

A COMMENT ON “ $\mathfrak{p} < \mathfrak{t}$ ”

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ABSTRACT. Dealing with the cardinal invariants \mathfrak{p} and \mathfrak{t} of the continuum we prove that

$$\mathfrak{m} = \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{t} = \aleph_2.$$

In other words, if \mathbf{MA}_{\aleph_1} (or a weak version of this) holds, then (of course $\aleph_2 \leq \mathfrak{p} \leq \mathfrak{t}$ and) $\mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{p} = \mathfrak{t}$. The proof is based on a criterion for $\mathfrak{p} < \mathfrak{t}$.

0. INTRODUCTION

We are interested in two cardinal invariants of the continuum, \mathfrak{p} and \mathfrak{t} . The cardinal \mathfrak{p} measures when a family of infinite subsets of ω with finite intersection property has a pseudo-intersection. A relative is \mathfrak{t} , which deals with towers, i.e., families well ordered by almost inclusion. These are closely related classical cardinal invariants. Rothberger [7], [8] proved (stated in our terminology) that $\mathfrak{p} \leq \mathfrak{t}$ and

$$\mathfrak{p} = \aleph_1 \Rightarrow \mathfrak{p} = \mathfrak{t},$$

and he asked if $\mathfrak{p} = \mathfrak{t}$.

Our main result is Corollary 2.5 stating that

$$\mathfrak{m} \geq \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{p} = \mathfrak{t},$$

where \mathfrak{m} is the minimal cardinal λ such that Martin Axiom for λ dense sets fails (i.e. $\neg\mathbf{MA}_\lambda$). Considering that $\mathfrak{m} \geq \aleph_1$ is a theorem (of ZFC), the parallelism with Rothberger’s theorem is clear. The reader may conclude that probably $\mathfrak{m} = \mathfrak{p} \Rightarrow \mathfrak{p} = \mathfrak{t}$; this is not unreasonable but we believe that eventually one should be able to show $\text{CON}(\mathfrak{m} = \lambda + \mathfrak{p} = \lambda + \mathfrak{t} = \lambda^+)$. In the first section we present a characterization of $\mathfrak{p} < \mathfrak{t}$ which is crucial for the proof of 2.5, and which also sheds some light on the strategy to approach the question of $\mathfrak{p} < \mathfrak{t}$ presented in [9].

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Notation Our notation is rather standard and compatible with that of classical textbooks (like Bartoszyński and Judach [3]). In forcing we keep the older convention that *a stronger condition is the larger one*.

- (1) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta \dots$) and also by i, j (with possible sub- and superscripts).
- (2) Cardinal numbers will be called $\kappa, \kappa_i, \lambda$.

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- (3) A bar above a letter denotes that the considered object is a sequence; usually \bar{X} will be $\langle X_i : i < \zeta \rangle$, where ζ is the length $\ell g(\bar{X})$ of \bar{X} . Sometimes our sequences will be indexed by a set of ordinals, say $S \subseteq \lambda$, and then \bar{X} will typically be $\langle X_\delta : \delta \in S \rangle$.
- (4) The set of all infinite subsets of the set ω of natural numbers is denoted by $[\omega]^{\aleph_0}$ and the relation of *almost inclusion* on $[\omega]^{\aleph_0}$ is denoted by \subseteq^* . Thus for $A, B \in [\omega]^{\aleph_0}$ we write $A \subseteq^* B$ if and only if $A \setminus B$ is finite.
- (5) The relations of *eventual dominance* on the Baire space ${}^\omega\omega$ are called \leq^* and $<^*$. Thus, for $f, g \in {}^\omega\omega$,
- $f \leq^* g$ if and only if $(\forall^\infty n < \omega)(f(n) \leq g(n))$ and
 - $f <^* g$ if and only if $(\forall^\infty n < \omega)(f(n) < g(n))$.

1. A CRITERION

In this section our aim is to prove Theorem 1.12 stating that $\mathfrak{p} < \mathfrak{t}$ implies the existence of a peculiar cut in $({}^\omega\omega, <^*)$. This also gives the background for our tries to get a progress on the consistency of $\mathfrak{p} < \mathfrak{t}$ in [9].

- Definition 1.1.** (1) We say that a set $A \in [\omega]^{\aleph_0}$ is a *pseudo-intersection* of a family $\mathcal{B} \subseteq [\omega]^{\aleph_0}$ if $A \subseteq^* B$ for all $B \in \mathcal{B}$.
- (2) A sequence $\langle X_\alpha : \alpha < \kappa \rangle \subseteq [\omega]^{\aleph_0}$ is a *tower* if $X_\beta \subseteq^* X_\alpha$ for $\alpha < \beta < \kappa$ but the family $\{X_\alpha : \alpha < \kappa\}$ has no pseudo-intersection.
- (3) \mathfrak{p} is the minimal cardinality of a family $\mathcal{B} \subseteq [\omega]^{\aleph_0}$ such that the intersection of any finite subcollection of \mathcal{B} is infinite but \mathcal{B} has no pseudo-intersection, and \mathfrak{t} is the smallest size of a tower.

A lot of results have been accumulated on these two cardinal invariants. For instance

- Bell [4] showed that \mathfrak{p} is the first cardinal μ for which $\mathbf{MA}_\mu(\sigma\text{-centered})$ fails,
- Szymański proved that \mathfrak{p} is regular (see, e.g., Fremlin [5, Proposition 21K]),
- Piotrowski and Szymański [6] showed that $\mathfrak{t} \leq \text{add}(\mathcal{M})$ (so also $\mathfrak{t} \leq \mathfrak{b}$).

For more results and discussion we refer the reader to [3, §1.3, §2.2].

Definition 1.2. We say that a family $\mathcal{B} \subseteq [\omega]^{\aleph_0}$ *exemplifies* \mathfrak{p} if:

- \mathcal{B} is closed under finite intersections (i.e., $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$), and
- \mathcal{B} has no pseudo-intersection and $|\mathcal{B}| = \mathfrak{p}$.

Proposition 1.3. *Assume $\mathfrak{p} < \mathfrak{t}$ and let \mathcal{B} exemplify \mathfrak{p} . Then there are a cardinal $\kappa = \text{cf}(\kappa) < \mathfrak{p}$ and a \subseteq^* -decreasing sequence $\langle A_i : i < \kappa \rangle \subseteq [\omega]^{\aleph_0}$ such that*

- (a) $A_i \cap B$ is infinite for every $i < \kappa$ and $B \in \mathcal{B}$, and
- (b) if A is a pseudo-intersection of $\{A_i : i < \kappa\}$, then for some $B \in \mathcal{B}$ the intersection $A \cap B$ is finite.

Proof. Fix an enumeration $\mathcal{B} = \{B_i : i < \mathfrak{p}\}$. By induction on $i < \mathfrak{p}$ we try to choose $A_i \in [\omega]^{\aleph_0}$ such that

- (α) $A_i \subseteq^* A_j$ whenever $j < i$,
- (β) $B \cap A_i$ is infinite for each $B \in \mathcal{B}$,
- (γ) if $i = j + 1$, then $A_i \subseteq B_j$.

If we succeed, then $\{A_i : i < \mathfrak{p}\}$ has no pseudo–intersection so $\mathfrak{t} \leq \mathfrak{p}$, a contradiction. So for some $i < \mathfrak{p}$ we cannot choose A_i . Easily i is a limit ordinal and let $\kappa = \text{cf}(i)$ (so $\kappa \leq i < \mathfrak{p}$). Pick an increasing sequence $\langle j_\varepsilon : \varepsilon < \kappa \rangle$ with limit i . Then $\langle A_{j_\varepsilon} : \varepsilon < \kappa \rangle$ is as required. \square

Remark 1.4. Concerning Proposition 1.3, let us note that Todorčević and Veličković used this idea in [10, Thm 1.5] to exhibit a σ –linked poset of size \mathfrak{p} that is not σ –centered.

Lemma 1.5. *Assume that*

- (i) $\bar{A} = \langle A_i : i < \delta \rangle$ is a sequence of members of $[\omega]^{\aleph_0}$, $\delta < \mathfrak{t}$,
- (ii) $\bar{B} = \langle B_n : n < \omega \rangle \subseteq [\omega]^{\aleph_0}$ is \subseteq^* –decreasing and
- (iii) for each $i < \delta$ and $n < \omega$ the intersection $A_i \cap B_n$ is infinite, and
- (iv) $(\forall i < j < \delta)(\exists n < \omega)(A_j \cap B_n \subseteq^* A_i \cap B_n)$.

Then for some $A \in [\omega]^{\aleph_0}$ we have

$$(\forall i < \delta)(A \subseteq^* A_i) \text{ and } (\forall n < \omega)(A \subseteq^* B_n).$$

Proof. Without loss of generality $B_{n+1} \subseteq B_n$ and $\emptyset = \bigcap \{B_n : n < \omega\}$ (as we may use $B'_n = \bigcap_{\ell \leq n} B_\ell \setminus \{0, \dots, n\}$). For each $i < \delta$ let $f_i \in {}^\omega \omega$ be defined by

$$f_i(n) = \min\{k \in B_n \cap A_i : k > f_i(m) \text{ for every } m < n\} + 1.$$

Since $\mathfrak{t} \leq \mathfrak{b}$, there is $f \in {}^\omega \omega$ such that $(\forall i < \delta)(f_i <^* f)$ and $n < f(n) < f(n+1)$ for $n < \omega$. Let

$$B^* = \bigcup \{(B_{n+1} \cap [n, f(n+1))) : n < \omega\}.$$

Then $B^* \in [\omega]^{\aleph_0}$ as for n large enough,

$$\min[B_{n+1} \setminus [0, n) \cap A_0] \leq f_0(n+1) < f(n+1).$$

Clearly for each $n < \omega$ we have $B^* \setminus [0, f(n)) \subseteq B_n$ and hence $B^* \subseteq^* B_n$. Moreover, $(\forall i < \delta)(A_i \cap B^* \in [\omega]^{\aleph_0})$ (as above) and $(\forall i < j < \delta)(A_j \cap B^* \subseteq^* A_i \cap B^*)$ (remember assumption (iv)). Now applying $\mathfrak{t} > \delta$ to $\langle A_i \cap B^* : i < \delta \rangle$ we get a pseudo–intersection A^* which is as required. \square

Definition 1.6. (1) Let \mathbf{S} be the family of all sequences $\bar{\eta} = \langle \eta_n : n \in B \rangle$ such that $B \in [\omega]^{\aleph_0}$, and for $n \in B$, $\eta_n \in {}^{[n, \omega)} 2$ for some $k \in (n, \omega)$. We let $\text{dom}(\bar{\eta}) = B$ and let $\text{set}(\bar{\eta}) = \bigcup \{\text{set}(\eta_n) : n \in \text{dom}(\bar{\eta})\}$, where $\text{set}(\eta_n) = \{\ell : \eta_n(\ell) = 1\}$.

(2) For $\bar{A} = \langle A_i : i < \alpha \rangle \subseteq [\omega]^{\aleph_0}$ let

$$\mathbf{S}_{\bar{A}} = \{\bar{\eta} \in \mathbf{S} : (\forall i < \alpha)(\text{set}(\bar{\eta}) \subseteq^* A_i) \text{ and } (\forall n \in \text{dom}(\bar{\eta}))(\text{set}(\eta_n) \neq \emptyset)\}.$$

(3) For $\bar{\eta}, \bar{\nu} \in \mathbf{S}$ let $\bar{\eta} \leq^* \bar{\nu}$ mean that for every n large enough,

$$n \in \text{dom}(\bar{\nu}) \Rightarrow n \in \text{dom}(\bar{\eta}) \wedge \eta_n \leq \nu_n$$

(where $\eta_n \leq \nu_n$ means “ η_n is an initial segment of ν_n ”).

(4) For $\bar{\eta}, \bar{\nu} \in \mathbf{S}$ let $\bar{\eta} \leq^{**} \bar{\nu}$ mean that for every $n \in \text{dom}(\bar{\nu})$ large enough, for some $m \in \text{dom}(\bar{\eta})$ we have $\eta_m \subseteq \nu_n$ (as functions).

(5) For $\bar{\eta} \in \mathbf{S}$ let $C_{\bar{\eta}} = \{\nu \in {}^\omega 2 : (\exists^\infty n)(\eta_n \subseteq \nu)\}$.

Observation 1.7. (1) If $\bar{\eta} \leq^* \bar{\nu}$, then $\bar{\eta} \leq^{**} \bar{\nu}$ which implies $C_{\bar{\nu}} \subseteq C_{\bar{\eta}}$.

(2) For every $\bar{\eta} \in \mathbf{S}$ and a meagre set $B \subseteq {}^\omega 2$ there is $\bar{\nu} \in \mathbf{S}$ such that $\bar{\eta} \leq^* \bar{\nu}$ and $C_{\bar{\nu}} \cap B = \emptyset$.

- Lemma 1.8.** (1) If $\bar{A} = \langle A_i : i < i^* \rangle \subseteq [\omega]^{\aleph_0}$ has finite intersection property and $i^* < \mathfrak{p}$, then $\mathbf{S}_{\bar{A}} \neq \emptyset$.
- (2) Every \leq^* -increasing sequence of members of \mathbf{S} of length $< \mathfrak{t}$ has an \leq^* -upper bound.
- (3) If $\bar{A} = \langle A_i : i < i^* \rangle \subseteq [\omega]^{\aleph_0}$ is \subseteq^* -decreasing and $i^* < \mathfrak{p}$, then every \leq^* -increasing sequence of members of $\mathbf{S}_{\bar{A}}$ of length $< \mathfrak{p}$ has an \leq^* -upper bound in $\mathbf{S}_{\bar{A}}$.

Proof. (1) Let $A \in [\omega]^{\aleph_0}$ be such that $(\forall i < i^*)(A \subseteq^* A_i)$ (exists as $i^* < \mathfrak{p}$). Let $k_n = \min(A \setminus (n+1))$, and let $\eta_n \in {}^{[n, k_n+1]}2$ be defined by $\eta_n(\ell)$ is 0 if $\ell \in [n, k_n)$ and is 1 if $\ell = k_n$. Then $\langle \eta_n : n < \omega \rangle \in \mathbf{S}_{\bar{A}}$.

(2) Let $\langle \bar{\eta}^\alpha : \alpha < \delta \rangle$ be a \leq^* -increasing sequence and $\delta < \mathfrak{t}$. Let $A_\alpha^* =: \text{dom}(\bar{\eta}^\alpha)$ for $\alpha < \delta$. Then $\langle A_\alpha^* : \alpha < \delta \rangle$ is a \subseteq^* -decreasing sequence of members of $[\omega]^{\aleph_0}$. As $\delta < \mathfrak{t}$ there is $A^* \in [\omega]^{\aleph_0}$ such that $\alpha < \delta \Rightarrow A^* \subseteq^* A_\alpha^*$. Now for $n < \omega$ we define

$$B_n = \bigcup \{ {}^{[m, k]}2 : m \in A^* \text{ and } n \leq m < k < \omega \},$$

and for $\alpha < \delta$ we define

$$A_\alpha = \{ \eta : \text{for some } n \in \text{dom}(\bar{\eta}^\alpha) \text{ we have } \eta_n^\alpha \trianglelefteq \eta \}.$$

One easily verifies that the assumptions of Proposition 1.5 are satisfied (well, replacing ω by B_0 !!). Let $A \subseteq B_0$ be given by the conclusion of 1.5, and put

$$A' = \{ n : \text{for some } \eta \in A \text{ we have } \eta \in \bigcup \{ {}^{[n, k]}2 : k \in (n, \omega) \} \}.$$

Plainly, the set A' is infinite. We let $\bar{\eta}^* = \langle \eta_n : n \in A' \rangle$ where η_n is any member of $A \cap B_n \setminus B_{n+1}$.

(3) Assume that $\bar{A} = \langle A_i : i < i^* \rangle \subseteq [\omega]^{\aleph_0}$ is \subseteq^* -decreasing, $i^* < \mathfrak{p}$, and $\langle \bar{\eta}^\alpha : \alpha < \delta \rangle \subseteq \mathbf{S}_{\bar{A}}$ is \leq^* -increasing, and $\delta < \mathfrak{p}$. Let us consider the following forcing notion \mathbb{P} .

A condition in \mathbb{P} is a quadruple $p = (\bar{\nu}, u, w, a) = (\bar{\nu}^p, u^p, w^p, a^p)$ such that

- (a) $u \in [\omega]^{< \aleph_0}$, $\bar{\nu} = \langle \nu_n : n \in u \rangle$, and for $n \in u$ we have:
 - $\nu_n \in {}^{[n, k_n]}2$ for some $k_n \in (n, \omega)$, and
 - $\text{set}(\nu_n) \neq \emptyset$,
- (b) $w \subseteq \delta$ is finite, and
- (c) $a \subseteq i^*$ is finite.

The order $\leq_{\mathbb{P}} = \leq$ of \mathbb{P} is given by $p \leq q$ if and only if $(p, q \in \mathbb{P}$ and)

- (i) $u^p \subseteq u^q$, $w^p \subseteq w^q$, $a^p \subseteq a^q$, and $\bar{\nu}^q \upharpoonright u^p = \bar{\nu}^p$,
- (ii) $\max(u^p) < \min(u^q \setminus u^p)$ and for $n \in u^q \setminus u^p$ we have
 - (iii) $(\forall \alpha \in w^p)(n \in \text{dom}(\bar{\eta}^\alpha) \wedge \eta_n^\alpha \triangleleft \nu_n^q)$,
 - (iv) $(\forall i \in a^p)(\text{set}(\nu_n^q) \subseteq A_i)$.

Plainly, \mathbb{P} is a σ -centered forcing notion and the sets

$$\mathcal{I}_m^{\alpha, i} = \{ p \in \mathbb{P} : \alpha \in w^p \wedge i \in a^p \wedge |u^p| > m \}$$

(for $\alpha < \delta$, $i < i^*$ and $m < \omega$) are open dense in \mathbb{P} . Since $|\delta| + |i^*| + \aleph_0 < \mathfrak{p}$, we may choose a directed set $G \subseteq \mathbb{P}$ meeting all the sets $\mathcal{I}_m^{\alpha, i}$. Putting $\bar{\nu} = \bigcup \{ \bar{\nu}^p : p \in G \}$ we will get an upper bound to $\langle \bar{\eta}^\alpha : \alpha < \delta \rangle$ in $\mathbf{S}_{\bar{A}}$. \square

Lemma 1.9. Assume that

- (i) $\mathfrak{p} < \mathfrak{t}$ and $\mathcal{B} = \{ B_\alpha : \alpha < \mathfrak{p} \}$ exemplifies \mathfrak{p} (see 1.2), and

- (ii) $\bar{A} = \langle A_i : i < \kappa \rangle \subseteq [\omega]^{\aleph_0}$ is \subseteq^* -decreasing, $\kappa < \mathfrak{p}$ and conditions (a)+(b) of Proposition 1.3 hold,
- (iii) $\text{pr} : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathfrak{p}$ is a bijection satisfying $\text{pr}(\alpha_1, \alpha_2) \geq \alpha_1, \alpha_2$.

Then we can find a sequence $\langle \bar{\eta}^\alpha : \alpha \leq \mathfrak{p} \rangle$ such that

- (a) $\bar{\eta}^\alpha \in \mathbf{S}_{\bar{A}}$ for $\alpha < \mathfrak{p}$ and $\bar{\eta}^\mathfrak{p} \in \mathbf{S}$ (sic!),
- (b) $\langle \bar{\eta}^\alpha : \alpha \leq \mathfrak{p} \rangle$ is \leq^* -increasing,
- (c) if $\alpha < \mathfrak{p}$ and $n \in \text{dom}(\bar{\eta}^{\alpha+1})$ is large enough, then $\text{set}(\eta_n^{\alpha+1}) \cap B_\alpha \neq \emptyset$ (hence $(\forall^\infty n \in \text{dom}(\bar{\eta}^\beta))(\text{set}(\eta_n^\beta) \cap B_\alpha \neq \emptyset)$ holds for every $\beta \in [\alpha + 1, \mathfrak{p}]$),
- (d) if $\alpha = \text{pr}(\beta, \gamma)$, then $\text{set}(\eta_n^{\alpha+1}) \cap B_\beta \neq \emptyset$ and $\text{set}(\eta_n^{\alpha+1}) \cap B_\gamma \neq \emptyset$ for $n \in \text{dom}(\bar{\eta}^{\alpha+1})$, and the truth values of

$$\min(\text{set}(\eta_n^{\alpha+1}) \cap B_\beta) < \min(\text{set}(\eta_n^{\alpha+1}) \cap B_\gamma)$$

are the same for all $n \in \text{dom}(\bar{\eta}^{\alpha+1})$,

- (e) in (d), if $\beta < \kappa$ we can replace B_β by A_β ; similarly with γ ; and if $\beta, \gamma < \kappa$ then we can replace both.

Proof. We choose $\bar{\eta}^\alpha$ by induction on α . For $\alpha = 0$ it is trivial, for α limit $< \mathfrak{p}$ we use Lemma 1.8(3) (and $|\alpha| < \mathfrak{p}$). At a successor stage $\alpha + 1$, we let β, γ be such that $\text{pr}(\beta, \gamma) = \alpha$ and we choose $B'_\alpha \in [\omega]^{\aleph_0}$ such that $B'_\alpha \subseteq B_\alpha \cap B_\beta \cap B_\gamma$ and $(\forall i < \kappa)(B'_\alpha \subseteq^* A_i)$. Next, for $n \in \text{dom}(\bar{\eta}^\alpha)$ we choose η'_n such that $\eta_n^\alpha \triangleleft \eta'_n$ and

$$\emptyset \neq \{\ell : \eta'_n(\ell) = 1 \text{ and } \ell g(\eta_n^\alpha) \leq \ell < \ell g(\eta'_n)\} \subseteq B'_\alpha.$$

Then we let $\bar{\eta}^{\alpha+1} = \langle \eta'_n : n \in \text{dom}(\bar{\eta}^\alpha) \rangle$. By shrinking the domain of $\bar{\eta}^{\alpha+1}$ there is no problem to take care of clause (d). It should be also clear that we may ensure clause (e) as well.

For $\alpha = \mathfrak{p}$, use 1.8(2). □

Definition 1.10. Let κ_1, κ_2 be infinite regular cardinals. A (κ_1, κ_2) -peculiar cut in ${}^\omega\omega$ is a pair $(\langle f_i : i < \kappa_1 \rangle, \langle f^\alpha : \alpha < \kappa_2 \rangle)$ of sequences of functions in ${}^\omega\omega$ such

- (α) $(\forall i < j < \kappa_1)(f_j <^* f_i)$,
- (β) $(\forall \alpha < \beta < \kappa_2)(f^\alpha <^* f^\beta)$,
- (γ) $(\forall i < \kappa_1)(\forall \alpha < \kappa_2)(f^\alpha <^* f_i)$,
- (δ) if $f : \omega \longrightarrow \omega$ is such that $(\forall i < \kappa_1)(f \leq^* f_i)$, then $f \leq^* f^\alpha$ for some $\alpha < \kappa_2$,
- (ε) if $f : \omega \longrightarrow \omega$ is such that $(\forall \alpha < \kappa_2)(f^\alpha \leq^* f)$, then $f_i \leq^* f$ for some $i < \kappa_1$.

Proposition 1.11. If $\kappa_2 < \mathfrak{b}$, then there is no (\aleph_0, κ_2) -peculiar cut.

Proof. Assume towards contradiction that $\mathfrak{b} > \kappa_2$ but there is an (\aleph_0, κ_2) -peculiar cut, say $(\langle f_i : i < \omega \rangle, \langle f^\alpha : \alpha < \kappa_2 \rangle)$ is such a cut. Let S be the family of all increasing sequences $\bar{n} = \langle n_i : i < \omega \rangle$ with $n_0 = 0$. For $\bar{n} \in S$ and $g \in {}^\omega\omega$ we say that \bar{n} obeys g if $(\forall i < \omega)(g(n_i) < n_{i+1})$. Also for $\bar{n} \in S$ define $h_{\bar{n}} \in {}^\omega\omega$ by

$$h_{\bar{n}} \upharpoonright [n_i, n_{i+1}) = f_i \upharpoonright [n_i, n_{i+1}) \quad \text{for } i < \omega.$$

Now, let $g^* \in {}^\omega\omega$ be an increasing function such that for every $n < \omega$ and $m \geq g^*(n)$ we have

$$f_{n+1}(m) < f_n(m) < \dots < f_1(m) < f_0(m).$$

Note that

$$(\square)_1 \text{ if } \bar{n} \in S \text{ obeys } g^*, \text{ then } (\forall i < \omega)(h_{\bar{n}} <^* f_i).$$

Now, for $\alpha < \kappa_2$ define $g^\alpha \in {}^\omega\omega$ by

$$(\square)_2 \quad g^\alpha(n) = \min \{k < \omega : k > n + 1 \wedge (\forall i \leq n)(\exists \ell \in [n, k])(f^\alpha(\ell) < f_i(\ell))\}.$$

Since $\kappa_2 < \mathfrak{b}$, we may choose $g \in {}^\omega\omega$ such that

$$g^* < g \quad \text{and} \quad (\forall \alpha < \kappa_2)(g^\alpha <^* g).$$

Pick $\bar{n} \in S$ which obeys g and consider the function $h_{\bar{n}}$. It follows from $(\square)_1$ that $h_{\bar{n}} <^* f_i$ for all $i < \omega$, so by the properties of an (\aleph_0, κ_2) -peculiar cut there is $\alpha < \kappa_2$ such that $h_{\bar{n}} \leq^* f^\alpha$. Then, for sufficiently large $i < \omega$ we have

- $h_{\bar{n}} \upharpoonright [n_i, n_{i+1}) = f_i \upharpoonright [n_i, n_{i+1}) \leq f^\alpha \upharpoonright [n_i, n_{i+1})$, and
- $n_i < g^\alpha(n_i) < g(n_i) < n_{i+1}$.

The latter implies that for some $\ell \in [n_i, n_{i+1})$ we have $f^\alpha(\ell) < f_i(\ell)$, contradicting the former. \square

Theorem 1.12. *Assume $\mathfrak{p} < \mathfrak{t}$. Then for some regular cardinal κ there exists a (κ, \mathfrak{p}) -peculiar cut in ${}^\omega\omega$ and $\aleph_1 \leq \kappa < \mathfrak{p}$.*

Proof. Use Proposition 1.3 and Lemma 1.9 to choose $\mathcal{B}, \kappa, \bar{A}, \text{pr}$ and $\langle \bar{\eta}^\alpha : \alpha \leq \mathfrak{p} \rangle$ so that:

- (i) $\mathcal{B} = \{B_\alpha : \alpha < \mathfrak{p}\}$ exemplifies \mathfrak{p} ,
- (ii) $\bar{A} = \langle A_i : i < \kappa \rangle \subseteq [\omega]^{\aleph_0}$ is \subseteq^* -decreasing, $\kappa = \text{cf}(\kappa) < \mathfrak{p}$ and conditions (a)+(b) of 1.3 hold,
- (iii) $\text{pr} : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ is a bijection satisfying $\text{pr}(\alpha_1, \alpha_2) \geq \alpha_1, \alpha_2$,
- (iv) the sequence $\langle \bar{\eta}^\alpha : \alpha \leq \mathfrak{p} \rangle$ satisfies conditions (a)–(e) of 1.9.

It is enough to find a suitable cut $\langle f_i : i < \kappa \rangle, \langle f^\alpha : \alpha < \mathfrak{p} \rangle \subseteq A^* \omega$ for some infinite $A^* \subseteq \omega$ (as by renaming, A^* is ω). Let

$$(v) \quad A^* = \text{dom}(\bar{\eta}^{\mathfrak{p}}),$$

$$(vi) \quad \text{for } i < \kappa \text{ we let } f_i : A^* \rightarrow \omega \text{ be defined by}$$

$$f_i(n) = \min \{ \ell : [\eta_n^{\mathfrak{p}}(n + \ell) = 1 \wedge n + \ell \notin A_i] \text{ or } \text{dom}(\eta_n^{\mathfrak{p}}) = [n, n + \ell) \},$$

$$(vii) \quad \text{for } \alpha < \mathfrak{p} \text{ we let } f^\alpha : A^* \rightarrow \omega \text{ be defined by}$$

$$f^\alpha(n) = \min \{ \ell + 1 : [\eta_n^{\mathfrak{p}}(n + \ell) = 1 \wedge n + \ell \in B_\alpha] \text{ or } \text{dom}(\eta_n^{\mathfrak{p}}) = [n, n + \ell) \}.$$

Note that (by the choice of f_i , i.e., clause (vi)):

$$(viii) \quad \bigcup \{ [n, n + f_i(n)) \cap \text{set}(\eta_n^{\mathfrak{p}}) : n \in A^* \} \subseteq^* A_i \text{ for every } i < \kappa.$$

Also,

$$(\otimes)_1^a \quad f_j \leq^* f_i \text{ for } i < j < \kappa.$$

[Why? Let $i < j < \kappa$. Then $A_j \subseteq^* A_i$ and hence for some n^* we have that $A_j \setminus n^* \subseteq A_i$. Therefore, for every $n \in A^* \setminus n^*$, in the definition of f_i, f_j in clause (vi), if ℓ can serve as a candidate for $f_i(n)$ then it can serve for $f_j(n)$, so (as we use the minimum there) $f_j(n) \leq f_i(n)$. Consequently $f_j \leq^* f_i$.]

Now, we want to argue that we may find a subsequence of $\langle f_i : i < \kappa \rangle$ which is $<^*$ -decreasing. For this it is enough to show that

$$(\otimes)_1^b \quad \text{for every } i < \kappa, \text{ for some } j \in (i, \kappa) \text{ we have } f_j <^* f_i.$$

So assume towards contradiction that for some $i(*) < \kappa$ we have

$$(\forall j)(i(*) < j < \kappa \Rightarrow \neg(f_j <^* f_{i(*)})).$$

For $j < \kappa$ put $B_j^* =: \{n \in A^* : f_j(n) \geq f_{i(*)}(n)\}$. Then $B_j^* \in [A^*]^{\aleph_0}$ is \subseteq^* -decreasing, so there is a pseudo-intersection B^* of $\langle B_j^* : j < \kappa \rangle$ (so $B^* \in [A^*]^{\aleph_0}$ and $(\forall j < \kappa)(B^* \subseteq^* B_j^*)$). Now, let $A' = \bigcup \{\text{set}(\eta_n^{\mathfrak{p}}) \cap [n, n + f_{i(*)}(n)] : n \in B^*\}$.

(*) A' is an infinite subset of ω .

[Why? Recall that by 1.9(a) we have $\bar{\eta}^0 \in \mathbf{S}_{\bar{A}}$ and hence $\text{set}(\bar{\eta}^0) \subseteq^* A_{i(*)}$ and $(\forall n \in \text{dom}(\bar{\eta}^0))(\text{set}(\eta_n^0) \neq \emptyset)$ (see Definition 1.6(2)). By 1.9(b) we know that for every large enough $n \in \text{dom}(\bar{\eta}^{\mathfrak{p}})$ we have $n \in \text{dom}(\bar{\eta}^0)$ and $\eta_n^0 \leq \eta_n^{\mathfrak{p}}$. For every large enough $n \in \text{dom}(\bar{\eta}^0)$ we have $\text{set}(\bar{\eta}^0) \setminus \{0, \dots, n-1\} \subseteq A_{i(*)}$, and hence for every large enough $n \in \text{dom}(\bar{\eta}^{\mathfrak{p}})$ we have $\eta_n^0 \leq \eta_n^{\mathfrak{p}}$ and $\emptyset \neq \text{set}(\eta_n^0) \subseteq A_{i(*)}$. Consequently, for large enough $n \in B^*$, $[n, n + f_{i(*)}(n)] \cap \text{set}(\eta_n^{\mathfrak{p}}) \neq \emptyset$ and we are done.]

(**) $A' \subseteq^* A_j$ for $j \in (i(*), \kappa)$ (and hence for all $j < \kappa$).

[As $f_j \upharpoonright B^* =^* f_{i(*)} \upharpoonright B^*$ for $j \in (i(*), \kappa)$.]

(***) $A' \cap B_\alpha$ is infinite for $\alpha < \mathfrak{p}$.

[Why? By clauses (c) + (a) of 1.9, for every large enough $n \in \text{dom}(\bar{\eta}^{\alpha+1})$ we have $\text{set}(\eta_n^{\alpha+1}) \cap B_\alpha \neq \emptyset$ and $\text{set}(\eta_n^{\alpha+1}) \subseteq A_{i(*)}$.]

Properties (*)–(***) contradict (b) of 1.3, finishing the proof of $(\otimes)_1^{\mathfrak{p}}$.

Thus passing to a subsequence if necessary, we may assume that

$(\otimes)_1^{\mathfrak{c}}$ the demand in (α) of 1.10 is satisfied, i.e., $f_j <^* f_i$ for $i < j < \kappa$.

Now,

$(\otimes)_2$ $(\forall i < \kappa)(\forall \alpha < \mathfrak{p})(f^\alpha <^* f_i)$.

[Why? Let $i < \kappa$, $\alpha < \mathfrak{p}$. For large enough $n \in A^*$ we have that $\text{set}(\eta_n^{\alpha+1}) \subseteq A_i$ and $\text{set}(\eta_n^{\alpha+1}) \cap B_\alpha \neq \emptyset$ and $\eta_n^{\alpha+1} \leq \eta_n^{\mathfrak{p}}$. Then for those n we have $f^\alpha(n) \leq f_i(n)$. Now, remembering $(\otimes)_1$, we may conclude that actually $f^\alpha <^* f_i$.]

$(\otimes)_3^{\mathfrak{a}}$ The set (of functions) $\{f_i : i < \kappa\} \cup \{f^\alpha : \alpha < \mathfrak{p}\}$ is linearly ordered by \leq^* ,

$(\otimes)_3^{\mathfrak{b}}$ in fact, if f', f'' are in the family then either $f' =^* f''$ or $f' <^* f''$ or $f'' <^* f'$.

[Why? By $(\otimes)_1$, $(\otimes)_2$ and clauses (d) + (e) of 1.9.]

Choose inductively a sequence $\bar{\alpha} = \langle \alpha(\varepsilon) : \varepsilon < \varepsilon^* \rangle \subseteq \mathfrak{p}$ such that:

- $\alpha(\varepsilon)$ is the minimal $\alpha \in \mathfrak{p} \setminus \{\alpha(\zeta) : \zeta < \varepsilon\}$ satisfying $(\forall \zeta < \varepsilon)(f^{\alpha(\zeta)} <^* f^\alpha)$, and
- we cannot choose $\alpha(\varepsilon^*)$.

We ignore (till (\otimes_7)) the question of the value of ε^* . Now,

$(\otimes)_4$ $\langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon^* \rangle$ satisfy clauses (α) – (γ) of 1.10.

[Why? By $(\otimes)_1$ – $(\otimes)_3$ and the choice of $\alpha(\varepsilon)$'s above.]

$(\otimes)_5$ $\langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon^* \rangle$ satisfy clause (δ) of 1.10.

[Why? Assume towards contradiction that $f : A^* \rightarrow \omega$ and

$$(\forall i < \kappa)(f \leq^* f_i) \text{ but } (\forall \varepsilon < \varepsilon^*)(\neg(f \leq^* f^{\alpha(\varepsilon)})).$$

Clearly, without loss of generality, we may assume that $[n, n + f(n)] \subseteq \text{dom}(\eta_n^{\mathfrak{p}})$ for $n \in A^*$. Let $A' = \bigcup \{[n, n + f(n)] \cap \text{set}(\eta_n^{\mathfrak{p}}) : n \in A^*\}$. Now for every $i < \kappa$, $A' \subseteq^* A_i$ because $f \leq^* f_i$ and by the definition of f_i . Also, for every $\alpha < \mathfrak{p}$, the intersection $A' \cap B_\alpha$ is infinite. Why? It follows from the choice of the sequence $\bar{\alpha}$ that for some $\varepsilon < \varepsilon^*$ we have $\neg(f^{\alpha(\varepsilon)} <^* f^\alpha)$, and thus $f^\alpha \leq^* f^{\alpha(\varepsilon)}$ (remember

(\otimes)₃. Hence, if $n \in A^*$ is large enough, then $f^\alpha(n) \leq f^{\alpha(\varepsilon)}(n)$ and for infinitely many $n \in A^*$ we have $f^\alpha(n) \leq f^{\alpha(\varepsilon)}(n) < f(n) \leq f_0(n) \leq |\text{dom}(\eta_n^{\mathfrak{p}})|$. For every such n we have $n + f^\alpha(n) - 1 \in A' \cap B_\alpha$. Together, A' contradicts clause (ii) of the choice of $\langle A_i : i < \kappa \rangle, \langle B_\alpha : \alpha < \mathfrak{p} \rangle$, specifically the property stated in 1.3(b).]

(\otimes)₆ $\langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon^* \rangle$ satisfy clause (ε) of 1.10.

[Why? Assume towards contradiction that $f : A^* \rightarrow \omega$, and

$$(\forall \varepsilon < \varepsilon^*)(f^{\alpha(\varepsilon)} \leq^* f) \text{ but } (\forall i < \kappa)(\neg(f_i \leq^* f)).$$

It follows from (\otimes)₁ (and the assumption above) that we may choose an infinite set $A^{**} \subseteq A^*$ such that $(\forall i < \kappa)((f \upharpoonright A^{**}) <^* (f_i \upharpoonright A^{**}))$. Let

$$A' = \bigcup \{[n, n + f(n)) \cap \text{set}(\eta_n^{\mathfrak{p}}) : n \in A^{**}\} \subseteq \omega.$$

Since $(f \upharpoonright A^{**}) <^* (f_i \upharpoonright A^{**})$ we easily see that $A' \subseteq^* A_i$ for all $i < \kappa$ (remember (viii)). Like in the justification for (\otimes)₅ above, if $\alpha < \mathfrak{p}$ then for some $\varepsilon < \varepsilon^*$ we have $f^\alpha \leq^* f^{\alpha(\varepsilon)}$ and we may conclude from our assumption towards contradiction that $f^\alpha \leq^* f$ for all $\alpha < \mathfrak{p}$. Like in (\otimes)₅ we conclude that for every $\alpha < \mathfrak{p}$ the intersection $A' \cap B_\alpha$ is infinite, contradicting the choice of $\langle A_i : i < \kappa \rangle, \langle B_\alpha : \alpha < \mathfrak{p} \rangle$.]

(\otimes)₇ $\varepsilon^* = \mathfrak{p}$.

[Why? The sequence $\langle \alpha(\varepsilon) : \varepsilon < \mathfrak{p} \rangle$ is an increasing sequence of ordinals $< \mathfrak{p}$, hence $\varepsilon^* \leq \mathfrak{p}$. If $\varepsilon^* < \mathfrak{p}$, then by the Bell theorem we get contradiction to (\otimes)₄–(\otimes)₆ above; cf. Proposition 2.1 below.]

So $\langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \mathfrak{p} \rangle$ are as required: clauses (α)–(γ) of 1.10 hold by (\otimes)₄, clause (δ) by \otimes ₅, clause (ε) by (\otimes)₆. Finally, since $\mathfrak{t} \leq \mathfrak{b}$, we may use Proposition 1.11 to conclude that (under our assumption $\mathfrak{p} < \mathfrak{t}$) there is no (\aleph_0, \mathfrak{p}) -peculiar cut and hence $\kappa \geq \aleph_1$. \square

Remark 1.13. The existence of (κ, \mathfrak{p}) -peculiar cuts for $\kappa < \mathfrak{p}$ is independent from “ZFC+ $\mathfrak{p} = \mathfrak{t}$ ”. We will address this issue in [9].

2. PECULIAR CUTS AND MA

Proposition 2.1. *Assume that $\kappa_1 \leq \kappa_2$ are infinite regular cardinals and there exists a (κ_1, κ_2) -peculiar cut in ${}^\omega\omega$. Then for some σ -centered forcing notion \mathbb{Q} of cardinality κ_1 and a sequence $\langle \mathcal{I}_\alpha : \alpha < \kappa_2 \rangle$ of open dense subsets of \mathbb{Q} , there is no directed $G \subseteq \mathbb{Q}$ such that $(\forall \alpha < \kappa_2)(G \cap \mathcal{I}_\alpha \neq \emptyset)$. Hence $\mathbf{MA}_{\kappa_2}(\sigma\text{-centered})$ fails and thus $\mathfrak{p} \leq \kappa_2$.*

Proof. Let $(\langle f_i : i < \kappa_1 \rangle, \langle f^\alpha : \alpha < \kappa_2 \rangle)$ be a (κ_1, κ_2) -peculiar cut in ${}^\omega\omega$. Define a forcing notion \mathbb{Q} as follows.

A condition in \mathbb{Q} is a pair $p = (\rho, u)$ such that $\rho \in {}^\omega\omega$ and $u \subseteq \kappa_1$ is finite.

The order $\leq_{\mathbb{Q}} = \leq$ of \mathbb{Q} is given by $(\rho_1, u_1) \leq (\rho_2, u_2)$ if and only if (both are in \mathbb{Q} and)

- (a) $\rho_1 \leq \rho_2$,
- (b) $u_1 \subseteq u_2$,
- (c) if $n \in [\ell g(\rho_1), \ell g(\rho_2))$ and $i \in u_1$, then $f_i(n) \geq \rho_2(n)$.

Plainly, \mathbb{Q} is a forcing notion of cardinality κ_1 . It is σ -centered as for each $\rho \in {}^\omega\omega$, the set $\{(\eta, u) \in \mathbb{Q} : \eta = \rho\}$ is directed.

For $j < \kappa_1$ let $\mathcal{I}_j = \{(\rho, u) \in \mathbb{Q} : j \in u\}$, and for $\alpha = \omega\beta + n < \kappa_2$ let

$$\mathcal{I}^\alpha = \{(\rho, u) \in \mathbb{Q} : (\exists m < \ell g(\rho))(m \geq n \wedge \rho(m) > f^\beta(m))\}.$$

Clearly $\mathcal{I}_j, \mathcal{I}^\alpha$ are dense open subsets of \mathbb{Q} . Suppose towards contradiction that there is a directed $G \subseteq \mathbb{Q}$ intersecting all $\mathcal{I}^\alpha, \mathcal{I}_j$ for $j < \kappa_1, \alpha < \kappa_2$. Put $g = \bigcup\{\rho : (\exists u)((\rho, u) \in G)\}$. Then

- g is a function, its domain is ω (as $G \cap \mathcal{I}^n \neq \emptyset$ for $n < \omega$), and
- $g \leq^* f_i$ (as $G \cap \mathcal{I}_i \neq \emptyset$), and
- $\{n < \omega : f^\alpha(n) < g(n)\}$ is infinite (as $G \cap \mathcal{I}^{\omega\alpha+n} \neq \emptyset$ for every n).

The properties of the function g clearly contradict our assumptions on $\langle f_i : i < \kappa_1 \rangle, \langle f^\alpha : \alpha < \kappa_2 \rangle$. \square

Corollary 2.2. *If there exists an (\aleph_0, κ_2) -peculiar cut, then $\text{cov}(\mathcal{M}) \leq \kappa_2$.*

Theorem 2.3. *Let $\text{cf}(\kappa_2) = \kappa_2 > \aleph_1$. Assume \mathbf{MA}_{\aleph_1} holds. Then there is no (\aleph_1, κ_2) -peculiar cut in ${}^\omega\omega$.*

Proof. Suppose towards contradiction that $\text{cf}(\kappa_2) = \kappa_2 > \aleph_1, (\langle f_i : i < \omega_1 \rangle, \langle f^\alpha : \alpha < \kappa_2 \rangle)$ is an (\aleph_1, κ_2) -peculiar cut and \mathbf{MA}_{\aleph_1} holds true. We define a forcing notion \mathbb{Q} as follows.

A condition in \mathbb{Q} is a pair $p = (u, \bar{\rho}) = (u^p, \bar{\rho}^p)$ such that

- (a) $u \subseteq \omega_1$ is finite, $\bar{\rho} = \langle \rho_i : i \in u \rangle = \langle \rho_i^p : i \in u \rangle$,
- (b) for some $n = n^p$, for all $i \in u$ we have $\rho_i \in {}^n\omega$,
- (c) for each $i \in u$ and $m < n^p$ we have $\rho_i(m) \leq f_i(m)$,
- (d) if $i_0 = \max(u)$ and $m \geq n^p$, then $f_{i_0}(m) > 2 \cdot |u^p| + 885$.
- (e) $\langle f_i \upharpoonright [n^p, \omega) : i \in u \rangle$ is $<$ -decreasing.

The order $\leq_{\mathbb{Q}} = \leq$ of \mathbb{Q} is given by $p \leq q$ if and only if $(p, q \in \mathbb{Q})$ and

- (f) $u^p \subseteq u^q$,
- (g) $\rho_i^p \leq \rho_i^q$ for every $i \in u^p$,
- (h) if $i < j$ are from u^p , then $\rho_i^q \upharpoonright [n^p, n^q) < \rho_j^q \upharpoonright [n^p, n^q)$,
- (i) if $i < j, i \in u^q \setminus u^p$ and $j \in u^p$, then for some $m \in [n^p, n^q)$ we have $f_j(m) < \rho_i^q(m)$.

Claim 2.3.1. *\mathbb{Q} is a ccc forcing notion of size \aleph_1 .*

Proof of the Claim. Plainly, the relation $\leq_{\mathbb{Q}}$ is transitive and $|\mathbb{Q}| = \aleph_1$. Let us argue that the forcing notion \mathbb{Q} satisfies the ccc.

Let $p_\varepsilon \in \mathbb{Q}$ for $\varepsilon < \omega_1$. Without loss of generality $\langle p_\varepsilon : \varepsilon < \omega_1 \rangle$ is without repetition. Applying the Δ -Lemma we can find an unbounded set $\mathcal{U} \subseteq \omega_1$ and $m(*) < n(*) < \omega$ and $n' < \omega$ such that for each $\varepsilon \in \mathcal{U}$ we have

- (i) $|u^{p_\varepsilon}| = n(*)$ and $n^{p_\varepsilon} = n'$; let $u^{p_\varepsilon} = \{\alpha_{\varepsilon, \ell} : \ell < n(*)\}$ and $\alpha_{\varepsilon, \ell}$ increases with ℓ , and
- (ii) $\alpha_{\varepsilon, \ell} = \alpha_\ell$ for $\ell < m(*)$ and $\rho_{\varepsilon, \ell} = \rho_\ell^*$ for $\ell < n(*)$, and
- (iii) if $\varepsilon < \zeta$ are from \mathcal{U} and $k, \ell \in [m(*), n(*)]$, then $\alpha_{\varepsilon, \ell} < \alpha_{\zeta, k}$.

Let $\varepsilon < \zeta$ be elements of \mathcal{U} such that $[\varepsilon, \zeta) \cap \mathcal{U}$ is infinite. Pick $k^* > n'$ such that for each $k \geq k^*$ we have

- the sequence $\langle f_\alpha(k) : \alpha \in \{\alpha_{\varepsilon, \ell} : \ell < n(*)\} \cup \{\alpha_{\zeta, \ell} : \ell < n(*)\} \rangle$ is strictly decreasing,
- $f_{\alpha_{\zeta, n(*)-1}}(k) > 885 \cdot (n(*) + 1)$, and
- $f_{\alpha_{\zeta, m(*)}}(k) + n(*) + 885 < f_{\alpha_{\varepsilon, n(*)-1}}(k)$.

(The choice is possible because $\langle f_i : i < \omega_1 \rangle$ is $<^*$ -decreasing and by the selection of ε, ζ we also have $\lim_{k \rightarrow \infty} (f_{\alpha_{\varepsilon, n(*)-1}}(k) - f_{\alpha_{\zeta, m(*)}}(k)) = \infty$.)

Now define $q = (u^q, \bar{\rho}^q)$ as follows:

- $u^q = u^{p_\varepsilon} \cup u^{p_\zeta}$, $n^q = k^* + 1$,
- if $n < n'$, $i \in u^{p_\varepsilon}$, then $\rho_i^q(n) = \rho_i^{p_\varepsilon}(n)$,
- if $n < n'$, $i \in u^{p_\zeta}$, then $\rho_i^q(n) = \rho_i^{p_\zeta}(n)$,
- if $i = \alpha_{\varepsilon, \ell}$, $\ell < n(*)$, $n \in [n', k^*]$, then $\rho_i^q(n) = \ell$, and if $j = \alpha_{\zeta, \ell}$, $m(*) \leq \ell < n(*)$, then $\rho_j^q(n) = n(*) + \ell + 1$,
- if $j = \alpha_{\zeta, \ell}$, $\ell < n(*)$, then $\rho_j^q(k^*) = \ell$, and if $i = \alpha_{\varepsilon, \ell}$, $m(*) \leq \ell < n(*)$, then $\rho_i^q(k^*) = f_{\alpha_{\zeta, m(*)}}(k^*) + \ell + 1$.

It is well defined (as $\rho_{\alpha_{\varepsilon, \ell}}^{p_\varepsilon} = \rho_{\alpha_{\zeta, \ell}}^{p_\zeta}$ for $\ell < m(*)$). Also $q \in \mathbb{Q}$. Lastly, one easily verifies that $p_\varepsilon \leq_{\mathbb{Q}} q$ and $p_\zeta \leq_{\mathbb{Q}} q$, so indeed \mathbb{Q} satisfies the ccc. \square

For $i < \omega_1$ and $n < \omega$ let

$$\mathcal{I}_{i,n} = \{p \in \mathbb{Q} : [u^p \not\subseteq i \text{ or for no } q \in \mathbb{Q} \text{ we have } p \leq_{\mathbb{Q}} q \wedge u^q \not\subseteq i] \text{ and } n^p \geq n\}.$$

Plainly,

(α) the sets $\mathcal{I}_{i,n}$ are open dense in \mathbb{Q} .

Also,

(β) for each $i < \omega_1$ there is $p_i^* \in \mathbb{Q}$ such that $u^{p_i^*} = \{i\}$.

It follows from 2.3.1 that

(δ) for some $i(*)$, $p_{i(*)}^* \Vdash_{\mathbb{Q}}$ “ $\{j < \omega_1 : p_j^* \in G\}$ is unbounded in ω_1 ”.

Note also that

(γ) if p is compatible with $p_{i(*)}^*$ and $p \in \mathcal{I}_{i,n}$ then $u_p \not\subseteq i$.

Since we have assumed \mathbf{MA}_{\aleph_1} and \mathbb{Q} satisfies the ccc (by 2.3.1), we may find a directed set $G \subseteq \mathbb{Q}$ such that $p_{i(*)}^* \in G$ and $\mathcal{I}_{i,n} \cap G \neq \emptyset$ for all $n < \omega$ and $i < \omega_1$. Note that then the set $\mathcal{U} := \bigcup \{u^p : p \in G\}$ is unbounded in ω_1 .

For $i \in \mathcal{U}$ let $g_i = \bigcup \{\rho_i^p : p \in G\}$. Clearly each $g_i \in {}^\omega \omega$ (as G is directed, $\mathcal{I}_{i,n} \cap G \neq \emptyset$ for $i < \omega_1$, $n < \omega$). Also $g_i \leq f_i$ by clause (c) of the definition of \mathbb{Q} , and $\langle g_i : i \in \mathcal{U} \rangle$ is $<^*$ -increasing by clause (h) of the definition of $\leq_{\mathbb{Q}}$. Hence for each $i < j$ from \mathcal{U} we have $g_i <^* g_j \leq^* f_j <^* f_i$. Thus by property 1.10(δ) of a peculiar cut, for every $i \in \mathcal{U}$ there is $\gamma(i) < \kappa_2$ such that $g_i <^* f^{\gamma(i)}$. Let $\gamma(*) = \sup\{\gamma(i) : i \in \mathcal{U}\}$. Then $\gamma(*) < \kappa_2$ (as $\kappa_2 = \text{cf}(\kappa_2) > \aleph_1$). Now, for each $i \in \mathcal{U}$ we have $g_i <^* f^{\gamma(*)} <^* f_i$, and thus for $i \in \mathcal{U}$ we may pick $n_i < \omega$ such that

$$n \in [n_i, \omega) \Rightarrow g_i(n) < f^{\gamma(*)}(n) < f_i(n).$$

For some n^* the set $\mathcal{U}_* = \{i \in \mathcal{U} : n_i = n^*\}$ is unbounded in ω_1 . Let $j \in \mathcal{U}_*$ be such that $\mathcal{U}_* \cap j$ is infinite. Pick $p \in G$ such that $j \in u^p$ and $n^p > n^*$ (remember $G \cap \mathcal{I}_{j, n^*+1} \neq \emptyset$ and G is directed). Since u^p is finite, we may choose $i \in \mathcal{U}_* \cap j \setminus u^p$, and then $q \in G$ such that $q \geq p$ and $i \in u^q$. It follows from clause (i) of the definition of the order \leq of \mathbb{Q} that there is $n \in [n^p, n^q)$ such that $f_j(n) < \rho_i^q(n) = g_i(n)$. Since $n > n^* = n_i = n_j$ we get $f_j(n) < g_i(n) < f^{\gamma(*)}(n) < f_j(n)$, a contradiction. \square

Remark 2.4. The proof of 2.3 actually used Hausdorff gaps on which much is known (see, e.g., Abraham and Shelah [1], [2]). More precisely, the proof could be presented as a two-step argument:

- (1) from \mathbf{MA}_{\aleph_1} one gets that every decreasing ω_1 -sequence is half-a-Hausdorff gap, and
- (2) if $\kappa_2 = \text{cf}(\kappa_2) > \aleph_1$, then the ω_1 -part of a peculiar (ω_1, κ_2) -cut cannot be half-a-Hausdorff gap.

Corollary 2.5. *If \mathbf{MA}_{\aleph_1} then $\mathfrak{p} = \aleph_2 \Leftrightarrow \mathfrak{t} = \aleph_2$. In other words,*

$$\mathfrak{m} = \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{t} = \aleph_2.$$

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