

ON DEPTH AND DEPTH⁺ OF BOOLEAN ALGEBRAS

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ABSTRACT. We show that the Depth⁺ of an ultraproduct of Boolean Algebras cannot jump over the Depth⁺ of every component by more than one cardinal. Consequently we have similar results for the Depth invariant.

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0. INTRODUCTION

Monk [2] has dealt systematically with cardinal invariants of Boolean algebras. In particular he dealt with the question how an invariant of an ultraproduct of a sequence of Boolean algebras relates to the ultraproduct of the sequence of the invariants of each of the Boolean algebras. That is the relationship of $\text{inv}(\prod_{\epsilon < \kappa} \mathbf{B}_\epsilon / D)$ with $\prod_{\epsilon < \kappa} \text{inv}(\mathbf{B}_\epsilon) / D$. One of the invariants he dealt with is the depth of a Boolean algebra, $\text{Depth}(\mathbf{B})$. We continue here [7] getting weaker results without “large cardinal axioms”. On related results see [1], [6], [3]. Further results on Depth and Depth^+ by the authors are contained in [4].

Recall:

Definition 0.1. Let \mathbf{B} be a Boolean Algebra.

$$\text{Depth}(\mathbf{B}) := \sup\{\theta : \exists \bar{b} = (b_\gamma : \gamma < \theta), \text{ increasing sequence in } \mathbf{B}\}$$

Dealing with questions of Depth , Saharon Shelah noticed that investigating a slight modification of Depth , namely - Depth^+ , might be helpful (see [7] for the behavior of Depth and Depth^+ above a compact cardinal).

Recall:

Definition 0.2. Let \mathbf{B} be a Boolean Algebra.

$$\text{Depth}^+(\mathbf{B}) := \sup\{\theta^+ : \exists \bar{b} = (b_\gamma : \gamma < \theta), \text{ increasing sequence in } \mathbf{B}\}$$

This article deals mainly with Depth^+ , in the aim to get results for the Depth . It follows [7], both - in the general ideas and in the method of the proof.

Let us take a look on the main claim of [7]:

Claim 0.3. *Assume*

- (a) $\kappa < \mu \leq \lambda$
- (b) μ is a compact cardinal
- (c) $\lambda = \text{cf}(\lambda)$
- (d) $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$
- (e) $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$, for every $i < \kappa$
- (f) $\mathbf{B} = \prod_{i < \kappa} \mathbf{B}_i / D$.

Then $\text{Depth}^+(\mathbf{B}) \leq \lambda$.

So, λ bounds the $\text{Depth}^+(\mathbf{B})$, where \mathbf{B} is an ultraproduct of the Boolean Algebras \mathbf{B}_i , if it bounds the Depth^+ of every \mathbf{B}_i . That requires some reasonable assumptions on λ , and also a pretty high price for that result - you should raise your view to a very large λ , above a compact cardinal. Now, the existence of large cardinals is an interesting philosophical question. You might think that adding a compact cardinal to your world is a natural extension of ZFC. But, mathematically, it is important to check what happens

without a compact cardinal (or below the compact, even if the compact cardinal exists).

In this article we drop the assumption of a compact cardinal. Consequently, we phrase a weaker conclusion. We prove that if λ bounds the \mathbf{Depth}^+ of every \mathbf{B}_i , then the \mathbf{Depth}^+ of \mathbf{B} cannot jump beyond λ^+ .

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1. BOUNDING DEPTH⁺

Notation 1.1. (a) κ, λ are infinite cardinals
 (b) D is an ultrafilter on κ
 (c) \mathbf{B}_i is a Boolean Algebra, for any $i < \kappa$
 (d) $\mathbf{B} = \prod_{i < \kappa} \mathbf{B}_i / D$.

We now state our main result:

Theorem 1.2. *Assume*

- (a) $\lambda = \text{cf}(\lambda)$
- (b) $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$
- (c) $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$, for every $i < \kappa$.

Then $\text{Depth}^+(\mathbf{B}) \leq \lambda^+$.

Remark 1.3. We can improve 1.2 (b), demanding only $\lambda^\kappa = \lambda$. We intend to give a detailed proof in a subsequent paper.

Corollary 1.4. *Assume*

- (a) $\lambda^\kappa = \lambda$
- (b) $\text{Depth}(\mathbf{B}_i) \leq \lambda$, for every $i < \kappa$.

Then $\text{Depth}(\mathbf{B}) \leq \lambda^+$.

Proof. By (b), $\text{Depth}^+(\mathbf{B}_i) \leq \lambda^+$ for every $i < \kappa$. By (a), $\alpha < \lambda^+ \Rightarrow |\alpha|^\kappa < \lambda^+$. Now, λ^+ is a regular cardinal, so the pair (κ, λ^+) satisfies the requirements of Theorem 1.2. So, $\text{Depth}^+(\mathbf{B}) \leq \lambda^{+2}$, and that means that $\text{Depth}(\mathbf{B}) \leq \lambda^+$. □_{1.4}

Remark 1.5. If λ is inaccessible (or even strong limit, with cofinality above κ), and $\text{Depth}(\mathbf{B}_i) < \lambda$ for every $i < \kappa$, you can easily verify that $\text{Depth}(\mathbf{B}) < \lambda$, using Theorem 1.2 and simple cardinal arithmetic.

Proof of Theorem 1.2:

Let $\langle M_\alpha : \alpha < \lambda^+ \rangle$ be a continuous and increasing sequence of elementary submodels of $(\mathcal{H}(\chi), \in)$ for sufficiently large χ with the following properties:

- (a) $(\forall \alpha < \lambda^+)(\|M_\alpha\| = \lambda)$
- (b) $(\forall \alpha < \lambda^+)(\lambda + 1 \subseteq M_\alpha)$
- (c) $(\forall \beta < \lambda^+)(\langle M_\alpha : \alpha \leq \beta \rangle \in M_{\beta+1})$.

Choose $\delta^* \in S_\chi^{\lambda^+} (= \{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\})$, such that $\delta^* = M_{\delta^*} \cap \lambda^+$. Assume toward a contradiction that $\langle a_\alpha : \alpha < \lambda^+ \rangle$ is an increasing sequence in \mathbf{B} . Let us write a_α as $\langle a_{i_i}^\alpha : i < \kappa \rangle / D$ for every $\alpha < \lambda^+$. We may assume that $\langle a_{i_i}^\alpha : \alpha < \lambda^+, i < \kappa \rangle \in M_0$.

We will try to create a set Z , in the Lemma below, with the following properties:

- (a) $Z \subseteq \lambda^+, |Z| = \lambda$
- (b) $\exists i_* \in \kappa$ such that for every $\alpha < \beta, \alpha, \beta \in Z$, we have $\mathbf{B}_{i_*} \models a_{i_*}^\alpha < a_{i_*}^\beta$

Since $|Z| = \lambda$, we have an increasing sequence of length λ in \mathbf{B}_{i^*} , so $\text{Depth}^+(\mathbf{B}_{i^*}) \geq \lambda^+$, contradicting the assumptions of the claim. $\square_{1.2}$

Lemma 1.6. *There exists Z as above.*

Proof. For every $\alpha < \beta < \lambda^+$, define:

$$A_{\alpha,\beta} = \{i < \kappa : \mathbf{B}_i \models a_i^\alpha < a_i^\beta\}$$

By the assumption, $A_{\alpha,\beta} \in D$ for all $\alpha < \beta < \lambda^+$. For all $\alpha < \delta^*$, Let A_α denote the set A_{α,δ^*} .

Let $\langle v_\alpha : \alpha < \lambda \rangle$ be increasing and continuous, such that for every $\alpha < \lambda$:

- (i) $v_\alpha \in [\delta^*]^{<\lambda}$, for every $\alpha < \lambda$,
- (ii) v_α has no last element, for every $\alpha < \lambda$,
- (iii) $\delta^* = \bigcup_{\alpha < \lambda} v_\alpha$.

Let $u \subseteq \delta^*$, $|u| \leq \kappa$. Define:

$$S_u = \{\beta < \delta^* : \beta > \sup(u) \text{ and } (\forall \alpha \in u)(A_{\alpha,\beta} = A_\alpha)\}.$$

Now define $C = \{\delta < \lambda : \delta \text{ is a limit ordinal and}$

$$(\forall \alpha < \delta)[(u \subseteq v_\alpha) \wedge (|u| \leq \kappa) \Rightarrow \sup(v_\delta) = \sup(S_u \cap \sup(v_\delta))]\}.$$

Since $\lambda = \text{cf}(\lambda)$ and $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$, and since $|v_\delta| < \lambda$ for all $\delta < \lambda$, C is a club set of λ .

The fact that $|D| = 2^\kappa < \text{cf}(\lambda) = \lambda$ implies that there exists $A_* \in D$ such that $S = \{\alpha < \lambda : \text{cf}(\alpha) > \kappa \text{ and } A_{\sup(v_\alpha)} = A_*\}$ is a stationary subset of λ .

C is a club and S is stationary, so $C \cap S$ is also stationary. Choose $\delta_0^1 = \min(C \cap S)$. Choose $\delta_{\epsilon+1}^1 \in C \cap S$ for every $\epsilon < \lambda$ such that $\epsilon < \zeta \Rightarrow \sup\{\delta_{\epsilon+1}^1 : \epsilon < \zeta\} < \delta_{\zeta+1}^1$. Define δ_ϵ^1 to be the limit of $\delta_{\gamma+1}^1$, when $\gamma < \epsilon$, for every limit $\epsilon < \lambda$. Since C is closed, we have:

- (a) $\{\delta_\epsilon^1 : \epsilon < \lambda\} \subseteq C$
- (b) $\langle \delta_\epsilon^1 : \epsilon < \lambda \rangle$ is increasing and continuous
- (c) $\delta_{\epsilon+1}^1 \in S$, for every $\epsilon < \lambda$

Lastly, define $\delta_\epsilon^2 = \sup(v_{\delta_\epsilon^1})$, for every $\epsilon < \lambda$. Define, for every $\epsilon < \lambda$, the following family:

$$\mathfrak{A}_\epsilon = \{S_u \cap \delta_{\epsilon+1}^2 \setminus \delta_\epsilon^2 : u \in [v_{\delta_{\epsilon+1}^2}]^{\leq \kappa}\}.$$

We get a family of non-empty sets, which is downward κ^+ -directed. So, there is a κ^+ -complete filter E_ϵ on $[\delta_\epsilon^2, \delta_{\epsilon+1}^2)$, with $\mathfrak{A}_\epsilon \subseteq E_\epsilon$, for every $\epsilon < \lambda$.

Define, for any $i < \kappa$ and $\epsilon < \lambda$, the sets $W_{\epsilon,i} \subseteq [\delta_\epsilon^2, \delta_{\epsilon+1}^2)$ and $B_\epsilon \subseteq \kappa$, by:

$$W_{\epsilon,i} := \{\beta : \delta_\epsilon^2 \leq \beta < \delta_{\epsilon+1}^2 \text{ and } i \in A_{\beta, \delta_{\epsilon+1}^2}\}$$

$$B_\epsilon := \{i < \kappa : W_{\epsilon,i} \in E_\epsilon^+\}.$$

Finally, take a look on $W_\epsilon := \bigcap\{\delta_\epsilon^2, \delta_{\epsilon+1}^2\} \setminus \bigcup\{W_{\epsilon,i} : i \in \kappa \setminus B_\epsilon\}$. For every $\epsilon < \lambda$, $W_\epsilon \in E_\epsilon$, since E_ϵ is κ^+ -complete, so clearly $W_\epsilon \neq \emptyset$.

Choose $\beta = \beta_\epsilon \in W_\epsilon$. If $i \in A_{\beta, \delta_{\epsilon+1}^2}$, then $W_{\epsilon, i} \in E_\epsilon^+$, so $A_{\beta, \delta_{\epsilon+1}^2} \subseteq B_\epsilon$ (by the definition of B_ϵ). But, $A_{\beta, \delta_{\epsilon+1}^2} \in D$, so $B_\epsilon \in D$, and consequently $A_* \cap B_\epsilon \in D$, for any $\epsilon < \lambda$.

Choose $i_\epsilon \in A_* \cap B_\epsilon$, for every $\epsilon < \lambda$. You choose λ i_ϵ -s from A_* , and $|A_*| = \kappa$, so we can arrange a fixed $i_* \in A_*$ such that the set $Y = \{\epsilon < \lambda : \epsilon \text{ is even ordinal, and } i_\epsilon = i_*\}$ has cardinality λ .

The last step will be as follows:

define $Z = \{\delta_{\epsilon+1}^2 : \epsilon \in Y\}$. Clearly, $Z \in [\delta^*]^\lambda \subseteq [\lambda^+]^\lambda$. We will show that for $\alpha < \beta$ from Z we get $\mathbf{B}_{i_*} \models a_{i_*}^\alpha < a_{i_*}^\beta$. The idea is that if $\alpha < \beta$ and $\alpha, \beta \in Z$, then $i_* \in A_{\alpha, \beta}$.

Why? Recall that $\alpha = \delta_{\epsilon+1}^2$ and $\beta = \delta_{\zeta+1}^2$, for some $\epsilon < \zeta < \lambda$ (that's the form of the members of Z). Define:

$$U_1 = S_{\{\delta_{\epsilon+1}^2\}} \cap [\delta_\zeta^2, \delta_{\zeta+1}^2) \in \mathfrak{A}_\zeta \subseteq E_\zeta.$$

$$U_2 = \{\gamma : \delta_\zeta^2 \leq \gamma < \delta_{\zeta+1}^2 \text{ and } i_* \in A_{\gamma, \delta_{\zeta+1}^2}\} \in E_\zeta^+.$$

So, $U_1 \cap U_2 \neq \emptyset$.

Choose $\iota \in U_1 \cap U_2$.

Now the following statements hold:

- (a) $\mathbf{B}_{i_*} \models a_{i_*}^\alpha < a_{i_*}^\iota$
 [Why? Well, $\iota \in U_1$, so $A_{\delta_{\epsilon+1, \iota}^2} = A_{\delta_{\epsilon+1}^2} = A_*$. But, $i_* \in A_*$, so $i_* \in A_{\delta_{\epsilon+1, \iota}^2}$, which means that $\mathbf{B}_{i_*} \models a_{i_*}^{\delta_{\epsilon+1}^2} (= a_{i_*}^\alpha) < a_{i_*}^\iota$.]
- (b) $\mathbf{B}_{i_*} \models a_{i_*}^\iota < a_{i_*}^\beta$
 [Why? Well, $\iota \in U_2$, so $i_* \in A_{\iota, \delta_{\zeta+1}^2}$, which means that $\mathbf{B}_{i_*} \models a_{i_*}^\iota < a_{i_*}^{\delta_{\zeta+1}^2} (= a_{i_*}^\beta)$.]
- (c) $\mathbf{B}_{i_*} \models a_{i_*}^\alpha < a_{i_*}^\beta$
 [Why? By (a)+(b)].

So, we are done. □_{1.6}

Without a compact cardinal, we may have a 'jump' of the Depth^+ in the ultraproduct of the Boolean Algebras (see [5, §5]). So, we can have $\kappa < \lambda$, $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$ for every $i < \kappa$, and $\text{Depth}^+(\mathbf{B}) = \lambda^+$. We can show that if there exists such an example for κ and λ , then you can create an example for every regular θ between κ and λ .

Claim 1.7. *Assume*

- (a) $\kappa < \lambda$, D is an ultrafilter on κ
- (b) $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$, for every $i < \kappa$
- (c) $\text{Depth}^+(\mathbf{B}) = \lambda^+$
- (d) $\theta \in \text{Reg} \cap [\kappa, \lambda)$.

Then there exist Boolean algebras \mathbf{C}_j , $j < \theta$, and a uniform ultrafilter E on θ , such that $\text{Depth}^+(\mathbf{C}_j) \leq \lambda$ for every $j < \theta$ and $\text{Depth}^+(\mathbf{C}) := \text{Depth}^+(\prod_{j < \theta} \mathbf{C}_j/E) = \lambda^+$.

Proof. Break θ into θ sets $(u_\alpha : \alpha < \theta)$ such that for every $\alpha < \theta$:

- (a) $|u_\alpha| = \kappa$,
- (b) $\bigcup_{\alpha < \theta} u_\alpha = \theta$,
- (c) $\alpha \neq \beta \Rightarrow u_\alpha \cap u_\beta = \emptyset$.

For every $\alpha < \theta$, let $f_\alpha : \kappa \rightarrow u_\alpha$ be one to one, onto and order preserving. Define D_α on u_α in the following way: If $A \subseteq u_\alpha$, then $A \in D_\alpha$ iff $f_\alpha^{-1}(A) \in D$. For θ itself, define a filter E_* on θ in the following way: If $A \subseteq \theta$, then $A \in E_*$ iff $A \cap u_\alpha \in D_\alpha$ for every (except, maybe $< \theta$ ordinals) $\alpha < \theta$. Now, choose any ultrafilter E on θ , such that $E_* \subseteq E$.

Define $\mathbf{C}_{f_\alpha(i)} = \mathbf{B}_i$, for every $\alpha < \theta$ and $i < \kappa$. You will get $(\mathbf{C}_j : j < \theta)$ such that $\text{Depth}^+(\mathbf{C}_j) \leq \lambda$ for every $j < \theta$. But, we will show that $\text{Depth}^+(\mathbf{C}) \geq \lambda^+$ (remember that $\mathbf{C} = \prod_{j < \theta} \mathbf{C}_j/E$).

Well, let $(a_\xi : \xi < \lambda)$ testify $\text{Depth}^+(\mathbf{B}) = \lambda^+$. Recall, a_ξ is $\langle a_i^\xi : i < \kappa \rangle / D$. We may write $f_\alpha(a_\xi)$ for $\langle f_\alpha(a_i^\xi) : i < \kappa \rangle / D_\alpha$, where $\alpha < \theta$. Clearly, $(f_\alpha(a_\xi) : \xi < \lambda)$ testifies $\text{Depth}^+(\mathbf{C}^\alpha) = \lambda^+$ where $\mathbf{C}^\alpha := \prod_{i < \kappa} \mathbf{C}_{f_\alpha(i)} / D_\alpha$.

Now, $\langle (f_\alpha(a_\xi) : \alpha < \theta) : \xi < \lambda \rangle / E$ is an increasing sequence in \mathbf{C} .

□_{1.7}

Remark 1.8. (1) Claim 1.7 applies, in a similar fashion, to the Depth invariant.

- (2) Claim 1.7 is useful for comparing $\text{Depth}(\mathbf{C})$ to $\prod_{j < \theta} \text{Depth}(\mathbf{C}_j)/E$, when $\lambda^\theta = \lambda$.

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