

GROUPWISE DENSITY CANNOT BE MUCH BIGGER THAN THE UNBOUNDED NUMBER

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ABSTRACT. We prove that \mathfrak{g} (the groupwise density number) is smaller or equal to \mathfrak{b}^+ , the successor of the minimal cardinality of an unbounded subset of ${}^\omega\omega$. This is true even for the version of \mathfrak{g} for groupwise dense ideals.

1. INTRODUCTION

In the present note we are interested in two cardinal characteristics of the continuum, the unbounded number \mathfrak{b} and the groupwise density number \mathfrak{g} . The former cardinal belongs to the oldest and most studied cardinal invariants of the continuum (see, e.g., van Douwen [9] and Bartoszyński and Judah [2]) and it is defined as follows.

Definition 1.1. (a) The partial order $\leq_{J_{\omega}^{\text{bd}}}$ on ${}^\omega\omega$ is defined by

$$f \leq_{J_{\omega}^{\text{bd}}} g \text{ if and only if } (\exists N < \omega)(\forall n > N)(f(n) \leq g(n)).$$

(b) The *unbounded number* \mathfrak{b} is defined by

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \text{ has no } \leq_{J_{\omega}^{\text{bd}}}\text{-upper bound in } {}^\omega\omega\}.$$

The groupwise density number \mathfrak{g} , introduced in Blass and Laflamme [4], is perhaps less popular but it has gained substantial importance in the realm of cardinal invariants. For instance, it has been studied in connection with the cofinality $\text{cf}(\text{Sym}(\omega))$ of the symmetric group on the set ω of all integers, see Thomas [8] or Brendle and Losada [5]. The cardinal \mathfrak{g} is defined as follows.

Definition 1.2. (a) We say that a family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is *groupwise dense* whenever:

- $B \subseteq A \in \mathcal{A}$, $B \in [\omega]^{\aleph_0}$ implies $B \in \mathcal{A}$, and
- for every increasing sequence $\langle m_i : i < \omega \rangle \in {}^\omega\omega$ there is an infinite set $\mathcal{U} \subseteq \omega$ such that $\bigcup\{[m_i, m_{i+1}) : i \in \mathcal{U}\} \in \mathcal{A}$.

(b) The *groupwise density number* \mathfrak{g} is defined as the minimal cardinal θ for which there is a sequence $\langle \mathcal{A}_\alpha : \alpha < \theta \rangle$ of groupwise dense subsets of $[\omega]^{\aleph_0}$ such that

$$(\forall B \in [\omega]^{\aleph_0})(\exists \alpha < \theta)(\forall A \in \mathcal{A}_\alpha)(B \not\subseteq^* A).$$

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(Recall that for infinite sets A and B , $A \subseteq^* B$ means $A \setminus B$ is finite.)

The unbounded number \mathfrak{b} and groupwise density number \mathfrak{g} can be in either order, see Blass [3] and more Mildenerger and Shelah [6], [7], the latter article gives a bound on \mathfrak{g} . However, as we show in Theorem 2.2, \mathfrak{g} cannot be bigger than \mathfrak{b}^+ .

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Notation: Our notation is rather standard and compatible with that of classical textbooks on Set Theory (like Bartoszyński and Judah [2]). We will keep the following rules concerning the use of symbols.

- (1) A, B, \mathcal{U} (with possible sub- and superscripts) denote subsets of ω , infinite if not said otherwise.
- (2) m, n, ℓ, k, i, j are natural numbers.
- (3) $\alpha, \beta, \gamma, \delta, \varepsilon, \xi, \zeta$ are ordinals, θ is a cardinal.

2. THE RESULT

Lemma 2.1. *For some cardinal $\theta \leq \mathfrak{b}$ there is a sequence $\langle B_{\zeta,t} : \zeta < \theta, t \in I_\zeta \rangle$ such that:*

- (a) $B_{\zeta,t} \in [\omega]^{\aleph_0}$
- (b) if $\zeta < \theta$ and $s \neq t$ are from I_ζ , then $B_{\zeta,s} \cap B_{\zeta,t}$ is finite (so $|I_\zeta| \leq 2^{\aleph_0}$),
- (c) for every $B \in [\omega]^{\aleph_0}$ the set

$$\{(\zeta, t) : \zeta < \theta \ \& \ t \in I_\zeta \ \& \ B_{\zeta,t} \cap B \text{ is infinite} \}$$

is of cardinality 2^{\aleph_0} .

Proof. This is a weak version of the celebrated base-tree theorem of Bohuslav Balcar and Petr Simon with $\theta = \mathfrak{h}$ which is known to be $\leq \mathfrak{b}$, see Balcar and Simon [1, 3.4, pg.350]. However, for the sake of completeness of our exposition, let us present a proof.

Let $\langle f_\zeta : \zeta < \mathfrak{b} \rangle$ be a $\leq_{J_{\mathfrak{b}^d}}$ -increasing sequence of members of ${}^\omega\omega$ with no $\leq_{J_{\mathfrak{b}^d}}$ -upper bound in ${}^\omega\omega$. Moreover we demand that each f_ζ is increasing (clearly, this does not change \mathfrak{b}). By induction on $\zeta < \mathfrak{b}$ choose sets \mathcal{T}_ζ and systems $\langle B_{\zeta,\eta} : \eta \in \mathcal{T}_{\zeta+1} \rangle$ such that:

- (i) $\mathcal{T}_\zeta \subseteq {}^\zeta(2^{\aleph_0})$ and if $\eta \in \mathcal{T}_{\zeta+1}$ then $B_{\zeta,\eta} \in [\omega]^{\aleph_0}$,
- (ii) if $\eta \in \mathcal{T}_\zeta$ and $\varepsilon < \zeta$, then $\eta \upharpoonright \varepsilon \in \mathcal{T}_\varepsilon$,
- (iii) if ζ is a limit ordinal, then

$$\mathcal{T}_\zeta = \{ \eta \in {}^\zeta(2^{\aleph_0}) : (\forall \varepsilon < \zeta)(\eta \upharpoonright \varepsilon \in \mathcal{T}_\varepsilon) \text{ and } (\exists A \in [\omega]^{\aleph_0})(\forall \varepsilon < \zeta)(A \subseteq^* B_{\varepsilon, \eta \upharpoonright (\varepsilon+1)}) \},$$

- (iv) if $\varepsilon < \zeta$ and $\eta \in \mathcal{T}_{\zeta+1}$, then $B_{\zeta,\eta} \subseteq^* B_{\varepsilon, \eta \upharpoonright (\varepsilon+1)}$,
- (v) for $\eta \in \mathcal{T}_{\zeta+1}$ and $m_1 < m_2$ from $B_{\zeta,\eta}$ we have $f_\zeta(m_1) < m_2$,
- (vi) if $\eta \in \mathcal{T}_\varepsilon$, then the set $\{B_{\varepsilon,\nu} : \eta \triangleleft \nu \in \mathcal{T}_{\varepsilon+1}\}$ is an infinite maximal subfamily of

$$\{A \in [\omega]^{\aleph_0} : (\forall \xi < \varepsilon)(A \subseteq^* B_{\xi, \eta \upharpoonright (\xi+1)})\}$$

consisting of pairwise almost disjoint sets.

It should be clear that the choice is possible. Note that for some limit $\zeta < \mathfrak{b}$ we may have $\mathcal{T}_\zeta = \emptyset$ (and then also $\mathcal{T}_\xi = \emptyset$ for $\xi > \zeta$). Also, if we define $\mathcal{T}_\mathfrak{b}$ as in (iii), then it will be empty (remember clause (v) and the choice of $\langle f_\zeta : \zeta < \mathfrak{b} \rangle$).

The lemma will readily follow from the following fact.

(*) For every $A \in [\omega]^{\aleph_0}$ there is $\xi < \mathfrak{b}$ such that

$$|\{\eta \in \mathcal{T}_{\xi+1} : B_{\xi,\eta} \cap A \text{ is infinite}\}| = 2^{\aleph_0}.$$

To show (*) let $A \in [\omega]^{\aleph_0}$ and define

$$S = \bigcup_{\zeta < \mathfrak{b}} \{\eta \in \mathcal{T}_\zeta : (\forall \varepsilon < \zeta)(A \cap B_{\varepsilon,\eta \upharpoonright (\varepsilon+1)} \text{ is infinite})\}.$$

Clearly S is closed under taking the initial segments and $\langle \rangle \in S$. By the “maximal” in clause (vi), we have that

(*)₁ if $\eta \in S \cap \mathcal{T}_\zeta$ where $\zeta < \mathfrak{b}$ is non-limit or $\text{cf}(\zeta) = \aleph_0$, then $(\exists \nu)(\eta \triangleleft \nu \in \mathcal{T}_{\zeta+1} \cap S)$.

Now,

(*)₂ if $\eta \in S$ and $\ell g(\eta)$ is non-limit or $\text{cf}(\ell g(\eta)) = \aleph_0$, then there are \triangleleft -incomparable $\nu_0, \nu_1 \in S$ extending η , i.e., $\eta \triangleleft \nu_0$ and $\eta \triangleleft \nu_1$.

[Why? As otherwise $S_\eta = \{\nu \in S : \eta \trianglelefteq \nu\}$ is linearly ordered by \triangleleft , so let $\rho = \bigcup S_\eta$. It follows from (*)₁ that $\ell g(\rho) > \ell g(\eta)$ is a limit ordinal (of uncountable cofinality). Moreover, by (iv)+(vi), we have that

$$\ell g(\eta) \leq \varepsilon < \ell g(\rho) \Rightarrow A \cap B_{\ell g(\eta), \rho \upharpoonright (\ell g(\eta)+1)} =^* A \cap B_{\varepsilon, \rho \upharpoonright (\varepsilon+1)}.$$

Hence, by (iii)+(ii), $\rho \in \mathcal{T}_{\ell g(\rho)}$ so necessarily $\ell g(\rho) < \mathfrak{b}$. Using (vi) again we may conclude that there is $\rho' \in S$ properly extending ρ , getting a contradiction.]

Consequently, we may find a system $\langle \eta_\rho : \rho \in {}^{\omega>2}$ $\subseteq S$ such that for every $\rho \in {}^{\omega>2}$:

- $k < \ell g(\rho) \Rightarrow \eta_{\rho \upharpoonright k} \triangleleft \eta_\rho$, and
- $\eta_{\rho \upharpoonright (0)}, \eta_{\rho \upharpoonright (1)}$ are \triangleleft -incomparable.

For $\rho \in {}^{\omega>2}$ let $\zeta(\rho) = \sup\{\ell g(\eta_\nu) : \rho \trianglelefteq \nu \in {}^{\omega>2}\}$. Pick ρ such that $\zeta(\rho)$ is the smallest possible (note that $\text{cf}(\zeta(\rho)) = \aleph_0$). Now it is possible to choose a perfect subtree T^* of ${}^{\omega>2}$ such that

$$\nu \in \lim(T^*) \Rightarrow \sup\{\ell g(\eta_{\nu \upharpoonright n}) : n < \omega\} = \zeta(\rho).$$

We finish by noting that for every $\nu \in \lim(T^*)$ we have that $\bigcup\{\eta_{\nu \upharpoonright n} : n < \omega\} \in \mathcal{T}_{\zeta(\rho)} \cap S$ and there is $\eta^* \in \mathcal{T}_{\zeta(\rho)+1} \cap S$ extending $\bigcup\{\eta_{\nu \upharpoonright n} : n < \omega\}$. \square

Theorem 2.2. $\mathfrak{g} \leq \mathfrak{b}^+$.

Proof. Assume towards contradiction that $\mathfrak{g} > \mathfrak{b}^+$.

Let $\langle f_\alpha : \alpha < \mathfrak{b} \rangle \subseteq {}^\omega \omega$ be an $\leq_{J_{\mathfrak{b}^d}}$ -increasing sequence with no $\leq_{J_{\mathfrak{b}^d}}$ -upper bound. We also demand that all functions f_α are increasing and $f_\alpha(n) > n$ for $n < \omega$. Fix a list $\langle \bar{m}_\xi : \xi < 2^{\aleph_0} \rangle$ of all sequences $\bar{m} = \langle m_i : i < \omega \rangle$ such that $0 = m_0$ and $m_i + 1 < m_{i+1}$.

For $\alpha < \mathfrak{b}$ we define:

- (*)₁ $n_{\alpha,0} = 0, n_{\alpha,i+1} = f_\alpha(n_{\alpha,i})$ (for $i < \omega$) and $\bar{n}_\alpha = \langle n_{\alpha,i} : i < \omega \rangle$;
- (*)₂ $\bar{n}_\alpha^0 = \langle 0, n_{\alpha,2}, n_{\alpha,4}, \dots \rangle = \langle n_{\alpha,i}^0 : i < \omega \rangle$ and $\bar{n}_\alpha^1 = \langle 0, n_{\alpha,3}, n_{\alpha,5}, n_{\alpha,7}, \dots \rangle = \langle n_{\alpha,i}^1 : i < \omega \rangle$.

Observe that

- (*)₃ if $\bar{m} \in {}^\omega \omega$ is increasing, then for every large enough $\alpha < \mathfrak{b}$ we have:
 - (α) $(\exists^\infty i < \omega)(m_{i+1} < f_\alpha(m_i))$, and hence

(β) for at least one $\ell \in \{0, 1\}$ we have

$$(\exists^\infty i < \omega)(\exists j < \omega)([m_i, m_{i+1}] \subseteq [n_{\alpha, j}^\ell, n_{\alpha, j+1}^\ell]).$$

Now, for $\xi < 2^{\aleph_0}$ we put:

- (*)₄ $\gamma(\xi) = \min\{\alpha < \mathfrak{b} : (\exists^\infty i < \omega)(f_\alpha(m_{\xi, i}) > m_{\xi, i+1})\}$;
- (*)₅ $\ell(\xi) = \min\{\ell \leq 1 : (\exists^\infty i < \omega)(\exists j < \omega)([m_{\xi, i}, m_{\xi, i+1}] \subseteq [n_{\gamma(\xi), j}^\ell, n_{\gamma(\xi), j+1}^\ell])\}$;
- (*)₆ $\mathcal{U}_\xi^1 = \{i < \omega : (\exists j < \omega)([m_{\xi, i}, m_{\xi, i+1}] \subseteq [n_{\gamma(\xi), j}^{\ell(\xi)}, n_{\gamma(\xi), j+1}^{\ell(\xi)}])\}$.

Note that $\gamma(\xi)$ is well defined by (α) of (*₃), and so also $\ell(\xi)$ is well defined (by (β) of (*₃)). Plainly, \mathcal{U}_ξ^1 is an infinite subset of ω . Now, for each $\xi < 2^{\aleph_0}$, we may choose \mathcal{U}_ξ^2 so that

- (*)₇ $\mathcal{U}_\xi^2 \subseteq \mathcal{U}_\xi^1$ is infinite and for any $i_1 < i_2$ from \mathcal{U}_ξ^2 we have

$$(\exists j < \omega)(m_{\xi, i_1+1} < n_{\gamma(\xi), j}^{\ell(\xi)} \ \& \ n_{\gamma(\xi), j+1}^{\ell(\xi)} < m_{\xi, i_2}).$$

Let a function $g_\xi : \mathcal{U}_\xi^2 \rightarrow \omega$ be such that

- (*)₈ $i \in \mathcal{U}_\xi^2 \ \& \ g_\xi(i) = j \Rightarrow [m_{\xi, i}, m_{\xi, i+1}] \subseteq [n_{\gamma(\xi), j}^{\ell(\xi)}, n_{\gamma(\xi), j+1}^{\ell(\xi)}]$.

Clearly, g_ξ is well defined and one-to-one. (This is very important, since it makes sure that the set $g_\xi[\mathcal{U}_\xi^2]$ is infinite.)

Fix a sequence $\bar{B} = \langle B_{\zeta, t} : \zeta < \theta, t \in I_\zeta \rangle$ given by Lemma 2.1 (so $\theta \leq \mathfrak{b}$ and \bar{B} satisfies the demands in (a)–(c) of 2.1). By clause 2.1(c), for every $\xi < 2^{\aleph_0}$, the set

$$\{(\zeta, t) : \zeta < \theta \text{ and } t \in I_\zeta \text{ and } B_{\zeta, t} \cap g_\xi[\mathcal{U}_\xi^2] \text{ is infinite} \}$$

has cardinality continuum.

Now, for each $\beta < \mathfrak{b}^+$ and $\xi < 2^{\aleph_0}$ we choose a pair $(\zeta_{\beta, \xi}, t_{\beta, \xi})$ such that

- (*)₉ $\zeta_{\beta, \xi} < \theta$ and $t_{\beta, \xi} \in I_{\zeta_{\beta, \xi}}$,
- (*)₁₀ $B_{\zeta_{\beta, \xi}, t_{\beta, \xi}} \cap g_\xi[\mathcal{U}_\xi^2]$ is infinite, and
- (*)₁₁ $t_{\beta, \xi} \notin \{t_{\alpha, \varepsilon} : \varepsilon < \xi \text{ or } \varepsilon = \xi \ \& \ \alpha < \beta\}$.

To carry out the choice we proceed by induction *first* on $\xi < 2^{\aleph_0}$, then on $\beta < \mathfrak{b}^+$. As there are 2^{\aleph_0} pairs (ζ, t) satisfying clauses (*₉) + (*₁₀) whereas clause (*₁₁) excludes $\leq (\mathfrak{b}^+ + |\xi|) \times \theta < 2^{\aleph_0}$ pairs (recalling that towards contradiction we are assuming $\mathfrak{b}^+ < \mathfrak{g} \leq 2^{\aleph_0}$), there is such a pair at each stage $(\beta, \xi) \in \mathfrak{b}^+ \times 2^{\aleph_0}$.

Lastly, for $\beta < \mathfrak{b}^+$ and $\xi < 2^{\aleph_0}$ we let

- (*)₁₂ $\mathcal{U}_{\beta, \xi} = g_\xi^{-1}[B_{\zeta_{\beta, \xi}, t_{\beta, \xi}}] \cap \mathcal{U}_\xi^2$

(it is an infinite subset of ω) and we put

- (*)₁₃ $A_{\beta, \xi}^+ = \bigcup\{[m_{\xi, i}, m_{\xi, i+1}] : i \in \mathcal{U}_{\beta, \xi}\}$, and
- (*)₁₄ $\mathcal{A}_\beta = \{A \in [\omega]^{\aleph_0} : \text{for some } \xi < 2^{\aleph_0} \text{ we have } A \subseteq A_{\beta, \xi}^+\}$.

By the choice of $\langle \bar{m}_\xi : \xi < 2^{\aleph_0} \rangle$, $A_{\beta, \xi}^+$ and \mathcal{A}_β one easily verifies that for each $\beta < \mathfrak{b}^+$:

- (*)₁₅ \mathcal{A}_β is a groupwise dense subset of $[\omega]^{\aleph_0}$.

Since we are assuming towards contradiction that $\mathfrak{g} > \mathfrak{b}^+$, there is an infinite $B \subseteq \omega$ such that

$$(\forall \beta < \mathfrak{b}^+)(\exists A \in \mathcal{A}_\beta)(B \subseteq^* A).$$

Hence for every $\beta < \mathfrak{b}^+$ we may choose $\xi(\beta) < 2^{\aleph_0}$ such that $B \subseteq^* A_{\beta, \xi(\beta)}^+$. Now, since $\gamma(\xi(\beta)) < \mathfrak{b}$ and $\zeta_{\beta, \xi(\beta)} < \theta \leq \mathfrak{b}$ and $\ell(\xi(\beta)) \in \{0, 1\}$, hence for some triple $(\gamma^*, \zeta^*, \ell^*)$ we have that

(\odot)₁ the set

$$W =: \{\beta < \mathfrak{b}^+ : (\gamma(\xi(\beta)), \zeta_{\beta, \xi(\beta)}, \ell(\xi(\beta))) = (\gamma^*, \zeta^*, \ell^*)\}$$

is unbounded in \mathfrak{b}^+ .

Note that if $\beta \in W$ then

$$\begin{aligned} (\odot)_2 \quad B \subseteq^* A_{\beta, \xi(\beta)}^+ &= \bigcup \{[m_{\xi(\beta), i}, m_{\xi(\beta), i+1}] : i \in \mathcal{U}_{\beta, \xi(\beta)}\} \subseteq \\ &\bigcup \{[n_{\gamma(\xi(\beta)), j}^{\ell(\xi(\beta))}, n_{\gamma(\xi(\beta)), j+1}^{\ell(\xi(\beta))}] : j = g_{\xi(\beta)}(i) \text{ for some } i \in \mathcal{U}_{\beta, \xi(\beta)}\} \subseteq \\ &\bigcup \{[n_{\gamma(\xi(\beta)), j}^{\ell(\xi(\beta))}, n_{\gamma(\xi(\beta)), j+1}^{\ell(\xi(\beta))}] : j \in B_{\zeta_{\beta, \xi(\beta)}, t_{\beta, \xi(\beta)}}\}. \end{aligned}$$

[Why? By the choice of $(\beta, \xi(\beta))$, by ($*$)₁₃, and by ($*$)₈ as $\text{Dom}(g_{\xi(\beta)}) \subseteq \mathcal{U}_{\beta, \xi(\beta)} \subseteq \mathcal{U}_{\beta, \xi(\beta)}^2$; also remember ($*$)₁₂.]

Also, for $\beta \in W$ we have $\ell(\xi(\beta)) = \ell^*$, $\gamma(\xi(\beta)) = \gamma^*$ and $\zeta(\beta, \xi(\beta)) = \zeta^*$, so it follows from (\odot)₂ that

$$(\odot)_3 \quad B \subseteq^* \bigcup \{[n_{\gamma^*, j}^{\ell^*}, n_{\gamma^*, j+1}^{\ell^*}] : j \in B_{\zeta^*, t_{\beta, \xi(\beta)}}\} \text{ for every } \beta \in W.$$

Consequently, if $\beta \neq \alpha$ are from W , then the sets

$$\bigcup \{[n_{\gamma^*, j}^{\ell^*}, n_{\gamma^*, j+1}^{\ell^*}] : j \in B_{\zeta^*, t_{\beta, \xi(\beta)}}\} \text{ and } \bigcup \{[n_{\gamma^*, j}^{\ell^*}, n_{\gamma^*, j+1}^{\ell^*}] : j \in B_{\zeta^*, t_{\alpha, \xi(\alpha)}}\}$$

are *not* almost disjoint. Hence, as $\langle n_{\gamma^*, j}^{\ell^*} : j < \omega \rangle$ is increasing, necessarily the sets $B_{\zeta^*, t_{\beta, \xi(\beta)}}$ and $B_{\zeta^*, t_{\alpha, \xi(\alpha)}}$ are not almost disjoint. So applying 2.1(b) we conclude that $t_{\beta, \xi(\beta)} = t_{\alpha, \xi(\alpha)}$. But this contradicts $\beta \neq \alpha$ by ($*$)₁₁, and we are done. \square

Definition 2.3. We define a cardinal characteristic \mathfrak{g}_f as the minimal cardinal θ for which there is a sequence $\langle \mathcal{I}_\alpha : \alpha < \theta \rangle$ of groupwise dense *ideals* of $\mathcal{P}(\omega)$ (i.e., $\mathcal{I}_\alpha \subseteq [\omega]^{\aleph_0}$ is groupwise dense and $\mathcal{I}_\alpha \cup [\omega]^{< \aleph_0}$ is an ideal of subsets of ω) such that

$$(\forall B \in [\omega]^{\aleph_0}) (\exists \alpha < \theta) (\forall A \in \mathcal{A}_\alpha) (B \not\subseteq^* A).$$

Observation 2.4. $2^{\aleph_0} \geq \mathfrak{g}_f \geq \mathfrak{g}$.

Theorem 2.5. $\mathfrak{g}_f \leq \mathfrak{b}^+$.

Proof. We repeat the proof of Theorem 2.2. However, for $\beta < \mathfrak{b}^+$ the family $\mathcal{A}_\beta \subseteq [\omega]^{\leq \aleph_0}$ does not have to be an ideal. So let \mathcal{I}_β be an ideal on $\mathcal{P}(\omega)$ generated by \mathcal{A}_β (so also \mathcal{I}_β is the ideal generated by $\{A_{\beta, \xi}^+ : \xi < 2^{\aleph_0}\} \cup [\omega]^{< \aleph_0}$). Lastly, let $\mathcal{I}'_\beta = \mathcal{I}_\beta \setminus [\omega]^{< \aleph_0}$.

Assume towards contradiction that $B \in [\omega]^{\aleph_0}$ is such that $(\forall \alpha < \mathfrak{b}^+) (\exists A \in \mathcal{I}_\alpha) (B \subseteq^* A)$. So for each $\beta < \mathfrak{b}^+$ we can find $k_\beta < \omega$ and $\xi(\beta, 0) < \xi(\beta, 1) < \dots < \xi(\beta, k_\beta) < 2^{\aleph_0}$ such that $B \subseteq^* \bigcup \{A_{\beta, \xi(\beta, k)}^+ : k \leq k_\beta\}$. Let D be a non-principal ultrafilter on ω to which B belongs. For each $\beta < \mathfrak{b}^+$ there is $k(\beta) \leq k_\beta$ such that $A_{\beta, \xi(\beta, k(\beta))}^+ \in D$. As in the proof there for some $(\gamma^*, \zeta^*, \ell^*, k^*, k^*)$ the following set is unbounded in \mathfrak{b}^+ :

$$W =: \{\beta < \mathfrak{b}^+ : k(\beta) = k^*, k_\beta = k^*, \gamma_{\xi(\beta, k^*)} = \gamma^*, \zeta_{\beta, \xi(\beta, k^*)} = \zeta^* \text{ and } \ell(\xi(\beta, k^*)) = \ell^*\}.$$

As there it follows that:

(\odot) if $\beta \in W$, then $\bigcup \{ [n_{\gamma^*, j}^{\ell^*}, n_{\gamma^*, j+1}^{\ell^*}) : j \in B_{\zeta^*, t_{\beta, \xi(\beta, k^*)}} \}$ belongs to D .

But for $\beta \neq \alpha \in W$ those sets are not almost disjoint whereas $(\zeta^*, t_{\beta, \xi(\beta, k^*)}) \neq (\zeta^*, t_{\alpha, \xi(\alpha, k^*)})$ are distinct, giving us a contradiction. \square

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