}, N_{a,\alpha^*}, N_{b,0}, N_{b,\alpha^*})$. By smoothness $M_{a,\alpha^*} \preceq N_{a,\alpha^*}$. So by the monotonicity of NF, $NF(M_{a,0}, M_{a,\alpha^*}, N_{b,0}, N_{b,\alpha^*})$.

The general case: By the proof of case c. We have only one problem: For ε limit, it is not clear why does $NF(M_{a,\varepsilon}, M_{a,\varepsilon+1}, N_{b,\varepsilon}, N_{b,\varepsilon+1})$, where we know $NF(M_{a,\varepsilon}, M_{a,\varepsilon+1}, N_{a,\varepsilon}, N_{a,\varepsilon+1}) \wedge NF(N_{a,\varepsilon}, N_{a,\varepsilon+1}, N_{b,\varepsilon}, N_{b,\varepsilon+1})$. Here we cannot use case b, because we do not know if $N_{b,\varepsilon}$ is decomposable over $N_{a,\varepsilon}$ and $N_{a,\varepsilon}$ is decomposable over $M_{a,\varepsilon}$. But we can use case c with $\alpha^* = 2$. $\dashv$

Proposition 5.5.3. *Let $\alpha^* \leq \lambda^+$. Let $\langle M_\varepsilon : \varepsilon \leq \alpha^* \rangle$ be a $\prec$-increasing continuous sequence of models in K such that for each $\varepsilon \leq \alpha^*$, if $\varepsilon < \lambda^+$ then M_ε is of cardinality λ (but if $\alpha^* = \lambda^+$ then the last model is of cardinality λ^+).*

(a) There is a $\prec$-increasing continuous sequence of models in K $\langle N_\varepsilon : \varepsilon \leq \alpha^* \rangle$ such that: $N_0 = M_0$, $M_\varepsilon \preceq N_\varepsilon$, $NF(M_\varepsilon, M_{\varepsilon+1}, N_\varepsilon, N_{\varepsilon+1})$ and $N_{\varepsilon+1}$ is decomposable over N_ε and over $M_{\varepsilon+1}$.

(b) Suppose $M^* \in K_\lambda$, $M^* \succ M_0$ and $M^* \bigcap M_{\alpha^*} = M_0$. Then there is a $\prec$-increasing continuous sequence of models in K $\langle N_\varepsilon : \varepsilon \leq \alpha^* \rangle$ such that:

$M^* \preceq N_0$, $M_\varepsilon \preceq N_\varepsilon$, $NF(M_\varepsilon, M_{\varepsilon+1}, N_\varepsilon, N_{\varepsilon+1})$, N_0 is decomposable over M_0 and $N_{\varepsilon+1}$ is decomposable over N_ε and over $M_{\varepsilon+1}$.

Proof. (a) We choose a pair $(N_\varepsilon, f_\varepsilon)$ by induction on $\varepsilon \leq \alpha^*$ such that:

(1) $\langle N_\varepsilon : \varepsilon \leq \alpha^* \rangle$ is an increasing continuous sequence of models in K_λ.
(2) $f_\varepsilon : M_\varepsilon \to N_\varepsilon$ is an embedding.
(3) $f_0 = id_{M_0}$.
(4) The sequence $\langle f_\varepsilon : \varepsilon \leq \alpha^* \rangle$ is increasing and continuous.
(5) For $\varepsilon < \alpha^*$, $NF(f_\varepsilon[M_\varepsilon], N_\varepsilon, f_{\varepsilon+1}[M_{\varepsilon+1}], N_{\varepsilon+1})$.
(6) For $\varepsilon < \alpha^*$, $N_{\varepsilon+1}$ is decomposable over N_ε and over $f_{\varepsilon+1}[M_{\varepsilon+1}]$.

Why can we carry out this construction? For $\varepsilon = 0$ or limit there is no problem. Suppose we chose $(N_\varepsilon, f_\varepsilon)$, how will we choose $(N_{\varepsilon+1}, f_{\varepsilon+1})$? By Theorem 5.3.7.a we can find $N^-_{\varepsilon+1}$ and $f_{\varepsilon+1}$ such that $NF(f_\varepsilon[M_\varepsilon], N_\varepsilon, f_{\varepsilon+1}[M_{\varepsilon+1}], N^-_{\varepsilon+1})$. Now by Proposition 3.2.4, we can find $N_{\varepsilon+1}$ such that $N^-_{\varepsilon+1} \preceq N_{\varepsilon+1}$ and $N_{\varepsilon+1}$ is decomposable over N_ε and over $f_{\varepsilon+1}[M_{\varepsilon+1}]$. Therefore we can carry out this construction.

Now, as in the proof of Proposition 3.1.8, without loss of generality, $f_\varepsilon = id_{M_\varepsilon}$ for every $\varepsilon \leq \alpha^*$ (because we can extend $f^{-1}_{\alpha^*}$ to a bijection g of N_{α^*} and take the sequence $\langle g[N_\varepsilon] : \varepsilon \leq \alpha^* \rangle$).

(b) It demands a tiny change in the proof: In the construction, $M^* \preceq N_0$ and it is decomposable over M_0. $\dashv$

Theorem 5.5.4. *$NF = NF_\lambda$ is the unique relation which satisfies $\bigotimes_{NF}$ and respects $\mathfrak{s}$.*

Proof. NF satisfies $\bigotimes_{NF}$: Clause a is clear. Clause b (the monotonicity) by Theorem 5.3.9.4. Clause c (the existence) by Theorem 5.3.7.a. Clause d (weak uniqueness) by Theorem 5.4.7. Clause e (symmetry) by Theorem 5.4.9. Clause f (long transitivity) by Theorem 5.5.1. By Theorem 5.4.10 NF respects $\mathfrak{s}$.

Suppose the relation R satisfies $\bigotimes_R$ and respects $\mathfrak{s}$. First we prove $NF(M_0, M_1, M_2, M_3) \Rightarrow R(M_0, M_1, M_2, M_3)$.

Case a: There is $a \in M_1 - M_0$ with $(M_0, M_1, a) \in K^{3,uq}$. Since NF respects $\mathfrak{s}$, $tp(a, M_2, M_3)$ does not fork over M_0. So since R respects $\mathfrak{s}$, by the definition of unique triples (see Definition 4.1.5), $R(M_0, M_1, M_2, M_3)$.

Case b: $NF^*(M_0, M_1, M_2, M_3)$. Since R satisfies long transitivity, and by Case a, $R(M_0, M_1, M_2, M_3)$.

The general case: Since R satisfies monotonicity, by Case b, $R(M_0, M_1, M_2, M_3)$. So we have proved that the relation NF is included in the relation R.

Conversely: Suppose $R(M_0, M_1, M_2, M_3)$. We have to prove that $NF(M_0, M_1, M_2, M_3)$. Since $\bigotimes_R$, R satisfies disjointness. So $M_1 \bigcap M_2 = M_0$. By $\bigotimes_{NF}$, for some model M_4 $NF(M_0, M_1, M_2, M_4)$. But by the first direction of the proof, $NF(M_0, M_1, M_2, M_4) \Rightarrow R(M_0, M_1, M_2, M_4)$, so $R(M_0, M_1, M_2, M_4)$. Since $\bigotimes_R$, R satisfies weak uniqueness, $R(M_0, M_1, M_2, M_3)$ and $R(M_0,$

$M_1, M_2, M_4)$, it follows that $(id_{M_1}, id_{M_2}, M_3) E_{M_0} (id_{M_1}, id_{M_2}, M_4)$. Therefore by Proposition 5.3.4 $NF(M_0, M_1, M_2, M_4)$ implies $NF(M_0, M_1, M_2, M_3)$, so $NF(M_0, M_1, M_2, M_3)$, as required. $\dashv$

6. A relation on K_{λ^+} that is based on the relation NF

6.1. **Introduction.** Recall that we want to derive from $\mathfrak{s}$ a good λ^+-frame. So first we have to define an AEC in λ^+ with amalgamation. Definition 6.1.4 presents the strong submodel relation on models of this AEC in λ^+.

Hypothesis 6.1.1. $\mathfrak{s}$ is a semi-good λ-frame with conjugation and $K^{3,uq}$ satisfies the existence property.

We will now define a notion for: a model of size λ is independent from a model of size λ^+ over a model of size λ in a model of size λ^+.

Definition 6.1.2. Define a 4-place relation $\widehat{NF}$ on K by $\widehat{NF}(N_0, N_1, M_0, M_1)$ iff the following hold:

(1) $n < 2 \Rightarrow N_n \in K_\lambda,\ M_n \in K_{\lambda^+}$.
(2) There is a pair of increasing continuous sequences $\langle N_{0,\alpha} : \alpha < \lambda^+\rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+\rangle$ such that for every α, $NF(N_{0,\alpha}, N_{1,\alpha}, N_{0,\alpha+1}, N_{1,\alpha+1})$ and for $n < 2$, $N_{0,n} = N_n,\ M_n = \bigcup\{N_{n,\alpha} : \alpha < \lambda^+\}$.

Theorem 6.1.3 (the $\widehat{NF}$-properties)**.**

(a) Disjointness: If $\widehat{NF}(N_0, N_1, M_0, M_1)$ then $N_1 \bigcap M_0 = N_0$.
(b) Monotonicity: Suppose $\widehat{NF}(N_0, N_1, M_0, M_1)$, $N_0 \preceq N_1^* \preceq N_1$, $N_1^* \bigcup M_0 \subseteq M_1^* \preceq M_1$ and $M_1^* \in K_{\lambda^+}$. Then $\widehat{NF}(N_0, N_1^*, M_0, M_1^*)$.
(c) Existence: Suppose $n < 2 \Rightarrow N_n \in K_\lambda$, $M_0 \in K_{\lambda^+}$, $N_0 \preceq N_1$, $N_0 \preceq M_0$, $N_1 \bigcap M_0 = N_0$. Then there is a model M_1 such that $\widehat{NF}(N_0, N_1, M_0, M_1)$.
(d) Weak Uniqueness: If $n < 2 \Rightarrow \widehat{NF}(N_0, N_1, M_0, M_{1,n})$, then there are M, f_0, f_1 such that f_n is an embedding of $M_{1,n}$ into M over $N_1 \bigcup M_0$.
(e) Respecting the frame: Suppose $\widehat{NF}(N_0, N_1, M_0, M_1), tp(a, N_0, M_0) \in S^{bs}(N_0)$. Then $tp(a, N_1, M_1)$ does not fork over N_0.

Proof. (a) Disjointness: Let $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{1,\varepsilon} : \varepsilon < \lambda^+\rangle$ be witnesses for $\widehat{NF}(N_0, N_1, M_0, M_1)$. Especially $\varepsilon < \lambda^+ \Rightarrow NF(N_{0,\varepsilon}, N_{1,\varepsilon}, N_{0,\varepsilon+1}, N_{1,\varepsilon+1})$. So by Proposition 5.3.6.1 $\varepsilon < \lambda^+ \Rightarrow N_{1,\varepsilon} \bigcap N_{0,\varepsilon+1} = N_{0,\varepsilon}$. So by the end of the proof of Theorem 3.3.5, $N_1 \bigcap M_0 = N_0$. Let $x \in N_1 \bigcap M_0$. So there is $\varepsilon < \lambda^+$ such that $x \in N_{0,\varepsilon}$. Denote $\varepsilon := \mathrm{Min}\{\varepsilon < \lambda^+ : x \in N_{0,\varepsilon}\}$. ε cannot be a limit ordinal as the sequence $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$ is continuous. If $\varepsilon = \zeta + 1$, then $x \in N_{0,\zeta+1} \bigcap N_1 \subseteq N_{0,\zeta+1} \bigcap N_{1,\zeta} = N_{0,\zeta}$, in contradiction to the minimality of ε. So ε must be equal to 0. Hence $x \in N_{0,0} = N_0$.
(b) Monotonicity: Let $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{1,\varepsilon} : \varepsilon < \lambda^+\rangle$ be witnesses for $\widehat{NF}(N_0, N_1, M_0, M_1)$. Let E be a club of λ^+ such that $0 \notin E$ and $\varepsilon \in E \Rightarrow N_{1,\varepsilon} \bigcap M_1^* \preceq N_{1,\varepsilon}$ [Why do we have such a club? Let E be a

club of λ^+ such that $0 \notin E$ and $\varepsilon \in E \Rightarrow N_{1,\varepsilon} \bigcap M_1^* \preceq M_1^*$. By the assumption, $M_1^* \preceq M_1$. So $\varepsilon \in E \Rightarrow N_{1,\varepsilon} \bigcap M_1^* \preceq M_1$. Now as $N_{1,\varepsilon} \preceq M_1$, by Axiom 1.0.3.1.e, $\varepsilon \in E \Rightarrow N_{1,\varepsilon} \bigcap M_1^* \preceq N_{1,\varepsilon}$]. We will prove that the sequences $\langle N_0 \rangle^\frown \langle N_{0,\varepsilon} : \varepsilon \in E \rangle$, $\langle N_1^* \rangle^\frown \langle N_{1,\varepsilon} \bigcap M_1^* : \varepsilon \in E \rangle$ witness that $\widehat{NF}(N_0, N_1^*, M_0, M_1^*)$. First, they are increasing [Why $\varepsilon < \zeta \wedge \{\varepsilon, \zeta\} \subseteq E \Rightarrow N_{1,\varepsilon} \bigcap M_1^* \preceq N_{1,\zeta} \bigcap M_1^*$? By the properties of E, $N_{1,\varepsilon} \bigcap M_1^* \preceq N_{1,\varepsilon}$. But $N_\varepsilon \preceq N_\zeta$. So $N_{1,\varepsilon} \bigcap M_1^* \preceq N_{1,\zeta}$. On the other hand, again by the properties of E, $N_{1,\varepsilon} \bigcap M_1^* \subseteq N_{1,\zeta} \bigcap M_1^* \preceq N_{1,\zeta}$. So by Axiom 1.0.3.1.e $N_{1,\varepsilon} \bigcap M_1^* \preceq N_{1,\zeta} \bigcap M_1^*$]. Secondly, we will prove that if $\varepsilon < \zeta$, $\{\varepsilon, \zeta\} \subseteq E$, then $NF(N_{0,\varepsilon}, N_{1,\varepsilon} \bigcap M_1^*, N_{0,\zeta}, N_{1,\zeta} \bigcap M_1^*)$. Fix such ε, ζ. By Theorem 5.5.1, (the long transitivity theorem), $NF(N_{0,\varepsilon}, N_{1,\varepsilon}, N_{0,\zeta}, N_{1,\zeta})$. By the properties of E and Axiom 1.0.3.1.e, $N_{0,\varepsilon} \preceq N_{1,\varepsilon} \bigcap M_1^* \preceq N_{1,\varepsilon}$, $N_{0,\zeta} \bigcup (N_{1,\varepsilon} \bigcap M_1^*) \subseteq N_{1,\zeta} \bigcap M_1^* \preceq N_{1,\zeta}$. Now by Theorem 5.3.9.5 (the monotonicity of NF), we have $NF(N_{0,\varepsilon}, N_{1,\varepsilon} \bigcap M_1^*, N_{0,\zeta}, N_{1,\zeta} \bigcap M_1^*)$.
(c) Existence: By Proposition 5.5.3.b.
(d) Weak Uniqueness: Since $\bigotimes_{NF}$ holds, it follows by Proposition 5.4.3.2 and Proposition 5.4.4. But we give another proof using Section 7: By Proposition 7.1.12.f, there is a model $M_{1,n}^+$ such that $M_{1,n} \prec_{\lambda^+}^+ M_{1,n}^+$. By Theorem 7.1.13.c, there is an isomorphism $f : M_{1,1}^+ \to M_{1,2}^+$ over $M_0 \bigcup N_1$. So $M_{1,2}^+, id_{M_{1,2}}, f \restriction M_{1,1}$ is a witness, as required.
(e) Let $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+ \rangle$, $\langle N_{1,\varepsilon} : \varepsilon < \lambda^+ \rangle$ be a witness for $\widehat{NF}(N_0, N_1, M_0, M_1)$. There is ε such that $a \in N_{0,\varepsilon}$. By Definition 6.1.2 (the definition of $\widehat{NF}$), we have $NF(N_0, N_1, N_{0,\varepsilon}, N_{1,\varepsilon})$. So the proposition is satisfied by Theorem 5.4.10 (the relation NF respects the frame). $\dashv$

Now we define a relation $\preceq_{\lambda^+}^{NF}$ on K_{λ^+}, that is based on the relation $\widehat{NF}$:

Definition 6.1.4. Suppose $M_0, M_1 \in K_{\lambda^+}$, $M_0 \preceq M_1$. Then $M_0 \preceq_{\lambda^+}^{NF} M_1$ when: there are $N_0, N_1 \in K_\lambda$ such that $\widehat{NF}(N_0, N_1, M_0, M_1)$.

Remark 6.1.5. If $M_0 \preceq_{\lambda^+}^{NF} M_1$ then $M_0 \preceq M_1$.

Proposition 6.1.6. $(K_{\lambda^+}, \preceq_{\lambda^+}^{NF})$ *satisfies the following properties:*

(a) *Suppose $M_0 \preceq M_1$, $n < 2 \Rightarrow M_n \in K_{\lambda^+}$. For $n < 2$, let $\langle N_{n,\varepsilon} : \varepsilon < \lambda^+ \rangle$ be a representation of M_n. Then $M_0 \preceq_{\lambda^+}^{NF} M_1$ iff there is a club $E \subseteq \lambda^+$ such that $(\varepsilon < \zeta \wedge \{\varepsilon, \zeta\} \subseteq E) \Rightarrow NF(N_{0,\varepsilon}, N_{0,\zeta}, N_{1,\varepsilon}, N_{1,\zeta})$.*
(b) *$\preceq_{\lambda^+}^{NF}$ is a partial order.*
(c) *If $M_0 \preceq M_1 \preceq M_2$ and $M_0 \preceq_{\lambda^+}^{NF} M_2$ then $M_0 \preceq_{\lambda^+}^{NF} M_1$.*
(d) *$(K_{\lambda^+}, \preceq_{\lambda^+}^{NF})$ satisfies Axiom c of AEC in λ^+, i.e.: If $\delta \in \lambda^{+2}$ is a limit ordinal and $\langle M_\alpha : \alpha < \delta \rangle$ is a $\preceq_{\lambda^+}^{NF}$-increasing continuous sequence, then $M_0 \preceq_{\lambda^+}^{NF} \bigcup \{M_\alpha : \alpha < \delta\}$ and obviously it is $\in K_{\lambda^+}$.*
(e) *K_{λ^+} has no $\preceq_{\lambda^+}^{NF}$-maximal model.*
(f) *If $(K_{\lambda^+}, \preceq_{\lambda^+}^{NF})$ satisfies smoothness (Definition 1.0.3.1.d), then it is an AEC in λ^+, (see Definition 1.0.3).*

(g) LST for $\preceq^{NF}_{\lambda^+}$: If $M_0 \preceq^{NF}_{\lambda^+} M_1$, $n < 2 \Rightarrow (A_n \subseteq M_n \wedge |A_n| \leq \lambda)$, then there are models $N_0, N_1 \in K_\lambda$ such that: $\widehat{NF}(N_0, N_1, M_0, M_1)$ and $n < 2 \Rightarrow A_n \subseteq N_n$.

Proof. (a) *One direction:* Let E be such a club. So $\langle N_{0,\varepsilon} : \varepsilon \in E\rangle$, $\langle N_{1,\varepsilon} : \varepsilon \in E\rangle$ witness that $M_0 \preceq^{NF}_{\lambda^+} M_1$ (Trace the definition of $\widehat{NF}$ (Definition 6.1.2) through the definition of NF (Definition 5.3.2) and NF^* (Definition 5.3.1) to see where the witnesses appear).

Conversely: Let $\langle M_{0,\alpha} : \alpha < \lambda^+\rangle$, $\langle M_{1,\alpha} : \alpha < \lambda^+\rangle$ be witnesses for $M_0 \preceq^{NF}_{\lambda^+} M_1$. Let E be a club such that $(n < 2 \wedge \varepsilon \in E) \Rightarrow M_{n,\alpha} = N_{n,\alpha}$. Suppose $\varepsilon < \zeta \wedge \{\varepsilon, \zeta\} \subseteq E$. We will prove $NF(N_{0,\varepsilon}, N_{1,\varepsilon}, N_{0,\zeta}, N_{1,\zeta})$, i.e., $NF(M_{0,\varepsilon}, M_{1,\varepsilon}, M_{0,\zeta}, M_{1,\zeta})$. The sequences $\langle M_{0,\alpha} : \varepsilon \leq \alpha \leq \zeta\rangle$, $\langle M_{1,\alpha} : \varepsilon \leq \alpha \leq \zeta\rangle$ are increasing and continuous. So by Theorem 5.5.1, (the long transitivity theorem) $NF(M_{0,\varepsilon}, M_{1,\varepsilon}, M_{0,\zeta}, M_{1,\zeta})$.

(b) The reflexivity is obvious. The antisymmetry is satisfied by the antisymmetry of the inclusion relation. The transitivity is satisfied by item a, Theorem 5.5.1 and the evidence that the intersection of two clubs is a club.

(c) For $n < 3$, let $\langle M_{n,\alpha} : \alpha < \lambda^+\rangle$ be a representation of M_n such that $\alpha < \lambda^+ \Rightarrow NF(M_{0,\alpha}, M_{0,\alpha+1}, M_{2,\alpha}, M_{2,\alpha+1})$. Let E be a club of λ^+ such that $\alpha \in E \Rightarrow M_{0,\alpha} \preceq M_{1,\alpha} \preceq M_{2,\alpha}$. By the monotonicity of NF, $\alpha \in E \Rightarrow NF(M_{0,\alpha}, M_{0,\alpha+1}, M_{1,\alpha}, M_{1,\alpha+1})$. The representations $\langle M_{0,\alpha} : \alpha \in E\rangle$, $\langle M_{1,\alpha} : \alpha \in E\rangle$ witness that $M_0 \preceq^{NF}_{\lambda^+} M_1$.

(d) Without loss of generality, $cf(\delta) = \delta$, so $\delta \leq \lambda^+$. Denote $M_\delta := \bigcup\{M_\alpha : \alpha < \delta\}$. For $\alpha < \delta$, let $\langle M_{\alpha,\varepsilon} : \varepsilon < \lambda^+\rangle$ be a representation of M_n. By item a for every α, there is a club $E_{\alpha,0} \subseteq \lambda^+$ such that $(\varepsilon < \zeta \wedge \{\varepsilon, \zeta\} \subseteq E_{\alpha,0}) \Rightarrow NF(M_{\alpha,\varepsilon}, M_{\alpha,\zeta}, M_{\alpha+1,\varepsilon}, M_{\alpha+1,\zeta})$. Let α be a limit ordinal. $\bigcup\{M_{\alpha,\varepsilon} : \varepsilon < \lambda^+\} = M_\alpha = \bigcup\{M_\beta : \beta < \alpha\} = \bigcup\{\bigcup\{M_{\beta,\varepsilon} : \varepsilon < \lambda^+\} : \beta < \alpha\} = \bigcup\{\bigcup\{M_{\beta,\varepsilon} : \beta < \alpha\} : \varepsilon < \lambda^+\}$. Every edge of this equivalences's sequence is a limit of a $\subseteq$-increasing continuous sequence of subsets of cardinality less than λ, and it is equal to M_α [Why is the sequence in the right edge, $\langle\bigcup\{M_{\beta,\varepsilon} : \beta < \alpha\} : \varepsilon < \lambda^+\rangle$ continuous? Let $\varepsilon < \lambda^+$ be a limit ordinal. Suppose $x \in \bigcup\{M_{\beta,\varepsilon} : \beta < \alpha\}$. Then there are ζ, β such that $x \in M_{\beta,\zeta}$. So $x \in \bigcup\{M_{\beta,\zeta} : \beta < \alpha\}$.] So there is a club $E_{\alpha,1} \subseteq \lambda^+$ such that $\varepsilon \in E_{\alpha,1} \Rightarrow M_{\alpha,\varepsilon} = \bigcup\{M_{\beta,\varepsilon} : \beta < \alpha\}$. For α limit, define $E_\alpha := E_{\alpha,0} \bigcap E_{\alpha,1}$, and for α not limit define $E_\alpha := E_{\alpha,0}$.

Case a: $\delta < \lambda^+$. Define $E := \bigcap\{E_\alpha : \alpha < \delta\}$. If $\varepsilon \in E$ then for $\alpha < \delta$, $\varepsilon \in E$, so $NF(M_{\alpha,\varepsilon}, M_{\alpha,Min(E-(\varepsilon+1))}, M_{\alpha+1,\varepsilon}, M_{\alpha+1,\mathrm{Min}(E-(\varepsilon+1))})$. So by Theorem 5.5.1 (the long transitivity theorem), $\varepsilon \in E \Rightarrow NF(M_{0,\varepsilon}, M_{0,\mathrm{Min}(E-(\varepsilon+1))}, M_{\delta,\varepsilon}, M_{\delta,\mathrm{Min}(E-(\varepsilon+1))})$. Hence $M_0 \preceq^{NF}_{\lambda^+} M_1$.

Case b: $\delta = \lambda^+$. Let $E := \{\varepsilon \in E : \varepsilon \text{ is a limit ordinal}, \alpha < \varepsilon \Rightarrow \varepsilon \in E_\alpha\}$. Denote $N_\varepsilon := \bigcup\{M_{\alpha,\varepsilon} : \alpha < \varepsilon\}$.

$$\begin{tikzcd}
M_0 \arrow[r, "id"] & M_\alpha \arrow[r, "id"] & M_\varepsilon \arrow[r, "id"] & M_\zeta \arrow[r, "id"] & M_{\lambda^+} \\
M_{0,\zeta} \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha,\zeta} \arrow[u, "id"] \arrow[r, "id"] & M_{\varepsilon,\zeta} \arrow[u, "id"] \arrow[r, "id"] & N_\zeta \arrow[u, "id"] & \\
M_{0,\varepsilon} \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha,\varepsilon} \arrow[u, "id"] \arrow[r, "id"] & N_\varepsilon \arrow[u, "id"] & & \\
M_{0,\alpha} \arrow[u, "id"] \arrow[r, "id"] & N_\alpha \arrow[u, "id"] & & & \\
M_{0,0} \arrow[u, "id"] & & & &
\end{tikzcd}$$

Claim 6.1.7. *For every $\varepsilon \in E$, the sequence $\langle M_{\alpha,\varepsilon} : \alpha < \varepsilon\rangle^\frown\langle N_\varepsilon\rangle$ is increasing and continuous (especially $N_\varepsilon \in K$).*

Proof. If $\varepsilon \in E$ is limit, then $\alpha < \varepsilon \Rightarrow \varepsilon \in E_{\alpha,1}$, so the sequence $\langle M_{\alpha,\varepsilon} : \alpha < \varepsilon\rangle$ is continuous. So it is sufficient to prove that $\alpha < \varepsilon \Rightarrow M_{\alpha,\varepsilon} \preceq M_{\alpha,\varepsilon+1}$. Suppose $\alpha < \varepsilon$. $\varepsilon \in E$, so $\varepsilon \in E_{\alpha,0}$. Hence $NF(M_{\alpha,\varepsilon}, M_{\alpha+1,\varepsilon}, M_{\alpha,\mathrm{Min}(E-(\varepsilon+1))}, M_{\alpha+1,\mathrm{Min}(E-(\varepsilon+1))})$, and especially $M_{\alpha,\varepsilon} \preceq M_{\alpha+1,\varepsilon}$. $\dashv$

Claim 6.1.8. *The sequence $\langle N_\varepsilon : \varepsilon \in E\rangle$ is $\preceq$-increasing.*

Proof. Suppose $\varepsilon < \zeta$, $\{\varepsilon, \zeta\} \subseteq E$. By (*), the sequences $\langle M_{\alpha,\varepsilon} : \alpha < \varepsilon\rangle^\frown\langle N_\varepsilon\rangle, \langle M_{\alpha,\zeta} : \alpha \leq \varepsilon\rangle$ are increasing and continuous. For every $\alpha \in \varepsilon$, the sequence $\langle M_{\alpha,\beta} : \beta < \lambda^+\rangle$ is a representation of M_α, and especially it is $\preceq$-increasing. So $(\forall \alpha \in \varepsilon) M_{\alpha,\varepsilon} \preceq M_{\alpha,\zeta}$. Hence by smoothness $N_\varepsilon \preceq M_{\varepsilon,\zeta}$. But by (*), $M_{\varepsilon,\zeta} \preceq N_\zeta$, so $N_\varepsilon \preceq N_\zeta$.]

$\dashv$

Claim 6.1.9. *The sequence $\langle N_\varepsilon : \varepsilon \in E\rangle$ is continuous.*

Proof. Suppose $\varepsilon = sup(E \bigcap \varepsilon)$. Let $x \in N_\varepsilon$. By the definition of N_ε, there is $\alpha < \varepsilon$ such that $x \in M_{\alpha,\varepsilon}$. ε is limit and the sequence $\langle M_{\alpha,\beta} : \beta \leq \varepsilon\rangle$ is continuous. So there is $\beta < \varepsilon$ such that $x \in M_{\alpha,\beta}$. $\varepsilon = sup(E \bigcap \varepsilon)$, so there is $\zeta \in (\beta, \varepsilon) \bigcap E$. $x \in M_{\alpha,\zeta}$ but by (*), $M_{\alpha,\zeta} \subseteq N_\zeta$, so $x \in N_\zeta$]. $\dashv$

Claim 6.1.10. $\bigcup\{N_\varepsilon : \varepsilon \in E\} = M_\delta$.

Proof. Clearly $\bigcup\{N_\varepsilon : \varepsilon \in E\} \subseteq M_\delta$. The other inclusion: Let $x \in M_\delta$. Then there is $\alpha < \delta$ such that $x \in M_\alpha$. So $(\exists \alpha, \beta) x \in M_{\alpha,\beta}$. So since $sup(E) = \delta$, there is $\zeta \in (\beta, \delta) \bigcap E$. So $x \in M_{\alpha,\zeta}$ which by (*) is $\subseteq N_\zeta$. So $x \in N_\zeta$]. $\dashv$

Claim 6.1.11. *If $\varepsilon < \zeta$, $\{\varepsilon, \zeta\} \subseteq E$ then $NF(M_{0,\varepsilon}, N_\varepsilon, M_{0,\zeta}, N_\zeta)$*

Proof. By the definition of E, $(\forall\alpha\in\varepsilon)\{\varepsilon,\zeta\}\subseteq E_\alpha$. So $(\forall\alpha\in\varepsilon)NF(M_{\alpha,\varepsilon}, M_{\alpha+1,\varepsilon}, M_{\alpha,\zeta}, M_{\alpha+1,\zeta})$. By (*), the sequences $\langle M_{\alpha,\varepsilon}:\alpha<\varepsilon\rangle^\frown\langle N_\varepsilon\rangle$, $\langle M_{\alpha,\zeta}:\alpha\leq\varepsilon\rangle$ are increasing and continuous. So by Theorem 5.5.1 (the long transitivity theorem), $NF(M_{0,\varepsilon}, N_\varepsilon, M_{0,\zeta}, M_{\varepsilon,\zeta})$. But by Claim 6.1.7, $M_{\varepsilon,\zeta}\prec N_\zeta$, so $NF(M_{0,\varepsilon}, N_\varepsilon, M_{0,\zeta}, N_\zeta)$]. $\dashv$

Now we return to the proof of Proposition 6.1.6. By Claims 6.1.8, 6.1.9, 6.1.10, the sequence $\langle N_\varepsilon:\varepsilon<\delta\rangle$ is a representation of M_δ. The sequence $\langle M_{0,\varepsilon}:\varepsilon<\lambda^+\rangle$ is a representation of M_0. Hence, by Claim 6.1.11 and item a, they witness that $M_0\preceq^{NF}_{\lambda^+} M_\delta$.

(e) By Proposition 6.1.3.c. Derived also by the existence proposition of the $\prec^+_{\lambda^+}$-extension, (Proposition 7.1.12.f), which we will prove later.

(f) We have actually proved it, (for example: Axiom 1.0.3.1.e by item c here and Axiom 1.0.3.1.c., by item d here).

(g) Let $\langle N_{0,\varepsilon}:\varepsilon<\lambda^+\rangle$, $\langle N_{1,\varepsilon}:\varepsilon<\lambda^+\rangle$ be witnesses for $M_0\preceq^{NF}_{\lambda^+} M_1$. By cardinality considerations, there is $\varepsilon\in\lambda^+$ such that for $n<2$ we have $A_n\subseteq N_{n,\varepsilon}$. But for every $\varepsilon<\lambda^+$, $\widehat{NF}(N_{0,\varepsilon}, N_{1,\varepsilon}, M_0, M_1)$. $\dashv$

A summary: We defined a relation $\preceq^{NF}_{\lambda^+}$ on K_{λ^+}, that is included in the relation $\preceq\restriction K_{\lambda^+}$. The restriction to the relation $\preceq^{NF}_{\lambda^+}$ enables to get the amalgamation property (see Theorem 7.1.18.c below). But it gives rise to a new problem: Does $(K_{\lambda^+},\preceq^{NF}_{\lambda^+})$ satisfies smoothness? We have proved that if $(K_{\lambda^+},\preceq^{NF}_{\lambda^+})$ satisfies smoothness, then it is an AEC in λ^+. The main aim of Sections 7,8,9 is to get smoothness. But for this we restrict ourselves to the saturated models in λ^+ over λ.

7. $\prec^+_{\lambda^+}$ AND SATURATED MODELS

7.1. **Introduction.** Now we restrict ourselves to K^{sat} (see Definition 7.1.2) in order to get smoothness. So we study the class $(K^{sat},\preceq^{NF}_{\lambda^+}\restriction K^{sat})$ ($\preceq^{NF}_{\lambda^+}$ is defined in Definition 6.1.4). We want to prove, under some model theoretic assumptions, that $(K^{sat},\preceq^{NF}_{\lambda^+}\restriction K^{sat})$ is an AEC in λ^+ and that it satisfies the amalgamation property.

Hypothesis 7.1.1. $\mathfrak{s}$ is a semi-good λ-frame with conjugation and $K^{3,uq}$ satisfies the existence property.

Definition 7.1.2. K^{sat} is the class of saturated models in λ^+ over λ.

Note that in the following theorem there is no set-theoretic hypothesis beyond ZFC.

Theorem 7.1.3. *If ($\mathfrak{s}$ is a semi-good λ-frame with conjugation, $K^{3,uq}$ satisfies the existence property and) $(K^{sat},\preceq^{NF}_{\lambda^+}\restriction K^{sat})$ does not satisfy smoothness (see Definition 1.0.3.1.d), then there are $2^{\lambda^{+2}}$ pairwise non-isomorphic models in $K_{\lambda^{+2}}$.*

How can we prove this theorem? First we find a relation $\prec^+_{\lambda^+}$ on K_{λ^+} such that:

(*) For every model M_0 in K_{λ^+}, there is a model M_1 such that $M_0 \prec^+_{\lambda^+} M_1$.
(**) If for $n = 1, 2$ $M_0 \prec^+_{\lambda^+} M_n$, then M_1, M_2 are isomorphic over M_0.
(***) If $\langle M_i : i \leq \alpha^* \rangle$ is an increasing continuous sequence, and $i < \alpha^* \Rightarrow M_i \prec^+_{\lambda^+} M_{i+1}$, then $M_0 \prec^+_{\lambda^+} M_{\alpha^*}$.

In Section 7 we study the properties of $\prec^+_{\lambda^+}$. Sections 8,9 are preparations for the proof of Theorem 7.1.3. A key theorem is Theorem 9.1.7: Suppose that there is an increasing continuous sequence $\langle M^*_\alpha : \alpha \leq \lambda+1 \rangle$ of models in K^{sat} such that: $\alpha < \beta < \lambda^+ \Rightarrow M^*_\alpha \prec^+_{\lambda^+} M^*_\beta \wedge M^*_\alpha \preceq^{NF}_{\lambda^+} M_{\lambda^++1}$ and $M^*_{\lambda^+} \npreceq^{NF} M^*_{\lambda^++1}$. Then for every $S \in S^{\lambda^{+2}}_{\lambda^+} := \{S : S$ is a stationary subset of λ^{+2} and $(\forall \alpha \in S) cf(\alpha) = \lambda^+\}$, there is a model M^S in $K_{\lambda^{+2}}$ such that $S(M^S) = S/D_{\lambda^{+2}}$. So there are $2^{\lambda^{+2}}$ pairwise non-isomorphic models in $K_{\lambda^{+2}}$.

Note that while $\prec^+_{\lambda^+}$ is a priori defined on K_{λ^+}, Proposition 7.1.6 shows that any $\prec^+_{\lambda^+}$ extension is saturated in λ^+ over λ, so in K^{sat}.

Definition 7.1.4. $\prec^+_{\lambda^+}$ is a 2-place relation on K_{λ^+}. For $M_0, M_1 \in K_{\lambda^+}$, $M_0 \prec^+_{\lambda^+} M_1$ if there are increasing continuous sequences of models in K_λ, $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N^\oplus_{1,\alpha} : \alpha < \lambda^+ \rangle$, and there is a club E of λ^+ such that:

(a) For $n = 0, 1$ $M_n = \bigcup \{N_{n,\alpha} : \alpha < \lambda^+\}$.
(b) $\alpha \in E \Rightarrow N_{0,\alpha} \preceq N_{1,\alpha} \preceq N^\oplus_{1,\alpha}$.
(c) If $\alpha < \beta$ and they are in E, then $NF(N_{0,\alpha}, N^\oplus_{1,\alpha}, N_{0,\beta}, N_{1,\beta})$.
(d) For every $\alpha \in E$, and every $p \in S^{bs}(N_{1,\alpha})$, there is an end-segment S of λ^+ such that for every $\beta \in S \bigcap E$ the model $N^\oplus_{1,\beta}$ realizes the non-forking extension of p to $N_{1,\beta}$.

In such a case $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N^\oplus_{1,\alpha} : \alpha < \lambda^+ \rangle$, E are said to be witnesses for $M_0 \prec^+_{\lambda^+} M_1$. Note that the $N_{1,\alpha}$ and the $N^{\bigoplus}_{1,\alpha}$ are an alternating chain that both union to M_1.

$$\begin{tikzcd}
M_0 \arrow[rrrrrr, "id"] & & & & & & M_1 \\
N_{0,3} \arrow[u, "id"] \arrow[rrrr, "id"] & & & & N_{1,3} \arrow[r, "id"] & N^{\oplus}_{1,3} \arrow[ur, "id"] & \\
N_{0,2} \arrow[u, "id"] \arrow[rrr, "id"] & & & N_{1,2} \arrow[r, "id"] & N^{\oplus}_{1,2} \arrow[u, "id"] & & \\
N_{0,1} \arrow[u, "id"] \arrow[rr, "id"] & & N_{1,1} \arrow[r, "id"] & N^{\oplus}_{1,1} \arrow[u, "id"] & & & \\
N_{0,0} \arrow[u, "id"] \arrow[r, "id"] & N_{1,0} \arrow[r, "id"] & N^{\oplus}_{1,0} \arrow[u, "id"] & & & &
\end{tikzcd}$$

By the following proposition if $M_0 \prec^+_{\lambda^+} M_1$, then we can find witnesses for it, with $E = \lambda^+$.

Proposition 7.1.5. *If*

(1) $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N^{\oplus}_{1,\alpha} : \alpha < \lambda^+ \rangle$, E *are witnesses for* $M_0 \prec^+_{\lambda^+} M_1$.

(2) For $\alpha \in E$, $M_{0,otp(\alpha \bigcap E)} = N_{0,\alpha}$, $M_{1,otp(\alpha \bigcap E)} = N_{1,\alpha}$, $M^{\oplus}_{1,otp(\alpha \bigcap E)} = N^{\oplus}_{1,\alpha}$.

Then $\langle M_{0,\beta} : \beta < \lambda^+ \rangle$, $\langle M_{1,\beta} : \beta < \lambda^+ \rangle$, $\langle M^{\oplus}_{1,\beta} : \beta < \lambda^+ \rangle$, λ^+ *are witnesses for* $M_0 \prec^+_{\lambda^+} M_1$.

Proof. Easy, so we prove Definition 7.1.4.c only. Suppose $\gamma_0 < \gamma_1$. We have to prove that $NF(M_{0,\gamma_0}, M^{\oplus}_{1,\gamma_0}, M_{0,\gamma_1}, M_{1,\gamma_1})$. There is a unique ordinal $\alpha \in E$ with $otp(\alpha \bigcap E) = \gamma_0$. So $M_{0,\gamma_0} = N_{0,\alpha} \wedge M^{\oplus}_{1,\gamma_0} = N^{\oplus}_{1,\alpha}$. Similarly there is a unique $\beta \in E$ such that $M_{0,\gamma_1} = N_{0,\beta} \wedge M_{1,\gamma_1} = N_{1,\beta}$. Now by clause b in the assumption, $NF(N_{0,\alpha}, N^{\oplus}_{1,\alpha}, N_{0,\beta}, N_{1,\beta})$, namely, $NF(M_{0,\gamma_0}, M^{\oplus}_{1,\gamma_0}, N_{0,\gamma_1}, N_{1,\gamma_1})$. $\dashv$

Proposition 7.1.6. *If* $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N^{\oplus}_{1,\alpha} : \alpha < \lambda^+ \rangle$, E *are witnesses for* $M_0 \prec^+_{\lambda^+} M_1$ *and* E^- *is a club of* λ^+ *with* $E^- \subseteq E$, *then* $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N^{\oplus}_{1,\alpha} : \alpha < \lambda^+ \rangle$, E^- *are witnesses for* $M_0 \prec^+_{\lambda^+} M_1$.

Proof. Trivial. $\dashv$

Proposition 7.1.7. *Suppose:*

(a) For $n = 1,2$ $NF(M_{0,0}, M_{0,1}, M_{n,0}, M_{n,1})$.
(b) $M_{1,0} \preceq N_0$, $M_{2,0} \preceq N_0$.
(c) $N_0 \bigcap M_{0,1} = M_{0,0}$.

Then for some model N_1 with $NF(M_{0,0}, M_{0,1}, N_0, N_1)$, we can assign to each $n \in \{1,2\}$ an embedding $f_n : M_{n,1} \to N_1$ over $M_{0,1} \bigcup M_{n,0}$ such that $NF(M_{n,0}, f_n[M_{n,1}], N_0, N_1)$.

$$\begin{tikzcd}
 & N_0 \arrow[rr, "id"] & & N_1 \\
M_{2,0} \arrow[ur, "id"] \arrow[rr, "id"] & & M_{2,1} \arrow[ur, "f_2"] & \\
 & M_{1,0} \arrow[uu, "id"] \arrow[rr, "id"] & & M_{1,1} \arrow[uu, "f_1"'] \\
M_{0,0} \arrow[uu, "id"] \arrow[ur, "id"] \arrow[rr, "id"] & & M_{0,1} \arrow[uu, "id"] \arrow[ur, "id"] &
\end{tikzcd}$$

Proof. For each $n \in \{1,2\}$ by Theorem 5.3.7 (the existence theorem for NF), we can find an amalgamation $(id_{N_0}, g_n, N_{n,1})$ of $N_0, M_{n,1}$ over $M_{n,0}$ with $NF(M_{n,0}, N_0, g_n[M_{n,1}], N_{n,1})$. But $NF(M_{0,0}, M_{n,0}, M_{0,1}, M_{n,1})$. So by Theorem 5.5.1 (the long transitivity theorem), $NF(M_{0,0}, N_0, g_n[M_{0,1}], N_{n,1})$. By Assumption c, $N_0 \bigcap M_{0,1} = M_{0,0}$. So by Theorem 5.4.7 (the weak uniqueness theorem), we can find h_1, h_2, N_1 such that the following hold:

(1) $h_n : N_{n,1} \to N_1$ is an embedding.
(2) $h_n \restriction N_0 = id_{N_0}$.
(3) $h_1 \circ g_1 \restriction M_{0,1} = h_1 \circ g_2 \restriction M_{0,1} = id_{M_{0,1}}$.

Now we define for $n = 1,2$ $f_n := h_n \circ g_n$. Why is f_n over $M_{0,1} \bigcup M_{n,0}$? By clause 3, $x \in M_{0,1} \Rightarrow f_n(x) = x$. Let $x \in M_{n,0}$. Then $g_n(x) = x$. By Assumption b, $M_{n,0} \subseteq N_0$, so $x \in N_0$. So by clause 2 $h_n(x) = x$. Hence $f_n(x) = h_n(g_n(x)) = h_n(x) = x$.

Claim 7.1.8. *$NF(M_{n,0}, f_n[M_{n,1}], N_0, N_1)$.*

Proof. $NF(M_{n,0}, N_0, g_n[M_{n,1}], N_{n,1})$. So by clauses 1,2 $NF(M_{n,0}, N_0, f_n[M_{n,1}], h_n[N_{n,1}])$. But $h_n[N_{n,1}] \preceq N_1$, so $NF(M_{n,0}, N_0, f_n[M_{n,1}], N_1)$. $\dashv$

Claim 7.1.9. *$NF(M_{0,0}, M_{0,1}, N_0, N_1)$.*

Proof. Since $NF(M_{1,0}, M_{1,1}, N_0, N_1)$, by Theorem 5.5.1 (the long transitivity theorem), it is enough to prove that $NF(M_{0,0}, M_{0,1}, M_{1,0}, f_1[M_{1,1}])$. But f_n is over $M_{0,1} \bigcup M_{1,0}$. Hence it follows by assumption a. $\dashv$

This completes the proof of Proposition 7.1.7 $\dashv$

Proposition 7.1.10.
(a) If $M_0 \prec^{+}_{\lambda^+} M_1$ then $M_0 \prec^{NF}_{\lambda^+} M_1$.

(b) If $M_0 \prec^+_{\lambda^+} M_1$ then $M_1 \in K^{sat}$.
(c) If $M_0 \preceq^{NF}_{\lambda^+} M_1 \prec^+_{\lambda^+} M_2$ then $M_0 \prec^+_{\lambda^+} M_2$.
(d) If $M_0 \prec^+_{\lambda^+} M_1 \prec^+_{\lambda^+} M_2$ then $M_0 \prec^+_{\lambda^+} M_2$.

Proof.

(a) If $\langle N_{0,\alpha} : \alpha < \lambda^+\rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+\rangle$, $\langle N_{2,\alpha} : \alpha < \lambda^+\rangle$, E witness that $M_0 \prec^+_{\lambda^+} M_1$, then $\langle N_{0,\alpha} : \alpha \in E\rangle$, $\langle N_{1,\alpha} : \alpha \in E\rangle$ witness that $\widehat{NF}(N_{0,0}, N_{1,0}, M_0, M_1)$. So $M_0 \preceq^{NF}_{\lambda^+} M_1$.
(b) By Theorem 2.5.8.2.
(c) Easy.
(d) By items a,c.

$\dashv$

Definition 7.1.11. The $\prec^+_{\lambda^+}$-*game* is a game between two players. It lasts λ^+ moves. In any move, the players choose models in K_λ with the following rules:

The 0 move: Player 1 chooses models $N_{0,0}, N_{1,0} \in K_\lambda$ with $N_{0,0} \preceq N_{1,0}$ and player 2 does not do anything.

The α move where α is limit: Player 1 must choose $N_{0,\alpha} := \bigcup\{N_{0,\beta} : \beta < \alpha\}$ and Player 2 must choose $N_{1,\alpha} := \bigcup\{N_{1,\beta} : \beta < \alpha\}$.

The $\alpha + 1$ move: Player 1 chooses a model $N_{0,\alpha+1}$ such that the following hold:

(1) $N_{0,\alpha} \preceq N_{0,\alpha+1}$.
(2) $N_{0,\alpha+1} \bigcap N_{1,\alpha} = N_{0,\alpha}$.

After player one chooses $N_{0,\alpha+1}$, player 2 has to choose $N_{1,\alpha+1}$ such that the following hold:

(1) $N_{1,\alpha} \preceq N_{1,\alpha+1}$.
(2) $NF(N_{0,\alpha}, N_{1,\alpha}, N_{0,\alpha+1}, N_{1,\alpha+1})$.

At the end of the game, player 2 wins the game if $\bigcup\{N_{0,\alpha} : \alpha < \lambda^+\} \prec^+_{\lambda^+} \bigcup\{N_{1,\alpha} : \alpha < \lambda^+\}$.

A *strategy for player 2* is a function F that assigns a model $N_{1,\alpha+1}$ to each triple $(\alpha, \langle N_{0,\beta} : \beta \leq \alpha + 1\rangle, \langle N_{1,\beta} : \beta \leq \alpha\rangle)$ that satisfies the following conditions:

(1) $\alpha < \lambda^+$.
(2) $\langle N_{0,\beta} : \beta \leq \alpha + 1\rangle$, $\langle N_{1,\beta} : \beta \leq \alpha\rangle$ are increasing continuous sequences of models in K_λ.
(3) $NF(N_{0,\alpha}, N_{1,\alpha}, N_{0,\alpha+1}, N_{1,\alpha+1})$ for $\beta < \alpha$.
(4) $N_{0,\alpha+1} \bigcap N_{1,\alpha} = N_{0,\alpha}$.

A *winning strategy for player 2* is a strategy for player 2, such that if player 2 acts by it, then he wins the game, no matter what player 1 does.

Proposition 7.1.12.

(a) For every $M_0 \in K_{\lambda^+}$, there is M_1 with $M_0 \prec^+_{\lambda^+} M_1$.

(b) If $M_0 \in K_{\lambda^+}$, $n < 2 \Rightarrow N_n \in K_\lambda$, $N_0 \prec M_0$, $N_0 \prec N_1$, $N_1 \bigcap M_0 = N_0$, then there is M_1 such that $M_0 \prec^+_{\lambda^+} M_1$ and $\widehat{NF}(N_0, N_1, M_0, M_1)$.
(c) Player 2 has a winning strategy in the $\prec^+_{\lambda^+}$-game.

Proof. (a) By c.
(b) By c.
(c) We describe a strategy: For $\alpha = 0$ player 2 has nothing to do, but he takes a paper and writes for himself: I define $N^{temp}_{1,0} := N_{1,0}$. For α limit, player 2 chooses $N_{1,\alpha} := \bigcup\{N_{1,\beta} : \beta < \alpha\}$ and writes for himself $N^{temp}_{1,\alpha} := N_{1,\alpha}$. In the $\alpha+1$ move, he writes for himself the following things:

(i) A model $N^{temp}_{1,\alpha+1}$ with $NF(N_{0,\alpha}, N_{1,\alpha}, N_{0,\alpha+1}, N^{temp}_{1,\alpha+1})$. By Theorem 5.3.7.a, it is possible.
(ii) A sequence of types $\langle p_{\alpha,\beta} : \beta < \lambda^+\rangle$ that includes $S^{bs}(N^{temp}_{1,\alpha})$.

Now player 2 chooses a model $N_{1,\alpha+1}$ such that the following hold:

(1) $N^{temp}_{1,\alpha+1} \preceq N_{1,\alpha+1}$.
(2) For each type in $p_{\gamma,\beta}$ with $\gamma < \alpha$, $\beta < \alpha$, $N_{1,\alpha+1}$ realizes the non-forking extension of $p_{\gamma,\beta}$ over $N^{temp}_{1,\alpha+1}$.

Why will player 2 win the game? By Definition 7.1.4, where the sequences $\langle N_{0,\alpha} : \alpha < \lambda^+\rangle$, $\langle N^{temp}_{1,\alpha} : \alpha < \lambda^+\rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+\rangle$ which appear here stand for the sequences $\langle N_{0,\alpha} : \alpha < \lambda^+\rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+\rangle$, $\langle N^{\oplus}_{1,\alpha} : \alpha < \lambda^+\rangle$ and λ^+ stands for E. $\dashv$

Roughly, the following theorem says that:
(a) The $\prec^+_{\lambda^+}$-extension is unique.
(b) Tameness: Every type over a model in K_{λ^+} is determined by its restrictions to submodels in K_λ.
(c) A preparation for symmetry.

Theorem 7.1.13. *Suppose for $n = 1, 2$ $M_0 \prec^+_{\lambda^+} M_n$, then:*
(a) M_1, M_2 are isomorphic over M_0.
(b) For every $a_1 \in M_1$, $a_2 \in M_2$, if for each $N \in K_\lambda$ with $N \preceq M_0$ $tp(a_1, N, M_1) = tp(a_2, N, M_2)$, then there is an isomorphism $f : M_1 \to M_2$ over M_0 with $f(a_1) = a_2$.
(c) Let $N^* \in K_\lambda$, $N_0 \preceq N^*$. If for $n = 1, 2$ $\widehat{NF}(N_0, N^*, M_0, M_n)$, then there is an isomorphism $f : M_1 \to M_2$ over $M_0 \bigcup N^*$.

The plan of the proof: We prove the three items simultaneously. The proof is similar to that of the uniqueness of the saturated model in λ^+ over λ. Suppose $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{1,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{\oplus}_{1,\varepsilon} : \varepsilon < \lambda^+\rangle$, λ^+ witness that $M_0 \prec^+_{\lambda^+} M_1$. So $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$ is a representation of M_0 and $\langle N_{1,0}, N^{\oplus}_{1,0}, N_{1,1}, N^{\oplus}_{1,1}, ...N_{1,\omega}, N^{\oplus}_{1,\omega}...\rangle$ is a representation of M_1. Suppose in addition that $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{2,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{\oplus}_{2,\varepsilon} : \varepsilon < \lambda^+\rangle$, λ^+ witness that $M_0 \prec^+_{\lambda^+} M_2$. We amalgamate M_1, M_2 over M_0 in λ^+ steps. In each step, we amalgamate the corresponding models in the representations of M_1, M_2 over the corresponding model in the representation of M_0. Now if (f_1, f_2, M_3) is an amalgamation of M_1, M_2 over M_0 and f_1, f_2 are onto M_3, then $f_2^{-1} \circ f_1$ is an isomorphism of M_1 into M_2 over M_0, as required. In odd steps, we choose the amalgamations such that at the end f_1, f_2 will be onto M_3, see requirement 8 below. In even steps we choose amalgamations with NF, see requirement 4 below.

Proof. Roughly, the following claim says that one representation of M_0 can serve as a part of the witness to both $M_0 \prec^+_{\lambda^+} M_1$ and $M_0 \prec^+_{\lambda^+} M_2$.

Claim 7.1.14. *There are sequences $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{1,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{\oplus}_{1,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{2,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{\oplus}_{2,\varepsilon} : \varepsilon < \lambda^+\rangle$ such that for $n = 1, 2$, $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$, $E = \lambda^+$, $\langle N^{\oplus}_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$ witnesses that $M_0 \prec^+_{\lambda^+} M_n$ (so $\bigcup\{N_{0,\varepsilon} : \varepsilon < \lambda^+\} = M_0$ and for $n = 1, 2$, $\bigcup\{N_{n,\varepsilon} : \varepsilon < \lambda^+\} = \bigcup\{N^{\oplus}_{n,\varepsilon} : \varepsilon < \lambda^+\} = M_n$).*

Proof. For $n = 1, 2$, we take witnesses $\langle N^{temp}_{0,n,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{temp}_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{\oplus,temp}_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$, E_n for $M_0 \prec^+_{\lambda^+} M_n$. Take a club E of λ^+ such that $E \subseteq E_1 \bigcap E_2$ and $\varepsilon \in E \Rightarrow N^{temp}_{0,1,\varepsilon} = N^{temp}_{0,2,\varepsilon}$. By Proposition 7.1.6 for $n = 1, 2$, $\langle N^{temp}_{0,n,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{temp}_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{\oplus,temp}_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$, E are witnesses for $M_0 \prec^+_{\lambda^+} M_n$. Define $N_{0,otp(\varepsilon \bigcap E)} := N^{temp}_{0,1,\varepsilon)}$. For $n = 1, 2$ and $\varepsilon \in E$, define $N_{n,otp(\varepsilon \bigcap E)} := N^{temp}_{n,\varepsilon}$. By Proposition 7.1.5 for $n = 1, 2$, $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$, $E = \lambda^+$, $\langle N^{\oplus}_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$ witness that $M_0 \prec^+_{\lambda^+} M_n$. ⊣

Let $\langle N_{0,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{1,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{\oplus}_{1,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N_{2,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^{\oplus}_{2,\varepsilon} : \varepsilon < \lambda^+\rangle$ be as in Claim 7.1.14. For item b, we require, in addition, that for $n = 1, 2$ $a_n \in N_{n,0}$ [Why can we do it? Take $\alpha < \lambda^+$ with $a_n \in N_{n,\alpha}$ for $n = 1, 2$ and replace $N_{n,\varepsilon}$ by $N_{n,\alpha+\varepsilon}$ in each one of the five sequences. Now rename the sequences]. For item c, we require in addition that for $n = 1, 2$, $NF(N_0, N^*, N_{0,0}, N_{n,0})$.

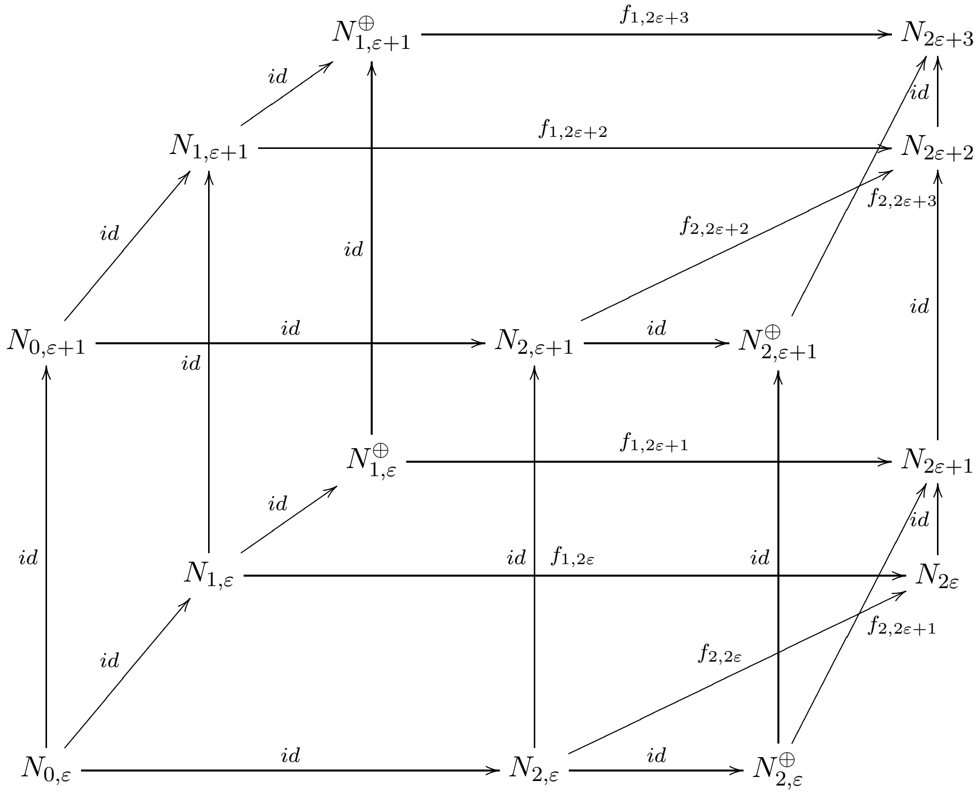
Define by induction on $\varepsilon \leq \lambda^+$ a triple $(N_\varepsilon, f_{1,\varepsilon}, f_{2,\varepsilon})$ such that:

(1) $\langle N_\varepsilon : \varepsilon \leq \lambda^+\rangle$ is an increasing continuous sequence of models in K_λ and for every $\varepsilon < \lambda^+$ $N_{2\varepsilon} \bigcap M_0 = N_{2\varepsilon+1} \bigcap M_0 = N_{0,\varepsilon}$.
(2) For item c we add: $f_{n,0} \upharpoonright N^*$ is the identity.
(3) For item b we add: $f_{1,0}(a_1) = f_{2,0}(a_2)$.
(4) $\varepsilon < \lambda^+ \Rightarrow NF(N_{0,\varepsilon}, N_{2\varepsilon+1}, N_{0,\varepsilon+1}, N_{2\varepsilon+2})$.
(5) For $n = 1, 2$, the sequence $\langle f_{n,\varepsilon} : \varepsilon \leq \lambda^+\rangle$ is increasing and continuous.
(6) For $\varepsilon < \lambda^+$, $f_{n,2\varepsilon}$ is an embedding of $N_{n,\varepsilon}$ to $N_{2\varepsilon}$ and $f_{n,2\varepsilon+1}$ is an embedding of $N^{\oplus}_{n,\varepsilon}$ to $N_{2\varepsilon+1}$.
(7) $f_{n,2\varepsilon} \upharpoonright N_{0,\varepsilon} = f_{n,2\varepsilon+1} \upharpoonright N_{0,\varepsilon}$ and it is the identity on $N_{0,\varepsilon}$.
(8) For every $\varepsilon < \lambda^+$, if for some $n \in \{1, 2\}$ $(*)_{n,\varepsilon}$ holds, then for some $m \in \{1, 2\}$ $(**)_{m,\varepsilon}$ holds, where:

$(*)_{n,\varepsilon}$ There is $p \in S^{bs}(N_{n,\varepsilon})$ such that p is realized in $N^{\oplus}_{n,\varepsilon}$ and $f_{n,2\varepsilon}(p)$ is realized in $N_{2\varepsilon}$.

$(**)_{m,\varepsilon}$, $f_{m,2\varepsilon+1}[N^{\oplus}_{m,\varepsilon}] \bigcap N_{2\varepsilon} \neq f_{m,2\varepsilon}[N_{m,\varepsilon}]$.

Note that requirement 4 is essentially a property of $N_{2\varepsilon+2}$ and $(**)_{m,\varepsilon}$ is essentially a property of $f_{m,2\varepsilon+1}$.



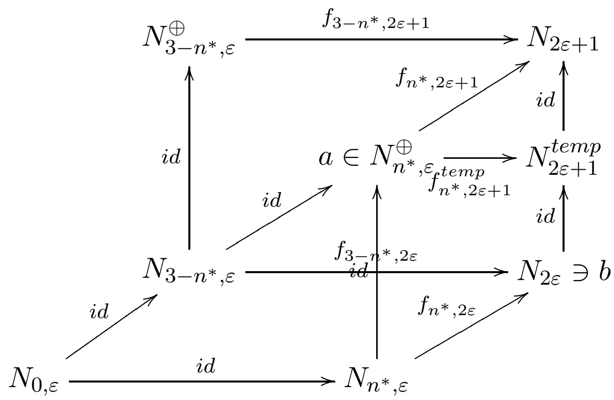
Why can we carry out the construction? For $\varepsilon = 0$, let $(f_{1,0}, f_{2,0}, N_0)$ be an amalgamation of $N_{1,0}, N_{2,0}$ over $N_{0,0}$, such that $N_0 \bigcap M_0 = N_{0,0}$ (i.e., we choose new elements for $N_0 - N_{0,0}$). In the proof of item b, by the definition of the equality between types, without loss of generality $f_{1,0}(a_1) = f_{2,0}(a_2)$, so 3 is satisfied. In the proof of item c, by Theorem 5.4.7 (the weak uniqueness theorem of NF), there is a joint embedding $f_{1,0}, f_{2,0}, N_0$ of $N_{1,0}, N_{2,0}$ over $N_{0,0} \bigcup N^*$. So 2 is satisfied.

For *limit* ε, define $N_\varepsilon = \bigcup\{N_\zeta : \zeta < \varepsilon\}$, $f_{n,\varepsilon} = \bigcup\{f_{n,\zeta} : \zeta < \varepsilon\}$. 5 is satisfied. 1 is satisfied by Axiom 1.0.3.1.c. 6 is satisfied by the continuity of the sequence $\langle N_{n,\varepsilon} : \varepsilon < \lambda^+\rangle$, and by smoothness (Definition 1.0.3.1.d). Clearly 7 is satisfied. 4,8 are irrelevant in the limit case.

The successor case: How can we construct $N_{2\varepsilon+1}, f_{1,2\varepsilon+1}, f_{2,2\varepsilon+1}$ and $N_{2\varepsilon+2}, f_{1,2\varepsilon+2}, f_{2,2\varepsilon+2}$, assuming we have constructed $N_{2\varepsilon}, f_{1,2\varepsilon}, f_{2,2\varepsilon}$?

The construction of $N_{2\varepsilon+1}, f_{1,2\varepsilon+1}, f_{2,2\varepsilon+1}$: Without loss of generality for some $n \in 1, 2$, we have $(*)_{n,\varepsilon}$ [Otherwise requirement 8 is irrelevant and we can use the existence of an amalgamation in $(K_\lambda, \preceq)$.] Fix n^* with $(*)_{n^*,\varepsilon}$. We are going to find $N_{2\varepsilon+1}, f_{n^*,2\varepsilon+1}, f_{3-n^*,2\varepsilon+1}$ with $(**)_{n^*,\varepsilon}$, namely, $f_{n^*,2\varepsilon+1}[N^{\oplus}_{n^*,\varepsilon}] \bigcap N_{2\varepsilon} \neq f_{n^*,2\varepsilon}[N_{n^*,\varepsilon}]$. Let p be a witness for $(*)_{n^*,\varepsilon}$, so for some a, b $tp(a, N_{n^*,\varepsilon}, N^{\oplus}_{n^*,\varepsilon}) = p$, $tp(b, f_{n^*,2\varepsilon}[N_{n^*,\varepsilon}], N_{2\varepsilon}) = f_{n^*,2\varepsilon}(p)$. So $tp(f_{n^*,2\varepsilon}(a), f_{n^*,2\varepsilon}[N_{n^*,\varepsilon}], f_{n^*,2\varepsilon}[N^{\oplus}_{n^*,\varepsilon}]) = tp(b, f_{n^*,2\varepsilon}[N_{n^*,\varepsilon}], N_{2\varepsilon})$. Hence by the definition of equality of types, for some $N^{temp}_{2\varepsilon+1}, f^{temp}_{n^*,2\varepsilon+1}$, the following hold:

(1) $N_{2\varepsilon} \preceq N^{temp}_{2\varepsilon+1}$.
(2) $f^{temp}_{n^*,2\varepsilon+1} : N^{\oplus}_{n^*,\varepsilon} \to N^{temp}_{2\varepsilon+1}$ is an embedding.
(3) $f_{n^*,2\varepsilon} \subseteq f^{temp}_{n^*,2\varepsilon+1}$.
(4) $f^{temp}_{n^*,2\varepsilon+1}(a) = b$.



Claim 7.1.15. $f^{temp}_{n^*,2\varepsilon+1}[N^{\oplus}_{n^*,\varepsilon}] \bigcap N_{2\varepsilon} \neq f^{temp}_{n^*,2\varepsilon}[N_{n^*,\varepsilon}]$.

Proof. $b \in N_{2\varepsilon}$. p is a basic type so it is a non-algebraic one. So $a \in N^{\oplus}_{n^*,\varepsilon} - N_{n^*,\varepsilon}$. Hence $b = f^{temp}_{n^*,2\varepsilon+1}(a) \in f^{temp}_{n^*,2\varepsilon+1}[N^{\oplus}_{n^*,\varepsilon}] - f^{temp}_{n^*,2\varepsilon+1}[N_{n^*,\varepsilon}]$. Therefore $b \in f^{temp}_{n^*,2\varepsilon+1}[N^{\oplus}_{n^*,\varepsilon}] \bigcap N_{2\varepsilon} - f^{temp}_{n^*,2\varepsilon}[N_{n^*,\varepsilon}]$. ⊣

As $(K_\lambda, \preceq)$ satisfies amalgamation, there are $N_{2\varepsilon+1}, f_{3-n^*,2\varepsilon+1}$ such that $N^{temp}_{2\varepsilon+1} \preceq N_{2\varepsilon+1}$ and $f_{3-n^*,2\varepsilon+1} : N^{\oplus}_{3-n^*,\varepsilon} \to N_{2\varepsilon+1}$ is an embedding that includes $f_{3-n^*,2\varepsilon}$. Now we define $f_{n^*,2\varepsilon+1} : N^{\oplus}_{n^*,\varepsilon} \to N_{2\varepsilon+1}$ by $f_{n^*,2\varepsilon+1}(x) = f^{temp}_{n^*,2\varepsilon+1}(x)$. By Claim 7.1.15, $(**)_{n^*}$ holds, so requirement 8 is satisfied. As for $m = 1, 2$, the embedding $f_{m,2\varepsilon+1}$ includes $f_{m,2\varepsilon}$, requirement 7 is satisfied. Without loss of generality, requirement 1 is satisfied. Requirement 4 is irrelevant in this case. Requirements 5,6 are satisfied.

The construction of $N_{2\varepsilon+2}, f_{n,2\varepsilon+2}$*:* By Proposition 7.1.7, there are $N_{2\varepsilon+2}$, $f_{1,2\varepsilon+2}, f_{2,2\varepsilon+2}$ such that: $NF(f_{n,2\varepsilon+1}[N^{\oplus}_{n,\varepsilon}], f_{n,2\varepsilon+2}[N_{n,\varepsilon+1}], N_{2\varepsilon+1}, N_{2\varepsilon+2})$, and the restriction of $f_{n,2\varepsilon+1}$ to $N_{0,\varepsilon}$ is the identity [Let $f^{+}_{n,2\varepsilon+1}$ be an injection of $N_{n,\varepsilon+1}$, $f_{n,2\varepsilon+1} \subseteq f^{+}_{n,2\varepsilon+1}$, and the restriction of $f^{+}_{n,2\varepsilon+1}$ to $N_{0,\varepsilon+1}$ is the identity. Substitute the models $N_{0,\varepsilon}, N_{0,\varepsilon+1}, f_{n,2\varepsilon+1}[N^{\oplus}_{n,\varepsilon}], N_{2\varepsilon+1}, f^{+}_{2\varepsilon+1}$ $[N_{n,\varepsilon+1}], N_{2\varepsilon+2}$ which appear here, for the models $M_{0,0}, M_{0,1}, M_{n,0}, N_0, M_{n,1}$,

N_1 which appear in Proposition 7.1.7, respectively. Assumption a of Proposition 7.1.7 (i.e., $NF(N_{0,\varepsilon}, N_{0,\varepsilon+1}, f_{n,2\varepsilon+1}[N^{\oplus}_{n,\varepsilon}], f^{+}_{n,2\varepsilon+1}[N_{n,\varepsilon+1}])$), is satisfied by Definition 7.1.4.a (recall that $f^{+}_{n,2\varepsilon+1}$ is an isomorphism over $N_{0,\varepsilon+1}$ and NF respects isomorphisms). Assumption b of Proposition 7.1.7 is satisfied by requirement 6 of the induction hypothesis. Assumption c of Proposition 7.1.7 is satisfied by requirement 4 of the induction hypothesis.] Hence we can carry out the construction.

Why is it sufficient? By clause 7, for $n = 1, 2$, $f_{n,\lambda^+} : M_n \to N_{\lambda^+}$ is an embedding over M_0.

Claim 7.1.16. $f_{1,\lambda^+}[M_1] = f_{2,\lambda^+}[M_2] = N_{\lambda^+}$.

Proof. Toward a contradiction, suppose there is $n \in \{1, 2\}$ such that $f_{n,\lambda^+}[M_n] \neq N_{\lambda^+}$. By Density (Theorem 2.6.8.1), there is an element b such that $tp(b, f_{n,\lambda^+}[M_n], N_{\lambda^+})$ is basic. $\langle f_{n,2\varepsilon}[N_{n,\varepsilon}] : \varepsilon < \lambda^+\rangle$ is a representation of $f_{n,\lambda^+}[M_n]$, so by Definition 2.6.1 there is $\varepsilon < \lambda^+$ such that for every $\zeta \in (\varepsilon, \lambda^+)$ the type $q_\zeta := tp(b, f_{n,2\zeta}[N_{n,\zeta}], N_{\lambda^+})$ does not fork over $f_{n,2\varepsilon}[N_{n,\varepsilon}]$. We choose this ε such that $b \in N_{2\varepsilon}$, (recall: $b \in N_{\lambda^+} = \bigcup\{N_\varepsilon : \varepsilon < \lambda^+\}$). So q_ζ is basic. Define $p_\zeta := f^{-1}_{n,2\zeta}(q_\zeta)$. So $p_\varepsilon \in S^{bs}(N_{n,\varepsilon})$. For every $\zeta \in (\varepsilon, \lambda^+)$, q_ζ is the non-forking extension of q_ε, so p_ζ is the non-forking extension of p_ε. Hence by Definition 7.1.4, there is an end segment $S^* \subseteq \lambda^+$ such that for $\zeta \in S^*$, p_ζ is realized in $N^{\oplus}_{2\zeta}$. But $q_\zeta = tp(b, f_{n,2\zeta}[N_{n,\zeta}], N_{2\zeta})$. So for every $\zeta \in S^*$, we have $(*)_{n,\zeta}$ (p_ζ is a witness for this). So by clause 8, there are $m \in \{1, 2\}$ and a stationary set $S^{**} \subseteq S^*$ such that for every $\zeta \in S^{**}$ we have $(**)_{m,\zeta}$, (there are no two non-stationary subsets which their union is an end segment of λ^+). The sequences $\langle N_{2\zeta} : \zeta \in S^{**}\rangle$, $\langle N_{m,\zeta} : \zeta \in S^{**}\rangle$, $\langle f_{m,2\zeta} : \zeta \in S^{**}\rangle$ are increasing and continuous. But by $(**)_{m,\zeta}$, we have $f_{m,2\zeta+1}[N^{\oplus}_{m,\zeta+1}] \bigcap N_{2\zeta} \neq f_{m,2\zeta}[N_{m,\zeta}]$, in contradiction to Proposition 1.0.30. $\dashv$

By Claim 7.1.16 $f^{-1}_{2,\lambda^+} \circ f_{1,\lambda^+}$ is an embedding of M_1 onto M_2 over M_0. In the proof of item b, we have to note that $f^{-1}_{2,\lambda^+} \circ f_{1,\lambda^+}(a_1) = f^{-1}_{2,0} \circ f_{1,0}(a_1) = a_2$ (by clause 3). In the proof of item c, we have to note that $f^{-1}_{2,\lambda^+} \circ f_{1,\lambda^+} \restriction N^* = f^{-1}_{2,0} \circ f_{1,0} \restriction N^*$ and by clause 3, it is the identity. $\dashv$

Corollary 7.1.17.

(a) *$(K_{\lambda^+}, \preceq^{NF}_{\lambda^+} \restriction K_{\lambda^+})$ satisfies the amalgamation property. So $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ satisfies the amalgamation property.*

(b) *Tameness: Let M_0, M_1, M_2 be models in K_{λ^+}, such that $M_0 \preceq M_1$, $M_0 \preceq M_2$. Suppose that for every $N \in K_\lambda$, $[N \preceq M_0] \Rightarrow tp(a_1, N, M_1) = tp(a_2, N, M_2)$. Then $tp(a_1, M_0, M_1) = tp(a_2, M_0, M_2)$.*

(c) *Suppose there is $N_0 \in K_\lambda$ such that for $n = 1, 2$ $tp(a_n, M_0, M_n)$ does not fork over N_0 and $tp(a_1, N_0, M_1) = tp(a_2, N_0, M_2)$. Then $tp(a_1, M_0, M_1) = tp(a_2, M_0, M_2)$].*

Proof.

(a) We could prove the amalgamation property without mentioning the relation $\prec^+_{\lambda^+}$. But we give a shorter proof, using Theorem 7.1.13. Suppose for $n = 1, 2$ $M_0 \prec^{NF}_{\lambda^+} M_n$. By Proposition 7.1.12.a, there is M^+_n such that $M_n \prec^+_{\lambda^+} M^+_n$. By Proposition 7.1.10.c, $M_0 \prec^+_{\lambda^+} M^+_n$. So by Theorem 7.1.13.c (the uniqueness of the $\prec^+_{\lambda^+}$-extension), there is an isomorphism $f : M^+_1 \to M^+_2$ over M_0. Hence $(f \restriction M_1, id_{M_2}, M^+_2)$ is an amalgamation of M_1, M_2 over M_0. The 'so' is by Proposition 7.1.10.a.

(b) Tameness: By Proposition 7.1.12.a, for $n = 1, 2$, there is M^+_n such that $M_n \prec^+_{\lambda^+} M^+_n$. By Theorem 7.1.13.b, there is an isomorphism $f : M^+_1 \to M^+_2$ over M_0, such that $f(a_1) = a_2$. So $(f \restriction M_1, id_{M_2}, M^+_2)$ witnesses that $tp(a_1, M_0, M_1) = tp(a_2, M_0, M_2)$.

(c) By item b. Note that if $N, N_0 \in K_\lambda$, $N \preceq N_0$ and $tp(a_1, N_0, M_1) = tp(a_2, N_0, M_2)$, then $tp(a_1, N, M_1) = tp(a_2, N, M_2)$

$\dashv$

Theorem 7.1.18.

(a) $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ satisfies Axiom c of AEC in λ^+ (see Definition 1.0.3.2.c).

(b) If $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ satisfies smoothness, then it is an AEC in λ^+.

(c) $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ satisfies the amalgamation property.

Proof. (a) Let $j < \lambda^{+2}$ and $\langle M_i : i < j\rangle$ be a $\preceq^{NF}_{\lambda^+}$-increasing continuous of models in K^{sat}. Let M_j be the union of this sequence. We prove that $M_j \in K^{sat}$ by induction on j. Let N be a model in K_λ such that $N \prec M_j$.

Case a: $\lambda < cf(j)$. In this case, for some $i < j$, $N \prec M_i$. Since M_i is full over N, of course so is M_j. Therefore $M_j \in K^{sat}$.

Case b: $cf(j) \leq \lambda$. Without loss of generality, $cf(j) = j$. So $|j| = j = cf(j) \leq \lambda$. Let $\langle N_{i,\alpha} : \alpha \in \lambda^+\rangle$ be a representation of M_i. For every $i < j$, let E_i be a club of λ^+ such that for $\alpha \in E_i$, $NF(N_{i,\alpha}, N_{i+1,\alpha}, N_{i,\alpha+1}, N_{i+1,\alpha+1})$ and if i is a limit ordinal, then $N_{i,\alpha} = \bigcup\{N_{\varepsilon,\alpha} : \varepsilon < i\}$. So $E := \bigcap\{E_i : i < j\}$ is a club set of λ^+ (because $|j| \leq \lambda$). Define $N_{j,\alpha} := \bigcup\{N_{i,\alpha} : i < j\}$. $\langle N_{j,\alpha} : \alpha \leq \lambda^+\rangle$ is a representation of M_j. Take $\alpha^* \in E$ such that $N \subseteq N_{j,\alpha^*}$. By Axiom 1.0.3.1.e, $N \preceq N_{j,\alpha^*}$, so it is sufficient to prove that M_j is saturated over N_{j,α^*}. Let $q \in S^{bs}(N_{j,\alpha^*})$. We will prove that q is realized in M_j. By the definition of E, the sequence $\langle N_{i,\alpha^*} : i < j\rangle$ is increasing and continuous, so by Definition 2.1.1.3.c (the local character) there is an ordinal $i < j$ such that q does not fork over N_{i,α^*}. M_i is saturated in λ^+ over λ, so there is $a \in M_i$ such that $tp(a, N_{i,\alpha^*}, M_i) = q \restriction N_{i,\alpha^*}$. By Definition 6.1.2, we have $\widehat{NF}(N_{i,\alpha^*}, N_{j,\alpha^*}, M_i, M_j)$. So by Theorem 6.1.3.e, ($\widehat{NF}$ respects $\mathfrak{s}$) $tp(a, N_{j,\alpha^*}, M_j)$ does not fork over N_{i,α^*}. Hence by Definition 2.1.1.3.d (the uniqueness of the non-forking extension) $tp(a, N_{j,\alpha^*}, M_j) = q$.

(b) The first part of Axiom c of AEC in λ^+ is item a here. Axioms b,e and the second part of Axiom c follow by Proposition 6.1.6.f.

(c) By Corollary 7.1.17.a. ⊣

8. Relative saturation

8.1. **Discussion:** This section is, like the previous, a preparation for the proof of Theorem 7.1.3. We study the relation $\preceq^{\otimes}_{\lambda^+}$, a kind of relative saturation. This relation is similar to 'closure of $\preceq^{NF}_{\lambda^+}$ under smoothness' (see Proposition 8.1.3.b). Theorem 9.1.13 says that non-equality between the relations $\preceq^{NF}_{\lambda^+}, \preceq^{\otimes}_{\lambda^+}$ is equivalent to non-smoothness and also to a strengthened version of non-smoothness.

Hypothesis 8.1.1. $\mathfrak{s}$ is a semi-good λ-frame with conjugation and $K^{3,uq}$ satisfies the existence property.

Definition 8.1.2. $\preceq^{\otimes}_{\lambda^+} := \{(M_0, M_1) : M_0, M_1 \in K^{sat}$, $M_0 \prec M_1$ and for every $N_0, N_1 \in K_\lambda$, if $N_0 \preceq M_0$, $N_0 \preceq N_1 \preceq M_1$ and $p \in S^{bs}(N_1)$ does not fork over N_0, then for some element $d \in M_0$ $tp(d, N_1, M_1) = p\}$.

Proposition 8.1.3.

(a) If $M_0 \in K^{sat}$ and $M_0 \preceq^{NF}_{\lambda^+} M_1$, then $M_0 \preceq^{\otimes}_{\lambda^+} M_1$.
(b) If $\langle M_\varepsilon : \varepsilon \leq \delta \rangle$ is a $\preceq^{NF}_{\lambda^+}$-increasing continuous sequence of models in K^{sat} and for every $\varepsilon \in \delta$, $M_\varepsilon \preceq^{NF}_{\lambda^+} M_{\delta+1}$, then $M_\delta \preceq^{\otimes}_{\lambda^+} M_{\delta+1}$.

Proof. (a) Suppose $M_0 \preceq^{NF}_{\lambda^+} M_1$ and $M_0 \in K^{sat}$. Let N_0, N_1 be models K_λ with $N_0 \preceq M_0$ and $N_0 \preceq N_1 \preceq M_1$ and let p be a type $S^{bs}(N_1)$ that does not fork over N_0. We have to find an element $d \in M_0$ with $tp(d, N_1, M_1) = p$. By Proposition 6.1.6.g (LST for $\preceq^{NF}_{\lambda^+}$), for some $N_0^+, N_1^+ \in K_\lambda$ $N_0 \preceq N_0^+$, $N_1 \preceq N_1^+$ and $\widehat{NF}(N_0^+, N_1^+, M_0, M_1)$. By Axiom 1.0.3.1.e, $N_0 \preceq N_0^+$ and $N_1 \preceq N_1^+$. Let q be the non-forking extension of p to N_1^+. Since $M_0 \in K^{sat}$ for some $d \in M_0$, $tp(d, N_0^+, M_0) = q \restriction N_0^+$. By Proposition 2.5.6 q does not fork over N_0, so by Definition 2.1.1.3.b (monotonicity) q does not fork over N_0^+. By Theorem 6.1.3, $\widehat{NF}$ respects $\mathfrak{s}$, so $tp(d, N_1^+, M_1)$ does not fork over N_1^+. So by Definition 2.1.1.3.b (uniqueness), $tp(d, N_1^+, M_1) = q$. Therefore $tp(d, N_1, M_1) = p$.

(b) Suppose $N_0, N_1 \in K_\lambda$, $N_0 \preceq N_\delta$, $N_0 \preceq N_1 \preceq M_{\delta+1}$ and $p \in S^{bs}(N_1)$ does not fork over N_0. We have to find an element $d \in M_\delta$ that realizes p. For every $\alpha \leq \delta + 1$, there is a representation $\langle N_{\alpha,\varepsilon} : \varepsilon < \lambda^+ \rangle$ of M_α. Without loss of generality, $cf(\delta) = \delta$.

Case a: $\delta = \lambda^+$. So for some $\alpha < \delta$, $N_0 \subseteq M_\alpha$ and we can use item a.

Case b: $\delta < \lambda^+$. For each $\alpha \in \delta$, let E_α be a club of λ^+ such that for each $\varepsilon \in E_\alpha$: $NF(N_{\alpha,\varepsilon}, N_{\alpha+1,\varepsilon}, N_{\alpha,\varepsilon+1}, N_{\alpha+1,\varepsilon+1})$ and if α is limit then $N_{\alpha,\varepsilon} = \bigcup\{N_{\beta,\varepsilon} : \beta < \alpha\}$. Let $E_\delta := \{\alpha \in \lambda^+ : N_{\delta,\varepsilon} \subseteq N_{\delta+1,\varepsilon},\ N_{\delta,\varepsilon} = \bigcup\{N_{\alpha,\varepsilon} : \alpha < \delta\}\}$. Denote $E := \bigcap\{E_\alpha : \alpha \leq \delta\}$. By cardinality considerations there is

$\varepsilon \in E$ such that for $n < 2$ $N_n \subseteq N_{\delta+n,\varepsilon}$, so by Axiom 1.0.3.1.e $N_n \preceq N_{\delta+n,\varepsilon}$.

$$\begin{tikzcd}
d \in M_\alpha \arrow[r, "id"] & M_\delta \arrow[r, "id"] & M_{\delta+1} & \\
N_{\alpha,\varepsilon} \arrow[u, "id"] \arrow[r, "id"] & N_{\delta,\varepsilon} \arrow[u, "id"] \arrow[r, "id"] & N_{\delta+1,\varepsilon} \arrow[u, "id"] & q \\
 & N_0 \arrow[u, "id"] \arrow[r, "id"] & N_1 \arrow[u, "id"] & p
\end{tikzcd}$$

Let $q \in S^{bs}(N_{\delta+1,\varepsilon})$ be the non-forking extension of p. By Proposition 2.5.6 (the transitivity proposition), q does not fork over N_0. By Definition 2.1.1.3.b (monotonicity), q does not fork over $N_{\delta,\varepsilon}$, so $q \upharpoonright N_{\delta,\varepsilon}$ is basic. As $\varepsilon \in E$, the sequence $\langle N_{\alpha,\varepsilon} : \alpha \leq \delta \rangle$ is increasing and continuous. So by Definition 2.1.1.3.c (local character), there is $\alpha < \delta$ such that $q \upharpoonright N_{\delta,\varepsilon}$ does not fork over $N_{\alpha,\varepsilon}$. So by Proposition 2.5.6, q does not fork over $N_{\alpha,\varepsilon}$. Since $M_\alpha \preceq^{NF}_{\lambda^+} M_{\delta+1}$ by item a for some $d \in M_\alpha$, $tp(d, N_{\delta+1,\varepsilon}, M_{\delta+1}) = q$. So $tp(d, N_1, M_{\delta+1}) = p$. $\dashv$

The following proposition is similar to the saturativity = model homogeneity lemma.

Proposition 8.1.4. *Suppose*

(1) $M_0 \preceq^{\otimes}_{\lambda^+} M_1$.
(2) For $n < 3$ $N_n \in K_\lambda$.
(3) $N_0 \preceq M_0$.
(4) $N_0 \preceq N_2$ *and* $N_0 \preceq N_1 \preceq M_1$.

Then for some $N_1^* \in K_\lambda$ *and an embedding* $f : N_2 \to M_0$ *the following hold:*

(a) $f \upharpoonright N_0 = id_{N_0}$.
(b) $NF(N_0, f[N_2], N_1, N_1^*)$.
(c) $N_1^* \preceq M_1$.

$$\begin{tikzcd}
M_0 \arrow[r, "id"] & M_1 \\
f[N_2] \arrow[u, "id"] \arrow[r, "id"] & N_1^* \arrow[u, "id"] \\
N_0 \arrow[u, "id"] \arrow[r, "id"] & N_1 \arrow[u, "id"]
\end{tikzcd}$$

Proof. We try to choose $N_{0,\varepsilon}, N_{1,\varepsilon}, N_{2,\varepsilon}, f_\varepsilon$ by induction on $\varepsilon < \lambda^+$ such that:

(1) For $n < 3$ $\langle N_{n,\varepsilon} : \varepsilon < \lambda^+ \rangle$ is an increasing continuous of models in K_λ.
(2) For $n < 3$ $N_{n,0} = N_n$, $f_0 = id_{N_0}$.

(3) For $\varepsilon < \lambda^+$, $N_{0,\varepsilon} \preceq M_0 \wedge N_{1,\varepsilon} \preceq M_1$.
(4) $\langle f_\varepsilon : \varepsilon < \lambda^+ \rangle$ is increasing and continuous.
(5) $f_\varepsilon : N_{0,\varepsilon} \to N_{2,\varepsilon}$ is an embedding over N_0.
(6) For every $\varepsilon \in \lambda^+$ there is a_ε such that $(N_{0,\varepsilon}, N_{0,\varepsilon+1}, a_\varepsilon)$ is a uniqueness triple, $f_{\varepsilon+1}(a_\varepsilon) \in N_{2,\varepsilon}$ and $tp(a_\varepsilon, N_{1,\varepsilon}, N_{1,\varepsilon+1})$ does not fork over $N_{0,\varepsilon}$.
(7) $N_{0,\varepsilon} \preceq N_{1,\varepsilon}$ (actually follows by 6).

$$\begin{tikzcd}
 & M_0 \arrow[r, "id"] & M_1 \\
N_{2,\varepsilon+1} & N_{0,\varepsilon+1} \arrow[l, "f_{\varepsilon+1}"'] \arrow[r, "id"] \arrow[u, "id"] & N_{1,\varepsilon+1} \arrow[u, "id"'] \\
N_{2,\varepsilon} \arrow[u, "id"] & N_{0,\varepsilon} \arrow[l, "f_\varepsilon"'] \arrow[r, "id"] \arrow[u, "id"] & N_{1,\varepsilon} \arrow[u, "id"'] \\
 & N_0 \arrow[ul, "id"] \arrow[r, "id"] \arrow[u, "id"] & N_1 \arrow[u, "id"']
\end{tikzcd}$$

By clauses 1,4,5 and particularly 6 and Proposition 1.0.30, we cannot succeed. Where will we get stuck? For $\varepsilon = 0$ or limit, we will not get stuck. Suppose we have defined $N_{0,\varepsilon}, N_{1,\varepsilon}, N_{2,\varepsilon}, f_\varepsilon$. By clause 5, $f_\varepsilon[N_{0,\varepsilon}] \preceq N_{2,\varepsilon}$.

Case a: $f_\varepsilon[N_{0,\varepsilon}] \neq N_{2,\varepsilon}$. In this case we can find $N_{0,\varepsilon+1}, N_{1,\varepsilon+1}, N_{2,\varepsilon+1}$, $f_{\varepsilon+1}$ such that clauses 1-7 above hold [By the existence of the basic types, there is $b \in N_{2,\varepsilon} - f_\varepsilon[N_{0,\varepsilon}]$ such that $p := tp(b, f_\varepsilon[N_{0,\varepsilon}], N_{2,\varepsilon})$ is basic. Let $q \in S^{bs}(N_{1,\varepsilon})$ be the non-forking extension of $f_\varepsilon^{-1}(p)$. As $M_0 \preceq^{\otimes}_{\lambda^+} M_1 \wedge (n < 2 \Rightarrow N_{n,\varepsilon} \preceq M_n^*) \wedge N_{0,\varepsilon} \preceq N_{1,\varepsilon} \in K_\lambda$, there is $a \in M_0$ which realizes q. So $tp(a, N_{0,\varepsilon}, M_0) = f_\varepsilon^{-1}(p)$. As $K^{3,uq}$ satisfies the existence property, we can find $N_{0,\varepsilon+1}$ such that $(N_{0,\varepsilon}, N_{0,\varepsilon+1}, a) \in K^{3,uq}$. As M_0 is saturated in λ^+ over λ, by Lemma 1.0.31 (the saturation = model homogeneity lemma), without loss of generality, $N_{0,\varepsilon+1} \preceq M_0$. Denote a as a_ε. Choose $N_{1,\varepsilon+1} \preceq M_1$ such that $N_{0,\varepsilon+1} \bigcup N_{1,\varepsilon} \subseteq N_{1,\varepsilon+1}$. By Axiom 1.0.3.1.e, $N_{0,\varepsilon+1} \preceq N_{1,\varepsilon+1} \wedge N_{1,\varepsilon} \preceq N_{1,\varepsilon+1}$. Now $f_\varepsilon(tp(a_\varepsilon, N_{0,\varepsilon}, N_{0,\varepsilon+1}) = p$. So there are $N_{2,\varepsilon+1}, f_{\varepsilon+1}$ such that: $N_{2,\varepsilon} \preceq N_{2,\varepsilon+1},\ f_{\varepsilon+1}(a_\varepsilon) = b,\ f_\varepsilon \subseteq f_{\varepsilon+1} : N_{0,\varepsilon+1} \to N_{2,\varepsilon+1}$.]

Case b: $f_\varepsilon[N_{0,\varepsilon}] = N_{2,\varepsilon}$. Hence $N_{1,\varepsilon}, f_\varepsilon^{-1} \restriction N_2$ witness that our proposition is true [By 6, Definition 5.3.2 and Definition 5.3.1, $\zeta < \varepsilon \Rightarrow NF(N_{0,\zeta}, N_{0,\zeta+1}, N_{1,\zeta}, N_{1,\zeta+1})$. So by Theorem 5.5.1 (the long transitivity theorem), $NF(N_0, N_{0,\varepsilon}, N_1, N_{1,\varepsilon})$. So by the monotonicity of NF, we have $NF(N_0, f_\varepsilon^{-1}[N_2], N_1, N_{1,\varepsilon})$. So clause b in the proposition is satisfied. Clauses a,c are satisfied by 5,3, respectively].

Let $\varepsilon + 1$ be the first ordinal, in which, we will get stuck. In other words, suppose we have defined $N_{0,\varepsilon}, N_{1,\varepsilon}, N_{2,\varepsilon}, f_\varepsilon$ and we cannot find models $N_{0,\varepsilon+1}, N_{1,\varepsilon+1}, N_{2,\varepsilon+1}, f_{\varepsilon+1}$ such that clauses 1-7 above hold. So case b holds and the proposition is proved. $\dashv$

Proposition 8.1.5. *If $M_0 \preceq M_1,\ n < 2 \Rightarrow (||M_n||) = \lambda^+ \wedge A_n \subseteq M_n \wedge |A_n| \leq \lambda)$, then there are models $N_0, N_1 \in K_\lambda$ such that: $n < 2 \Rightarrow A_n \subseteq N_n \preceq M_n$ and $N_1 \bigcap M_0 = N_0$ (so of course $N_0 \preceq N_1$).*

Proof. Standard. $\dashv$

$M_1^* \preceq^{\otimes}_{\lambda^+} M_2^*$ does not imply $M_1^* \preceq^{NF}_{\lambda^+} M_2^*$, but we are able to construct useful approximations to the M_i^*.

Proposition 8.1.6. *If $M_1^* \preceq^{\otimes}_{\lambda^+} M_2^*$, then there is an increasing continuous sequence of models in K^{sat}, $\langle M_\varepsilon : \varepsilon \leq \lambda^+ + 1\rangle$ such that:*

(a) $M_{\lambda^+} = M_1^*,\ M_{\lambda^+ +1} = M_2^*$.
(b) $\varepsilon < \lambda^+ \Rightarrow M_\varepsilon \prec^+_{\lambda^+} M_{\varepsilon+1}$.
(c) $\varepsilon < \lambda^+ \Rightarrow M_\varepsilon \preceq^{NF}_{\lambda^+} M_2^*$.

Proof. By Proposition 7.1.12.c, there is a winning strategy for player 2 in the $\prec^+_{\lambda^+}$-game. Let F be such a winning strategy. Enumerate M_2^* by $\{a_\varepsilon : \varepsilon < \lambda^+\}$. We construct $\langle N_{\alpha,\varepsilon} : \varepsilon \leq \alpha\rangle,\ N_\alpha$ by induction on α such that the following hold:

(1) For each $\varepsilon \leq \alpha$, $N_{\alpha,\varepsilon} \in K_\lambda$ and $N_{\alpha,\varepsilon} \preceq M_1^*$.
(2) $\langle N_{\alpha,\varepsilon} : \varepsilon \leq \alpha < \lambda^+\rangle$ is increasing continuous in the variables α, ε.
(3) $\langle N_\alpha : \alpha < \lambda^+\rangle$ is an increasing continuous sequence of models in K_λ.
(4) $N_{\alpha,\alpha} \preceq N_\alpha \preceq M_2^*$.
(5) If $\alpha + 1$ is odd, then for each $\varepsilon \leq \alpha$, $N_{\alpha+1,\varepsilon+1}$ is isomorphic to $F(\langle N_{\beta,\varepsilon} : \varepsilon + 1 \leq \beta \leq \alpha + 1\rangle,\ \langle N_{\beta,\varepsilon+1} : \varepsilon + 1 \leq \beta \leq \alpha\rangle)$ over $N_{\alpha,\varepsilon+1} \bigcup N_{\alpha+1,\varepsilon}$.
(6) If $\alpha + 1$ is odd, then $NF(N_{\alpha,\alpha}, N_\alpha, N_{\alpha+1,\alpha+1}, N_{\alpha+1})$
(7) $a_\alpha \in N_{2\alpha+2}$.
(8) $N_{2\alpha} \bigcap M_1^* \subseteq N_{2\alpha,2\alpha}$.
(9) If $\alpha + 1$ is odd then $N_{\alpha+1,\alpha+1} = N_{\alpha+1,\alpha}$.
(10) If $\alpha + 1$ is odd then $N_{\alpha+1,0} \bigcap N_\alpha = N_{\alpha,0},\ N_{\alpha+1,0} \neq N_{\alpha,0}$.
(11) If $\alpha + 1$ is even then for each $\varepsilon \leq \alpha$ $N_{\alpha+1,\varepsilon} = N_{\alpha,\varepsilon}$.

$$\begin{tikzcd}
M_\varepsilon \arrow[r, "id"] & M_{\varepsilon+1} \arrow[r, "id"] & M_\alpha \arrow[r, "id"] & M_{\lambda^+} = M_1^* \arrow[r, "id"] & M_{\lambda^+ +1} = M_2^* \\
N_{\alpha,\varepsilon} \arrow[u, "id"] \arrow[r, "id"] & N_{\alpha,\varepsilon+1} \arrow[u, "id"] \arrow[r, "id"] & N_{\alpha,\alpha} \arrow[u, "id"] \arrow[rr, "id"] & & N_\alpha \arrow[u, "id"] \\
N_{\varepsilon+1,\varepsilon} \arrow[u, "id"] \arrow[r, "id"] & N_{\varepsilon+1,\varepsilon+1} \arrow[u, "id"] \arrow[rrr, "id"] & & & N_{\varepsilon+1} \arrow[u, "id"] \\
N_{\varepsilon,\varepsilon} \arrow[u, "id"] \arrow[rrrr, "id"] & & & & N_\varepsilon \arrow[u, "id"]
\end{tikzcd}$$

[Explanation: $N_{\alpha,\alpha}$, N_α are approximations for M_1^*, M_2^*, respectively. $N_{\alpha,\varepsilon}$ is an approximation for M_ε. When $\alpha+1$ is even, we increase the approximations of M_1^*, M_2^*, such that at the end we will have $M_2^* \subseteq \bigcup\{N_\alpha : \alpha < \lambda^+\}$, $M_1^* = \bigcup\{N_{\alpha,\alpha} : \alpha < \lambda^+\}$ by 7,8, respectively. When $\alpha+1$ is odd, we increase the approximations of M_ε (mainly by clause 10). Clause 11 says that in even steps the approximations to M_ε do not increase. Clause 5 insures, that at the end, we will have $M_\varepsilon \prec^+_{\lambda^+} M_{\varepsilon+1}$. Clause 6 insures, that at the end requirement c will be satisfied. The point of the proof is, that we could not demand 6 for every α, (as otherwise we prove $M_1^* \preceq^{NF}_{\lambda^+} M_2^*$, which might be wrong). But we succeed to prove that $NF(N_{\alpha,\varepsilon}, N_\alpha, N_{\alpha+1,\varepsilon}, N_{\alpha+1})$ so $M_\varepsilon \preceq^{NF}_{\lambda^+} M_2^*$].

Why can we carry out the construction? We construct by induction on α. For limit α, by clauses 2,3 there is no freedom. Clauses 1,4 are satisfied by smoothness, clauses 5,6,7,9,10,11 are irrelevant and clause 8 is satisfied. For $\alpha = 0$ we choose $N_0, N_{0,0}$ by Proposition 8.1.5. Suppose we have defined $\langle N_{\alpha,\varepsilon} : \varepsilon \le \alpha\rangle$, N_α. What will we do in step $\alpha+1$?
Case a: $\alpha+1$ is even. For $\varepsilon \le \alpha$ define $N_{\alpha+1,\varepsilon} := N_{\alpha,\varepsilon}$. By Proposition 8.1.5 there are $N_{\alpha+1}$, $N_{\alpha+1,\alpha+1}$ as required, especially clauses 7,8 are satisfied.
Case b: $\alpha+1$ is odd. Define $N^{temp}_{\alpha+1,\varepsilon}$ by induction on $\varepsilon \le \alpha$ such that:

(1) $\langle N^{temp}_{\alpha+1,\varepsilon} : \varepsilon \le \alpha\rangle$ is an $\preceq$-increasing continuous sequence.
(2) $N^{temp}_{\alpha+1,\varepsilon+1} = F(\langle N_{\beta,\varepsilon} : \varepsilon+1 \le \beta \le \alpha\rangle^\frown\langle N^{temp}_{\alpha+1,\varepsilon}\rangle,\ \langle N_{\beta,\varepsilon+1} : \varepsilon+1 \le \beta < \alpha\rangle)$.
(3) $N_{\alpha,0} \preceq N^{temp}_{\alpha+1,0}$.

Now by Proposition 8.1.4, there are $N_{\alpha+1}$ and an embedding $g : N^{temp}_{\alpha+1,\alpha} \to M_1^*$ over $N_{\alpha,\alpha}$ such that we have $NF(N_{\alpha,\alpha}, N_\alpha, g[N^{temp}_{\alpha+1,\alpha}], N_{\alpha+1})$. For every $\varepsilon \le \alpha$, define $N_{\alpha+1,\varepsilon} := g[N^{temp}_{\alpha+1,\varepsilon}]$. Now define $N_{\alpha+1,\alpha+1} := N_{\alpha+1,\alpha}$. So we can carry out the construction.

Why is it sufficient? For $\varepsilon < \lambda^+$ define $M_\varepsilon := \bigcup\{N_{\alpha,\varepsilon} : \varepsilon \le \alpha < \lambda^+\}$. Define $M_{\lambda^+} := \bigcup\{M_\varepsilon : \varepsilon < \lambda^+\}$, $M_{\lambda^++1} := \bigcup\{N_\alpha : \alpha < \lambda^+\}$. We will prove that the sequence $\langle M_\varepsilon : 0 < \varepsilon < \lambda^+ + 1\rangle$ satisfies requirements a,b,c:

(a) By 3,4,7 $M_{\lambda^++1} = M_2^*$. Why is $M_{\lambda^+} = M_1^*$? By 1 $M_{\lambda^+} \subseteq M_1^*$. Let $x \in M_1^*$. Then $x \in M_2^* = M_{\lambda^++1}$. So by the definition of M_{λ^++1} and 3, there is α such that $x \in N_{2\alpha}$. So by 8 $x \in N_{2\alpha,2\alpha}$. But by the definitions of $M_\varepsilon, M_{\lambda^+}$, $N_{2\alpha,2\alpha} \subseteq M_{2\alpha} \subseteq M_{\lambda^+}$.

(b) By 2,10 $|M_0| = \lambda^+$. By 2 and smoothness, the sequence $\langle M_\varepsilon : \varepsilon < \lambda^+\rangle$ is $\preceq$-increasing and continuous. So $|M_\varepsilon| = \lambda^+$. Does $\varepsilon < \lambda^+ \Rightarrow M_\varepsilon \in K^{sat}$? Not exactly, but we can prove by induction on ε that $0 < \varepsilon < \lambda^+ \Rightarrow (M_\varepsilon \in K^{sat} \wedge M_\varepsilon \prec^+_{\lambda^+} M_{\varepsilon+1})$: For $\varepsilon = 0$ by 10. For limit ε by Theorem 7.1.18.a. For ε successor by 5 and Proposition 7.1.10.b. So requirement b is satisfied.

(c) The sequences $\langle N_{\alpha,\varepsilon} : \varepsilon \le \alpha < \lambda^+\rangle$, $\langle N_\alpha : \varepsilon \le \alpha < \lambda^+\rangle$ are representations of M_ε, M_{λ^++1}, respectively. Let $\alpha \in \lambda^+$. We will prove

$NF(N_{\alpha,\varepsilon}, N_\alpha, N_{\alpha+1,\varepsilon}, N_{\alpha+1})$. If $\alpha+1$ is even, this is satisfied by clause 11. So let $\alpha+1$ be odd. By 6 we have: (*) $NF(N_{\alpha,\alpha}, N_\alpha, N_{\alpha+1,\alpha+1}, N_{\alpha+1})$. By 5 and Theorem 5.5.1 (the transitivity of NF), $NF(N_{\alpha,\varepsilon}, N_{\alpha,\alpha}, N_{\alpha+1,\varepsilon}, N_{\alpha+1,\alpha})$ [Why? By 5 (and Proposition 7.1.12.c), $\forall\zeta \in [\varepsilon,\alpha) NF(N_{\alpha,\zeta}, N_{\alpha,\zeta+1}, N_{\alpha+1,\zeta}, N_{\alpha+1,\zeta+1})$. The sequences $\langle N_{\alpha,\zeta} : \zeta \in [\varepsilon,\alpha)\rangle$, $\langle N_{\alpha+1,\zeta} : \zeta \in [\varepsilon,\alpha)\rangle$ are increasing and continuous. So by Theorem 5.5.1 (the long transitivity theorem), $NF(N_{\alpha,\varepsilon}, N_{\alpha,\alpha}, N_{\alpha+1,\varepsilon}, N_{\alpha+1,\alpha})$. So by the monotonicity of NF, we have: (**) $NF(N_{\alpha,\varepsilon}, N_{\alpha,\alpha}, N_{\alpha+1,\varepsilon}, N_{\alpha+1,\alpha+1})$]. Now by (*),(**) and Theorem 5.5.1 $NF(N_{\alpha,\varepsilon}, N_{\alpha+1,\varepsilon}, N_\alpha, N_{\alpha+1})$. Note that we use here freely Theorem 5.4.9 (the symmetry theorem of NF). $\dashv$

9. Non-smoothness implies non-structure

9.1. **Introduction.**

Hypothesis 9.1.1. $\mathfrak{s}$ is a semi-good λ-frame with conjugation and $K^{3,uq}$ satisfies the existence property.

Definition 9.1.2. Let $\bar{M} = \langle M_\alpha : \alpha < \alpha^*\rangle$ be an increasing sequence of models in K_{λ^+}. We say that $\bar{M}$ is $\preceq^{NF}_{\lambda^+}$*-increasing in the successor ordinals* if $\beta < \gamma < \alpha^* \Rightarrow M_{\beta+1} \preceq^{NF}_{\lambda^+} M_{\gamma+1}$.

Definition 9.1.3. Let $\alpha \leq \lambda^{+2}$ and let $\bar{M} = \langle M_\alpha : \alpha < \lambda^{+2}\rangle$ be an $\preceq^{NF}_{\lambda^+}$-increasing in the successor ordinals and continuous sequence with union M. Define $T(\bar{M}) =: \{\delta \in \lambda^{+2} : \exists\alpha \in (\delta, \lambda^{+2})\ M_\delta \npreceq^{NF} M_\alpha\}$. Define $T(M) =: T(\bar{M})/D_{\lambda^{+2}}$ where $D_{\lambda^{+2}}$ is the club filter on λ^{+2}. (By Proposition 9.1.5, $T(M)$ does not depend on the representation $\bar{M}$).

Proposition 9.1.4. *Let $\bar{M} = \langle M_\alpha : \alpha < \lambda^{+2}\rangle$ be a $\preceq^{NF}_{\lambda^+}$-increasing in the successor ordinals and continuous sequence. Then:*
(a) For each $\alpha < \lambda^{+2}$, $M_\alpha \preceq^{NF}_{\lambda^+} M_{\alpha+1} \Leftrightarrow [(\forall\beta \in (\alpha, \lambda^{+2}))M_\alpha \preceq^{NF}_{\lambda^+} M_\beta]$.
(b) $T(\bar{M}) = \{\delta \in \lambda^{+2} : \forall\alpha \in (\delta, \lambda^{+2})\ M_\delta \npreceq^{NF} M_\alpha\}$.

Proof.
(a) Easy (by Proposition 6.1.6.c).
(b) By item a.

⊣

Proposition 9.1.5. *Suppose:*
(1) The sequences $\bar{M}^1 := \langle M_{\alpha,1} : \alpha < \lambda^{+2}\rangle$, $\bar{M}^2 := \langle M_{\alpha,1} : \alpha < \lambda^{+2}\rangle$ *are* $\preceq^{NF}_{\lambda^+}$*-increasing in the successor ordinals and continuous.*
(2) $M_1 = \bigcup\{M_{\alpha,1} : \alpha < \lambda^{+2}\}$ *and* $M_2 = \bigcup\{M_{\alpha,2} : \alpha < \lambda^{+2}\}$.
(3) M_1, M_2 *are isomorphic.*

Then $T(\bar{M}^1)/D_{\lambda^{+2}} = T(\bar{M}^2)/D_{\lambda^{+2}}$.

Proof. Let $f : M_1 \to M_2$ be an isomorphism. Define $E := \{\alpha \in \lambda^{+2} : f[M_{1,\alpha}] = M_{2,\alpha}\}$. So $T(\langle M_{\alpha,1} : \alpha \in E\rangle) = T(\langle f[M_{\alpha,1}] : \alpha \in E\rangle) = T(\langle M_{\alpha,2} : \alpha \in E\rangle)$. By Proposition 9.1.4.b $T(\langle M_{\alpha,1} : \alpha \in E\rangle) = T(\bar{M}^1) \bigcap E$ and $T(\langle M_{\alpha,2} : \alpha \in E\rangle) = T(\bar{M}^2) \bigcap E$. Hence $T(\bar{M}^1) \bigcap E = T(\bar{M}^2) \bigcap E$. ⊣

Proposition 9.1.6. *Assume that we can assign to each* $S \in S^{\lambda^{+2}}_{\lambda^+} := \{S : S$ *is a stationary subset of* λ^{+2} *and* $(\forall \alpha \in S) cf(\alpha) = \lambda^+\}$*, a model* $M^S \in K_{\lambda^{+2}}$ *with* $T(M^S) = S/D_{\lambda^{+2}}$ *(especially it is defined).*

Then there are $2^{\lambda^{+2}}$ *non-isomorphic models in* $K_{\lambda^{+2}}$.

Proof. Since $|S^{\lambda^{+2}}_{\lambda^+}| = 2^{\lambda^{+2}}$ it follows by Proposition 9.1.5. ⊣

The following theorem says that there is a kind of a witness for non-$\preceq^{NF}_{\lambda^+}$-smoothness, such that if it holds, then there are $2^{\lambda^{+2}}$ non-isomorphic models in $K_{\lambda^{+2}}$.

Theorem 9.1.7. *Suppose that there is an increasing continuous sequence* $\langle M^*_\alpha : \alpha \leq \lambda^+ + 1\rangle$ *of models in* K^{sat} *such that for each* α, β *with* $\alpha < \beta < \lambda^+$*, we have* $M^*_\alpha \prec^+_{\lambda^+} M^*_\beta \preceq^{NF}_{\lambda^+} M^*_{\lambda^+ + 1}$ *but* $M^*_{\lambda^+} \npreceq^{NF} M^*_{\lambda^+ + 1}$.

Then there are $2^{\lambda^{+2}}$ *pairwise non-isomorphic models in* $K_{\lambda^{+2}}$.

Proof. By Proposition 9.1.6, it is enough to assign to each $S \in S^{\lambda^{+2}}_{\lambda^+}$ a model $M^S \in K_{\lambda^{+2}}$ with $T(M^S) = S/D_{\lambda^{+2}}$. Let S be a stationary subset of λ^{+2} such that $\alpha \in S \Rightarrow cf(\alpha) = \lambda^+$. We will choose a model M_β by induction on $\beta < \lambda^{+2}$ such that:
(1) $M_\beta \in K^{sat}$.
(2) The sequence $\langle M_\beta : \beta < \lambda^{+2}\rangle$ is continuous.
(3) $\beta \in \lambda^{+2} - S \Rightarrow M_\beta \prec^+_{\lambda^+} M_{\beta+1}$.
(4) If $\beta \in S$ then $(M_\beta, M_{\beta+1}) \cong (M^*_{\lambda^+}, M^*_{\lambda^+ + 1})$.
(5) For each $\beta < \lambda^{+2}$ $M_\beta \preceq^{NF}_{\lambda^+} M_{\beta+1} \Leftrightarrow \beta \notin S$.

Note that clause 5 is the crucial point and it actually follows by clauses 3,4.

[Why is it possible to choose M_β? For $\beta = 0$ we choose a model $M_0 \in K^{sat}$. For limit ordinal β, define $M_\beta = \bigcup\{M_\gamma : \gamma < \beta\}$. What will we do in the $\beta+1$ step? Clause 5 follows by clauses 3,4. So it is enough to find $M_{\beta+1}$ which satisfies clauses 3,4.

case a: $\beta \notin S$. In this case we choose $M_{\beta+1}$ such that $M_\beta \prec^+_{\lambda^+} M_{\beta+1}$ (see Proposition 7.1.12.a).

case b: $\beta \in S$. Since $M_\beta, M^*_{\lambda^+}$ are saturated in λ^+ over λ, they are isomorphic. Hence we can find $M_{\beta+1}$ satisfying clause 4].

Define $M^S := \bigcup\{M_\alpha : \alpha < \lambda^{+2}\}$. It remains to prove that $T(M^S) = S/D_{\lambda^{+2}}$ (especially $T(M^S)$ is defined). But if $T(\langle M_\alpha : \alpha < \lambda^{+2}\rangle)$ is defined then by clause 5, $T(M^S) = T(\langle M_\alpha : \alpha < \lambda^{+2}\rangle)/D_{\lambda^{+2}} = S/D_{\lambda^{+2}}$. So it is enough to prove that it is defined, namely, to prove that for each α, β with $\alpha < \beta < \lambda^{+2}$ we have $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_{\beta+1}$. But it is easier to prove the following stronger claim:

Claim 9.1.8. *For every $\beta \leq \lambda^+$ $(*)_\beta$: For each α with $\alpha < \beta$, the following hold:*

(1) $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_{\beta+1}$.
(2) If $\beta \notin S$ then $M_{\alpha+1} \prec^+ M_{\beta+1}$.

Proof. $(*)_0$ is vacuous.

Why does $(*)_\beta \Rightarrow (*)_{\beta+1}$ hold? Fix $\alpha < \beta+1$. We prove that $M_{\alpha+1} \prec^+_{\lambda^+} M_{\beta+2}$. By clause 3, $M_{\beta+1} \prec^+ M_{\beta+2}$. So, if $\alpha = \beta$ then $M_{\alpha+1} \prec^+_{\lambda^+} M_{\beta+2}$. So without loss of generality, $\alpha < \beta$. By $(*)_\beta$ $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_{\beta+1}$. But $M_{\beta+1} \prec^+_{\lambda^+} M_{\beta+2}$. So by Proposition 7.1.10.c, $M_{\alpha+1} \prec^+_{\lambda^+} M_{\beta+2}$. This establishes $(*)_{\beta+1}$.

Assume that δ is a limit ordinal and $(*)_\beta$ holds for each β with $\beta < \delta$. We have to prove $(*)_\delta$. Let $\langle \gamma(\varepsilon) : \varepsilon < cf(\delta)\rangle$ be an increasing continuous of ordinals with limit δ, such that for every ε, $\gamma(\varepsilon+1)$ is a successor of a successor ordinal. Note that for every $\varepsilon < \mathrm{cf}(\delta)$ $\gamma_\varepsilon \notin S$, because $cf(\gamma_\varepsilon) < \mathrm{cf}(\delta) \leq \lambda^+$. Consider the sequence $\langle M_{\gamma_\varepsilon} : \varepsilon < cf(\delta)\rangle$.

Claim 9.1.9. *$M_{\gamma_\varepsilon} \prec^+ M_{\gamma_{\varepsilon+1}}$ for each $\varepsilon < cf(\delta)$.*

Proof. Since $\gamma_\varepsilon \notin S$, by clause 3, $M_{\gamma_\varepsilon} \prec^+_{\lambda^+} M_{\gamma_\varepsilon+1}$. If $\gamma_{\varepsilon+1} = \gamma_\varepsilon + 1$, then the claim is proved. Assume $\gamma_{\varepsilon+1} > \gamma_\varepsilon + 1$. $\gamma_{\varepsilon+1} = \zeta + 1$ for some successor ζ. $\zeta \notin S$. So by $(*)_\zeta.2$, $M_{\gamma_\varepsilon+1} \prec^+_{\lambda^+} M_{\zeta+1} = M_{\gamma_{\varepsilon+1}}$. So $M_{\gamma_\varepsilon} \prec^+_{\lambda^+} M_{\gamma_\varepsilon+1} \prec^+_{\lambda^+} M_{\gamma_{\varepsilon+1}}$. Hence by Proposition 7.1.10.d $M_{\gamma_\varepsilon} \prec^+_{\lambda^+} M_{\gamma_{\varepsilon+1}}$. ⊣

Claim 9.1.10. *The sequence $\langle M_{\gamma_\varepsilon} : \varepsilon < cf(\delta)\rangle^\frown\langle M_\delta\rangle$ is continuous.*

Proof. Take $\delta' \in \{\gamma_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\} \bigcup \{\delta\}$ and take $x \in M_{\delta'}$. We have to find $\varepsilon < cf(\delta)$ such that $\gamma_\varepsilon < \delta'$ and $x \in M_{\gamma_\varepsilon}$. By clause 2 the sequence $\langle M_\beta : \beta < \lambda^{+2}\rangle$ is continuous, so for some $\beta < \delta'$ $x \in M_\beta$. The ordinals sequence $\langle \gamma_\varepsilon : \varepsilon < cf(\delta)\rangle^\frown\langle\delta\rangle$ is increasing and continuous. Hence for some $\varepsilon < cf(\delta)$ with $\beta < \gamma_\varepsilon < \delta'$. Since $M_\beta \subseteq M_{\gamma_\varepsilon}$, $x \in M_{\gamma_\varepsilon}$. ⊣

Claim 9.1.11. *$M_{\gamma_\varepsilon} \preceq^{NF}_{\lambda^+} M_\delta$ for each $\varepsilon < \mathrm{cf}(\delta)$.*

Proof. By Proposition 6.1.6.d (and Claim 9.1.9, Claim 9.1.10 and Proposition 7.1.10.a). ⊣

Now we return to the proof of $(*)_\delta$. Fix $\alpha < \delta$.

Claim 9.1.12. *$M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_{\gamma_{\varepsilon+1}}$ for some $\varepsilon < cf(\delta)$.*

Proof. Take $\varepsilon < cf(\delta)$ with $\alpha + 1 < \gamma_{\varepsilon+1}$. $\gamma_{\varepsilon+1} = \zeta + 1$ for some ζ. So by $(*)_\zeta.1$, $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_{\zeta+1} = M_{\gamma_{\varepsilon+1}}$. $\dashv$

Case a: $\delta \notin S$. In this case by clause 4, $M_\delta \prec^+_{\lambda^+} M_{\delta+1}$. So by Proposition 7.1.10.c, it is enough to prove that $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_\delta$. By Claim 9.1.12 $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_{\gamma_{\varepsilon+1}}$ for some ε. By Claim 9.1.11, $M_{\gamma_{\varepsilon+1}} \preceq^{NF}_{\lambda^+} M_\delta$. So by Proposition 6.1.6.b, $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_\delta$.

Case b: $\delta \in S$. In this case we have to prove that $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_{\delta+1}$. We choose f_α by induction on $\alpha \le \lambda^+$ such that:

(1) For every $\alpha \le \lambda^+$, $f_\alpha : M^*_\alpha \to M_{\gamma_\alpha}$ is an isomorphism.
(2) $\langle f_\alpha : \alpha \le \lambda^+ \rangle$ is an increasing continuous sequence of isomorphisms.

There is no problem to carry out this induction [Why? We can choose f_0 by Theorem 1.0.32, (the uniqueness of the saturated model in λ^+ over λ). $M^*_\alpha \prec^+_{\lambda^+} M^*_{\alpha+1}$. By Claim 9.1.8 $M_{\gamma_\alpha} \prec^+_{\lambda^+} M_{\gamma_{\alpha+1}}$. So by Theorem 7.1.13.a, for every α, we can find $f_{\alpha+1}$. For α limit take union].

Now by clause 4, $(M_\delta, M_{\delta+1}) \cong (M^*_{\lambda^+}, M^*_{\lambda^++1})$. So we can find an isomorphism $f : M_{\lambda^++1} \to M_{\delta+1}$ that extends f_{λ^+}. For every $\varepsilon < \lambda^+$, $M^*_\varepsilon \preceq^{NF}_{\lambda^+} M^*_{\lambda^++1}$, so $M_{\gamma_\varepsilon} = f[M^*_\varepsilon] \preceq^{NF}_{\lambda^+} f[M^*_{\lambda^++1}] = M_{\delta+1}$. So $M_{\gamma_\varepsilon} \preceq M_{\delta+1}$ for each $\varepsilon < cf(\delta)$. Hence $M_{\gamma_{\varepsilon+1}} \preceq^{NF}_{\lambda^+} M_{\delta+1}$ for each $\varepsilon < cf(\delta)$. But by Claim 9.1.12 for some $\varepsilon < cf(\delta)$ $M_{\alpha+1} \preceq^{NF}_{\lambda^+} M_{\gamma_{\varepsilon+1}}$. Therefore by Proposition 6.1.6.b, $M_{\alpha+1} \preceq^{NF} M_{\delta+1}$. $\dashv$

$\dashv$

Theorem 9.1.13. *The following conditions are equivalent:*

(a) $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ does not satisfy smoothness.
(b) There are $M^*_1, M^*_2 \in K^{sat}$ such that $M^*_1 \preceq^{\otimes} M^*_2$ but $M^*_1 \npreceq^{NF} M^*_2$.
(c) There is a sequence $\langle M_\varepsilon : \varepsilon \le \lambda^+ + 1 \rangle$ of models in K^{sat} such that for each ε, ζ with $\varepsilon < \zeta \le \lambda^+ + 1$, we have $\varepsilon \ne \lambda^+ \Leftrightarrow M_\varepsilon \prec^+_{\lambda^+} M_\zeta \Leftrightarrow M_\varepsilon \preceq^{NF}_{\lambda^+} M_\zeta$.

Proof. $c \Rightarrow a$ is clear. $b \Rightarrow c$ holds by Proposition 8.1.6. $a \Rightarrow b$ holds by Proposition 8.1.3.b. $\dashv$

Now we can prove Theorem 7.1.3, but first we recall it: If $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ does not satisfy smoothness, then there are $2^{\lambda^{+2}}$ pairwise non-isomorphic models in $K_{\lambda^{+2}}$.

Proof. Condition a of Theorem 9.1.13 is satisfied, so condition c is satisfied, too. Hence by Theorem 9.1.7 we have the conclusion of the theorem. $\dashv$

10. A good λ^+-frame

10.1. **Discussion.** In Definitions 2.6.1, 2.6.2 and 2.6.4, we expanded the definition of the non-forking relation and basic types to models in $K_{>\lambda}$. In Theorem 2.6.8 we proved some axioms of a good frame for this expansions. Here we are going to prove the other axioms. So why are Sections 3-9 needed? In other words, what are the difficulties in proving that $\mathfrak{s}^+$ (defined below) is a good λ^+-frame? The main problem is that amalgamation may not hold in $(K_{\lambda^+}, \preceq\restriction K_{\lambda^+})$. Now we can solve this problem by restricting the relation $\preceq_{K_{\lambda^+}}$ to the relation $\preceq^{NF}_{\lambda^+}$. But then we lose smoothness. We solve this problem, showing that if we restrict to the class of saturated models in λ^+ over λ then non-smoothness of $\preceq^{NF}_{\lambda^+}$ implies many models. Now the relation $\prec^+_{\lambda^+}$ and tameness enable us to prove the remaining axioms.

Definition 10.1.1. Let $\mathfrak{s}$ be a semi-good λ-frame. We say that $\mathfrak{s}$ is *successful* when:

(1) $K^{3,uq}$ satisfies the existence property.
(2) satisfies smoothness.

Hypothesis 10.1.2. $\mathfrak{s}$ is a successful semi-good λ-frame with conjugation.

We recall that the types in this paper are classes of triples under some equivalence relation. But this relation depends on the partial order, that we define on the class of models, see Definition 1.0.24. For $M_0, M_1 \in K_{\lambda^+}$, when we write $tp(a, M, N)$, we mean to the partial order $\preceq$. But when we want to consider the partial order $\preceq^{NF}_{\lambda^+}$, we have to write it explicitly.

Definition 10.1.3. For $M_0, M_1 \in K^{sat}$ and $a \in M_1 - M_0$, we define

$$tp^+(a, M_0, M_1) := tp_{((K^{sat})^{up}, (\preceq^{NF}_{\lambda^+}\restriction K^{sat})^{up})}(a, M_0, M_1).$$

(About '*sat*' see Definition 7.1.2 and about '*up*' see Definition 1.0.16.)

Proposition 10.1.4. *For every M_0, M_1, M_2 with $M_0 \preceq^{NF}_{\lambda^+} M_1 \wedge M_0 \preceq^{NF}_{\lambda^+} M_2$ and every a_1, a_2 with $a \in M_1 - M_0 \wedge a_2 \in M_2 - M_0$:*

$$tp^+(a_1, M_0, M_1) = tp^+(a_2, M_0, M_2) \Leftrightarrow tp(a_1, M_0, M_1) = tp(a_2, M_0, M_2).$$

Proof. The first direction: Suppose $tp^+(a_1, M_0, M_1) = tp^+(a_2, M_0, M_2)$. By Theorem 7.1.18.c, $(K^{sat}, \preceq^{NF}_{\lambda^+}\restriction K^{sat})$ satisfies the amalgamation property. So there are f_1, f_2, M_3 such that: $M_0 \preceq^{NF}_{\lambda^+} M_3$, $f_n : M_n \to M_3$ is a $\preceq^{NF}_{\lambda^+}$-embedding over M_0 and $f_1(a_1) = f_2(a_2)$. But $K^{sat} \subseteq K$, and the relation $\preceq^{NF}_{\lambda^+}$ is included in the relation $\preceq$, so the amalgamation (f_1, f_2, M_3) witnesses that $tp(a_1, M_0, M_1) = tp(a_2, M_0, M_2)$.

The second direction: Suppose $tp(a_1, M_0, M_1) = tp(a_2, M_0, M_2)$. Take an amalgamation (f_1, f_2, M_3) of M_1, M_2 over M_0 with $f_1(a_1) = f_2(a_2)$. For each $N \in K_\lambda$ with $N \preceq M_0$ $tp(f_1(a_1), N, f_1[M_1]) = tp(f_2(a_2), N, f_2[M_2])$. So by Theorem 7.1.13.b, $tp^+(a_1, M_0, M_1) = tp^+(a_2, M_0, M_2)$. $\dashv$

Although we defined restriction of types in Definition 1.0.24.3, the following definition is needed for tp^+:

Definition 10.1.5. For $p = tp^+(a, M_0, M_1)$ and $N \in K_\lambda$ with $N \preceq M_0$, we define $p \restriction N := tp(a, N, M_1)$.

The following definition is based on Definition 2.6.1.

Definition 10.1.6. $\mathfrak{s}^+ := ((K^{sat})^{up}, (\preceq^{NF}_{\lambda^+} \restriction K^{sat})^{up}, \mathfrak{s}^{bs,+}, \overset{+}{\bigcup})$, where:

(1) For each $M \in K^{sat}$, we define $S^{bs,+}(M) := \{tp^+(a, M, N) : \{M, N\} \subseteq K^{sat},\ M \preceq^{NF}_{\lambda^+} N,\ tp(a, M, N) \in S^{bs}_{>\lambda}\}$

(2) $\overset{+}{\bigcup}$ is defined by: $tp^+(a, M_1, M_2)$ does not fork over M_0 if $\{M_0, M_1, M_2\} \subseteq K^{sat},\ M_0 \preceq^{NF}_{\lambda^+} M_1 \preceq^{NF}_{\lambda^+} M_2$ and $tp(a, M_1, M_2)$ does not fork over M_0 in the sense of Definition 2.6.2.

Proposition 10.1.7.

(a) S^{bs} is well-defined: It does not depend on the triple (M_0, M_1, a) that represents the type.

(b) $\overset{+}{\bigcup}$ is well-defined: It does not depend on the triple (M_0, M_1, a) that represents the type.

Proof. By Proposition 10.1.4. $\dashv$

Proposition 10.1.8. *Let $\mathfrak{s}$ be a successful semi-good λ-frame with conjugation.*

(1) $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ satisfies Axiom c of AEC in λ^+ (i.e., Definition 1.0.3.2.c).

(2) $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ is an AEC in λ^+.

(3) $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ satisfies the amalgamation property.

Proof. By Theorem 7.1.18 and hypothesis 10.1.2. $\dashv$

Theorem 10.1.9. *Let $\mathfrak{s}$ be a successful semi-good λ-frame with conjugation. Then $\mathfrak{s}^+$ is a good λ^+-frame.*

(So although in λ we have almost stability only, we get good λ^+-frame, so stability!)

Proof. By Proposition 10.1.8, $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ is an AEC in λ^+ with amalgamation. So by Fact 1.0.18, $((K^{sat})^{up}, (\preceq^{NF}_{\lambda^+} \restriction K^{sat})^{up})$ is an AEC with LST number λ^+. By Theorem 1.0.32, K^{sat} is categorical. So $(K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat})$ satisfies the joint embedding property. By Proposition 7.1.12.a and Proposition 7.1.10.a, there is no $\preceq^{NF}_{\lambda^+}$-maximal model in K^{sat}. What about the axioms of the basic types and the non-forking relation? By Theorem 2.6.8, the following axioms are satisfied: Density, monotonicity, local character and continuity.

Proposition 10.1.10. $\mathfrak{s}^+$ *satisfies basic stability.*

Proof. Let $M \in K^{sat}$. $M \in K_{\lambda^+}$, so it has a representation $\langle N_\alpha : \alpha \in \lambda^+ \rangle$ (each N_α is of cardinality λ). For $p \in S^{bs,+}(M)$ define (α_p, q_p) by: α_p is the minimal ordinal in λ^+ such that p does not fork over N_α. $q_p =: p \restriction N_{\alpha_p}$. For every $\alpha \in \lambda^+$ by Definition 2.1.3 (semi-good λ-frame), we have $|S^{bs}(N_\alpha)| \leq \lambda^+$, so $|(\alpha_p, q_p) : p \in S^{bs,+}(M)| \leq \lambda^+ \times \lambda^+ = \lambda^+$. So it is sufficient to prove that the function $p \to (\alpha_p, q_p)$ is an injection. For every $p_1, p_2 \in S^{bs,+}(M)$ if $\alpha_{p_1} = \alpha_{p_2} \wedge q_{p_1} = q_{p_2}$. Therefore by Corollary 7.1.17.c (tameness) $p_1 = p_2$. $\dashv$

Proposition 10.1.11. $\mathfrak{s}^+$ *satisfies uniqueness in the sense of Definition 2.1.1.3.d.*

Proof.
1) By the proof of Corollary 7.1.17.c (tameness).
2) Suppose $n < 2 \Rightarrow M_n \in K^{sat}$, $M_0 \preceq M_1$, $p, q \in S^{bs,+}(M_1)$, $p \restriction M_0 = q \restriction M_0$ and p, q do not fork over M_0. By the definition of $\overset{+}{\bigcup}$, there are $N_p, N_q \in K_\lambda$, such that $N_p \preceq M_0$, $N_q \preceq M_0$, p does not fork over N_p and q does not fork over N_q. As $LST(K, \preceq) \leq \lambda$, there is a model $N \in K_\lambda$ with $N_p \bigcup N_q \subseteq N \preceq M_0$. By Axiom 1.0.3.1.e, $N_p \preceq N$ and $N_q \preceq N$. By Theorem 2.6.8(2) (monotonicity), p, q do not fork over N. By the assumption $p \restriction M_0 = q \restriction M_0$, so $p \restriction N = q \restriction N$. Hence by item 1, $p = q$. $\dashv$

Proposition 10.1.12. $\mathfrak{s}^+$ *satisfies symmetry in the sense of Definition 2.1.1.3.e.*

Proof.

$$\begin{tikzcd}
 & M_2 \arrow[rrr, "id"] & & & M_4 \\
 & & & M_3 \arrow[ur, "id"] & \\
M_0 \arrow[uur, "id"] \arrow[rr, "id"] & & M_1 \arrow[ur, "id"] & & \\
 & N_2 \arrow[uuu, "id"] \arrow[ur, "id"] \arrow[rrr, "id"] & & & N_4 \arrow[uuu, "id"] \\
 & & & N_3 \arrow[uuu, "id"] \arrow[ur, "id"] & \\
N_0 \arrow[uuu, "id"] \arrow[uur, "id"] \arrow[rr, "id"] & & N_1 \arrow[uuu, "id"] \arrow[ur, "id"] & &
\end{tikzcd}$$

Suppose 1-5 where:
(1) $\{M_0, M_1, M_3\} \subseteq K^{sat}$.
(2) $M_0 \preceq^{NF}_{\lambda^+} M_1 \preceq^{NF}_{\lambda^+} M_3$.
(3) $a_1 \in M_1$.
(4) $tp(a_1, M_0, M_3) \in S^{bs,+}(M_0)$.

(5) $a_2 \in M_3$ and $tp(a_2, M_1, M_3)$ does not fork over M_0.

Step a: We choose models $N_0, N_1, N_3 \in K_\lambda$ which satisfy 6-12 where:
(6) $n \in \{0, 1, 3\} \Rightarrow N_n \preceq M_n$ and $N_0 \preceq N_1 \preceq N_3$.
(7) $tp(a_2, M_1, M_3)$ does not fork over N_0.
(8) $tp(a_1, M_0, M_3)$ does not fork over N_0.
(9) $a_1 \in N_1$.
(10) $a_2 \in N_3$.
(11) $\widehat{NF}(N_0, N_1, M_0, M_1)$.
(12) $\widehat{NF}(N_1, N_3, M_1, M_3)$.
(Why is it possible? By 2, there are representations $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N_{1,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N^*_{1,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N_{3,\alpha} : \alpha < \lambda^+ \rangle$ of M_0, M_1, M_1, M_3, respectively, such that: $\alpha < \lambda^+ \Rightarrow NF(N_{0,\alpha}, N_{1,\alpha}, N_{0,\alpha+1}, N_{1,\alpha+1})$, $NF(N^*_{1,\alpha}, N_{3,\alpha}, N^*_{1,\alpha+1}, N_{3,\alpha+1})$. Let E be a club of λ^+ such that $\alpha \in E \Rightarrow N_{1,\alpha} = \mathcal{N}^*_{1,\alpha}$. Choose $\alpha \in E$ big enough such that 7,8,9,10 will satisfied for $N_0 = N_{0,\alpha}$ $N_1 = N_{1,\alpha}$, $N_3 = N_{3,\alpha}$)

Step b: [We use the symmetry axiom] By 6,8 we have:
(13) $tp(a_1, N_0, N_3) \in S^{bs}(N_0)$.
by 6,7 we have:
(14) $tp(a_2, N_1, N_3)$ does not fork over N_0.
Now by Definition 2.1.1.3.e (symmetry) there are $N^*_2, N^*_4 \in K_\lambda$ which satisfy 15-18:
(15) $N_0 \preceq N^*_2 \preceq N^*_4$.
(16) $N_3 \preceq N^*_4$.
(17) $a_2 \in N^*_2$.
(18) $tp(a_1, N^*_2, N^*_4)$ does not fork over N_0.

Step c: [Move everything to K^{sat}] We choose f which satisfies 19,20:
(19) f is an injection, $dom(f) = N^*_4$ and $f \restriction N_3$ is the identity.
(20) $f[N^*_4] \bigcap M_3 = N_3$.
Define $N_4 := f[N^*_4]$, $N_2 := f[N^*_2]$. By the existence proposition of the $\prec^+_{\lambda^+}$-extensions (Proposition 7.1.12.b), there is $M_4 \in K_\lambda$ which satisfies 21,22:
(21) $\widehat{NF}(N_3, N_4, M_3, M_4)$.
(22) $M_3 \prec^+_{\lambda^+} M_4$.
By 20 (mainly) we know:
(23) $N_2 \bigcap M_0 = N_0$.
(Why? By 15 and the definitions of f, N_2, we have $N_0 \preceq N_2$. By 6, $N_0 \preceq M_0$. Let $x \in N_2 \bigcap M_0$. By 2,15 $x \in N_4 \bigcap M_3$. So by 20, $x \in N_3$. So $x \in N_3 \bigcap M_1$. Hence by 12, $x \in N_1$. So $x \in N_1 \bigcap M_0$. Hence by 11, we have $x \in N_0$). So by the existence proposition of $\widehat{NF}$ (Proposition 6.1.3.c), there is $M_2 \in K^{sat}$ such that:
(24) $\widehat{NF}(N_0, N_2, M_0, M_2)$.

Without loss of generality, $N_4 \bigcap M_2 = N_2$ as $M_0 \bigcap N_4 = N_0$. By Proposition 7.1.12.b there is $M_6 \in K^{sat}$ which satisfies 25,26:
(25) $M_2 \prec^+_{\lambda^+} M_6$.
(26) $\widehat{NF}(N_2, N_4, M_2, M_6)$.

Step d: We will prove 27,28:
(27) $tp(a_1, M_2, M_6)$ does not fork over N_0.
(28) There is an isomorphism $g : M_6 \to M_4$ over $M_0 \bigcup N_2$.
Then we will conclude:
(29) $tp(a_1, g[M_2], M_4)$ does not fork over M_0.
By 25, Proposition 7.1.10.c and 24 we have:
(30) $M_0 \prec^+_{\lambda^+} M_6$.
By 24,25 and Theorem 6.1.3.b (monotonicity):
(31) $NF(N_0, N_2, M_0, M_6)$.
By 24,26,28 and the transitivity of the relation $\widehat{NF}$, we have:
(32)$NF(N_0, N_2, M_0, M_4)$.
By 2,22 and Proposition 7.1.10.c:
(33) $M_0 \prec^+_{\lambda^+} M_4$.
By 30-33 and Theorem 7.1.13.c, we know 28. By 26, and Theorem 6.1.3.e (respecting the frame):
(34) $tp(a_1, M_2, M_6)$ does not fork over N_2. By 18 (and 12,9,19):
(35) $tp(a_1, N_2, N_4)$ does not fork over N_0. By 26 $N_4 \preceq M_6$, so by Theorem 2.6.8(3) (the transitivity of the non-forking relation), we have:
(27) $tp(a_1, M_2, M_6)$ does not fork over N_0.

Step e: It remains to prove
(36) $a_2 \in g[M_2]$. By 28, g is an isomorphism over N_2, so it is sufficient to prove $a_2 \in N_2$. By 17 $a_2 \in N_2^*$. So by 10,19 $a_2 \in N_2$. $\dashv$

By the following proposition, $\mathfrak{s}^+$ satisfies extension in the sense of Definition 2.1.1.3.f.

Proposition 10.1.13.

(1) If $N \preceq M \in K^{sat}$, $p \in S^{bs}(N)$, $N \in K_\lambda$, then there is $q \in S^{bs,+}(M)$ such that $q \restriction N = p$ and q does not fork over N.
(2) If $\{M_0, M_1\} \subseteq K^{sat}$, $M_0 \preceq^{NF}_{\lambda^+} M_1$, $p \in S^{bs,+}(M_0)$, then there is an extension of p to $S^{bs,+}(M_1)$.

Proof.

(1) Let a, N_1 be such that $tp(a, N, N_1) = p$. By Theorem 6.1.3.c, without loss of generality, there is a model M_1 such that $\widehat{NF}(N, N_1, M, M_1)$. By Theorem 6.1.3.e, $q := tp(a, M, M_1)$ does not fork over N.
(2) By the definition of $S^{bs,+}$, there is a model $N \in K_\lambda$ such that $N \preceq M_0$ and p does not fork over N. By item (1), there is $q \in S^{bs,+}(M_1)$ which does not fork over N, and $q \restriction N = p \restriction N$. q does not fork

over M_0 as it does not fork over N. So it is sufficient to prove that $q_0 := q \upharpoonright M_0 = p$. By Theorem 2.6.8.2 (monotonicity), q_0 does not fork over N. $q_0 \upharpoonright N = q \upharpoonright N = p \upharpoonright N$. Hence by Corollary 7.1.17.c (tameness) $p = q_0$.

$\dashv$

This completes the proof of Theorem 10.1.9. $\dashv$

11. Conclusions

Definition 11.0.14. Let λ be a cardinal and let n be a natural number. We define λ^{+n} as the n-th cardinal after λ: $\lambda^{+0} = \lambda$ and $\lambda^{+(n+1)}$ is the successor cardinal of λ^{+n}.

11.1. **Proof of the main theorem.** Now we can prove Theorem 1.0.1:

Theorem 11.1.1. *Suppose:*

(1) $\mathfrak{s} = (K, \preceq, S^{bs}, \bigcup)$ *is a semi-good* λ*-frame with conjugation.*
(2) $K^{3,uq}$ *is dense with respect to* $\preceq_{bs}$.
(3) $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$.

Then

(1) There is a good λ^+*-frame* $\mathfrak{s}^+ = ((K^{sat}, \preceq^{NF}_{\lambda^+} \upharpoonright K^{sat})^{up}, S^{bs,+}, \overset{+}{\bigcup})$, *such that* $K^{sat} \subseteq K_{\lambda^+}$ *and the relation* $\preceq^{NF}_{\lambda^+} \upharpoonright K^{sat}$ *is included in the relation* $\preceq \upharpoonright K^{sat}$.
(2) $\mathfrak{s}^+$ *satisfies the conjugation property.*
(3) There is a model in K *of cardinality* λ^{+2}.
(4) There is a model in K *of cardinality* λ^{+3}.

A reader might wonder: does this really work with no assumption on the number of models in K of cardinality λ^+? So how do you get amalgamation (in K_λ)?

The point is that we assume amalgamation implicitly, it is hidden in the definition of a semi-good frame.

Proof. (1) $K^{3,uq}$ is dense with respect to $\preceq_{bs}$. $\mathfrak{s}$ satisfies the conjugation property, so by Proposition 4.1.12, $K^{3,uq}$ satisfies the existence property. By clause 3 of our assumption, $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$. Hence by Theorem 7.1.3, $(K^{sat}, \preceq^{NF}_{\lambda^+} \upharpoonright K^{sat})$ satisfies smoothness, i.e., $\mathfrak{s}$ is successful (Definition 10.1.1). So Hypothesis 10.1.2 is satisfied. Therefore by Theorem 10.1.9, $\mathfrak{s}^+$ is a good λ^+-frame. Obviously $K^{sat} \subseteq K_{\lambda^+}$ and $\preceq^{NF}_{\lambda^+}$ is included in the relation $\preceq \upharpoonright K_{\lambda^+}$.

(2) Why does $\mathfrak{s}^+$ have conjugation? Suppose $M_0 \preceq^{NF}_{\lambda^+} M_1$, $\{M_0, M_1\} \subseteq K^{sat}$ and $p \in S^{bs,+}(M_1)$ does not fork over M_0. By the definition of $\overset{+}{\bigcup}$, there is $N \in K_\lambda$ such that $N \preceq M_0$ and p does not fork over N.

$$\begin{tikzcd} p \restriction M_0 & f(p \restriction M_0) = p \\ M_0 \arrow[r, "id", "f"'] & M_1 \\ N \arrow[u, "id"] & \end{tikzcd}$$

By Theorem 1.0.32.a (the uniqueness of the saturated model), there is an isomorphism $f : M_0 \to M_1$ over N. By Theorem 2.6.8(2) (monotonicity), $p \restriction M_0$ does not fork over N. So $f(p \restriction M_0)$ does not fork over N. But also p does not fork over N and $f(p \restriction M_0) \restriction N = (p \restriction M_0) \restriction N = p \restriction N$. [Why do we have the first equality? There are M_0^+, f^+, a such that $p \restriction M_0 = tp(a, M_0, M_0^+)$ and $f \subseteq f^+$, $dom(f^+) = M_0^+$. So $(p \restriction M_0) \restriction N = tp(a, N, M_0^+) = tp(f^+(a), N, f^+[M_0^+]) = tp(f^+(a), M_1, f^+[M_0^+]) \restriction N = f(p \restriction M_0) \restriction N$]. By Proposition 10.1.11(1), $\mathfrak{s}^+$ satisfies uniqueness in the sense of Definition 2.1.1.3.d. So $f(p \restriction M_0) = p$.

(3) By Proposition 3.1.9(2).

(4) Substitute $\mathfrak{s}^+$ instead of $\mathfrak{s}$ in Proposition 3.1.9(2). $\dashv$

Now we want to present a conjecture that motivates the hypothesis that $K^{3,uq}$ is dense with respect to $\preceq_{bs}$. In order to state the conjecture, we have to give the following definitions.

First we define the ideal of weak diamond. It was firstly defined in [DvSh]. An introduction to the weak diamond appear in Appendix C of [Ba].

Definition 11.1.2. Let λ be an infinite cardinal. We define $WDmId(\lambda) := \{A \subseteq \lambda :$ for some $F :^{<\lambda} \lambda \to 2$ for every $c : A \to 2$ for some $\eta : \lambda \to \lambda$ the set $\{\delta \in A : F(\eta \restriction \delta) = c(\delta)\}$ is not stationery$\}$.

Definition 11.1.3. Let μ be a cardinal, λ be a regular uncountable cardinal and $\mathbb{I}$ a normal ideal on λ. $\mathbb{I}$ is said to be *not saturated* in μ when: There is a sequence $\langle A_i : i < \mu \rangle$ such that $A_i \subseteq \lambda$, $A_i \notin \mathbb{I}$ for $i < \mu$ and $A_i \bigcap A_j \in \mathbb{I}$ for $i \neq j \in \mu$.

In the last chapter of [Sh:h], Shelah almost proved the following conjecture for good frames. In [JrSh 966] we did more. The pattern of the proof for this conjecture but with syntactic types is in [Sh 87b] and Chapter 23 of [Ba].

Conjecture 11.1.4. *Let $\mathfrak{s}$ be a semi-good λ-frame. Assume that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$ and $WDmId(\lambda^+)$ is not saturated in λ^+. If $K^{3,uq}_{\mathfrak{s}}$ is not dense with respect to $\preceq_{bs}$, then $I(\lambda^{+2}, K) = 2^{\lambda^{+2}}$.*

In the following theorem, we replace the assumption that $K^{3,uq}$ is dense with respect to $\preceq_{bs}$ (that appear in Theorem 11.1.1), by assumptions that imply that $K^{3,uq}$ is dense with respect to $\preceq_{bs}$. This theorem is the inductive step for Corollary 11.1.6.

Theorem 11.1.5. *Suppose:*

(1) $\mathfrak{s} = (K, \preceq, S^{bs}, \bigcup)$ *is a semi-good* λ*-frame with conjugation.*
(2) $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$.
(3) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$, *and* $WDmId(\lambda^+)$ *is not saturated in* λ^{+2}.
(4) *Conjecture 11.1.4.*

Then

(1) *There is a good* λ^+*-frame* $\mathfrak{s}^+ = (K^{sat}, \preceq^{NF}_{\lambda^+} \restriction K^{sat}, S^{bs,+}, \overset{+}{\bigcup})$, *such that* $K^{sat} \subseteq K_{\lambda^+}$ *and the relation* $\preceq^{NF}_{\lambda^+} \restriction K^{sat}$ *is included in the relation* $\preceq \restriction K^{sat}$.
(2) $\mathfrak{s}^+$ *satisfies the conjugation property.*
(3) *There is a model in* K *of cardinality* λ^{+2}.
(4) *There is a model in* K *of cardinality* λ^{+3}.

Proof. By assumptions (2),(3) and Conjecture 11.1.4, $K^{3,uq}$ is dense with respect to $\preceq_{bs}$. Now use Theorem 11.1.1. $\dashv$

Corollary 11.1.6. *Suppose:*

(1) $n < \omega$.
(2) $\mathfrak{s} = (K, \preceq, S^{bs}, \bigcup)$ *is a semi-good* λ*-frame with conjugation.*
(3) $m < n \Rightarrow I(\lambda^{+(2+m)}, K) < 2^{\lambda^{+(2+m)}}$.
(4) $2^{\lambda^{+m}} < 2^{\lambda^{+(m+1)}}$ *for every* $m < n+1$ *and* $WDmId(\lambda^{+1+m})$ *is not saturated in* $\lambda^{+(2+m)}$ *for every* $m < n$.
(5) *Conjecture 11.1.4.*

then *there is a good* λ^{+n}*-frame* $\mathfrak{s}^n =: ((K^n, \leq^n), S^{bs,+n}, \overset{+n}{\bigcup})$, *such that:*

(1) $K^n_{\lambda^{+n}} \subseteq K_{\lambda^{+n}}$ *and the relation* $\leq^n$ *is included in the relation* $\preceq^k \restriction K^n$.
(2) $\mathfrak{s}^n$ *satisfies the conjugation property.*
(3) *There is a model in* K^n *of cardinality* $\lambda^{+(2+n)}$.

Proof. By induction on n, using Theorem 11.1.5. $\dashv$

Now we prove Theorem 1.0.2:

Theorem 11.1.7. *Suppose:*

(1) $\mathfrak{s} = (K, \preceq, S^{bs}, \bigcup)$ *is a semi-good* λ*-frame with conjugation.*
(2) $m < \omega \Rightarrow I(\lambda^{+(2+m)}, K) < 2^{\lambda^{+(2+m)}}$.
(3) $2^{\lambda^{+m}} < 2^{\lambda^{+m+1}}$ *and for every* $m < \omega$, $WDmId(\lambda^{+1+m})$ *is not saturated in* $\lambda^{+(2+m)}$.
(4) *Conjecture 11.1.4.*

Then *there is a model in* K^n *of cardinality* λ^{+n} *for every* $n < \omega$.

Proof. By Corollary 11.1.6. $\dashv$

12. Comparison to [Sh:h].II

A reader who knows [Sh:h].II, might ask about the main problems in doing this work. As in [Sh:h].II, there is a wide use of brimmed extensions (i.e., using stability); we had to find alternatives.

First the relation NF is defined in [Sh:h].II using brimness, so we found a natural definition (maybe an easier one) which is equivalent to the definition in [Sh:h].II, but not using brimness.

Another problem was proving conjugation (see Definition 2.5.5). But in the main examples there is conjugation, so it is reasonable to assume conjugation.

Another problem was to find a relation $\prec^{+}_{\lambda^+}$ on K^{sat} which satisfies the required properties (see the discussion before Definition 7.1.4). [Sh:h].II essentially uses brimness. But as the needed relation is on models of cardinality λ^+, we can find such a relation, using just almost stability.

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