$\aleph_N$-FREE ABELIAN GROUP WITH NO NON-ZERO HOMOMORPHISM TO $\mathbb{Z}$ SH883

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Abstract. We, for any natural n, construct an $\aleph_n$-free abelian groups which have few homomorphisms to $\mathbb{Z}$. For this we use "$\aleph_n$-free $(n+1)$-dimensional black boxes". The method is hopefully relevant to other constructions of $\aleph_n$-free abelian groups.

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Annotated Content

§1 Constructing $\aleph_{k(*)+1}$-free Abelian group

[We introduce "$\mathbf{x}$ is a combinatorial $k(*)$-parameter". We also give a short cut for getting only "there is a non-Whitehead $\aleph_{k(*)+1}$-free non-free abelian group" (this is from 1.6 on). This is similar to [Sh 771, §5], so proofs are put in an appendix, except 1.14, note that 1.14(3) really belongs to §3.]

§2 Black boxes

[We prove that we have black boxes in this context, see 2.1; it is based on the simple black box. Now 2.3 belongs to the short cut.]

§3 Constructing abelian groups from combinatorial parameter

[For $\mathbf{x} \in K^{\rm cb}_{k(*)+1}$ we define a class $\mathscr{G}_{\mathbf{x}}$ of abelian groups constructed from it and a black box. We prove they are all $\aleph_{k(*)+1}$-free of cardinality $|\Gamma|^{\mathbf{x}} + \aleph_0$ and some $G \in \mathscr{G}_{\mathbf{x}}$ satisfies $\text{Hom}(G, \mathbb{Z}) = \{0\}$.]

§4 Appendix 1

[We give adaptation of the proofs from [Sh 771, §5] with the relevant changes.]

§0 Introduction

For regular $\theta = \aleph_n$ we look for a θ-free abelian group G with $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$. We first construct G and a pure subgroup $\mathbb{Z}z \subseteq G$ which is not a direct summand. If instead "not direct product" we ask "not free" so naturally of cardinality θ, we know much: see [EM02].

We can ask further questions on abelian groups, their endormorphism rings, similarly on modules; naturally questions whose answer is known when we demand $\aleph_1$-free instead $\aleph_n$-free; see [GbTl06] . But we feel those two cases can serve as a base for significant number of such problems (or we can immitate the proofs). Also these cases are reasonable for sorting out the set theoretical situation. Why not $\theta = \aleph_\omega$ and higher cardinals? (there are more reasonable cardinals for which such results are not excluded), we do not fully know: note that also in previous questions historically this was harder.

Note that there is such an abelian group of cardinality $\aleph_1$, by [Sh:98, §4] and see more in Göbel-Shelah-Struüngman [GShS 785]. However, if MA then $\aleph_2 < 2^{\aleph_0} \Rightarrow$ any $\aleph_2$-free abelian group of cardinality $< 2^{\aleph_0}$ fail the question.

The groups we construct are in a sense complete, like ${}^\omega\mathbb{Z}$. They are close to the ones from [Sh 771, §5] but there $S = \{0, 1\}$ as there we are interested in Borel abelian groups. See earlier [Sh 161], see representations of [Sh 161] in [Sh 523, §3], [EM02].

However we still like to have $\theta = \aleph_\omega$, i.e. $\aleph_\omega$-free abelian groups. Concerning this we continue in [Sh 898].

We thank Ester Sternfield and Rüdiger Göbel for corrections.

We shall use freely the well known theorem saying

0.1 Theorem. A subgroup of a free abelian group is a free abelian group.

0.2 Definition. 1) $\Pr(\lambda, \kappa)$: means that for some $\bar{G}$ we have:

- (a) $\bar{G} = \langle G_\alpha : \alpha \leq \kappa + 1 \rangle$
- (b) $\bar{G}$ is an increasing continuous sequence of free abelian groups
- (c) $|G_{\kappa+1}| \leq \lambda$,
- (d) $G_{\kappa+1}/G_\alpha$ is free for $\alpha < \kappa$,
- (e) $G_0 = \{0\}$
- (f) G_β/G_α is free if $\alpha \leq \beta \leq \kappa$
- (g) some $h \in \mathrm{Hom}(G_\kappa; \mathbb{Z})$ cannot be extended to $\hat{h} \in \mathrm{Hom}(G_{\kappa+1}, \mathbb{Z})$.

2) We let $\Pr^-(\lambda, \theta, \kappa)$ be defined as above, only replacing "$G_{\kappa+1}/G_\alpha$ is free for $\alpha < \kappa$" by "$G_{\kappa+1}/G_\kappa$ is θ-free.

§1 Constructing $\aleph_{k(*)+1}$-free abelian groups

1.1 Definition. 1) We say $\mathbf{x}$ is a combinatorial parameter if $\mathbf{x} = (k, S, \Lambda) = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$ and they satisfy clauses (a)-(c)

- (a) $k < \omega$
- (b) S is a set (in [Sh 771], $S = \{0, 1\}$),
- (c) $\Lambda \subseteq {}^{k+1}({}^{\omega}S)$ and for simplicity $|\Lambda| \geq \aleph_0$ if not said otherwise.

1A) We say $\mathbf{x}$ is an abelian group k-parameter when $\mathbf{x} = (k, S, \Lambda, \mathbf{a})$ such that (a),(b),(c) from part (1) and:

- (d) $\mathbf{a}$ is a function from $\Lambda \times \omega$ to $\mathbb{Z}$.

2) Let $\mathbf{x} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$ or $\mathbf{x} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}}, \mathbf{a}^{\mathbf{x}})$. A parameter is a k-parameter for some k and $K^{\mathrm{cb}}_{k(*)}/K^{\mathrm{gr}}_{k(*)}$ is the class of combinatorial/abelian group $k(*)$-parameters.

3) We may write $\mathbf{a}^{\mathbf{x}}_{\bar{\eta},n}$ instead $\mathbf{a}^{\mathbf{x}}(\eta, n)$. Let $w_{k,m} = w(k,m) = \{\ell \leq k : \ell \neq m\}$.
4) We say $\mathbf{x}$ is full when $\Lambda^{\mathbf{x}} = {}^{k(*)}({}^{\omega}S)$.
5) If $\Lambda \subseteq \Lambda^{\mathbf{x}}$ let $\mathbf{x} \restriction \Lambda$ be $(k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda)$ or $(k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda, \mathbf{a} \restriction (\Lambda \times \omega))$ as suitable. We may write $\mathbf{x} = (\mathbf{y}, \mathbf{a})$ if $\mathbf{a} = \mathbf{a}^{\mathbf{x}}, \mathbf{y} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$.

1.2 Convention. If $\mathbf{x}$ is clear from the context we may write k or $k(*), S, \Lambda, \mathbf{a}$ instead of $k^{\mathbf{x}}, S^{\mathbf{s}}, \Lambda^{\mathbf{x}}, \mathbf{a}^{\mathbf{x}}$.

A variant of the above is

1.3 Definition. 1) For $\bar{S} = \langle S_m : m \leq k \rangle$ we define when $\mathbf{x}$ is a $\bar{S}$-parameter: $\bar{\eta} \in \Lambda^{\mathbf{x}} \wedge m \leq k^{\mathbf{x}} \Rightarrow \eta_m \in {}^{\omega}(S_m)$.
2) We say $\bar{\alpha}$ is a $(\mathbf{x}, \bar{\chi})$-black box or $\bar{\alpha}$ witness $\mathrm{Qr}(\mathbf{x}, \bar{\chi})$ when:

- (a) $\bar{\chi} = \langle \chi_m : m \leq k^{\mathbf{x}} \rangle$
- (b) $\bar{\alpha} = \langle \bar{\alpha}_{\bar{\eta}} : \bar{\eta} \in \Lambda^{\mathbf{x}} \rangle$
- (c) $\bar{\alpha}_{\bar{\eta}} = \langle \alpha_{\bar{\eta},m,n} : m \leq k^{\mathbf{x}}, n < \omega \rangle$ and $\alpha_{\bar{\eta},m,n} < \chi_m$
- (d) if $h_m : \Lambda^{\mathbf{x}}_m \to \chi_m$ for $m \leq k^{\mathbf{x}}$ <u>then</u> for some $\bar{\eta} \in \Lambda^{\mathbf{x}}$ we have: $m \leq k^{\mathbf{x}} \wedge n < \omega \Rightarrow h_m(\bar{\eta} \upharpoonleft \langle m, n \rangle) = \alpha_{\bar{\eta},m,n}$, see clause (a) of Definition 1.4 below on "$\bar{\eta} \upharpoonleft \langle m, n \rangle$" and $\Lambda^{\mathbf{x}}_m$.

2A) We may replace $\bar{\chi}$ by χ if $\bar{\chi} = \langle \chi : \ell \le k^{\mathbf{x}} \rangle$. We may replace $\mathbf{x}$ by $\Lambda^{\mathbf{x}}$ (so say $\mathrm{Qr}(\Lambda^{\mathbf{x}}, \bar{\chi})$ or say $\bar{\alpha}$ is a $(\Lambda, \bar{\chi})$-black box).
3) We say a parameter $\mathbf{x}$ is $\bar{S}$-full or $\mathbf{x}$ is a full $(\bar{S}, k)$-parameter when $\Lambda^{\mathbf{x}} = \prod\limits_{m \le k} {}^{\omega}(S_m)$.

1.4 Definition. For a $k(*)$-parameter $\mathbf{x}$ and for $m \le k(*)$ let

- (a) $\Lambda^{\mathbf{x}}_m = \Lambda_{\mathbf{x},m} = \{\bar{\eta} : \bar{\eta} = \langle \eta_\ell : \ell \le k(*) \rangle$ and $\eta_m \in {}^{\omega>}S$ and $\ell \le k(*) \wedge \ell \ne m \Rightarrow \eta_\ell \in {}^{\omega}S$ and for some $\bar{\eta}' \in \Lambda$ we have $n < \omega, \bar{\eta} = \bar{\eta}' \upharpoonleft \langle m, n \rangle\}$ where $\bar{\eta} = \bar{\eta}' \upharpoonleft \langle m, n \rangle$ means $\eta_m = \eta'_m \restriction n$ and $\ell \le k(*) \wedge \ell \ne m \Rightarrow \eta_\ell = \eta'_\ell\}$
- (b) $\Lambda^{\mathbf{x}}_{\le k(*)}$ is $\cup\{\Lambda^{\mathbf{x}}_m : m \le k(*)\}$
- (c) $m(\bar{\eta}) = m$ if $\bar{\eta} \in \Lambda^{\mathbf{x}}_m$.

1.5 Definition. 1) We say a combinatorial $k(*)$-parameter $\mathbf{x}$ is free <u>when</u> there is a list $\langle \bar{\eta}^\alpha : \alpha < \alpha(*) \rangle$ of $\Lambda^{\mathbf{x}}$ such that for every α for some $m \le k(*)$ and some $n < \omega$ we have

$(*)$ $\bar{\eta}^\alpha_m \upharpoonleft \langle m, n \rangle \notin \{\eta^\beta_m \upharpoonleft \langle m, n \rangle : \beta < \alpha\}$.

2) We say a combinatorial k-parameter $\mathbf{x}$ is θ-free when $\mathbf{x} \restriction \Lambda = (k, S^{\mathbf{x}}, \Lambda)$ is free for every $\Lambda \subseteq \Lambda^{\mathbf{x}}$ of cardinality $< \theta$.

Remark. 1) We can require in $(*)$ even $(\exists^\infty n)[\eta^\alpha_m(n) \notin \cup\{\eta^\beta_\ell(n') : \ell \le k, \beta < \alpha, n' < \omega\}]$.

At present this seems an immaterial change.

1.6 Definition. For $k(*) < \omega$ and an abelian group $k(*)$-parameter $\mathbf{x}$ we define an abelian group $G = G_{\mathbf{x}}$ as follows: it is generated by $\{x_{\bar{\eta}} : m \le k(*)$ and $\bar{\eta} \in \Lambda^{\mathbf{x}}_m\} \cup \{y_{\bar{\eta},n} : n < \omega$ and $\bar{\eta} \in \Lambda^{\mathbf{x}}\} \cup \{z\}$ freely except the equations:

$\boxtimes_{\bar{\eta},n}$ $(n!) y_{\bar{\eta},n+1} = y_{\bar{\eta},n} + \mathbf{a}^{\mathbf{x}}_{\bar{\eta},n} z + \sum\{x_{\bar{\eta} \upharpoonleft <m,n>} : m \le k(*)\}$.

1.7 Explanation. A canonical example of a non-free group is $(\mathbb{Q}, +)$. Other examples are related to it after we divide by something. The y's here play the role of providing (hidden) copies of $\mathbb{Q}$. What about x's? For $\bar{\eta} \in \Lambda$ we consider $\langle y_{\bar{\eta},n} : n < \omega \rangle$, as a candidate to represent $(\mathbb{Q}, +), k(*) + 1$, "opportunities" to avoid having $(\mathbb{Q}, +)$ as a quotient, say by dividing K by a subgroup generated by some of the x's.

This is used to prove $G_{\mathbf{x}}$ is not free even not $\aleph_{k(*)+2}$-free, which is necessary. But for each $m \le k(*)$ if $\langle x_{\bar\eta\restriction(m,n)} : n < \omega\rangle$ are not in K, or at least $x_{\eta\restriction(m,n)}$ for n large enough then $\mathbb{Q}$ is not represented using $\langle y_{\bar\eta,n} : n < \omega\rangle$; so we have $k(*)+1$ "opportunities" to avoid having $\langle y_{\bar\eta,n} : n < \omega\rangle$ represent $(\mathbb{Q},+)$ in the quotient, one for each infinite cardinal $\le \aleph_{k(*)}$. This helps us prove $\aleph_{k(*)}$-freeness. More specifically, for each $m(*) \le k(*)$ if $H \subseteq G$ is the subgroup which is generated by $X = \{x_{\bar\eta\restriction<m,n>} : m \ne m(*)$ and $\bar\eta \in {}^{k(*)+1}({}^{\omega}S)$ and $m \le k(*)\}$, still in G/H the set $\{y_{\bar\eta,n} : n < \omega\}$ does not generate a copy of $\mathbb{Q}$, as witnessed by $\{x_{\bar\eta\restriction<m(*),n>} : n < \omega\}$.

As a warm up we note:

1.8 Claim. *For $k(*) < \omega$ and $k(*)$-parameter $\mathbf{x}$ the abelian group $G_{\mathbf{x}}$ is an $\aleph_1$-free abelian group.*

Now systematically

1.9 Definition. Let $\mathbf{x}$ be a $k(*)$-parameter.
1) For $U \subseteq {}^{\omega}S$ let $G_U = G_U^{\mathbf{x}}$ be the subgroup of G generated by $Y_U = Y_U^{\mathbf{x}} = \{z\} \cup \{y_{\bar\eta,n} : \bar\eta \in \Lambda \cap {}^{k(*)+1}(U)$ and $n < \omega\} \cup \{x_{\bar\eta\restriction<m,n>} : m \le k(*)$ and $\bar\eta \in \Lambda \cap {}^{(k(*)+1)}(U)$ and $n < \omega\}$. Let $G_U^+ = G_U^{\mathbf{x},+}$ be the divisible hull of G_U and $G^+ = G^+_{({}^{\omega}S)}$.
2) For $U \subseteq {}^{\omega}S$ and finite $u \subseteq {}^{\omega}S$ let $G_{U,u}$ be the subgroup[1] of G generated by $\cup\{G_{U\cup(u\setminus\{\eta\})} : \eta \in u\}$; and for $\bar\eta \in {}^{k(*)\ge}U$ let $G_{U,\bar\eta}$ be the subgroup of G generated by $\cup\{G_{U\cup\{\eta_k : k < \ell g(\bar\eta) \text{ and } k \ne \ell\}} : \ell < \ell g(\bar\eta)\}$.
3) For $U \subseteq {}^{\omega}S$ let $\Xi_U = \Xi_U^{\mathbf{x}} = \{$the equation $\boxtimes_{\bar\eta,n} : \bar\eta \in \Lambda \cap {}^{k(*)+1}U$ and $n < \omega\}$. Let $\Xi_{U,u} = \Xi^{\mathbf{x}}_{U,u} = \cup\{\Xi_{U\cup(u\setminus\{\beta\})} : \beta \in u\}$.

1.10 Claim. *Let $\mathbf{x} \in K_{k(*)}$.*
0) If $U_1 \subseteq U_2 \subseteq {}^{\omega}S$ <u>then</u> $G^+_{U_1} \subseteq G^+_{U_2} \subseteq G^+$.
1) For any $n(*) < \omega$, the abelian group G_U^+ (which is a vector space over $\mathbb{Q}$), has the basis $Y_U^{n(*)} := \{z\} \cup \{y_{\bar\eta,n(*)} : \bar\eta \in \Lambda \cap {}^{k(*)+1}(U)\} \cup \{x_{\bar\eta\restriction<m,n>} : m \le k(*), \bar\eta \in \Lambda \cap {}^{k(*)+1}(U)$ and $n < \omega\}$.
2) For $U \subseteq {}^{\omega}S$ the abelian group G_U is generated by Y_U freely (as an abelian group) except the set Ξ_U of equations.
3) If $m(*) < \omega$ and $U_m \subseteq {}^{\omega}S$ for $m < m(*)$ <u>then</u> the subgroup $G_{U_0} + \ldots + G_{U_{m(*)-1}}$ of G is generated by $Y_{U_0} \cup Y_{U_1} \cup \ldots \cup Y_{U_{m(*)-1}}$ freely (as an abelian group) except the equations in $\Xi_{U_0} \cup \Xi_{U_1} \cup \ldots \cup \Xi_{U_{m(*)-1}}$.
3A) Moreover $G/(G_{U_0} + \ldots + G_{U_{m(*)+1}})$ is $\aleph_1$-free provided that

$\circledast$ *if $\eta_0, \ldots, \eta_{k(*)} \in \cup\{U_m : m < m(*)\}$ are such that*

[1] note that if $u = \{\eta\}$ then $G_{U,u} = G_U$

$(\forall \ell \le k(*))(\exists m < m(*))[\{\eta_0, \dots, \eta_{k(*)}\} \backslash \{\eta_\ell\} \subseteq U_m)$
then for some $m < m(*)$ *we have* $\{\eta_0, \dots, \eta_{k(*)}\} \subseteq U_m$.

4) If $m(*) \le k(*)$ *and* $U_\ell = U \backslash U'_\ell$ *for* $\ell < m(*)$ *and* $\langle U'_\ell : \ell < m(*)\rangle$ *are pairwise disjoint then* $\circledast$ *holds.*
5) $G_{U,u} \subseteq G_{U \cup u}$ *if* $U \subseteq {}^\omega S$ *and* $u \subseteq {}^\omega S \backslash U$ *is finite; moreover* $G_{U,u} \subseteq_{\rm pr} G_{U \cup u} \subseteq_{\rm pr} G$.
6) If $\langle U_\alpha : \alpha < \alpha(*)\rangle$ *is* $\subseteq$*-increasing continuous then also* $\langle G_{U_\alpha} : \alpha < \alpha(*)\rangle$ *is* $\subseteq$*-increasing continuous.*
7) If $U_1 \subseteq U_2 \subseteq U \subseteq {}^\omega S$ *and* $u \subseteq {}^\omega S \backslash U$ *is finite,* $|u| < k(*)$ *and* $U_2 \backslash U_1 = \{\eta\}$ *and* $v = u \cup \{\eta\}$ *then* $(G_{U,u} + G_{U_2 \cup u})/(G_{U,u} + G_{U_1 \cup u})$ *is isomorphic to* $G_{U_1 \cup v}/G_{U_1,v}$.
8) If $U \subseteq {}^\omega S$ *and* $u \subseteq {}^\omega S \backslash U$ *has* $\le k(*)$ *members then* $(G_{U,u} + G_u)/G_{U,u}$ *is isomorphic to* $G_u/G_{\emptyset,u}$.

1.11 Discussion: For the reader's benefit we write what the group $G_{\mathbf{x}}$ is for the case $k(*) = 0$. So, omitting constant indexes and replacing sequences of length one by the unique entry we get that it is generated by $y_{\eta,n}$ (for $\eta \in {}^\omega S, n < \omega$) and x_ν (for $\nu \in {}^{\omega >} S$) freely as an abelian group except the equations $(n!) y_{\eta,n+1} = y_{\eta,n} + x_{\eta \restriction n}$. Note that if K is the countable subgroup generated by $\{x_\nu : \nu \in {}^{\omega >}2\}$ then G/K is a divisible group of cardinality continuum hence G is not free. So G is $\aleph_1$-free but not free.

Now we have the abelian group version of freeness, the positive results in 1.12, 1.13 and the negative results in 1.13.

1.12 The Freeness Claim. *Let* $\mathbf{x} \in K_{k(*)}$.
1) The abelian group $G_{U \cup u}/G_{U,u}$ *is free if* $U \subseteq {}^\omega S, u \subseteq {}^\omega S \backslash U$ *and* $|u| \le k \le k(*)$ *and* $|U| \le \aleph_{k(*)-k}$.
2) If $U \subseteq {}^\omega S$ *and* $|U| \le \aleph_{k(*)}$, *then* G_U *is free.*

1.13 Claim. *1) If* $\mathbf{x}$ *is a combinatorial* $k(*)$*-parameter then* $\mathbf{x}$ *is* $\aleph_{k(*)+1}$*-free.*
2) If $\mathbf{x}$ *is an abelian group* $k(*)$*-parameter and* $(k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$ *is free, then* $G_{\mathbf{x}}$ *is free.*

Proof. 1) Easily follows by (2).
2) Similar and follows from 3.2 as easily G belongs to $\mathscr{G}_{(k(*),S^{\mathbf{x}},\Lambda^{\mathbf{x}})}$, see Definition 3.3.

1.14 Claim. *Assume* $\mathbf{x} \in K^{\mathrm{cb}}_{k(*)}$ *is full (i.e.* $\Lambda^{\mathbf{x}} = {}^{k(*)+1}({}^{\omega}(S^{\mathbf{x}}))$*).*
1) If $U \subseteq {}^{\omega}S$ *and* $|U| \geq (|S| + \aleph_0)^{+(k(*)+1)}$*, the* $(k(*)+1)$*-th successor of* $|S| + \aleph_0$*. Then* $G^{\mathbf{x}}_U$ *is not free.*
2) If $|S^{\mathbf{x}}| \geq \aleph_{k(*)+1}$ *then* $G_{\mathbf{x}}$ *is not free.*
3) Assume $\mathbf{x} \in K^{\mathrm{cb}}_{k(*)}, |S^{\mathbf{x}}_\ell| + \lambda_\ell < \lambda_{\ell+1}$ *for* $\ell < k(*)$ *and* $|\Lambda^{\mathbf{x}}| \geq \lambda_{k(*)}$ *and* $G \in \mathscr{G}_{\mathbf{x}}$ *(see Definition 3.3) then* G *is not free.*

Proof. 1) Let $\aleph_\alpha = |S|$. Assume toward contradiction that G_U is free and let χ be large enough; for notational simplicity assume $|U| = \aleph_{\alpha+k(*)+1}$, this is O.K. as a subgroup of a free abelian group is a free abelian group. We choose N_ℓ by downward induction on $\ell \leq k(*)$ such that

- (a) N_ℓ is an elementary submodel[2] of $(\mathscr{H}(\chi), \in, <^*_\chi)$
- (b) $\|N_\ell\| = |N_\ell \cap \aleph_{\alpha+k(*)}| = \aleph_{\alpha+\ell}$ and $\{\zeta : \zeta \leq \aleph_{\alpha+\ell}\} \subseteq N_\ell$
- (c) $\langle x_{\bar\eta} : \bar\eta \in \Lambda^{\mathbf{x}}_{\leq k(*)}\rangle, \langle y_{\bar\eta,n} : \bar\eta \in \Lambda^{\mathbf{x}}$ and $n < \omega\rangle, U$ and G_U belong to N_ℓ and $N_{\ell+1}, \ldots, N_{k(*)} \in N_\ell$.

Let $G_\ell = G_U \cap N_\ell$, a subgroup of G_U. Now

$(*)_0$ $G_U/(\Sigma\{G_\ell : \ell \leq k(*)\})$ is a free (abelian) group
[easy or see [Sh 52], that is:
as G_U is free we can prove by induction on $k \leq k(*)+1$ then $G/(\Sigma\{G_{k(*)+1-\ell} : \ell < k\})$ is free, for $k = 0$ this is the assumption toward contradiction, the induction step is by Ax VI in [Sh 52] for abelian groups and for $k = k(*)+1$ we get the desired conclusion.]

Now

$(*)_1$ letting U^1_ℓ be U for $\ell = k(*)+1$ and $\bigcap\limits_{m=\ell}^{k(*)} (N_m \cap U)$ for $\ell \leq k(*)$; we have:
U^1_ℓ has cardinality $\aleph_{\alpha+\ell}$ for $\ell \leq k(*)+1$
[Why? By downward induction on ℓ. For $\ell = k(*)+1$ this holds by an assumption. For $\ell = k(*)$ this holds by clause (b). For $\ell < k(*)$ this holds by the choice of N_ℓ as the set $\bigcap\limits_{m=\ell+1}^{k(*)} (N_m \cap U)$ has cardinality $\aleph_{\alpha+\ell+1} \geq \aleph_\ell$ and belong to N_ℓ and clause (b) above.]

[2] $\mathscr{H}(\chi)$ is $\{x$: the transitive closure of x has cardinality $< \chi\}$ and $<^*_\chi$ is a well ordering of $\mathscr{H}(\chi)$

$(*)_2$ $U^2_\ell =: U^1_{\ell+1} \backslash (N_\ell \cap U)$ has cardinality $\aleph_{\alpha+\ell+1}$ for $\ell \le k(*)$
[Why? As $|U^1_{\ell+1}| = \aleph_{\ell+1} > \aleph_\ell = \|N_\ell\| \ge |N_\ell \cap U|$.]

$(*)_3$ for $m < \ell \le k(*)$ the set $U^3_{m,\ell} =: U^2_\ell \cap \bigcap\limits_{r=m}^{\ell-1} N_r$ has cardinality $\aleph_{\alpha+m}$
[Why? By downward induction on m. For $m = \ell - 1$ as $U^2_\ell \in N_m$ and $|U^2_\ell| = \aleph_{\alpha+\ell+1}$ and clause (b). For $m < \ell - 1$ similarly.]

Now for $\ell = 0$ choose $\eta^*_\ell \in U^2_\ell$, possible by $(*)_2$ above. Then for $\ell > 0, \ell \le k(*)$ choose $\eta^*_\ell \in U^3_{0,\ell}$. This is possible by $(*)_3$. So clearly

$(*)_4$ $\eta^*_\ell \in U$ and $\eta^*_\ell \in N_m \cap U \Leftrightarrow \ell \ne m$ for $\ell, m \le k(*)$.
[Why? If $\ell = 0$, then by its choice, $\eta^*_\ell \in U^2_\ell$, hence by the definition of U^2_ℓ in $(*)_2$ we have $\eta^*_\ell \notin N_\ell$, and $\eta^*_\ell \in U^1_{\ell+1}$ hence $\eta^*_\ell \in N_{\ell+1} \cap \ldots \cap N_{k(*)}$ by $(*)_1$ so $(*)_4$ holds for $\ell = 0$. If $\ell > 0$ then by its choice, $\eta^*_\ell \in U^3_{0,\ell}$ but $U^3_{m,\ell} \subseteq U^2_\ell$ by $(*)_3$ so $\eta^*_\ell \in U^2_\ell$ hence as before $\eta^*_\ell \in N_{\ell+1} \cap \ldots \cap N_{k(*)}$ and $\eta^*_\ell \notin N_\ell$. Also by $(*)_3$ we have $\eta^*_\ell \in \bigcap\limits_{r=0}^{\ell-1} N_\ell$ so $(*)_4$ really holds.]

Let $\bar\eta^* = \langle \eta^*_\ell : \ell \le k(*) \rangle$ and let G' be the subgroup of G_U generated by $\{x_{\bar\eta \restriction <m,n>} : m \le k(*)$ and $\bar\eta \in {}^{k(*)+1}U$ and $n < \omega\} \cup \{y_{\bar\eta,n} : \bar\eta \in {}^{k(*)+1}U$ but $\bar\eta \ne \bar\eta^*$ and $n < \omega\}$. Easily $G_\ell \subseteq G'$ recalling $G_\ell = N_\ell \cap G_U$ hence $\Sigma\{G_\ell : \ell \le k(*)\} \subseteq G'$, but $y_{\bar\eta^*,0} \notin G'$ hence

$(*)_5$ $y_{\bar\eta^*,0} \notin \sum\{G_\ell : \ell \le k(*)\}$.

But for every n

$(*)_6$ $\bar n! y_{\bar\eta^*,n+1} - y_{\bar\eta^*,n} = \Sigma\{x_{\bar\eta^* \restriction <m,n>} : m \le k(*)\} \in \Sigma\{G_\ell : \ell \le k(*)\}$.
[Why? $x_{\bar\eta^* \restriction <m,n>} \in G_m$ as $\bar\eta^* \restriction (k(*)) + 1 \backslash \{m\}) \in N_m$ by $(*)_4$.]

We can conclude that in $G_U / \sum\{G_\ell : \ell \le k(*)\}$, the element $y_{\bar\eta^*,0} + \sum\{G_\ell : \ell \le k(*)\}$ is not zero (by $(*)_5$) but is divisible by every natural number by $(*)_6$. This contradicts $(*)_0$ so we are done.
2),3) Left to the reader. $\square_{1.14}$

§2 Black Boxes

2.1 Claim. *1) Assume* $k(*) < \omega, \chi = \chi^{\aleph_0}, \lambda = \beth_{k(*)}(\chi)$ *and* $S = \lambda, \Lambda = {}^{k(*)+1}({}^\omega S)$ *or just* $S_\ell = \lambda_\ell = \chi_\ell, \beth_\ell(\chi) = \lambda_\ell^{\aleph_0} = \chi_\ell$ *for* $\ell \le k(*)$ *and* $\Lambda = \prod\limits_{\ell \le k(*)} {}^\omega(S_\ell)$ *and* $\mathbf{x} = (k(*), \lambda, \Lambda)$ *so* $\mathbf{x}$ *is a full combinatorial* $\langle S_\ell : \ell \le k(*)\rangle$*-parameter. Then* Λ *has a* χ*-black box, i.e.* $\mathrm{Qr}(\Lambda_{\mathbf{x}^{k(*)}}, \chi)$*, see Definition 1.3.*
2) Moreover, $\mathbf{x}$ *has the* $\langle \chi_\ell : \ell \le k(*)\rangle$*-black box, i.e. for every* $\bar B = \langle B_{\bar\eta} : \bar\eta \in \Lambda^{\mathbf{x}}_{\le k(*)}\rangle$ *satisfying clause (c) below we can find* $\langle h_{\bar\eta} : \bar\eta \in \Lambda\rangle$ *such that:*

- (a) $h_{\bar\eta}$ *is a function with domain* $\{\bar\eta \restriction \langle m,n\rangle : m \le k(*), 2 \le n < \omega\}$
- (b) $h_{\bar\eta}(\bar\eta \restriction \langle m,n\rangle) \in B_{\bar\eta \restriction <m,n>}$
- (c) $B_{\bar\eta \restriction \langle m,n\rangle}$ *is a set of cardinality* χ_m
- (d) *if* h *is a function with domain* $\Lambda^{\mathbf{x}}_{\le k(*)}$*, see Definition 1.4 such that* $h(\bar\eta \restriction \langle m,n\rangle) \in B_{(\bar\eta \restriction <m,n>)}$ *for* $\bar\eta \in \Lambda, m \le k(*), n < \omega$ *and* $\alpha_\ell < \lambda_\ell$ *for* $\ell \le k(*)$ *then for some* $\bar\eta \in \Lambda^{\mathbf{x}}, h_{\bar\eta} \subseteq h$ *and* $\eta_\ell(0) = \alpha_\ell$ *for* $\ell \le k(*)$.

3) Assume $\chi_\ell = \lambda_\ell^{\aleph_0}, \chi_{\ell+1} = \chi_{\ell+1}^{\chi_\ell}$ *for* $\ell \le k(*)$*. If* $S_\ell = \lambda_\ell$ *for simplicity, for* $\ell \le k(*), \mathbf{x}$ *is a full combinatorial* $(\bar S, k(*))$*-parameter, and* $|B_{\bar\eta \restriction <m,n>}| \le \chi_{k(*)}$ *for* $\bar\eta \in \Lambda^{\mathbf{x}}$ *then we can find* $\langle h_{\bar\eta} : \bar\eta \in \Lambda^{\mathbf{x}}\rangle$ *as in part (2), moreover such that:*

- (e) *if* $\bar\eta \in \Lambda$ *then* η_ℓ *is increasing*
- (f) *if* λ_ℓ *is regular then we can in clause (d) above add: if* E_ℓ *is a club of* λ_ℓ *for* $\ell \le k(*)$ *then we can demand: if* $\bar\eta \in \Lambda^{\mathbf{x}}$ *then for each* ℓ *for some* $\alpha^*_\ell < \lambda_\ell$ *we have* $\eta_\ell \in {}^\omega(E_\ell \cup \{\alpha^*_\ell\})$
- (g) *if* λ_ℓ *is singular of uncountable cofinality,* $\lambda_\ell = \Sigma\{\lambda_{\ell,i} : i < \mathrm{cf}(\lambda_\ell)\}$, $\mathrm{cf}(\lambda_{i,\ell}) = \lambda_{i,\ell}$ *increasing with* i *we can add: if* $u_\ell \subseteq \mathrm{cf}(\lambda_\ell)$ *is unbounded,* $E_{\ell,i}$ *a club of* $\lambda_{\ell,i}$ *then* $\eta_\ell \in {}^\omega(E_{i,\ell} \cup \{\alpha^*_\ell\})$ *for some* $i \in u_\ell$.

Proof. Part (1) follows form part (2) which follows from part (3), so let us prove part (3). To uniformize the notation in 2.1(1), i.e. 1.3(2) and 2.1(2),(3) we shall denote:

$\odot_1$ $h_{\bar\eta}(\bar\eta \restriction \langle m,n\rangle) = \alpha^{k(*)}_{\bar\eta,m,n}$.

Note that without loss of generality[3] $\bar\nu \in \Lambda^{\mathbf{x}}_{\le k(*)} \Rightarrow B_{\bar\nu} = |B_{\bar\nu}|$, i.e. without loss of generality $\bar\eta \in \Lambda^{\mathbf{x}} \wedge n < \omega \wedge m \le k(*) \Rightarrow B_{\eta 1 <m,n>} = \chi_n$ and we use $\alpha^{k(*)}_{\bar\eta,m,n} = h_{\bar\eta}(\bar\eta \upharpoonleft \langle m,n\rangle)$ for $\bar\eta \in \Lambda_{\mathbf{x}}, m \le k(*)$ and $n < \omega$. We prove part (3) by induction on $k(*)$. Let $\Lambda_k = \Lambda^{\mathbf{x}}$ and without loss of generality $S_\ell = \lambda_\ell$.

Case 1: $k(*) = 0$.

By the simple black box, see [Sh 300, III,§4], or better [Sh:e, VI,§2], see below for details on such a proof.

Case 2: $k(*) = k+1$.

Let

$\odot_2$ $\alpha^k = \langle \alpha^k_{\bar\eta,m,n} : \bar\eta \in \Lambda_k, n < \omega, m \le k\rangle$ witness parts (2),(3) for k, i.e. for $\mathbf{x}^k$, hence no need to assume $\mathbf{x}^k$ is full.

So $\lambda = \lambda_{k(*)}, \chi = \chi_{k(*)}$ and let $\mathbf{H} = \{h : h$ is a function from Λ_k to $\chi\}$. So $|\mathbf{H}| \le (\lambda)^{\lambda_k^{\aleph_0}} = \chi$. By the simple black box, see below, we can find $\langle \bar h_\eta : \eta \in {}^\omega\lambda\rangle$ such that

$\odot_3$ (α) $\bar h_\eta = \langle h_{\eta,n} : n < \omega\rangle$ and $h_{\eta,n} \in \mathbf{H}$ for $\eta \in {}^\omega\lambda$

(β) if $\bar f = \langle f_\nu : \nu \in {}^{\omega>}\lambda\rangle$ and $f_\nu \in \mathbf{H}$ for every such ν and $\alpha < \lambda$ and $\rho \in {}^{\omega>}\lambda$ is increasing then for some increasing $\eta \in {}^\omega\lambda$ we have $\rho \triangleleft \eta$ and $n < \omega \Rightarrow h_{\eta,n} = f_{\eta\restriction n}$

(γ) if $\mathrm{cf}(\lambda) > \aleph_0$ and E is a club of λ then we can add $\cup\{\eta(n) : n < \omega\} \in E$.

[Why? First assume $\chi = \lambda$. Let $\langle \bar g_\alpha = \langle g_{\alpha,\ell} : \ell < n_\alpha\rangle : \alpha < \lambda\rangle$ enumerate ${}^{\omega>}\mathbf{H}$ such that for each $\bar g \in {}^{\omega>}\mathbf{H}$ the set $\{\alpha < \lambda : \bar g_\alpha = \bar g\}$ is unbounded in λ. Now for $\eta \in {}^\omega\lambda$ and $n < \omega$ let $h_{\eta,n} = g_{\eta(k),n}$ for every k large enough if well defined and $g_{\eta\restriction(n+1),n}$ otherwise. So clause (α) of $\odot_3$ holds and as for clause (β) of $\odot_3$, let $\bar f = \langle f_\nu : \nu \in {}^{\omega>}\lambda\rangle$ be given, $f_\nu \in \mathbf{H}$.

Assume $\rho \in {}^{\omega>}\lambda$ is increasing. We choose α_n by induction on $n < \omega$ such that:

$\odot_4$ (α) $\alpha_n = \rho(n)$ if $n < \ell g(\rho)$

[3] Why? (As doubts were cast we shall elaborate.) For $\bar\eta \in \Lambda_{\le k(*)}$ let $B'_{\bar\eta} = \{i : i < |B_{\bar\eta}|\}$ for $\bar\eta \in \Lambda_{\le k(*)}$ and let $g_{\bar\eta}$ be a one-to-one function from B'_η onto $B_{\bar\eta}$. Now assume that $\langle h'_{\bar\eta} : \bar\eta \in \Lambda_{\le k(*)}\rangle$ is as required in the claim for $\langle B'_{\bar\eta} : \bar\eta \in \Lambda\rangle$ and define a fucntion h_η with domain $\mathrm{Dom}(h'_\eta) = \{\bar\eta \restriction \langle m,n\rangle : m \le k(*)$ and $n < \omega\}$ such that $h_{\bar\eta}(\bar\eta \upharpoonleft \langle m,n\rangle) = g_\eta(h_\eta(\bar\eta \upharpoonleft \langle m,n\rangle)) \in B_{\bar\eta 1 <m,n>}$ for $\bar\eta \in \Lambda, m \le k(*), n < \omega$. Define the function h' with domain $\Lambda_{\le k(*)}$ by $h'(\bar\eta) = g^{-1}_{\bar\eta} \circ h$, so h' is well defined with domain $\lambda_{\le k(*)}$ such that $h'(\bar\eta) \in B'_{\bar\eta}$. By the choice of $\langle h'_{\bar\eta} : \bar\eta \in \Lambda_{\le k(*)}\rangle$ there is $\bar\eta \in \Lambda$ such that $m \le k(*) \wedge n < \omega \Rightarrow h'(\bar\eta \upharpoonleft \langle m,n\rangle) = h'(\bar\eta \upharpoonleft \langle m,n\rangle)$. But by the choice of $h_{\bar\eta}, h'$ we have $m \le k(*) \wedge n < \omega \Rightarrow h_{\bar\eta}(\bar\eta \upharpoonleft \langle m,n\rangle) = g^{-1}_{\bar\eta}(h'_{\bar\eta}(\bar\eta \upharpoonleft \langle m,n\rangle)) = g^{-1}_{\bar\eta}(h'(\bar\eta \upharpoonleft \langle m,n\rangle) = h(\bar\eta \upharpoonleft \langle m,n\rangle)$ as required.

(β) $\alpha_n < \lambda$ and $\alpha_n > \alpha_m$ if $n = m+1$

(γ) if $n \geq \ell g(\rho)$ then α_n satisfies $\bar{g}_{\alpha_n} = \langle f_{\langle \alpha_\ell : \ell < m \rangle} : m \leq n \rangle$.

Now $\eta =: \langle \alpha_n : n < \omega \rangle$ is as required in clause (β) of $\odot_3$; to get also clause (γ) of $\odot_3$ we should add in clause (β) of $\odot_4$ then $\alpha_n > \min(E \backslash \alpha_m)$.

Second, if $\chi > \lambda$ but still $\chi \leq \lambda^{\aleph_0}$, let $\langle \bar{g}_\alpha : \alpha < \chi^{\aleph_0} \rangle$ list ${}^{\omega >}\mathbf{H}$, and let $\mathbf{h}_n : \chi \to \lambda$ for $n < \omega$ be such[4] that $\alpha < \beta < \chi \Rightarrow (\forall^\infty n)(\mathbf{h}_n(\alpha) \neq \mathbf{h}_n(\beta))$ and let cd: $\lambda \to {}^{\omega >}\lambda$ be one to one onto. Now for $\eta \in {}^\omega \lambda$ and $n < \omega$ let $h_{\eta,n}$ be g_α where α is the unique ordinal $\alpha < \chi$ such that for every $k < \omega$ large enough $(\mathrm{cd}(\eta(k)))(n) = \mathbf{h}_n(\alpha)$ so in particular $\langle \ell g(\mathrm{cd}(\eta(k)) : k < \omega \rangle$ is going to infinity or $h_{\eta,n}$ is not well defined; in fact, we can use only the case $\ell g(\mathrm{cd}(\eta(k)) = k$; stipulating $h_{\eta,n} \in {}^\omega\{0\}$ when not defined. So we have defined $\langle h_{\eta,n} : \eta \in {}^\omega \lambda, n < \omega \rangle$. Now we immitate the previous argument: clause (β) of $\circledast_2$ holds.

Next we shall define $\bar{\alpha}^{k(*)} = \langle \alpha^{k(*)}_{\bar{\eta},m,n} : \bar{\eta} \in \Lambda_{k+1}, m \leq k(*), n < \omega \rangle$ as required; so let $\bar{\eta} = \langle \eta_\ell : \ell \leq k(*) \rangle \in \Lambda_{k(*)}$ we define $\bar{\alpha}^{k(*)}_{\bar{\eta}} = \langle \alpha^{k(*)}_{\bar{\eta},m,n} : m \leq k(*), n < \omega \rangle$ as follows:

$\odot_5$ if $\eta_{k(*)} \in {}^\omega \lambda$ and $\langle \eta_0, \dots, \eta_{k(*)-1} \rangle \in \Lambda_k$ <u>then</u> for $m \leq k(*)$ and $n < \omega$

(α) if $m = k(*)$ then $\alpha^{k(*)}_{\bar{\eta},m,n} = h_{\eta_{k(*)},n}(\langle \eta_0, \dots, \eta_{k(*)-1} \rangle) < \lambda_m$

(β) if $m < k(*)$, i.e. $m \leq k$ then $\alpha^{k(*)}_{\bar{\eta},m,n} = \alpha^k_{\bar{\eta} \restriction k(*),m,n} < \lambda_m$.

Clearly $\alpha^{k(*)}_{\bar{\eta},m,n} < \lambda_m$ in all cases, as required, (in clause (a),(b),(c) of 2.1(2) and (e) of 2.1(3). But we still have to prove that $\langle \bar{\alpha}^{k(*)}_{\bar{\eta},m,n} : \bar{\eta} \in \Lambda^{k+1}, m \leq k(*), n < \omega \rangle$ witness $\mathrm{Qr}(\mathbf{x}^{k(*)}, \chi)$, see Definition 1.3(2) this suffices for 2.1(2), little more is needed for 2.1(3); just using (γ) of $\odot_3$ and the induction hypothesis.

Why does this hold? Let h be a function with domain $\Lambda^{\mathbf{x}^{k(*)}}_{\leq k(*)}$ as in part (3) and $\alpha^*_\ell < \lambda_\ell$ for $\ell \leq k(*)$.

For $\nu \in {}^{\omega >}\lambda$ let $f_\nu : \Lambda_k \to \lambda = \lambda_{k(*)}$ be defined by: $f_\nu(\langle \eta_\ell : \ell \leq k \rangle) =: h(\langle \eta_\ell : \ell \leq k \rangle \hat{\ } \langle \nu \rangle)$. So by $\odot_3$ above for some increasing $\eta^*_{k(*)} \in {}^\omega \lambda$ we have $\eta^*_{k(*)}(0) = \alpha^*_{k(*)}$ and

$\odot_6$ $n < \omega \Rightarrow f_{\eta^*_{k(*)} \restriction n} = h_{\eta^*_{k(*)},n}$.

Now substituting the definition of $\bar{f}$ we have

$\odot_7$ $\langle \eta_0, \dots, \eta_k \rangle \in \Lambda_k \wedge n < \omega \Rightarrow h_{\eta^*_{k(*)},n}(\eta_0, \dots, \eta_k) = h(\langle \eta_0, \dots, \eta_k, \eta^*_{\eta(*)} \restriction n \rangle)$.

[4]recall $(\forall^\infty N)$ means "for every large enough $n < \omega$"

Substituting the definition of $\bar{\alpha}^k$ we have

$\odot_8$ if $\langle \eta_0, \dots, \eta_k \rangle \in \Lambda_k$ and $n < \omega$ then $\alpha^{k(*)}_{<\eta_0,\dots,\eta_k,\eta^*_{k(*)}>} = h(\langle \eta_0, \dots, \eta_k, \eta^*_{k(*)} \restriction n \rangle)$.

Now we define a function h' with domain $\Lambda^{\mathbf{x}^k}_{\le k}$ by: if $\bar{\eta} \in \Lambda^{\mathbf{x}^k}_{\le k}$ then $h'(\bar{\eta}) = h(\bar{\eta}\hat{\ }\langle \eta^*_{k(*)} \rangle)$.

So by the choice of $\bar{\alpha}^k$ in $\odot_2$ we can find $\langle \eta^*_0, \dots, \eta^*_k \rangle \in \Lambda_k$ with no repetitions such that $\eta^*_\ell(0) = \alpha^*_\ell$ for $\ell \le k$ and in $\odot_2$

$\odot_9$ $m \le k \wedge n < \omega \Rightarrow \alpha^k_{\langle \eta^*_0,\dots,\eta^*_k \rangle, m,n} = h'(\langle \eta^*_0, \dots, \eta^*_k \rangle \upharpoonright (m,n) \rangle)$.

Let $\bar{\eta}^* = \langle \eta^*_0, \dots, \eta^*_k, \eta^*_{k+1} \rangle, \bar{\eta}' = \langle \eta^*_0, \dots, \eta^*_i \rangle$.

Note that

$\odot_{10}$ if $m \le k, n < \omega$ then $h'(\bar{\eta}' \upharpoonright \langle k, m \rangle) = h((\bar{\eta}' \upharpoonright \langle m, n \rangle)\hat{\ }\langle \eta^*_{k(*)} \rangle) = h(\bar{\eta}^* \upharpoonright \langle m, n \rangle)$.

Now by $\odot_9 + \odot_{10}$ and $\odot_5(\beta)$ this means

$\odot_{11}$ if $m \le k$ and $n < \omega$ then $\alpha^{k(*)}_{\bar{\eta}^*,m,n} = h(\bar{\eta}^* \upharpoonright \langle k, m \rangle)$.

So by putting together $\odot_8 + \odot_{11}$ we are clearly done, i.e. we can check that $\langle \eta^*_0, \dots, \eta^*_k, \eta^*_{k(*)} \rangle$ is as required. $\square_{2.1}$

2.2 Conclusion. For every $k < \omega$ there is an $\aleph_{k+1}$-free abelian group G of cardinality $\beth_{k+1}$ and pure (non-zero) subgroup $\mathbb{Z}z \subseteq G$ such that $\mathbb{Z}z$ is not a direct summand of G.

Proof. Let $\chi = 2^{\aleph_0}$ and $\mathbf{x}$ be a combinatorial k-parmeter as guaranteed by 2.1. Now by 2.3(2) below we can expand $\mathbf{x}$ to an abelian group k-parameter, so $G_{\mathbf{x}}$ is as required.

2.3 Claim. *1) If $\mathbf{x}$ is a combinatorial k-parameter such that* $\mathrm{Qr}(\mathbf{x}, 2^{\aleph_0})$ *<u>then</u> for some $\mathbf{a}, \mathbf{y} := (\mathbf{x}, \mathbf{a})$ is an abelian group k-parameter such that $h \in \mathrm{Hom}(G_{\mathbf{y}}, \mathbb{Z}) \Rightarrow h(z) = 0$.*
2) For every k there is an $\aleph_{k+1}$-free abelian group G of cardinality $\beth_{k+1}$ and $z \in G$ a pure $z \in G$ as above.

Proof. 1) Let $\bar{\alpha}$ witness $\mathrm{Qr}(\mathbf{x}, 2^{\aleph_0})$. We define a function ι:Ord $\to \mathbb{Z}$ by: $\iota(\alpha)$ in α if $\alpha < \omega$, is $-n$ if $\alpha = \omega + n < \omega + \omega$ and is zero otherwise. For each $\bar{\eta} \in \Lambda^{\mathbf{x}}$ we shall choose a sequence $\langle \mathbf{a}_{\bar\eta,n} : n < \omega \rangle$ of integers such that for any $b \in \mathbb{Z} \backslash \{0\}$ for no $\bar c \in {}^\omega\mathbb{Z}$ do we have:

$\boxtimes_{\bar\eta}$ for each $n < \omega$ we have

$$n! c_{n+1} = c_n + \mathbf{a}_{\bar\eta,n} b + \Sigma\{\iota(\alpha_{\bar\eta,m,n}) : m \le k(*)\}.$$

This is easy: for each pair $(b, c_0) \in \mathbb{Z} \times \mathbb{Z}$ the set of $\langle \mathbf{a}_n : n < \omega \rangle \in {}^\omega\mathbb{Z}$ such that there is at least one sequence (and always at most one sequence) $\langle c_0, c_1, c_2, \ldots \rangle$ of integers such that $\boxtimes_{\bar\eta}$ holds for them, is meagre, even no-where dense so the choice of $\langle \mathbf{a}_{\bar\eta,n} : n < \omega \rangle$ is possible.

Now toward contradiction assume that h is a homomorphism from $G_{\mathbf{x}}$ to $z\mathbb{Z}$ such that $h(z) = bz, b \in \mathbb{Z}\backslash\{0\}$. We define $h' : \Lambda^{\mathbf{x}}_{\le k} \to \chi$ by $h'(\bar\eta) = n$ if $n < \omega$ and $h(x_{\bar\eta}) = nz$ and $h'(\bar\eta) = \omega + n$ if $n < \omega$ and $h(x_{\bar\eta}) = (-n)z$.

By the choice of $\bar\alpha$, for some $\bar\eta \in \Lambda^{\mathbf{x}}$ we have: $m \le k \wedge n < \omega \Rightarrow h'(\bar\eta \restriction \langle m, n \rangle) = \alpha_{\bar\eta,m,n}$. Hence $h(x_{\bar\eta \restriction (m,n)}) = \iota(\alpha_{\bar\eta,m,n})z$ for $m \le k, n < \omega$.

Let $c_n \in \mathbb{Z}$ be such that $h(y_{\bar\eta,n}) = c_n z$. Now the equation $\boxtimes_{\bar\eta,n}$ in Definition 1.6 is mapped to the n-th equation in $\boxtimes_{\bar\eta}$, so an obvious contradiction.

2) By part (1) and 2.2. $\square_{2.3}$

2.4 Remark. 1) We can replace χ by a set of cardinality χ in Definition 1.3. Using $\mathbb{Z}z$ instead of χ simplify the notation in the proof of 2.3.

2) We have not tried to save in the cardinality of G in 2.3(2), using as basic of the induction the abelian group of cardinality $\aleph_0$ or $\aleph_1$.

2.5 Claim. *1) If $\chi_0 = \chi_0^{\aleph_0}, \chi_{m+1} = 2^{\chi_m}$ and $\lambda_m = \chi_m$ for $m \le k$ for the $\bar\chi$-full combinatorial k-parameter $\mathbf{x}$, the $(\mathbf{x}, \bar\chi)$-black box exist.*

2.6 Conclusion. Assume $\mu_0 < \ldots < \mu_{k(*)}$ are strong limit of cofinality $\aleph_0$ (or $\mu_0 = \aleph_0$), $\lambda_\ell = \mu_\ell^+, \chi_\ell = 2^{\mu_\ell}$.

<u>Then</u> in 2.1 for $\bar\eta \in \Lambda^{\mathbf{x}}$ we can let $h_{\bar\eta,m}$ has domain $\{\bar\nu \in \Lambda^{\mathbf{x}}_m : [\nu_\ell = \eta_\ell$ for $\ell = m+1, \ldots, k(*)\}$.

§3 Constructing abelian groups from combinatorial parameters

3.1 Definition. 1) We say F is a μ-regressive function on a combinatorial parameter $\mathbf{x} \in K^{\rm cb}_{k(*)}$ when $S^{\mathbf{x}}$ is a set of ordinals and:

(a) $\mathrm{Dom}(F)$ is $\Lambda^{\mathbf{x}}$

(b) $\mathrm{Rang}(F) \subseteq [\Lambda^{\mathbf{x}} \cup \Lambda^{\mathbf{x}}_{\le k(*)}]^{\le \aleph_0}$

(c) for every $\bar\eta \in \Lambda^{\mathbf{x}}$ and $m \le k(*)$ we[5] have $\sup \mathrm{Rang}(\eta_m) > \sup(\cup\{\mathrm{Rang}(\nu_m) : \bar\nu \in F(\bar\eta)\})$; note $\bar\nu_\ell \in \Lambda^{\mathbf{x}}$ or $\bar\nu \in \Lambda^{\mathbf{x}}_{\le k(*)}$ as $F(\bar\eta)$ is a set of such objects.

1A) We say F is finitary when $F(\bar\eta)$ is finite for every $\bar\eta$.
1B) We say F is simple if $\eta_{k(*)}(0)$ determined $F(\bar\eta)$ for $\bar\eta \in \Lambda^{\mathbf{x}}$.
2) For $\mathbf{x}, F$ as above and $\Lambda \subseteq \Lambda^{\mathbf{x}}$ we say that Λ is free for $(\mathbf{x}, F)$ when: $\Lambda \subseteq \Lambda^{\mathbf{x}}$ and there is a sequence $\langle \bar\eta^\alpha : \alpha < \alpha(*)\rangle$ listing $\Lambda' = \Lambda \cup \bigcup\{F(\bar\eta) : \bar\eta \in \Lambda\}$ and sequence $\langle \ell_\alpha : \alpha < \alpha(*)\rangle$ such that

(a) $\ell_\alpha \le k(*)$

(b) if $\alpha < \alpha(*)$ and $\bar\eta^\alpha \in \Lambda$ then $F(\bar\eta^\alpha) \subseteq \{\bar\eta^\beta : \beta < \alpha\} \cup \{\bar\eta^\gamma \upharpoonright \langle m, n\rangle : \gamma < \alpha$ is such that $\bar\eta^\gamma \in \Lambda^{\mathbf{x}}$ and $n < \omega, m \le k(*)\}$

(c) if $\alpha < \alpha(*)$ and $\bar\eta^\alpha \in \Lambda$ then for some $n < \omega$ we have $\bar\eta^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\bar\eta^\beta \upharpoonright \langle \ell_\alpha, n\rangle : \beta < \alpha, \eta^\beta \in \Lambda\} \cup \{\bar\eta^\beta : \beta < \alpha\}$.

3) We say $\mathbf{x}$ is θ-free for F is $(\mathbf{x}, F)$ is μ-free <u>when</u> $\mathbf{x}, F$ are as in part (1) and every $\Lambda \subseteq \Lambda^{\mathbf{x}}$ of cardinality $< \theta$ is free for $(\mathbf{x}, F)$.

3.2 Claim. *1) If $\mathbf{x} \in K^{\rm cb}_{k(*)}$ and F is a regressive function on $\mathbf{x}$ <u>then</u> $(\mathbf{x}, F)$ is $\aleph_{k(*)+1}$-free provided that F is finitary or simple.*
2) In addition: if $k \le k(*), \Lambda \subseteq \Lambda^{\mathbf{x}}$ has cardinality $\le \aleph_k$ and $\bar u = \langle u_{\bar\eta} : \bar\eta \in \Lambda\rangle$ satisfies $u_{\bar\eta} \subseteq \{0, \dots, k(*)\}, |u_\eta| > k$, <u>then</u> we can find $\langle \bar\eta^\alpha : \alpha < \aleph_k\rangle, \langle \ell_\alpha : \alpha < \aleph_k\rangle, \langle n_\alpha : \alpha < \aleph_k\rangle$ such that:

(a) $\Lambda \subseteq \{\bar\eta^\alpha : \alpha < \aleph_k\}$

(b) *if $\bar\eta_\alpha \in \Lambda^{\mathbf{x}}$ then $\ell_\alpha \in u_{\bar\eta^\alpha}, n_\alpha < \omega$*

(c) $\bar\eta^\alpha \upharpoonright \langle \ell_\alpha, n_\alpha\rangle \notin \{\bar\eta^\beta \upharpoonright \langle \ell_\alpha, n_\alpha\rangle : \beta < \alpha\} \cup \{\bar\eta^\beta : \beta < \alpha\}$.

Remark. We may wonder:

[5] actually, suffice to have it for $\ell = k(*)$

Ruedeger Question: Assume $F(\bar\eta) \in [\Lambda_{\le k(*)}]^{\le\aleph_0}$ for $\bar\eta \in \Lambda^{\mathbf{x}}$ is as in Definition 3.1. Is this O.K. in the proof of 3.2, particularly Case 1?

Answer: Seems not. Assume $\bar\nu \ne \bar\rho \in \Lambda$ and

(A) $u_{\bar\rho} = \{\ell_1\}, F(\bar\nu) = \{\bar\rho \upharpoonright \langle \ell_1, n\rangle : n < \omega\}$
(B) $u_{\bar\nu} = \{\ell_2\}, F(\bar\rho) = \{\bar\nu \upharpoonright \langle \ell_2, n\rangle : n < \omega\}$.

So if $(\nu, \bar\rho) = (\eta_{\alpha_4}, \eta_{\alpha_2})$, we have $\alpha_0 \ne \alpha_1$ as $\bar\nu \ne \bar\rho, \neg(\alpha_1 < \alpha_2)$ by (B), and $\neg(\alpha_2 < \alpha_1)$ by (A).

Proof. 1) Follows by part (2) for the case $k = k(*), u_{\bar\eta} = \{0, \dots, k(*)\}$ for every $\bar\eta \in \Lambda$.
2) So we are assuming $\mathbf{x} \in K^{\rm cb}_{k(*)}, F$ is a regressive function on $\mathbf{x}$ which is finitary or simple, $k \le k(*), \Lambda \subseteq \Lambda^{\mathbf{x}}$ has cardinality $\le \aleph_k$ and without loss of generality Λ is closed under $\bar\eta \mapsto F(\bar\eta) \cap \Lambda^{\mathbf{x}}$. We prove this by induction on k.

Case 1: $k = 0$.

Subcase 1A: Ignoring F.

Let $\langle \bar\eta^\alpha : \alpha < |\Lambda|\rangle$ list Λ with no repetitions (so $\alpha < |\Lambda| \Rightarrow \alpha < \aleph_k = \aleph_0$). Now $\alpha < |\Lambda| \Rightarrow u_{\bar\eta^\alpha} \ne \emptyset$ and let $\ell_\alpha = \min(u_{\bar\eta^\alpha}) \le k(*)$. Hence for each $\alpha < |\Lambda|$ we know that $\beta < \alpha \Rightarrow \bar\eta^\beta \ne \bar\eta^\alpha$, hence for some $n = n_{\alpha,\beta} < \omega$ we have $\bar\eta^\beta \upharpoonright \langle \ell_\alpha, n_{\alpha,\beta}\rangle \ne \bar\eta^\alpha \upharpoonright \langle \ell_\alpha, n_{\alpha,\beta}\rangle$.

Let $n_\alpha = \sup\{n_{\alpha,\beta} : \beta < \alpha\rangle$, it is $< \omega$ as $\alpha < \omega$. Now $\langle (\ell_\alpha, n_\alpha) : \alpha < |\Lambda|\rangle$ is as required.

Subcase 1B: $\bar\eta \in \Lambda \Rightarrow F(\bar\eta)$ is finite[6].

Let $\langle \eta^\alpha : \alpha < |\Lambda|\rangle$ list Λ, we choose w_j by induction on $j \le j(*), j(*) \le \omega$ such that:

$\circledast$ (a) $w_j \subseteq |\Lambda|$ is finite for $j < \omega$
(b) $j \in w_{j+1}$
(c) if $\alpha \in w_j$ then $F(\bar\eta^\alpha) \cap \Lambda \subseteq \{\bar\eta^\alpha : \beta \in w_j\}$
(d) $w_{j(*)} = |\Lambda|$ and $w_0 = \emptyset$
(e) $w_j \subseteq w_{j+1}$
(f) if $j(*) = \omega$ then $w_{j(*)} = \cup\{w_j : j < j(*)\}$.

[6]If we assume for $\bar\eta \in \Lambda \Rightarrow F(\bar\eta) \subseteq \Lambda_{\le k(*)}$ then any list $\langle \bar\eta^\alpha : \alpha < |\Lambda|\rangle$ with no repetitions and $\bar\ell = \langle \ell_\alpha : \alpha < |\Lambda|\rangle, \ell_\alpha \in u_{\bar\eta^\alpha}$ will do. Why? Because $Y_\alpha := \cup\{F(\bar\eta^\beta) : \beta < \alpha\}$ is a finite subset of $\Lambda_{\le k(*)}$. Now for $\alpha < |\Lambda|$ the set $u^1_\alpha := \{n < \omega : \bar\eta^\alpha \upharpoonright \langle \ell_\alpha, n\rangle$ belongs to $Y_\alpha\}$ is finite, and also for each $\beta < \alpha$ the set $u^r, Y_{\alpha,\beta} := \{n < \omega : \bar\eta^\alpha \upharpoonright \langle \ell_\alpha, n\rangle = \bar\eta^\beta \upharpoonright \langle \ell_\alpha, n\rangle\}$ is finite. As α is finite we can find $n = n_\alpha \in \omega \backslash Y_\alpha \backslash \cup \{Y_{\alpha,\beta} : \beta < \alpha\}$. Now $\langle n_\alpha : \alpha < |\Lambda|\rangle$ is as required.

No problem to do this; for clause (c) use "F is regressive, the ordinals well ordered but we elaborate. Assume that the finite $w_j \subseteq |\Lambda|$ has been chosen. We define $w_{j,m}$ by induction on m such that $w_{j,m} \subseteq |\Lambda|$ is finite and $\subseteq$-increasing with m. For $m = 0$ let $w_{j,m} = w_j \cup \{\alpha\}$. If $w_{j,m}$ is defined let

$$\begin{aligned} w_{j,m+1} = w_{j,m} \cup \{\beta < |\Lambda| : & \text{ for some } \alpha \in w_{j,m} \text{ we have} \\ & \bar{\eta}^\beta \in F(\bar{\eta}^\alpha) \cap \Lambda\}. \end{aligned}$$

As $w_{j,m}$ is finite and $\subseteq |\Lambda|$ and each $F(\bar{\eta}^\alpha)$ is finite and $\subseteq \{\bar{\eta}^\gamma : \gamma < |\Lambda|\}$ clearly $w_{j,m+1}$ is finite $\subseteq |\Lambda|$.

Lastly, we let w_{j+1} be $\cup\{w_{j,m} : m < \omega\}$. If it is finite we have carried the inductive step on j. If not, then $\langle w_{j,m} : m < \omega\rangle$ is $\subset$-increasing and we let $\gamma_{j,m} = \sup\{\eta_{\alpha,0}(i) : i < \omega, \alpha \in w_{j,m+1} \backslash w_{j+m}\}$ and it suffices to prove

$(*)$ $\gamma_{j,m} > \gamma_{j,m+1}$ (both are ordinals!).

Why $(*)$ is true? As by the definition of $\gamma_{j,m+1}$ for some $i_* < \omega$ and $\beta_* \in w_{j,m+2} \backslash w_{j,m+1}$ we have $\eta_{\beta,0}(i_*) = \gamma_{j,m+1}$. By the definition of $w_{j,m+2}$ as $\beta_* \notin w_{j,m+1}$, there is $\alpha_* \in w_{j,m+1}$ such that $\bar{\eta}^\beta \in F(\bar{\eta}^\alpha) \cap \Lambda$.

As $\beta_* \notin w_{j,m+1}$ necessarily $\alpha_* \notin w_{j,m}$ hence by the definition of $\gamma_{j,m}$ we know that $(\forall i < \omega)(\eta_{\alpha,0}(i) < \gamma_{j,m})$. By clause (c) of Definition 3.1(1) as $\bar{\eta}^\beta \in F(\bar{\eta}^\alpha)$ we know that $\eta_{\beta,0}(i_*) < \sup\{\eta_{\alpha,0}(i) : i < \omega\}$. By the last two sentences we are done proving $(*)$, so we are done defining w_{j+1} hence we finish justifying $\circledast$.

Now let $\langle \beta(j,i) : i < i^*_j\rangle$ list $w_{j+1} \backslash w_j$ such that: if $i_1, i_2 < i^*_j$ and $\bar{\eta}^{\beta(j,i_1)} \in F(\bar{\eta}^{\beta(j,i_2)})$ then $i_1 < i_2$; we prove existence by F being regressive. Let $\langle \bar{\nu}_{j,i} : i < i^{**}_j\rangle$ list $\cup\{F(\bar{\eta}^\alpha) : \alpha \in w_{j+1} \backslash w_j\} \backslash \Lambda^{\mathbf{x}} \backslash \{F(\bar{\eta}^\alpha) : \alpha \in w_j\}$.

Let $\alpha^*_j = \Sigma\{i^{**}_{j(1)} + i^*_{j(1)} : j(1) < j\}$. Now we choose $\bar{\rho}_\varepsilon$ for $\varepsilon < \alpha^*_j$ for $j < j(*)$ as follows:

(a) $\rho_{\alpha^*_j + i} = \nu_{j,i}$ if $i < i^{**}_j$

(b) $\bar{\rho}_{\alpha^*_j + i^{**}_j + i} = \bar{\eta}^{\beta(j,i)}$ if $i < i^*_j$.

Lastly, we choose $n_{\alpha_j + i} < \omega$ for $i < i^*_j$ as in case 1A.

Now check.

<u>Subcase 1C</u>: F is simple.

Note that $F(\bar{\eta})$ when defined is determined by $\eta_{k(*)}(0)$ and is included in $\{\bar{\nu} \in \Lambda^{\mathbf{x}}_{\le k(*)} \cup \Lambda^{\mathbf{x}} : \sup \mathrm{Rang}(\nu_{k(*)}) < \eta_{k(*)}(0)\}$. So let $u = \{\eta_{k(*)}(0) : \bar{\eta} \in \Lambda\}$ and $u^* = u \cup \{\sup(u) + 1\}$ and for $\alpha \in u$ let $\Lambda_\alpha = \{\bar{\eta} \in \Lambda : \eta_{k(*)}(0) = \alpha\}$ and for $\alpha \in u^*$

let $\Lambda_{<\alpha} = \cup\{\Lambda_\beta : \beta \in u\}$. Now by induction on $\beta \in u^*$ we choose $\langle(\bar\eta^\varepsilon, \ell_\varepsilon) : \varepsilon < \varepsilon_\beta\rangle$ such that it is a required for $\Lambda_{<\beta}$. For $\beta = \min(u)$ this is trivial and if $\mathrm{otp}(u \cap \beta)$ is a limit ordinal this is obvious. So assume $\alpha = \max(u \cap \beta)$, we use Subcase 1A on Λ_α, and combine them naturally promising $\ell_\alpha = k(*) \Rightarrow n_\alpha > 1$.

<u>Case 2</u>: $k = k_* + 1$ and $|\Lambda| = \aleph_k$.

Let $\langle \Lambda_\varepsilon : \varepsilon < \aleph_k\rangle$ be $\subseteq$-increasing continuous with union Λ, $|\Lambda_{1+\varepsilon}| = \aleph_{k_*}, \Lambda_0 = \emptyset$, each Λ_ε closed enough, mainly:

$\circledast_1$ if $\bar\eta^i \in \Lambda_\varepsilon$ for $i < i(*) < \omega, \bar\rho \in \Lambda$ and $\{\rho_\ell : \ell \le k(*)\} \subseteq \{\eta^i_\ell : \ell \le k(*), i < i(*)\}$ <u>then</u> $\bar\rho \in \Lambda_\varepsilon$

$\circledast_2$ Λ_ε is closed under $\bar\eta \mapsto F(\bar\eta) \cap \Lambda^{\mathbf{x}}$.

Next

$\odot$ if $\varepsilon < \aleph_k, \bar\eta \in \Lambda_{\varepsilon+1} \backslash \Lambda_\varepsilon$ then $u'_{\bar\eta} = \{\ell \in u_{\bar\eta}$: for every or just some $n < \omega$ for some $\bar\nu \in \Lambda_\varepsilon$ we have $\bar\eta \upharpoonright \langle \ell, n\rangle = \bar\nu \upharpoonright \langle \ell, n\rangle\}$ has at most one member.

[Why? So assume toward contradiction that $\bar\eta \in \Lambda_{\varepsilon+1}$ and $\ell(1) \ne \ell(2)$ belong to $u'_{\bar\eta}$. Hence by the definition of $u'_{\bar\eta}$ there are $\bar\nu^1, \bar\nu^2 \in \Lambda_\varepsilon$ and $n_1, n_2 < \omega$ such that $\bar\eta \upharpoonright \langle \ell_1, n_1\rangle \in \bar\nu^1 \upharpoonright \langle \ell_1, n_1\rangle$ and $\bar\eta \upharpoonright \langle \ell_1, n_2\rangle = \bar\nu^2 \upharpoonright \langle \ell_2, n_2\rangle$. Now $m \le k(*) \Rightarrow$ for some $i \in \{1,2\}, m \le \ell_i \Rightarrow$ for some $i \in \{1,2\}, \eta_m$ is $(\bar\eta \upharpoonright \langle \ell_i, n_i\rangle)_m \Rightarrow \eta_m \in \{\rho_\ell : \bar\rho \in \Lambda_\varepsilon\}$. Hence $\{\eta_\ell : \ell \le k(*)\} \subseteq \{\rho_\ell : \ell \le k(*)$ and $\bar\rho \in \Lambda_\varepsilon\}$. So by $\circledast_1$ we have $\bar\eta \in \Lambda_\varepsilon$, so we are done.]

Apply the induction hypothesis to $\Lambda_{\varepsilon+1} \backslash \Lambda_\varepsilon$ for each ε and get $\langle(\bar\eta^{\varepsilon,\alpha}, \ell_{\varepsilon,\alpha,n_{\varepsilon,\alpha}}) : \alpha < \alpha(\varepsilon)\rangle$ such that $\bar\eta^{\varepsilon,\alpha} \upharpoonright \langle \ell^\varepsilon_{\varepsilon,\ell}, n_{\varepsilon,\alpha}\rangle \notin \{\bar\eta^{\varepsilon,\beta} \upharpoonright \langle \ell_{\varepsilon,\beta}, n_{\varepsilon,\beta}\rangle : \beta < \alpha\rangle$.

Let $\alpha_* = \Sigma\{\alpha(\varepsilon) : \varepsilon < |\Lambda|\}$ and $\alpha = \Sigma\{\alpha(\zeta) : \zeta < \varepsilon\} + \beta, \beta < \alpha(\varepsilon)$ let $\eta^\alpha = \eta^{\varepsilon,\beta}, \ell_\alpha = \ell_{\varepsilon,\beta}, \eta_\alpha = \eta_{\varepsilon,\beta}$. I.e. we combine but for $\Lambda_{\varepsilon+1} \backslash \Lambda_\varepsilon$ we use $\langle u_{\bar\eta} \backslash u'_{\bar\eta} : \bar\eta \in \Lambda_{\varepsilon+1} \backslash \Lambda_\varepsilon\rangle$, so $|u_{\bar\eta} \backslash u'_{\bar\eta}| \ge k - 1 = k_*$. $\square_{3.2}$

3.3 Definition. For a combinatorial parameter $\mathbf{x}$ we define $\mathscr{G}_{\mathbf{x}}$, the class of abelian groups derived from $\mathbf{x}$ as follows: $G \in \mathscr{G}_{\mathbf{x}}$ if there is a simple (or finitary) regressive F on $\Lambda^{\mathbf{x}}$ and G is generated by $\{y_{\bar\eta,n} : \eta \in \Lambda^{\mathbf{x}}, n < \omega\} \cup \{x_{\bar\eta} : \bar\eta \in \Lambda^{\mathbf{x}}_{\le k(*)}\}$ freely except

$$\boxtimes_{\bar\eta,n} \quad (n!) y_{\bar\eta,n+1} = y_{\bar\eta,n} + b_{\bar\eta,n} z_{\bar\eta,n} + \sum\{x_{\bar\eta \upharpoonright <m,n>} : m \le k(*)\}$$

where

$\odot$ (a) $b_{\bar\eta,n} \in \mathbb{Z}$

(b) $z_{\bar\eta,n}$ is a linear combination of

$$\{x_{\bar\nu} : \bar\nu \in F(\bar\eta)\backslash\Lambda^{\mathbf{x}}\} \cup \{y_{\bar\eta,n} : \bar\eta \in F(\bar\eta) \cap \Lambda^{\mathbf{x}} \text{ and } (\forall m \le k(*))(\bar\eta \restriction \langle m,n\rangle) \in F(\bar\eta)\}.$$

3.4 Claim. *If* $\mathbf{x} \in K^{\rm cb}_{k(*)}$ *and* $G \in \mathscr{G}_{\mathbf{x}}$ *(i.e. G is an abelian group derived from* $\mathbf{x}$*),* *<u>then</u> G is $\aleph_{k(*)+1}$-free.*

Proof. We use claim 3.2. So let H be a subgroup of G of cardinality $\le \aleph_{k(*)}$. We can find Λ such that

- $(*)$ (a) $\Lambda \subseteq \Lambda^{\mathbf{x}}$ has cardinality $\le \aleph_{k(*)}$
  - (b) every equation which $X_\Lambda = \{x_{\bar\eta\restriction<m,n>}, y_{\bar\eta,n} : m \le k(*), n < \omega, \bar\eta \in \Lambda\}$ satisfies in G, is implied by the equations from $\Gamma_\Lambda = \cup\{\boxtimes_{\bar\eta,n}: \bar\eta \in \Lambda\}$
  - (c) $H \subseteq G_\Lambda = \langle x_{\bar\eta\restriction<m,n>}, y_{\bar\eta,n} : \bar\eta \in \Lambda, m \le k(*), n < \omega\rangle_G$
  - (d) if $\bar\eta \in \Lambda$ then $F(\bar\eta)$ is included in $\Lambda \cup \{\bar\nu \restriction (\ell,n) : \bar\nu \in \Lambda, \ell \le k(*)$ and $n < \omega\}$.

So it sufices to prove that G_Λ is a free (abelian) group.

Let the sequence $\langle(\bar\eta^\alpha, \ell_\alpha) : \alpha < \alpha(*)\rangle$ be as proved to exist in 3.2. Let $\mathscr{U} = \{\alpha < \alpha(*) : \bar\eta^\alpha \in \Lambda\} \cup \{\alpha(*)\}$ and for $\alpha \le \alpha(*)$ let $X^0_\alpha = \{x_{\bar\eta^\beta\restriction<m,n>} : \beta \in \alpha \cap \mathscr{U}, m \le k(*)$ and $n < \omega\}$ and $X^1_\alpha = X^0_\alpha \cup \{\bar\eta^\beta : \beta \in \alpha\backslash\mathscr{U}\}$. So for each $\alpha \in \mathscr{U}$ there is $\bar n_\alpha = \langle n_{\alpha,\ell} : \ell \in v_\alpha\rangle$ such that: $\ell_\alpha \in v_\alpha \subseteq \{0,\dots,k(*)\}, n_{\alpha,\ell} < \omega$ and $X^1_{\alpha+1}\backslash X^1_\alpha = \{x_{\bar\eta\restriction<\ell,n>} : \ell \in v_\alpha$ and $n \in [n_{\alpha,\ell},\omega)\}$.

For $\alpha \le \alpha(*)$ let $G_{\Lambda,\alpha} = \langle\{y_{\bar\eta^\beta,n}, x_{\bar\nu} : \beta \in \mathscr{U} \cap \alpha$ and $\bar\nu \in X^1_\beta\}\rangle_{G_\Lambda}$. Clearly $\langle G_{\Lambda,\alpha} : \alpha \le \alpha(*)\rangle$ is purely increasing continuous with union G_Λ, and $G_{\Lambda,0} = \{0\}$. So it suffices to prove that $G_{\Lambda,\alpha+1}/G_{\Lambda,\alpha}$ is free. If $\alpha \notin \mathscr{U}$ the quotient is trivially a free group, and if $\alpha \in \mathscr{U}$ we can use $\ell_\alpha \in v_\alpha$ to prove that it is free giving a basis. $\square_{3.4}$

3.5 Conclusion. For every $k(*) < \omega$ there is an $\aleph_{k(*)+1}$-free abelian group G of cardinality $\lambda = \beth_{k(*)+1}$ such that $\mathrm{Hom}(G,\mathbb{Z}) = \{0\}$.

Proof. We use $\mathbf{x}$ and $\langle h_{\bar\eta} : \bar\eta \in \Lambda^{\mathbf{x}}\rangle$ from 2.1(3), and we shall choose $G \in \mathscr{G}_{\mathbf{x}}$. So G is $\aleph_{k(*)+1}$-free by 3.4.

Let $\mathscr{S} = \{\langle (a_i, \bar\eta_i) : i < i_1\rangle\char`\^\langle (b_j, \bar\nu_j, n_j) : j < j_1\rangle : i_1 < \omega, a_i \in \mathbb{Z}, \bar\eta_i \in \Lambda^{\mathbf{x}}_{\le k(*)}$ and $j_1 < \omega, b_j \in \mathbb{Z}, \nu_j \in \Lambda^{\mathbf{x}}, n_j < \omega\}$ (actually $\mathscr{S} = \Lambda^{\mathbf{x}}_{\le k(*)}$ suffice noting $\bar\nu_j = \langle \nu_{j,\ell} : \ell \le k(*)\rangle$).

So $|\mathscr{S}| = \lambda_{k(*)}$ and let $\bar p$ be such that:

(a) $\bar p = \langle p^\alpha : \alpha < \lambda\rangle$

(b) $\bar p$ lists $\mathscr{S}$

(c) $p^\alpha = \langle (a^\alpha_i, \bar\eta^\alpha_i) : i < i_\alpha\rangle\char`\^\langle (b^\alpha_j, \bar\nu^\alpha_j, n^\alpha_j) : j < j_\alpha\rangle$ so $\bar\nu^\alpha_j = \langle \nu^\alpha_{j,\ell} : \ell \le k(*)\rangle$

(d) sup $\mathrm{Rang}(\eta^\alpha_{i,k(*)}) < \alpha$, sup $\mathrm{Rang}(\nu^\alpha_{j,k(*)}) < \alpha$ if $i < i_\alpha, j < j_\alpha$.

Now to apply Definition 3.3 we have to choose z_α (for Definition 3.3) as $\Sigma\{a^\alpha_i x_{\bar\eta_i} : i < i_\alpha\} + \Sigma\{b^\alpha_j y_{\bar\nu^\alpha_j, n^\alpha_j} : j < j_\alpha\}$ and $z_{\bar\eta} = z_{\bar\eta,n} = z_{\eta_{k(*)}(0)}$ for $\bar\eta \in \Lambda^{\mathbf{x}}, n < \omega$ then for $\bar\eta \in \Lambda^{\mathbf{x}}$ we choose $\langle b_{\bar\eta,n} : n < \omega\rangle \in {}^\omega\mathbb{Z}$ such that:

⊛ there is no function h from $\{z_{\bar\eta}\} \cup \{y_{\bar\eta,n} : n < \omega\} \cup \{x_{\bar\eta\restriction<m,n>} : m \le k(*), n < \omega\}$ into $\mathbb{Z}$ satisfying

⊛ (a) $h(z_{\bar\eta}) \ne 0$ and

(b) $h(x_{\bar\eta\restriction<m,n>}) = h_{\bar\eta}(\bar\eta \restriction \langle m,n\rangle)$ for $m \le k(*), n < \omega$

(c) for every n sn

$(*)_n$ $\quad n! h(y_{\bar\eta,n+1}) = h(y_{\bar\eta,n}) + b_{\bar\eta,n} h(z_{\bar\eta}) + \Sigma\{\{x_{\bar\eta\restriction<m,n>}):$ $m \le k(*)\}$.

E.g. for each $\rho \in {}^\omega 2$ we can try $b^\rho_n = \rho(n)$ and assume toward contradiction that for each $\rho \in {}^\omega 2$ there is h_ρ as above. Hence for some $c \in \mathbb{Z}\backslash\{0\}$ the set $\{\rho \in {}^\omega 2 : h_\rho(z_{\bar\eta}) = c\}$ is uncountable. So we can find $\rho_1 \ne \rho_2$ such that $h_{\rho_1} = c = h_{\rho_2}(x_\nu)$ and $\rho_1 \restriction (|c| + 7) = \rho_2 \restriction (|c| + 7)$. So for some $n \ge |c| + 7, \rho_1 \restriction n = \rho_2 \restriction n$ and $\rho_1(n) \ne \rho_2(n)$. Now consider the equation $(*)_n$ for $h_{\bar\rho_1}$ and $h_{\bar\rho_2}$, subtract them and get $(\rho_1(n) - \rho_2(n))c$ is divisible by $n!$, clear contradiction.

So $G \in \mathscr{G}_{\mathbf{x}}$ is well defined and is $\aleph_{k(*)+1}$-free by 3.4. Suppose $h \in \mathrm{Hom}(G, \mathbb{Z})$ is non-zero, so for some $\alpha < \lambda_{k(*)}, h(z_\alpha) \ne 0$ (actually as $G^1 = \langle \{x_{\bar\nu} : \bar\nu \in \Lambda^{\mathbf{x}}_{\le k(*)}\}\rangle_G$ is a subgroup such that G/G^1 is divisible necessarily $h \restriction G^1$ is not zero hence in 2.1(2) for some $\bar\nu \in \Lambda^{\mathbf{x}}_{\le k(*)}$ we have $h(x_{\bar\nu}) \ne 0$). So by the choice of $\langle h_{\bar\eta} : \bar\eta \in \Lambda\rangle$ for some $\bar\eta \in \Lambda^{\mathbf{x}}, \eta_{k(*)}(0) = \alpha$ and we have $h_{\bar\eta} = h \restriction \{x_{\bar\eta\restriction<m,n>} : m \le k(*), n < \omega\}$. By ⊛ we clearly get a contradiction. $\square_{3.5}$

Remark. We can give more details as in the proof of 2.3.

3.6 Conclusion. For every $n \le m < \omega$ there is a purely increasing sequence $\langle G_\alpha : \alpha \le \omega_n + 1\rangle$ of abelian groups, $G_\alpha, G_\beta/G_\alpha$ are free for $\alpha < \beta \le \omega_n$ and $G_{\omega_n+1}/G_{\omega_n}$ is $\aleph_n$-free and for some $h \in \mathrm{Hom}(G_\kappa, \mathbb{Z})$ has no extension in $\mathrm{Hom}(G_{\omega_n+1}, \mathbb{Z})$.

Proof. Let G, z be as in 2.2. So also $G/\mathbb{Z}z$ is $\aleph_n$-free. Let $G_\alpha = \langle\{z\}\rangle_G$ for $\alpha \le \omega_2, G_{\omega_n+1} = G$.

§4 Appendix 1

4.1 Notation. If $\bar\eta^* \in \Lambda^{\mathbf{x}}_m$ and $\bar\eta = \bar\eta^* \restriction \{\ell \le k(*) : \ell \ne m\}$ and $\nu = \eta^*_m$ then let $x_{m,\bar\eta,\nu} := x_{\bar\eta^*}$. (See proof of 1.12).

Proof of 1.8. Let $U \subseteq {}^\omega S$ be countable (and infinite) and define G'_U like G restricting ourselves to $\eta_\ell \in U$; by the Löwenheim-Skolem argument it suffices to prove that G'_U is a free abelian group. List $\Lambda \cap {}^{k(*)+1}U$ without repetitions as $\langle \bar\eta_t : t < t^* \le \omega\rangle$, and choose $s_t < \omega$ by induction on $t < \omega$ such that $[r < t \ \& \ \bar\eta_r \restriction k(*) = \bar\eta_t \restriction k(*) \Rightarrow \emptyset = \{\eta_{t,k(*)} \restriction \ell : \ell \in [s_t,\omega)\} \cap \{\eta_{r,k(*)} \restriction \ell : \ell \in [s_r,\omega)\}]$.

Let

$$Y_1 = \{x_{m,\bar\eta,\nu} : m < k(*), \bar\eta \in {}^{k(*)+1\backslash\{m\}}U \text{ and } \nu \in {}^{\omega>}2\}$$

$$Y_2 = \Big\{x_{m,\bar\eta,\nu} : m = k(*), \bar\eta \in {}^{k(*)}U \text{ and for no } t < t^* \text{ do we have } \bar\eta = \bar\eta_t \restriction k(*) \ \& \ \nu \in \{\eta_{t,k(*)} \restriction \ell : s_t \le \ell < \omega\}\Big\}$$

$$Y_3 = \{y_{\bar\eta_t,n} : t < t^* \text{ and } n \in [s_t,\omega)\}.$$

Now

$(*)_1$ $Y_1 \cup Y_2 \cup Y_3 \cup \{z\}$ generates G'_U.

[Why? Let G' be the subgroup of G'_U which $Y_1 \cup Y_2 \cup Y_3$ generates. First we prove by induction on $n < \omega$ that for $\bar\eta \in {}^{k(*)}U$ and $\nu \in {}^nS$ we have $x_{k(*),\bar\eta,\nu} \in G'$. If $x_{k(*),\bar\eta,\nu} \in Y_2$ this is clear; otherwise, by the definition of Y_2 for some $\ell < \omega$ (in fact $\ell = n$) and $t < \omega$ such that $\ell \ge s_t$ we have $\bar\eta = \bar\eta_t \restriction k(*), \nu = \eta_{t,k(*)} \restriction \ell$.

Now

(a) $y_{\bar\eta_{t,\ell+1}}, y_{\bar\eta_{t,\ell}}$ are in $Y_3 \subseteq G'$.

Hence by the equation $\boxtimes_{\bar\eta,n}$ in Definition 1.6, clearly $x_{k(*),\bar\eta,\nu} \in G'$. So as $Y_1 \subseteq G' \subseteq G'_U$, all the generators of the form $x_{k(*),\bar\eta,\nu}$ with each $\eta_\ell \in U$ are in G'.

Next note that

(b) $x_{m,\bar\eta_t \restriction \{i \le k(*) : i \ne m\},\nu}$ belong to $Y_1 \subseteq G'$ if $m < k(*)$.

Now for each $t < \omega$ we prove that all the generators $y_{\bar\eta_t,n}$ are in G'. If $n \geq s_t$ then clearly $y_{\bar\eta_t,n} \in Y_3 \subseteq G'$. So it suffices to prove this for $n \leq s_t$ by downward induction on n; for $n = s_t$ by an earlier sentence, for $n < s_t$ by $\boxtimes_{\bar\eta,n}$. Together all the generators are in this subgroup so we are done.]

$(*)_2$ $Y_1 \cup Y_2 \cup Y_3 \cup \{z\}$ generates G'_U freely.
[Why? Translate the equations, see more in [Sh 771, §5].]

$\square_{1.8}$

Proof of 1.10. 0), 1) Obvious.
2),3),4) Follows.
5) Let $\langle \eta_\ell : \ell < m(*)\rangle$ list $u, U_\ell = U \cup (u\backslash\{\eta_\ell\})$ so $G_{U,u} = G_{U_0^+} \ldots + G_{U_{m(*)-1}}$. First, $G_{U,u} \subseteq G_{U\cup u}$ follows by the definitions. Second, we deal with proving $G_{U,u} \subseteq_{\rm pr} G_{U\cup u}$. So assume $z^* \in G, a^* \in \mathbb{Z}$ and a^*z^* belongs to $G_{U_0} + \ldots + G_{U_{m(*)}}$ so it has the form $\Sigma\{b_i x_{\bar\eta'\restriction <m_i,n_i>} : i < i(*)\} + \Sigma\{c_j y_{\bar\eta_j,n_j} : j < j(*)\} + az$ with $i(*) < \omega, j(*) < \omega$ and $a^*, b_i, c_j \in \mathbb{Z}$ and $\nu_i, \bar\eta^i, \bar\eta_j$ are suitable sequences of members of $U_{\ell(i)}, U_{\ell(i)}, U_{k(j)}$ respectively where $\ell(i), k(j) < m(*)$. We continue as in [Sh 771].
6) Easy.
7) Clearly $U_1 \cup v = U_2 \cup u$ hence $G_{U_1\cup u} \subseteq G_{U_1\cup v} = G_{U_2\cup u}$ hence $G_{U,u} + G_{U_1\cup u}$ is a subgroup of $G_{U,u} + G_{U_2\cup u}$, so the first quotient makes sense.

Hence $(G_{U,u} + G_{U_2\cup u})/(G_{U,u} + G_{U_1\cup u})$ is isomorphic to $G_{U_2\cup u}/(G_{U_2\cup u} \cap (G_{U,u} + G_{U_1\cup u}))$. Now $G_{U_1,v} \subseteq G_{U_1\cup v} = G_{U_2\cup v} \subseteq G_{U,u} + G_{U_2,u}$ and $G_{U_1,v} \subseteq G_{U,v} = G_{U,v\backslash U} = G_{U,u} \subseteq G_{U,u} + G_{U_2,u}$. Together $G_{U_1,v}$ is included in their intersection, i.e. $G_{U_2\cup u} \cap (G_{U,u} + G_{U_1\cup u})$ include $G_{U_1,v}$ and using part (1) both has the same divisible hull inside G^+. But as $G_{U_1,v}$ is a pure subgroup of G by part (5) hence of $G_{U_1\cup v}$. So necessarily $G_{U_1\cup u} \cap (G_{U,u} + G_{U_1,u}) = G_{U_1,v}$, so as $G_{U_2\cup u} = G_{U_1\cup v}$ we are done.
8) See [Sh 771, §5]. $\square_{1.10}$

Proof of 1.12. 1) We prove this by induction on $|U|$; without loss of generality $|u| = k$ as also $k' = |u|$ satisfies the requirements.

Case 1: U is countable.

So let $\{\nu^*_\ell : \ell < k\}$ list u be with no repetitions, now if $k = 0$, i.e. $u = \emptyset$ then $G_{U\cup u} = G_U = G_{U,u}$ so the conclusion is trivial. Hence we assume $u \neq \emptyset$, and let $u_\ell := u\backslash\{\nu^*_\ell\}$ for $\ell < k$.

Let $\langle \bar\eta_t : t < t^* \leq \omega\rangle$ list with no repetitions the set $\Lambda_{U,u} := \{\bar\eta \in \Lambda^{\mathbf{x}} \cap {}^{k(*)+1}(U \cup u)$: for no $\ell < k$ does $\bar\eta \in {}^{k(*)+1}(U \cup u_\ell)\}$. Now comes a crucial point: let $t < t^*$, for each $\ell < k$ for some $r_{t,\ell} \leq k(*)$ we have $\eta_{t,r_{t,\ell}} = \nu^*_\ell$ by the definition of $\Lambda_{U,u}$, so

$|\{r_{t,\ell} : \ell < k\}| = k < k(*)+1$ hence for some $m_t \le k(*)$ we have $\ell < k \Rightarrow r_{t,\ell} \ne m_t$ so for each $\ell < k$ the sequence $\bar\eta_t \restriction (k(*)+1 \backslash \{m_t\})$ is not from $\{\langle \rho_s : s \le k(*)$ and $s \ne m_t\rangle : \rho_s \in {}^\omega(U \cup u_\ell)$ for every $s \le k(*)$ such that $s \ne m_t\}$.

For each $t < t^*$ we define $J(t) = \{m \le k(*)$: the set $\{\eta_{t,s} : s \le k(*)\ \&\ s \ne m\}$ is included in $U \cup u_\ell$ for no $\ell \le k\}$. So $m_t \in J(t) \subseteq \{0, \dots, k(*)\}$ and $m \in J(t) \Rightarrow \bar\eta_t \restriction \{j \le k(*) : j \ne m\} \notin {}^{k(*)+1\backslash\{m\}}(U \cup u_\ell)$ for every $\ell \le k$. For $m \le k(*)$ let $\bar\eta'_{t,m} := \bar\eta_t \restriction \{j \le k(*) : j \ne m\}$ and $\bar\eta'_t := \bar\eta'_{t,m_t}$. Now we can choose $s_t < \omega$ by induction on $t < t^*$ such that

(*) if $t_1 < t, m \le k(*)$ and $\bar\eta'_{t_1,m} = \bar\eta'_{t,m}$, then $\eta_{t,m} \restriction s_t \notin \{\eta_{t_1,m} \restriction \ell : \ell < \omega\}$.

Let $Y^* = \{x_{m,\bar\eta,\nu} \in G_{U\cup u} : x_{m,\bar\eta,\nu} \notin G_{U\cup u_\ell}$ for $\ell < k\} \cup \{y_{\bar\eta,n} \in G_{U\cup u} : y_{\bar\eta,n} \notin G_{U\cup u_\ell}$ for $\ell < k\}$.
Let

$$Y_1 = \{x_{m,\bar\eta,\nu} \in Y^*\text{: for no } t < t^* \text{ do we have } m = m_t\ \&\ \bar\eta = \bar\eta'_t\}.$$

$$\begin{aligned} Y_2 = \{x_{m,\bar\eta,\nu} \in Y^* : x_{m,\bar\eta,\nu} \notin Y_1 &\text{ but for no} \\ &t < t^* \text{ do we have } m = m_t\ \&\ \bar\eta = \bar\eta'_t\ \& \\ &\eta_{t,m_t} \restriction s_t \trianglelefteq \nu \triangleleft \eta_{t,m_t}\} \end{aligned}$$

$$Y_3 = \{y_{\bar\eta,n} : y_{\bar\eta,n} \in Y^* \text{ and } n \in [s_t, \omega) \text{ for the } t < t^* \text{ such that } \bar\eta = \bar\eta_t\}.$$

Now the desired conclusion follows from

$(*)_1$ $\{y + G_{U,u} : y \in Y_1 \cup Y_2 \cup Y_3\}$ generates $G_{U\cup u}/G_{U,u}$

$(*)_2$ $\{y + G_{U,u} : y \in Y_1 \cup Y_2 \cup Y_3\}$ generates $G_{U\cup u}/G_{U,u}$ freely.

Proof of $(*)_1$. It suffices to check that all the generators of $G_{U\cup u}$ belong to $G'_{U\cup u} =: \langle Y_1 \cup Y_2 \cup Y_3 \cup G_{U,u}\rangle_G$.

First consider $x = x_{m,\bar\eta,\nu}$ where $\eta \in {}^{k(*)+1}(U \cup u), m \le k(*)$ and $\nu \in {}^nS$ for some $n < \omega$. If $x \notin Y^*$ then $x \in G_{U,u_\ell}$ for some $\ell < k$ but $G_{U\cup u_\ell} \subseteq G_{U,u} \subseteq G'_{U\cup u}$ so we are done, hence assume $x \in Y^*$. If $x \in Y_1 \cup Y_2 \cup Y_3$ we are done so assume $x \notin Y_1 \cup Y_2 \cup Y_3$. As $x \notin Y_1$ for some $t < t^*$ we have $m = m_t\ \&\ \bar\eta = \eta'_t$. As $x \notin Y_2$, clearly for some t as above we have $\eta_{t,m_t} \restriction s_t \trianglelefteq \nu \triangleleft \eta_{t,m_t}$. Hence by Definition 1.6 the equation $\boxtimes_{\bar\eta_t,n}$ from Definition 1.6 holds, now $y_{\bar\eta_t,n}, y_{\bar\eta_t,n+1} \in Y_3 \subseteq G'_{U\cup u}$. So in order to deduce from the equation that $x = x_{\bar\eta'_t \upharpoonright <m_t,n>}$ belongs to $G'_{U\cup u}$, it suffices to show that $x_{\bar\eta'_{t,j}\upharpoonright<j,n>} \in G'_{U\cup u}$ for each $j \le k(*), j \ne m_t$. But each such $x_{\bar\eta'_{t,j}\upharpoonright<j,n>}$ belong to $G'_{U\cup u}$ as it belongs to $Y_1 \cup Y_2$.

[Why? Otherwise necessarily for some $r < t^*$ we have $j = m_r, \bar\eta'_{t,j} = \bar\eta'_{r,m_r}$ and $\eta_{r,m_r} \restriction s_r \trianglelefteq \eta_t \restriction n \triangleleft \eta_{r,m_r}$ so $n \geq s_r$ and as said above $n \geq s_t$. Clearly $r \neq t$ as $m_r = j \neq m_t$, now as $\bar\eta'_{t,m_r} = \bar\eta'_{r,m_r}$ and $\bar\eta_t \neq \bar\eta_r$ (as $t \neq r$) clearly $\eta_{t,m_r} \neq \eta_{r,m_r}$. Also $\neg(r < t)$ by $(*)$ above applied with r, t here standing for t_1, t there as $\eta_{r,m_r} \restriction s_r \trianglelefteq \eta_{t,j} \restriction n \triangleleft \eta_{r,m_r}$. Lastly for if $t < r$, again $(*)$ applied with t, r here standing for t_1, t there as $n \geq m_t$ gives contradiction.]
So indeed $x \in G'_{U\cup u}$.

Second consider $y = y_{\bar\eta,n} \in G_{U\cup u}$, if $y \notin Y^*$ then $y \in G_{U,u} \subseteq G'_{U\cup u}$, so assume $y \in Y^*$. If $y \in Y_3$ we are done, so assume $y \notin Y_3$, so for some $t, \bar\eta = \bar\eta_t$ and $n < s_t$. We prove by downward induction on $s \leq s_t$ that $y_{\bar\eta,s} \in G'_{U\cup u}$, this clearly suffices. For $s = s_t$ we have $y_{\bar\eta,s} \in Y_3 \subseteq G'_{U\cup u}$; and if $y_{\bar\eta,s+1} \in G'_{U\cup u}$ use the equation $\boxtimes_{\bar\eta_t,s}$ from 1.6, in the equation $y_{\bar\eta,s+1} \in G'_{U\cup u}$ and the x's appearing in the equation belong to $G'_{U\cup u}$ by the earlier part of the proof (of $(*)_1$) so necessarily $y_{\bar\eta,s} \in G'_{U\cup u}$, so we are done.

Proof of $(*)_2$. We rewrite the equations in the new variables recalling that $G_{U\cup u}$ is generated by the relevant variables freely except the equations of $\boxtimes_{\bar\eta,n}$ from Definition 1.6. After rewriting, all the equations disappear.

Case 2: U is uncountable.

As $\aleph_1 \leq |U| \leq \aleph_{k(*)-k}$, necessarily $k < k(*)$.

Let $U = \{\rho_\alpha : \alpha < \mu\}$ where $\mu = |U|$, list U with no repetitions. Now for each $\alpha \leq |U|$ let $U_\alpha := \{\rho_\beta : \beta < \alpha\}$ and if $\alpha < |U|$ then $u_\alpha = u \cup \{\rho_\alpha\}$. Now

- $\odot_1$ $\langle (G_{U,u} + G_{U_\alpha\cup u})/G_{U,u} : \alpha < |U|\rangle$ is an increasing continuous sequence of subgroups of $G_{U\cup u}/G_{U,u}$.
  [Why? By 1.10(6).]
- $\odot_2$ $G_{U,u} + G_{U_0\cup u}/G_{U,u}$ is free.
  [Why? This is $(G_{U,u} + G_{\emptyset\cup u})/G_{U,u} = (G_{U,u} + G_u)/G_{U,u}$ which by 1.10(8) is isomorphic to $G_u/G_{\emptyset,u}$ which is free by Case 1.]

Hence it suffices to prove that for each $\alpha < |U|$ the group $(G_{U,u}+G_{U_{\alpha+1}\cup u})/(G_{U,u}+G_{U_\alpha\cup u})$ is free. But easily

- $\odot_3$ this group is isomorphic to $G_{U_\alpha\cup u_\alpha}/G_{U_\alpha,u_\alpha}$.
  [Why? By 1.10(7) with $U_\alpha, U_{\alpha+1}, U, \rho_\alpha, u$ here standing for U_1, U_2, U, η, u there.]
- $\odot_4$ $G_{U_\alpha\cup u_\alpha}/G_{U_\alpha,u_\alpha}$ is free.
  [Why? By the induction hypothesis, as $\aleph_0 + |U_\alpha| < |U| \leq \aleph_{k(*)-(k+1)}$ and $|u_\alpha| = k+1 \leq k(*)$.]

2) If $k(*) = 0$ just use 1.8, so assume $k(*) \geq 1$. Now the proof is similar to (but easier than) the proof of case (2) inside the proof of part (1) above.

$\square_{1.12}$

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