THE ERDÖS-RADO ARROW FOR SINGULAR

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Abstract. We prove that if $\text{cf}(\lambda) > \aleph_0$ and $2^{\text{cf}(\lambda)} < \lambda$ then $\lambda \rightarrow (\lambda, \omega + 1)^2$ in ZFC.

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0. INTRODUCTION

For regular uncountable \( \kappa \), the Erdös-Dushnik-Miller theorem, Theorem 11.3 of [1], states that \( \kappa \rightarrow (\kappa, \omega + 1)^2 \). For singular cardinals, \( \kappa \), they were only able to obtain the weaker result, Theorem 11.1 of [1], that \( \kappa \rightarrow (\kappa, \omega + 1)^2 \). It is not hard to see that if \( \text{cf}(\kappa) = \omega \) then \( \kappa \nrightarrow (\kappa, \omega + 1)^2 \). If \( \text{cf}(\kappa) > \omega \) and \( \kappa \) is a strong limit cardinal, then it follows from the General Canonization Lemma, Lemma 28.1 in [1], that \( \kappa \rightarrow (\kappa, \omega + 1)^2 \). Question 11.4 of [1] is whether this holds without the assumption that \( \kappa \) is a strong limit cardinal, e.g., whether, in ZFC,

\[(1) \; \aleph_1 \rightarrow (\aleph_1, \omega + 1)^2.\]

In [5] it was proved that \( \aleph_0 < \kappa = \text{cf}(\lambda) \) and \( 2^\kappa < \lambda \) then \( \kappa \rightarrow (\lambda, \omega + 1)^2 \) if \( 2^\text{cf}(\lambda) < \lambda \) and there is a nice filter on \( \kappa \), (see [3, Ch.V]; follows from suitable failures of SCH). Also proved there are consistency results when \( 2^\text{cf}(\lambda) = \lambda \).

Here continuing [5] but not relying on it, we eliminate the extra assumption, i.e., we prove (in ZFC)

**Theorem 0.1.** If \( \aleph_0 < \kappa = \text{cf}(\lambda) \) and \( 2^\kappa < \lambda \) then \( \kappa \rightarrow (\lambda, \omega + 1)^2 \).

Before starting the proof, let us recall the well known definition:

**Definition 0.2.** Let \( D \) be an \( \aleph_1 \)-complete filter on \( Y \), and \( f \in Y^{\text{Ord}} \), and \( \alpha \in \text{Ord} \cup \{ \infty \} \).

We define when \( \text{rk}_D(f) = \alpha \) by induction on \( \alpha \) (it is well known that \( \text{rk}_D(f) < \infty \)):

\[(*) \; \text{rk}_D(f) = \alpha \; \text{iff} \; \beta < \alpha \Rightarrow \text{rk}_D(f) \neq \beta, \; \text{and for every} \; g \in Y^{\text{Ord}} \; \text{satisfying} \; g <_D f, \; \text{there is} \; \beta < \alpha \; \text{such that} \; \text{rk}_D(g) = \beta.\]

Notice that we will use normal filters on \( \kappa = \text{cf}(\kappa) > \aleph_0 \), so the demand of \( \aleph_1 \)-completeness in the definition, holds for us.

Recall also

**Definition 0.3.** Assume \( Y, D, f \) are as in definition 0.2.

\[ J[f, D] = \{ Z \subseteq Y : Y \setminus Z \in D \; \text{or} \; \text{rk}_{D+(Y \setminus Z)}(f) > \text{rk}_D(f) \} \]

Lastly, we quote the next claim (the definition 0.3 and claim are from [2], and explicitly [4](5.8(2), 5.9)):

**Claim 0.4.** Assume \( \kappa > \aleph_0 \) is realized, and \( D \) is a \( \kappa \)-complete (a normal) filter on \( Y \).

Then \( J[f, D] \) is a \( \kappa \)-complete (a normal) ideal on \( Y \) disjoint to \( D \) for any \( f \in Y^{\text{Ord}} \).
1. The proof

In this section we prove Theorem 0.1 of the Introduction, which, for convenience, we now restate.

**Theorem 1.1.** If $\aleph_0 < \kappa = \text{cf}(\lambda)$, $2^\kappa < \lambda$ then $\lambda \rightarrow (\lambda, \omega + 1)^2$.

**Proof.**

**Stage A** We know that $\aleph_0 < \kappa = \text{cf}(\lambda) < \lambda$, $2^\kappa < \lambda$ We will show that $\lambda \rightarrow (\lambda, \omega + 1)^2$.

So, towards a contradiction, suppose that

$$(*)_1 c : [\lambda]^2 \rightarrow \{\text{red}, \text{green}\}$$

but has no red set of cardinality $\lambda$ and no green set of order type $\omega + 1$.

Choose $\bar{\lambda}$ such that:

$$(*)_2 \bar{\lambda} = (\lambda_i : i < \kappa)$$

is increasing and continuous with limit $\lambda$, and for $i = 0$ or $i$ a successor ordinal, $\lambda_i$ is a successor cardinal. We also let $\Delta_0 = \lambda_0$ and for $i < \kappa$, $\Delta_{i+1} = [\lambda_i, \lambda_{i+1})$. For $\alpha < \lambda$ we will let $i(\alpha)$ be the unique $i < \kappa$ such that $\alpha \in \Delta_i$.

We can clearly assume, in addition, that

$$(*)_3 \lambda_0 > 2^\kappa, \text{ for } i < \kappa, \lambda_{i+1} \geq \lambda_i^{++}, \text{ and that each } \Delta_i \text{ is homogeneously red for } c.$$

The last is justified by the Erdős-Dushnik-Miller theorem for $\lambda_{i+1}$, i.e., as $\lambda_{i+1} \rightarrow (\lambda_{i+1}, \omega + 1)^2$ because $\lambda_{i+1}$ is regular.

**Stage B:** For $0 < i < \kappa$, we define $\text{Seq}_i$ to be $\{\langle \alpha_0, ..., \alpha_{n-1} : i(\alpha_0) < ... < i(\alpha_{n-1}), i \rangle \}$.

For $i < \kappa$ and $\langle \alpha_0, ..., \alpha_{n-1} = \bar{\alpha} \rangle \in \text{Seq}_i$ we say $\bar{\alpha} \in T^\chi$ iff $\{\alpha_0, ..., \alpha_{n-1}, \chi\}$ is homogeneously green for $c$. Note that an infinite increasing branch in $T^\chi$ violates the non-existence of a green set of order type $\omega + 1$, so,

$$(*)_4 T^\chi \text{ is well-founded, that is we cannot find } \eta_0 < \eta_1 < ... < \eta_n \triangleq ...$$

Therefore the following definition of a rank function, $\text{rk}^\chi$, on $\text{Seq}_i$ can be carried out.

If $\eta \in \text{Seq}_i \setminus T^\chi$ then $\text{rk}^\chi(\eta) = -1$. We define $\text{rk}^\chi : \text{Seq}_i \rightarrow \text{Ord} \cup \{-1\}$ as follows by induction on the ordinal $\xi$, we have $\text{rk}^\chi(\bar{\alpha}) = \xi$ iff for all $\epsilon < \xi$, $\text{rk}^\chi(\bar{\alpha})$ was not defined as $\epsilon$ but there is $\beta$ such that $\text{rk}^\chi(\bar{\alpha} \setminus \langle \beta \rangle) \geq \epsilon$.

Of course, if $\xi$ is a successor ordinal, it is enough to check for $\epsilon = \xi - 1$, and for limit ordinals, $\delta$, if for all $\xi < \delta$, $\text{rk}^\chi(\bar{\alpha}) \geq \xi$, then $\text{rk}^\chi(\bar{\alpha}) \geq \delta$.

In fact, it is clear that the range of $\text{rk}^\chi$ is a proper initial segment of $\mu^+_i$, where $\mu_i := \text{card}(\bigcup \{\Delta_\epsilon : \epsilon < i\})$, and so, in particular, the range of $\text{rk}^\chi$ has cardinality at most $\lambda_i$. Note that $\lambda_{i+1} \geq \lambda_i^{++} > \mu^+_i$.

Now we can choose $B_i$, an end-segment of $\Delta_i$ such that for all $\bar{\alpha} \in \text{Seq}_i$ and all $0 \leq \gamma < \mu^+_i$, if there is $\zeta \in B_i$ such that $\text{rk}^\chi(\bar{\alpha}) = \gamma$, then there are $\lambda_{i+1}$ such $\zeta$-s.

Recall that $\Delta_i$ and therefore also $B_i$ are of order type $\lambda_{i+1}$, which is a successor cardinal $> \mu^+_i > |\text{Seq}_i|$ hence such $B_i$ exists. Everything is now in place for the main definition.
Stage C: \((\bar{\alpha}, Z, D, f) \in K\) iff

1. \(D\) is a normal filter on \(\kappa\),
2. \(f : \kappa \to \text{Ord}\),
3. \(Z \in D\)
4. for some \(0 < i < \kappa\) we have \(\bar{\alpha} \in \text{Seq}_i\) and \(Z\) is disjoint to \(i + 1\) and for every \(j \in Z\) (hence \(j > i\)) there is \(\zeta \in B_j\) such that \(rk^C(\bar{\alpha}) = f(j)\)

(since, in particular, \(\bar{\alpha} \in T\zeta\)).

Stage D: Note that \(K \neq \emptyset\), since if we choose \(\zeta_j \in B_j\), for \(j < \kappa\), take \(Z = \kappa \setminus \{0\}\), \(\bar{\alpha}\) the empty sequence, choose \(D\) to be any normal filter on \(\kappa\) and define \(f\) by \(f(j) = rk^C(\bar{\alpha})\), then \((\bar{\alpha}, Z, D, f) \in K\).

Now clearly by 0.2, among the quadruples \((\bar{\alpha}, Z, D, f) \in K\), there is one with \(rk_D(f)\) minimal. So, fix one such quadruple, and denote it by \((\bar{\alpha}^*, Z^*, D^*, f^*)\). Let \(D^*_1\) be the filter on \(\kappa\) dual to \(J[f^*, D^*]\), so by claim 0.4 it is a normal filter on \(\kappa\) extending \(D^*\).

For \(j \in Z^*\), set \(C_j = \{\zeta \in B_j : rk^C(\bar{\alpha}^*) = f^*(j)\}\). Thus by the choice of \(B_j\) we know that \(\text{card}(C_j) = \lambda_{j+1}\), and for every \(\zeta \in C_j\) the set \((\text{Rang}(\bar{\alpha}^*) \cup \{\zeta\})\) is homogeneously green under the colouring \(c\). Now: suppose \(j \in Z^*\). For every \(Y \in Z^* \setminus (j + 1)\) and \(\zeta \in C_j\), let \(C^+_Y(\zeta) = \{\xi \in C_Y : c(\{\zeta, \xi\}) = \text{green}\}\).

Also, let \(Z^+(\zeta) = \{Y \in Z^* \setminus (j + 1) : \text{card}(C^+_Y(\zeta)) = \lambda_{Y+1}\}\).

Stage E: For \(j \in Z^*\) and \(\zeta \in C_j\), let \(Y(\zeta) = Z^* \setminus Z^+(\zeta)\). Since \(\lambda_0 > 2^\kappa\) and \(\lambda_{j+1} > \lambda_0\) is regular, for each \(j \in Z^*\) there are \(Y = Y_j \subseteq \kappa\) and \(C_j' \subseteq C_j\) with \(\text{card}(C_j') = \lambda_{j+1}\) such that \(\zeta \in C_j' \Rightarrow Y(\zeta) = Y_j\).

Let \(\hat{Z} = \{j \in Z^* : Y_j \in D^*_1\}\). Now the proof split to two cases.

**Case 1:** \(\hat{Z} \neq \emptyset\) mod \(D^*_1\)

Define \(Y^* = \{\bar{\zeta} \in \hat{Z} : \text{for every } i \in \hat{Z} \cap j, \text{we have } j \in Y_i\}\). Notice that \(Y^*\) is the intersection of \(\hat{Z}\) with the diagonal intersection of \(\kappa\) sets from \(D^*_1\)

(since \(i \in \hat{Z} \Rightarrow Y_i \in D^*_1\)), hence (by the normality of \(D^*_1\)) \(Y^* \neq \emptyset\) mod \(D^*_1\).

But then, as we will see soon, by shrinking the \(C_j'\) for \(j \in Y^*\), we can get a homogeneous red set of cardinality \(\lambda\), which is contrary to the assumption toward contradiction.

We define \(\bar{C}_j\) for \(j \in Y^*\) by induction on \(j\) such that \(\bar{C}_j\) is a subset of \(C_j'\) of cardinality \(\lambda_{j+1}\). Now, for \(j \in Y^*\), let \(\bar{C}_j\) be the set of \(\xi \in C_j'\) such that for every \(i \in Y^* \cap j\) and every \(\zeta \in \bar{C}_i\) we have \(\xi \notin C^+_Y(\zeta)\). So, in fact, \(\bar{C}_j\) has cardinality \(\lambda_{j+1}\) as it is the result of removing < \(\lambda_{j+1}\) elements from \(C_j'\) where \(|C_j'| = \lambda_{j+1}\) by its choice. Indeed, the number of such pairs \((i, \zeta)\) is \(\leq \lambda_j\) and: for \(i \in Y^* \cap j\) and \(\zeta \in \bar{C}_i:\

(a) \(j \in Y_i\) [Why? by the definition of \(Y^*\) as \(j \in Y^*\)]
(b) \(\zeta \in C'_j\) [Why? as \(\zeta \in \bar{C}_i\) and \(\bar{C}_i \subseteq C'_j\) by the induction hypothesis]
(c) \(Y(\zeta) = Y_i\) [Why? as by (b) we have \(\zeta \in C'_j\) and the choice of \(C'_j\)]
(d) \(j \in Y(\zeta)\) [Why? by (a)+(c)]
(e) \(j \notin Z^+(\zeta)\) [Why? by (d) and the choice of \(Y(\zeta)\) as \(Z^* \setminus Z^+(\zeta)\)]
(f) \( C_j^+ (\zeta) \) has cardinality \(< \lambda_{j+1} \) [Why? by (e) and the choice of \( Z^+(\zeta) \), as \( j \in \hat{Z} \subseteq Z^+ \)].

So \( \hat{C}_j \) is a well defined subset of \( C_j' \) of cardinality \( \lambda_{j+1} \) for every \( j \in Y^* \).

But then, clearly the union of the \( \hat{C}_j \) for \( j \in Y^* \), call it \( \hat{C} \) satisfies:

(a) it has cardinality \( \lambda \) [as \( j \in Y^* \Rightarrow |\hat{C}_j| = \lambda_{j+1} \) and \( \sup (Y^*) = \kappa \) as \( Y^* \neq \emptyset \mod D_1^* \)]

(b) \( c[|\hat{C}_j|^2 \) is constantly red [as we are assuming (**3)]

(\( \gamma \)) if \( i < j \) are from \( Y^* \) and \( \zeta \in \hat{C}_i, \xi \in \hat{C}_j \) then \( c\{\zeta, \xi\} = \text{red} \) [as \( \xi \notin C_j^+(\zeta) \)]

So \( \hat{C} \) has cardinality \( \lambda \) and is homogeneously red. This concludes the proof in the case \( \hat{Z} \neq 0 \mod D_1^* \)

**Case 2:** \( \hat{Z} = 0 \mod D_1^* \).

In that case there are \( i \in Z^*, \beta \in C_i \) such that \( Z^+(\beta) \neq 0 \mod D_1^* \) [Why? well, \( Z^* \in D^* \subseteq D_1^* \) and \( \hat{Z} = 0 \mod D_1^* \), hence \( Z^* \setminus \hat{Z} \neq \emptyset \).

Choose \( i \in Z^* \setminus \hat{Z} \). By the definition of \( \hat{Z}, Y_i \notin D_1^* \). So, if \( \beta \in C_i' \) then \( Y(\beta) = Y_i \notin D_1^* \) and choose \( \beta \in C_i' \), so \( Y(\beta) \notin D_1^* \) hence by the definition of \( Y(\beta) \) we have \( Z^* \setminus Z^+(\beta) = Y(\beta) \notin D_1^* \). Since \( Z^* \in D_1^* \), we conclude that \( Z^+(\beta) \neq 0 \mod D_1^* \).

Let \( \alpha' = \alpha^* \setminus (\beta), Z' = Z^+(\beta), D' = D^* + Z' \), it is a normal filter by the previous sentence as \( D^* \subseteq D_1^* \) and lastly we define \( f' \in \text{"Ord} \)

(a) if \( j \in Z' \) then \( f'(j) = \text{Min}\{\text{rk}(\alpha') : \gamma \in C_j^+(\beta) \subseteq B_j\} \)

(b) otherwise \( f'(j) = 0 \)

Clearly

(\( \alpha' \), \( Z', D', f' \)) \( \in K \), and

(\( \beta \)) \( f' <_{D'} f^* \)

[Why? as \( Z' \in D' \) and if \( j \in Z' \) then for some \( \gamma \in C_j^+(\beta) \) we have \( f'(j) = \text{rk}(\alpha') = \text{rk}(\alpha^* \setminus (\beta)) \) which by the definition of \( \text{rk} \) is \( \leq \text{rk}(\alpha^*) = f^*(j) \), recalling (4) from stage C.]

hence

(\( \gamma \)) \( \text{rk}_{D'}(f') < \text{rk}_{D^*}(f^*) \)

[Why? see Definition 0.2].

(0.2)

But \( \text{rk}_{D'}(f^*) = \text{rk}_{D^*}(f^*) \) as \( Z' = Z^+(\beta) \neq 0 \mod D_1^* \) by the definition of \( D_1^* \) as extending the filter dual to \( J[f^*, D^*] \), see Definition 0.3. Hence \( \text{rk}_{D'}(f') < \) \( \text{rk}_{D^*}(f^*) \), so we get a contradiction to the choice of \( (\alpha^*, Z^*, D^*, f^*) \).

Clearly at least one of the two cases holds, so we are done. \( \square \)
REFERENCES


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