# ABSTRACT ELEMENTARY CLASSES NEAR $\aleph_1$
# SH88R

SAHARON SHELAH

Abstract. We prove in ZFC, that no $\psi \in \mathbb{L}_{\omega_1,\omega}[\mathbf{Q}]$ have unique model of uncountable cardinality, this confirms the Baldwin conjecture. But we analyze this in more general terms. We introduce and investigate a.e.c. and also versions of limit models, and prove some basic properties like representation by PC class, for any a.e.c. For $\mathrm{PC}_{\aleph_0}$-representable a.e.c. we investigate the conclusion of having not too many non-isomorphic models in $\aleph_1$ and $\aleph_2$, but have to assume $2^{\aleph_0} < 2^{\aleph_1}$ and even $2^{\aleph_1} < 2^{\aleph_2}$.





I would like to thank Alice Leonhardt for the beautiful typing.
This research was partially supported by the United States Israel Binational Science Foundation (BSF) and the NSF. First Typed - 04/May/18. Latest version - 2015/Jan/23; 2015/Jan/5.

## 0. Introduction

In [Sh:48], proving a conjecture of Baldwin, we show that ($\mathbf{Q}$ here stands for the quantifier $\mathbf{Q}^{\mathrm{car}}_{\geq\aleph_1}$, there are uncountably many)

$(*)_1$ no $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ has a unique uncountable model up to isomorphism

by showing that

$(*)_2$ categoricity (of $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$) in $\aleph_1$ implies the existence of a model of $\psi$ of cardinality $\aleph_2$ (so $\psi$ has $\geq 2$ non-isomorphism models).

Unfortunately, both $(*)_1$ and $(*)_2$ were not proved in ZFC because diamond on $\aleph_1$ was assumed. In [Sh:87a] and [Sh:87b] this set theoretic assumption was weakened to $2^{\aleph_0} < 2^{\aleph_1}$; here we shall prove it in ZFC (see §3). However, for getting the conclusion from the weaker model theoretic assumption $\dot{I}(\aleph_1, \psi) < 2^{\aleph_1}$ as there, we still need $2^{\aleph_0} < 2^{\aleph_1}$.

The main result of [Sh:87a], [Sh:87b] was:

$(*)_3$ if $n > 0, 2^{\aleph_0} < 2^{\aleph_1} < \ldots < 2^{\aleph_n}, \psi \in \mathbb{L}_{\omega_1,\omega}, 1 \leq \dot{I}(\aleph_\ell, \psi) < \mu_{\mathrm{wd}}(\aleph_\ell)$ for $\ell \leq n, \ell \geq 1$ (where $\mu_{\mathrm{wd}}(\aleph_\ell)$ is usually $2^{\aleph_\ell}$ and always $> 2^{\aleph_{\ell-1}}$, see 0.6 below) <u>then</u> $\psi$ has a model of cardinality $\aleph_{n+1}$

$(*)_4$ if $2^{\aleph_0} < 2^{\aleph_1} < \ldots < 2^{\aleph_n} < 2^{\aleph_{n+1}} < \ldots$ and $\psi \in \mathbb{L}_{\omega_1,\omega}, 1 \leq \dot{I}(\aleph_\ell, \psi) < \mu_{\mathrm{wd}}(\aleph_\ell)$ for $\ell < \omega$ <u>then</u> $\psi$ has a model in every infinite cardinal (and satisfies Łos Conjecture), (note that $(*)_3$ for $n = 1$, assuming $\diamondsuit_{\aleph_1}$ was proved in [Sh:48]).

In $(*)_4$, it is proved that without loss of generality $\mathfrak{K}$ is excellent; this means in particular that $K$ is the class of atomic models of some countable first order $T$. The point is that an excellent class $\mathfrak{K}$ is similar to the class of models of an $\aleph_0$-stable first order $T$. In particular the set of relevant types, $\mathbf{S}_{\mathfrak{K}}(A, M)$ is defined as $\{p(x) : p(x)$ a complete type over $A$ in $M$ in the first order sense such that $p \restriction B$ is isolated for every finite $B \subseteq A\}$. <u>But</u> we better restrict ourselves to "nice $A$", that is $A$ which is the universe of some $N \prec M$ or $A = N_1 \cup N_2$ where $N_0, N_1, N_2$ are in stable amalgamation or $\cup\{N_u : u \in \mathcal{P} \subseteq \mathcal{P}(n)\}$ for some (so called) stable system $\langle N_u : u \in \mathcal{P}\rangle$; on stable such systems in the stable first order case see [Sh:c, XII,§5]. So types are quite like the first order case. In particular we say $M \in \mathfrak{K}$ is $\lambda$-full <u>when</u>: if $p \in \mathbf{S}_{\mathfrak{K}}(A, M), A$ as above, $|A| < \lambda$ implies $p$ is realized in $M$; this is the replacement of $\lambda$-saturated for that context.

Why in [Sh:87a] and [Sh:87b], $\psi$ was assumed to be just in $\mathbb{L}_{\omega_1,\omega}$ and not more generally in $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$? Mainly because we feel that in [Sh:48], the logic $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ was incidental. We delay the search for the right context to this sequel. So here we are working in a.e.c., "abstract elementary class" (so no logic is present in the context) which are formally like elementary classes, i.e. $(\mathrm{Mod}_T, \prec), T$ first order but note the absence of amalgamation, still they have closure under union of increasing chains. It is $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ where $\leq_{\mathfrak{K}}$ is the "abstract" notion of elementary submodel. So if $\mathcal{L}$ is a fragment of $\mathbb{L}_{\infty,\omega}(\tau)$ (for a fixed vocabulary), $T \subseteq \mathcal{L}$ a theory included in $\mathcal{L}$, and we let $K = \{M : M \models T\}, M \leq_{\mathfrak{K}} N$ if and only if $M \prec_{\mathcal{L}} N$, we get such a class; if $\mathcal{L}$ is countable then $\mathfrak{K}$ has L.S.T. number $\aleph_0$. So the class of models of $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ is not represented directly, but can be with minor adaptation; see 3.19(2). Surprisingly (and by not so hard proof), every a.e.c. $\mathfrak{K}$ can be represented as a pseudo elementary class if we allow omitting types, (see 1.10). We introduce a relative of saturated models (for stable first order $T$) and full models (for excellent classes, see [Sh:87a] and [Sh:87b]): limit models; really several variants of this notion. See Definition 3.3. The strongest and most important variant is "$M \in K_\lambda$ superlimit" which means: $M$ is universal (under $\leq_{\mathfrak{K}}$), $(\exists N)(M \leq_{\mathfrak{K}} N \wedge M \neq N)$ and if $M_i \cong M$ for $i < \delta \leq \|M\|$ and $M_i$ is $\leq_{\mathfrak{K}}$-increasing then $\bigcup_{i<\delta} M_i \cong M$. If we restrict ourselves to $\delta$'s of cofinality $\kappa$ we get $(\lambda,\kappa)$-superlimit. Such $M$ exists for a first order $T$ for some pairs $\lambda,\kappa$. In particular (see more in [Sh:868])

$(*)_5$ for every $\lambda \geq 2^{|T|} + \beth_\omega$, a superlimit model of $T$ of cardinality $\lambda$ exists if and only if $T$ is superstable (by [Sh:868, 3.1]).

Moreover

$(*)_6$ "almost always"; for $\lambda \geq 2^{|T|} + \kappa, \kappa = \text{cf}(\kappa)$ (for simplicity) we have: a $(\lambda,\kappa)$-superlimit model exists iff $T$ is stable in $\lambda \& \kappa \geq \kappa(T)$ or $\lambda = \lambda^{<\kappa}$.

But we can prove something under those circumstances: if $K$ is categorical in $\lambda$ or just have a superlimit model $M^*$ in $\lambda$, but the $\lambda$-amalgamation property fails for $M^*$ and $2^\lambda < 2^{\lambda^+}$ then $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ (see 3.9). With some reasonable restrictions on $\lambda$ and $K$, we can prove e.g. $\dot{I}(\lambda, K) = \dot{I}(\lambda^+, K) = 1 \Rightarrow \dot{I}(\lambda^{++}, K) \geq 1$, (see 3.12, 3.14).

However, our long term main aim was to do the parallel of [Sh:87a] and [Sh:87b] in the present context, i.e., for an a.e.c. $\mathfrak{K}$ and it is natural to assume $\mathfrak{K}$ is $\text{PC}_{\aleph_0}$, here we prepare the ground.

Sections 4,5 present work toward this goal (§5 assuming $2^{\aleph_0} < 2^{\aleph_1}$; §4 without it). We should note that dealing with superlimit models rather than full ones make problems, as well as the fact that the class is not necessarily elementary in some reasonable logics. Because of the second we were driven to use formulas which hold "generically", are "forced" instead of are satisfied, and "the type $\bar{a}$ materialize" instead of realize and $\text{gtp}(\bar{a}, N, M)$ instead of $\text{tp}(\bar{a}, N, M)$. We also (necessarily) encounter the case "$\mathbf{D}(N)$ of cardinality $\aleph_1$ for $N \in K_{\aleph_0}$", see 5.2, 5.4(6). Because of the first, the scenario for getting a full model in $\aleph_1$ (which can be adapted to $(\aleph_1, \{\aleph_1\})$-superlimit - see 5.18) does not seem to be enough for getting superlimit models in $\aleph_1$ (see 5.45).

We had felt that arriving at enough conclusions on the models of cardinality $\aleph_1$ to start dealing with models of cardinality $\aleph_2$, will be a strong indication that we can complete the generalization of [Sh:87a] and [Sh:87b], so getting superlimits in $\aleph_1$ is the culmination of this paper and a natural stopping point. Trying to do the rest (of the parallel to [Sh:87a] and [Sh:87b]) was delayed.

Much remains to be done,

{88r-0.0}

0.1. **Problem.** 1) Prove $(*)_3, (*)_4$ in our context.
2) Parallel results in ZFC; e.g. prove $(*)_3$ for $n = 1, 2^{\aleph_0} = 2^{\aleph_1}$.
Note that if $2^{\aleph_0} = 2^{\aleph_1}$, assuming $1 \leq \dot{I}(\aleph_1, K) < 2^{\aleph_1}$ give really less model theoretic

consequences, as new phenomena arise (see §6). See §4 (and its concluding remarks).
3) Construct examples; e.g. (an a.e.c.) $\mathfrak{K}$ (or $\psi \in \mathbb{L}_{\omega_1,\omega}$), categorical in $\aleph_0, \aleph_1, \dots, \aleph_n$ but not in $\aleph_{n+1}$.
4) If $\mathfrak{K}$ is a $\mathrm{PC}_\lambda$ class, categorical in $\lambda, \lambda^+$, does it necessarily have a model in $\lambda^{++}$?

See the book's introduction [Sh:E53] on the progress on those problems in particular on [Sh:576], redone here in [Sh:46]. The direct motivation for [Sh:576] was that Grossberg asked me (Oct. 1994) some questions in this neighborhood (mainly 0.1(4)), in particular:

- $(*)$ assume $K = \mathrm{Mod}(T)$, (i.e. $K$ is the class of models of $T$), $T \subseteq L_{\omega_1,\omega}, |T| = \lambda, I(\lambda, K) = 1$ and $1 \le I(\lambda^+, K) < 2^{\lambda^+}$. Does it follow that $I(\lambda^{++}, K) > 0$?

We think of this as a test problem and much prefer a model theoretic to a set theoretic solution. This is closely related to 0.1(4) above and to 3.12 (where we assume categoricity in $\lambda^+$, do not require $2^\lambda < 2^{\lambda^+}$ <u>but</u> take $\lambda = \aleph_0$ or some similar cases) and 5.30(4) (and see 5.2 and 4.8 on the assumptions) (there we require $2^\lambda < 2^{\lambda^+}, 1 \le I(\lambda^+, K) < 2^{\lambda^+}$ <u>and</u> $\lambda = \aleph_0$).

Problem [Sh:576, 0.1] was stated a posteriori but is, I think, the real problem, it says:

- $(**)$ Can we have some (not necessarily much) classification theory for reasonable non-first order classes $\mathfrak{K}$ of models, with no uses of even traces of compactness and only mild set theoretic assumptions?

This is a revised version of [Sh:88] which continues [Sh:87a], [Sh:87b] but do not use them. The paper [Sh:88] and the present chapter relies on [Sh:48] only when deducing results on $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$; it improves some of its early results and extends the context. The work on [Sh:88] was done in 1977, and a preprint was circulated. Before the paper had appeared, a user-friendly expository article of Makowsky [Mak85] represent, give background and explain the easy parts of the paper. In [Sh:88] the author have corrected and replaced some proofs and added mainly §6. See more in [Sh:F709].

We thank Rami Grossberg for lots of work in the early eighties on previous versions, i.e. [Sh:88], which improved this paper, and the writing up of an earlier version of §6 and Assaf Hasson on helpful comments in 2002 and Alex Usvyatsov for very careful reading, corrections and comments and Adi Jarden and Alon Siton on help in the final stages.

* * *

On history and background on $\mathbb{L}_{\omega_1,\omega}, \mathbb{L}_{\infty,\omega}$ and the quantifier $\mathbf{Q}$ see [Kei71]. On $(D,\lambda)$-sequence-homogeneous (which 2.2 - 2.5 here generalized) see Keisler-Morley [KM67], this is defined in 2.3(5), and 2.5 is from there. Theorem 3.9 is similar to [Sh:87a, 2.7] and [Sh:87b, 6.3].

0.2. **Remark.** On non-splitting used here in 5.6 see [Sh:3], [Sh:c, Ch.I,Def.2.6,p.11] or [Sh:48].
We finish §0 by some necessary quotation.

By [Kei70] and [Mor70],

{88r-0.1}

0.3. **Claim.** *1) Assume that* $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ *has a model* $M$ *in which* $\{\mathrm{tp}_\Delta(\bar a, \emptyset, M) : \bar a \in M\}$ *is uncountable where* $\Delta \subseteq \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ *is countable, then* $\psi$ *has* $2^{\aleph_1}$ *pairwise non-isomorphic models of cardinality* $\aleph_1$*, in fact we can find models* $M_\alpha$ *of* $\psi$ *of cardinality* $\aleph_1$ *for* $\alpha < 2^{\aleph_1}$ *such that* $\{\mathrm{tp}_\Delta(a; \emptyset, M_\alpha) : a \in M_\alpha\}$ *are pairwise distinct where* $\mathrm{tp}_\Delta(\bar a, A, M) = \{\varphi(\bar x, \bar b) : \varphi(\bar x, \bar y) \in \Delta$ *and* $M \models \varphi[\bar a, \bar b]$ *and* $\bar b \in {}^{\omega>}A\}$.
*2) If* $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q}), \Delta \subseteq \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ *is countable and* $\{\mathrm{tp}_\Delta(\bar a, \emptyset, M) : \bar a \in {}^{\omega>}M$ *and* $M$ *is a model of* $\psi\}$ *is uncountable, then it has cardinality* $2^{\aleph_0}$.

Also note

{88r-0.9}

0.4. **Observation.** Assume ($\tau$ is a vocabulary and)

(a) $K$ is a family of $\tau$-models of cardinality $\lambda$

(b) $\mu > \lambda^\kappa$

(c) $\{(M, \bar a) : M \in K$ and $\bar a \in {}^\kappa M\}$ has $\geq \mu$ members up to isomorphism.

Then $K$ has $\geq \mu$ models up to isomorphisms (similarly for $= \mu$).

*Proof.* See [Sh:a, VIII,1.3] or just check by cardinal arithmetic. $\square_{0.4}$

Further

{88r-0.2}

0.5. **Claim.** *1) Assume* $\lambda$ *is regular uncountable,* $M_0$ *is a model with countable vocabulary and* $T = \mathrm{Th}_{\mathbb{L}}(M_0)$, $<$ *a binary predicate from* $\tau(T)$ *and* $(P^{M_0}, <^{M_0}) = (\lambda, <)$. *Then every countable model* $M$ *of* $T$ *has an end extension, i.e.,* $M \prec N$ *and* $P^M \neq P^N$ *and* $a \in P^N \wedge b \in P^M \wedge a <^N b \Rightarrow a \in M$.
*2) Moreover, we can further demand* $(P^N, <^N)$ *is non-well ordered and we can demand* $|P^N| = \aleph_1, (P^N, <^N)$ *is* $\aleph_1$*-like (which means that it has cardinality* $\aleph_1$ *but every (proper) initial segment has cardinality* $< \aleph_1$*); and we can demand* $N$ *is countable.*
*3) Moreover, we can add the demand that in* $(P^N, <^N)$ *there is a first element in* $P^N \backslash P^M$ *and we can add the demand: in* $(P^N, <^N)$*, there is no first element in* $P^N \backslash P^M$.

*Proof.* 1),2) Keisler [Kei70].
3) By [Sh:43] and independently Schmerl [Sch76]. $\square_{0.5}$

By Devlin-Shelah [DvSh:65], and [Sh:f, Ap,§1] (the so-called weak diamond).

{88r-0.wD

0.6. **Theorem.** *Assume that* $2^\lambda < 2^{\lambda^+}$.
*1) There is a normal ideal* $\mathrm{WDmId}_{\lambda^+}$ *on* $\lambda^+$ *and* $\lambda^+ \notin \mathrm{WDmId}_{\lambda^+}$*, of course, (the members are called small set) such that: if* $S \in (\mathrm{WDmId}_{\lambda^+})^+$ *(e.g.,* $S = \lambda^+$*) and* $\mathbf{c} : {}^{\lambda^+>}(\lambda^+) \to \{0,1\}$*, then there is* $\bar\ell = \langle \ell_\alpha : \alpha < \lambda^+\rangle \in {}^{\lambda^+}2$ *such that for every* $\eta \in {}^{\lambda^+}(\lambda^+)$ *the set* $\{\delta \in S : \mathbf{c}(\eta \restriction \delta) = \ell_\alpha\}$ *is stationary; we call* $\bar\ell$ *a weak diamond sequence (for the colouring*

**c** *and the stationary set* $S$*).*
*2)* $\mu_* = \mu_{\rm wd}(\lambda^+)$*, the cardinal defined by* $(*)$ *below, is* $> 2^\lambda$ *(we do not say* $\geq 2^{\lambda^+}$*!)*

$(*)$ $(\alpha)$ *if* $\mu < \mu_*$ *and* $\mathbf{c}_\varepsilon$ *for* $\varepsilon < \mu$ *is as above then we can find* $\bar\ell$ *as in part (1) for all the* $\mathbf{c}_\varepsilon$*'s simultaneously*

$(\beta)$ $\mu_*$ *is maximal such that clause* $(\alpha)$ *holds.*

*3)* $\mu_* = \mu_{unif}(\lambda^+, 2^\lambda)$ *satisfies* $\mu_*^{\aleph_0} = 2^{\lambda^+}$ *and moreover* $\lambda \geq \beth_\omega \Rightarrow \mu_* = 2^\lambda$ *where* $\mu_{unif}(\lambda^+, \chi)$ *is the first cardinal* $\mu$ *such that we can find* $\langle \mathbf{c}_\alpha : \alpha < \mu\rangle$ *such that:*

- $(a)$ $\mathbf{c}_\alpha$ *is a function from* ${}^{\lambda^+ >}(\lambda^+)$ *to* $\chi$
- $(b)$ *there is no* $\rho \in {}^{\lambda^+}\chi$ *such that for every* $\alpha < \mu$ *for some* $\eta \in {}^{\lambda^+}(\lambda^+)$ *the set* $\{\delta < \lambda : \mathbf{c}_\alpha(\eta \restriction \delta) \neq \rho(\delta)\}$ *is stationary (so* $\mu_{\rm wd}(\lambda^+) = \mu_{unif}(\lambda^+, 2)$*).*

*See more in* [Sh:838, §0,§9] *and hopefully in* [Sh:E45].

The following are used in §2.

{88r-0.5}

0.7. **Definition.** 1) For a regular uncountable cardinal $\lambda$ let $\check{I}[\lambda] = \{S \subseteq \lambda$: some pair $(E, \bar a)$ witnesses $S \in \check{I}(\lambda)$, see below$\}$.
2) We say that $(E, u)$ is a witness for $S \in \check{I}[\lambda]$ <u>if</u>:

- $(a)$ $E$ is a club of the regular cardinal $\lambda$
- $(b)$ $u = \langle u_\alpha : \alpha < \lambda\rangle, a_\alpha \subseteq \alpha$ and $\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha$
- $(c)$ for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of $\delta$ of order-type $< \delta$ (and $\delta$ is a limit ordinal).

By [Sh:420] and [Sh:E12]

{88r-0.6}

0.8. **Claim.** *Let* $\lambda$ *be regular uncountable.*
*1) If* $S \in \check{I}[\lambda]$ <u>*then*</u> *we can find a witness* $(E, \bar a)$ *for* $S \in \check{I}[\lambda]$ *such that:*

- $(a)$ $\delta \in S \cap E \Rightarrow \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta)$
- $(b)$ *if* $\alpha \notin S$ *then* $\mathrm{otp}(a_\alpha) < \mathrm{cf}(\delta)$ *for some* $\delta \in S \cap E$*.*

*2)* $S \in \check{I}[\lambda]$ <u>*iff*</u> *there is a pair* $(E, \bar{\mathcal{P}})$ *such that:*

- $(a)$ $E$ *is a club of the regular uncountable* $\lambda$
- $(b)$ $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$*, where* $\mathcal{P}_\alpha \subseteq \{u : u \subseteq \alpha\}$ *has cardinality* $< \lambda$
- $(c)$ *if* $\alpha < \beta < \lambda$ *and* $\alpha \in u \in \mathcal{P}_\beta$ *then* $u \cap \alpha \in \mathcal{P}_\alpha$
- $(d)$ *if* $\delta \in E \cap S$ *then some* $u \in \mathcal{P}_\delta$ *is an unbounded subset of* $\delta$ *(and* $\delta$ *is a limit ordinal).*

## 1. Axioms and simple properties for classes of models

{88r-1.1}

1.1. **Context.** 1) Here in §1-§5, $\tau$ is a vocabulary, $K$ will be a class of $\tau$-models and $\le_{\mathfrak{K}}$ a two-place relation on the models in $K$. We do not always strictly distinguish between $K$ and $\mathfrak{K} = (K, \le_{\mathfrak{K}})$. We shall assume that $K, \le_{\mathfrak{K}}$ are fixed; and usually we assume that $\mathfrak{K}$ is an a.e.c. (abstract elementary class) which means that the following axioms hold.
2) For a logic $\mathcal{L}$ let $M \prec_{\mathcal{L}} N$ mean $M$ is an elementary submodel of $N$ for the language $\mathcal{L}(\tau_M)$ and $\tau_M \subseteq \tau_N$, i.e., if $\varphi(\bar{x}) \in \mathcal{L}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ then $M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}]$; similarly $M \prec_L N$ for $L$ a language, i.e. a set of formulas in some $\mathcal{L}(\tau_M)$. So $M \prec N$ in the usual sense means $M \prec_{\mathbb{L}} N$ as $\mathbb{L}$ is first order logic and $M \subseteq N$ means $M$ is a submodel of $N$.

{88r-1.2}

1.2. **Definition.** 1) We say $\mathfrak{K}$ is a a.e.c. with L.S.T. number $\lambda(\mathfrak{K}) =$ LST$(\mathfrak{K})$ <u>if</u>:
<u>Ax 0</u>: The holding of $M \in K, N \le_{\mathfrak{K}} M$ depend on $N, M$ only up to isomorphism, i.e. $[M \in K, M \cong N \Rightarrow N \in K]$ and [if $N \le_{\mathfrak{K}} M$ and $f$ is an isomorphism from $M$ onto the $\tau$-model $M', f \restriction N$ is an isomorphism from $N$ onto $N'$ <u>then</u> $N' \le_{\mathfrak{K}} M'$].

<u>Ax I</u>: if $M \le_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. $M$ is a submodel of $N$).

<u>Ax II</u>: $M_0 \le_{\mathfrak{K}} M_1 \le_{\mathfrak{K}} M_2$ implies $M_0 \le_{\mathfrak{K}} M_2$ and $M \le_{\mathfrak{K}} M$ for $M \in K$.

<u>Ax III</u>: If $\lambda$ is a regular cardinal, $M_i (i < \lambda)$ is a $\le_{\mathfrak{K}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \le_{\mathfrak{K}} M_j$) and continuous (i.e. for $\delta < \lambda, M_\delta = \bigcup\limits_{i<\delta} M_i$) <u>then</u> $M_0 \le_{\mathfrak{K}} \bigcup\limits_{i<\lambda} M_i$.

<u>Ax IV</u>: If $\lambda$ is a regular cardinal and $M_i$ (for $i < \lambda$) is $\le_{\mathfrak{K}}$-increasing continuous and $M_i \le_{\mathfrak{K}} N$ for $i < \lambda$ <u>then</u> $\bigcup\limits_{i<\lambda} M_i \le_{\mathfrak{K}} N$.

<u>Ax V</u>: If $N_0 \subseteq N_1 \le_{\mathfrak{K}} M$ and $N_0 \le_{\mathfrak{K}} M$ <u>then</u> $N_0 \le_{\mathfrak{K}} N_1$.

<u>Ax VI</u>: If $A \subseteq N \in K$ and $|A| \le$ LST$(\mathfrak{K})$ then for some $M \le_{\mathfrak{K}} N, A \subseteq |M|$ and $\|M\| \le$ LST$(\mathfrak{K})$ (and LST$(\mathfrak{K})$ is the minimal infinite cardinal satisfying this axiom which is $\ge |\tau|$; the $\ge |\tau|$ is for notational simplicity).
2) We say $\mathfrak{K}$ is a weak [1] a.e.c. <u>if</u> above we omit clause IV.

1.3. **Remark.** Note that AxV holds for $\prec_{\mathcal{L}}$ for any logic $\mathcal{L}$.

<u>Notation</u>: Let $K_\lambda = \{M \in K : \|M\| = \lambda\}$ and $K_{<\lambda} = \bigcup\limits_{\mu<\lambda} K_\mu$ and $\mathfrak{K}_\lambda = (K_\lambda, \le_{\mathfrak{K}} \restriction K_\lambda)$ and similarly $\mathfrak{K}_{<\lambda}, K_{\le\lambda}, \mathfrak{K}_{\ge\lambda}, K_{\ge\lambda}$. Recall $\mathbb{L}$ is first order logic.

[1] this is not really investigated here

1.4. **Definition.** The embedding $f : N \to M$ is called a $\leq_{\mathfrak{K}}$-embedding if the range of $f$ is the universe of a model $N' \leq_{\mathfrak{K}} M$ (so $f : N \to N'$ is an isomorphism onto). {88r-1.3}

{88r-1.4}

1.5. **Definition.** Let $T_1$ be a theory in $\mathcal{L}(\tau_1), \Gamma$ a set of types in $\mathcal{L}(\tau_1)$ for some logic $\mathcal{L}$, usually first order.
1) $\mathrm{EC}(T_1, \Gamma) = \{M : M$ an $\tau_1$-model of $T_1$ which omits every $p \in \Gamma\}$.
We implicitly use that $\tau_1$ is reconstructible from $T_1, \Gamma$. A problem may arise only if some symbols from $\tau_1$ are not mentioned in $T_1$ and in $\Gamma$, so we may write $\mathrm{EC}(T_1, \Gamma, \tau_1)$, but usually we ignore this point.
2) For $\tau \subseteq \tau_1$ we let $\mathrm{PC}(T_1, \Gamma, \tau) = \mathrm{PC}_\tau(T_1, \Gamma) = \{M : M$ is a $\tau$-reduct of some $M_1 \in \mathrm{EC}(T_1, \Gamma)\}$.
3) We say that $K$, a class of $\tau$-models, is a $\mathrm{PC}^\mu_\lambda$ or $\mathrm{PC}_{\lambda,\mu}$ class when for some $T_1, \Gamma_1, \tau_1$ we have $\tau \subseteq \tau_1, T_1$ a first order theory in the vocabulary $\tau_1, \Gamma_1$ a set of types in $\mathbb{L}(\tau_1), K = \mathrm{PC}_\tau(T_1, \Gamma_1)$ and $|T_1| \leq \lambda, |\Gamma_1| \leq \mu$.
4) We say $\mathfrak{K}$ is $\mathrm{PC}^\mu_\lambda$ or $\mathrm{PC}_{\lambda,\mu}$ if for some $(T_1, \Gamma_1, \tau_1), (T_2, \Gamma_2, \tau_2)$ as in part (3) we have $K = \mathrm{PC}(T_1, \Gamma_1, \tau)$ and $\{(M, N) : M \leq_{\mathfrak{K}} N$ hence $M, N \in K\} = \mathrm{PC}(T_2, \Gamma_2, \tau')$ where $\tau' = \tau \cup \{P\} \subseteq \tau_2, P$ a new one-place predicate, so $|\tau_\ell| \leq \lambda, |\Gamma_\ell| \leq \mu$ for $\ell = 1, 2$.
If $\mu = \lambda$ we may omit $\mu$.
5) In (4) we may say "$\mathfrak{K}$ is $(\lambda, \mu)$-presentable" and if $\lambda = \mu$ we may say "$\mathfrak{K}$ is $\lambda$-presentable".

{88r-1.5}

1.6. **Example.** If $T \subseteq \mathbb{L}(\tau), \Gamma$ a set of types in $\mathbb{L}(\tau)$, then $K := \mathrm{EC}(T, \Gamma), \leq_{\mathfrak{K}} := \prec_{\mathbb{L}}$ form an a.e.c. with LST-number $\leq |T| + |\tau| + \aleph_0$, that is, satisfy the Axioms from 1.2 (for $\mathrm{LST}(\mathfrak{K}) := |\tau| + \aleph_0$).

{88r-1.6}

1.7. **Observation.** Let $I$ be a directed set (i.e. partially ordered by $\leq$, such that any two elements have a common upper bound).
1) If $M_t$ is defined for $t \in I$ and $t \leq s \in I$ implies $M_t \leq_{\mathfrak{K}} M_s$ then $\bigcup_{s \in I} M_s \in K$ and for every $t \in I$ we have $M_t \leq_{\mathfrak{K}} \bigcup_{s \in I} M_s$.
2) If in addition $t \in I$ implies $M_t \leq_{\mathfrak{K}} N$ then $\bigcup_{s \in I} M_s \leq_{\mathfrak{K}} N$.

*Proof.* By induction on $|I|$ (simultaneously for (1) and (2)).

If $I$ is finite, then $I$ has a maximal element $t(0)$, hence $\bigcup_{t \in I} M_t = M_{t(0)}$, so there is nothing to prove.
So suppose $|I| = \mu$ and we have proved the assertion when $|I| < \mu$. Let $\lambda = \mathrm{cf}(\mu)$ so $\lambda$ is a regular cardinal; hence we can find $I_\alpha$ (for $\alpha < \lambda$) such that $|I_\alpha| < |I|, \alpha < \beta < \lambda$

implies $I_\alpha \subseteq I_\beta \subseteq I, \bigcup_{\alpha<\lambda} I_\alpha = I$, for limit $\delta < \lambda, I_\delta = \bigcup_{\alpha<\delta} I_\alpha$ and each $I_\alpha$ is directed and non-empty; this is trivial when $\lambda > \aleph_0$ and obvious otherwise. Let $M^\alpha = \bigcup_{t \in I_\alpha} M_t$; so by the induction hypothesis on (1) we know that $t \in I_\alpha$ implies $M_t \leq_{\mathfrak{K}} M^\alpha$. If $\alpha < \beta$ then $t \in I_\alpha$ implies $t \in I_\beta$ hence $M_t \leq_{\mathfrak{K}} M^\beta$; hence by the induction hypothesis on (2) applied to $\langle M_t : t \in I_\alpha \rangle, M_\beta$ we have $M^\alpha = \bigcup_{t \in I_\alpha} M_t \leq_{\mathfrak{K}} M^\beta$. So by Ax III, applied to $\langle M^\alpha : \alpha < \lambda \rangle$ we have $M^\alpha \leq_{\mathfrak{K}} \bigcup_{\beta<\lambda} M^\beta = \bigcup_{t \in I} M_t$, and as $t \in I_\alpha$ implies $M_t \leq_{\mathfrak{K}} M^\alpha$, by Ax II, $t \in I$ implies $M_t \leq_{\mathfrak{K}} \bigcup_{s \in I} M_s$. So we have finished proving part (1) for the case $|I| = \mu$. To prove (2) in this case note that for each $\alpha < \lambda, \langle M_t : t \in I_\alpha \rangle$ is $\leq_{\mathfrak{K}}$-directed and $t \in I_\alpha \Rightarrow M_t \leq_{\mathfrak{K}} N$, so clearly by the induction hypothesis for (2) we have $M^\alpha := \cup\{M_t : t \in I_\alpha\}$ is $\leq_{\mathfrak{K}} N$. So $\alpha < \lambda \Rightarrow M^\alpha \leq_{\mathfrak{K}} N$ and as proved above $\langle M^\alpha : \alpha < \lambda \rangle$ is $\leq_{\mathfrak{K}}$-increasing and obviously it is continuous, hence by Ax IV, $\bigcup_{s \in I} M_s = \bigcup_{\alpha<\lambda} M^\alpha \leq_{\mathfrak{K}} N$. $\square_{1.7}$

{88r-1.7}

1.8. **Lemma.** *Let $\tau_1 = \tau \cup \{F_i^n : i < \mathrm{LST}(\mathfrak{K}), n < \omega\}, F_i^n$ an $n$-place function symbol (assuming, of course, $F_i^n \notin \tau$).*

*Every model $M$ (in $K$) can be expanded to an $\tau_1$-model $M_1$ such that:*

(A) *$M_{\bar{a}} \leq_{\mathfrak{K}} M$ when $n < \omega, \bar{a} \in {}^n|M|$ and where $M_{\bar{a}}$ is the submodel of $M$ with universe $\{F_i^n(\bar{a}) : i < \mathrm{LST}(\mathfrak{K})\}$*

(B) *if $\bar{a} \in {}^n|M|$ then $\|M_{\bar{a}}\| \leq \mathrm{LST}(\mathfrak{K})$*

(C) *if $\bar{b}$ is a subsequence of a permutation of $\bar{a}$, then $M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}}$*

(D) *for every $N_1 \subseteq M_1$ we have $N_1 \restriction \tau \leq_{\mathfrak{K}} M$.*

*Proof.* We define by induction on $n$, the values of $M_{\bar{a}}$ and of $F_i^n(\bar{a})$ for every $i < \mathrm{LST}(\mathfrak{K}), \bar{a} \in {}^n|M|$ such that $F_i^n$ is symmetric, i.e. preserved under permuting its variables. Arriving to $n$, for each $\bar{a} \in {}^nM$ by Ax VI there is an $M_{\bar{a}} \leq_{\mathfrak{K}} M$ such that $\|M_{\bar{a}}\| \leq \mathrm{LST}(\mathfrak{K}), |M_{\bar{a}}|$ include $\cup\{M_{\bar{b}} : \bar{b}$ a subsequence of $\bar{a}$ of length $< n\} \cup \bar{a}$ and $M_{\bar{a}}$ does not depend on the order of $\bar{a}$. Let $|M_{\bar{a}}| = \{c_i : i < i_0 \leq \mathrm{LST}(\mathfrak{K})\}$ and define $F_i^n(\bar{a}) = c_i$ for $i < i_0$ and $c_0$ for $i_0 \leq i < \mathrm{LST}(\mathfrak{K})$.

Clearly our conditions are satisfied; in particular, if $\bar{b}$ is a subsequence of $\bar{a}, M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}}$ by Ax V and clause (D) holds by 1.7 and Ax IV. $\square_{1.8}$

{88r-1.7A

1.9. **Remark.** 1) This is the "main" place we use Ax V,VI; it seems that we use it rarely, e.g., in 2.12 which is not used later. It is clear that we can omit Ax V if we strengthen somewhat Ax VI for the proofs above.

2) Note that in 1.8, we do not require that $M_{\bar a}$ is closed under the functions $(F^n_i)^{M_1}$. By a different bookkeeping we can have it: renaming $\tau_{1,\varepsilon} = \tau \cup \{F^n_i : i < \mathrm{LST}(\mathfrak{K}) \times \varepsilon, n < \omega\}$ for $\varepsilon \le \omega$ and we choose a $\tau_{1,n}$-expansion $M_{1,n}$ of $M$ such that $m < n \Rightarrow M_{1,n} \restriction \tau_{1,m} = M_{1,m}$. Let $M_{1,0} = M$, and if $M_{1,n}$ is defined, choose for every $\bar a \in {}^{\omega>}(M_{1,n})$ a (non-empty) subset $A^{1,n}_{\bar a}$ of $M_{1,n}$ of cardinality $\le \mathrm{LST}(\mathfrak{K})$ such that $A^{1,n}_{\bar a}$ is closed under the functions of $M_{1,n}$ and $M \restriction A^{1,n}_{\bar a} \le_{\mathfrak{K}} M$, let $A^{1,n}_{\bar a} = \{c_{\bar a,i} : i \in [\mathrm{LST}(\mathfrak{K}) \times n, \mathrm{LST}(\mathfrak{K}) \times (n+1))$ and define $M_{1,n+1}$ by letting $(F^m_i)^{M_{1,n+1}}(\bar a) = c_{\bar a,i}$. Let $M_1 = M_{1,\omega}$ be the $\tau_\omega$-model with the universe of $M$ such that $n < \omega \Rightarrow M_1 \restriction \tau_{1,n} = M_{1,n}$.
3) Actually $M_{1,1}$ suffices if we expand it by making every term $\tau(\bar x)$ equal to some function $F(\bar x)$.
4) Alternatively demand for $n > 0$ that $F^n_i(\bar a)$ is $F^{|u|}_i(\bar a \restriction u), u = \{i < n : a_i \notin \{a_j : j < i\}$.

{88r-1.8}

1.10. **Lemma.** *1) $\mathfrak{K}$ is $(\mathrm{LST}(\mathfrak{K}), 2^{LST(\mathfrak{K})})$-presentable.*
*2) There is a set $\Gamma$ of types in $\mathbb{L}(\tau_1)$ in fact complete quantifier free (where $\tau_1$ is from Lemma 1.8) such that $K = \mathrm{PC}_\tau(\emptyset, \Gamma)$.*
*3) For the $\Gamma$ from part (2), if $M_1 \subseteq N_1 \in \mathrm{EC}(\emptyset, \Gamma)$ and $M, N$ are the $\tau$-reducts of $M_1, N_1$ respectively then $M \le_{\mathfrak{K}} N$.*
*4) For the $\Gamma$ from part (2), we have $\{(M, N) : M \le_{\mathfrak{K}} N$ so $N, M \in K\} = \{(M_1 \restriction \tau, N_1 \restriction \tau) : M_1 \subseteq N_1$ are both from $\mathrm{PC}_\Gamma(\emptyset, \Gamma)\}$.*

*Proof.* 1) By part (2) the first half of "$\mathfrak{K}$ is $(\mathrm{LST}(\mathfrak{K}), 2^{\mathrm{LST}(\mathfrak{K})})$-presentable holds". The second part will be proved with part (4).
2) Let $\Gamma_n$ be the set of complete quantifier free $n$-types $p(x_0, \dots, x_{n-1})$ in $\mathbb{L}(\tau_1)$ such that: if $M_1$ is a $\tau_1$-model, $\bar a$ realizes $p$ in $M_1$ and $M$ is the $\tau$-reduct of $M_1$, then $M_{\bar a} \in K$ and $M_{\bar b} \le_{\mathfrak{K}} M_{\bar a}$ for any subsequence $\bar b$ of any permutation of $\bar a$; where $M_{\bar c}(\bar c \in {}^m|M_1|)$ is the submodel of $M$ whose universe is $\{F^m_i(\bar c) : i < \mathrm{LST}(\mathfrak{K})\}$. Clearly there are such submodels (when $K \ne \emptyset$).

Let $\Gamma$ be the set of $p$ which, for some $n$, are complete quantifier free $n$-types (in $\mathbb{L}(\tau_1)$) which do not belong to $\Gamma_n$. By 1.7(1) we have $\mathrm{PC}_\tau(\emptyset, \Gamma) \subseteq K$ and by 1.8 $K \subseteq \mathrm{PC}_\tau(\emptyset, \Gamma)$.
3) Similar to the proof of (2) using 1.7(2).
4) The inclusion $\supseteq$ holds by part (3); so let us prove the other direction. Given $N \le_{\mathfrak{K}} M$ we apply the proof of 1.8 to $M$, but demand further $\bar a \in {}^nN \Rightarrow M_{\bar a} \subseteq N$; simply add this demand to the choice of the $M_{\bar a}$'s (hence of the $F^n_i$'s). We still have a debt from part (1).

We let $\Gamma'_n$ be the set of complete quantifier free $n$-types in $\tau'_1 := \tau_1 \cup \{P\}$ ($P$ a new unary predicate), $p(x_0, \dots, x_{n-1})$ such that:

- $(*)$ if $M_1$ is an $\tau'_1$-model, $\bar a$ realizes $p$ in $M_1, M$ the $\tau$-reduct of $M_1$, <u>then</u>
  - $(\alpha)$ $M_{\bar b} \le_{\mathfrak{K}} M_{\bar a}$ for any subsequence $\bar b$ of $\bar a$ where $M_{\bar c}$ (for $\bar c \in |M_1|$) is the submodel of $M$ whose universe is $\{(F^m_i)^{M_1}(\bar c) : i < \mathrm{LST}(\mathfrak{K})\}$, where $m = \ell g(\bar c)$ (and there are such models),
  - $(\beta)$ $\bar b \subseteq P^{M_1} \Rightarrow M_{\bar b} \subseteq P^{M_1}$ for $\bar b \subseteq \bar a$.

We leave the rest to the reader (alternatively, use $\mathrm{PC}_{\tau_1'}(T', \Gamma), T'$ saying "$P$ is closed under all the functions $F_i^n$). $\square_{1.10}$

By the proof of 1.10(4).

{88r-1.9}

1.11. **Conclusion.** The $\tau_1$ and $\Gamma$ from 1.10 (so $|\tau_1| \leq \mathrm{LST}(\mathfrak{K})$) satisfy: for any $M \in K$ and any $\tau_1$-expansion $M_1$ of $M$ which is in $\mathrm{EC}_{\tau_1}(\emptyset, \Gamma)$

(a) $N_1 \prec_{\mathbb{L}} M_1 \Rightarrow N_1 \subseteq M_1 \Rightarrow N_1 \restriction \tau \leq_{\mathfrak{K}} M$

(b) $N_1 \prec_{\mathbb{L}} N_2 \prec_{\mathbb{L}} M_1 \Rightarrow N_1 \subseteq N_2 \subseteq M_1 \Rightarrow N_1 \restriction \tau \leq_{\mathfrak{K}} N_2 \restriction \tau$

(c) if $M \leq_{\mathfrak{K}} N$ then there is a $\tau_1$-expansion $N_1$ of $N$ from $\mathrm{EC}_{\tau_1}(\emptyset, \Gamma)$ which extends $M_1$.

{88r-1.10

1.12. **Conclusion.** If for every $\alpha < (2^{\mathrm{LST}(\mathfrak{K})})^+, \mathfrak{K}$ has a model of cardinality $\geq \beth_\alpha$ <u>then</u> $K$ has a model in every cardinality $\geq \mathrm{LST}(\mathfrak{K})$.

*Proof.* Use 1.10 and the classical upper bound on value of the Hanf number for: first order theory and omitting any set of types, for languages of cardinality $\mathrm{LST}(\mathfrak{K})$ (see, e.g., [Sh:c, VII,5.3,5.5]). $\square_{1.12}$

{88r-1a.1

1.13. **Conclusion.** Assume that $\mathfrak{K}$ is an a.e.c., $\mu = |\tau_{\mathfrak{K}}| + \mathrm{LST}(\mathfrak{K})$ and for simplicity $\tau_{\mathfrak{K}} \subseteq \mu$ or just $\tau_{\mathfrak{K}} \subseteq \mathbf{L}_\mu$, recalling $\mathbf{L}$ is the constructible universe of Göbel. If $\lambda > \mu$ and $\mathfrak{A} \prec (\mathcal{H}(\chi, \in)$ and $\mu + 1 \subseteq \mathfrak{A}$ and $\mathfrak{K} \in \mathfrak{A}$ which means $\{(M,N) : M \leq_{\mathfrak{K}} N$ has universe $\subseteq \mu\} \in \mathfrak{A}$ <u>then</u>:

(a) $M \in \mathfrak{K} \cap K \Rightarrow M \restriction \mathfrak{A} \leq_{\mathfrak{K}} M$

(b) if $M \leq_{\mathfrak{K}} N$ so both belongs to $K$ and $M, N \in \mathfrak{A}$ then $M \restriction \mathfrak{A} \leq_{\mathfrak{K}} N \restriction \mathfrak{A}$

(c) if $\mathfrak{A} \prec \mathfrak{B}$ and $[b <_{\mathfrak{B}} \mu \Rightarrow b \in \mathfrak{A}]$ and $\mathfrak{B} \models$ "$M \in K$" <u>then</u> $M[\mathfrak{B}] \in K$

(d) similarly for $\mathfrak{B} \models$ "$M \leq_{\mathfrak{K}} N$"
where

$(*)_1$ if $M \in \mathfrak{A}$ then $M \restriction \mathfrak{A}$ is the submodel of $M$ with universe $|M| \cap |\mathfrak{A}|$

$(*)_2$ if $\mathfrak{B} \models$ "$M \in \mathfrak{K}$" then $M[\mathfrak{B}]$ is the following $\tau_K$-model:

(a) it has universe $\{b \in \mathfrak{B} : \mathfrak{B} \models$ "$b$ an element of the model $M$"$\}$

(b) for any $m$-place predicate $Q$ of $\tau, Q^M = \{\langle b_0, \dots, b_{m-1}\rangle : \mathfrak{B} \models$ "$M \models Q[b_0, \dots, b_{m-1}]$"$\}$

(c) for any $m$-place function symbol $G$ of $\tau$, similarly.

*Proof.* Should be clear. $\square_{1.13}$

1.14. **Remark.** 1) Clearly $\{\mu : \mu \geq \text{ LST}(\mathfrak{K}) \text{ and } K_\mu \neq 0\}$ is an initial segment of the class of cardinals $\geq$ LST$(\mathfrak{K})$.
2) For every cardinal $\kappa(\geq \aleph_0)$ and ordinal $\alpha < (2^\kappa)^+$ there is an a.e.c. $\mathfrak{K}$ such that: LST$(\mathfrak{K}) = \kappa = |\tau_{\mathfrak{K}}|$ and $\mathfrak{K}$ has a model of cardinality $\lambda$ iff $\lambda \in [\kappa, \beth_\alpha(\kappa))$. This follows by [Sh:c, VII,§5,p.432] in particular [Sh:c, VII,5.5](6), because

- $(a)$ if a vocabulary of cardinality $\leq \kappa$ and $T \subseteq \mathbb{L}(\tau)$ and $\Gamma$ a set of $(\mathbb{L}(\tau), < \omega)$-types then $K = \{M : M \text{ a } \tau\text{-model of } T \text{ omitting every } \in \Gamma\}$ and $\leq_{\mathfrak{K}} = \prec \restriction K$ form an a.e.c. (we can use $\Gamma$ a set of quantifier free types, $T = \emptyset$), with LST$((\mathfrak{K}, \leq_{\mathfrak{K}}) \leq \kappa$
- $(b)$ if $\{c_i \neq c_j : i < j < \kappa\} \subseteq T$ then $K$ above has no model of cardinality $< \kappa$.

3) More on such theorems see [Sh:394].
4) We can phrase 1.13 "for any $\mathfrak{B}$ in appropriate EC$(T_1, \Gamma_1)$", but the present formulation is the way we use it.

## 2. Amalgamation properties and homogeneity

{88r-2.0}

2.1. **Context.** $\mathfrak{K}$ is an a.e.c.

The main theorem 2.9, the existence and uniqueness of the model-homogeneous models, is a generalization of Jonsson [Jón56], [Jón60] to the present context. The result on the upper bound $2^{2^{\aleph_0+|\tau|}}$ for the number of $D$-sequence homogeneous universal-models of cardinality is of Keisler-Morley [KM67]. Earlier there were serious good reasons to concentrate on sequence-homogeneous models, but here we deal with the model-homogeneous case. From 2.14 to the end we consider what we can say when we omit smoothness, i.e. AxIV of Definition 1.2.

{88r-2.1}

2.2. **Definition.** 1) $\mathbb{D}(M) := \{N/\cong : N \le_{\mathfrak{K}} M, \|N\| \le \text{ LST}(\mathfrak{K})\}$.
2) $\mathbb{D}(\mathfrak{K}) := \{N/\cong : N \in K, \|N\| \le \text{ LST}(\mathfrak{K})\}$.
3) $D(M) = \{\text{tp}_{\mathbb{L}(\tau_M)}(\bar{a}, \emptyset, M) : \bar{a} \in {}^{\omega>}M\}$.

{88r-2.2}

2.3. **Definition.** Let $\lambda >$ LST$(\mathfrak{K})$.
1) A model $M$ is $\lambda$-model-homogeneous when: <u>if</u> $N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} M, \|N_1\| < \lambda, f$ an $\le_{\mathfrak{K}}$-embedding of $N_0$ into $M$, <u>then</u> some $\le_{\mathfrak{K}}$-embedding $f' : N_1 \to M$ extends $f$.
1A) A model $M$ is $(\mathbb{D}, \lambda)$-model-homogeneous <u>if</u> $\mathbb{D} = \mathbb{D}(M)$ and $M$ is a $\lambda$-model homogeneous.
1B) Adding "above $\mu$" means in $\mathfrak{K}_{\ge\mu}$.
2) $M$ is $\lambda$-strongly model-homogeneous <u>if</u>: for every $N \in K_{<\lambda}$ such that $N \le_{\mathfrak{K}} M$ and a $\le_{\mathfrak{K}}$-embedding $f : N \to M$ there exists an automorphism $g$ of $M$ extending $f$.
3) $M$ is $\lambda$-model universal homogeneous (for $\mathfrak{K}$) <u>when</u>: $\lambda >$ LST$(\mathfrak{K})$, every[2] $N \in K_{\text{LST}(\mathfrak{K})}$ is $\le_{\mathfrak{K}}$-embeddable into $M$ and for every $N_\ell \in K_{<\lambda}$ (for $\ell = 0, 1$) such that $N_0 \le_{\mathfrak{K}} N_1$ and $\le_{\mathfrak{K}}$-embedding $f : N_0 \to M$ there exists a $\le_{\mathfrak{K}}$-embedding $g : N_1 \to M$ extending $f$ (unlike (1), we do not demand that $N_1$ is $\le_{\mathfrak{K}}$-embeddable into $M$; the universal is related to $\lambda$, it does not imply $M$ is universal).
4) For each of the above three properties and the one below, if $M$ has cardinality $\lambda$ and has the $\lambda$-property then we may say for short that $M$ has the property (i.e. omitting $\lambda$).
5) $M$ is $(D, \lambda)$-sequence-homogeneous <u>if</u>:

- $(a)$ $D = D(M) = \{\text{tp}_{\mathbb{L}(\tau_M)}(\bar{a}, \emptyset, M) : \bar{a} \in |M|$, i.e., $\bar{a}$ a finite sequence from $M\}$ and
- $(b)$ if $a_i \in M$ for $i \le \alpha < \lambda, b_j \in M$ for $j < \alpha$ and $\text{tp}_{\mathbb{L}(\tau M)}(\langle a_i : i < \alpha\rangle, \emptyset, M) = \text{tp}_{\mathbb{L}(\tau_M)}(\langle b_i : i < \alpha\rangle, \emptyset, M)$, <u>then</u> for some $b_\alpha \in M$, $\text{tp}_{\mathbb{L}(\tau_M)}(\langle a_i : i \le \alpha\rangle, \emptyset, M) = \text{tp}_{\mathbb{L}(\tau_M)}(\langle b_i : i \le \alpha\rangle, \emptyset, M)$.

5A) In (5) we omit $D$ when $D = \{\text{tp}_{\mathbb{L}(\tau_K)}(\bar{a}, \emptyset, N) : \bar{a} \in {}^nN$ where $n < \omega$ and $M \prec_{\mathbb{L}} N\}$.
6) We omit the "model/sequence", when which one is clear from the context, i.e., if $D$ is as in 2.2(3) = 2.3(5)(a), $(D, \lambda)$-homogeneous means $(D, \lambda)$-sequence-homogeneous: if $\mathbb{D}$ is as in Definition 2.2(1), $(\mathbb{D}, \lambda)$-homogeneous means $(\mathbb{D}, \lambda)$-model-homogeneous, if not obvious

[2]in fact, $N \in K_{\le\lambda}$ is O.K. by 2.5(2)

we mean the model version.
7) $M$ is $\lambda$-universal <u>when</u> every $N \in K_\lambda$ can be $\leq_{\mathfrak{K}}$-embedded into it. Similarly $(< \lambda)$-universal, $(\leq \lambda)$-universal.

{88r-2.3}

2.4. **Claim.** *Assume $N$ is $\lambda$-model-homogeneous and $\mathbb{D}(M) \subseteq \mathbb{D}(N)$, (and $\mathrm{LST}(\mathfrak{K}) < \lambda$, of course).*
*1) If $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M, \|M_0\| < \lambda, \|M_1\| \leq \lambda$ and $f$ is a $\leq_{\mathfrak{K}}$-embedding of $M_0$ into $N$, <u>then</u> we can extend $f$ to a $\leq_{\mathfrak{K}}$-embedding of $M_1$ into $N$.*
*2) If $M_1 \leq_{\mathfrak{K}} M, \|M_1\| \leq \lambda$ <u>then</u> there is a $\leq_{\mathfrak{K}}$-embedding of $M_1$ into $N$.*

*Proof.* We prove by induction on $\mu \leq \lambda$ simultaneously that:

- $(i)_\mu$ for every $M_1 \leq_{\mathfrak{K}} M, \|M_1\| \leq \mu$ (yes! not $< \mu$) there is a $\leq_{\mathfrak{K}}$-embedding of $M_1$ into $N$
- $(ii)_\mu$ if $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M, \|M_1\| \leq \mu, \|M_0\| < \lambda$ then any $\leq_{\mathfrak{K}}$-embedding $f$ of $M_0$ into $N$ can be extended to a $\leq_{\mathfrak{K}}$-embedding of $M_1$ into $N$.

Clearly $(i)_\lambda$ is part (2) and $(ii)_\lambda$ is part (1) so this is enough. $\square$

*Proof of $(i)_\mu$.* If $\mu \leq \mathrm{LST}(\mathfrak{K})$, this follows by $\mathbb{D}(M) \subseteq \mathbb{D}(N)$.
If $\mu > \mathrm{LST}(\mathfrak{K})$, then by 1.11 we can find $\bar{M}_1 = \langle M_1^\alpha : \alpha < \mu \rangle$ such that $M_1 = \bigcup\limits_{\alpha<\mu} M_1^\alpha$ and $\alpha < \mu \Rightarrow M_1^\alpha \leq_{\mathfrak{K}} M_1$ and $M_1^\alpha$ is $\leq_{\mathfrak{K}}$-increasing continuous with $\alpha$ and $\alpha < \mu \Rightarrow \|M_1^\alpha\| < \mu$. We define by induction on $\alpha$, a $\leq_{\mathfrak{K}}$-embedding $f_\alpha : M_1^\alpha \to N$, such that for $\beta < \alpha, f_\alpha$ extend $f_\beta$. For $\alpha = 0$ we can define $f_\alpha$ by $(i)_{\chi(0)}$ which holds as by the induction hypothesis, where $\chi(\beta) := \|M_1^\beta\|$. We next define $f_\alpha$ for $\alpha = \gamma + 1$: by $(ii)_{\chi(\alpha)}$ which holds by the induction hypothesis there is a $\leq_{\mathfrak{K}}$-embedding $f_\alpha$ of $M_1^\alpha$ into $N$ extending $f_\gamma$.

Lastly, for limit $\alpha$ we let $f_\alpha = \bigcup\limits_{\beta<\alpha} f_\beta$, it is a $\leq_{\mathfrak{K}}$-embedding into $N$ by 1.7. So we finish the induction and $\bigcup\limits_{\alpha<\mu} f_\alpha$ is as required. $\square$

*Proof of $(ii)_\mu$.* First, assume that $\mu = \lambda$ so we have proved $(ii)_\theta$ for $\theta < \lambda$ and $\|M_1\| = \lambda > \|M_0\|$, so $\mathrm{LST}(\mathfrak{K}) < \mu = \lambda$ hence we can find $\langle M_1^\alpha : \alpha < \mu \rangle$ as in the proof of $(i)_\mu$ such that $M_1^0 = M_0$ and let $\chi(\beta) = \|M_1^\beta\|$. Now we define $f_\beta$ by induction on $\beta \leq \mu$ such that $f_\beta$ is a $\leq_{\mathfrak{K}}$-embedding of $M_\beta^1$ into $N$ and $f_\beta$ is increasing continuous in $\beta$ and $f_0 = f$. We can do this as in the proof of $(i)_\mu$ by $(ii)_{\chi(\alpha)}$ for $\alpha < \mu$.

Second, assume $\|M_1\| < \lambda$. Let $g$ be a $\leq_{\mathfrak{K}}$-embedding of $M_1$ into $N$, it exists by $(i)_\mu$ which we have just proved. Let $g$ be onto $N_1' \leq_{\mathfrak{K}} N$, and let $g \restriction M_0$ be onto $N_0' \leq_{\mathfrak{K}} N_1'$, and let $f$ be onto $N_0 \leq_{\mathfrak{K}} N$. So clearly $h : N_0' \to N_0$ defined by $h(g(a)) = f(a)$ for $a \in |M_0|$, is an isomorphism from $N_0'$ onto $N_0$. So $N_0, N_0', N_1' \leq_{\mathfrak{K}} N$. As $\|M_1\| < \lambda$ clearly $\|N_1'\| < \lambda$

so (by the assumption "$N$ is $\lambda$-model-homogeneous", see Definition 2.3(1)) we can extend $h$ to an isomorphism $h'$ from $N_1'$ onto some $N_1 \leq_{\mathfrak{K}} N$, so $h' \circ g : M_1 \to N$ is as required. $\square_{2.4}$

{88r-2.4}

2.5. **Conclusion.** 1) If $M, N$ are model-homogeneous, of the same cardinality ($>$ LST($\mathfrak{K}$)) and $\mathbb{D}(M) = \mathbb{D}(N)$ <u>then</u> $M, N$ are isomorphic. Moreover, if $M_0 \leq_{\mathfrak{K}} M, \|M_0\| < \|M\|$, <u>then</u> any $\leq_{\mathfrak{K}}$-embedding of $M_0$ into $N$ can be extended to an isomorphism from $M$ onto $N$.

2) The number of model-homogeneous models from $\mathfrak{K}$ of cardinality $\lambda$ is $\leq 2^{2^{\mathrm{LST}(\mathfrak{K})}}$; if in Definition 1.2, AxVI, in the definition of LST($\mathfrak{K}$) we omit $|\tau| \leq$ LST($\mathfrak{K}$), the bound is $2^{2^{\mathrm{LST}(\mathfrak{K})+|\tau(\mathfrak{K})|}}$.

3) If $M$ is $\lambda$-model-homogeneous and $\mathbb{D}(M) = \mathbb{D}(\mathfrak{K})$ <u>then</u> $M$ is ($\leq \lambda$)-universal, i.e. every model $N$ (in $K$) of cardinality $\leq \lambda$, has a $\leq_{\mathfrak{K}}$-embedding into $M$. So if $\mathbb{D}(M) = \mathbb{D}(\mathfrak{K})$ <u>then</u>: $M$ is $\lambda$-model universal homogeneous (see Definition 2.3(3)) iff $M$ is a $\lambda$-model-homogeneous iff $M$ is $(\lambda, \mathbb{D}(\mathfrak{K}))$-homogeneous.

4) If $M$ is $\lambda$-model-homogeneous <u>then</u> it is $\lambda$-universal for $\{N \in K_\lambda : \mathbb{D}(N) \subseteq \mathbb{D}(M)\}$.

5) If $M$ is $(D, \lambda)$-sequence-homogeneous, ($\lambda >$ LST($\mathfrak{K}$)) <u>then</u> $M$ is a $\lambda$-model homogeneous.

6) For $\lambda >$ LST($\mathfrak{K}$), $M$ is $\lambda$-model universal homogeneous <u>iff</u> $M$ is $\lambda$-model-homogeneous and ($\leq$ LST($\mathfrak{K}$))-universal.

*Proof.* 1) Immediate by 2.4(1), using the standard hence and forth argument.

2) The number of models (in $K$) of power $\leq$ LST($\mathfrak{K}$) is, up to isomorphism, $\leq 2^{\mathrm{LST}(\mathfrak{K})}$ (recalling that we are assuming $|\tau(\mathfrak{K})| \leq$ LST($\mathfrak{K}$)). Hence the number of possible $\mathbb{D}(M)$ is $\leq 2^{2^{\mathrm{LST}(\mathfrak{K})}}$. So by 2.5(1) we are done.

3),4),5) Immediate. $\square_{2.5}$

{88r-2.4A

2.6. **Remark.** The results parallel to 2.5(1)-(4) for $\lambda$-sequence homogeneous models and $D(M)$ hold, too.

{88r-2.5}

2.7. **Definition.** 1) A model $M$ has the $(\lambda, \mu)$-amalgamation property (= am.p., in $\mathfrak{K}$, of course) <u>if</u>: for every $M_1, M_2$ such that $\|M_1\| = \lambda, \|M_2\| = \mu, M \leq_{\mathfrak{K}} M_1$ and $M \leq_{\mathfrak{K}} M_2$, there is a model $N$ and $\leq_{\mathfrak{K}}$-embeddings $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ such that $f_1 \restriction |M| = f_2 \restriction |M|$. Now the meaning of e.g. the $(\leq \lambda, < \mu)$-amalgamation property should be clear. Always $\lambda, \mu \geq$ LST($\mathfrak{K}$) (and, of course, if we use $< \mu, \mu >$ LST($\mathfrak{K}$)).

1A) In part (1) we add the adjective "disjoint" <u>when</u> $f_1(M_1) \cap f_2(M_2) = M$. Similarly in (2) below.

2) $\mathfrak{K}$ has the $(\kappa, \lambda, \mu)$-amalgamation property <u>if</u> every model $M$ (in $K$) of cardinality $\kappa$ has the $(\lambda, \mu)$-amalgamation property. The $(\kappa, \lambda)$-amalgamation property for $\mathfrak{K}$ means just the $(\kappa, \kappa, \lambda)$-amalgamation property. The $\kappa$-amalgamation property for $\mathfrak{K}$ is just the $(\kappa, \kappa, \kappa)$-amalgamation property.

3) $\mathfrak{K}$ has the $(\lambda, \mu)$-JEP (joint embedding property) if for any $M_1 \in K, M_2 \in K$ of cardinality $\lambda, \mu$ respectively there is $N \in K$ into which $M_1$ and $M_2$ are $\leq_{\mathfrak{K}}$-embeddable.
4) The $\lambda$-JEP is the $(\lambda, \lambda)$-JEP.
5) The amalgamation property means the $(\kappa, \lambda, \mu)$-amalgamation property for every $\lambda, \mu \geq \kappa (\geq \text{LST}(\mathfrak{K}))$.
6) The JEP means the $(\lambda, \mu)$-JEP for every $\lambda, \mu \geq \text{LST}(\mathfrak{K})$.

2.8. **Remark.** Clearly in 2.7, parts (1), (2) first sentence, (3),(5), the roles of $\lambda, \mu$ are symmetric.

{88r-2.6}

2.9. **Theorem.** *1) If* $\text{LST}(\mathfrak{K}) < \kappa \leq \lambda, \lambda = \lambda^{<\kappa}, K_\lambda \neq \emptyset$ *and* $\mathfrak{K}$ *has the* $(<\kappa, \lambda)$-*amalgamation property* <u>*then*</u> *for every model* $M$ *of cardinality* $\lambda$, *there is a* $\kappa$-*model-homogeneous model* $N$ *of cardinality* $\lambda$ *satisfying* $M \leq_{\mathfrak{K}} N$. *If* $\kappa = \lambda$, *alternatively the* $(<\kappa, <\lambda)$-*amalgamation property suffices.*
*2) So in (1) if* $\kappa = \lambda$, *there is a universal, model-homogeneous model of cardinality* $\lambda$, *provided that for some* $M \in K_{\leq\lambda}, \mathbb{D}(M) = \mathbb{D}(\mathfrak{K})$ *or just* $\mathfrak{K}$ *has the* $\text{LST}(\mathfrak{K})$-JEP.
*3) If* $\mathfrak{K}$ *has the amalgamation property and the* $\text{LST}(\mathfrak{K})$-JEP, <u>*then*</u> $\mathfrak{K}$ *has the* JEP.

{88r-2.6A}

2.10. **Remark.** 1) The last assumption of 2.9(2) holds, e.g., if $(\leq \text{LST}(\mathfrak{K}), < 2^{\text{LST}(\mathfrak{K})})$-JEP holds and $|\mathbb{D}(\mathfrak{K})| \leq \lambda$.
2) If for some $M \in K, \mathbb{D}(M) = \mathbb{D}(\mathfrak{K})$ <u>then</u> we can have such $M$ of cardinality $\leq 2^{\text{LST}(\mathfrak{K})}$.
3) We can in 2.9 replace the assumption "$(<\kappa, \lambda)$-amalgamation property" by "$(<\kappa, <\lambda)$-amalgamation property" if, e.g., no $M \in K_{<\lambda}$ is maximal.

*Proof.* Immediate; in (1) note that if $\kappa$ is singular then necessarily $\lambda > \kappa \& \lambda = \lambda^\kappa = \lambda^{<\kappa^+}$ so we can replace $\kappa$ by $\kappa^+$. $\square_{2.9}$

{88r-2.6B}

2.11. **Remark.** Also the corresponding converses hold.

{88r-2.7}

2.12. **Lemma.** *1) If* $\text{LST}(\mathfrak{K}) \leq \kappa$ *and* $\mathfrak{K}$ *has the* $\kappa$-*amalgamation property* <u>*then*</u> $\mathfrak{K}$ *has the* $(\kappa, \kappa^+)$-*amalgamation property and even the* $(\kappa, \kappa^+, \kappa^+)$-*amalgamation property.*
*2) If* $\kappa \leq \mu \leq \lambda$ *and* $\mathfrak{K}$ *has the* $(\kappa, \mu)$-*amalgamation property and the* $(\mu, \lambda)$-*amalgamation property* <u>*then*</u> $\mathfrak{K}$ *has the* $(\kappa, \lambda)$-*amalgamation property. If* $\mathfrak{K}$ *has the* $(\kappa, \mu, \mu)$ *and the* $(\mu, \lambda)$-*amalgamation property,* <u>*then*</u> $\mathfrak{K}$ *has the* $(\kappa, \lambda, \mu)$-*amalgamation property.*
*3) If* $\lambda_i (i \leq \alpha)$ *is increasing and continuous,* $\text{LST}(\mathfrak{K}) \leq \lambda_0$ *and for every* $i < \alpha, \mathfrak{K}$ *has the* $(\lambda_i, \mu + \lambda_i, \lambda_{i+1})$-*amalgamation property* <u>*then*</u> $\mathfrak{K}$ *has the* $(\lambda_0, \mu + \lambda_0, \lambda_\alpha)$-*amalgamation property.*
*4) If* $\kappa \leq \mu_1 \leq \mu$ *and for every* $M, \|M\| = \mu_1$, *there is* $N, M \leq_{\mathfrak{K}} N, \|N\| = \mu$, <u>*then*</u> *the* $(\kappa, \mu, \lambda)$-*amalgamation property (for* $\mathfrak{K}$*) implies the* $(\kappa, \mu_1, \lambda)$-*amalgamation property (for*

*$\mathfrak{K}$).*
*5) Similarly with the disjoint amalgamation version.*

*Proof.* Straightforward, e.g.
3) So assume $M_0 \in K_{\lambda_0}, M_0 \le_{\mathfrak{K}} M_1 \in K_{\mu+\lambda_0}$ and $M_0 \le_{\mathfrak{K}} M_2 \in K_{\lambda_\alpha}$ and for variety we prove for the disjoint amalgamation version (see part (5)). By e.g. 1.11 we can find an $\le_{\mathfrak{K}}$-increasing continuous sequence $\langle M_{2,i} : i \le \alpha\rangle$ such that $M_{2,0} = M_0, M_{2,\alpha} = M_2$ and $M_{2,i} \in K_{\lambda_i}$ for $i \le \alpha$.

Without loss of generality $M_1 \cap M_2 = M_0$. We now choose $M_{1,i}$ by induction on $i \le \alpha$ such that:

- $(*)$ $(a)$ $\langle M_{1,j} : j \le i\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous
  - $(b)$ $M_{1,i} = M_1$ if $i = 0$
  - $(c)$ $M_{1,i} \in K_{\mu+\lambda_i}$
  - $(d)$ $M_{2,i} \le_{\mathfrak{K}} M_{1,i}$
  - $(e)$ $M_{2,i} \cap M_{1,\alpha} = M_{1,i}$.

For $i = 0$ see clause (b), for $i$ limit take union, for $i = j+1$ apply the disjoint $(\lambda_j, \mu+\lambda_j, \lambda_i)$-amalgamation to $M_{2,j}, M_{1,j}, M_{2,j+1}$. For $i = \alpha$ we are done. $\square_{2.12}$

{88r-2.8}

2.13. **Conclusion.** If $\mathrm{LST}(\mathfrak{K}) \le \chi_1 < \chi_2$ and $\mathfrak{K}$ has the $\kappa$-amalgamation property whenever $\chi_1 \le \kappa < \chi_2$ <u>then</u> $\mathfrak{K}$ has the $(\kappa, \lambda, \mu)$-amalgamation property whenever $\chi_1 \le \kappa \le \lambda \le \chi_2, \kappa \le \mu \le \chi_2$ and $\kappa < \chi_2$.

* * *

It may be interesting to note that even waiving AX IV we can say something.

{88r-2.9}

2.14. **Context.** For the remainder of this section $\mathfrak{K}$ is just a weak a.e.c., i.e., Ax IV is not assumed.

{88r-2.10

2.15. **Definition.** Let $M \in K$ have cardinality $\lambda$, a regular uncountable cardinal $>$ $\mathrm{LST}(\mathfrak{K})$. We say $M$ is <u>smooth</u> if there is a sequence $\langle M_i : i < \lambda\rangle$ with $M_i$ being $\le_{\mathfrak{K}}$-increasing continuous, $M_i \le_{\mathfrak{K}} M$ and $\|M_i\| < \lambda$ for $i < \lambda$ and $M = \bigcup\limits_{i<\lambda} M_i$.

{88r-2.10

2.16. **Remark.** We can define $S/\mathcal{D}$-smooth, for $S$ a subset of $\mathcal{P}(\lambda), \mathcal{D}$ a filter on $\mathcal{P}(\lambda)$, that is: $M \in K_\lambda$ is $(S/\mathcal{D})$-smooth when for every one-to-one function $f$ from $|M|$ onto $\lambda$ the set $\{u \in \mathcal{P}(\lambda) : M \restriction \{a : f(a) \in u\} \le_{\mathfrak{K}} M\} \in D$. Usually we demand that for every permutation $f$ on $\lambda, \{u \subseteq \lambda$: $u$ is closed under $f\} \in \mathcal{D}$, and usually we demand that $\mathcal{D}$ is a normal $\mathrm{LST}(\mathfrak{K})^+$-complete filter).

{88r-2.11

2.17. **Claim.** *Assume that $\lambda = \lambda^{<\lambda} > |\tau_K|, \mathfrak{K}_{<\lambda}$ has no maximal member and $\mathfrak{K}$ has $(<\lambda, <\lambda, <\lambda)$-amalgamation property and* $\mathrm{LST}(\mathfrak{K}) < \lambda$ *or at least assume in the $(<\lambda, <\lambda, <\lambda)$-amalgamation demand that the resulting model has cardinality $< \lambda$. Then $\mathfrak{K}_\lambda$ has a smooth model-homogeneous member.*

*Proof.* Same proof. $\square_{2.17}$

{88r-2.11}

2.18. **Lemma.** *If $M, N \in K_\lambda(\lambda > \mathrm{LST}(\mathfrak{K}))$ are smooth, model-homogeneous and $\mathbb{D}(M) = \mathbb{D}(N)$ then $M \cong N$.*

*Proof.* By the hence and forth argument, left to the reader (the set of approximations is $\{f : f$ isomorphism from some $M' \leq_{\mathfrak{K}} M$ of cardinality $< \lambda$ onto some $N' \leq_{\mathfrak{K}} N\}$ but note that not for any increasing continuous sequence of approximations is the union an approximation). $\square_{2.18}$

{88r-2.11A}

2.19. **Remark.** It is reasonable to consider

(∗) if $M \in K_\lambda, (\lambda >$ LST$(\mathfrak{K}))$ is smooth and model-homogeneous and $N \in K_\lambda$ is smooth, $\mathbb{D}(N) \subseteq \mathbb{D}(M)$ then $N$ can be $\leq_{\mathfrak{K}}$-embedded into $M$.

This can be proved in the context of universal classes (e.g. $AxFr_1$ from [Sh:300b]).

{88r-2.12}

2.20. **Fact.** 1) If $\mathfrak{K}_i = (K_i, <_i)$ is a (weak) a.e.c., i.e. with $\lambda_i =$ LST$(K_i, \leq_i)$ where $\lambda_i \geq \aleph_0$ for $i < \alpha, i < \alpha \Rightarrow \tau_{K_i} = \tau$ and $K = \bigcap\limits_{i<\alpha} K_i$ and $\leq$ is defined by $M \leq N$ if and only if for every $i < \alpha, M \leq_i N$ then $\mathfrak{K} = (K, \leq)$ is a [weak] a.e.c. with $\mathrm{LST}(\mathfrak{K}) \leq \sum\limits_{i<\alpha} \lambda_i$.

2) Concerning AxI-V, we can omit some of them in the assumption and still get the rest in the conclusion. But for AxVI we need in addition to assume AxV + AxIV$_\theta$ for at least one $\theta =$ cf$(\theta) \leq \sum\limits_{i<\alpha} \lambda_i$.

*Proof.* Easy. $\square_{2.20}$

{88r-2.13}

2.21. **Example.** Consider the class $K$ of norm spaces over the reals with $M \leq_{\mathfrak{K}} N$ iff $M \subseteq N$ and $M$ is complete inside $N$. Now $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is a weak a.e.c. with $\mathrm{LST}(\mathfrak{K}) = 2^{\aleph_0}$ and it is as required in 2.17.

## 3. Limit models and other results

In this section we introduce various variants of limit models (the most important are the superlimit ones). We prove that if $\mathfrak{K}$ has a superlimit model $M^*$ of cardinality $\lambda$ for which the $\lambda$-amalgamation property fails and $2^\lambda < 2^{\lambda^+}$ then $\dot{I}(\lambda, K) = 2^\lambda$ (see 3.9). We later prove that if $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ is categorical in $\aleph_1$ then it has model in $\aleph_2$ see 3.19(2). This finally solves Baldwin's problem (see §0). In fact we prove an essentially more general result on a.e.c. and $\lambda$ (see 3.12, 3.14).

The reader can read 3.3(1),(1A),(1B) ignore the other definitions, and continue with 3.8(2),(5) and everything from 3.9 (interpreting all variants as superlimits).
You may wonder can we prove the parallel to Baldwin conjecture in $\lambda^+$ if $\lambda > \aleph_0$; it is

$\circledast_\lambda$ if $\mathfrak{K}$ is $\lambda$-presentable a.e.c. with $\mathrm{LST}(\mathfrak{K}) = \lambda$, categorical in $\lambda^+$ then $K_{\lambda^{++}} \neq \emptyset$.

This is false when $\mathrm{cf}(\lambda) > \aleph_0$.

3.1. **Context.** $\mathfrak{K}$ is an a.e.c. {88r-3.0}

3.2. **Example.** Let $\lambda$ be given and $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ be defined by {88r-3.0.1}

$$K = \{(A,<) : (A,<) \text{ a well order of order type } \leq \lambda^+\}$$
$$\leq_{\mathfrak{K}} = \{(M,N) : M,N \in K \text{ and } N \text{ is an end extension of } M\}.$$

Now

- $(a)$ $\mathfrak{K}$ is an abstract elementary class with $\mathrm{LST}(\mathfrak{K}) = \lambda$ and $\mathfrak{K}$ categorical in $\lambda^+$
- $(b)$ if $\lambda$ has cofinality $\geq \aleph_1$ then $\mathfrak{K}$ is $\lambda$-presentable (see, e.g., [Sh:c, VII,§5] and history there); by clause (a) it is always $(\lambda, 2^\lambda)$-presentable,
- $(c)$ $\mathfrak{K}$ has no model of cardinality $> \lambda^+$.

Note that if we are dealing with classes which are categorical (or just simple in some sense), we have a good chance to find limit models and they are useful in constructions.

3.3. **Definition.** Let $\lambda$ be a cardinal $\geq$ $\mathrm{LST}(\mathfrak{K})$. For parts 3) - 7) but not 8), for simplifying the presentation we assume the axiom of global choice (alternatively, we restrict ourselves to models with universe an ordinal $< \lambda^+$). {88r-3.1}

1) $M \in K_\lambda$ is locally superlimit (for $\mathfrak{K}$) if:

- $(a)$ for every $N \in K_\lambda$ such that $M \leq_{\mathfrak{K}} N$ there is $M' \in K_\lambda$ isomorphic to $M$ such that $N \leq_{\mathfrak{K}} M'$ and $N \neq M'$
- $(b)$ if $\delta < \lambda^+$ is a limit ordinal and $\langle M_i : i < \delta\rangle$ is $\leq_{\mathfrak{K}}$-increasing sequence and $M_i \cong M$ for $i < \delta$ then $\bigcup_{i<\delta} M_i \cong M$.

1A) $M \in K_\lambda$ is globally superlimit if (a) +(b) and

- $(c)$ $M$ is universal in $\mathfrak{K}_\lambda$, i.e., any $N \in K_\lambda$ can be $\leq_{\mathfrak{K}}$-embedded into $M$.

1B) Just superlimit means globally. Similarly with the other notions below we define the global version as adding clause (c) from (1A) and the default version is the global one. (Note that in the local version we can restrict our class to $\{N \in K_\lambda : M$ can be $\leq_{\mathfrak{K}}$-embedded into $N\}$ and get the global one).

2) For $\Theta \subseteq \{\mu : \aleph_0 \leq \mu < \lambda, \mu \text{ regular}\}$, $M \in K_\lambda$ is locally $(\lambda, \Theta)$-superlimit <u>if</u>:

- $(a)$ as in part (1) above
- $(b)$ if $\langle M_i : i \leq \mu\rangle$ is $\leq_{\mathfrak{K}}$-increasing, $M_i \cong M$ for $i < \mu$ and $\mu \in \Theta$ then $\cup\{M_i : i < \mu\} \cong M$.

2A) If $\Theta$ is a singleton, say $\Theta = \{\theta\}$, we may say that $M$ is locally $(\lambda, \theta)$-superlimit.

3) Let $S \subseteq \lambda^+$ be stationary. $M \in K_\lambda$ is called locally $S$-strongly limit or locally $(\lambda, S)$-strongly limit <u>when</u> for some function: $\mathbf{F} : K_\lambda \to K_\lambda$ we have:

- $(\alpha)$ for $N \in K_\lambda$ we have $N \leq_{\mathfrak{K}} \mathbf{F}(N)$
- $(\beta)$ if $\delta \in S$ is a limit ordinal and $\langle M_i : i < \delta\rangle$ is a $\leq_{\mathfrak{K}}$-increasing continuous sequence [3] in $K_\lambda$ and $M_0 \cong M$ and $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$, <u>then</u> $M \cong \cup\{M_i : i < \delta\}$
- $(\gamma)$ if $M \leq_{\mathfrak{K}} M_1 \in K_\lambda$ then there is $N$ such that $M_1 <_{\mathfrak{K}} N \in K_\lambda$.

4) Let $S \subseteq \lambda^+$ be stationary. $M \in K_\lambda$ is called locally $S$-limit or locally $(\lambda, S)$-limit <u>if</u> for some function $\mathbf{F} : K_\lambda \to K_\lambda$ we have:

- $(\alpha)$ for every $N \in K_\lambda$ we have $N \leq_{\mathfrak{K}} \mathbf{F}(N)$
- $(\beta)$ if $\langle M_i : i < \lambda^+\rangle$ is a $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of $K_\lambda, M_0 \cong M, \mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$ <u>then</u> for some closed unbounded [4] subset $C$ of $\lambda^+$,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

- $(\gamma)$ if $M \leq_{\mathfrak{K}} M_1 \in K_\lambda$ then there is $N, M_1 <_{\mathfrak{K}} N \in K_\lambda$.

5) We define "locally $S$-weakly limit", "locally $S$-medium limit" like "locally $S$-limit", "locally $S$-strongly limit" respectively by demanding that the domain of $\mathbf{F}$ is the family of $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of $\mathfrak{K}_{<\lambda}$ of length $< \lambda$ and replacing "$\mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$" by "$M_{i+1} \leq_{\mathfrak{K}} \mathbf{F}(\langle M_j : j \leq i+1\rangle) \leq_{\mathfrak{K}} M_{i+2}$". We replace "limit" by "limit$^-$" if "$\mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$", "$M_{i+1} \leq_{\mathfrak{K}} \mathbf{F}(\langle M_j : j \leq i+1\rangle) \leq_{\mathfrak{K}} M_{i+2}$" are replaced by "$\mathbf{F}(M_i) \leq_{\mathfrak{K}} M_{i+1}$", "$M_i \leq_{\mathfrak{K}} \mathbf{F}(\langle M_j : j \leq i\rangle) \leq_{\mathfrak{K}} M_{i+1}$" respectively.

6) If $S = \lambda^+$ then we omit $S$ (in parts (3), (4), (5)).

7) For $\Theta \subseteq \{\mu : \aleph_0 \leq \mu \leq \lambda$ and $\mu$ is regular$\}, M$ is locally $(\lambda, \Theta)$-strongly limit if $M$ is locally $\{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$-strongly limit. Similarly for the other notions (where $\Theta \subseteq \{\mu : \mu \text{ regular } \leq \lambda\}$. If we do not write $\lambda$ we mean $\lambda = \|M\|$. Let locally $(\lambda, \theta)$-strongly limit mean locally $(\lambda, \theta)$-strongly limit.

8) We say that $M \in K_\lambda$ is invariantly strong limit when in part (3) we demand that $\mathbf{F}$ is just a subset of $\{(M, N)/\cong : M \leq_{\mathfrak{K}} N$ are from $K_\lambda\}$ and in clause (b) of part (3) we

[3] no loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $\mathbf{F}$, i.e., only $\mathbf{F}(M')$ for $M' \cong M$ are required

[4] we can use a filter as a parameter

replace "$\mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$" by "$(\exists N)(M_{i+1} \leq_{\mathfrak{K}} N \leq_{\mathfrak{K}} M_{i+2} \wedge ((M_{i+1}, N)/\cong) \in \mathbf{F})$" but abusing notation we still write $N = \mathbf{F}(M)$ instead $((M,N)/\cong) \in \mathbf{F}$. Similarly with the other notions, so if $\mathbf{F}$ acts on suitable $\leq_{\mathfrak{K}}$-increasing sequence of models then we use the isomorphic type of $\bar{M}\hat{}\langle N\rangle$.

3.4. **Obvious implication diagram.** For $\Theta, S_1$ as in 3.3(7) and $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$ is a stationary subset of $\lambda^+$: {88r-3.1A}

$$\begin{array}{ccccc}
 & & \text{superlimit } = (\lambda, \{\mu : \mu \leq \lambda \text{ regular}\})\text{-superlimit} & & \\
 & & \downarrow & & \\
 & & (\lambda,\Theta)\text{-superlimit} & & \\
 & & \downarrow & & \\
 & & S_1\text{-strongly limit} & & \\
 & \swarrow & & \searrow & \\
S_1\text{-medium limit} & & & & S_1\text{-limit} \\
 & \searrow & & \swarrow & \\
 & & S_1\text{-weakly limit} & &
\end{array}$$

3.5. **Lemma.** *0) All the properties are preserved if $S$ is replaced by a subset and if $\mathfrak{K}$ has the $\lambda$-JEP, the local and global version in Definition 3.3 are equivalent.* {88r-3.2}
*1) If $S_i \subseteq \lambda^+$ for $i < \lambda^+, S = \{\alpha < \lambda^+ : (\exists i < \alpha)\alpha \in S_i\}$ and $S_i \cap i = \emptyset$ for $i < \lambda$ <u>then</u>: $M$ is $S_i$-strongly limit for each $i < \lambda$ if and only if $M$ is $S$-strongly limit.*
*2) Suppose $\kappa \leq \lambda$ is regular and $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa\}$ is a stationary set and $M \in K_\lambda$ <u>then</u> the following are equivalent:*

- (a) *$M$ is $S$-strongly limit*
- (b) *$M$ is $(\lambda, \{\kappa\})$-strongly limit*
- (c) *$M \in \mathfrak{K}_\lambda$ is $\leq_{\mathfrak{K}}$-universal not $<_{\mathfrak{K}}$-maximal and there is a function $\mathbf{F} : K_\lambda \to K_\lambda$ satisfying $(\forall N \in K_\lambda)[N \leq_{\mathfrak{K}} \mathbf{F}(N)]$ such that if $M_i \in K_\lambda$ for $i < \kappa, [i < j \Rightarrow M_i \leq_{\mathfrak{K}} M_j], \mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$ and $M_0 \cong M$ <u>then</u> $\bigcup_{i<\kappa} M_i \cong M$.*

*2A) If $S \subseteq \lambda^+, \Theta = \{\mathrm{cf}(\delta) : \delta \in S\}$ <u>then</u> $M$ is $S$-strongly limit iff clause (c) in part (2) above holds for every $\kappa \in \Theta$.*
*3) In part (1) we can replace "strongly limit" by "limit", "medium limit" and "weakly limit".*
*4) Suppose $\kappa \leq \lambda$ is regular, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa\}$ is a stationary set which belongs to $\check{I}[\lambda]$ (see 0.7, 0.8 above) and $M \in K_\lambda$. The following are equivalent*

- (a) *$M$ is $S$-medium limit in $\mathfrak{K}_\lambda$*
- (b) *$M \in K_\lambda$ is $\leq_{\mathfrak{K}}$-universal not maximal and there is a function $\mathbf{F}$ from $_{\alpha<\kappa} \to \bigcup^\alpha (K_\lambda)$ to $K$ such that*
  - ($\alpha$) *for any $\leq_{\mathfrak{K}}$-increasing $\langle M_i : i \leq \alpha\rangle$ if $M_0 = M, \alpha < \kappa, M_i$ is $\leq_{\mathfrak{K}}$-increasing, $M_i \in K_\lambda$, then $M_\alpha \leq_{\mathfrak{K}} \mathbf{F}(\langle M_i : i \leq \alpha\rangle)$*

($\beta$) *if* $\langle M_i : i < \kappa\rangle$ *is* $\le_{\mathfrak{K}}$*-increasing,* $M_0 = M, M_i \in K_\lambda$ *and for* $i < \kappa$ *we have* $M_{i+1} \le_{\mathfrak{K}} \mathbf{F}(\langle M_j : j \le i+1\rangle) \le_{\mathfrak{K}} M_{i+2}$ *then* $\bigcup\limits_{i<\kappa} M_i \cong M$.

*Proof.* 0) Trivial.
1) Recall that in Definition 3.3(3), clause (b) we use $\mathbf{F}$ only on $M_{i+1}$; (see the proof of (2A) below, second part).
2) For (c) $\Rightarrow$ (a) note that the demands on the sequence are "local", $M_{i+1} \le_{\mathfrak{K}} \mathbf{F}(M_{i+1}) \le_{\mathfrak{K}} M_{i+2}$, (whereas in part (4) they are "global").
2A) First assume that $M$ is $S$-strongly limit and let $\mathbf{F}$ witness it. Suppose $\kappa \in \Theta$, so we choose $\delta_\kappa \in S$ with $\mathrm{cf}(\delta_\kappa) = \kappa$ and let $\langle \alpha_i : i < \kappa\rangle$ be increasing continuous with limit $\delta, \alpha_0 = 0, \alpha_{i+1}$ a successor of a successor ordinal for each $i < \kappa$. We now define $\mathbf{F}_\kappa$ as follows: to define $\mathbf{F}_\kappa(M)$ we define $\mathbf{F}_{\kappa,\alpha}$ for $\alpha \le \delta$ by induction on $\alpha \le \delta$. Let:

($a$) if $\alpha = 0$ then $\mathbf{F}_{\kappa,0}(M) = M$

($b$) if $\alpha = \beta + 1$ then $\mathbf{F}_{\kappa,\alpha}(M) = \mathbf{F}(\mathbf{F}_{\kappa,\beta}(M))$

($c$) if $\alpha \le \delta$ a limit ordinal then $\mathbf{F}_{\kappa,\alpha}(M) = \cup\{\mathbf{F}_{\kappa,\beta}(M) : \beta < \alpha\}$.

Lastly, let $\mathbf{F}_\kappa(M)$ be $\mathbf{F}_{\kappa,\delta}(M)$.

Now suppose $\langle N_i : i \le \kappa\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, $N_i \in K_\lambda$ and $\mathbf{F}_\kappa(N_{i+1}) \le_{\mathfrak{K}} N_{i+2}$ for $i < \kappa$ and we should prove $N_\kappa \cong M$. Now we can find $\langle M_j : j < \lambda^+\rangle$ such that it obeys $\mathbf{F}$ and $M_{\alpha_i} = N_i$ for $i < \kappa$; so clearly we are done.

Second, assume that for each $\kappa \in \Theta$, clause (c) of 3.5(2) holds and let $\mathbf{F}_\kappa$ exemplify this. Let $\langle \kappa_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ list $\Theta$ so $\varepsilon(*) < \lambda^+$ and define $\mathbf{F}$ as follows. For any $M \in \mathfrak{K}$ choose $M_{[\varepsilon]}$ by induction on $\varepsilon \le \varepsilon(*)$ as follows: $M_{[0]} = M, M_{[\varepsilon+1]} = \mathbf{F}_{\kappa_\varepsilon}(M_{[\varepsilon]})$ and for $\varepsilon$ limit ordinal let $M_{[\varepsilon]} = \cup\{M_{[\zeta]} : \zeta < \varepsilon\}$. Lastly, let $\mathbf{F}[M] = M_{[\varepsilon(*)]}$. Now check.
3) No new point.
4) First note that ($a$) $\Rightarrow$ ($b$) should be clear. Second, we prove that ($b$) $\Rightarrow$ ($a$) so let $\mathbf{F}$ witness that clause (b) holds. Let $E, \langle u_\alpha : \alpha < \lambda\rangle$ witness that $S \in \check{I}[\lambda]$, i.e.

$(*)_1$ ($a$) $E$ a club of $\lambda$

($b$) $u_\alpha \subseteq \alpha$ and $\mathrm{otp}(u_\alpha) \le \kappa$ for $\alpha < \lambda$

($c$) if $\alpha \in S \cap E$ then $\alpha = \sup(u_\alpha)$ and $\mathrm{otp}(u_\alpha) = \kappa$

($d$) if $\alpha \in \lambda \backslash S \cap E$ then $\mathrm{otp}(u_\alpha) < \kappa$

($e$) if $\alpha \in u_\beta$ then $u_\alpha = u_\beta \cap \alpha$.

We can add

$(*)_2$ ($f$) if $\beta \in u_\alpha$ then $\beta$ has the form $3\gamma + 1$.

Let $\langle \alpha_\varepsilon : \varepsilon < \lambda\rangle$ list $E$ in increasing order and without loss of generality $\alpha_0 = 0, \alpha_{1+\varepsilon}$ is a limit ordinal (note that only the limit ordinals of $S$ count).

To define $\mathbf{F}'$ as required we shall deal with the requirement according to whether $\delta \in S$ is "easy", i.e. $\delta \notin E$ so $\delta \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}]$ for some $\varepsilon < \lambda^+$ so after $\alpha_\varepsilon$ we can "take care of it", or $\delta$ is "hard", i.e. $\delta \in E$ so we use the $\alpha \in u_\delta$.

We choose $\langle e_\delta : \delta \in S\backslash E\rangle$ such that $\delta \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}] \cap S$ implies $e_\delta \subseteq \delta = \sup(e_\delta)$ and $\min(e_\delta) > \alpha_\varepsilon$, $\mathrm{otp}(e_\delta) = \kappa, e_\delta$ is closed and $\alpha \in e_\delta \Rightarrow \alpha = \sup(e_\delta \cap \alpha) \vee (\alpha \in \{3\gamma + 2 : \gamma < \delta\})$. If $\delta \in S \cap E$ let $e_\delta$ be the closure of $u_\delta$. Let $\langle \gamma_{\delta,\zeta} : \zeta < \kappa\rangle$ list $e_\delta$ in increasing order.

We now define a function $\mathbf{F}'$ so let $\langle M_j : j \le i+1\rangle$ be given and let $\alpha_\varepsilon \le i < \alpha_{\varepsilon+1}$. We fix $\varepsilon$ so $(\alpha_\varepsilon, \alpha_{\varepsilon+1})$ and now define $\mathbf{F}'(\langle M_j : j \le i+1\rangle)$ by induction on $i \in [\alpha_\varepsilon, \alpha_{\varepsilon+1})$ assuming that if $\alpha_\varepsilon \le j'+1 < i+1$ then $\mathbf{F}'(\langle M_j : j \le j'+1\rangle) \le_{\mathfrak{K}} M_{j'+2}$ and further there is $\bar N^{j'+1} = \langle N_{j'+1,\xi} : \xi < \alpha_{\varepsilon+1}\rangle$ such that the following holds:

$(*)_3$ $\bar N^{j'+1}$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous, $M_{j'+1} \le_{\mathfrak{K}} N_{j'+1,0}$ and $N_{j'+1,\xi} \le_{\mathfrak{K}_\lambda} M_{j'+2}$

$(*)_4$ if $\delta \in (S\backslash E) \cap (\alpha_{\varepsilon+1}\backslash\alpha_\varepsilon), j'+1 = \gamma_{\delta,\zeta}$ (so necessarily $j'+1 \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}), j'+1 \in \{3\gamma+2 : \gamma < \lambda\}, \zeta$ is a successor ordinal) then let $\bar N^*_{\delta,j'} = \langle N^*_{\delta,j',\zeta'} : \zeta' \le \zeta\rangle$ be the following sequence of length $\zeta+1, N^*_{\delta,j',\zeta'}$ is $N_{\gamma_{\delta,\zeta'},\zeta'}$ if $\zeta'$ is a successor ordinal and is $M_{\gamma_{\delta,\zeta'}}$ if $\zeta'$ is limit or zero, and we demand $\mathbf{F}(\langle N^*_{\delta,j',\zeta'} : \zeta' \le \zeta\rangle) \le_{\mathfrak{K}} N_{j'+1,\zeta+1}$

$(*)_5$ if $j'+1 \in u_\delta$ for some $\delta \in S\cap E$ hence $j'+1 \in \{3\gamma+1 : \gamma < \delta\}$ and $\zeta = \mathrm{otp}(u_{j'+1}) < \kappa$ and $f_\varepsilon$ is the one-to-one order preserving function from $\zeta+1$ onto $c\ell(u_{j'+1} \cup \{j'+1\})$ and $\zeta'$ is a successor, <u>then</u> $\mathbf{F}(\langle M_{\alpha_{f_\varepsilon(\zeta')}} : \zeta' \le \zeta\rangle) \le_{\mathfrak{K}} M_{\alpha_\varepsilon+1}$.

This implicitly defines $\mathbf{F}'$. Now $\mathbf{F}'$ is as required: $M_i \cong M$ when $i < \lambda$, $\mathrm{cf}(i) = \kappa$ by $(*)_4$ when $(\exists\varepsilon)(\alpha_\varepsilon < i < \alpha_{\varepsilon+1})$ and by $(*)_5$ when $(\exists\varepsilon)(i = \alpha_\varepsilon)$. $\square_{3.5}$

{88r-3.3}

3.6. **Lemma.** *Let $T$ be a first order complete theory, $K$ its class of models and $\le_{\mathfrak{K}} = \prec_{\mathbb{L}}$.*
*1) If $\lambda$ is regular, $M$ a saturated model of $T$ of cardinality $\lambda$, <u>then</u> $M$ is $(\lambda, \{\lambda\})$-superlimit.*
*2) If $T$ is stable, and $M$ a saturated model of $T$ of cardinality $\lambda$ <u>then</u> $M$ is $(\lambda, \{\mu : \kappa(T) \le \mu \le \lambda$ and $\mu$ is regular$\})$-superlimit (on $\kappa(T)$-see* [Sh:c, III,§3]*). (Note that by* [Sh:c] *if $\lambda$ is singular and $T$ has a saturated model of cardinality $\lambda$ <u>then</u> $T$ is stable and $\mathrm{cf}(\lambda) \ge \kappa(T)$).*
*3) If $T$ is stable, $\lambda$ singular $> \kappa(T), M$ a special model of $T$ of cardinality $\lambda, S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ is stationary and $S \in \check{I}[\lambda]$ (see above 0.7, 0.8) <u>then</u> $M$ is $(\lambda, S)$-medium limit.*

3.7. **Remark.** See more in [Sh:868].

*Proof.* 1) Because if $M_i$ is a $\lambda$-saturated model of $T$ for $i < \delta$, $\mathrm{cf}(\delta) \ge \lambda$, <u>then</u> $\bigcup\limits_{i<\delta} M_i$ is $\lambda$-saturated. Remembering the uniqueness of a $\lambda$-saturated model of $T$ of cardinality $\lambda$ we finish.
2) Use [Sh:c, III,3.11]: if $M_i$ is a $\lambda$-saturated model of $T, \langle M_i : i < \delta\rangle$ increasing $\mathrm{cf}(\delta) \ge \kappa(T)$ <u>then</u> $\bigcup\limits_{i<\delta} M_i$ is $\lambda$-saturated.
3) Should be clear by now. $\square_{3.6}$

{88r-3.4}

3.8. **Claim.** *1) If $M_\ell \in K_\lambda$ are $S_\ell$-weakly limit and $S_0 \cap S_1$ is stationary, __then__ $M_0 \cong M_1$, provided $\kappa$ has $(\lambda, \lambda)$-JEP.*
*2) $K$ has at most one locally weakly limit model of cardinality $\lambda$ provided $K$ has $(\lambda, \lambda)$-JEP.*
*3) If $M \in K_\lambda$ __then__ $\{S \subseteq \lambda^+ : M$ is $S$-weakly limit or $S$ not stationary$\}$ is a normal ideal over $\lambda^+$.*
*Instead "$S$-weakly limit", also "$S$-medium limit", "$S$-limit", "$S$-strongly limit" can be used.*
*4) In Definition 3.3 without loss of generality $\mathbf{F}(N) \cong M$ or $\mathbf{F}(\bar{M}) \cong M$ according to the case (and we can add $N <_{\mathfrak{K}} \mathbf{F}(N)$, etc.)*
*5) If $K$ is categorical in $\lambda$, __then__ the $M \in K_\lambda$ is superlimit provided that $K_{\lambda^+} \neq \emptyset$ (or, what is equivalent, $M$ has a proper $\leq_{\mathfrak{K}}$-extension).*

*Proof.* Easy.
1) E.g., let $\mathbf{F}_\ell$ witness that $M_\ell$ is $S_\ell$-weakly limit. We can choose $(M^0_\alpha, M^1_\alpha)$ by induction on $\alpha$ such that: $\langle M^\ell_\beta : \beta \leq \alpha \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous for $\ell = 0, 1, M^0_\alpha \leq_{\mathfrak{K}} M^1_{\alpha+1}, M^1_\alpha \leq_{\mathfrak{K}} M^0_{\alpha+1}$ and $\mathbf{F}_\ell(\langle M^\ell_\beta : \beta \leq \alpha+1 \rangle) \leq M^\ell_{\alpha+2}$. So for some club $E_\ell$ of $\lambda^+, \delta \in S_\ell \cap E_\ell \Rightarrow M^\ell_\delta \cong M_\ell$ for $\ell = 0, 1$. But $S_0 \cap S_1$ is stationary hence there is a limit ordinal $\delta \in S_0 \cap S_1 \cap E_0 \cap E_1$, hence $M_0 \cong M^0_\delta = M^1_\delta \cong M_1$ as required. $\square_{3.8}$

{88r-3.5}

3.9. **Theorem.** *If $2^\lambda < 2^{\lambda^+}, M \in K_\lambda$ superlimit, $S = \lambda^+$ __or__ $M$ is $S$-weakly limit, $S$ is not small (see Definition 0.6) and $M$ does not have the $\lambda$-amalgamation property (in $\mathfrak{K}$) __then__ $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$, moreover there is no universal member in $\mathfrak{K}_{\lambda^+}$ and $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow \dot{I}\dot{E}(\lambda^+, K) = 2^{\lambda^+}$, that is there are $2^{\lambda^+}$ models $M \in K_{\lambda^+}$ no one $\leq_{\mathfrak{K}}$-embeddable into another.*

{88r-3.5A}

3.10. **Remark.** 0) So in 3.9, if $K$ is categorical in $\lambda$ __then__ it has $\lambda$-amalgamation.
1) We can define a superlimit for a family of models, i.e., when $\mathbf{N} = \{N_t : t \in I\} \subseteq \mathfrak{K}_\lambda$ is superlimit (i.e., if $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing, $i < \delta \Rightarrow M_i \in \mathfrak{K}_\lambda, \delta$ a limit ordinal $< \lambda^+, M_\delta = \cup\{M_i : i < \delta\}$ __then__ $\bigwedge\limits_{i<\delta} \bigvee\limits_{t \in I} M_i \cong N_t \Rightarrow \bigvee\limits_{t\in I} M_\delta \cong N_t$ (and the other variants).
Of course, the family is $\subseteq K_\lambda$ and is not empty. Essentially everything generalizes __but__ in 3.9 the hypothesis should be stronger: the family should satisfy that any member does not have the amalgamation property. E.g. $\mathbf{N} = \mathfrak{K}_\lambda$, (and we can reduce the general case to this by changing $\mathfrak{K}$). But this complicates the situation, and the gain is not clear, so we do not elaborate this.
2) We can many times (and in particular in 3.9) strengthen "there is no $\leq_{\mathfrak{K}}$-universal $M \in K_{\lambda^+}$" to "there is no $M \in K_\mu$ into which every $N \in K_{\lambda^+}$ can be $\leq_{\mathfrak{K}}$-embedded" for $\mu$ not too large. We need $\neg$ Unif$(\lambda^+, S, 2, \mu)$, (see [Sh:f, AP,§1]).

*Proof.* Let $\mathbf{F}$ be as in Definition 3.3(5) for $M$. We now choose by induction on $\alpha < \lambda^+$, models $M_\eta$ for $\eta \in {}^\alpha 2$ such that:

$\circledast_1$ $(i)$ $M_\eta \in K_\lambda, M_{<>} = M$,

$(ii)$ if $\beta < \alpha$ and $\eta \in {}^\alpha 2$ __then__ $M_{\eta \restriction \beta} \leq_{\mathfrak{K}} M_\eta$

$(iii)$ if $i + 2 \leq \alpha$ and $\eta \in {}^\alpha 2$, __then__ $(\mathbf{F}(\langle M_{\eta \restriction j} : j \leq i+1 \rangle)) \leq_{\mathfrak{K}} M_{\eta \restriction (i+2)}$

$(iv)$ if $\alpha = \beta + 1$ and $\beta$ non-limit, $\eta \in {}^\alpha 2$, __then__ $M_{\eta \restriction \beta} \neq M_\eta$

$(v)$ if $\alpha < \lambda$ is a limit ordinal and $\eta \in {}^\alpha 2$ __then__:

$(a)$ $M_\eta = \cup \{ M_{\eta \restriction \beta} : \beta < \ell g(\eta) \}$ and

$(b)$ if $M_\eta$ fails the $\lambda$-amalgamation property __then__ $M_{\eta \hat{} <0>}, M_{\eta \hat{} <1>}$

cannot be amalgamated over $M_\eta$, i.e. for no $N$ do we have:

$M_\eta \leq_{\mathfrak{K}} N \in K$ and $M_{\eta \hat{} <0>}, M_{<\eta \hat{} <1>}$ can be $\leq_{\mathfrak{K}}$-embedded

into $N$ over $M_\eta$.

For $\alpha = 0, \alpha$ limit, we have no problem, for $\alpha + 1, \alpha$ limit: if $M_\eta$ fails the $\lambda$-amalgamation property - use its definition, otherwise let $M_{\eta \hat{} <1>} = M_\eta = M_{\eta \hat{} <0>}$; for $\alpha + 1, \alpha$ non-limit - use $\mathbf{F}$ to guaranteee clause (iii), and then for clause (iv) use clause $(\gamma)$ of Definition 3.3(5), i.e., 3.3(4).

Let for $\eta \in {}^{\lambda^+} 2, M_\eta = \bigcup\limits_{\alpha < \lambda^+} M_{\eta \restriction \alpha}$. By changing names we can assume that

$\circledast_1$ $(vi)$ for $\eta \in {}^\alpha 2 (\alpha < \lambda^+)$ the universe of $M_\eta$ is an ordinal $< \lambda^+$ (or even

$\subseteq \lambda \times (1 + \ell g(\eta))$ and we could even demand equality).

So (by clause (iv)) for $\eta \in {}^{\lambda^+} 2, M_\eta$ has universe $\lambda^+$.

First, why is there no universal member in $\mathfrak{K}_{\lambda^+}$? If $N \in K_{\lambda^+}$ is universal (by $\leq_{\mathfrak{K}}$, of course), without loss of generality its universe is $\lambda^+$. For $\eta \in {}^{\lambda^+} 2$ as $M_\eta \in K_{\lambda^+}$, there is a $\leq_{\mathfrak{K}}$-embedding $f_\eta$ of $M_\eta$ into $N$. So $f_\eta$ is a function from $\lambda^+$ to $\lambda^+$. Let $\eta \in {}^{\lambda^+} 2$, by the choice of $\mathbf{F}$ and of $\langle M_{\eta \restriction \alpha} : \alpha < \lambda^+ \rangle$ there is a closed unbounded $C_\eta \subseteq \lambda^+$ such that $\alpha \in S \cap C_\eta \Rightarrow M_{\eta \restriction \alpha} \cong M$, hence $M_{\eta \restriction \alpha}$ fails the $\lambda$-amalgamation property. Without loss of generality for $\delta \in C_\eta, M_{\eta \restriction \delta}$ has universe $\delta$. Now by 0.6, if $\langle (f_\rho, C_\rho) : \rho \in {}^{\lambda^+} 2 \rangle$ satisfies that for each $\rho \in {}^{\lambda^+} 2, f_\rho : \lambda^+ \to \lambda^+$ and $C_\rho \subseteq \lambda^+$ is closed unbounded __then__ for some $\eta \neq \nu \in {}^{\lambda^+} 2$ and $\delta \in C_\eta \cap S$ we have $\eta \restriction \delta = \nu \restriction \delta, \eta(\delta) \neq \nu(\delta)$ and $f_\eta \restriction \delta = f_\nu \restriction \delta$.
[Why? For every $\delta < \lambda^+, \rho \in {}^\delta 2$ and $f : \delta \to \lambda^+$ we define $\mathbf{c}(\rho, f) \in 2$ as follows: it is 1 iff there is $\nu \in {}^{\lambda^+} 2$ such that $\rho = \nu \restriction \delta \& f = f_\nu \restriction \delta \& \nu(\delta) = 0$ and is 0 otherwise. So some $\eta \in {}^{\lambda^+} 2$ is a weak diamond sequence for the colouring $\mathbf{c}$ and the stationary set $S$. Now $C_\eta, f_\eta$ are well defined and $S' = \{ \delta \in S : \delta$ limit and $\eta(\delta) = \mathbf{c}(\eta \restriction \delta, f \restriction \delta) \}$ is a stationary subset of $\lambda^+$, so we can choose $\delta \in S' \cap C_\eta$. If $\eta(\delta) = 0$, then $\mathbf{c}(\eta \restriction \delta, f \restriction \delta) = 0$ by the choice of $S'$ but $\eta$ witness that $\mathbf{c}(\eta \restriction \delta, f \restriction \delta)$ is 1, standing for $\nu$ there. If $\eta(\delta) = 1$ there is $\nu$ witnessing $\mathbf{c}(\eta \restriction \delta, f_\eta \restriction \delta) = 1$, in particular $\nu(\delta) = 0$, so $\eta, \nu, \eta \restriction \delta$, are as required.]

Now as $\delta \in S \cap C_\eta \subseteq C_\eta$ it follows that $M_{\eta \restriction \delta} \cong M$ hence $M_{\eta \restriction \delta}$ fails the $\lambda$-amalgamation property. Also $M_{\eta \restriction \delta}$ has universe $\delta$ as $\delta \in C_\eta$ and $M_{\eta \restriction \delta} = M_{\nu \restriction \delta}$ as $\eta \restriction \delta = \nu \restriction \delta$.

So $f_\eta \restriction M_{\eta \restriction \delta} = f_\eta \restriction \delta = f_\nu \restriction \delta = f_\nu \restriction M_{\nu \restriction \delta}$. So $f_\eta \restriction M_{\eta \restriction (\delta+1)}, f_\nu \restriction M_{\nu \restriction (\delta+1)}$ show that $M_{\eta \restriction (\delta+1)}, M_{\nu \restriction (\delta+1)}$, can be amalgamated over $M_{\eta \restriction \delta}$ contradicting clause (v)(b) of the construction, i.e. of $\circledast$. So there is no $\leq_{\mathfrak{K}}$-universal $N \in \mathfrak{K}_{\lambda^+}$.

It takes some more effort to get $2^{\lambda^+}$ pairwise non-isomorphic models (rather than just quite many).

Case A [5]: There is $M^* \in K_\lambda, M \leq_{\mathfrak{K}} M^*$ such that for every $N$ satisfying $M^* \leq_{\mathfrak{K}} N \in K_\lambda$ there are $N^1, N^2 \in K_\lambda$ such that $N \leq_{\mathfrak{K}} N^1, N \leq_{\mathfrak{K}} N^2$ and $N^2, N^1$ cannot be $\leq_{\mathfrak{K}}$-amalgamated over $M^*$ (not just $N$). In this case we do not need "$M$ is $S$-weakly limit".

We redefine $M_\eta, \eta \in {}^\alpha 2, \alpha < \lambda^+$ such that:

$\circledast_2$ (a) $\nu \triangleleft \eta \in {}^\alpha 2 \Rightarrow M_\nu \leq_{\mathfrak{K}} M_\eta \in K_\lambda$:

(b) if $\alpha = 0, M_{<>} = M^*$;

(c) if $\alpha$ limit and $\eta \in {}^\alpha 2$ then $M_\eta = \bigcup\limits_{\beta<\alpha} M_{\eta \restriction \beta}$;

(d) if $\eta \in {}^\beta 2, \alpha = \beta + 1$, use the assumption for $N = M_\eta$, now obviously the $(N^1, N^2)$ there satisfies $N^1 \neq N$ and $N^2 \neq N$, so we can have $M_\eta <_{\mathfrak{K}} M_{\eta\hat{\ }<1>} \in K_\lambda, M_\eta <_{\mathfrak{K}} M_{\eta\hat{\ }<0>} \in K_\lambda$, such that $M_{\eta\hat{\ }<0>}, M_{\eta\hat{\ }<1>}$ cannot be amalgamated over $M^*$.

Obviously, the models $M_\eta = \bigcup\limits_{\alpha<\lambda^+} M_{\eta \restriction \alpha}$, for $\eta \in {}^{\lambda^+}2$ are pairwise non-isomorphic over $M^*$ and by 0.4 as $2^\lambda < 2^{\lambda^+}$ we finish proving $\dot{I}(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$.

Note also that for each $\eta \in {}^{\lambda^+}2$ the set $\{\nu \in {}^{\lambda^+}2 : M_\nu$ can be $\leq_{\mathfrak{K}}$-embedded into $M_\eta\}$ has cardinality $\leq |\{f : f$ a $\leq_{\mathfrak{K}}$-embedding of $M^*$ into $M_\eta\}| \leq 2^\lambda$. So if $(2^\lambda)^+ < 2^{\lambda^+}$, then by Hajnal free subset theorem ([Haj62]), there are $2^{\lambda^+}$ models $M_\eta \in K_{\lambda^+} (\eta \in {}^{\lambda^+}2)$ no one $\leq_{\mathfrak{K}}$-embeddable into another. Case B: Not Case A.

Now we return to the first construction, but we can add

(vii) if $\eta \in {}^{(\alpha+1)}2$, then if $M_\eta \leq_{\mathfrak{K}} N^1, N^2$ both in $K_\lambda$, then $N^1, N^2$ can be $\leq_{\mathfrak{K}}$-amalgamated over $M_{\eta \restriction \alpha}$.

As $\{W \subseteq \lambda^+ : W$ is small$\}$ is a normal ideal (see 0.6), (and it is on a successor cardinal) it is well known that we can find $\lambda^+$ pairwise disjoint non-small $S_\zeta \subseteq S$ for $\zeta < \lambda^+$. We define a colouring (= function) $\mathbf{c}$:

[5]we can make it a separate claim

$\circledast_3$ (a) $\mathbf{c}(\eta,\nu,f)$ will be defined <u>iff</u> for some limit ordinal $\delta < \lambda^+, \eta \in {}^{\delta}2, \nu \in {}^{\delta}2$

and $f$ is a function from $\delta$ to $\lambda^+$

(b) $\mathbf{c}(\eta,\nu,f) = 1$ <u>iff</u> the triple $(\eta,\nu,f)$ belongs to the domain of $\mathbf{c}$ (i.e., is

as in (a)) and $M_\eta, M_\nu$ have universe $\delta, f$ is a $\leq_{\mathfrak{K}}$-embedding of $M_\eta$

into $M_\nu$ and for some $\rho, \nu\hat{\ } < 0 > \triangleleft \rho \in {}^{\lambda^+}2$ the function $f$ can be

extended to a $\leq_{\mathfrak{K}}$-embedding of $M_{\eta\hat{\ }<0>}$ into $M_\rho$

(c) $\mathbf{c}(\eta,\nu,f)$ is zero <u>iff</u> it is defined but is $\neq 1$.

For each $\zeta$, as $S_\zeta$ is not small, by simple coding, for every $\zeta < \lambda^+$ there is $h_\zeta : S_\zeta \to \{0,1\}$ such that:

$(*)_\zeta$ for every $\eta \in {}^{\lambda^+}2, \nu \in {}^{\lambda^+}2$ and $f : \lambda^+ \to \lambda^+$, for a stationary set of $\delta \in S_\zeta$

$$\mathbf{c}(\eta \restriction \delta, \nu \restriction \delta, f \restriction \delta) = h_\zeta(\delta).$$

Now for every $W \subseteq \lambda^+$ we define $\eta_W \in {}^{\lambda^+}2$ as follows:

$\eta_W(\alpha)$ is $h_\zeta(\alpha)$, if $\zeta \in W$ and $\alpha \in S_\zeta$ (note that there is at most one $\zeta$)

$\eta_W(\alpha)$ is zero if there is no such $\zeta$.

Now we can show (chasing the definitions) that

$\circledast_4$ if $W(1), W(2) \subseteq \lambda^+, W(1) \nsubseteq W(2)$, <u>then</u> $M_{\eta_{W(1)}}$ cannot be $\leq_{\mathfrak{K}}$-embedded into $M_{\eta_{W(2)}}$.

This clearly suffices.

Why is $\circledast_4$ true? Suppose $W(1) \nsubseteq W(2)$, let $\zeta \in W(1) \backslash W(2)$ and toward contradiction let $f$ be a $\leq_{\mathfrak{K}}$-embedding of $M_{\eta_{W(1)}}$ into $M_{\eta_{W(2)}}$, so $E = \{\delta : M_{\eta_{W(1)}\restriction\delta}, M_{\eta_{W(2)}\restriction\delta}$ have universe $\delta$ and $f \restriction \delta$ is a $\leq_{\mathfrak{K}}$-embedding of $M_{\eta_{W(1)}\restriction\delta}$ into $M_{\eta_{W(2)}\restriction\delta}\}$ is a club of $\lambda^+$. Hence by the choice of $\mathbf{c}$ and $h_\zeta$ there is $\delta \in E \cap S_\zeta$ such that

$\boxtimes$ $\mathbf{c}(\eta_{W(1)} \restriction \delta, \eta_{W(2)} \restriction \delta, f \restriction \delta) = h_\zeta(\delta)$ and $M_{\eta_{w(1)}\restriction\delta}$ is not an amalgamation base.

Now the proof splits to two cases. <u>Case 1</u>: $h_\zeta(\delta) = 0$.

So $\eta_{W(1)}(\delta) = 0 = \eta_{W(2)}(\delta)$ and by clause (b) of $\circledast_3$ above, i.e., the definition of $\mathbf{c}$ we have the objects $\eta_{W(1)}, \eta_{W(2)}, f \restriction M_{\eta_{W(1)}\hat{\ }<0>} = f \restriction M_{\eta_{W(1)}\restriction(\delta+1)}$ witness that $\mathbf{c}(\eta_{W(1)} \restriction \delta, \eta_{W(2)} \restriction \delta, f \restriction \delta) = 1$, contradiction. <u>Case 2</u>: $h_\zeta(\delta) = 1$.

So $\eta_{W(1)}(\delta) = 1, \eta_{W(2)}(\delta) = 0, \mathbf{c}(\eta_{W(1)} \restriction \delta, \eta_{W(2)} \restriction \delta, f \restriction \delta) = 1$. By the definition of $\mathbf{c}$, we can find $\nu$ such that $(\eta_{W(2)} \restriction \delta)\hat{\ } < 0 > \trianglelefteq \nu \in {}^{\lambda^+}2$ and a $\leq_{\mathfrak{K}}$-embedding $g$ of $M_{(\eta_{W(1)}\restriction\delta)\hat{\ }<0>}$ into $M_\nu$.

For some $\alpha \in (\delta, \lambda^+), f$ embeds $M_{\eta_{W(1)}\restriction(\delta+1)} = M_{(\eta_{W(1)}\restriction\delta)\hat{\ }<1>}$ into $M_{\eta_{W(2)}\restriction\alpha}$ and $g$ embeds $M_{(\eta_{W(1)}\restriction\delta)\hat{\ }<0>}$ into $M_{\nu\restriction\alpha}$.

As $\eta_{W(2)} \restriction \delta\hat{\ } < 0 > \triangleleft \nu \restriction \alpha$ and $\eta_{W(2)} \restriction \delta\hat{\ } < 0 > \triangleleft \eta_{W(2)} \restriction \alpha$ by clause (vii) above there are $f_1, g_1$ and $N \in K_\lambda$ such that

(a) $M_{\eta_{W(2)}\restriction\delta} \le_{\mathfrak{K}} N$

(b) $f_1$ is a $\le_{\mathfrak{K}}$-embedding of $M_{\eta_{W(2)}\restriction\alpha}$ into $N$ over $M_{\eta_{W(2)}\restriction\delta}$

(c) $g_1$ is a $\le_{\mathfrak{K}}$-embedding of $M_{\nu\restriction\alpha}$ into $N$ over $M_{\eta_{W(2)}\restriction\delta}$.

So

$(b)^*$ $f_1 \circ f$ is a $\le_{\mathfrak{K}}$-embedding of $M_{(\eta_{W(1)}\restriction\delta)\char`\^<1>}$ into $N$

$(c)^*$ $g_1 \circ g$ is a $\le_{\mathfrak{K}}$-embedding of $M_{(\eta_{W(1)}\restriction\delta)\char`\^<0>}$ into $N$

$(d)^*$ $f_1 \circ f, g_1 \circ g$ extend $f \restriction \delta : M_{\eta_{W(1)}\restriction\delta} \to N$ (both).

So together we get a contradiction to assumption $(*)_1(d)$. $\square_{3.9}$

{88r-3.6}

3.11. **Theorem.** *1) Assume one of the following cases occurs:*

$(a)_1$ *$\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$ (hence $\mathrm{LST}(\mathfrak{K}) = \aleph_0$) and $1 \le \dot{I}(\aleph_1, \mathfrak{K}) < 2^{\aleph_1}$*
*or*

$(a)_2$ *$\mathfrak{K}$ has models of arbitrarily large cardinality, $\mathrm{LST}(\mathfrak{K}) = \aleph_0$ and $\dot{I}(\aleph_1, \mathfrak{K}) < 2^{\aleph_1}$.*

*Then there is an a.e.c. $\mathfrak{K}_1$ such that*

(A) *$M \in K_1 \Rightarrow M \in K$ and $M \le_{\mathfrak{K}_1} N \Rightarrow M \le_{\mathfrak{K}} N$ and $\mathrm{LST}(\mathfrak{K}_1) = \mathrm{LST}(\mathfrak{K})(= \aleph_0)$*

(B) *if $K$ has models of arbitrarily large cardinality then so does $K_1$*

(C) *$\mathfrak{K}_1$ is $\mathrm{PC}_{\aleph_0}$*

(D) *$(K_1)_{\aleph_1} \ne \emptyset$*

(E) *all models of $K_1$ are $\mathbb{L}_{\infty,\omega}$-equivalent and $M \le_{\mathfrak{K}_1} N \Leftrightarrow M \prec_{\mathbb{L}_{\infty,\omega}} N \& M \le_{\mathfrak{K}} N$ and $K_1$ is categorical in $\aleph_0$ and $M_* \in (K_1)_{\aleph_0} \Rightarrow K_1 = \{N \in K : N \equiv_{\mathbb{L}_{\infty,\omega}(\tau_K)} M_*\}$*

(F) *if $\mathfrak{K}$ is categorical in $\aleph_1$ then $(K_1)_\lambda = K_\lambda$ for every $\lambda > \aleph_0$; moreover $\le_{\mathfrak{K}_1} = \le_{\mathfrak{K}} \restriction (K_1)_{\ge\aleph_1}$.*

*2) If in (1) we add $\mathrm{LST}(\mathfrak{K})$ names to formulas in $\mathbb{L}_{\infty,\omega}$ (i.e. to a set of representatios up to equivalence) then we can assume each member of $K$ is $\aleph_0$-sequence-homogeneous. The vocabulary remains countable, in fact, for some countable first order theory $T$, the models of $K$ are the atomic models of $T$ (in the first order sense) and $\le_{\mathfrak{K}}$ becomes $\subseteq$ (being a submodel).*

*Proof.* Like [Sh:48, 2.3,2.5] (using 2.20 here for $\alpha = 2$). E.g. why, if $K$ is categorical in $\aleph_1$ then $\le_{\mathfrak{K}_1} = \le_{\mathfrak{K}} \restriction (K_1)_{\ge\aleph_1}$? We have to prove that if $M \le_{\mathfrak{K}} N$ are uncountable then $M \prec_{\mathbb{L}_{\infty,\omega}(\tau_K)} N$. But there is $M_* \in K_{\aleph_0}$ such that $K_1 = \{M' \in K : M' \equiv_{\mathbb{L}_{\infty,\omega}} M_*\}$ and $(K_1)_{\aleph_1} = K_{\aleph_1} \ne \emptyset$, so it suffices to prove $M \prec_{\mathbb{L}_{\omega_1,\omega}(T)} N$, so assume this is a counterexample so for some $\varphi(x,\bar{y}) \in \mathbb{L}_{\omega_1,\omega}(\tau)$ and $\bar{a} \in {}^{\ell g(\bar{y})}M, b \in N$ we have $N \models \varphi[b,\bar{a}]$ but for no $b' \in M$ do we have $N \models \varphi[b',\bar{a}]$ and without loss of generality the quantifier depth of $\varphi(x,\bar{y}), \gamma$ is minimal (for all such pairs $(M,N)$). Let $\Delta_\gamma = \{\psi(\bar{z}) \in \mathbb{L}_{\omega_1,\omega}(\tau_K) : \psi$ has quantifier depth $\le \gamma\}$ hence $M' \le_{\mathfrak{K}} N', M' \in K_{>\aleph_0} \Rightarrow M' \prec_{\Delta_\gamma} N'$. Also without loss of generality $\|M\| = \|N\| = \aleph_1$. Now choose $M_\alpha \in K_{\aleph_1}$ by induction on $\alpha < \omega_2$, which is $\le_{\mathfrak{K}}$-increasing continuous (hence $\prec_{\Delta_\gamma}$ increases) and for each $\alpha$ there is an isomorphism $f_\alpha$ from $N$ onto

$M_{\alpha+1}$ mapping $M$ onto $M_\alpha$, recalling the categoricity. By Fodor lemma for some $\alpha < \beta$ we have $f_\alpha(\bar{a}) = f_\beta(\bar{a})$, so $f_\beta^{-1}(f_\alpha(b))$ contradict the choice of $\varphi(x, \bar{y}), b, \bar{a}$. $\square_{3.11}$

We arrive to the main theorem of this section.

{88r-3.7}

3.12. **Theorem.** *Suppose $\mathfrak{K}$ and $\lambda$ satisfy the following conditions:*

$(A)$ *$\mathfrak{K}$ has a superlimit member $M^*$ of cardinality $\lambda, \lambda \geq \mathrm{LST}(\mathfrak{K})$, (if $K$ is categorical in $\lambda$, then by assumption $(B)$ below there is such $M^*$; really invariantly $\lambda^+$-strongly limit suffice if (d) of $(*)$ of 3.13(2) below holds, see Definition 3.3)*

$(B)$ *$\mathfrak{K}$ is categorical in $\lambda^+$*

$(C)$ $(\alpha)$ *$\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}, \lambda = \aleph_0$ or*

$(\beta)$ *$\mathfrak{K} = \mathrm{PC}_\lambda, \lambda = \beth_\delta, \mathrm{cf}(\delta) = \aleph_0$ or*

$(\gamma)$ *$\lambda = \aleph_1, \mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$ or*

$(\delta)$ *$\mathfrak{K}$ is $\mathrm{PC}_\mu, \lambda \geq \beth_{(2^\mu)^+}$; not useful for 3.12, still it too implies $(*)_{\lambda,\mu}$ in 3.13.*

*Then $K$ has a model of cardinality $\lambda^{++}$.*

{88r-3.7A

3.13. **Remark.** 1) If $\lambda = \aleph_0$ we can wave hypothesis (A) by the previous theorem 3.11.
2) Hypothesis (C) can be replaced by (giving a stronger theorem):

$(*)_{\lambda,\mu}(a)$ $\mathfrak{K}$ is $\mathrm{PC}_\mu$ and

$(b)$ any $\psi \in \mathbb{L}_{\mu^+,\omega}$ which has a model $M$ of order-type $\lambda^+, |P^M| = \lambda$, has a non-well-ordered model $N$ of cardinality $\lambda$

$(c)$ $\{M \in K_\lambda : M \cong M^*\}$ is $\mathrm{PC}_\mu$ (among models in $K_\lambda$) and

$(d)$ for some $\mathbf{F}$ witnessing "$M^*$ is invariantly $\lambda$-strongly limit", that is the class $\{(M, \mathbf{F}(M)) : M \in K_\lambda\}$ is $\mathrm{PC}_\mu$ (if $M^*$ is superlimit this clause is not required as $\mathbf{F}$ = the identity on $K_\lambda$ is O.K.)

3) It is well known, see e.g. [Sh:c, VII,§5] that hypothesis (C) implies $(*)_{\lambda,\mu}$ from part (2), see more [GrSh:259].

*Proof.* By 3.13(3) we can assume $(*)_{\lambda,\mu}$ from 3.13(2).

Stage a: It suffices to find $N_0 \leq_{\mathfrak{K}} N_1, \|N_0\| = \lambda^+, N_0 \neq N_1$.

Why? We define by induction on $\alpha < \lambda^{++}$ a model $N_\alpha \in K_{\lambda^+}$ such that $\beta < \alpha$ implies $N_\beta \leq_{\mathfrak{K}} N_\alpha$ and $N_\beta \neq N_\alpha$. Clearly $N_0, N_1$ are defined (without loss of generality $\|N_1\| = \lambda^+$ as $\lambda \geq \mathrm{LST}(\mathfrak{K})$, also otherwise we already have the desired conclusion), for limit $\delta < \lambda^{++}$ the model $\bigcup_{\alpha<\delta} N_\alpha$ is as required. For $\alpha = \beta + 1$, by the $\lambda^+$-categoricity, $N_0$ is isomorphic to $N_\beta$, say by $f$ and we define $N_{\beta+1}$ such that $f$ can be extended to an isomorphism from $N_1$ onto $N_{\beta+1}$, so clearly $N_{\beta+1}$ is as required. Now $\bigcup_{\alpha<\lambda^{++}} N_\alpha \in K_{\lambda^{++}}$ is as required. Hence the following theorem completes the proof of 3.12 (use $\mathbf{F}$ = the identity for the superlimit case). $\square_{3.12}$

We can find $N_0 \leq_{\mathfrak{K}} N_1, N_0 \neq N_1$ such that $N_0, N_1 \in K^{\mathbf{F}}_{\lambda^+}$, when the following clauses hold:

{88r-3.8}

3.14. **Theorem.** *Suppose the following clauses:*

- $(A)$ *$\mathfrak{K}$ has an invariantly $\lambda$-strongly limit member $M^*$ of cardinality $\lambda$, as exemplified by $\mathbf{F}: K_\lambda \to K_\lambda$ and $\mathfrak{K}_\lambda$ has the* JEP *(see Definition 3.3)*
- $(B)$ $\dot{I}(\lambda^+, K_{\lambda^+}) < 2^{\lambda^+}$ *or even just* $\dot{I}(\lambda^+, K^{\mathbf{F}}_{\lambda^+}) < 2^{\lambda^+}$ *(or just* $\dot{I}\dot{E}(\lambda^+, K^{\mathbf{F}}_{\lambda^+}) < 2^{\lambda^+}$ *(see below))*
- $(C)$ *$\mathfrak{K}$ is a* $\mathrm{PC}_\mu$ *class, as well as* $\mathbf{F}$, *i.e., $K'$ is* $\mathrm{PC}_\mu$ *where $K'$ is a class closed under an isomorphism of $(\tau_{\mathfrak{K}} \cup \{P\})$-models, $P$ a unary predicate such that* $K'_\lambda = \{(N, M) : N = \mathbf{F}(M)\}$
- $(D)$ $\mu = \lambda = \aleph_0$ *or* $\mu = \lambda = \beth_\delta, \mathrm{cf}(\delta) = \aleph_0$ *or* $\mu = \aleph_0, \lambda = \aleph_1$ *or just* $(*)_{\lambda,\mu}(c)$ *from 3.13(2)*
- $(E)$ *$K$ categorical in $\lambda$ or at least there is* $\psi \in \mathbb{L}_{\omega_1,\omega}(\tau^+)$ *such that* $(M^*/\cong) = \{M \restriction \tau_{\mathfrak{K}} : M \models \psi, \|M\| = \lambda\}$.

where

{88r-3.8.1}

3.15. **Definition.** Assume $\mathbf{F} : K_\lambda \to K_\lambda$ satisfies $M \leq_{\mathfrak{K}} \mathbf{F}(M)$ for $M \in K_\lambda$ or more generally $\mathbf{F} \subseteq \{(M, N) : M \leq_{\mathfrak{K}} N$ are from $K_\lambda\}$ satisfies $(\forall M \in K_\lambda)(\exists N)((M, N) \in \mathbf{F})$ or just $(\forall M \in K_\lambda)(\exists N_0, N_1)[(N_0, N_1) \in \mathbf{F} \wedge M \leq_{\mathfrak{K}} N_0 \leq_{\mathfrak{K}} N_1]$. Then we let $K^{\mathbf{F}}_{\lambda^+} := \{\bigcup\limits_{i<\lambda^+} M_i : M_i \in K_\lambda, \langle M_i : i < \lambda^+\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous not eventually constant and $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$ or $(M_{i+1}, M_{i+2}) \in \mathbf{F}\}$ for $i < \lambda$.

{88r-3.8.3}

3.16. **Remark.** 1) As the sequence in the definition of $K^{\mathbf{F}}_{\lambda^+}$ is $\leq_{\mathfrak{K}}$-increasing and the sequence is not eventually constant (which follows if $(M, N) \in \mathbf{F} \Rightarrow M \neq N$), necessarily $K^{\mathbf{F}}_{\lambda^+} \subseteq \mathfrak{K}_{\lambda^+}$.
2) Theorem 3.14 is good for classes which are not exactly a.e.c., see, e.g., 3.19.

Considering $K^{\mathbf{F}}_{\lambda^+}$ we may note that the proofs of some earlier claims give more. In particular (before proving 3.14), similarly to 3.9:

{88r-3.8.4}

3.17. **Claim.** *Assume that*

- $(a)$ $2^\lambda < 2^{\lambda^+}$
- $(b)$ *$\mathfrak{K}$ is an a.e.c. and* $\mathrm{LST}(\mathfrak{K}) \leq \lambda$
- $(c)$ *$M \in K_\lambda$ is $S$-weakly limit, $S$ not small (see Definition 0.6)*
- $(d)$ *$M$ does not have the amalgamation property in $\mathfrak{K}$ (= is an amalgamation base)*
- $(e)$ *$\mathbf{F}$ is as in 3.15.*

<u>Then</u> $\dot{I}(\lambda^+, K^{\mathbf{F}}_{\lambda^+}) = 2^{\lambda^+}$.

*Proof.* To avoid confusion rename $\mathbf{F}$ of clause (e) as $\mathbf{F}_1$, and choose $\mathbf{F}_2$ which exemplifies "$M$ is $S$-weakly limit", i.e., as in Definition 3.3(5). Now we define $\mathbf{F}'$ with the same domain as $\mathbf{F}_2$ by $\mathbf{F}'(\langle M_j : j \le i\rangle) = \mathbf{F}_1(\mathbf{F}_2(\langle M_j : j \le i\rangle)$, and continue as in the proof of 3.9 noting that $\mathbf{F}'$ works as well there.

The sequence of models $\langle M_\eta : \eta \in {}^{\lambda^+}2\rangle$ we got there are from $K^{\mathbf{F}_1}_{\lambda^+}$ (so witness that $\dot{I}(\lambda^+, K^{\mathbf{F}_1}_{\lambda^+}) = 2^{\lambda^+}$) because:

(*) if the sequence $\langle M_\alpha : \alpha < \lambda^+\rangle, M_\alpha \in \mathfrak{K}_\lambda$ for $\alpha < \lambda^+$ is $\le_{\mathfrak{K}}$-increasing continuous and $\mathbf{F}'(\langle M_j : j \le i+1\rangle) \le_{\mathfrak{K}} M_{i+2}$ then $\cup\{M_\alpha : \alpha < \lambda^+\} \in K^{\mathbf{F}_1}_{\lambda^+}$.

$\square_{3.17}$

Also similarly to 3.11 we can prove:

3.18. **Claim.** *Assume $\mathfrak{K}$ is a $\mathrm{PC}_{\aleph_0}$ and $\mathbf{F}$ a $\mathrm{PC}_{\aleph_0}$ is as in 3.15. If $1 \le \dot{I}(\aleph_1, K^{\mathbf{F}}_{\aleph_1}) < 2^{\aleph_1}$ <u>then</u> the conclusion of 3.11 above holds.* {88r-3.8.

*Proof of 3.14.* (Hence of 3.12). The reader may do well to read it with $\mathbf{F}$ = the identity in mind.

<u>Stage b</u>: We now try to find $N_0, N_1$ as mentioned in stage (a) above by approximations of cardinality $\lambda$. A triple will denote here $(M, N, a)$ satisfying $M, N \cong M^*$ (see hypothesis $(A)$), $M \le_{\mathfrak{K}} N$ and $a \in N\backslash M$. Let $<$ be the following partial order among this family of triples: $(M, N, a) < (M', N', a')$ if $a = a', N \le_{\mathfrak{K}} N', M \le_{\mathfrak{K}} M', M \ne M'$ and moreover $(\exists N'')[N \le_{\mathfrak{K}} N'' \& \mathbf{F}(N'') \le_{\mathfrak{K}} N']$ and $(\exists M'')[M \le_{\mathfrak{K}} M'' \& \mathbf{F}(M'') \le_{\mathfrak{K}} M']$. (It is tempting to omit $a$ and require $M = M' \cap N$, but this apparently does not work as we do know if disjoint amalgamation $\mathfrak{K}_{\aleph_0}$ exist).

We first note that there is at least one triple (as $M^*$ has a proper elementary extension which is isomorphic to it, because it is a limit model by clause (A) of the assumption). <u>Stage c</u>: We show that if there is no maximal triple, our conclusions follows.

We choose by induction on $\alpha$ a triple $(M_\alpha, N_\alpha, a)$ increasing by $<$. For $\alpha = 0$ see the end of previous stage, for $\alpha = \beta + 1$, we can define $(M_\alpha, N_\alpha, a)$ by the hypothesis of this stage. For limit $\delta < \lambda^+, (M_\delta, N_\delta, a)$ will be $(\bigcup\limits_{\alpha<\delta} M_\alpha, \bigcup\limits_{\alpha<\delta} N_\alpha, a)$ (notice $M_\delta \le_{\mathfrak{K}} N_\delta$ by AxIV of 1.2 and $M_\delta, N_\delta$ are isomorphic to $M^*$ by the choice of $\mathbf{F}$ and the definition of order on the family of triples). Now similarly $M = \bigcup\limits_{\alpha<\lambda^+} M_\alpha \le_{\mathfrak{K}} N = \bigcup\limits_{\alpha<\lambda^+} N_\alpha$ are both from $\mathfrak{K}^{\mathbf{F}}_{\lambda^+}$ and the element $a$ exemplifies $M \ne N$, so by Stage (a) we finish.
Recall

$\circledast$ if $(M, N, a)$ is a maximal triple then there is no triple $(M', N', a)$ such that $M' \leq_{\mathfrak{K}} N', M <_{\mathfrak{K}} M', N \leq_{\mathfrak{K}} N', a \in N' \backslash M'$ and $(\exists M'')(M \leq_{\mathfrak{K}} M'' \leq_{\mathfrak{K}} \mathbf{F}(M'') \leq_{\mathfrak{K}} M')$ and $(\exists N'')(N \leq_{\mathfrak{K}} N'' \leq_{\mathfrak{K}} \mathbf{F}(N'') \leq_{\mathfrak{K}} N')$.

Stage d: There are $M_i \cong M^*$ for $i \leq \omega$ such that $[i < j \leq \omega \Rightarrow M_j <_{\mathfrak{K}} M_i], i < \omega \Rightarrow \mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_i$ and $|M_\omega| = \bigcap_{n<\omega} |M_n|$ and note that $M_i$ is $\lambda^+$-strongly limit.

This stage is dedicated to proving this statement. As $M^*$ is superlimit (or just strongly limit), there is an $\leq_{\mathfrak{K}}$-increasing continuous sequence $\langle M_i : i < \lambda^+\rangle, M_i \cong M^*$ and $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$. (Note that this is true also for limit models as we can restrict ourselves to a club of $i$'s). So without loss of generality $\bigcup_{i<\lambda^+} M_i$ has universe $\lambda^+, M_0$ has universe $\lambda$.

Define a model $\mathfrak{B}$.

Its universe is $\lambda^+$. Relations and Functions:

(a) those of $\bigcup_{i<\lambda^+} M_i$

(b) $R$-two place: $aRi$ if and only if $a \in M_i$

(c) $P$ (monadic relation) $P = \lambda$ which is the universe of $M_0$

(d) $g$, a two-place function such that for each $i, g(i, -)$ is an isomorphism from $M_0$ onto $M_i$

(e) $<$ (two-place relation) - the usual ordering (on the ordinals $< \lambda^+$)

(f) relations with parameter $i$ witnessing $M_i \leq_{\mathfrak{K}} \bigcup_{j<\lambda^+} M_j$ (we can instead make functions witnessing $M \in K$ as in 1.10 (the strong version) and have: each $M_i$ is closed under them))

(g) relations with parameter $i$ witnessing each $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2}$ and $M_{i+1} \neq M_{i+2}$ (including $(M_{i+1}, \mathbf{F}(M_{i+1})) \in \mathbf{F}$)

(h) if $\mu = \lambda$, also individual constant for each $a \in M_0$.

Let $\psi \in \mathbb{L}_{\mu^+,\omega}$ describe this, in particular for clauses (f), (g) use clause (C) of the assumptions. So $\psi$ has a non-well ordered model $\mathfrak{B}^*, |P^{\mathfrak{B}^*}| = \lambda$ (by clause (D) of the assumption see 3.13(2)+(3)). So let

$$\mathfrak{B}^* \models \text{“}a_{n+1} < a_n\text{” for } n < \omega.$$

Let for $a \in \mathfrak{B}^*, A_a = \{x \in \mathfrak{B}^* : \mathfrak{B}^* \models xRa\}$

$$M_a = (\mathfrak{B}^* \restriction \tau_{\mathfrak{K}}) \restriction A_a.$$

Easily $M_a \leq_{\mathfrak{K}} (\mathfrak{B}^* \restriction \tau_{\mathfrak{K}})$ (use clause (f)) and $\|M_a\| = \lambda$. In fact $M_a$ is superlimit or just isomorphic to $M^*$ if $\mu = \lambda$, as $\psi$ includes the diagram of $M_0 = M^*$, having names for all members, and if $\mu < \lambda$ see assumption (E). So $M_{a_n} \leq_{\mathfrak{K}} \mathfrak{B}^* \restriction \tau_{\mathfrak{K}}, M_{a_{n+1}} \subseteq M_{a_n}$ hence $M_{a_{n+1}} \leq_{\mathfrak{K}} M_{a_n}$ by Ax V. Let $M_n := M_{a_n}$. Let $I = \{b \in \mathfrak{B}^* : \bigwedge\limits_{n<\omega} [\mathfrak{B}^* \models b < a_n]\}$.

Also as for $b \in I, M_b <_{\mathfrak{K}} \mathfrak{B}^* \restriction \tau_{\mathfrak{K}}$ and $M_{b_1} <_{\mathfrak{K}} M_{b_2}$ for $b_1 <^{\mathfrak{B}^*} b_2$, by Ax IV clearly $M_\omega := (\mathfrak{B}^* \restriction (\tau_{\mathfrak{K}})) \restriction \bigcup\limits_{b\in I} A_b$ satisfies $M_\omega \leq_{\mathfrak{K}} \mathfrak{B}^* \restriction \tau_{\mathfrak{K}}$ hence $M_\omega \leq_{\mathfrak{K}} M_n$ for $n < \omega$. Obviously $M_\omega \subseteq \bigcap\limits_{n<\omega} M_n$ and equality holds as $\psi$ guarantee

$(*)$ for every $y \in \mathfrak{B}^*$ there is a minimal $x \in \mathfrak{B}^*$ such that $y \in M_x$.

As each $M_b$ is isomorphic to $M^*$, of cardinality $\lambda$, also $M_\omega$ is. Stage e: Suppose that there is a maximal triple, then we shall show $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ and moreover $\dot{I}(\lambda^+, K^{\mathbf{F}}_{\lambda^+}) = 2^{\lambda^+}$, and so we shall get a contradiction to assumption (B).

So there is a maximal triple $(M^0, N^0, a)$. Hence by the uniqueness of the limit model for each $M \in K_\lambda$ which is isomorphic to $M^*$ hence to $M^0$ there are $N, a$ satisfying $M \leq_{\mathfrak{K}} N \cong M^* \in K_\lambda, a \in N \backslash M$ such that: if $M <_{\mathfrak{K}} M' \leq_{\mathfrak{K}} N' \in \mathfrak{K}_\lambda, N <_{\mathfrak{K}} N', (\exists M'')(M \leq_{\mathfrak{K}} M'' \leq_{\mathfrak{K}} \mathbf{F}(M'') \leq_{\mathfrak{K}} M' \cong M^*)$ and $(\exists N'')(N \leq_{\mathfrak{K}} N'' \leq_{\mathfrak{K}} \mathbf{F}(N'') \leq_{\mathfrak{K}} N' \cong M^*)$ then $a \in M'$. (That is, in some sense $a$ is algebraic over $M$). We can waive $(\exists N'')(N \leq_{\mathfrak{K}} N'' \leq_{\mathfrak{K}} \mathbf{F}(N'') \leq_{\mathfrak{K}} N' \cong M^*)$ as by the definition of strongly limit there is $N'_* \cong M^*$ such that $\mathbf{F}(N') \leq_{\mathfrak{K}} N'_*$. On the other hand by Stage d

$(*)_1$ for each $M \in K_\lambda$ isomorphic to $M^*$ there are $M'_n (n < \omega)$ such that $M \leq_{\mathfrak{K}} M'_{n+1} <_{\mathfrak{K}} M'_n \in K_\lambda, M'_n \cong M^*$ and $\mathbf{F}(M'_{n+1}) \leq_{\mathfrak{K}} M'_n$ and $\bigcap\limits_{n<\omega} M'_n = M$.

For notational simplicity: $M \in K_\lambda, |M|$ an ordinal $\Rightarrow |\mathbf{F}(M)|$ an ordinal.

Now for each $S \subseteq \lambda^+$ we define by induction on $\alpha \leq \lambda^+, M^S_\alpha$, increasing (by $<_{\mathfrak{K}}$) and continuous with universe an ordinal $< \lambda^+$ such that $M^S_\alpha \cong M^*$ and if $\beta + 2 \leq \alpha$ then $\mathbf{F}(M_{\beta+1}) \leq_{\mathfrak{K}} M_{\beta+1}$. Let $M^S_0 = M^*$ and for limit $\delta < \lambda^+$ and let $M^S_\delta = \bigcup\limits_{\alpha<\delta} M^S_\alpha$; by the induction assumption and the choice of $M^*, \mathbf{F}$ clearly $M^S_\delta$ is isomorphic to $M^*$. For $\alpha = \beta + 1, \beta$ successor let $M^S_\alpha$ be such that $\mathbf{F}(M^S_\beta) <_{\mathfrak{K}} M^S_\alpha \cong M^*$. So we are left with the case $\alpha = \delta + 1, \delta$ limit or zero.

Now if $\delta \in S$ hence $M^S_\delta \cong M^*$, choose $M_{\delta+1}, a^S_\delta$ such that $(M^S_{\delta+1}, M^S_\delta, a^S_\delta)$ is a maximal triple (possible as by the hypothesis of this case there is a maximal triple, and there is a unique strong limit model). If $\delta \notin S$ we choose $M^{S,n}_\delta \in K_\lambda$ for $n < \omega$ (not used) such that $M^S_\delta <_{\mathfrak{K}} M^{S,n+1}_\delta \leq_{\mathfrak{K}} M^{S,n}_\delta$ and $\mathbf{F}(M^{S,n+1}_\delta) \leq_{\mathfrak{K}} M^{S,n}_\delta$ for $n < \omega$ and $M^S_\delta = \bigcap\limits_{n<\omega} M^{S,n}_\delta$ and $M^{S,n}_\delta \cong M^*$; and let $M^S_{\delta+1} = M^{S,0}_\delta$ (again possible as $M_\delta \cong M^*$ and an $(*)_1$ above).
Lastly, let $M^S = \bigcup\limits_\alpha M^S_\alpha$.

Now clearly it suffices to prove that if $S^0, S^1 \subseteq \lambda^+, S^1 \backslash S^0$, is stationary then $M^{S^1} \ncong M^{S^0}$. So suppose $f$ is a $\leq_{\mathfrak{K}}$-embedding from $M^{S^1}$ onto $M^{S^0}$ or just into $M^{S^0}$. Then

$E^2 = \{\delta < \lambda^+ : M^{S^1}_\delta, M^{S^0}_\delta$ each has universe $\delta$ and $i < \lambda^+$ implies $[i < \delta \Leftrightarrow f(i) < \delta]\}$ is a closed unbounded subset of $\lambda^+$, hence there is a limit ordinal $\delta \in (S^1 \backslash S^0) \cap E^2$. Let us look at $f(a^{S^1}_\delta)$; as $\delta \in S^1, a^{S^1}_\delta$ is well defined, also $a^{S_1}_\delta \in M^{S^1}_{\delta+1} \backslash M^{S^1}_\delta$, as $\delta \in E^2$ it follows that $f(a^{S^1}_\delta) \not< \delta$ hence $f(a^{S^1}_\delta)$ belongs to $M^{S^0} \backslash M^{S^0}_\delta$ but $M^{S^0}_\delta = \bigcap\limits_{n<\omega} M^{S^0,n}_\delta$ (as $\delta \notin S^0$).

Hence for some $n, f(a^{S^1}_\delta) \notin M^{S^0,n}_\delta$. Let $\beta \in (\delta, \lambda^+)$ be large enough such that $f(M^{S^1}_{\delta+1}) \subseteq M^{S^0}_\beta$. But then $f(M^{S^1}_\delta) \leq_{\mathfrak{K}} M^{S^0,n}_\delta \leq_{\mathfrak{K}} M^{S^0}_\beta$ and $f(M^{S^1}_{\delta+1}) \leq_{\mathfrak{K}} M^{S^0}_\beta$ and $a^{S^1}_\delta \notin f^{-1}(M^{S^0,n}_\delta)$. Now $(f(M^{S^1}_\delta)), f(M^{S^1}_{\delta+1}), f(a^{S^1}_\delta))$ has the same properties as $(M^{S^1}_\delta, M^{S^1}_{\delta+1}, a^{S^1}_\delta)$ because if $f$ is an isomorphism from $M'$ onto $M'' \in K_\lambda$ then we can extend $f$ to an isomorphism from $\mathbf{F}(M')$ onto $\mathbf{F}(M'')$ (i.e., the "invariant"). But $(f(M^{S^1}_\delta), f(M^{S^1}_{\delta+1}), f(a^{S^1}_\delta)) < (M^{S^0,n}_\delta, M^{S^0}_\beta, f(a^{S^1}_\delta))$, contradiction. So we are done. $\square_{3.12}$

{88r-3.9}

3.19. **Conclusion.** 1) If $\mathrm{LST}(\mathfrak{K}) = \aleph_0, K$ is $\mathrm{PC}_{\aleph_0}$ and $\dot{I}(\aleph_1, K) = 1$, then $K$ has a model of cardinality $\aleph_2$.
2) If $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ ($\mathbf{Q}$ is the quantifier "there are uncountably many") has one and only one model of cardinality $\aleph_1$ up to isomorphism then $\psi$ has a model in $\aleph_2$.

*Proof.* 1) By 3.11 we get suitable $\mathfrak{K}_1$ (as in its conclusion) and by 3.12 the class $\mathfrak{K}_1$ has a model in $\aleph_2$, hence $\mathfrak{K}$ has a model in $\aleph_2$.
2) We can replace $\psi$ by a countable theory $T \subseteq \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$.
Let $L$ be a fragment of $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$ in which $T$ is included (e.g., $L$ is the closure of $T \cup$(the atomic formulas) under subformulas, $\neg, \wedge, (\exists x), (\mathbf{Q}x)$; in particular $L$ includes, of course, first order logic). By [Sh:48], without loss of generality $T$ "says" that every formula $\varphi(x_0, \ldots, x_{n-1})$ of $L$ is equivalent to an atomic formula (i.e., $P(x_0, \ldots, x_{n-1}), P$ a predicate) and every type realized in model of $T$ is isolated (i.e., every model is atomic), and $T$ is complete in $L$. Let

$$\begin{aligned} K = \{M : & M \text{ an atomic } \tau(T)\text{-model of } T \cap \mathbb{L} \text{ and if } M \models P[\bar{a}] \\ & \text{and } (\forall \bar{x})[P(\bar{x}) \equiv \neg(\mathbf{Q}y)R(y,\bar{x})] \in T \\ & \text{then } \{b : M \models R[b,\bar{a}]\} \text{ is countable}\} \end{aligned}$$

$M \leq_{\mathfrak{K}} N$ iff $M \leq^* N$ which means :(a) $M \prec_{\mathbb{L}} N$
(b) if $M \models P(\bar{a})$ and $\forall \bar{x}[P(\bar{x}) \equiv \neg \mathbf{Q} y R(y,\bar{x})] \in T$ then for no $b \in N \backslash M$ do we have $N \models R[b,\bar{a}]$.

So $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is categorical in $\aleph_0$, is an a.e.c. and is $\mathrm{PC}_{\aleph_0}$. Let $\mathbf{F}$ be (see 3.3(8)) such that for $M \in K_{\aleph_0}, N = \mathbf{F}(M)$ iff: $M <^{**} N$ which says $M \leq_{\mathfrak{K}} N \in K_{\aleph_0}$ and if $\bar{a} \in M, M \models P[\bar{a}], \forall \bar{x}[P(\bar{x}) \equiv \mathbf{Q} y R(y,\bar{x})] \in T$, then for some $b \in N \backslash M$ we have $N \models R[b,\bar{a}]$. So $\mathbf{F}$ is invariant.

Note that every $M \in K^{\mathbf{F}}_{\aleph_1}$ is a model of $\psi$. So 3.14 gives that some $M \in K^{\mathbf{F}}_{\aleph_1}$ has a proper extension in $K^{\mathbf{F}}_{\aleph_1}$.
The rest should be easy, just as in stage (a) of the proof of 3.12. $\square_{3.19}$

{88r-3.9A

3.20. **Question.** 1) Under the assumptions of 3.19(2), can we get $M \in K_{\aleph_2}$, such that: if $M \models P[\bar{a}], \forall \bar{x}[P(\bar{x}) \equiv (\mathbf{Q}y)R(y,\bar{x})] \in T$ then $\{b \in M : M \models R[b,\bar{a}]\}$ has cardinality $\aleph_2$? Note that in the proof of 3.14 we show that no triple is maximal.

{88r-3.9.

3.21. **Remark.** 1) We could have used multi-valued $\mathbf{F}$ then in the proof above $N = \mathbf{F}(M)$ just means the demand there.
2) To answer 3.20, i.e., to prove the existence of $M \in K_{\aleph_2}$ as above we have to prove:

- $(*)_1$ there are $N, N_i \in K^{\mathbf{F}}_{\aleph_1}$ for $i < \omega_1$ and $N \leq_{\mathfrak{K}} N_i$ such that if $N \models P[\bar{a}]$ and the sentence $(\forall \bar{x})(P(\bar{x}) \equiv (\mathbf{Q}y)R(y,\bar{x})]$ belongs to $T$, <u>then</u> for some $i < \omega_1$ there is $b_* \in N_i \backslash N$ such that $N_i \models R[b,\bar{a}]$.
  Clearly
- $(*)_2$ the existence of $N, N_i$ as in $(*)_1$ is equivalent to "$\psi^*$ has a model" for some $\psi^* \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ which is defined from $T, \leq_{\mathfrak{K}}$.

Hence

- $(*)_3$ it is enough to prove that for some forcing notion $\mathbb{P}$ in $\mathbf{V}^{\mathbb{P}}$ there are $N, N_i$ as in $(*)_1$.

There are some natural c.c.c. forcing notions tailor-made for this

- $(*)_4$ consider the class of triples $(M,N,a)$ such that $M \leq_{\mathfrak{K}} N \in K_{\aleph_0}, \bar{a} \in {}^{\omega>}N, \ell < \ell g(\bar{a}) \Rightarrow a_\ell \notin M$, order as in the proof of 3.14. By the same proof there is no maximal triple.

3) We can restrict ourselves in $(*)_2$ to

$$\{R(y,\bar{a}) : \bar{a} \in {}^{\ell g(\bar{x})}N \text{ and } \bar{a} \text{ realizes a type } p(\bar{x})\}.$$

Also we may demand $i < \omega_1 \Rightarrow N_i = N_0$ and we may try to force such a sequence of models (or pairs) and there is a natural forcing. By absoluteness it is enough to prove that it satisfies the c.c.c.

{88r-3.9B

3.22. **Problem.** If $\mathfrak{K}$ is $\mathrm{PC}_\lambda, K$ categorical in $\lambda$ and $\lambda^+$, does it necessarily have a model in $\lambda^{++}$?

3.23. **Remark.** The problem is proving $(*)$ of 3.13.

{88r-3.9C

3.24. **Question.** Assume $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$ is complete in $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$, is categorical in $\aleph_1$, has an uncountable model $M, \bar{a} \in {}^n M$ and $\varphi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$ axiomatizes the $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$-theory of $(M, \bar{a})$. Is $\varphi$ categorical in $\aleph_1$?

{88r-3.9D

3.25. **Question.** Can we weaken the demand on $M^*$ in 3.14 to "$M^*$ is a $\lambda^+$-limit model"?

## 4. Forcing and categoricity

The main aim in this section is, for $\mathfrak{K}$ as in §1 with $\mathrm{LST}(\mathfrak{K}) = \aleph_0$, to find what we can deduce from $1 \le \dot{I}(\aleph_1, K) < 2^{\aleph_1}$, first without assuming $2^{\aleph_0} < 2^{\aleph_1}$.

We can build a model of cardinality $\aleph_1$ by an $\omega_1$-sequence of countable approximations. Among those, there are models which are the union of a quite generic $<_{\mathfrak{K}}$-increasing sequence $\langle N_i : i < \omega_1 \rangle$ of countable models, so it is natural to look at them (e.g. if $\mathfrak{K}$ is categorical in $\aleph_1$, every model in $K_{\aleph_1}$ is like that). We say on such models that they are quite generic. More exactly, we look at countable models and figure out properties of the quite generic models in $\mathfrak{K}_{\aleph_1}$. The main results are 4.13(a),(f). Note that the case $2^{\aleph_0} = 2^{\aleph_1}$, though in general making our work harder, can be utilized positively - see 4.11.

A central notion is (e.g.) "the type which $\bar{a} \in {}^{\omega>}(N_1)$ materializes in $(N_1, N_0)$", $N_0 \le_{\mathfrak{K}} N_1 \in K_{\aleph_0}$. This is as the name indicates, the type materialized in $N_1^+$, which is $N_1$ expanded by $P^{N_1^+} = N_0$; it consists of the set of formulas forced (in the model theoretic sense started by Robinson) to satisfy; here forced is defined thinking on $(K_{\aleph_0}, \le_{\aleph_0})$ so models in $K_{\aleph_1}$ can be constructed as the union of quite generic $<_{\mathfrak{K}}$-increasing $\omega_1$-sequence. As we would like to build models of cardinality $\aleph_1$ by such sequence, the "materialize" in $(N_1, N_0)$ becomes realized in the (quite generic) $N \in K_{\aleph_1}$; but most of our work is in $K_{\aleph_0}$. This is also a way to express $\mathbf{Q}$ speaking on countable models.

By the hypothesis 4.8 justified by §3, the $\mathbb{L}_{\infty,\omega}(\tau_{\mathfrak{K}})$-theory of $M \in K$ is clear, in particular has elimination of quantifiers hence $M \le_{\mathfrak{K}} N \Rightarrow M \prec_{\mathbb{L}_{\infty,\omega}} N$, but for $\bar{N} = \langle N_\alpha : \alpha < \omega_1 \rangle$ as above we would like to understand $(N_\beta, N_\alpha)$ for $\alpha < \beta$ (from the point of view of $N, \bar{N}$ is not reconstructible, but its behaviour on a club is). Toward a parallel analysis of such pairs we again analyze them by $\langle L^0_\alpha : \alpha < \omega_1 \rangle$ (similarly to [Mor70]).

4.1. *Convention.* We fix $\lambda >$ LST$(\mathfrak{K})$ as well as the a.e.c. $\mathfrak{K}$. {88r-4.0}

The main case below is here $\lambda = \aleph_1, \kappa = \aleph_0$. For $\lambda >$ LST$(\mathfrak{K})$

4.2. **Definition.** For $\lambda >$ LST$(\mathfrak{K})$ and $N_* \in K_{<\lambda}$ and $\mu, \kappa$ satisfying $\lambda \ge \kappa \ge \aleph_0, \mu \ge \kappa$ and let {88r-4.1}
1) $\mathbb{L}^0_{\mu,\kappa}$ be first order logic enriched by conjunctions (and disjunctions) of length $< \mu$, homogeneous strings of existential quantifiers or of universal quantifiers of length $< \kappa$, and the cardinality quantifier $\mathbf{Q}$ interpreted as $\exists^{\ge\lambda}$. But we apply those operations such that any formula has $< \kappa$ free variables, and the non-logical symbols are from $\tau(\mathfrak{K})$ so actually we should write $\mathbb{L}^0_{\mu,\kappa}(\tau_{\mathfrak{K}})$ but we may "forget" to say this when clear; the syntax does not depend on $\lambda$ but we shall mention it in the definition of satisfaction.
2) For a logic $\mathcal{L}$ and $A_i, A \subseteq N_*$ for $i < \alpha, \alpha < \lambda$ let $\mathcal{L}(N_*, A_i; A)_{i<\alpha}$ be the language, with the logic $\mathcal{L}$, and with the vocabulary $\tau_{N_*,\bar{A},A}$ where $\bar{A} = \langle A_i : i < \alpha \rangle$ and $\tau_{N_*,\bar{A};A}$ consists of $\tau(K)$, the predicates $x \in N_*$ and $x \in A_i$ for $i < \alpha$ and the individual constants $c$ for $c \in A$. (If $A = \emptyset$, we may omit the $A$; if we omit $N_*$ then "$x \in N_*$" is omitted, if the sequence of the $A_i$ is omitted then the "$x \in A_i$" are omitted, so $\mathcal{L}()$ means having the vocabulary

$\tau(K)$). So $\mathcal{L}(N_*, A_i; A)_{i<\alpha}$ formally should have been written $\mathcal{L}(\tau_{N_*,\bar{A};A})$.
3) $\mathbb{L}^1_{\mu,\kappa}$ is defined is as in part (1), but we have also variables (and quantification) over relations of cardinality $< \lambda$. Let $\mathbb{L}^{-1}_{\mu,\kappa}$ be as in part (1) but not allowing the cardinality quantifier $\mathbf{Q}$; this is the classical logic $\mathbb{L}_{\mu,\kappa}$.
4) $(N, N_*, A_i; A)_{i<\alpha}$ is the model $N$ expanded to a $\tau_{N_*,\bar{A};A}$-model by monadic predicates for $N_*, A_i (i < \alpha)$ and individual constants for every $c \in A$.
5) For "$x \in N_*$", "$x \in A_i$" we use the predicates $P, P_i$ respectively, so we may write $\mathcal{L}(\tau+P)$ instead $\mathcal{L}(N_*)$, but writing $\mathcal{L}(N_*)$ we fix the interpretation of $P$.
Let $\tau^{+\alpha} = \tau \cup \{P, P_\beta : \beta < \alpha\}$ and if $L = \mathcal{L}(\tau^{+0})$, i.e., for $\alpha = 0$ then $L(N)$ means $L$ but we fix the interpretation of $P$ as $N$, i.e., $|N|$, the set of elements of $N$.
Let $L(N_*, N_i)_{i\in u}$ where $u$ a set of $< \kappa$ ordinals means the language $L$ in the vocabulary $T \cup \{P, P_i : i \in u\}$ when we fix the interpretation of $P$ as $N_*$ and of $P_{\mathrm{otp}(u\cap\alpha)}$ as $N_\alpha$.

{88r-4.2}

4.3. **Definition.** 1) For $N_* \in K_{<\lambda}$ and $\varphi(x_0, \ldots) \in \mathbb{L}^1_{\mu,\kappa}(N_*, \bar{A}; A)$ we define by induction on $\varphi$ when $N_0 \Vdash^\lambda_{\mathfrak{K}} \varphi[a_0, \ldots]$ where $N_* \le_{\mathfrak{K}} N_0 \in K_{<\lambda}, a_0, \ldots$ are elements of $N_0$ or appropriate relations over it, depending on the kind of $x_i$. Pedentically we should write $(N_0, N_*, \bar{A}; A) \Vdash^\lambda_{\mathfrak{K}} \varphi[a_0, \ldots]$; and we may do it when not clear from the context.

For $\varphi$ atomic this means $N_0 \models \varphi[a_0, \ldots]$. For $\varphi = \bigwedge\limits_i \varphi_i$ this means

$$N_0 \Vdash^\lambda_{\mathfrak{K}} \varphi_i[a_0, \ldots] \text{ for each } i.$$

For $\varphi = \exists \bar{x} \psi(\bar{x}, a_0, \ldots)$ this means that for every $N_1$ satisfying $N_0 \le_{\mathfrak{K}} N_1 \in K_{<\lambda}$ there is $N_2$ satisfying $N_1 \le_{\mathfrak{K}} N_2 \in K_{<\lambda}$ and $\bar{b}$ from $N_2$ of the appropriate length (and kind) such that $N_2 \Vdash^\lambda_{\mathfrak{K}} \psi[\bar{b}, a]$.

For $\varphi = \neg\psi$ this means that for no $N_1$ do we have $N_0 \le_{\mathfrak{K}} N_1 \in K_{<\lambda}$ and $N_1 \Vdash^\lambda_{\mathfrak{K}} \psi[a_0, \ldots]$.

For $\varphi(x_0, \ldots) = (\mathbf{Q}y)\psi(y, x_0, \ldots)$ this means that for every $N_1$ satisfying $N_0 \le_{\mathfrak{K}} N_1 \in K_{<\lambda}$ there is $N_2$ satisfying $N_0 \le_{\mathfrak{K}} N_2 \in K_{<\lambda}$ and $a \in N_2 \backslash N_1$ such that $N_2 \Vdash^\lambda_{\mathfrak{K}} \psi[a, a_0, \ldots]$.
2) In part (1) if $\varphi \in \mathbb{L}^1_{\mu,\kappa}(N_*)$ we can omit the demand "$N_* \le_{\mathfrak{K}} N$" similarly below.
3) For a language $L \subseteq \mathbb{L}^1_{\mu,\kappa}(N_*, \bar{A}; A)$ and a model $N$ satisfying $N_* \le_{\mathfrak{K}} N \in K_{<\lambda}$ and a sequence $\bar{a} \in {}^{\lambda>}N$ the $L$-generic type of $\bar{a}$ in $N$ is $\mathrm{gtp}(\bar{a}; N_*, \bar{A}; A; N) = \{\varphi(\bar{x}) \in L : N \Vdash^\lambda_{\mathfrak{K}} \varphi[\bar{a}]\}$.
4) Let $\mathrm{gtp}^\lambda_L(\bar{a}; N_*, \bar{A}; A; N)$ where $N_* \le_{\mathfrak{K}} N \in K_\lambda$ and $L \subseteq \mathcal{L}(N_*, \bar{A}; A)$ be $\{\varphi(\bar{x}) : \varphi \in \mathcal{L}(N_*, \bar{A}; A)$ and for some $N' \in K_{<\lambda}$ we have $N \le_{\mathfrak{K}} N' \le_{\mathfrak{K}} N$ and $N' \Vdash^\lambda_{\mathfrak{K}} \varphi[\bar{a}]\}$; we may omit $\bar{A}, A$ (and omit $\lambda$ if clear from the context) and may write $\mathcal{L}$ instead of $L = \mathcal{L}(N_*, \bar{A}; A)$; but note Definition 5.5.
5) We say "$\bar{a}$ materializes $p$ (or $\varphi$)" if $p$ (or $\{\varphi\}$) is a subset of the $L$-generic type of $\bar{a}$ in $N$.

{88r-4.3}

4.4. **Definition.** Let $N_i (i < \lambda)$ be an increasing (by $\le_{\mathfrak{K}}$) continuous sequence, $N = \bigcup\limits_{i<\lambda} N_i, \|N_i\| < \lambda$ and $L^* \subseteq \bigcup\limits_{\alpha<\kappa} \mathbb{L}^1_{\infty,\kappa}(\tau^{+\alpha})$.

1) $N$ is $L^*$-generic, if for any formula $\varphi(x_0,\dots) \in L^* \cap \mathbb{L}^1_{\infty,\kappa}(\tau_{\mathfrak{K}})$ and $a_0,\dots \in N$ we have: $N \models \varphi[a_0,\dots] \Leftrightarrow$ for some $\alpha < \lambda, N_\alpha \Vdash^\lambda_{\mathfrak{K}} \varphi[a_0,\dots]$.
2) The $\le_{\mathfrak{K}}$-presentation $\langle N_i : i < \lambda\rangle$ of $N$ is $L^*$-generic <u>when</u> for any $\alpha < \lambda$ of cofinality $\ge \kappa$ and $\psi(x_0,\dots) \in L^*(N_\alpha, N_i)_{i\in I}$ satisfying $I \subseteq \alpha, |I| < \kappa$ and $a_0,\dots \in N$ we have:

$$N \models \psi[a_0,\dots] \Leftrightarrow \text{ for some } \gamma < \lambda, N_\gamma \Vdash^\lambda_{\mathfrak{K}} \psi[a_0,\dots]$$

and for each $\beta \ge \alpha$, with cofinality $\ge \kappa, N_\beta$ is almost $L^*(N_\alpha, N_i)_{i\in I}$-generic (see part (5)).
3) $N$ is strongly $L^*$-generic <u>if</u> it has an $L^*$-generic presentation (in this case, if $\lambda$ is regular, then for any presentation $\langle N_i : i < \lambda\rangle$ of $N$ there is a closed unbounded $E \subseteq \lambda$ such that $\langle N_i : i \in E\rangle$ is an $L^*$-generic presentation).
4) We say that $N \in K_{<\lambda}$ is pseudo $L^*$-generic <u>if</u>

(a) for every $\varphi(\bar{x}) = \exists\bar{y}\psi(\bar{x},\bar{y}) \in L^*$, if $N \Vdash^\lambda_{\mathfrak{K}} \varphi(\bar{a})$ then for some $\bar{b}, N \Vdash^\lambda_{\mathfrak{K}} \psi(\bar{a},\bar{b})$

(b) for every $\bar{a} \in N, \bar{a}$ materializes in $N$ some complete $L^*$-type.

5) We add "almost" to any of the above defined notions <u>when</u>: for $\Vdash^\lambda_{\mathfrak{K}}$, the inductive definitions of satisfaction works except possibly for $\mathbf{Q}$ (e.g., $N \Vdash^\lambda_{\mathfrak{K}} \exists x\varphi(x,\dots)$ iff for some $a \in N, N \Vdash^\lambda_{\mathfrak{K}} \varphi(a,\dots)$).

{88r-4.3A}

4.5. **Remark.** 1) Notice we can choose $N_i = N_0 = N$, so $\|N\| < \lambda$. In particular almost (and pseudo) $L^*$-generic models of cardinality $< \lambda$ may well exist.
2) Here we concentrate on $\lambda = \aleph_1$ and fragments of $\mathbb{L}^0_{\infty,\omega}$ (mainly $\mathbb{L}^0_{\omega_1,\omega}$ and its countable fragments).
3) There are obvious implications, and forcing is preserved by isomorphism and replacing $N(\in K_{<\lambda})$ by $N', N \le_{\mathfrak{K}} N' \in K_{<\lambda}$.

There are obvious theorems on the existence of generic models, e.g.,

{88r-4.4}

4.6. **Theorem.** *1) Assume* $N_0 \in K_{<\lambda}, \lambda = \mu^+, \mu^{<\kappa} = \mu, L \subseteq \bigcup\limits_{\alpha<\kappa} \mathbb{L}_{\infty,\kappa}(\tau^{+\alpha})$ *and* $L$ *is closed under subformulas and* $|L| < \lambda$. *<u>Then</u> there are* $N_i (i < \lambda)$ *such that* $\langle N_i : i < \lambda\rangle$ *is an* $L$*-generic representation of* $N = \bigcup\limits_{i<\lambda} N_i$, *(hence* $N$ *is strongly* $L$*-generic).*
*2) In part (1),* $N \in K_\lambda$ *if no* $N', N_0 \le_{\mathfrak{K}} N' \in K_{<\lambda}$ *is* $\le_{\mathfrak{K}}$*-maximal.*

*Proof.* Straightforward. $\square_{4.6}$

{88r-4.4A}

4.7. **Remark.** 1) If $L = \bigcup\limits_{i<\lambda} L_i, |L_i| < \lambda$, then we can get "$\langle N_i : j < i < \lambda\rangle$ is an $L_j$-generic representation of $N$ for each $j < \lambda$".
2) When we speak on "complete $L$-type $p$" we mean $p = p(x_0,\dots,x_{n-1})$ for some $n$.

From time to time we add some hypothesis and prove a series of claims; such that the hypothesis holds, at least without loss of generality, in the case we are interested in. We are mainly interested in the case $\dot{I}(\aleph_1,\mathfrak{K}) < 2^{\aleph_1}$, etc., so by 3.11, 3.18 it is reasonable to make:

{88r-4.5}

4.8. **Hypothesis.** $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$, $\le_{\mathfrak{K}}$ refines $\mathbb{L}_{\infty,\omega}$ and $\mathfrak{K}$ is categorical in $\aleph_0$ and $1 \le \dot{I}(\aleph_1,K)$ and $\dot{I}(\aleph_1,K^{\mathbf{F}}_{\aleph_1}) < 2^{\aleph_1}$ where $K^{\mathbf{F}}_{\aleph_1}$ is as in Definition 3.15 and is $\mathrm{PC}_{\aleph_0}$ or just $\mathbf{K}^{\mathbf{F}}_{\aleph_1} = \{M \restriction \tau_{\mathfrak{K}} : M \models \psi\}$ for some $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ (if $\mathbf{F}$ is invariant, this follows).

{88r-4.5.3}

4.9. **Remark.** 0) We can add: every $M \in K_{\aleph_0}$ is atomic (model of $\mathrm{Th}_{\mathbb{L}}(M)$).
1) Usually below we ignore the case $\dot{I}(\aleph_1,\mathfrak{K}) < 2^{\aleph_0}$ as the proof is the same.
2) We can deal similarly with the case $1 \le \dot{I}(\aleph_1,K') < 2^{\aleph_0}$ where $\mathfrak{K}_{\aleph_1} \subseteq K'_{\aleph_1} \subseteq \{M \in \mathfrak{K}_{\aleph_1} : M$ is strongly $L_*$-generic$\}$ and $K'$ is $\mathrm{PC}_{\aleph_0}$ (or less: $\{M \restriction \tau_{\mathfrak{K}} : M$ a model of $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau^*)\}$).
3) Can we use $\mathbf{F}$ a function with domain $K_{\aleph_0}$ such that $M \le_{\mathfrak{K}} \mathbf{F}(M_0) \in K_{\aleph_0}$ for $M \in K_{\aleph_0}$ without the extra assumptions or even $\mathbf{F} : \{\bar{M} = \langle M_i : i \le \alpha\rangle$ is $\le_{\mathfrak{K}_{\aleph_0}}$-increasing continous$\} \to \mathfrak{K}_{\aleph_0}$ such that $M_\alpha \le_{\mathfrak{K}} \mathbf{F}(M_i : i \le \alpha\rangle)$? We cannot use the non-definability of well ordering (see 3.11(3)); (as in the proof of (f) of 4.13).

{88r-4.6}

4.10. **Claim.** *1) If $\bar{a} \in N \in K_{\aleph_0}$ and $\varphi(\bar{x}) \in \mathbb{L}^0_{\infty,\omega}(\tau^{+0})$ (so $\bar{a}$ is a finite sequence) <u>then</u> $(N,N) \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi[\bar{a}]$ or $(N,N) \Vdash^{\aleph_1}_{\mathfrak{K}} \neg\varphi[\bar{a}]$ (i.e. $P$ is interpreted as $N$).*
*2) If $(N,N) \Vdash^{\aleph_1}_{\mathfrak{K}} \exists\bar{x} \wedge p(\bar{x})$, where $p(\bar{x})$ is a not necessarily complete $n$-type ($n = \ell g(\bar{x})$) in $L$ where $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$ is countable, <u>then</u> for some complete $n$-type $q$ in $L$ extending $p$ we have $(N,N) \Vdash^{\aleph_1}_{\mathfrak{K}} \exists\bar{x} \wedge q(\bar{x})$.*

*Proof.* 1) Suppose not, for each $S \subseteq \omega_1$, we define by induction on $\alpha$, $N^S_\alpha \in K_{\aleph_0}(\alpha < \omega_1)$, increasing (by $\le_{\mathfrak{K}}$) and continuous, $N^S_0 = N$ and for limit $\alpha$, $N^S_\alpha = \bigcup\limits_{\beta<\alpha} N^S_\beta$. For $\alpha = 2\beta+1$ remember that $(N^S_\beta,\bar{a}) \cong (N,\bar{a})$ because $N = N_0 \le_{\mathfrak{K}} N^S_\beta$ hence $N_0 \prec_{\mathbb{L}_{\infty,\omega}} N^S_\beta \in K_{\aleph_0}$ hence $(N^S_\beta,\bar{a}) \equiv_{\mathbb{L}_{\infty,\omega}} (N,\bar{a})$ hence they are isomorphic. So $(N^S_\beta,N^S_\beta)$ forces ($\Vdash^{\aleph_1}_{\mathfrak{K}}$) neither $\varphi[\bar{a}]$ nor $\neg\varphi[\bar{a}]$. So there are $M_\ell$ (for $\ell = 0,1$) such that $N^S_\beta \le_{\mathfrak{K}} M_\ell \in K_{\aleph_0}$ and $(M_0,N^S_\beta) \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi[\bar{a}]$ but $(M_1,N^S_\beta) \Vdash^{\aleph_1}_{\mathfrak{K}} \neg\varphi[\bar{a}]$. Now if $\beta \in S$ we let $N^S_\alpha = M_0$ and if $\beta \notin S$ we let $N^S_\alpha = M_1$.

Lastly, $M_{2\beta+2} = \mathbf{F}(M_{2\beta+1})$ recalling $\mathbf{F}$ is from 4.8. Let $N^S = \bigcup\limits_{\alpha<\omega_1} N^S_\alpha$. Now if $S(0)\backslash S(1)$ is stationary then $(N^{S(0)},\bar{a}) \ncong (N^{S(1)},\bar{a})$. Why? Because if $f : N^{S(0)} \to N^{S(1)}$ is an isomorphism from $N^{S(0)}$ onto $N^{S(1)}$ mapping $\bar{a}$ to $\bar{a}$ then for some closed unbounded set $E \subseteq \omega_1$, we have: if $\alpha \in E$ then $f$ maps $N^{S(0)}_\alpha$ onto $N^{S(1)}_\alpha$, so choose some $\alpha \in E \cap S(0)\backslash S(1)$ and choose $\beta \in E\backslash(\alpha+1)$. Now $(N^{S(0)}_{\alpha+1},N^{S(0)}_\alpha) \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi[\bar{a}]$, hence $(N^{S(0)}_\beta,N^{S(0)}_\alpha) \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi[\bar{a}]$, and similarly $(N^{S(1)}_\beta,N^{S(1)}_\alpha) \Vdash^{\aleph_1}_{\mathfrak{K}} \neg\varphi(\bar{a})$, but $f \restriction N^{S(0)}_\beta$ is an isomorphism from $N^{S(0)}_\beta$ onto

$N_\beta^{S(1)}$ mapping $N_\alpha^{S(0)}$ onto $N_\alpha^{S(1)}$ and $\bar a$ to itself and we get a contradiction. By 0.4, we get $\dot I(\aleph_1, K) = 2^{\aleph_1}$, contradiction.

2) Easy by 4.6 and part (1). In detail, if $N \le_{\mathfrak K} M_1 \in \mathfrak K_{\aleph_0}$ then by the definition of $\Vdash_{\mathfrak K}^{\aleph_1}$ and the assumption we can find $(M_2, \bar a)$ satisfying $M_1 \le_{\mathfrak K} M_2 \in \mathfrak K_{\aleph_0}$ and $\bar a \in M_2$ such that $(M_2, N) \Vdash_{\mathfrak K}^{\aleph_1} \wedge p(\bar a)$. As $L$ is countable and the definition of $\Vdash_{\mathfrak K}^{\aleph_1}$ without loss of generality for every formula $\varphi(\bar x) \in L, (M_2, N) \Vdash_{\mathfrak K}^{\aleph_1} \varphi[\bar a]$ or $(M_2, N) \Vdash_{\mathfrak K}^{\aleph_1} \neg\varphi[\bar a]$. (Why? Simply let $\langle \varphi_n(\bar x) : n < \omega\rangle$ list the formulas $\varphi(\bar x) \in L$ and choose $M_{2,n} \in \mathfrak K_{\aleph_0}$ by induction on $n$ such that $M_{2,0} = M_2, M_{2,n} \le_{\mathfrak K} M_{2,n+1}$ such that $(M_{2,n+1}, N) \Vdash_{\mathfrak K}^{\aleph_1} \varphi_n(\bar x)$ or $(M_{2,n+1}, N) \Vdash_{\mathfrak K}^{\aleph_1} \neg\varphi_n(\bar x)$; now replace $M_2$ by $\cup\{M_{2,n} : n < \omega\}$). Recalling Definition 4.3(4), let $q = \mathrm{gtp}_{L(N)}(\bar a, N, M_2)$, it is a complete $(L(N), n)$-type. So clearly $(M_2, N) \Vdash_{\mathfrak K}^{\aleph_1} (\exists \bar x) \wedge q(\bar x)$. Now apply the proof of part (1) to the formula $(\exists \bar x) \wedge q(\bar x)$ so we are done. $\square_{4.10}$

4.11. **Claim.** *For each countable $L \subseteq \mathbb L^0_{\omega_1,\omega}(\tau^{+0})$ and $N \in K_{\aleph_0}$ the number of complete $L(N)$-types $p$ (with no parameters) such that $N \Vdash_{\mathfrak K}^{\aleph_1} (\exists \bar x) \wedge p(\bar x)$, is countable.* {88r-4.7}

*Proof.* At first glance it seemed that 0.3 will imply this trivially. However, here we need the parameter $N$ as an interpretation of the predicate $P$ and if $2^{\aleph_0} = 2^{\aleph_1}$ there are too many choices. So we shall deal with "every $N_\alpha$ in some presentation". Suppose the conclusion fails. First we choose by induction $N_\alpha$ (for $\alpha < \omega_1$) such that:

$\circledast$ (i) $N_\alpha \in K_{\aleph_0}$ is $\le_{\mathfrak K}$-increasing and $\langle N_\alpha : \alpha < \omega_1\rangle$ is $L$-generic
(ii) for each $\beta < \alpha$, there is $a^\beta_\alpha \in N_{\alpha+1} \backslash N_\alpha$ materializing an $L(N_\beta)$-type not materialized in $N_\alpha$, (i.e. in $(N_\alpha, N_\beta)$; see Definition 4.3(2) on materialize), (possible by 4.10 and our assumption toward contradiction)
(iii) $|N_\alpha| = \omega\alpha$
(iv) for $\alpha < \beta, N_\beta$ is pseudo $L(N_\alpha)$-generic and $\mathbf F(N_{2\beta+1}) \le_{\mathfrak K} N_{2\beta+2}$.

Now let $N = \cup\{N_\alpha : \alpha < \omega_1\}$ and we expand $N$ by all relevant information: the order $<$ on the countable ordinals, $c (c \in N_0)$, enough "set theory", "witness" for $N_\beta \le_{\mathfrak K} N_\alpha$ for $\beta < \alpha$ and the 2-place functions $F, F(\beta, \alpha) = a^\beta_\alpha$ and lastly witnesses of $\mathbf F(N_{2\beta+1}) \le_{\mathfrak K} N_{2\beta+2}$ recalling $\mathbf F$ is quite definable by Definition 4.8 and names for all formulas in $L(N_\alpha)$ (with $\alpha$ as a parameter), i.e., the relations $R_{\varphi(\bar x)} = \{\langle\alpha\rangle\hat{\ }\bar a : \alpha < \omega_1, \bar a \in {}^{\ell g(x)}N$ and for every $\beta < \omega_1$ large enough $(N_\beta, N_\alpha) \Vdash_{\mathfrak K}^{\aleph_1}$ "$\varphi(\bar a)$"$\}$ for $\varphi(\bar x) \in L$. Clearly for every $\alpha < \omega_1, \varphi(\bar x) \in L(N_\alpha)$ and $\bar a \in {}^{\ell g(\bar x)}N$ we have $(N, N_\alpha) \models \varphi[\bar a]$ iff for every $\beta < \omega_1$ large eough we have $(N_\beta, N_\alpha) \Vdash_{\mathfrak K}^{\aleph_1}$ "$\varphi[\bar a]$". We get a model $\mathfrak B$ with countable vocabulary and $\psi \in \mathbb L_{\omega_1,\omega}(\mathbf Q)$ expressing all this. By 0.3(1) applied to the case $\Delta = L$, there are models $\mathfrak B_i$ (for $i < 2^{\aleph_1}$) of cardinality $\aleph_1$ (note $N_0 \le_{\mathfrak K} \mathfrak B \restriction \tau_{\mathfrak K}$), so that the set of $L(N_0)$-types realizes in $N^i$ (the $\tau(K)$-reduct of $\mathfrak B_i$) are distinct for distinct $i$'s. So $(N^i, c)_{c \in N_0}$ are pairwise non-isomorphic. If $2^{\aleph_0} < 2^{\aleph_1}$ we finish by 0.4.

So we can assume $2^{\aleph_0} = 2^{\aleph_1}$. In $N$, uncountably many complete $L(N_0)$-n-types are realized hence by 0.3(2) the set $\{p : p$ a complete $L(N_0) - m$-type is realized in some $N', N_0 \le_{\mathfrak K} N' \in \mathfrak K_{\aleph_1}$ for some $m < \omega\}$ has cardinality continuum, hence by 4.10 the set of complete $L(N_0)$-types $p = p(x)$ such that $(N_0, N_0) \Vdash^{\aleph_1}_{\mathfrak{K}} \exists \bar{x} \wedge p(\bar{x})$ has cardinality $2^{\aleph_0}$. So we choose by induction on $\alpha < 2^{\aleph_0}$ a sequence $\langle N^\alpha_i, a^\alpha_i : i < \omega_1 \rangle$ such that:

(a) $N^\alpha_i \in \mathfrak{K}_{\aleph_0}$

(b) $N^\alpha_{i_0} \le_{\mathfrak{K}} N^\alpha_i$ for $i_0 < i < \omega_1$

(c) $a^\alpha_i \in N^\alpha_{i+1} \backslash N^\alpha_i$ materialize a complete $L(N^\alpha_i)$-type $p^\alpha_i$

(d) if $j < \omega_1$ is a limit ordinal then $N^\alpha_j = \cup \{N^\alpha_i : i < j\}$

(e) $p^\alpha_i \notin \{\mathrm{gtp}(\bar{a}; N^\beta_{j_1}; N^\beta_{j_2}) : j_1 < j_2 < \omega_1, \bar{a} \in {}^{\omega >}(N^\beta_{j_2})$ and $\beta < \alpha\}$ (see Definition 4.3(4))

(f) $\mathbf{F}(N_{2\beta+1}) \le_{\mathfrak{K}} N_{2\beta+2}$.

As $\aleph_1 < 2^{\aleph_1} = 2^{\aleph_0}$ this is possible, i.e., in clause (e) we should find a type which is not in a set of $\le \aleph_1 \times |\alpha| < 2^{\aleph_0}$ types, as the number of possibilities is $2^{\aleph_0}$; let $N_\alpha = \cup\{N^\alpha_i : i < \omega_1\}$ for $\alpha < 2^{\aleph_0}$, clearly $N_\alpha \in K_{\aleph_1}$. Now toward contradiction if $\beta < \alpha < 2^{\aleph_0}$ and $N_\alpha \cong N_\beta$ then there is an isomorphism $f$ from $N_\alpha$ onto $N_\beta$; necessarily $f$ maps $N^\alpha_i$ onto $N^\beta_i$ for a club of $i$. For any such $i, p^\alpha_i \in \mathrm{gtp}_L(f(\bar{a}^\alpha_i); N^\beta_i; N^\beta_j)$ for $j$ large enough, contradiction. $\square_{4.11}$

{88r-4.7A}

4.12. **Remark.** In the proof of 4.11(2), we can fix $m$ and we can combine the two cases, when for $N \in K^{\mathbf{F}}_{\aleph_1}$ represent by $\langle N_\alpha : \alpha < \omega_1 \rangle$ we consider $\mathbf{P}_N = \{p : p$ a complete $L - m$-type such that for a club of $\alpha < \omega_1$ for some $\beta \in (\alpha, \omega_1)$ and $\bar{a} \in {}^m(N_\beta)$ materialize $p$ in $(N_\beta, N_\alpha)\}$, can replace "club" by "stationarily many". That is we can prove that $\{\mathbf{P}_N : N \in K^{\mathbf{F}}_{\aleph_1}\}$ has cardinality $2^{\aleph_1}$.



{88r-4.8}

4.13. **Lemma.** *1) There are countable $L^0_\alpha \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$ for $\alpha < \omega_1$ increasing continuous in $\alpha$, closed under finitary operations and subformulas such that, letting $L^0_{<\omega_1} = \cup\{L^0_\alpha : \alpha < \omega_1\}$ we have (some clauses do not metion the $L^0_\alpha$'s):*

(a) *for each $N \in K_{\aleph_0}$ and every complete $L^0_\alpha(N)$-type $p(\bar{x})$ we have $N \Vdash^{\aleph_1}_{\mathfrak{K}} (\exists \bar{x}) \wedge p(\bar{x}) \Rightarrow \wedge p \in L^0_{\alpha+1}(N)$.
Hence for every $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-formula $\psi(\bar{x})$ there are formulas $\varphi_n(\bar{x}) \in L^0_{<\omega_1}$ for $n < \omega$ such that $(N, N) \Vdash^{\aleph_1}_{\mathfrak{K}} (\forall \bar{x})[\psi(\bar{x}) \equiv \bigvee\limits_n \varphi_n(\bar{x})]$*

(b) *for every $N_0 \le_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ there is $N_2, N_1 \le_{\mathfrak{K}} N_2 \in K_{\aleph_0}$, such that for every $\bar{a} \in N_2$ and $\varphi(\bar{x}) \in \mathbb{L}^0_{\omega_1,\omega}(N_0)$, of course with $\ell g(\bar{a}) = \ell g(\bar{x}) < \omega$, we have $(N_2, N_0) \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi[\bar{a}]$ or $(N_2, N_0) \Vdash^{\aleph_1}_{\mathfrak{K}} \neg\varphi[\bar{a}]$*

(c) *If $N \le_{\mathfrak{K}} N_\ell \in K_{\aleph_0} (\ell = 1, 2), \bar{a}_\ell \in N_\ell$ and the $L^0_{<\omega_1}(N)$-generic types of $\bar{a}_\ell$ in $N_\ell$ are equal (though they are not necessarily complete; i.e., for every $\varphi(\bar{x}) \in L^0_{<\omega_1}(N)$ we have $N_1 \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi(\bar{a}_1)$ iff $N_2 \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi[\bar{a}_2]$), <u>then</u> so are the $\mathbb{L}^0_{\infty,\omega}(N)$-generic types. In fact, there is $M, N \le_{\mathfrak{K}} M$ and $\le_{\mathfrak{K}}$-embeddings $f_\ell : N_\ell \to M$ such that $f_\ell$ maps $N$ onto itself and $f_1(\bar{a}_1) = f_2(\bar{a}_2)$ though we do not claim $f_1 \restriction N = f_2 \restriction N$. Also if*

*$N_1 = N_2$ then there is $M \in K_{\aleph_0}$ which $\leq_{\mathfrak{K}}$-extends $N_1$ and an automorphism $f$ of $M$ mapping $N$ onto itself and $\bar{a}_1$ to $a_2$.*

(d) *For each $N \in K_{\aleph_0}$ and complete $\mathbb{L}^0_{\omega_1,\omega}(N)$-type $p(\bar{x})$, the class $K^1 := \{(N, M, \bar{a}) : M \in K_{\aleph_0}, N \leq_{\mathfrak{K}} M$ and for some $M', M \leq_{\mathfrak{K}} M' \in K_{\aleph_0}$ and $\bar{a}$ materialize $p$ in $(M; N)\}$ is a $PC_{\aleph_0}$-class.*

(e) *for any complete $\mathbb{L}^{-1}_{\omega_1,\omega}(N)$-type $p(\bar{x})$ for some complete $\mathbb{L}^0_{\omega_1,\omega}(N)$-type $q_p$, if $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}, \bar{a} \in M$ and $\bar{a}$ materialize $p$ in $(M, N)$, <u>then</u> $\bar{a}$ materialize $q_p$ in $(M, N)$; on $\mathbb{L}^0, \mathbb{L}^{-1}$ see Definition 4.2(1),(3)*

(f) *the number of complete $\mathbb{L}^0_{\omega_1,\omega}(N)$-types $p$ which for some $\bar{a}, M$ we have $\bar{a} \in {}^{\omega>}M, M \in K_{\aleph_0}, N \leq_{\mathfrak{K}} M$ and $\bar{a}$ materialize in $(M, N)$ is $\leq \aleph_1$*

(g) *if in clause (f) we get that there are $\aleph_1$ such types then $\dot{I}(\aleph_1, K) \geq \aleph_1$*

(h) *let $L^{-1}_\alpha := L^0_\alpha \cap \mathbb{L}^{-1}_{\omega_1,\omega}(\tau^{+0})$ then the parallel clauses to (a)-(g) holds.*

*2) Clause (e) means that*

(i) *assume further that $N_0 \leq_{\mathfrak{K}} N_\ell \in K_{\aleph_0}$ for $\ell = 1, 2$ and $\bar{a}_\ell \in N_\ell$ and the $L^{-1}_{<\omega_1}(N)$-type which $\bar{a}_1$ materializes in $N_1$ is equal to the $L^{-1}_{<\omega_1}(N)$-type which $\bar{a}_2$ materializes in $N_2$. <u>Then</u> we can find $N^+_1, N^+_2$ such that $N_\ell \leq_{\mathfrak{K}} N^+_\ell \in K_{\aleph_0}$ for $\ell = 1, 2$ and isomorphism $f$ from $N^+_1$ onto $N^+_2$ mapping $N$ onto itself and $\bar{a}_1$ to $\bar{a}_2$.*

4.14. **Remark.** 1) We cannot get rid of the case of $\aleph_1$ types (but see 5.23, 5.30) by the following variant of a well known example of Morley [Mor70] for $\dot{I}(\aleph_0, K) = \aleph_2$. For let $K = \{(A, E, <) : E$ an equivalence relation on $A$, each $E$-equivalence class is countable, $x < y \Rightarrow xEy$ and on each $E$-equivalence class $<$ is a 1-transitive linear order, i.e. $xEy \Rightarrow (x/E, <, x) \cong (y/E, <, y)\}$ and $M \leq_{\mathfrak{K}} N$ if $M \subseteq N$ and $[x \in M \wedge y \in N \wedge xEy \Rightarrow y \in M]$. By the analysis of such countable linear orders, each $(a/E^M, <)$ up to isomorphism is determined by $(\alpha, \ell) \in \omega_1 \times 2$. For appropriate $\mathbf{F}$, if $M = \mathbf{F}(N), a \in N$ and $I$ is an interval of $(a/E^N, <^N)$ which is 1-transitive then for some $b \in M \backslash N, (b/E^M, <^M)$ is isomorphic to $(I, <^N)$. This is enough. {88r-4.8A
2) In clauses (c),(i) of 4.13 the mapping are not necessarily the identity on $N$. In clause (i) the assumption is apparently weaker (those by its conclusion the assumption of (c) holds).
3) Note that clause (f) of 4.13 does not follow from clause (a) as there may be $\aleph_1$-Kurepa trees.
4) In clause (c) of 4.13 for the second sentence we can weaken the assumption: if $\varphi(\bar{x}) \in L^0_{<\omega_1}(N)$ and $(N_1; N) \nVdash^{\aleph_1}_{\mathfrak{K}} \varphi(\bar{a}_1)$ then $(N_2, N) \nVdash^{\aleph_1}_{\mathfrak{K}} \varphi(\bar{a}_2)$. This is enough to get the $M_{1,\alpha}, M_{2,\alpha}$ from the proof. (Why? For each $\alpha < \omega_1$, there are $M_{1,\alpha}$ such that $N_1 \leq_{\mathfrak{K}} M_{1,\alpha} \in K_{\aleph_0}$ and a complete $L^0_\alpha - \ell g(\bar{a}_i)$-type $p_*(\bar{x})$ such that $(M_{1,\alpha}, N) \Vdash \wedge p_*(\bar{a}_1)$. But $\neg \wedge p_1(\bar{x}) \in L_{\alpha+1}$ and obviously $(N_1, N) \nVdash \neg \wedge p_*(\bar{a}_1)$ hence $(N_2, N) \nVdash^{\aleph_1}_{\mathfrak{K}} \neg \wedge p_*(\bar{a}_2)$ hence there is $M_{2,\alpha}$ such that $N_2 \leq_{\mathfrak{K}} M_{2,\alpha} \in K_{\aleph_0}$ and $(M_{2,\alpha}; N) \Vdash^{\aleph_1}_{\mathfrak{K}} \wedge p_*[\bar{a}_2]$. Now continue as in the proof below).

4.15. **Remark.** We can prove clause (b) and the last sentence in clause (c) of 4.13 directly not mentioning the $L^0_\alpha$-s.

*Proof.* Note that proving clause (e) we say "repeat the proof of clause (a),(b),(c),(d) for $L^{-1}_{\omega,\omega}$".

Clause $(a)$: We choose $L^0_\alpha$ by induction on $\alpha$ using 4.11. The second phrase is proved by induction on the depth of the formula using 4.10.

Clause $(b)$: By iterating $\omega$ times, it suffices to prove this for each $\bar a \in N_1$, so again by iterating $\omega$ times it suffices to prove this for a fix $\bar a \in N_1$. If the conclusion fails we can define by induction on $n < \omega$ for every $\eta \in {}^n 2$, a model $M_\eta$ and $\varphi_\eta(\bar x) \in \mathbb{L}^0_{\omega_1,\omega}(N)$ such that:

- $(i)$ $M_{<>} = N_1$
- $(ii)$ $M_\eta \le_{\mathfrak{K}} M_{\eta\hat{}<\ell>} \in K_{\aleph_0}$ for $\ell = 0,1$
- $(iii)$ $(M_\eta, N) \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi_\eta(\bar a)$
- $(iv)$ $\varphi_{\eta\hat{}<1>}(\bar x) = \neg\varphi_{\eta\hat{}<0>}(\bar x)$.

Now for $\eta \in {}^\omega 2$, let $M_\eta = \bigcup\limits_{n<\omega} M_{\eta\restriction n}$. Clearly for $\eta \in {}^\omega 2$ we have $M_\eta \Vdash^{\aleph_1}_{\mathfrak{K}} (\exists \bar x)[\bigwedge\limits_{n<\omega} \varphi_{\eta\restriction n}(\bar x)]$ and, after slight work, we get contradiction to 4.11 + 4.10.

Clause $(c)$: In general by clause (a) for each $\alpha < \omega_1$ we can find $M^\alpha_\ell \in K_{\aleph_1}$ for $\ell = 1,2$ such that $N_\ell \le_{\mathfrak{K}} M^\alpha_\ell$ and $(M^\alpha_1, \bar a_1), (M^\alpha_2, \bar a_2)$ are $L^0_\alpha(N)$-equivalent and without loss of generality each of $N, N_\ell, M^\alpha_\ell$ have universe an ordinal $< \omega_1$. Let $\mathfrak{A} = (\mathcal{H}(\aleph_2), N, N_1, N_2, \langle M^\alpha_1 : \alpha < \omega_1\rangle, \langle M^\alpha_2 : \alpha < \omega_1\rangle)$ let $\mathfrak{A}_1 \prec \mathfrak{A}$ be countable and recalling 0.5(3) find a non-well ordered countable model $\mathfrak{A}_2$, which is an end extension of $\mathfrak{A}_1$ for $\omega^{\mathfrak{A}_1}_1$, hence $\omega^{\mathfrak{A}_2} = \omega$ so $N^{\mathfrak{A}_2} = N, N^{\mathfrak{A}_2}_\ell = N_\ell$ for $\ell = 1,2$. For $x \in (\omega_1)^{\mathfrak{A}_2} \backslash \mathfrak{A}_1$ let $M^x_\ell = (M^x_\ell)^{\mathfrak{A}_2}$ so $N_\ell \le_{\mathfrak{K}} M^x_\ell \in K_{\aleph_0}$. Now there are $x_n$ such that $\mathfrak{A}_2 \models$ "$x_{n+1} < x_n$ are countable ordinals"; so using the hence and forth argument $(M^{x_0}_1, \bar a_1, N) \cong (M^{x_0}_2, \bar a_2, N)$.
[Why? Let $\mathcal{F}_n = \{(\bar b^1, \bar b^2) : \bar b^\ell \in {}^n(M^{x_0}_\ell)$ and $\mathfrak{A}_2 \models \mathrm{gtp}_{L^0_{x_n}}(\bar a^1\hat{}\bar b^1, N; M^{x_0}_1) = \mathrm{gtp}_{L^0_{x_n}}(\bar a^2\hat{}\bar b^2; N; M^{x_0}_2)\}$. Clearly $(<>,<>) \in \mathcal{F}_0$ and if $(\bar b^1, \bar b^2) \in \mathcal{F}_n, \ell \in \{1,2\}$ and $b^\ell_n \in M^{x_0}_\ell$ then there is $b^{3-\ell}_n \in M^{x_0}_{3-\ell}$ such that $(\bar b^1\hat{}\langle b^1_n\rangle, \bar b^2\hat{}\langle b^2_n\rangle) \in \mathcal{F}_{n+1}$. As $M^{x_0}_1, M^{x_0}_2$ are countable we can find an isomorphism.]
But this is as required in the second phrase of (c).

We still have to prove the first phrase. For this we prove by induction on the ordinal $\alpha$ that

- $\circledast^1_\alpha$ if for $\ell = 1,2, \bar a_\ell \in {}^{\omega>}(N_\ell)$ materialize in $(N_\ell, N_*)$ a complete $L^0_{<\alpha}$-type $p(\bar x)$ not depending on $\ell$ and $\varphi(\bar x) \in \mathbb{L}^0_{\infty,\omega}(N_*)$ has quantifier depth $< \alpha$ then: $\ell \in \{1,2\} \Rightarrow (N_\ell, N_*) \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi(\bar a_\ell)$ <u>or</u> $\ell \in \{1,2\} \Rightarrow (N_\ell, N_*) \Vdash^{\aleph_1}_{\mathfrak{K}} \neg\varphi(\bar a_\ell)$.

For countable $N \le_{\mathfrak{K}} M$ and $\bar a \in {}^{\omega>}N$

- $\odot_1$ let $\mathbf{P}_\alpha(N, M, \bar a) = \{\mathrm{gtp}_{L^0_{<\alpha}}(\bar a; N; M^\perp) : M \le_{\mathfrak{K}} M^+ \in K_{\aleph_0}$ and $\mathrm{gtp}_{L^0_\alpha}(\bar a; N; M^+)$ is a complete $L^0_\alpha$-type$\}$.

Now

$\odot_2$ for $\beta < \alpha < \omega_1$, from $\mathrm{gtp}_{L^0_\alpha}(\bar{a}; N; M)$ we can complete $\mathbf{P}_\beta(N, M, \bar{a})$

$\odot_3$ for $\alpha < \omega_1$, from $\mathbf{P}_\beta(N, M, \bar{a})$ we can compute $\mathrm{gtp}_{L^0_\alpha}(\bar{a}; N; M)$

$\odot_4$ assume $N \leq_{\mathfrak{K}} M$ are countable and $\bar{a} \in {}^{\omega >}M$; for $\varphi(\bar{x}) \in L^0_{\omega_1,\omega}(N)$ of quantifier depth $< \alpha$ we have:
$\varphi(\bar{x}) \in \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}(N)}(\bar{a}; N; M)$, <u>iff</u> for every $q(\bar{x}) \in \mathbf{P}_\alpha(N, M, \bar{a}), \varphi(\bar{x})$ belong to the type computed implicitly in $\circledast_\alpha$, i.e. if $q(\bar{x}) = \mathrm{gtp}_{L^0_{<\alpha}}(\bar{a}'; N'; M')$ then $(N', M') \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi(\bar{x})$.

Those three should be clear and gives the desired conclusion. Also the last sentence is easy.

<u>Clause $(d)$</u>: Let $N_0 \leq_{\mathfrak{K}} M_0 \in K_{\aleph_0}$ and $\bar{a}_0 \in M_0$ be such that $(M_0, N_0) \Vdash^{\aleph_1}_{\mathfrak{K}} \bigwedge\limits_{\varphi(\bar{x}) \in p} \varphi[\bar{a}_0]$, (if it does not exist, the set of triples is empty). Let $K'' := \{(N, M, \bar{a}) : M \in K_{\aleph_0}, N \in K_{\aleph_0}, N \leq_{\mathfrak{K}} M$, and there are $M'' \in K_{\aleph_0}, M \leq_{\mathfrak{K}} M''$ and $\leq_{\mathfrak{K}}$-embedding $f : M_0 \to M''$, such that $f(N_0) = N, g(\bar{a}_0) = \bar{a}\}$. Clearly it is a $\mathrm{PC}_{\aleph_0}$ class. Also $M_0 \leq_{\mathfrak{K}} M' \in K_{\aleph_0} \Rightarrow \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}(N_0)}(\bar{a}; N_0; M_0) = \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}(N_0)}(\bar{a}; N_0, M')$.

Now first if $(N, M, \bar{a}) \in K''$ let $(M'', f)$ witness this so by applying clause (b) of 4.13 $\mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(\bar{a}; N; M) \subseteq \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(\bar{a}; N; M'') = \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(\bar{a}; N; f(M_0)) = \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(a_0; N_0; M_0) = p$ so $(N, M, \bar{a}) \in K^1$.

Second, if $(N, M, \bar{a}) \in K^1$ let $f_0$ be an isomorphism from $M_0$ onto $M_0$. Let $(M_1, f_1)$ be such that $N_0 \leq_{\mathfrak{K}} M_1 \in K_{\aleph_0}, f_1 \supseteq f_0$ is an isomorphism from $M_1$ onto $M$ and $\bar{a}_1 = f_i^{-1}(\bar{a})$ hence $p = \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(\bar{a}_1; N_0; M_1)$ and we apply clause (c) of 4.13 with $N_0, M_0, \bar{a}_0, M_1, \bar{a}_1$ here standing for $N, M_1, \bar{a}_1, M_2, \bar{a}_2$ there and can finish easily.

<u>Clause $(e)$</u>: We can define $\langle L^{-1}_\alpha : \alpha < \omega_1\rangle$ satisfying the parallel of Clause $(a)$ and repeat the proofs of clauses (b),(c) and we are done.

<u>Clause $(f)$</u>: Suppose this fails.

The proof splits to two cases.

<u>Case A</u>: $2^{\aleph_0} = 2^{\aleph_1}$.

We shall prove $\dot{I}(\aleph_1, K) \geq 2^{\aleph_0}$, thus, (as $2^{\aleph_0} = 2^{\aleph_1}$) contradicting Hypothesis 4.8.
Let $p_i$ (for $i < \omega_2$) be distinct complete $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-types such that for each $i, p_i$ is materialized in some pair $(M; N)$, so $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$ (they exist by the assumption that (f) fails). For each $i < \omega_2$ we define $N_{i,\alpha}, \xi_{i,\alpha}$ (for $\alpha < \omega_1$) and $\bar{a}_{i,\alpha}$ such that:

$\boxtimes_1$ $(i)$ $N_{i,\alpha} \in K_{\aleph_0}$ has universe $\omega(1 + \alpha), N_{0,0} = N$

$(ii)$ $\langle N_{i,\alpha} : \alpha < \omega_1\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous

$(iii)$ $\bar{a}_{i,\alpha} \in N_{i,\alpha+1}, \bar{a}_{i,\alpha}$ materialize $p_i$ in $(N_{i,\alpha+1}, N_{i,\alpha})$

$(iv)$ for every $\alpha < \beta < \omega_1$ and $\bar{a} \in {}^{\omega >}(N_{i,\beta})$, the sequence $\bar{a}$ materialize

in $(N_{i,\beta}, N_{i,\alpha})$ a complete $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-type

$(v)$ $\xi_{i,\alpha} < \omega_1$ is strictly increasing continuous in $\alpha$

$(vi)$ for $\alpha < \beta, N_{i,\beta}$ is pseudo $L^0_\beta(N_{i,\alpha})$-generic, see 4.4(4) and take care

of $\mathbf{Q}$, i.e., if $\gamma < \beta, p(y,\bar{x})$ a complete $L^0_\gamma$-type and

$(N_{i,\beta}, N_{i,\alpha}) \Vdash^{\aleph_1}_{\mathfrak{K}} (\mathbf{Q}y) \wedge p(y,\bar{a})$

then for some $b \in N_{i,\beta+1} \backslash N_{i,\beta}$ we have $(N_{i,\beta+1}, N_{i,\alpha}) \Vdash^{\aleph_1}_{\mathfrak{K}} \wedge p(b,\bar{a})$

$(vii)$ if $\alpha < \beta$ and $\bar{a}, \bar{b} \in N_{\beta-1}$ materialize different $\mathbb{L}^0_{\omega_1,\omega}(N_{i,\alpha})$-types in

$N_{i,\beta}$, <u>then</u> $\bar{a}, \bar{b}$ realize different $(\mathbb{L}_{\omega_1,\omega}(\tau^{+0}) \cap L^{-1}_{\xi_{i,\beta+1}})(N_\alpha)$-types in

$N_{i,\beta}$

$(viii)$ $N_i = \cup\{N_{i,\alpha} : \alpha < \omega_1\}$

$(ix)$ if $\alpha_\ell < \beta$ for $\ell = 1,2, \gamma < \beta, n < \omega$ and $\bar{a}_1 \in {}^n(N_{i,\beta})$ then for some

$\bar{a}_2 \in {}^n(N_{i,\beta})$ we have $\mathrm{gtp}_{L^0_\gamma}(\bar{a}_1; N_{i,\alpha_1}; N_{i,\beta}) =$

$\mathrm{gtp}_{L^0_\gamma}(\bar{a}_2; N_{i,\alpha_2}; N_{i,\beta})$

$(ix)^+$ moreover, if $n < \omega, \gamma_1 < \gamma_2 < \beta, \alpha_\ell < \beta, \bar{a}_\ell \in {}^n(N_{i,\beta})$ for $\ell = 1,2$

and $\mathrm{gtp}_{L^0_{\gamma_2}}(\bar{a}_1; N_{i,\alpha-1}; N_{i,\beta}) =$ $\mathrm{gtp}_{L^0_{\gamma_2}}(\bar{a}_2; N_{i,\alpha_2}; N_{i,\beta})$ and

$b_1 \in N_{i,\beta}$ <u>then</u> for some $b_2 \in N_{i,\beta}$ we have $\mathrm{gtp}_{L^0_{\gamma_1}}(\bar{a}_1{}^\frown\langle b_1\rangle; N_{i,\alpha_1}; N_{i,\beta})$

$= \mathrm{gtp}_{L^0_{\gamma_1}}(\bar{a}{}^\frown\langle b_2\rangle; N_{i,\alpha_2}; N_{i,\beta})$.

This is possible by the earlier claims. By clause (e) of 4.13 clearly

$\boxtimes_2$ the pair $(N_i, N_0)$ is $L^{-1}_{<\omega_1}(\tau^{+0})$-homogeneous.

We could below use $D_i$ a set of complete $L^0_{\delta(i)}$-types, the only problem is that the countable $(D_i, \aleph_0)$-homogeneous models have to be redefined using "materialized" instead "realized". As it is we need to use clause (e) to translate the results on $L^0_{\delta(i)}$ to $L^{-1}_{\delta(i)}$.

Let $\tau^* = \{\in, Q_1, Q_2\} \cup \{c_\ell : \ell < 5\}, c_\ell$ an individual constant and $\mathfrak{A}^*_i$ be $(\mathcal{H}(\aleph_2), \in)$ expanded to a $\tau^*$-model, by predicates for $K, \leq_{\mathfrak{K}}$ with $Q^{\mathfrak{A}^*_i}_1 = K \cap \mathcal{H}(\aleph_2), Q^{\mathfrak{A}^*}_2 = \{(M,N) : M \leq_{\mathfrak{K}} N$ both in $\mathcal{H}(\aleph_2)\}, c^{\mathfrak{A}^*_i}_0, \ldots, c^{\mathfrak{A}^*_i}_4$ being $\{\langle N_{i,\alpha} : \alpha < \omega_1\rangle\}, \langle \xi_{i,\alpha} : \alpha < \omega_1\rangle, \{\langle \bar{a}_{i,\alpha} : \alpha < \omega_1\rangle\}, N_i$ and $\{i\}$ respectively.

Let $\mathfrak{A}_i$ be a countable elementary submodel of $\mathfrak{A}^*_i$ so $|\mathfrak{A}_i| \cap \omega_1$ is an ordinal $\delta(i) < \omega_1$. It is also clear that $c^{\mathfrak{A}_i}_3$ is $N_{i,\delta(i)}$ as $c^{\mathfrak{A}^*_i}_3 = N_i$. As $\mathfrak{A}_i$ is defined for $i < \omega_2$, for some unbounded $S \subseteq \omega_2$ and $\delta < \omega_1$, for every $i \in S, \delta(i) = \delta$ and for $i,j \in S$, some sequence from $N_j$ materializes $p_i$ in the pair $(N_j, N_{j,\delta(j)})$ iff $i = j$. For $i \in S$ let $D_i = \{p : p$ is a complete $L^{-1}_{\delta(i)}$-type materialized in $(N_{i,\delta(i)}, N_{i,0})\}$. Because of the $\xi_{i,\alpha}$'s choice and $\boxtimes_2$ the pair $(N_{i,\delta}, N_0)$

is $(D_i, \aleph_0)$-homogeneous and $D_i$ is a countable set of complete $L^{-1}_\delta$-types. Note that by the choice of $S, i \neq j (\in S) \Rightarrow D_i \neq D_j$.

Let $\Gamma = \{D : D$ a countable set of complete $L^{-1}_\delta$-types, such that for some model $\mathfrak{A} = \mathfrak{A}_D$ of $\bigcap\limits_{i \in S} \mathrm{Th}_{\mathbb{L}_{\omega,\omega}}(\mathfrak{A}_i)$, with $\{a : \mathfrak{A}_D \models$ "a countable ordinal$\} = \delta$ (and the usual order) we have $D = \{\{\varphi(\bar{x}) : \varphi(\bar{x}) \in L^{-1}_\delta$ and $\mathfrak{A}_D \models$ "$(N; N_0) \Vdash^{\aleph_1}_{\mathfrak{K}} \varphi[\bar{a}]$"$\} : \bar{a} \in N$ where $N = c_3^{\mathfrak{A}_D}\}\}$.

So $D_i \in \Gamma$ for $i < \omega_2$, hence $\Gamma$ is uncountable.

By standard descriptive set theory $\Gamma$ (is an analytic set hence) has cardinality continuum. So let $D(\zeta) \in \Gamma$ be distinct for $\zeta < 2^{\aleph_0}$. For each $\zeta$, let $\mathfrak{A}^0_{D(\zeta)}$ be as in the definition of $\Gamma$. We define by induction on $\alpha < \omega_1, \mathfrak{A}^\alpha_{D(\zeta)}$ such that

- $(\alpha)$ $\mathfrak{A}^\alpha_{D(\zeta)}$ is countable
- $(\beta)$ $\alpha < \beta \Rightarrow \mathfrak{A}^\alpha_{D(\zeta)} \prec_{\mathbb{L}_{\omega,\omega}} \mathfrak{A}^\beta_{D(\zeta)}$
- $(\gamma)$ for limit $\alpha$ we have $\mathfrak{A}^\alpha_{D(\zeta)} = \bigcup\limits_{\beta < \alpha} \mathfrak{A}^\beta_{D(\zeta)}$
- $(\delta)$ if $d \in \mathfrak{A}^{\alpha+1}_{D(\zeta)} \backslash \mathfrak{A}^\alpha_{D(\zeta)}, \mathfrak{A}^{\alpha+1}_{D(\zeta)} \models$ "$d$ a countable ordinal" <u>then</u> for $a \in \mathfrak{A}^\alpha_{D(\zeta)}$ we have $\mathfrak{A}^{\alpha+1}_{D(\zeta)} \models$ "if $a$ is a countable ordinal then $a < d$"
- $(\varepsilon)$ for $\alpha = 0$ in clause $(\delta)$ there is no minimal such $d$
- $(\zeta)$ for every $\alpha$ there is $d_{\zeta,\alpha} \in \mathfrak{A}^{\alpha+1}_{D(\zeta)} \backslash \mathfrak{A}^\alpha_{D(\zeta)}$ satisfying $\mathfrak{A}^{\alpha+1}_{D(\zeta)} \models$ "$d_{\zeta,\alpha}$ a countable ordinal" and for $\alpha \neq 0$ it is minimal.

without loss of generality

$(*)$ $(\mathcal{H}(\aleph_1)^{\mathfrak{A}^0_{D(\zeta)}}, \in^{\mathfrak{A}^0_{D(\zeta)}})$ is equal to its Mostowski collapse (and $\mathbb{L}_{\omega_1,\omega}(N) \subseteq \mathcal{H}(\aleph_1)$).

(We could have fixed also $\mathrm{otp}(\mathfrak{A}_i \cap \omega_2)$, hence ensure also $(\mathfrak{A}^0_{D(\zeta)}, \in^{\mathfrak{A}^0_{D(\zeta)}})$ is equal to its Mostowski collapse).

Let $M_{\zeta,\alpha}$ be the $d_{\zeta,\alpha}$-th member of the $\omega_1$-sequence of models in $\mathfrak{A}^\beta_{D(\zeta)}$ for $\beta > \alpha$ (remember $c_0^{\mathfrak{A}^*_i} = \langle N_{i,\alpha} : \alpha < \omega_1 \rangle$). Let $M_\zeta = \bigcup\limits_{\alpha < \omega_1} M_{\zeta,\alpha}$. By absoluteness from $\mathfrak{A}^\beta_{D(\zeta)}$ we have $M_{\zeta,\alpha} \leq_{\mathfrak{K}} M_{\zeta,\beta} \in K_{\aleph_0}$. Now

$(*)$ $0 < \alpha < \beta, (M_{\zeta,\beta}, M_{\zeta,\alpha})$ is $(D(\zeta), \aleph_0)$-homogeneous.

[Why? Assume $\mathfrak{A}^\alpha_{D(\zeta)} \models$ "$d_1 < d_2$ are countable ordinals $> \gamma$" when $\gamma < \delta$. Now if $\bar{a}, \bar{b} \in {}^{\omega >}(N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2})$ and $[\gamma < \delta \Rightarrow \mathrm{gtp}_{L^0_\gamma}(\bar{a}; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_1}; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2}) = \mathrm{gtp}_{L^0_\gamma}(\bar{b}; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_1}; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2})]$ then also $\mathfrak{A}^\alpha_{D(\zeta)}$ satisfies this but "$\mathfrak{A}^\alpha_{D(\zeta)}$ thinks that the countable ordinals are well ordered" <u>hence</u> for some $d, \mathfrak{A}^\alpha_{D(\zeta)} \models$ "$d$ is a countable ordinal $> \gamma$" for each $\gamma < \delta$ and we have $\mathfrak{A}^\alpha_{D(\zeta)} \models$ "$\mathrm{gtp}_{L^0_d}(\bar{a}; N_{d_1}; N_{d_2}) = \mathrm{gtp}_{L^0_d}(\bar{a}; N_{d_1}; N_{d_2})$". Hence if $\mathfrak{A}^\alpha_{D(\zeta)} \models$ "$d' < d$" then for every $a \in N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2}$ for some $b \in N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2}$ we have

$$\mathfrak{A}^\alpha_{D(\zeta)} \models \text{"}\mathrm{gtp}_{L^0_d}(\bar{a}\hat{\ }\langle a \rangle; N_{d_1}; N_{d_2}) = \mathrm{gtp}_{L^0_d}(\bar{b}\hat{\ }\langle b \rangle; N_{d_1}; N_{d_2})\text{"}$$

hence $\text{gtp}_{L^0_\gamma}(\bar{a}\hat{\ }\langle a\rangle; N^{\mathfrak{A}^\alpha_{D(\zeta)}}; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2}) =$ $\text{gtp}(\bar{b}\hat{\ }\langle b\rangle; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_1}; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2})$.

Also we can replace $L^0_\delta$ by $L^{-1}_\delta$. By clause (x) of $\boxtimes_1$ the set $\{\text{gtp}_{L^0_\delta}(\bar{a}; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_1}; N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2}) : a \in {}^{\omega>}(N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2})\}$ is $D_i$.

So $(N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2}, N^{\mathfrak{A}^\alpha_{D(\zeta)}}_{d_2})$ is $(D_i, \aleph_0)$-homogenous.

So from the isomorphism type of $M_\zeta$ we can compute $D(\zeta)$. So $\zeta \neq \xi \Rightarrow M_\zeta \not\cong M_\xi$. As $M_\zeta \in K_{\aleph_1}$ we finish. Case B: $2^{\aleph_0} < 2^{\aleph_1}$.

By 3.9, $\mathfrak{K}$ has the $\aleph_0$-amalgamation property. So clearly if $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}, \bar{a} \in M$, then $\bar{a}$ materializes in $(M, N)$ a complete $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-type. We would now like to use descriptive set theory.

We represent a complete $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-type materialized in some $(N, M)$ by a real, by representing the isomorphism type of some $(N, M, \bar{a}), N \leq_{\mathfrak{K}} M \in K_{\aleph_0}, \bar{a} \in M$. The set of representatives is analytic recalling $\mathfrak{K}$ is $\text{PC}_{\aleph_0}$, and the equivalence relation is $\Sigma^1_1$. [As $(N_1, M_1, \bar{a}_1), (N_2, M_2, \bar{a}_2)$ represents the same type if and only if for some $(N, M), N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$, there are $\leq_{\mathfrak{K}}$-embeddings $f_1 : M_1 \to M, f_2 : M_2 \to M$ such that $f_1(N_1) = f_2(N_2) = N$ and $f_1(\bar{a}) = f_2(\bar{a})$.]

By Burgess [Bur78] (or see [Sh:202]) as there are $> \aleph_1$ equivalence classes, there is a perfect set of representation, pairwise representing different types.

From this we easily get that without loss of generality that their restriction to some $L^0_\alpha$ are distinct, contradicting part (a).

Clause $(g)$: Easy by the proof of clause (f), Case A above but much simpler as in 4.12.

Clause $(h)$: As in the proof of clause (e).
2) Should by clear by now. $\square_{4.13}$

4.16. **Remark.** 1) Note that in the proof of Clause (f) of 4.13, in Case (A) we get many types too but it was not clear whether we can make the $N_\zeta$ to be generic enough, to get the contradiction we got in Case (B) but this is not crucial here.
2) We may like to replace $\mathbb{L}^0_{\omega_1,\omega}$ by $\mathbb{L}^1_{\omega_1,\omega}$ in 4.10, 4.11 and 4.13 (except that, for our benefit, in 4.13(e), we may retain the definition of $L^1(N)$). We lose the ability to build $L$-generic models in $K_{\aleph_1}$ (as the number of (even unary) relations on $N \in K_{\aleph_0}$ is $2^{\aleph_0}$, which may be $> \aleph_1$). However, we can say "$\bar{a}$ materializes in $N \in K_{\aleph_0}$ the type $p = p(\bar{x})$ which is a complete type in $\mathbb{L}^1_{\omega_1,\omega}(N_n, N_{n-1}, \ldots, N_0)$; where $N_0 \leq_{\mathfrak{K}} \ldots \leq_{\mathfrak{K}} N_n \leq_{\mathfrak{K}} N, N_\ell$ countable)".
[Why? Let some $N^1, \bar{a}^1$ be as above, $\bar{a}^1$ materialize $p$ in $(N^1, N_n, \ldots, N_0)$ then this holds for $(N, \bar{a})$ iff for some $N', f$ we have $N \leq_{\mathfrak{K}} N' \in K_{\aleph_1}$ and $f$ is an isomorphism from $N^1$ onto $N''$ mapping $\bar{a}^1$ to $\bar{a}$ and $N_\ell$ to $N_\ell$ for $\ell \leq n$. If there is no such pair $(N^1, \bar{a}^1)$ this is trivial.]
We can get something on formulas.

This suffices for 4.10.

{88r-4.9}

4.17. **Concluding remarks for §4.** 0) We can get more information on the case $1 \leq \dot{I}(\aleph_1, K) < 2^{\aleph_1}$ (and the case $1 \leq \dot{I}(\aleph_1, K^{\mathbf{F}}_{\aleph_1}) < 2^{\aleph_1}$, etc.).
1) As in 3.9, there is no difficulty in getting the results of this section for the class of models of $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ because using $(K, \leq_{\mathfrak{K}})$ from the proof of 3.19(2) in all constructions we get many non-isomorphic models for appropriate $\mathbf{F}$, as in 4.9(2).
2) For generic enough $N \in K_{\aleph_1}$ with $\leq_{\mathfrak{K}}$-representation $\langle N_\alpha : \alpha < \omega_1 \rangle$, we have determined the $N_\alpha$'s (by having that without loss of generality $K$ is categorical in $\aleph_0$). In this section we have shown that for some club $E$ of $\omega_1$, for all $\alpha < \beta$ from $E$ the isomorphism type of $(N_\beta, N_\alpha)$ essentially [6] is unique. We can continue the analysis, e.g., deal with sequences $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} \ldots \leq_{\mathfrak{K}} N_k \in K_{\aleph_0}$ such that $N_{\ell+1}$ is pseudo $L^0_\alpha(N_\ell, N_{\ell-1}, \ldots, N_0)$-generic. We can prove by induction on $k$ that for any countable $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^{+k})$ for some $\alpha$, any strong $L$-generic $N \in K_{\aleph_1}$ is $L$-determined. That is, for any $\langle N_\alpha : \alpha < \omega_1 \rangle, N_\alpha \leq_{\mathfrak{K}} N$ countable $\leq_{\mathfrak{K}}$-increasing continuous with union $N$, for some club $E$ for all $\alpha_0 < \ldots < \alpha_k$ from $N$ the isomorphic type of $\langle N_{\alpha_k}, N_{\alpha_k}, \ldots, N_{\alpha_0} \rangle$ is the same; i.e., determining for $\mathbb{L}_{\infty,\omega}(aa)$.
3) We can do the same for stronger logics, let us elaborate.
Let us define a logic $\mathcal{L}^*$. It has as variable
variables for elements $x_1, x_2$...and
variables for filters $\mathcal{Y}_1, \mathcal{Y}_2$... The atomic formulas are:

(i) the usual ones
(ii) $x \in \operatorname{Dom}(\mathcal{Y})$.

The logical operations are:

($a$) $\wedge$ conjunction, $\neg$ negation
($b$) $(\exists x)$ existential quantification where $x$ is individual variable
($c$) the quantifier aa acting on variables $\mathcal{Y}$ so we can form $(aa\, \mathcal{Y})\varphi$
($d$) the quantification $(\exists x \in \operatorname{Dom}(\mathcal{Y}))\varphi$
($e$) the quantification $(\exists^f x \in \operatorname{Dom}(\mathcal{Y}))\varphi$.

It should be clear what are the free variables of a formula $\varphi$. The variable $\mathcal{Y}$ vary on pairs (a countable set, a filter on the set). Now in $\exists x[\varphi, \mathcal{Y}], (\exists x \in \operatorname{Dom}(\mathcal{Y}))\varphi, (\exists^f x \in \operatorname{Dom}(\mathcal{Y}))\varphi, x$ is bounded but not $\mathcal{Y}$ and in $aa\mathcal{Y}, \mathcal{Y}$ is bounded. The satisfaction relation is defined as usual plus

($\alpha$) $M \models (\exists x \in \operatorname{Dom}(\mathcal{Y})\varphi(x, \mathcal{Y}, \bar{a})$ if and only if for some $b$ from the domain of $\mathcal{Y}, M \models \varphi[b, \mathcal{Y}, \bar{a}]$
($\beta$) $M \models \exists^f x \in \operatorname{Dom}(\mathcal{Y})\varphi(x, \mathcal{Y}_{\bar{a}})$ if and only if $\{x \in \operatorname{Dom}(\mathcal{Y}) : \models \varphi(x, \mathcal{Y}, \bar{a})\} \in \mathcal{Y}$
($\gamma$) $M \models (aa\, \mathcal{Y}, \bar{a})\varphi(\mathcal{Y})$ if and only if there is a function $\mathbf{F}$ from ${}^{\omega>}([M]^{<\aleph_1}) \rightarrow [M]^{<\aleph_1}$ such that: if $A_n \subseteq M, |A_n| \leq \aleph_0, A_n \subseteq A_{n+1}$ and $\mathbf{F}(A_0, \ldots, A_n) \subseteq A_{n+1}$ <u>then</u>

[6] why only essentially? as the number of relevant complete types can be $\aleph_1$; we can get rid of this by shrinking $\mathfrak{K}$

modified:2015-01-25

(88r) revision:2015-01-23

$M \models \varphi[\mathcal{Y}_{\langle A_n : n<\omega\rangle}, \bar{a}]$ where $\mathcal{Y}_{\langle A_n : n<\omega\rangle}$ is the filter on $\bigcup\limits_{n<\omega} A_n$, generated by $\{\cup\{A_n : n < \omega\} \backslash A_\ell : \ell < \omega\}$.

4) We, of course, can define $\mathcal{L}^*_{\mu,\kappa}$ (extending $\mathbb{L}_{\mu,\ell}$). As we like to analyze models in $\aleph_1$, it is most natural to deal with $\mathcal{L}^*_{\omega_1,\omega}$. We can prove that (if $1 \le \dot{I}(\aleph_1, \mathfrak{K}) < 2^{\aleph_1}$) the quantifier $aa\,\mathcal{Y}$ is determined on $K_{\aleph_1}$ (i.e., for almost all $\mathcal{Y}, \varphi(\mathcal{Y})$ iff not for almost all $\mathcal{Y}, \neg\varphi(\mathcal{Y})$.

5) The logic from (3) strengthens the stationary logic $\mathbb{L}(aa)$, see [Sh:43], [BKM78].

Not so strongly: looking at $\mathrm{PC}_{\aleph_0}$ class for $\mathbb{L}_{\omega_1,\omega}(aa)$ (i.e., $\{M \restriction \tau : M$ a model of $\psi$ of cardinal $\aleph_1\})$, we can assume that $\psi \vdash$ " $<$ is an $\aleph_1$-like order". Now we can express $\varphi \in \mathcal{L}^*_{\omega_1,\omega}$, but the determinacy tells us more. Also we can continue to define higher variables $\mathcal{Y}$.

## 5. There is a superlimit model in $\aleph_1$

Here we make

5.1. **Hypothesis.** Like 4.8, but also $2^{\aleph_0} < 2^{\aleph_1}$. {88r-5.0}

(Note that we can assume that $K_{\aleph_0}$ is the class of atomic models of a first order complete countable theory).

This section is the deepest (of this paper = chapter). The main difficulties are proving the facts which are obvious in the context of [Sh:48]. So while it was easy to show that every $p \in \mathbf{D}^*(N)$ is definable over a finite set ($\mathbf{D}^*(N)$ is defined below), it was not clear to me how to prove that if you extend the type $p$ to $q \in \mathbf{D}^*(M)$ where $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$, by the same definition, then $q \models p$ (remember $p, q$ are types materialized not realized, and at this point in the paper we still do not have the tools to replace the models by uncountable generic enough models). So we rather have to show that failure is a non-structure property, i.e., implies existence of many models.

Also symmetry of stable amalgamation becomes much more complicated. We prove existence of stable amalgamation by four stages (5.29,5.30(3),5.34,5.37). The symmetry is proved as a consequence of uniqueness of one sided amalgamation, (so it cannot be used in its proof). Originally the intention was the culmination of the section to be the existence of a superlimit models in $\aleph_1$ (5.45). This seems a natural stopping point as it seems reasonable to expect that the next step should be phrasing the induction on $n$, i.e., dealing with $\aleph_n$ and $\mathcal{P}(n-\ell)$-diagrams of models of power $\aleph_\ell$ as in [Sh:87a], [Sh:87b]; (so this is done in [Sh:705]). But less is needed in [Sh:600].

5.2. **Definition.** We define functions $\mathbf{D}, \mathbf{D}^*$ with domain $K_{\aleph_0}$. {88r-5.1}

1) For $N \in K_{\aleph_0}$ let $\mathbf{D}(N) = \{p : p$ is a complete $\mathbb{L}^0_{\omega_1,\omega}(N)$-type over $N$ such that for some $\bar{a} \in M \in K_{\aleph_0}, N \leq_{\mathfrak{K}} M$ and $\bar{a}$ materializes $p$ in $(M, N)\}$, (i.e. the members of $p$ have the form $\varphi(\bar{x}, \bar{a})$, ($\bar{x}$ finite and fixed for $p$) $\bar{a}$ a finite sequence from $N$ and $\varphi \in \mathbb{L}^0_{\omega_1,\omega}(N)$).

2) For $N \in K_{\aleph_0}$ let $\mathbf{D}^*(N) = \{p : p$ a complete $\mathbb{L}^0_{\omega_1,\omega}(N;N)$-type such that for some $\bar{a} \in M \in K_{\aleph_0}, N \leq_{\mathfrak{K}} M$ and $\bar{a}$ materializes $p$ in $(M, N; N)\}$.

3) For $p(\bar{x}, \bar{y}) \in \mathbf{D}(N)$ let $p(\bar{x}, \bar{y}) \restriction \bar{x} \in \mathbf{D}(N)$ be defined naturally; i.e. if for some $M, N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$ and $\bar{a} \restriction \bar{b} \in {}^{\ell g(\bar{x}\hat{\ }\bar{y})}M$ materializing $p(\bar{x}, \bar{y})$ such that $\ell g(\bar{x}) = \ell g(\bar{a})$, the sequence $\bar{a}$ materializes $p(\bar{x}, \bar{y}) \restriction x \in \mathbf{D}(N)$. Similarly for permuting the variables.

5.3. **Explanation.** 0) Recall that any formula in $\mathbb{L}^0_{\omega_1,\omega}(N)$ has finitely many free variables. {88r-5.1A}

1) So for every finite $\bar{b} \in N$ and $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}^0_{\omega_1,\omega}(N)$, if $p \in \mathbf{D}(N)$, then $\varphi(\bar{x}, \bar{b}) \in p$ or $\neg\varphi(\bar{x}, \bar{b}) \in p$.

2) But a formula from $p \in \mathbf{D}^*(N)$ may have all $c \in N$ as parameters whereas a formula from $p \in \mathbf{D}(N)$ can mention only finitely many members of $N$.

5.4. **Lemma.** 1) $\mathfrak{K}$ *has the* $\aleph_0$-*amalgamation property.* {88r-5.2}

2) *If* $N_* \leq_{\mathfrak{K}} N \in K_{\aleph_0}$, $A_i \subseteq N_*$ *for* $i \leq n$ *then for every sentence* $\psi \in \mathbb{L}^1_{\infty,\omega}(N_*, A_n, \dots, A_1; A_0)$ *we have*

$$N \Vdash^{\aleph_1}_{\mathfrak{K}} \psi \text{ or } N \Vdash^{\aleph_1}_{\mathfrak{K}} \neg\psi.$$

3) *If* $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$, *then every* $\bar{a} \in M$ *materializes in* $(M, N; N)$ *one and only one type from* $\mathbf{D}^*(N)$ *and also materializes in* $(M, N)$ *one and only one type from* $\mathbf{D}(N)$. *Also for every* $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$ *and* $q \in \mathbf{D}^*(N)$ *for some* $M', M \leq_{\mathfrak{K}} M' \in K_{\aleph_0}$ *and some* $\bar{b} \in M'$ *materializes* $q$ *in* $(M; N)$.

4) *For every* $N \in K_{\aleph_0}$ *and countable* $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(N; N)$ *the number of complete* $L(N; N)$*-types* $p$ *such that* $N \Vdash^{\aleph_1}_{\mathfrak{K}}$ "$(\exists \bar{x}) \wedge p$" *is countable; note that pedantically* $L \subseteq \mathbb{L}_{\omega_1,\omega}(\tau^+ \cup \{c : c \in N\})$ *and we restrict ourselves to models* $M$ *such that* $P^M = |N|, c^M = c$.

5) *For* $N \in K_{\aleph_0}$ *there are countable* $L^0_\alpha \subseteq \mathbb{L}^0_{\omega_1,\omega}(N; N)$ *for* $\alpha < \omega_1$ *increasing continuous in* $\alpha$, *closed under finitary operations (and subformulas) such that:*

$(*)$ *for each complete* $L^0_\alpha$*-type* $p$ *we have*

$$[N \Vdash^{\aleph_1}_{\mathfrak{K}} \exists \bar{x} \wedge p \Rightarrow \wedge p \in L^0_{\alpha+1}].$$

*Hence for every* $\mathbb{L}^0_{\omega_1,\omega}(N; N)$ *formula* $\psi(\bar{x})$ *for some* $\varphi_n(\bar{x}) \in \bigcup\limits_{\alpha<\omega} L^0_\alpha$ *for* $n < \omega$ *for every* $N \in K_{\aleph_0}$

$$(N, N) \Vdash^{\aleph_1}_{\mathfrak{K}} (\forall \bar{x})[\psi(\bar{x}) \equiv \bigvee_{n<\omega} \varphi_n(\bar{x})].$$

6) *For* $N \in K_{\aleph_0}$ *we have* $|\mathbf{D}^*(N)| \leq \aleph_1$ *and* $|\mathbf{D}(N)| \leq \aleph_1$.

7) *If* $p \in \mathbf{D}^*(N)$ *then there is* $q$ *such that: if* $N \leq_{\mathfrak{K}} M \in K_\lambda, \bar{a} \in M$ *materializes* $p$ *in* $(M; N)$ *then the complete* $\mathbb{L}^0_{\infty,\omega}(N)$*-type which* $\bar{a}$ *realizes in* $M$ *over* $N$ *is* $q$*; also* $q$ *belongs to* $\mathbf{D}(N)$ *and is unique. Moreover, we can replace* $q$ *by the complete* $\mathbb{L}^{-1}_{\omega_1,\omega}(N)$*-type which* $\bar{a}$ *materializes in* $M$. *Similarly for* $\mathbf{D}(N), \mathbb{L}^0_{\infty,\omega}(N), \mathbb{L}^{-1}_{\omega_1,\omega}(N)$.

8) *If* $n < \omega$ *and* $\bar{b}, \bar{c} \in {}^n N$ *realize the same* $\mathbb{L}_{\omega_1,\omega}(\tau)$*-type in* $N$ *then they materialize the same* $\mathbb{L}^1_{\omega_1,\omega}(\tau^{+0})$*-type in* $(N, N)$.

9) *If* $f$ *is an isomorphism from* $N_1 \in K_{\aleph_0}$ *onto* $N_2 \in K_{\aleph_0}$ *then* $f$ *induces a one to one function from* $\mathbf{D}(N_1)$ *onto* $\mathbf{D}(N_2)$ *and from* $\mathbf{D}^*(N_1)$ *onto* $\mathbf{D}^*(N_2)$.

*Proof.* 1) By 3.9.
2) By 1).
3) By 2) and 1).
4) Like the proof of 4.11 (just easier).

5) Like the proof of 4.13(a).
6) Like the proof of 4.13(f) (recalling 0.4).
7) Clear as in $p \in \mathbf{D}^*(N)$ we allow more formulas than for $q \in \mathbf{D}(N)$.
8),9) Easy, too. $\square_{5.4}$

We shall use from now on a variant of gtp (in Definition 4.3(4) we define $\mathrm{gtp}_L(\bar{a}; N_*, \bar{A}; A; N)$.

{88r-5.2.7}

5.5. **Definition.** 1) If $N_0 \le_{\mathfrak{K}} N_1 \in K_{\aleph_0}, \bar{a} \in N_1$, $\mathrm{gtp}(\bar{a}, N_0, N_1)$ is the $p \in \mathbf{D}(N_0)$ such that $(N_1, N_0) \Vdash^{\aleph_1}_{\mathfrak{K}} \wedge p[\bar{a}]$. So $\bar{a}$ materializes (but does not necessarily realize) $\mathrm{gtp}(\bar{a}, N_0, N_1)$. We may omit $N_1$ when clear from context. We define $\mathrm{gtp}^*(\bar{a}, N_0, N_1) \in \mathbf{D}^*(N_0)$ similarly.
2) We say $p = \mathrm{gtp}^*(\bar{b}, N_0, N_1)$ is definable over $\bar{a} \in N_0$ <u>if</u> $\mathrm{gtp}(\bar{b}, N_0, N_1) = p^- := \{\varphi(\bar{x}, \bar{a}) \in p : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}^0_{\omega_1,\omega}(N_0)$ and $\bar{a} \in {}^{\ell g(\bar{y})}(N_0) \subseteq {}^{\omega >}(N_0)\}$ is definable over $\bar{a}$ (see Definition 5.7 below, note that $p \mapsto p^-$ is a one-to-one mapping from $\mathbf{D}^*(N_0)$ onto $\mathbf{D}(N_0)$ by 5.9(1) below). So stationarization is defined for $p \in \mathbf{D}^*(N_0)$, too, after we know 5.9(1).

{88r-5.4}

5.6. **Claim.** *1) Each $p \in \mathbf{D}(N)$ does not $(\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0}), \mathbb{L}_{\omega_1,\omega}(\tau))$-split (see Definition 5.7 below; also see more below) over some finite subset $C$ of $N$, hence p is definable over it. Moreover, letting $\bar{c}$ list $C$ there is a function $g_p$ satisfying $g_p(\varphi(\bar{x}, \bar{y}))$ is $\psi_{p,\varphi}(\bar{y}, \bar{z}) \in \mathbb{L}_{\omega_1,\omega}(\tau)$ such that for each $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}^0_{\omega_1,\omega}(N)$ and $\bar{a} \in N$ we have $[\varphi(\bar{x}, \bar{a}) \in p \Leftrightarrow N \models \psi_{p,\varphi}(\bar{a}, \bar{c})]$, (in particular, $\mathbf{Q}$ is "not necessary").*
*2) Every automorphism of $N$ maps $\mathbf{D}(N)$ onto itself and each $p \in \mathbf{D}(N)$ has at most $\aleph_0$ possible images; we may also call them conjugates. So if $g$ is an isomorphism from $N_0 \in K_{\aleph_0}$ onto $N_1 \in K_{\aleph_0}$ then $g(\mathbf{D}(N_0)) = \mathbf{D}(N_1)$.*
*3) If $N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ and $\bar{a} \in N_1$ then* $\mathrm{gtp}(\bar{a}, N_0, N_1) = \mathrm{gtp}(\bar{a}, N_0, N_2)$.

Before we prove 5.6:

{88r-5.4.1}

5.7. **Definition.** Assume

- $(a)$ $N$ is a model
- $(b)$ $\Delta_1$ is a set of formulas (possibly in a vocabulary $\nsubseteq \tau_N$) closed under negation
- $(c)$ $\Delta_2$ is a set of formulas in the vocabulary $\tau = \tau_N$
- $(d)$ $p$ is a $(\Delta_1, n)$-type over $N$ (i.e., each member has the form $\varphi(\bar{x}, \bar{a}), \bar{a}$ from $N, \varphi(\bar{x}, \bar{y})$ from $\Delta_1, \bar{x} = \langle x_\ell : \ell < n\rangle$; no more is required (we may allow other formulas but they are irrelevant)
- $(e)$ $A \subseteq N$.

0) We say $p$ is a complete $\Delta_1$-type over $B$ when:

- $(i)$ $B \subseteq N$
- $(ii)$ $\varphi(\bar{x}, \bar{b}) \in p \Rightarrow \bar{b} \subseteq A \wedge \varphi(\bar{x}, \bar{y}) \in \Delta_1$
- $(iii)$ if $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ and $\bar{b} \in {}^{\ell g(\bar{y})}A$ then $\varphi(\bar{x}, \bar{b}) \in p$ or $\neg\varphi(\bar{x}, \bar{b}) \in p$.

The default value here for $\Delta_1$ is $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{K}})$.
1) We say that $p$ does $(\Delta_1, \Delta_2)$-split over $A$ when there are $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ and $\bar{b}, \bar{c} \in {}^{\ell g(\bar{y})}N$ such that

($\alpha$) $\varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p$

($\beta$) $\bar{b}, \bar{c}$ realize the same $\Delta_2$-type over $A$.

2) We say that $p$ is $(\Delta_1, \Delta_2)$-definable over $A$ <u>when</u>: for every formula $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ there is a formula $\psi(\bar{y}, \bar{z}) \in \Delta_2$ and $\bar{c} \in {}^{\ell g(\bar{z})}A$ such that

$$\varphi(\bar{x}, \bar{b}) \in p \Rightarrow N \models \psi[\bar{b}, \bar{c}]$$

$$\neg\varphi(\bar{x}, \bar{b}) \in p \Rightarrow N \models \neg\psi[\bar{b}, \bar{c}]$$

(in the case $p$ is complete over $B, \bar{b} \subseteq B$ we get "iff").
3) Above we may write $\Delta_2$ instead of $(\Delta_1, \Delta_2)$ when this holds for every $\Delta_1$ (equivalently $\Delta_1$ is $\{\varphi(\bar{x}, \bar{y}) : \varphi(\bar{x}, \bar{a}) \in p\}$).

{88r-5.4.2}

5.8. **Observation.** Assume

$(a), (b), (c), (d), (e)$ as in 5.7 and in addition
$(d)^+$ $p$ is a complete $(\Delta_1, n)$-type over $N$, i.e., if

$$\varphi(\bar{x}, \bar{y}) \in \Delta_1, \bar{d} \in {}^{\ell g(\bar{y})}N, \bar{x} = \langle x_\ell : \ell < n\rangle$$

then $\varphi(\bar{x}, \bar{d}) \in p$ or $\neg\varphi(\bar{x}, \bar{d}) \in d$. <u>Then</u> the following conditions are equivalent:

($\alpha$) $p$ does not $(\Delta_1, \Delta_2)$-splits over $A$

($\beta$) there is a sequence of $\langle g_{\varphi(\bar{x},\bar{y})} : \varphi(\bar{x}, \bar{y}) \in \Delta_1\rangle$ of functions such that:

 ($i$) $g_{\varphi(\bar{x},\bar{y})}$ is a function with domain including $\{\mathrm{tp}_{\Delta_2}(\bar{b}, A, N)$: $\bar{b} \in {}^{\ell g(\bar{y})}N\}$

 ($ii$) the values of $g_{\varphi(\bar{x},\bar{y})}$ are truth values

 ($iii$) if $\varphi(\bar{x}, \bar{y}) \in \Delta_1, \bar{b} \in {}^{\ell g(\bar{y})}N$ and $q = \mathrm{tp}_{\Delta_2}(\bar{b}, A, N)$ <u>then</u>:

 $\varphi(\bar{x}, \bar{b}) \in p \Rightarrow g_{\varphi(\bar{x},\bar{y})}(q) = \text{true}$, and

 $\neg\varphi(\bar{x}, \bar{b}) \in p \Rightarrow g_{\varphi(\bar{x},\bar{y})}(q) = \text{false}$.

*Proof of 5.8.* Reflect on the definitions. $\square_{5.8}$

*Proof of 5.6.* 1) Clearly the second sentence follows from the first, so we shall prove the first. Assume this fails. Let $(M, \bar{a})$ be such that $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$ the sequence $\bar{a} \in M$ materializes $p$ and clearly for every $\bar{b} \in M, (M, N) \Vdash \wedge q[\bar{b}]$ for some $q(\bar{x}) \in \mathbf{D}(N)$ and let $\langle b^*_\ell : \ell < \omega \rangle$ list $N$. We choose by induction on $n, \langle C^0_\eta, C^1_\eta, f_\eta, \bar{a}^0_\eta, \bar{a}^1_\eta : \eta \in {}^n 2 \rangle$ such that

- $(a)$ $C^\ell_\eta$ is a finite subset of $N$ for $\ell < 2, \eta \in {}^n 2$
- $(b)$ $f_\eta$ is an automorphism of $N$ mapping $C^0_\eta$ onto $C^1_\eta$
- $(c)$ $\{b^*_{\ell g(\eta)}\} \cup C^0_\eta \cup C^1_\eta \subseteq C^0_{\eta\hat{}<\ell>} \cap C^1_{\eta\hat{}<\ell>}$ for $\ell = 0, 1$
- $(d)$ $\bar{a}^0_\eta, \bar{a}^1_\eta \in N$ realize in $N$ the same $\mathbb{L}_{\omega_1,\omega}(\tau)$-type over $C^0_\eta \cup C^1_\eta \cup \{b^*_{\ell g(\eta)}\}$ in $(M, N)$ but $\bar{a}\hat{}\bar{a}^0_\eta, \bar{a}\hat{}\bar{a}^1_\eta$ do not materialize the same $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$ in $(M, N)$ (this exemplifies splitting), so $\varphi_\eta(\bar{x}, \bar{y}_\eta)$ belongs to the first, $\neg\varphi_\eta(\bar{x}, \bar{y}_\eta)$ belongs to the second (where $\ell g(\bar{x}) = \ell g(\bar{a}), \ell g(\bar{y}_\eta) = \ell g(\bar{a}^0_\eta)$)
- $(e)$ $f_{\eta\hat{}<0>}(\bar{a}^0_\eta) = \bar{a}^1_\eta, f_{\eta\hat{}<1>}(\bar{a}^1_\eta) = \bar{a}^1_\eta$
- $(f)$ $f_\eta \restriction C^0_\eta \subseteq f_{\eta\hat{}<\ell>}$ for $\ell = 0, 1$
- $(g)$ $\bar{a}^0_\eta\hat{}\bar{a}^1_\eta \subseteq C^0_{\eta\hat{}<\ell>} \cap C^1_{\eta\hat{}<\ell>}$.

For $n = 0$ let $C^0_\eta, C^1_\eta = \emptyset, f_\eta = \text{ id}_N$. Recall that $K_{\aleph_0}$ is categorical in $\aleph_0$ and $N$ is countable, hence if $n < \omega, \bar{b}', \bar{b}'' \in {}^n N$ realize the same $\mathbb{L}_{\omega_1,\omega}(\tau)$-type over a finite subset $B$ of $N$, then some automorphism of $N$ over $B$ maps $\bar{b}'$ to $\bar{b}''$ by a theorem of Scott (see [Kei71]). If $(C^0_\eta, C^1_\eta, f_\eta)$ are defined and satisfies clauses (a), (b) we recall that by our assumption toward contradiction as $C^0_\eta \cup C^1_\eta \cup \{b^*_{\ell g(\eta)}\}$ is a finite subset of $N$, there are $\bar{a}^0_\eta, \bar{a}^1_\eta \in {}^{\omega>}N$ as required in clause (d) again. So clearly there are automorphisms $f_{\eta\hat{}<0>}, f_{\eta\hat{}<1>}$ extending $f_\eta \restriction C^0_\eta$ such that $f_{\eta\hat{}<0>}(\bar{a}^0_\eta) = \bar{a}^1_\eta, f_{\eta\hat{}<1>}(\bar{a}^1_\eta) = \bar{a}^1_\eta$ as required in clause (e), (f).

Lastly, choose $C^0_{\eta\hat{}<\ell>} = C^0_\eta \cup C^1_\eta \cup f^{-1}_{\eta\hat{}<f>}(C^0_\eta) \cup \{b^*_{\ell g(\eta)}, f^{-1}_{\eta\hat{}<\ell>}(b^*_{\ell g(\eta)}),$
$\bar{a}^0_\eta\hat{}\bar{a}^1_\eta, f^{-1}_{\eta\hat{}<\ell>}(\bar{a}^0_\eta\hat{}\bar{a}^1_\eta)\}$ and $C^1_{\eta\hat{}<\ell>} = f_{\eta\hat{}<\ell>}(C^0_{\eta\hat{}<\ell>})$.

Having carried the induction, for every $\eta \in {}^\omega 2$ clearly $f_\eta = \cup\{f_{\eta\restriction n} \restriction C^0_\eta : n < \omega\}$ is an automorphism of $N$.

[Why? As $\langle f_{\eta\restriction n} \restriction C^0_{\eta\restriction n} : n < \omega \rangle$ is an increasing sequence of functions by clauses (b) + (c) + (f), the union $f_\eta$ is a partial function; as in addition each $f_\eta$ is an automorphism of $N$ by clause (b), also $f_\eta$ is a partial automorphism of $N$. Recalling $\langle b^*_\ell : \ell < n \rangle$ list $N$, clearly $f_\eta$ have domain $N$ by clause (c) and as $f_{\eta\restriction n}(C^0_{\eta\restriction n}) = C^1_{\eta\restriction n}$ the union $f_\eta$ has range $N$ by clause (c).] Hence for some $M_\eta \in K_{\aleph_0}$ there is an isomorphism $f^+_\eta$ from $M$ onto $M_\eta$ extending $f$. Now for some $p_\eta \in \mathbf{D}(N), f_\eta(\bar{a})$ materialize $p_\eta$ in $(M_\eta, N)$. Choose a countable $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^+)$ which include $\{\varphi_\eta(\bar{x}, \bar{y}_\eta) : \eta \in {}^{\omega>}2\}$. Easily if $\eta\hat{}\langle \ell \rangle \triangleleft \eta_\ell \in {}^\omega 2$ for $\ell = 0, 1$ then $\varphi(\bar{x}, \bar{a}^1_\eta) \in p_0, \neg\varphi(\bar{x}, \bar{a}^1_\eta) \in p_1$. So $\eta \neq \nu \in {}^\omega 2 \Rightarrow p_\eta \cap L \neq p_\nu \cap L$ by clauses (d) + (e), contradiction to 5.4(4) as we can use $\leq \aleph_0$ formulas to distinguish.

2) Follows.

3) Trivial. $\square_{5.6}$

5.9. **Claim.** *1) Suppose $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ and $N_1$ forces that $\bar{a}, \bar{b}$ (in $N_1$) realize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$, <u>then</u> $N_1$ forces that they realize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0; N_0)$-type; (the* {88r-5.5}

*inverse is trivial).*
*1A) Suppose* $N_0 \subseteq_{\mathfrak{K}} N_\ell \in K_{\aleph_0}$ *and* $\bar{a}_\ell \in {}^{\omega>}(N_\ell)$ *for* $\ell = 1,2$ *and* $\text{gtp}(\bar{a}_1, N_0, N_1) = \text{gtp}(\bar{a}_2, N_0, N_1)$ *then we can find* $(N_1^+, N_2^+, f)$ *such that* $N_1 \leq_{\mathfrak{K}} N_1^+ \in K_{\aleph_0}, N_2 \leq_{\mathfrak{K}} N_2^+ \in K_{\aleph_0}$ *and* $f$ *is an isomorphism from* $N_1^+$ *onto* $N_2^+$ *over* $N_0$ *mappnig* $\bar{a}_1$ *to* $\bar{a}_2$*.*
*2) If* $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ *and* $\bar{a}, \bar{b} \in N_2$ *(remember* $N_2$ *determines the complete* $\mathbb{L}^0_{\omega_1,\omega}(N_1)$*-generic types of* $\bar{a}, \bar{b}$*) then from the* $\mathbb{L}^0_{\omega_1,\omega}(N_1)$*-generic type of* $\bar{a}$ *over* $N_1$ *we can compute the* $\mathbb{L}^0_{\omega_1,\omega}(N_0)$*-generic type of* $\bar{a}$ *over* $N_0$ *(hence if the* $\mathbb{L}^0_{\omega_1,\omega}(N_1)$*-generic types of* $\bar{a}, \bar{b}$ *over* $N_1$ *are equal, then so are the* $\mathbb{L}^0_{\omega_1,\omega}(N_0)$*-generic types of* $\bar{a}, \bar{b}$ *over* $N_0$*).*
*3) For every* $N_a \in K_{\aleph_0}$ *there is a one-to-one function* $f$ *from* $\mathbf{D}(N)$ *onto* $\mathbf{D}^*(N)$ *such that: if* $N \subseteq_{\mathfrak{K}} M \in K_{\aleph_0}$ *and* $\bar{a} \in {}^{\omega>}M$ *then* $f(\text{gtp}(\bar{a}, N, M)) = \text{gtp}_{\mathbb{L}_{\omega_1,\omega}(N;N)}(\bar{a}; N; N; M)$*.*

5.10. **Remark.** 1) So there is no essential difference between $\mathbf{D}(N)$ and $\mathbf{D}^*(N)$.
2) Recall that in a formula of $\mathbb{L}^0_{\omega_1,\omega}(N_0; N_0)$ all $c \in N_0$ may appear as individual constants.

*Proof.* 1) We shall prove there are $N_2$ such that $N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ and an automorphism of $N_2$ over $N_0$ taking $\bar{a}$ to $\bar{b}$; this clearly suffices; and we prove the existence of such $N_2$, of course, by hence and forth arguments. We shall use 5.4(2) freely. So by renaming and symmetry, it suffices to prove that

- $(*)$ if $m < \omega, N_0 \leq_{\mathfrak{K}} N_0$ and $\bar{a}, \bar{b} \in {}^m(N_1)$ materialize the same $\mathbb{L}^0_{\infty,\omega}(N_0)$-type over $N_0$ then for every $c \in N_1$, there are $N_2$ and $d \in N_2$ such that $\bar{a}\hat{\ } < c >, \bar{b}\hat{\ } < d >$ materialize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$.

However, by the previous claim 5.4 for some $\bar{a}^* \in {}^{\omega>}(N_0)$ the $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$ that $\bar{a}\hat{\ } < c >$ materialize in $(N_1, N_0)$ does not $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-split over $\bar{a}^*$. Now $\bar{a}, \bar{b}$ materialize in $(N_1, N_0)$ the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$ hence $\bar{a}^*\hat{\ }\bar{a}, \bar{a}^*\hat{\ }\bar{b}$ materialize in $(N_1, N_0)$ the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type. Hence there is $N_2, N_1 \leq_{\mathfrak{K}} N_2 \in K_0$ and an automorphism $f$ of $N_2$ mapping $N_0$ onto $N_1$ and mapping $\bar{a}^*\hat{\ }\bar{a}$ to $\bar{a}^*\hat{\ }\bar{b}$ (but possibly $f \restriction N_0 \neq \text{id}_{N_0}$), this holds by the last sentence in 4.13(c). Let $d = f(c)$, hence if $\bar{a}\hat{\ } < c >, \bar{b}\hat{\ } < d >$ materialize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type in $(N_2, N_0)$ then they materialize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$ in $(N_2, N_0)$.
1A) Similarly to part (1).
2) Clearly it suffices to prove the “hence ” part. By the assumption and proof of 5.9(1) there are $N_3$ satisfying $N_2 \leq_{\mathfrak{K}} N_3 \in K_{\aleph_0}$ and $f$ an automorphism of $N_3$ over $N_1$ taking $\bar{a}$ to $\bar{b}$. Now the conclusion follows.
3) Should be clear. $\square_{5.9}$

{88r-5.6.1}

5.11. **Definition.** 1) We say that $\mathbf{D}_*$ is a $\mathfrak{K}$-diagram function when

- $(a)$ $\mathbf{D}_*$ is a function with domain $K_{\aleph_0}$ (later we shall lift it to $K$)
- $(b)$ $\mathbf{D}_*(N) \subseteq \mathbf{D}(N)$ and has at least one non-algebraic member for $N \in K_{\aleph_0}$

(c) if $N_1, N_2 \in K_{\aleph_0}$ and $f$ is an isomorphism from $N_1$ onto $N_2$ then $f$ maps $\mathbf{D}_*(N_1)$ onto $\mathbf{D}_*(N_2)$, this applies in particular to an automorphism of $N \in K_{\aleph_0}$.

1A) Such $\mathbf{D}_*$ is called weakly good when:

(d) ($\alpha$) $\mathbf{D}_*(N)$ is closed under subtypes, that is: if $p(\bar{x}) \in \mathbf{D}_*(N), \bar{x} = \langle x_\ell : \ell < m\rangle, \pi$ is a function from $\{0,\dots,m-1\}$ into $\{0,\dots,n-1\}$ then some (necessarily unique) $\bar{q}(\langle x_0,\dots,x_{n-1}\rangle) \in \mathbf{D}_*(N)$ is equal to $\{\varphi(\langle x_0,\dots,\bar{x}_{n-1}\rangle) : \varphi(x_{\pi(0)},\dots,x_{\pi(m-1)}) \in p(\bar{x})\}$

($\beta$) if $N \le_{\mathfrak{K}} M \in K_{\aleph_0}, \bar{a}_1, \bar{b}_1 \in {}^{\omega>}N, \bar{a}_2 \in {}^{\ell g(\bar{a}_1)}M$ and $(M,\bar{a}_1) \cong (M,\bar{a}_2)$ and $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a}_2;N;M) \in \mathbf{D}(N)$ then for some $M^+, \bar{b}_2$ we have $M \le_{\mathfrak{K}} M^+ \in K_{\aleph_0}, \bar{b}_2 \in {}^{\ell g(\bar{b}_1)}(M^+), (M^+,\bar{a}_1{}^\frown\bar{b}_1) \cong (M^+,\bar{a}_2{}^\frown\bar{b}_2)$ and $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a}_2{}^\frown\bar{b}_2;N;M^+)$

($\gamma$) if $N \le_{\mathfrak{K}} M \in K_{\aleph_0}, \bar{a} \in {}^{\omega>}M$ and $\bar{b} \in {}^{\omega>}N$ and $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a};N;M) \in \mathbf{D}(N)$ then $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a}{}^\frown\bar{b};N;M) \in \mathbf{D}(N)$.

2) Such $\mathbf{D}_*$ is called countable if $N \in K_{\aleph_0} \Rightarrow |\mathbf{D}_*(N)| \le \aleph_0$.

3) Such $\mathbf{D}_*$ is called good when it is weakly good (i.e., clause (d) holds) and

(e) $\mathbf{D}_*(N)$ has amalgamation (i.e., if $p_0(\bar{x}), p_1(\bar{x},\bar{y}), p_2(\bar{x},\bar{z}) \in \mathbf{D}_*(N)$ and $p_0 \subseteq p_1 \cap p_2$ then there is $q(\bar{x},\bar{y},\bar{z}) \in \mathbf{D}_*(N)$ which includes $p_1(\bar{x},\bar{y}) \cup p_2(\bar{x},\bar{z})$).

4) Such $\mathbf{D}_*$ is called very good if it is good and:

(f) $N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} N_2 \in K_{\aleph_0}, \bar{a}_0 \subseteq \bar{a}_1 \subseteq \bar{a}_2$ and $\bar{a}_\ell \subseteq N_\ell$ for $\ell = 0,1,2$ and $\mathrm{gtp}(\bar{a}_{\ell+1},N_\ell,N_{\ell+1})$ is definable over $\bar{a}_\ell$ and belongs to $\mathbf{D}_*(N_\ell)$ for $\ell = 0,1$ then $\mathrm{gtp}(\bar{a}_2,N_0,N_2)$ belongs to $\mathbf{D}_*(N_0)$ and is definable over $\bar{a}_0$.

{88r-5.6.

5.12. **Remark.** 1) Note that if $\mathbf{D}$ is a weakly good $\mathfrak{K}$-diagram function and $N \in K_{\aleph_0}$ and $p \in \mathbf{D}(N)$ then we can find $(M,\bar{a})$ such that $N \le_{\mathfrak{K}} M \in K_{\aleph_0}, \bar{a} \in {}^{\omega>}M, p = \mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a};N;M)$ and for every $\bar{b} \in {}^{\omega>}M$ the type $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{b};N;M)$ belongs to $\mathbf{D}(N)$.
2) If moreover, $\mathbf{D}$ is a good $\mathfrak{K}$-diagram function then we can demand above that $M$ is $(\mathbf{D}(N),\aleph_0)^*$-homogeneous, see Definition 5.15(1) below.
3) On very good $\mathbf{D}$ see 5.13(2).
4) The $\mathbf{D}_\alpha$'s in 5.13 below are very good $\mathfrak{K}$-diagrams and for us it sufices to have then the properties mentioned above, so we do not elaborate.

{88r-5.6}

5.13. **Fact.** 1) There are $\mathbf{D}_\alpha, \mathbf{D}^*_\alpha$ for $\alpha < \omega_1$, functions with domain $K_{\aleph_0}$ such that:

(a) for $N \in K_{\aleph_0}, \mathbf{D}_\alpha(N), \mathbf{D}^*_\alpha(N)$ is a countable subset of $\mathbf{D}(N), \mathbf{D}^*(N)$ respetively

(b) for each $N \in K_{\aleph_0}, \langle \mathbf{D}_\alpha(N) : \alpha < \omega_1\rangle$ as well as $\langle \mathbf{D}^*_\alpha(N) : \alpha < \omega_1\rangle$ are increasing continuous

(c) $\mathbf{D}(N) = \bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(N)$ and $\mathbf{D}^*(N) = \bigcup_{\alpha<\omega_1} \mathbf{D}^*_\alpha(N)$

(*d*) if $N_1, N_2 \in K_{\aleph_0}, f$ is an isomorphism from $N_1$ onto $N_2$ then $f$ maps $\mathbf{D}_\alpha(N_1)$ onto $\mathbf{D}_\alpha(N_2)$ and $\mathbf{D}^*_\alpha(N_1)$ onto $\mathbf{D}^*_\alpha(N_2)$ for $\alpha < \omega_1$

(*e*) for every $\alpha < \omega_1$ and $N \in K_{\aleph_0}$ there is a $(\mathbf{D}_\alpha(N), \aleph_0)^*$-homogeneous model (see below Definition 5.15(1)) (obviously it is unique up to isomorphism over $N$)

(*f*) if $N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} N_2 \in K_{\aleph_0}, N_2$ is $(\mathbf{D}_\alpha(N_1), \aleph_0)^*$-homogeneous (see Definition 5.15(1) below) and $N_1$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous or just $(\mathbf{D}_\beta(N_0), \aleph_0)^*$-homogeneous for some $\beta \le \alpha$ or just $\bar{b} \in {}^{\omega >}(N_1) \Rightarrow \operatorname{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{b}; N_0; N_1) \in \mathbf{D}(N_0)$ then $N_2$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous

$(f)^+$ if $\langle \alpha_\varepsilon : \varepsilon \le \zeta \rangle$ is increasing continuous sequence of countable ordinals, $\zeta > 0$ and $\langle N_\varepsilon : \varepsilon \le \zeta \rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, $N_\varepsilon \in \mathfrak{K}_{\aleph_0}$, for every $\bar{a} \in N_{\varepsilon+1}$, $\operatorname{gtp}(\bar{a}, N_\varepsilon, N_{\varepsilon+1}) \in \mathbf{D}_\alpha(N_\varepsilon)$ and for every $\xi < \zeta$ for some $\varepsilon \in [\xi, \zeta)$, $N_{\varepsilon+1}$ is $(\mathbf{D}_{\alpha_\varepsilon}(N_\varepsilon), \aleph_0)^*$-homogeneous then $N_\zeta$ is $(\mathbf{D}_{\alpha_\zeta}(N_0), \aleph_0)^*$-homogeneous

(*g*) $N_1$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous if and only if $N_1$ is $(\mathbf{D}^*_\alpha(N_0), \aleph_0)^*$-homogeneous where $N_0 \le_{\mathfrak{K}} N_1 \in K_{\aleph_0}$

(*h*) $\mathbf{D}_\alpha$ is a very good countable $\mathfrak{K}$-diagram function.

2) If $\mathbf{D}$ is very good then clauses (d),(e),(f),(f)$^+$ hold for it (and also (g), defining $\mathbf{D}^*$ as $f''(\mathbf{D}), f$ from 5.17(3).

{88r-5.6.4}

5.14. **Remark.** 1) We can add

(*h*) if $\mathfrak{K}, <^*$ are as derived from the $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ in the proof of 3.19(2) then we can add: if $N_0 \le_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ and every $p \in \mathbf{D}_0(N_0)$ is materialized in $N_1$ then $N_0 <^* N_1$.

2) So our results apply to $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$, too.
3) So it follows that if $\langle N_i : i \le \alpha \rangle$ is $\le_{\mathfrak{K}}$-increasing in $K_{\aleph_0}, N_{i+1}$ is $(\mathbf{D}_{\beta_i}(N_0), \aleph_0)^*$-homogeneous and $\langle \beta_i : i < \alpha \rangle$ is non-decreasing with supremum $\beta$ then $N_\alpha$ is $(\mathbf{D}_\beta, \aleph_0)^*$-homogeneous.
4) So by 5.13(1)(h) each $\mathbf{D}_\alpha$ is very good and countable.

*Proof of 5.13.* First, $\mathbf{D}$ is a $\mathfrak{K}$-diagram function by Definition 5.2 and 5.4(9). As $\mathbf{D}(N)$ has cardinality $\le \aleph_1$ by 5.4(6) we can find a sequence $\langle \mathbf{D}_\alpha : \alpha < \omega_1 \rangle$ such that

⊛ (*a*) $\mathbf{D}_\alpha$ is a countable $\mathfrak{K}$-diagram function

(*b*) for every $N \in K_{\aleph_0}$ the sequence $\langle \mathbf{D}_\alpha(N) : \alpha < \omega_1 \rangle$ is increasing

continuous with union $\mathbf{D}(N)$.

Second, $\mathbf{D}$ is very good (clause (f) of 5.11 holds obviously but to prove that it reflects to $\mathbf{D}_\alpha$ for a cub of $\alpha < \omega_1$ we need 5.23 below, no vicious circle; the other - easier).

Third, note that for each of the demands (d),(e),(f) from Definition 5.11, for a club of $\delta < \omega_1, \mathbf{D}_\delta$ satisfies it. So without loss of generality each $\mathbf{D}_\alpha$ is very good.

The parts on $\mathbf{D}^*_\alpha$ follow by 5.9, and see 5.17(1) below which does not rely on 5.13-5.16 (and see proof of 5.19). $\square_{5.13}$

{88r-5.7}

5.15. **Definition.** Assume $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ and $\mathbf{D}_*$ is a $\mathfrak{K}$-diagram.
1) We say that $(N_1, N_0)$ or just $N_1$ is $(\mathbf{D}_*(N_0), \aleph_0)^*$-homogeneous over $N_0$ (but we may omit the "over $N_0$") if:

- $(a)$ every $\bar{a} \in N_1$ materializes in $(N_1, N_0)$ over $N_0$ some $p \in \mathbf{D}_*(N_0)$ and every $q \in \mathbf{D}_\alpha(N_0)$ is materialized in $(N_0, N_1)$ by some $\bar{b} \in N_1$
- $(b)$ if $\bar{a}, \bar{b} \in N_1, \bar{a}, \bar{b}$ materialize in $(N_1, N_0)$ the same type over $N_0$ and $c \in N_1$ <u>then</u> for some $d \in N_1$ sequence $\bar{a}\hat{\ } < c >, \bar{b}\hat{\ } < d >$ materialize in $(N_1, N_0)$ the same type from $\mathbf{D}_*(N_0)$.

2) Similarly for $(\mathbf{D}^*_*(N_0), \aleph_0)^*$-homogeneity, pedantically we have to say $(N_1, N_0; N_0)$ is $(\mathbf{D}^*(N), \aleph_0)^*$-homogeneous, but normally say $N_1$ is.

{88r-5.7A

5.16. **Remark.** 1) Now this is meaningful only for $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$, but later it becomes meaningful for any $N \leq_{\mathfrak{K}} M \in K$.
2) Uniqueness for such countable models hold in this context too.

Now by 5.9.

{88r-5.8}

5.17. **Conclusion.** If $(N_1, N_0)$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous <u>then</u> $N_1$, i.e. $(N_1, N_0, c)_{c \in N_0}$ is $(\mathbf{D}^*_\alpha(N_0), \aleph_0)^*$-homogeneous.

*Proof.* This is easy by 5.9(1) and clause (g) of 5.13. $\square_{5.17}$

{88r-5.9}

5.18. **Lemma.** *There is $N^* \in K_{\aleph_1}$ such that $N^* = \bigcup\limits_{\alpha < \omega_1} N_\alpha$ and $N_\alpha \in K_{\aleph_0}$ is $\leq_{\mathfrak{K}}$-increasing continuous with $\alpha$ and $N_{\alpha+1}$ is $(\mathbf{D}_{\alpha+1}(N_\alpha), \aleph_0)^*$-homogeneous for $\alpha < \omega_1$.*

*Proof.* Should be clear. $\square_{5.18}$

{88r-5.10

5.19. **Theorem.** *The $N^* \in K_{\aleph_1}$ from 5.18 is unique (even not depending on the choice of $\mathbf{D}_\alpha(N)$'s), is universal and is, $(\mathbb{D}(\mathfrak{K}), \aleph_1)$-model-homogeneous hence model-homogeneous (for $\mathfrak{K}$).*

*Proof.* Uniqueness: For $\ell = 0, 1$ let $N^\ell_\alpha, \mathbf{D}^\ell_\alpha$ $(\alpha < \omega_1)$ be as in 5.13, 5.18 and we should prove $\bigcup_{\alpha<\omega_1} N^0_\alpha \cong \bigcup_{\alpha<\omega_1} N^1_\alpha$; because of clause (g) of 5.13 it does not matter if we use the $\mathbf{D}$ or $\mathbf{D}^*$ version. As $\mathbf{D}^\ell_\alpha$ $(\alpha < \omega_1)$ is increasing and continuous, $|\mathbf{D}^\ell_\alpha(N)| \le \aleph_0$ and $\bigcup_{\alpha<\omega_1} \mathbf{D}^\ell_\alpha(N) = \mathbf{D}(N)$ for every $N \in K_{\aleph_0}$ and the $\mathbf{D}^\ell_\alpha$'s commute with isomorphisms, clearly there is a closed unbounded $E \subseteq \omega_1$ consisting of limit ordinals, such that $\alpha \in E \Rightarrow \mathbf{D}^0_\alpha = \mathbf{D}^1_\alpha$. Let $E = \{\alpha(i) : i < \omega_1\}, \alpha(i)$ increasing and continuous. Now we define by induction on $i < \omega_1$, an isomorphism $f_i$ from $N^0_{\alpha(i)}$ on $N^1_{\alpha(i)}$, increasing with $i$. For $i = 0$ use the $\aleph_0$-categoricity of $K$ and for limit $i$ let $f_i = \bigcup_{j<i} f_j$. Suppose $f_i$ is defined, then by clause (d) of 5.13 the function $f_i$ maps $\mathbf{D}^0_{\alpha(i+1)}(N^0_{\alpha(i)})$ onto $\mathbf{D}^0_{\alpha(i+1)}(N^1_{\alpha(i)})$ and by the choice of $E, \mathbf{D}^0_{\alpha(i+1)} = \mathbf{D}^1_{\alpha(i+1)}$. By the assumption on the $N^\ell_\alpha$ and clause (f)$^+$ of 5.13, $N^\ell_{\alpha(i+1)}$ is $(\mathbf{D}^\ell_{\alpha(i+1)}(N^\ell_{\alpha(i)}), \aleph_0)^*$-homogeneous. Summing up those facts and 5.13(e) we see that we can extend $f_i$ to an isomorphism $f_{i+1}$ from $N^0_{\alpha(i+1)}$ onto $N^1_{\alpha(i+1)}$.

Now $\bigcup_{i<\omega_1} f_i$ is the required isomorphism. Universality: Let $M \in K_{\aleph_1}$, so $M = \bigcup_{\alpha<\omega_1} M_\alpha, M_\alpha$ is $\le_{\mathfrak{K}}$-increasing continuous and $\|M_\alpha\| \le \aleph_0$. We now define $f_\alpha, N_\alpha, \gamma_\alpha$ by induction on $\alpha < \omega_1$ such that: $\gamma_\alpha \in [\alpha, \omega_1)$ is increasing continuous with $\alpha, f_\alpha$ is a $\le_{\mathfrak{K}}$-embedding of $M_\alpha$ into $N_\alpha \in K_{\aleph_0}, N_\alpha$ is $\le_{\mathfrak{K}}$-increasing continuous, $f_\alpha$ is increasing and continuous, and for $\beta < \alpha, N_{\beta+1}$ is $(\mathbf{D}_{\gamma_{\beta+1}}(N_\beta), \aleph_0)^*$-homogeneous. For $\alpha = 0$ let $N_\alpha = M_\alpha$ and $f_\alpha = \mathrm{id}_{N_\alpha}$. For $\alpha$ limit use unions. Let $\alpha = \beta + 1$, we use the $\aleph_0$-amalgmation property (which holds by 3.9, 4.8). So there is a pair $(f_\alpha, N'_\alpha)$ such that $N_\beta \le_{\mathfrak{K}} N'_\alpha \in K_{\aleph_0}$ and $f_\alpha$ is a $\le_{\mathfrak{K}}$-embedding of $M_\alpha$ into $N'_\alpha$ extending $f_\beta$. The set $\{\mathrm{gtp}(\bar{a}, N_\beta, N'_\alpha), \bar{a} \in {}^{\omega>}(N'_\alpha)\}$ is a countable subset of $\mathbf{D}(N_\beta)$ hence is $\subseteq \mathbf{D}_{\gamma_\alpha}(N_\beta)$ for some $\gamma \in (\gamma_\beta, \omega_1)$. By 5.13(1)(c) there is $N_\alpha$ which $\le_{\mathfrak{K}}$-extends $N'_\alpha$ and is $(\mathbf{D}_{\gamma_\alpha}(N'_\alpha), \aleph_0)^*$-homogeneous; by 5.13(1)(f) we are done. So $f = \cup\{f_\alpha : \alpha < \omega_1\}$ embeds $M$ into $N = \cup\{N_\alpha : \alpha < \omega_1\}$ which is isomorphic to $N^*$ by the uniqueness. So the universality follows from the uniqueness. $(\mathbb{D}(\mathfrak{K}), \aleph_1)$-Model-homogeneity: So let $\langle N_\alpha : \alpha < \omega_1\rangle, \mathbf{D}_\alpha, N^*$ be as in 5.13, 5.18 and we are given $(M_0, M_1, M^+_0, f)$ such that $M_0 \le_{\mathfrak{K}} M^+_0 \in K_{\aleph_0}, M_1 \le_{\mathfrak{K}} N^*, f$ an isomorphism from $M_0$ onto $M_1$. For some $\gamma < \omega_1$ we have $M_1 \le_{\mathfrak{K}} N_\gamma$.

Now $\{\mathrm{gtp}(\bar{a}, M_0, M^+_0) : \bar{a} \in {}^{\omega>}(M^+_0)\}$ is a countable subset of $\mathbf{D}(M_0)$ hence $\subseteq \mathbf{D}_{\gamma_0}(M_0)$ for some $\gamma_0 < \omega_1$; also $\{\mathrm{gtp}(\bar{a}, M_1, N_\gamma) : \bar{a} \in {}^{\omega>}(N_\gamma)\}$ is a countable subset of $\mathbf{D}(M_1)$ hence $\subseteq \mathbf{D}_{\gamma_1}(M_1)$ for some $\gamma_1 < \omega_1$. Let $\beta = \max\{\gamma, \gamma_0, \gamma_1\}$ and let $M^*_0 \in K_{\aleph_0}$ be $(\mathbf{D}_\beta(M^+_0), \aleph_0)^*$-homogeneous so $M^+_0 \le_{\mathfrak{K}} M^*_0$, exists by 5.13(1)(e), hence $M^*_0 \in K_{\aleph_0}$ is $(\mathbf{D}_\beta(M_0), \aleph_0)^*$-homogeneous by 5.13(1)(f) because $\beta \ge \gamma_0$. Now $N_\beta$ is $(\mathbf{D}(N_\gamma), \aleph_0)^*$-homogeneous by 5.13(1), so as $\beta \ge \gamma_1$ is follows that $N_\beta$ is $(\mathbf{D}_\gamma(M_1), \aleph_0)^*$-homogeneous.

By 5.13(1)(d),(e) we can extend $f$ to an isomorphic $g$ from $M^*_0$ onto $N_\beta$, so $g \restriction M^+_0$ is a $\le_{\mathfrak{K}}$-embedding of $M^+_0$ into $N$.

We can deduce "$N^*$ is a model-homogeneous directly; let $M_0, M_1 \leq_{\mathfrak{K}} N^*$ be countable, and $f$ is isomorphic from $M_0$ onto $M_1$. Let $\gamma < \omega_1$ be such that $M_0, M_1 \leq_{\mathfrak{K}} N_\gamma$, Let $\gamma_\ell$ be such that $\{\mathrm{gtp}(\bar{a}, M_\ell, N_\gamma) : \bar{a} \in {}^{\omega>}(N_\gamma)\} \subseteq \mathbf{D}_{\gamma_\ell}(M_\ell)$ for $\ell = 0,1$ and let $\beta = \max\{\gamma, \gamma_0, \gamma_1\} + 1$. As above $N_\beta$ is $(\mathbf{D}_\beta(M_\ell), \aleph_0)^*$-homogeneous, and now we choose an automorphism $f_\alpha$ of $N_\alpha$ increasing with $\alpha$ and extended $f$ for $\alpha \in [\beta, \omega_1)$ by induction. Now $\cup\{f_\alpha : \alpha \in (\beta, \omega_1)\}$ is an automorphism of $N^*$ extending $f$. $\square_{5.19}$

5.20. **Definition.** 1) If $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ and $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 1,2$ and they are definable in the same way (see Definition 5.7 (and 5.6), so in particular both do not split over the same finite subset of $N_0$), <u>then</u> we call $p_1$ the stationarization of $p_0$ over $N_1$. {88r-5.11
2) For $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0,1$ let $p_1 \models p_0$ mean that $N_0 \leq_{\mathfrak{K}} N_1$ and if $N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ and $\bar{a} \in N_2$ materializes $p_1$ then it materializes $p_0$.

5.21. **Remark.** It is easy to justify the uniqueness implied by "<u>the</u> stationarization". {88r-5.11

Observe

5.22. **Claim.** *If $p_\ell = \mathrm{gtp}(\bar{a}, N_\ell, N_2)$ for $\ell = 0,1$ and $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ <u>then</u> $p_1 \models p_0$.* {88r-5.12

*Proof.* Easy. $\square_{5.22}$

5.23. **Claim.** *1) Suppose $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}, \bar{a}_i \in N_i$, (for $i = 0,1,2$), $\bar{a}_0 \subseteq \bar{a}_1 \subseteq \bar{a}_2$, i.e. the ranges increase, $\mathrm{gtp}(\bar{a}_1, N_0, N_1)$ is definable over $\bar{a}_0$ and $\mathrm{gtp}(\bar{a}_2, N_1, N_2)$ is definable over $\bar{a}_1$. <u>Then</u> $\mathrm{gtp}(\bar{a}_2, N_0, N_2)$ is definable over $\bar{a}_0$. Moreover, the definition depends only on the definitions mentioned previously.* {88r-5.16
*2) If $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} N_2$ and $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0,1,2$ and $p_{\ell+1}$ is the stationarization of $p_\ell$ over $N_{\ell+1}$ for $\ell = 0,1$, <u>then</u> $p_2$ is the stationarization of $p_0$ over $N_2$.*

*Proof.* 1) So we have to prove that $\mathrm{gtp}(\bar{a}_2, N_0, N_2)$ does not split over $\bar{a}_0$. Let $n < \omega$ and $\bar{b}, \bar{c} \in {}^n N_0$ realize the same type in $N_0$ over $\bar{a}_0$ (in the logic $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{K}})$, or even first order logic when every $N \in K_{\aleph_0}$ is atomic). Now also $\bar{b}\hat{\ }\bar{a}_1, \bar{c}\hat{\ }\bar{a}_1$ materialize the same $\mathbb{L}_{\omega_1,\omega}(N_0)$-type in $N_1$ hence they realize the same $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{K}})$-type (recall 5.4(8)). Hence $\bar{b}, \bar{c}$ realize the same $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{K}})$-type in $N_1$ over $\bar{a}_1$ in $N_1$. But $\mathrm{gtp}(\bar{a}_2, N_0, N_2)$ does not split over $\bar{a}_1$, so by the previous sentence we get that $\bar{b}\hat{\ }\bar{a}_2, \bar{c}\hat{\ }\bar{a}_2$ materializes the same $\mathbb{L}_{\omega_1,\omega}(N_0)$-type in $N_2$.
2) Easy. The "moreover" is proved similarly. $\square_{5.23}$

5.24. **Lemma.** *Suppose $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}, p_\ell \in \mathbf{D}(N_\ell)$ and $p_1$ is a stationarization of $p_0$ over $N_1$, <u>then</u> $p_1 \models p_0$, i.e., every sequence materializing $p_1$ materializes $p_0$ in any $N_2$ such that $N_1 \leq_{\mathfrak{K}} N_2$.* {88r-5.12

5.25. **Remark.** 1) In [Sh:48], [Sh:87a], [Sh:87b] and [Sh:c] the parallel proof of the claims were totally trivial, but here we need to invoke $\dot{I}(\aleph_1, K) < 2^{\aleph_1}$.
2) A particular case can be proved in the context of §4.

*Proof.* So suppose $N_0, N_1, p_0, p_1$ contradict the claim and let $\bar{a}^* \in N_0$ be such that $p_0$ is definable over $\bar{a}^*$ so $p_1$, too. By 5.13(e)+(f) there are $\delta < \omega_1$ and $N_2 \in K_{\aleph_0}$ satisfying $N_1 \leq_{\mathfrak{K}} N_2$ such that $N_2$ is $(\mathbf{D}^*_\delta(N_\ell), \aleph_0)^*$-homogeneous for $\ell = 0, 1$. We can find $p_2 \in \mathbf{D}(N_2)$ which is the stationarization of $p_0, p_1$. It is enough to prove that $p_2 \models p_1$.
[Why? First, note that there is an automorphism $f$ of $N_2$ which maps $N_1$ onto $N_0$ and $f(\bar{a}^*) = \bar{a}^*$ hence $f(p_2) = p_2, f(p_1) = p_0$ hence $p_2 \models p_0$. Now assume that $N_1 \leq_{\mathfrak{K}} N_1^+ \in K_{\aleph_0}$ and $\bar{a}_1 \in N_1^+$ materializes $p_1$ clearly we can find $N_2^+, \bar{a}_2$ such that $N_2 \leq_{\mathfrak{K}} N_2^+ \in K_{\aleph_0}$ and $\bar{a}_2 \in N_2^+$ which materializes $p_2$, as we are assuming $p_2 \models p_1$ it also materializes $p_1$ hence there are $N_3, f$ such that $N_1^+ \leq_{\mathfrak{K}} N_3 \in K_{\aleph_0}$ and $f$ is a $\leq_{\mathfrak{K}}$-embedding of $N_2^+$ into $N_3$ over $N_1$ mapping $\bar{a}_2$ to $\bar{a}_1$. But $p_2 \models p_0$ (see above) hence $f(\bar{a}_2) = \bar{a}_1$ materializes $p_0$ and $p_1$, too.]

So without loss of generality for some $\delta$

$\circledast$ $N_1$ is $(\mathbf{D}^*_\delta(N_0), \aleph_0)^*$-homogeneous over $N_0$.

For $N \in K_{\aleph_0}, N_0 \leq_{\mathfrak{K}} N$, let $p_N$ be the stationarization of $p$ over $N$; so

$\boxtimes_1$ if $N_0 \leq_{\mathfrak{K}} N \in K_{\aleph_0}$ then $p_N$ is definable over $\bar{a}^*$.

Without loss of generality the universes of $N_0, N_1$ are $\omega, \omega \times 2$ respectively.

Now we choose by induction on $\alpha$ a model $N_\alpha \in K_{\aleph_0}\, (\alpha < \omega_1), |N_\alpha| = \omega(1+\alpha), [\beta < \alpha \Rightarrow N_\beta \leq_{\mathfrak{K}} N_\alpha]$; $N_0, N_1$ are the ones mentioned in the claim and $\bar{a}_\alpha \in N_{\alpha+1}$ materializes the stationarization $p_\alpha \in \mathbf{D}^*_\delta(N_\alpha)$ of $p_0$ over $N_\alpha$ and for $\alpha < \beta, N_\beta$ is $(\mathbf{D}^*_\delta(N_\alpha), \aleph_0)$-homogeneous (see 5.13(f),(f)$^+$). Recalling that $\mathfrak{K}$ is categorical in $\aleph_0$ (and uniqueness over $N_0$ of $(\mathbf{D}_\delta(N_0), \aleph_0)^*$-homogeneous models) we have $\alpha > \beta \Rightarrow (N_\alpha, N_\beta) \cong (N_1, N_0)$ so, recalling $\circledast$ clearly $\bar{a}_\alpha$ does not materialize $p_{N_\beta}$ (in $N_{\alpha+1}$). Let $N = \cup\{N_\alpha : \alpha < \omega_1\}$. Let $\mathfrak{B}$ be $(\mathcal{H}(\aleph_2), \in)$ expanded by $N, K \cap \mathcal{H}(\aleph_2), \leq_{\mathfrak{K}} \restriction \mathcal{H}(\aleph_2)$ and anything else which is necessary. Let $\mathfrak{B}^-$ be a countable elementary submodel of $\mathfrak{B}$ to which $\langle N_\alpha : \alpha < \omega_1\rangle, N$ belong and let $\delta(*) = \mathfrak{B}^- \cap \omega_1$. For any stationary co-stationary $S \subseteq \omega_1$, let $\mathfrak{B}_S$ be a model which is

- ($\alpha$) $\mathfrak{B}_S$ an elementary extension of $\mathfrak{B}^-$
- ($\beta$) $\mathfrak{B}_S$ is an end extension of $\mathfrak{B}^-$ for $\omega_1$, that is, if $\mathfrak{B}_S \models$ "$s < t$ are countable ordinals" and $t \in \mathfrak{B}^-$ then $s \in \mathfrak{B}^-$
- ($\gamma$) among the $\mathfrak{B}_S$-countable ordinals not in $\mathfrak{B}^-$ there is no first one
- ($\delta$) "the set of countable ordinals" of $\mathfrak{B}_S$ is $I_S, I_S = \bigcup\limits_{\alpha<\omega_1} I^S_\alpha$, even $I^S_0$ is not well ordered, each $I_\alpha$ a countable initial segment of $I_S, \alpha < \beta \Rightarrow I^S_\alpha \subseteq I^S_\beta \wedge I^S_\alpha \neq I^S_\beta$
- ($\varepsilon$) $I_S \backslash I^S_\alpha$ has a first element if and only if $\alpha \in S$ and then we call it $s(\alpha)$.

In particular $\omega$ and finite sets are standard in $\mathfrak{B}_S$. For $s \in I_S, N_s[\mathfrak{B}_s] := N_s^{\mathfrak{B}_S}$ is defined naturally, and so is $N^S = N^{\mathfrak{B}_S}$; clearly $N_s^{\mathfrak{B}_S} \in K_{\aleph_0}$ is $\leq_{\mathfrak{K}}$-increasing with $s \in I$ as those definitions are $\Sigma^1_1$ (as $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$). Let $N^S_\alpha = \bigcup\limits_{s \in I_\alpha} N^{\mathfrak{B}_S}_s$ and let $s+1$ be the successor of $s$ in $I_S$.

So

$\boxplus$ if $\mathfrak{B}_S \models$ "$s < t$ are countable ordinals", then $(N^{\mathfrak{B}_S}_t, N^{\mathfrak{B}_S}_s)$ is $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_s), \aleph_0)^*$-homogeneous and if $s \in I_\alpha$ then $N^S_\alpha$ is $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_1), \aleph_0)^*$-homogeneous.

If $\alpha \in S$ then clearly the type $p = p_{N^S_\alpha}$ satisfies (using absoluteness from $\mathfrak{B}_S$ because $N^S_\alpha$ is definable in $\mathfrak{B}_S$ as $N^{\mathfrak{B}_S}_{s(\alpha)}$):

$(a)$ $p$ is materialized in $N^S$ (i.e. in $N^S_\beta$ for a club of $\beta \in S$)

but by the assumption toward contradiction

$(b)$ for a closed unbounded $E \subseteq \omega_1$ for no $\beta \in E \cap S, \beta > \alpha^*$ and $\gamma \in (\beta, \omega_1)$ does a sequence from $N^S$ materialize both $p = p_{N^S_\alpha}$ and its stationarization $p_{N^S_\beta}$ over $N^S_\beta$ in $N^S_\gamma$ (again remember $N^S_\alpha = N^{\mathfrak{B}_S}_{s(\alpha)}$ because $\alpha \in S$)

and similarly

$(c)$ for a closed unbounded set of $\beta > \alpha, N^S_\beta$ is $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous.

We shall prove that every $\alpha < \omega_1$,

$\boxdot$ if $\alpha \notin S$ then $\alpha$ cannot satisfy the statement (c) above.

This is sufficient because if $S(1), S(2) \subseteq \omega_1$ are stationary co-stationary, $f$ is an isomorphism from $N^{S(1)}$ onto $N^{S(2)}$ mapping $\bar{a}^*$ to itself, then for a closed unbounded set $E \subseteq \omega_1$, for each $\alpha < \omega_1$ the function $f$ maps $N^{S(1)}_\alpha$ onto $N^{S(2)}_\alpha$, hence the property above is preserved, hence $S(1) \cap E = S(2) \cap E$. But there is a sequence $\langle S_i : i < 2^{\aleph_1} \rangle$ of subsets of $\omega_1$ such that for $i \neq j$ the set $S_i \backslash S_j$ is stationary. So by 0.4 we have $\dot{I}(\aleph_1, K) = 2^{\aleph_1}$, contradiction.

So suppose $\alpha \in \omega_1 \backslash S, p = p_{N^S_\alpha}$ and clause (c) above hold; but obviously $(c) \Rightarrow (a)$, recalling $p_0 \in \mathbf{D}_\delta(N_0)$ hence $p_{N^S_\alpha} \in \mathbf{D}_\delta(N^S_\alpha)$ so let $\bar{a} \in N^S$ materialize $p$ in $N^S$ and we shall get a contradiction.

There are elements $0 = t(0) < t(1) < \ldots < t(k)$ of $I^S$ and $\bar{a}_0 \in N_0 = N^{\mathfrak{B}_S}_{t(0)}, \bar{a}_{\ell+1} \in N^{\mathfrak{B}_S}_{t(\ell)+1}$ such that $\bar{a} \subseteq \bar{a}_k, \bar{a}^* \subseteq \bar{a}_0, \bar{a}_\ell \subseteq \bar{a}_{\ell+1}$ and $\mathrm{gtp}(\bar{a}_{\ell+1}, N^{\mathfrak{B}_S}_{t(\ell)}, N^{\mathfrak{B}_S}_{t(\ell+1)})$ is definable over $\bar{a}_\ell$ and if $t(\ell+1)$ is a successor (in $I_S$) then it is the successor of $t(\ell)$ and if limit in $I^S$ then $\bar{a}_\ell = \bar{a}_{\ell+1}$.

[Why do they exist? Because of the sentence saying that for every $\bar{a}$ we can find such $k, t(\ell)(\ell \leq k), \bar{a}_\ell(\ell \leq k)$ as above is satisfied by $\mathfrak{B}$ and involve parameters which belong to $\mathfrak{B}^-$ hence to $\mathfrak{B}_S$, etc., so $\mathfrak{B}_S$ inherits it (and finiteness is absolute from $\mathfrak{B}_S$)]. It follows that $\mathrm{gtp}(\bar{a}, N^{\mathfrak{B}_S}_{t(\ell)}, N^{\mathfrak{B}_S}_{t(k)})$ is definable over $\bar{a}_\ell$ for each $\ell < k$.

Clearly $t(0) = 0 \in I_\alpha$ but $t(k) \notin I_\alpha$ (otherwise $t(k)+1 \in I_\alpha$ hence $\bar{a} \in N^{\mathfrak{B}_S}_{t(k)+1} \leq_{\mathfrak{K}} N^S_\alpha$, impossible as $p$ is a non-algebraic type over $N^{\mathfrak{B}_S}_\alpha$). Hence for some $\ell$ we have $t(\ell) \in I_\alpha, t(\ell+1) \notin I_\alpha$. By the construction $t(\ell+1)$ is limit (in $I^S$) hence $\bar{a}_{\ell+1} = \bar{a}_\ell$. As $\alpha \notin S$ we can choose $t(*) \in I_S \backslash I^S_\alpha, t(*) < t(\ell+1)$. As we are assuming (toward contradiction) that $\alpha, p$ satisfy clause (c), for some $\beta \in S, s(\beta)$ is well defined and $s(\beta) > t(k)$ (on the definition of $s(\gamma)$ for $\gamma \in S$ see clause $(\varepsilon)$ above) and $N^S_\beta$ is $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous. Now $N^{\mathfrak{B}_S}_{s(\beta)} = N^S_\beta, N^{\mathfrak{B}_S}_{t(\ell+1)}$ are isomorphic over $N_{t(*)}$ (being both $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_{t(*)}), \aleph_0)^*$-homogeneous by the choice of $\mathfrak{B}_S$, see $\boxplus$ above).

So as $N^S_\alpha \le_{\mathfrak{K}} N^{\mathfrak{B}_S}_{t(\ell+1)} \le_{\mathfrak{K}} N^{\mathfrak{B}_S}_{s(\beta)} = N^S_\beta$ and, as said above, $N^S_\beta$ is $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous also $N^{\mathfrak{B}_S}_{t(\ell+1)}$ is $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous, too, hence $(N^{\mathfrak{B}_S}_{t(\ell+1)}, N^S_\alpha, \bar a^*) \cong (N_1, N_0, \bar a^*)$.

As, by $\boxplus$ above, clearly $N^S_\alpha, N^{\mathfrak{B}_S}_{t(*)}$ are $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_{t(\ell)+1}), \aleph_0)^*$-homogeneous there is an isomorphism $f_0$ from $N^S_\alpha$ onto $N^{\mathfrak{B}_S}_{t(*)}$ over $N^{\mathfrak{B}_S}_{t(\ell)+1}$. As $N^{\mathfrak{B}_S}_{t(\ell+1)}$ is $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_{t(*)}), \aleph_0)^*$-homogeneous and $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous by the previous paragraph (where we use $\beta$) we can extend $f_0$ to an automorphism $f_1$ of $N^{\mathfrak{B}_S}_{t(\ell+1)}$. Let $\gamma \in S \cap E$ satisfy $s(\gamma) \ge t(k) + 1$. As $\mathrm{gtp}(\bar a_k, N^{\mathfrak{B}_S}_{t(\ell+1)}, N^S_\gamma)$ is definable over $\bar a_\ell = \bar a_{\ell+1}$ and $\bar a_\ell = f_0(\bar a_\ell) = f_1(\bar a_\ell)$ (as $\bar a_\ell \in N^{\mathfrak{B}_S}_{t(\ell)+1}$) and $N^S_{\gamma+1}$ is $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_{t(\ell+1)}), \aleph_0)^*$-homogeneous, we can extend $f_1$ to an automorphism $f_2$ of $N^S_\gamma$ satisfying $f_2(\bar a_k) = \bar a_k$.

Notice that by the choice of $\langle \bar a_\ell : \ell \le k\rangle$ and $\langle t(\ell) : \ell \le k\rangle$ it follows that for any $m < k$, $\mathrm{gtp}(\bar a_k, N_{t(m)}, N_{t(k)+1})$ does not split over $\bar a_m$ hence is definable over it by 5.23, and recall that we know that $\bar a_\ell = \bar a_{\ell+1}$.

So there is in $N^S$ a sequence materializing both $\mathrm{gtp}(\bar a, N^S_\alpha, N^S_\gamma) = p_{N^S_\alpha}$ and its stationarization over $N^S_{t(\ell+1)}$: just $\bar a (\subseteq \bar a_k)$ (so use $f_2$).

This contradicts the assumption as $(N_1, N_0, \bar a^*) \cong (N^{\mathfrak{B}_S}_{t(\ell+1)}, N^S_\alpha, \bar a^*)$. $\square_{5.24}$

The following claim 5.26(5)-(9) and Definition 5.27 are closely related.

{88r-5.13}

5.26. **Claim.** *1) If $\bar a \in N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} N_2 \in K_{\aleph_0}, \bar b \in N_2, p_1 = \mathrm{gtp}(\bar b, N_1, N_2)$ is definable over $\bar a \in N_0$, <u>then</u> $p_0 = \mathrm{gtp}(\bar b, N_0, N_2)$ is definable in the same way over $\bar a$, hence $\mathrm{gtp}(\bar b, N_1, N_2)$ is its stationarization.*
*2) For a fixed countable $M \in K_{\aleph_0}$ to have a common stationarization in $\mathbf{D}(N')$ for some $N'$ satisfying $M \le_{\mathfrak{K}} N'$ or $N' \le_{\mathfrak{K}} M$ is an equivalence relation over $\{p$: for some $N \le_{\mathfrak{K}} M, p \in \mathbf{D}(N)\}$ (and we can choose the common stationarization in $\mathbf{D}(M)$ as a representative). So if $N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} N_2 \in K_{\aleph_0}, p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0,1,2$ and $p_1, p_2$ are stationarizations of $p_0$ then $p_2 \models p_1$.*
*3) If $N_\alpha \in K_{\aleph_0}$ $(\alpha \le \omega+1)$ is $\le_{\mathfrak{K}}$-increasing and continuous and $\bar a \in N_{\omega+1}$ <u>then</u> for some $n < \omega$, for every $k$ we have: $n < k \le \alpha \le \omega$ implies $\mathrm{gtp}(\bar a, N_\alpha, N_{\omega+1})$ is the stationarization of $\mathrm{gtp}(\bar a, N_k, N_{\omega+1})$.*
*4) If $N \le_{\mathfrak{K}} M \in K, N \in K_{\aleph_0}, \bar a \in M$ <u>then</u> for all $M' \in K_{\aleph_0}$, satisfying $\bar a \in M', N \le_{\mathfrak{K}} M' \le_{\mathfrak{K}} M, \mathrm{gtp}(\bar a, N, M')$ is the same, we call it $\mathrm{gtp}(\bar a, N, M)$ (the new point is that $M$ is not necessarily countable. This is compatible with Definition 5.27(c) being a special case).*
*5) Suppose $N_0 \le_{\mathfrak{K}} N_1$ (in $K$), $\bar a \in N_1$, <u>then</u> there is a countable $M \le_{\mathfrak{K}} N_0$, such that for every countable $M'$ satisfying $M \le_{\mathfrak{K}} M' \le_{\mathfrak{K}} N_0$ we have $\mathrm{gtp}(\bar a, M', N_1)$ is the stationarization of $\mathrm{gtp}(\bar a, M, N_1)$. Moreover there is a finite $A \subseteq N_0$ such that any countable $M \le_{\mathfrak{K}} N_0$ which includes $A$ is O.K. So $\mathrm{gtp}(\bar a, N_0, N_1)$ from 5.27(c) is well defined and $\in \mathbf{D}(N_0)$ and is definable over some finite $A \subseteq N_0$.*

*6) The parallel of Part (3) holds for $N_\alpha \in K$, too, and any limit ordinal instead of $\omega$. That is if $\langle N_\alpha : \alpha \le \delta + 1\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous and $\bar{a} \in N_{\delta+1}$, then for some $\alpha < \delta$ and countable $M \le_{\mathfrak{K}} N_\alpha$ we have: $M \le_{\mathfrak{K}} M' \le_{\mathfrak{K}} M_\delta \Rightarrow \mathrm{gtp}(\bar{a}, M', M_\delta)$ is the stationarization of $\mathrm{gtp}(\bar{a}, M, M_\delta)$; similarly for every $p \in \mathbf{D}(N_\delta)$.*
*7) If $N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} N_2 \le_{\mathfrak{K}} N_3 \le_{\mathfrak{K}} N_4$ and $\bar{a} \in N_4$ and $\mathrm{gtp}(\bar{a}, N_3, N_4)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N_0, N_4)$ then $\mathrm{gtp}(\bar{a}, N_2, N_4)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N_1, N_3)$. Also if $\bar{b}'$ satisfies $\mathrm{Rang}(\bar{b}') \subseteq \mathrm{Rang}(\bar{a})$ and $\mathrm{gtp}(\bar{a}, N_2, N_4)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N_1, N_4)$ then this holds also for $\bar{b}$. We can replace $\mathrm{gtp}(\bar{a}, N_3, N_4)$ by $p \in \mathbf{D}(N_4)$.*
*8) If $N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ and $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0, 1, 2$ and $p_{\ell+1}$ is the stationarization of $p_\ell$ for $\ell = 0, 1$ then $p_2$ is the stationarization of $p_0$.*
*9) If $\langle M_\alpha : \alpha \le \delta + 1\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, $\delta$ a limit ordinal and $\bar{a} \in {}^{\omega>}(M_{\delta+1})$ then*

- *(a) for some $\alpha < \delta$ we have $\mathrm{gtp}(\bar{a}, M_\beta, M_{\delta+1})$ is the stationarization of $\mathrm{gtp}(\bar{a}, M_\alpha, M_{\delta+1})$ whenever $\beta \in [\alpha, \delta)$*
- *(b) if $\mathrm{gtp}(\bar{a}, M_\alpha, M_{\delta+1})$ is the stationarization of $\mathrm{gtp}(\bar{a}, M_0, M_{\delta+1})$ for every $\alpha < \delta$ then this holds for $\alpha = \delta$, too.*

*10) If $\langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, $\delta$ a limit ordinal and $p_\delta \in \mathbf{D}(M_\delta)$, then for some $\alpha < \beta$ there is $p_\alpha \in \mathbf{D}(M_\alpha)$ such that $p_\delta$ is the stationarization of $p_\alpha$.*
*11) Those definitions in 5.27 are compatible with the ones for countable models.*
*12) $\mathrm{gtp}(\bar{a}, N, M)$ (where $\bar{a} \in M, N \le_{\mathfrak{K}} M$ are both is $K$) is the stationarization over $N$ of $\mathrm{gtp}(\bar{a}, N', M)$ for every large enough countable $N' \le_{\mathfrak{K}} N$, see 5.26(5).*

*Proof.* 1) As we can replace $N_2$ by any $N_2'$ satisfying $N_2 \le_{\mathfrak{K}} N_2' \in K_{\aleph_0}$,. without loss of generality for some $\alpha$, $N_2$ is $(\mathbf{D}^*_\alpha(N_0), \aleph_0)^*$-homogeneous and $(\mathbf{D}^*_\alpha(N_1), \aleph_0)^*$-homogeneous. Let $p_2 \in \mathbf{D}(N_2)$ be the stationarization of $p_1$ over $N_2$.

So by 5.24 we get $p_2 \models p_1$. On the other hand, clearly there is an isomorphism $f_0$ from $N_0$ onto $N_1$ such that $f_0(\bar{a}) = \bar{a}$; and by the assumption above on $N_2$, $f_0$ can be extended to an automorphism $f_1$ of $N_2$.

Note that $f_1$ maps $p_0 = \mathrm{gtp}(\bar{b}, N_0, N_2)$ to $p_0' := \mathrm{gtp}(f_1(\bar{b}), f_1(N_0), N_2)$ and maps $p_2$ to itself as $f_0(\bar{a}) = \bar{a}$.

Now $p_1 \models p_0$ (by the choices of $p_1, p_0$) and $p_2 \models p_1$ by 5.9(1), together $p_2 \models p_0$. As $f_1(p_2) = p_2, f_1(p_0) = p_0'$ it follows that $p_2 \models p_0'$. As also $p_2 \models p_1$ and $p_0', p_1 \in \mathbf{D}(N_1)$ it follows that $p_0' = p_1$ hence $p_1, p_0'$ have the same definition over $\bar{a}$, but now also $p_0 \in \mathbf{D}(N_0), p_0' \in \mathbf{D}(N_1)$ have the same definition over $\bar{a}$ (using $f_1$), together also $p_1, p_0$ have the same definition over $\bar{a}$, which means that $p_1$ is the stationarization of $p_0$ over $N_1$ and we are done.
2) Trivial.
3) By part (1).
4) Easy.
5) By (3) and (4).
6)-12) Easy by now. $\square_{5.26}$

{88r-5.14}

5.27. **Definition.** By 5.26(5) the type $\mathrm{gtp}(\bar a, M, N)$ can be reasonably defined when $M \le_{\mathfrak{K}} N, \bar a \in {}^{\omega>}N$ and we can define $\mathbf{D}(N)$ and $\mathbf{D}_*(N)$, $\mathrm{gtp}(\bar a, N, M)$ and stationarization for not necessarily countable $N$ and $N \le_{\mathfrak{K}} M \in K$. Everything still holds, except that maybe some $p$'s are not materialized in any $\le_{\mathfrak{K}}$-extension of $N$.

More formally

- $(a)$ if $N \le_{\mathfrak{K}} M$ and $N \in K_{\aleph_0}$ and $p \in \mathbf{D}(N)$ then the stationarization of $p$ over $M$ is $\cup\{q : N_1 \in K_{\aleph_0}$ satisfies $N_1 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} M$ and $q$ is the stationarization of $p \in \mathbf{D}(N_1)\}$
- $(b)$ if $M \in \mathfrak{K}$ then $\mathbf{D}(M) = \{q$: for some countable $N \le_{\mathfrak{K}} M$ and $p \in \mathbf{D}(N)$ the type $q$ is the stationarization of $p$ over $M\}$, similarly for $\mathbf{D}_*$, a $\mathfrak{K}$-diagram
- $(c)$ if $N \le_{\mathfrak{K}} M$ and $\bar a \in {}^{\omega>}M$ then $\mathrm{gtp}(\bar a, N, M)$ is defined as $\cup\{\mathrm{gtp}(\bar a, N', M') : N_0 \le_{\mathfrak{K}} N' \le_{\mathfrak{K}} M' \in K_{\aleph_0}, M' \le_{\mathfrak{K}} M, N' \le_{\mathfrak{K}} N\}$ for every countable large enough $N_0 \le_{\mathfrak{K}} N$; it is well defined and belongs to $\mathbf{D}(N)$ by 5.26(5) and we say $\bar a$ materializes $\mathrm{gtp}(\bar a, N, M)$ in $M$
- $(d)$ if $N \in \mathfrak{K}, N \le_{\mathfrak{K}} M$ and $p \in \mathbf{D}(N)$ is definable over the countable $N_0 \le_{\mathfrak{K}} N$ equivalently is the stationarization of some $p' \in \mathbf{D}(N_0)$, <u>then</u> the stationarization of $p$ over $M$ is the stationarization of $p'$ over $M$, see clause (a), equivalently $\cup\{p_{M_0} : N_0 \le_{\mathfrak{K}} M_0 \le_{\mathfrak{K}} M, M_0$ is countable$\}$ where $p_{M_0}$ is the stationarization of $p' \in \mathbf{D}(N_0)$ over $M_0$; it belongs to $\mathbf{D}(N_0)$
- $(e)$ if $p(\bar x, \bar y) \in \mathbf{D}(M)$ then $p(\bar x, \bar y) \restriction \bar x \in \mathbf{D}(M)$ is naturally defined; 5.2(3) similarly for permuting the variables
- $(f)$ for $N \le_{\mathfrak{K}} M$ we say that $M$ is $(\mathbf{D}(N), \aleph_0)^*$-homogeneous <u>when</u> for every $p(\bar x, \bar y) \in \mathbf{D}(N)$ and $\bar a \in {}^{\ell g(\bar x)}(M)$ materializing $p(\bar x, \bar y) \restriction x$ in $M$ there is $\bar b \in {}^{\ell g(\bar y)}M$ such that $\bar a {}^\frown \bar b$ materializes $p(\bar x, \bar y)$ in $M$.



5.28. **Remark.** Claim 5.29 below strengthens 3.9, it is a step toward non-forking amalgamation.

{88r-5.15}

5.29. **Claim.** *Suppose $N_0 \le_{\mathfrak{K}} N_1 \in K_{\aleph_0}, N_0 \le_{\mathfrak{K}} N_2 \in K_{\aleph_0}, \bar a \in N_1$. <u>Then</u> we can find $M, N_0 \le_{\mathfrak{K}} M \in K_{\aleph_0}$ and $\le_{\mathfrak{K}}$-embeddings $f_\ell$ of $N_\ell$ into $M$ over $N_0$ (for $\ell = 1, 2$) such that $\mathrm{gtp}(f_1(\bar a), f_2(N_2), M)$ is a stationarization of $p_0 = \mathrm{gtp}(\bar a, N_0, N_1)$ (so $f_1(\bar a) \notin f_2(N_2)$).*

*Proof.* Let $p_2 \in \mathbf{D}(N_2)$ be the stationarization of $p_0$. Clearly we can find an $\alpha < \omega_1$ (in fact, a closed unbounded set of $\alpha$'s) and $N_1', N_2'$ from $K_{\aleph_0}$ which are $(D^*_\alpha(N_0), \aleph_0)^*$-homogeneous and $N_\ell \le_{\mathfrak{K}} N_\ell'$ (for $\ell = 1, 2$) and some $\bar b \in N_2'$ materializing $p_2$. But by 5.24, $\bar b$ materializes $p_0$ hence there is an isomorphism $f$ from $N_1'$ onto $N_2'$ over $N_0$ satisfying $f(\bar a) = \bar b$, recalling 5.9(1A). Now let $M = N_2', f_1 = f \restriction N_1, f_2 =$ id. $\square_{5.29}$

{88r-5.17}

5.30. **Claim.** *1) For any* $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_1}$ *so* $N_0 \in K_{\leq \aleph_1}$, *there is* $N_2$ *such that* $N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_1}$ *and* $N_2$ *is* $(\mathbf{D}(N_0), \aleph_0)^*$*-homogeneous.*
*2) Also 5.29 holds for* $N_2 \in K_{\aleph_1}$ *(but still* $N_0, N_1 \in K_{\aleph_0}$*).*
*3) If* $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ *and* $N_0 \leq_{\mathfrak{K}} N_2 \in K_{\leq \aleph_1}$ *then we can find* $M \in K_{\leq \aleph_1}$ *and* $\leq_{\mathfrak{K}}$*-embeddings* $f_1, f_2$ *of* $N_1, N_2$ *into* $M$ *over* $N_0$ *respectively such that* $\text{gtp}(f_1(\bar{c}), f_2(N_2), M)$ *is a stationarization of* $\text{gtp}(\bar{c}, N_0, N_1)$ *for every* $\bar{c} \in N_1$, *hence* $f_1(N_1) \cap f_2(N_2) = N_0$.
*4)* $K_{\aleph_2} \neq \emptyset$.

5.31. **Remark.** 1) Note that 5.30(3) is another step toward stable amalgamation.
2) Note that 5.30(3) strengthen 5.30(2) hence 5.29.

*Proof.* 1) As we can iterate $\leq_{\mathfrak{K}}$-increasing $N_1$ in $K_{\aleph_1}$, it is enough to prove that: if $p(\bar{x}, \bar{y}) \in \mathbf{D}(N_0)$ and $\bar{a} \in N_1$ materializes $p(\bar{x}, \bar{y}) \restriction \bar{x}$ in $(N_1, N_0)$ then for some $N_2 \in K_{\aleph_1}, N_1 \leq_{\mathfrak{K}} N_2$ and for some $\bar{b} \in N_2$ the sequence $\bar{a}\hat{\ }\bar{b}$ materializes $p(\bar{x}, \bar{y})$ in $(N_2, N_0)$. Let $M_0 \leq_{\mathfrak{K}} N_0$ be countable and $q \in \mathbf{D}(M_0)$ be such that $p(\bar{x}, \bar{y})$ a stationarization of $q$. Without loss of generality if $N_0$ is countable then $M_0 = N_0$. Note that the case $N_0 = M_0$ is easier. Choose $M_i (0 < i < \omega_1)$ such that $M_i \leq_{\mathfrak{K}} N_1, N_1 = \bigcup\limits_{i<\omega_1} M_i, \langle M_i : i < \omega_1 \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous sequence of countable models, $M_0 \cup \bar{a} \subseteq M_1$. As $\langle M_i \cap N_0 : i < \omega_1 \rangle$ is an increasing continuous sequence of countable sets with union $N_0$ clearly for a club of $i < \omega_1, M_i \cap N_0 \leq_{\mathfrak{K}} N_0$ hence $M_i \cap N_0 \leq_{\mathfrak{K}} M_i$. So without loss of generality $i < \omega_1 \Rightarrow M_i \cap N_0 \leq_{\mathfrak{K}} N_0, M_i$. For every $\bar{c} \in N_1$ there is a countable $N_{0,\bar{c}}$ such that $M_0 \leq_{\mathfrak{K}} N_{0,\bar{c}} \leq_{\mathfrak{K}} N_0$ and: if $N_{0,\bar{c}} \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} N_0$ and $N' \in K_{\aleph_0}$ then $\text{gtp}(\bar{c}, N', N_1)$ is the stationarization of $\text{gtp}(\bar{c}, N_{0,\bar{c}}, N_1)$. Without loss of generality $\bar{c} \in M_i \Rightarrow N_{0,\bar{c}} \subseteq M_i$ hence

$(*)$ for every $\bar{c} \in M_i$, $\text{gtp}(\bar{c}, N_0, N_1)$ is a stationarization of $\text{gtp}(\bar{c}, N_0 \cap M_i, M_i)$.

We can find $M_1^* \in K_{\aleph_0}$ satisfying $M_1 \leq_{\mathfrak{K}} M_1^*$ and $\bar{b} \in M_1^*$ such that $q = \text{gtp}(\bar{a}\hat{\ }\bar{b}, M_0, M_1^*)$. We can find $\bar{a}_2, \bar{a}_1, \bar{a}_0$ such that $\bar{a}_0 \in M_1 \cap N_0, \bar{a}_1 \in M_1, \bar{a}_2 \in M_1^*, \bar{b} \subseteq \bar{a}_2, \bar{a} \subseteq \bar{a}_1$ and $\bar{a}_0 \trianglelefteq \bar{a}_1 \trianglelefteq \bar{a}_2$ and $\text{gtp}(\bar{a}_2, M_1, M_1^*)$, $\text{gtp}(\bar{a}_1, M_1 \cap N_0, M_1)$ are definable over $\bar{a}_1, \bar{a}_0$, respectively. Now we define $f_j, M_j^*, 1 \leq j < \omega_1$ by induction on $i$ such that:

- $(i)$ $\langle M_i^* : 1 \leq i \leq j \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous
- $(ii)$ $M_j^*$ is countable, $M_1^*$ already given
- $(iii)$ $f_j$ is a $\leq_{\mathfrak{K}}$-embedding of $M_j$ into $M_j^*$
- $(iv)$ $f_1$ is the identity on $M_1$
- $(v)$ $f_j$ is increasing continuous with $j$
- $(vi)$ $\text{gtp}(\bar{a}_2, f_j(M_j), M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}_2, M_1, M_1^*)$ (so definable over $\bar{a}_1$).

For $j = 1$ we have it letting $f_j^* = \text{id}_{M_1}$.
For $j > 1$ successor, use 5.29 to define $(M_j, f_j)$ such that $\text{grp}(\bar{a}_2, f_j(M_j), M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}_2, f_{j-1}(M_{j-1}), M_{j-1}^*)$. So clauses (i)-(v) clearly holds. Clause (vi) follows

by 5.26(8).
For $j$ limit: let $M^*_j = \bigcup\limits_{1 \le i < j} M^*_i$ and $f_j = \cup\{f_i : 1 \le i < j\}$, condition (vi) holds by 5.26(3).

By renaming without loss of generality $f_j = \mathrm{id}_{M_j}$ for $j \in [1, \omega_1)$.
By $(*)$ we get that $\mathrm{gtp}(\bar a_1, N_0 \cap M_j, M^*_j) = \mathrm{gtp}(\bar a_1, N_0 \cap M_j, M_j)$ is definable over $\bar a_0$ (as this holds for $j = 1$). Combining this and clause (vi), by 5.23(1) we get for every $j \ge 1$, that $\mathrm{gtp}(\bar a_2, N_0 \cap M_j, M^*_j)$ is the stationarization of $\mathrm{gtp}(\bar a_2, N_0 \cap M_1, M^*_1)$. Hence by the choice of $\bar a_2, \bar a_1, a_0$ and 5.26(7), easily $\mathrm{gtp}(\bar a\,\hat{}\,\bar b, N_0 \cap M_j, M^*_j)$ is the stationarization of $\mathrm{gtp}(\bar a\,\hat{}\,\bar b, N_0 \cap M_1, M^*_1)$ hence of $\mathrm{gtp}(\bar a\,\hat{}\,\bar b, M_0, M^*_1)$. Let $N_2 = \cup\{M^*_j : j \in [1, \omega_1)\}$, clearly $N_1 \le_{\mathfrak{K}} N_2 \in K_{\aleph_1}$.

So by 5.26(9), clause (c) and the first sentence in the proof, we finish.
2) Similar proof [7] (or use the proof of part (3)).
3) Without loss of generality $N_2 \cong N^*$ from 5.18 (as we can replace $N_2$ by an extension so use 5.19 and 5.26(7)).

Also (by 5.30(1)) there is $M, N_2 \le_{\mathfrak{K}} M \in K_{\aleph_1}$ such that $M$ is $(\mathbf{D}(N_2), \aleph_0)^*$-homogeneous. As $N_1$ is countable there is $\alpha < \omega_1$ such that for every $\bar c \in N_1$, $\mathrm{gtp}(\bar c, N_0, N_1) \in \mathbf{D}_\alpha(N_0)$. Let $M = \bigcup\limits_{i < \omega_1} M_i$ with $M_i \in K_{\aleph_0}$ being $\le_{\mathfrak{K}}$-increasing continuous. So for some $i$ we have $\alpha < i < \omega_1, M_i \cap N_2 \le_{\mathfrak{K}} M$ and (recalling 5.26(6)) for every $\bar c \in M_i$, $\mathrm{gtp}(\bar c, N_2, M)$ is stationarization of $\mathrm{gtp}(\bar c, N_2 \cap M_i, M_i)$ and $M_i$ is $(\mathbf{D}_i(N_2 \cap M_i), \aleph_0)^*$-homogeneous. Now we can find an isomorphism $f_0$ from $N_0$ onto $N_2 \cap M_i$ (as $K$ is $\aleph_0$-categorical) and extend it to an automorphism $f_2$ of $N_2$ (by 5.19-model homogeneity). Also there is $N'_1$ such that $N_1 \le_{\mathfrak{K}} N'_1 \in K_{\aleph_0}$ and $N'_1$ is $(\mathbf{D}_i(N_1), \aleph_0)^*$-homogeneous, hence is $(\mathbf{D}_i(N_0), \aleph_0)^*$-homogeneous (by the choice of $\alpha$ as $\alpha < i$ see 5.13(f)), hence there is an isomorphism $f'_1$ from $N'_1$ onto $M_i$ extending $f_0$. Now $f_0, f'_1 \restriction N_1, f_2, M$ show that amalgamation as required exists (we just change names).
4) Immediate, use 1) or 2) or 3) $\omega_2$-times. $\square_{5.30}$

{88r-5.18}

5.32. **Definition.** For any $\mathbf{D}_* = \mathbf{D}_\alpha$ for some $\alpha < \omega_1$ (or just any very good $\mathfrak{K}$-diagram $\mathbf{D}_*$, see 5.11, i.e., satisfies the demands on each $\mathbf{D}_\alpha$ in 5.13) we define:
1) $M \le_{\mathbf{D}_*} N$ if $M \le_{\mathfrak{K}} N$ and for every $\bar a \in N$

$$\mathrm{gtp}(\bar a, M, N) \in \mathbf{D}_*(M).$$

2) $K_{\mathbf{D}_*}$ is the class of $M \in K$ which are the union of a family of countable submodels, which is directed by $\le_{\mathbf{D}_*}$.
3) $\mathfrak{K}_{\mathbf{D}_*} = (K_{\mathbf{D}_*} \le_{\mathbf{D}_*})$, or pedantically $(K_{\mathbf{D}_*}, \le_{\mathbf{D}_*} \restriction K_{\mathbf{D}_*})$.

[7] here $N_1 \in K_{\aleph_1}$ is O.K.; similar to 2.12(1)

{88r-5.19}

5.33. **Claim.** *Let* $\mathbf{D}_*$ *be countable and as in 5.32.*
*1) The pair* $(K_{\mathbf{D}_*}, \le_{\mathbf{D}_*})$ *is an* $\aleph_0$*-presentable a.e.c., that is it satisfies all the axioms from 1.2(1) and is* $\mathrm{PC}_{\aleph_0}$.
*2) Also for* $(K_{\mathbf{D}_*}, \le_{\mathbf{D}_*})$*, we get* $\mathbf{D}(N)$ *countable and equal to* $\mathbf{D}_*(N)$ *for every countable* $N \in K_{\mathbf{D}_*}$.

*Proof.* 1) Obviously $K_{\mathbf{D}_*}$ is a class of $\tau$-models and $\le_{\mathbf{D}_*}$ is a two-place relation on $K_{D_*}$; also they are preserved by isomorphisms. About being $\mathrm{PC}_{\aleph_0}$ note that

- $\circledast_1$ $M \in K_{\mathbf{D}_*}$ iff $M \in K$ and for some model $\mathfrak{B}$ with universe $|M|$ and countable vocabulary, for every countable $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \mathfrak{B}$ we have $M \restriction \mathfrak{B}_1 \le_{\mathbf{D}_*} M \restriction \mathfrak{B}_2$ <u>iff</u> there is a directed partial order and $\langle M_t : t \in I\rangle$ such that $M_t \in K_{\aleph_0}$ and $s <_I t \Rightarrow M_s \le_{\mathfrak{K}} M_t$ and $\bar a \subseteq M_t \Rightarrow \mathrm{gtp}(\bar a, M_s, M_t) \in \mathbf{D}_*(M_s)$
- $\circledast_2$ similarly for $M \le_{\mathbf{D}_*} N$.

<u>Ax I</u>: If $M \le_{\mathbf{D}_*} N$ then $M \le_{\mathfrak{K}} N$ hence $M \subseteq N$. <u>Ax II</u>: The transitivity of $\le_{\mathbf{D}_*}$ holds

by 5.11(4), 5.23(1) + Definition 5.27 (works as $\mathbf{D}_*$ is closed enough or use clause (f) of 5.13). The demand $M \le_{\mathbf{D}_*} M$ is trivial [8]. <u>Ax III</u>: Assume $\langle M_i : i < \lambda\rangle$ is $\le_{\mathbf{D}_*}$-

increasing continuous and $M = \cup\{M_i : i < \lambda\}$. As $\mathfrak{K}$ is an a.e.c. clearly $M \in K$ and $i < \lambda \Rightarrow M_i \le_{\mathfrak{K}} M$. Also for each $i < \lambda$ and $\bar a \in M$ for some $j \in (i, \lambda)$ we have $\bar a \in M_j$ hence $\mathrm{gtp}(\bar a, M_i, M_j) \in \mathbf{D}_*(M_i)$ but recalling 5.26(7) it follows that $\mathrm{gtp}(\bar a, M_i, M) = \mathrm{gtp}(\bar a, M_i, M_j) \in \mathbf{D}_*(M_i)$. So $i < \lambda \Rightarrow M_i \le_{\mathbf{D}_*} M$. By applying $\circledast_1$ to every $M_i$ and coding we can easily show that $M \in K_{\mathbf{D}_*}$ thus finishing. <u>Ax IV</u>: Assume $\langle M_i : i < \lambda\rangle, M$ are

as above and $i < \lambda \Rightarrow M_i \le_{\mathbf{D}_*} N$. To prove $M \le_{\mathbf{D}_*} N$ note that as $\mathfrak{K}$ is an a.e.c., we have $M \le_{\mathfrak{K}} N$ and consider $\bar a \in N$. By 5.26(6) for some $i < \lambda$, $\mathrm{gtp}(\bar a, M, N)$ is the stationarization of $\mathrm{gtp}(\bar a, M_i, N)$ but the latter belongs to $\mathbf{D}_*(M_i)$ hence $\mathrm{gtp}(\bar a, M, N) \in \mathbf{D}_*(M)$ as required. <u>Ax V</u>: By $\circledast_2$ this is translated to the case $N_0, N_1, M \in K_{\aleph_0}$ but then

it holds easily. <u>Ax VI</u>: By $\circledast_1 + \circledast_2$ + Ax VI for $\mathfrak{K}$.

2) So we replace $\mathfrak{K}$ by $\mathfrak{K}' = \mathfrak{K}_{\mathbf{D}_*}$ and easily all that we need for $\mathbf{D}$ for $\mathfrak{K}'$ is satisfied by $\mathbf{D}_*$ (actually repeating the works in §5 till now on $\mathfrak{K}'$ we get it) noting that

- $\circledast$ if $M_0 \le_{\mathbf{D}_*} M_\ell \in K_{\aleph_0}$ for $\ell = 1$ and $\mathrm{gtp}(\bar a_1, M_0, M_1) = \mathrm{gtp}(\bar a_2, M_0, M_2)$ <u>then</u> there is a triple $(M_1^+, M_2^+, f)$ such that $M_\ell \le_{\mathbf{D}_*} M_\ell^+ \in K_{\aleph_0}, M_\ell^+$ is $(\mathbf{D}(M_i), \aleph_0)^*$-homogeneous for $i = 0, \ell$ and $f$ is an isomorphism from $M_1^+$ onto $M_2^+$ over $M_0$ mapping $\bar a_1$ to $a_2$.

This by:

[8] recall that $M \restriction \mathfrak{B} = M \restriction \{a \in M : a \in \mathfrak{B}\}$

modified:2015-01-25

(88r) revision:2015-01-23

$\circledast_1$ if $M_0 \le_{\mathbf{D}_*} M_1 \le_{\mathbf{D}_*} M_2$ and $\bar a \in M_1$ then $\mathrm{gtp}(\bar a, M_0, M_1) = \mathrm{gtp}(\bar a, M_0, M_2) \in \mathbf{D}_*(M_0)$

$\circledast_2$ if $M_0 \in K_{\aleph_0}$ then for some $M_1 \in K_{\aleph_0}$ we have $M_0 \le_{\mathbf{D}_*} M_2$ and $M_1$ is $(\mathbf{D}_*(M_0), \aleph_0)^*)$-homogeneous

$\circledast_3$ if $M_0 \le_{\mathbf{D}_*} M_1 \le_{\mathbf{D}_*} M_2$ and $M_2$ is $(\mathbf{d}_*(M_1), \aleph_0)^*$-homogeneous then $M_2$ is $(\mathbf{D}_*(M_0), \aleph_0)^*$-homogeneous

$\circledast_4$ if $M_0 \le_{\mathbf{D}_*} M_\ell \in K_{\aleph_0}$, $\mathrm{gtp}(\bar a_1, M_0, M_1) = \mathrm{gtp}(\bar a_2, M_0, M_2)$ then there is an isomorphism from $M_1$ onto $M_2$ over $M_0$ mapping $\bar a_1$ to $\bar a_2$.

$\square_{5.33}$

{88r-5.20}

5.34. **Claim.** *Suppose $N_0 \le_{\mathfrak{K}} N_\ell \in K_{\aleph_0}$ ($\ell = 1, 2$) and $\bar c \in N_2$, then there is $M, N_0 \le_{\mathfrak{K}} M$ and $\le_{\mathfrak{K}}$-embeddings $f_\ell$ of $N_\ell$ into $M$ over $N_0$ for $\ell = 1, 2$ such that*

(i) *for every $\bar a \in N_1, \mathrm{gtp}(f_1(\bar a), f_2(N_2), M)$ is a stationarization of $\mathrm{gtp}(\bar a, N_0, N_1)$*

(ii) *$\mathrm{gtp}(f_2(\bar c), f_1(N_1), M)$ is a stationarization of $\mathrm{gtp}(\bar c, N_0, N_2)$.*

5.35. **Remark.** This is one more step toward stable amalgamation: in 5.29 we have gotten it for one $\bar a \in N_1$, in 5.30(3) for every $\bar a \in N_1$, which gives disjoint amalgamation.

*Proof.* Clearly we can for $\ell = 1, 2$ replace $N_\ell$ by any $N'_\ell, N_\ell \le_{\mathfrak{K}} N'_\ell \in K_{\aleph_0}$, and without loss of generality $N_0 = N_1 \cap N_2$. By 5.30(3) there is $N_3 \in K_{\aleph_0}$ such that $N_\ell \le_{\mathfrak{K}} N_3$ for $\ell < 3$ and $\bar a \in {}^{\omega>}(N_1) \Rightarrow \mathrm{gtp}(\bar a, N_2, N_3)$ is the stationarization of $\mathrm{gtp}(\bar a, N_0, N_1)$. So we can assume that for some $\mathbf{D}_\alpha$ as in Definition 5.32 we have $N_\ell$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous for $\ell = 1, 2$. As in the proof of 5.24, we can find a countable linear order $I$, such that every element $s \in I$ has an immediate successor $s + 1, 0$ is first element and $I^*$ has a subset isomorphic to the rationals (follow really) and models $M_s \in K_{\aleph_0}$, (for $s \in I$) such that $s < t \Rightarrow M_s \le_{\mathfrak{K}} M_t$ and $M_t$ is $(\mathbf{D}_\alpha(M_s), \aleph_0)$-homogeneous when $s <_I t$, etc. So by 5.26(3) for every initial segment $J$ of $I$ and $t \in I$ such that $J < t$, that is, $(\forall s \in J)(s <_I t)$, if $J$ has no last element and $I \backslash J$ has no first element then $M_t$ is $(\mathbf{D}_\alpha(M_J), \aleph_0)^*$-homogeneous, where $M_J = \bigcup\limits_{s \in J} M_s = \bigcap\limits_{t \in I \backslash J} M_t$. We let $N_0^J = M_J, N_1^J = M_I$ and $N_2^J$ be a $(\mathbf{D}_\alpha(N_0^J), \aleph_0)^*$-homogeneous model satisfying $N_0^J \le_{\mathfrak{K}} N_2^J$ and without loss of generality $N_1^J \cap N_2^J = N_0^J$. Also easily there is $N'_0 <_{\mathfrak{K}} N_0$ such that $\mathrm{gtp}(\bar c, N_0, N_1)$ is definable over some $\bar c_0 \subseteq N'_0$ and $N_0$ is $(\mathbf{D}_\alpha(N'_0), \aleph_0)$-homogeneous. Clearly the triples $(N_0, N_1, N_2), (N_0^J, N_1^J, N_2^J)$ are isomorphic and let $f_0^J, f_1^J, f_2^J$ be appropriate isomorphisms such that $f_0^J \subseteq f_1^J, f_2^J$ and without loss of generality $f_0^J(N'_0) = M_0$. Now by 5.30(3), there is $M^J \in K_{\aleph_0}$ satisfying $N_\ell^J \le_{\mathfrak{K}} M^J$ ($\ell = 0, 1, 2$) such that for every $\bar a \in N_1^J$, $\mathrm{gtp}(\bar a, N_2^J, M^J)$ is the stationarization of $\mathrm{gtp}(\bar a, N_0^J, N_1^J)$ and there are $N_3 \in K_{\aleph_0}, N_\ell \le_{\mathfrak{K}} N_3$ for $\ell = 0, 1, 2$ and an isomorphism $f_3^J \supseteq f_1^J \cup f_2^J$ from $N_3$ onto $M^J$.

Suppose our conclusion fails, then $\mathrm{gtp}(f_2^J(\bar c), N_1^J, M^J)$ is not the stationarization of $\mathrm{gtp}(f_2^J(\bar c), N_0^J, M^J)$. Moreover, as in the proof of 5.24, $t \in I \backslash J \Rightarrow M_I = N_1^J, M_t$ are isomorphic over $N^J_0 = M_J$, hence we can replace $N^J_1$ by $M_t$ for any $t \in I\backslash J$ so as we assume that our conclusion fails, $t \in I\backslash J \Rightarrow$ $\mathrm{gtp}(f^J_2(\bar c), M_t, M^J)$ is not a stationarization of $\mathrm{gtp}(f^J_2(\bar c), N^J_0, M^J)$ and the latter is the stationarization of $\mathrm{gtp}(f^J_2(\bar c), M_0, M^J)$. Let $p_J =$ $\mathrm{gtp}(f^J_2(\bar c), N^J_1, M^J) =$ $\mathrm{gtp}(\bar c, M_I, M^J)$; all this was done for any appropriate $J$. So it is easy to check that $J_1 \neq J_2 \Rightarrow p_{J_1} \neq p_{J_2}$, but as $I^* \subseteq I \& |I| = \aleph_0$, we have continuum many such $J$'s hence such $p_J$'s. If CH fails, we are done. Otherwise, note that moreover, we can ensure that for $J_1 \neq J_2$ as above there is an automorphism of $M_I$ taking $p_{J_1}$ to $p_{J_2}$, hence for some $\beta < \omega_1, \{p_J : J \text{ as above}\} \subseteq \mathbf{D}_\beta(M_I)$, i.e., $(f^{J_2}_1) \circ (f^{J_1}_1)^{-1}$ maps one to the other, contradiction by clause (d) of 5.13. (Alternatively repeat the proof of 5.24. More elaborately by the way $\mathbf{D}_\alpha$ was chosen, Claim 5.30(3) holds for $\mathfrak{K}_{\mathbf{D}_*}$ hence without loss of generality $M^J$ is $(\mathbf{D}_\alpha(N_1), \aleph_0)$-homogeneous and so without loss of generality for some $t(*) \in I\backslash J, N^J_1 = M_{t(*)}), N^J = M_{t(*)+1}$ and we get a contradiction as in the proof of 5.24 (i.e., the choice of $\langle \bar a_\ell : \ell \leq \ell(*)\rangle$ there[9]). $\square_{5.34}$

{88r-5.21

5.36. **Definition.** 1) $\mathfrak{K}$ has the symmetry property when the following holds: if $N_0 \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} N_3\,(\ell = 1,2)$ and for every $\bar a \in N_1$, $\mathrm{gtp}(\bar a, N_2, N_3)$ is the stationarization of $\mathrm{gtp}(\bar a, N_0, N_3)$, then for every $\bar b \in N_2$, $\mathrm{gtp}(\bar b, N_1, N_3)$ is the stationarization of $\mathrm{gtp}(\bar b, N_0, N_3)$.
2) If $N_0, N_1, N_2 \leq_{\mathfrak{K}} N_3$ satisfies the assumption and conclusion of part (1) we say that $N_1, N_2$ are in stable amalgamation over $N_0$ inside $N_3$ (or in two-sided stable amalgamation over $N_0$ inside $N_3$). If only the hypothesis of (1) holds we say they are in a one sided stable amalgamation over $N_0$ inside $N_3$ (then the order of $(N_1, N_2)$ is important).
3) We say that $\mathfrak{K}$ has unique [one sided] amalgamation when: if $N_0 \leq_{\mathfrak{K}} N_\ell \in K_{\aleph_0}$ for $\ell = 1,2$ then $N_1, N_2$ has unique [one sided] stable amalgamation, see part (4).
4) We say $N_1, N_2$ have a unique [one sided] stable amalgamation over $N_0$, where for notational simplicity $N_1 \cap N_2 = N_0$, provided that: if $(*)$, i.e. clauses (a)-(d) below hold then $(**)$ below holds, where:

- $(*)$ $(a)$ $N_1 \leq_{\mathfrak{K}} N_3, N_2 \leq_{\mathfrak{K}} N_3$ and $(N_1, N_2)$ in [one sided] stable amalgamation inside $N_3$ over $N_0$ and $\|N_3\| \leq \|N_1\| + \|N_2\|$
  - $(b)$ $M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3$ for $\ell = 1,2$ and $(M_1, M_2)$ are in [one sided] stable amalgamation inside $M_3$ over $M_0$ (hence $M_1 \cap M_2 = M_0$)
  - $(c)$ $f_\ell$ is an isomorphism from $N_\ell$ onto $M_\ell$ for $\ell = 0,1,2$
  - $(d)$ $f_0 \subseteq f_1$ and $f_0 \subseteq f_2$
- $(**)$ we can find $M'_3, M_3 \leq_{\mathfrak{K}} M'_3$ and $f_3$, a $\leq_{\mathfrak{K}}$-embedding of $N_3$ into $M'_3$ extending $f_1 \cup f_2$.

We at last get the existence of stable amalgamation (to which we earlier get approximations).

{88r-5.22

5.37. **Claim.** *For any $N_0 \leq_{\mathfrak{K}} N_1, N_2$, all from $K_{\aleph_0}$, we can find $M, N_0 \leq_{\mathfrak{K}} M \in K_{\aleph_0}$ and $\leq_{\mathfrak{K}}$-embeddings $f_1, f_2$ of $N_1, N_2$ respectively over $N_0$ into $N$ such that $N_0, f_1(N_1), f_2(N_1)$ are in stable amalgamation.*

[9] A third way is to use forcing and absoluteness to use the case CH fail

5.38. **Remark.** In the proof we could have "inverted the tables" and used $\bar{c}_\zeta$ in the $\omega_1$ direction.

*Proof.* We define by induction on $\zeta < \omega_1, \langle M^\zeta_\alpha : \alpha < \omega_1\rangle$ and $\bar{c}_\zeta$ such that:

(*i*) $\langle M^\zeta_\alpha : \alpha < \omega_1\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous and $M^\zeta_\alpha \in K_{\aleph_0}$

(*ii*) for $\alpha < \zeta, M^\zeta_\alpha = M^\alpha_\alpha$ and $\xi < \zeta \& \alpha < \omega_1 \Rightarrow M^\xi_\alpha \le_{\mathfrak{K}} M^\zeta_\alpha$

(*iii*) for $\zeta$ limit, $M^\zeta_\alpha = \bigcup\limits_{\xi<\zeta} M^\xi_\alpha$

(*iv*) for $\zeta \le \alpha < \omega_1, \zeta$ non-limit $M^\zeta_{\alpha+1}$ is $(\mathbf{D}_{\alpha+1}(M^\zeta_\alpha), \aleph_0)^*$-homogeneous

(*v*) for every $\bar{c} \in M^\zeta_{\alpha+1}$, $\mathrm{gtp}(\bar{c}, M^{\zeta+1}_\alpha, M^{\zeta+1}_{\alpha+1})$ is a stationarization of $\mathrm{gtp}(\bar{c}, M^\zeta_\alpha, M^\zeta_{\alpha+1})$

(*vi*) $\bar{c}_\zeta \in M^{\zeta+1}_{\zeta+1}$ and for $\zeta + 1 < \alpha < \omega_1$, $\mathrm{gtp}(\bar{c}_\zeta, M^\zeta_\alpha, M^{\zeta+1}_\alpha)$ is the stationarization of $\mathrm{gtp}(\bar{c}_\zeta, M^\zeta_{\zeta+1}, M^{\zeta+1}_{\zeta+1})$

(*vii*) for every $p \in \mathbf{D}(M^\xi_\alpha)$ for some $\zeta$ satisfying $\xi+\alpha < \zeta < \omega_1$ we have $\mathrm{gtp}(\bar{c}_\zeta, M^\zeta_{\zeta+1}, M^{\zeta+1}_{\zeta+1})$ is a stationarization of $p$.

There is no problem doing this (by 5.34 and as in earlier constructions); in limit stages we use local character 5.26(3) and $\mathbf{D}_\alpha$ being closed under stationarization.
Now easily for a thin enough closed unbounded set of $E \subseteq \omega_1$, for every $\zeta \in E$ we have

$(*)_\zeta(a)$ $M^\zeta_\zeta$ is $(\mathbf{D}_\zeta(M^0_\zeta), \aleph_0)^*$-homogeneous

(*b*) for every $\bar{c} \in M^\zeta_\zeta$, $\mathrm{gtp}(\bar{c}, \bigcup\limits_{\alpha<\omega_1} M^0_\alpha, \bigcup\limits_{\xi<\omega_1} M^\xi_\xi)$ is a stationarization of $\mathrm{gtp}(\bar{c}, M^0_\zeta, M^\zeta_\zeta)$

(*c*) for every $\bar{c} \in M^0_{\zeta+1}$, $\mathrm{gtp}(\bar{c}, M^{\zeta+1}_\zeta, M^{\zeta+1}_{\zeta+1})$ is a stationarization of gtp $(\bar{c}, M^0_\zeta, M^0_{\zeta+1})$.

[Why? Clause (c) holds by clause (v) of the construction (as $\langle M^\zeta_\varepsilon : \varepsilon \le \zeta\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous). Clause (b) holds as $E$ is thin enough, i.e., is proved as in earlier constructions (i.e., see the proof of $(*)$ in the proof of 5.30(1)). As for Clause (a), first note that by clauses (i),(ii),(iii) the sequence $\langle M^\zeta_\varepsilon : \varepsilon \le \zeta\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous. By clause (vi) we have $\varepsilon < \zeta \Rightarrow$ $\mathrm{gtp}(\bar{c}_\varepsilon, M^\varepsilon_\zeta, M^{\varepsilon+1}_\zeta)$ does not fork over $M^\varepsilon_\zeta$ and clause (vii) of the construction we have: if $p \in \mathbf{D}_\zeta(M^\zeta_\varepsilon), \varepsilon < \zeta$ then for some $\xi \in (\varepsilon, \zeta)$, $\mathrm{gtp}(\bar{c}_\xi, M^\zeta_\xi, M^\zeta_{\xi+1})$ is a non-forking extension of $p$. As $E$ is thin enough we have $\bar{d} \in M^\zeta_\zeta \Rightarrow$ $\mathrm{gtp}(\bar{d}, M^\zeta_0, M^\zeta_\zeta) \in \mathbf{D}_\zeta(M^\zeta_0)$. Together it is easy to get clause (a), e.g., see 5.47.]

So as in the proof of 5.30(3) we can finish (choose $\zeta \in E, f_0$ an isomorphism from $N_0$ onto $M^0_\zeta, f_1 \supseteq f_0$ is an $\le_{\mathfrak{K}}$-embedding of $N_1$ into $M^\zeta_\zeta$ and $f_2 \supseteq f_0$ a $\le_{\mathfrak{K}}$-embedding of $N_2$ into $M^0_{\zeta+1}$). $\square_{5.37}$

5.39. **Remark.** Note that in [Sh:600] we use only the results up to this point.

{88r-5.23}

5.40. **Theorem.** *1) Suppose in addition to the hypothesis of this section that $2^{\aleph_1} < 2^{\aleph_2}$ and the club ideal on $\aleph_1$ is not $\aleph_2$-saturated and $\dot I(\aleph_2, K) < 2^{\aleph_2}$ or just $\dot I(\aleph_2, K(\aleph_1\text{-saturated}))$ $< 2^{\aleph_2}$. Then $\mathfrak{K}$ has the symmetry property.*
*2) Assume $2^{\aleph_1} < 2^{\aleph_2}$ and $\dot I(\aleph_2, K(\aleph_1\text{-saturated})) < \mu_{unif}(\aleph_2, 2^{\aleph_1})$; this number is always $> 2^{\aleph_1}$, usually $2^{\aleph_2}$, see 0.6. Then $\mathfrak{K}$ has the symmetry property and stable amalgamation in $K_{\aleph_0}$ is unique (we know that it always exists and it follows by (1) + (2) that one sided amalgamation is unique).*

{88r-5.23

5.41. **Discussion.** 1) This certainly gives a desirable conclusion. However, part (2) is not used so we shall return to it in [Sh:838].

More elaborately, in [Sh:838, 4.1], in the "lean version of [Sh:838], see reading plan A in [Sh:838, §0], assuming the weak diamond ideal is not $\aleph_2$-saturated we prove 5.40(2) hence we also prove a slight weaker version of 5.40(1), replacing "$\dot I(\aleph_2, K)(\aleph_1\text{-saturated}) < 2^{\aleph_2}$" by $\dot I(\aleph_2, K(\aleph_1\text{-saturated})) < \mu_{\text{unif}}(\aleph_2, 2^{\aleph_1})$. !!

Better, in [Sh:838, 4.40] we prove 5.40(2) fully. Still, the proof given below of part (1) is not covered presently by [Sh:838] and it gives nicer reasons for non-isomorphisms (essentially diferent natural invariants). !!

2) As for part (1), we can avoid using it (except in 5.45 below). More fully, in [Sh:600, §3] dealing with $\mathfrak{K}$ as here by [Sh:600, 3.4] for every $\alpha < \omega_1$ we derive a good $\aleph_0$-frame $\mathfrak{s}_\alpha$ with $\mathfrak{K}^{\mathfrak{s}_\alpha} = \mathfrak{K}_{\mathbf{D}_\alpha}$ (if we would have liked to derive a good $\aleph_1$-frame we would need 5.40). !!

Then in [Sh:705] if $\mathfrak{s}$ is successful (holds, e.g. if $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}$ and $\dot I(\aleph_2, \mathfrak{K}^{\mathfrak{s}_\alpha}) < 2^{\aleph_2}$ and $\mathrm{WDmId}_{\aleph_1}$ is not $\aleph_2$-saturated) then we derive the successor $\mathfrak{s}^+_\alpha$, a good $\aleph_1$-frame with $K^{\mathfrak{s}^+_\alpha} \subseteq \{M \in K^{\mathfrak{s}_\alpha}_{\aleph_1} : M$ is $\aleph_1$-saturated for $K^{\mathfrak{s}_\alpha}\}$, and $\mathfrak{s}^+_\alpha$ is even good$^+$ (see Claim [Sh:705, !!
1.6](2) and Definition [Sh:705, 1.3]). This suffices for the main conclusions of [Sh:600, §9] !!
and end of [Sh:705, §12].

3) Still we may wonder is $\leq_{\mathfrak{s}^+_\alpha} = \leq_{\mathfrak{K}} \restriction \mathfrak{K}_{\mathfrak{s}^+_\alpha}$? If $\mathfrak{s}_\alpha$ is good$^+$ then the answer is yes (see [Sh:705, !!
1.6](1)). That is, the present theorem 5.40 is used in [Sh:705, §1] to prove $\mathfrak{s}$ is "good$^+$", really this is proved in 5.45. In fact part (1) of 5.40 is enough to prove that $\mathfrak{s}_{\mathbf{D}_*}$ is good$^+$, see [Sh:705, 1.5](1A). !!

3) The proof of 5.40(1) gives that if $\mathfrak{K}$ fails the symmetry property then $\dot I(\aleph_2, K) \geq 2^{\aleph_1}$ even if $2^{\aleph_1} = 2^{\aleph_2}$ and do not use $2^{\aleph_0} = 2^{\aleph_1}$ directly (but use earlier results of §5). The case "$\mathcal{D}_{\aleph_1}$ is $\aleph_2$-saturated, $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}, \dot I(\aleph_2, \aleph_2) < \mu_{\text{unif}}(\aleph_2, 2^{\aleph_2})$" is covered in [Sh:838].

*Proof.* 1) So in the first part toward contradiction we can assume that $K^4 \neq \emptyset$ where $K^4$ is the class of quadruple $\bar N = (N_0, N_1, N_2, N_3)$ such that $N_1, N_2$ are one sided stably amalgamated over $N_0$ inside $N_3$ but $N_2, N_1$ are not. Hence there is $\bar c \in N_2$ such that $\mathrm{gtp}(\bar c, N_1, N_3)$ is not the stationarization of $\mathrm{gtp}(\bar c, N_0, N_2) = \mathrm{gtp}(\bar c, N_0, N_3)$. We define a two-place relation $\leq$ on $K^4$ by $\bar N^1 \leq \bar N^2$ iff $N^1_0 = N^2_0, N^1_\ell \leq_{\mathfrak{K}} N^2_\ell$ for $\ell = 0, 1, 2$ and $\bar a \in N^1_1 \Rightarrow \mathrm{gtp}(\bar a, N^2_2, N^2_3)$ is definable over some $\bar b \in N^1_0$. Easily this is a partial order and $K^4$

is closed under union of increasing countable sequences. Hence without loss of generality for some $\mathbf{D}_*, \bar{N}^*$

$(*)$ $(a)$ $\mathbf{D}_* \in \{\mathbf{D}_\alpha : \alpha < \omega_1\}$
$(b)$ $\bar{N}^* \in K^4$
$(c)$ $N^*_\ell$ is $(\mathbf{D}_*(N^*_0), \aleph_0)^*$-homogeneous over $N^*_0$ for $\ell = 1, 2$
$(d)$ $N^*_3$ is $(\mathbf{D}_*(N^*_\ell), \aleph_0)^*$-homogeneous over $N^*_\ell$ for $\ell = 1, 2$

$\square$

So we have proved

{88r-5.23.8} 5.42. **Observation.** To prove 5.40, we can assume that $\mathbf{D} = \mathbf{D}_\alpha$ for $\alpha < \omega_1$, i.e., $\mathbf{D}$ is countable.

*Continuation of the proof of 5.40.* A problem is that we still have not proven the existence of a superlimit model of $K$ of cardinality $\aleph_1$ though we have a candidate $N^*$ from 5.18. So we use $N^*$, but to ensure we get it at limit ordinals (in the induction on $\alpha < \aleph_2$), we have to take a stationary $S_0 \subseteq \omega_1$ with $\omega_1 \backslash S_0$ not small, i.e., $\omega_1 \backslash S_0$ does not belong to the ideal $\mathrm{WDmId}_{\aleph_1}$ from Theorem 0.6 and "devote" it to ensure this, using 5.37. $\square_{5.40}$

The point of using $S_0$ is as follows (this is supposed to help to understand the quotation from [Sh:838]):

{88r-5.23A} 5.43. **Definition.** 1) Let $K^{\mathrm{qt}} = \{\bar{N} : \bar{N} = \langle N_\alpha : \alpha < \omega_1\rangle$ be $\leq_{\mathfrak{K}}$-increasing continuous, $N_\alpha \in K_{\aleph_0}, N_{\alpha+1}$ is $(\mathbf{D}_\alpha(N_\alpha), \aleph_0)^*$-homogeneous$\}$.
2) On $K^{\mathrm{qt}}$ we define a two-place relation $<^a_S$ (for $S \subseteq \omega_1$) as follows:
$\bar{N}^1 <^a_S \bar{N}^2$ if and only if for some closed unbounded $E \subseteq \omega_1$

$(a)$ for every $\alpha \in C$ we have $N^1_\alpha \leq_{\mathfrak{K}} N^2_\alpha$ and $N^1_{\alpha+1} \leq_{\mathfrak{K}} N^2_{\alpha+1}$
$(b)$ for every $\alpha < \beta$ from $E$ we have $N^2_\beta \cap \bigcup\limits_{\alpha<\omega_1} N^1_\alpha = N^1_\beta$ and $N^1_\beta, N^2_\alpha$ are in one sided stable amalgamation over $N^1_\alpha$ inside $N^2_\beta$, i.e. if $\bar{a} \in N^1_\beta$ then $\mathrm{gtp}(\bar{a}, N^2_\alpha, N^2_\beta)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N^1_\alpha, N^1_\beta)$
$(c)$ if $\alpha \in S \cap C$ then $N^2_\alpha, N^1_{\alpha+1}$ are in stable amalgamation over $N^1_\alpha$ inside $N^2_{\alpha+1}$.

{88r-5.23B} 5.44. **Fact.** 0) The two-place relation $<^a_S$ defined in 5.43 are partial orders on $K^{\mathrm{qt}}$ for $n < \omega$.
1) If $\bar{N}^n \leq^a_{S_0} \bar{N}^{n+1}$ and let $E_n$ exemplify this (as in the Definition 5.43) and let $E_\omega = \bigcap\limits_{n<\omega} E_n, E'_\omega = \{\alpha, \alpha+1 : \alpha \in C_\omega\}$ and let $N^\omega_\alpha = \bigcup\limits_{n<\omega} N^n_\beta$ when $\beta = \mathrm{Min}[E'_\omega \backslash \alpha]$. Then $\langle N^\omega_\alpha : \alpha < \omega_1\rangle \in K_{<\aleph_1}$ and $\bar{N}^n \leq^a_{S_0} \langle N^\omega_\alpha : \alpha < \omega_1\rangle$ for $n < \omega$.
2) If $\langle \bar{N}^\varepsilon : \varepsilon < \omega_1\rangle$ is $<^a_S$-increasing and $N^\varepsilon = \cup\{N^\varepsilon_\alpha : \alpha < \omega_1\} \in K_{\aleph_1}$ is $\leq_{\mathfrak{K}}$-increasing

continuous, the club $E_{\varepsilon,\zeta}$ witness $\bar{N}^\varepsilon \le \bar{N}^\zeta$ for $\varepsilon < \zeta < \aleph_1$ and $\langle N_\alpha : \alpha < \omega_1\rangle$ a $\le_{\mathfrak{K}}$-representation of $N$, and for a club of $\alpha < \aleph_1, N_\alpha = \cup\{N^\varepsilon_\alpha : \varepsilon < \alpha\}, N_{\alpha+1} = \cup\{N^\varepsilon_{\alpha+1} : \varepsilon < \alpha\}$ then $\varepsilon < \omega_1 \Rightarrow \bar{N}^\varepsilon \le^a_{S_0} \bar{N}$.

*Proof.* Should be easy by now.

Continuation of the proof of 5.40: it is done as follows.

There is $\langle S_\varepsilon : \varepsilon < \omega_1\rangle$ such that $S_\varepsilon \subseteq \omega_1, \zeta < \varepsilon \Rightarrow S_\zeta \cap S_\varepsilon$ countable and $S_0, S_{\varepsilon+1}\backslash S_\varepsilon \in (\mathcal{D}_{\omega_1})^+$, possible by an assumption.

Now for any $u \subseteq \omega_2$ we choose $N^u_\varepsilon, N^u_\varepsilon$ by induction on $\varepsilon < \omega_2$ such that

$\circledast$ (a) $\bar{N}^u_\varepsilon = \langle N^u_{\varepsilon,\alpha} : \alpha < \omega_1\rangle \in K^{\rm qt}$

(b) $N^u_\varepsilon = \cup\{N^u_{\varepsilon,\alpha} : \alpha < \omega_1\} \in K_{\aleph_1}$

(c) for $\zeta < \varepsilon$ we have $\bar{N}^u_\zeta <^1_{S_\xi} \bar{N}^u_\varepsilon$ when $\xi \notin [\zeta,\varepsilon) \cap u$ (we can use $S'_{[\zeta,\varepsilon)}$, the compliment of the diagonal union of $\{\langle S_\xi : \varepsilon \in [\zeta,\varepsilon)\rangle \cap u\}$

(d) we can demand continuity as defined implicitly in Fact 5.44

(e) for each $\varepsilon \in u$ for a club of $\alpha < \omega_1$ if $\alpha \in S_\varepsilon$ then $N^u_{\varepsilon+1,\alpha}, N^u_{\varepsilon,\alpha+1}$ are not in stable amalgamation over $N^u_{\varepsilon,\alpha}$ inside $N^u_{\varepsilon+1,\alpha+1}$ (though is in one side).

Lastly, let $N^u = \cup\{N^u_\varepsilon : \varepsilon < \omega_1\} \in K_{\aleph_2}$. Now we can prove that if $u, v \subseteq \omega_2$ and $N^u \approx N^v$ then for some club $C$ of $\omega_2, u \cap C = v \cap C$. So we can easily get $\dot{I}(\aleph_2, \mathfrak{K}) = 2^{\aleph_2}$ and even $\dot{I}(\aleph_2, \mathfrak{K}(\aleph_1\text{-saturated})) = 2^{\aleph_2}$. $\square_{5.40}$

5.45. **Theorem.** *Suppose $\mathfrak{K}$ has the symmetry property (holds if the assumption of 5.40(1) hold). Then $\mathfrak{K}$ has a superlimit model in $\aleph_1$.* {88r-5.24

*Proof.* We have a candidate $N^*$ from 5.18. So let $\langle N_i : i < \delta\rangle$ be $\le_{\mathfrak{K}}$-increasing, $N_i \cong N^*$ and without loss of generality $\delta = \mathrm{cf}(\delta)$. If $\delta = \omega_1$ this is very easy. If $\delta = \omega$, let $N_\omega = \bigcup\limits_{i<\omega} N_i$ and for each $i \le \omega$ let $\langle N^\alpha_i : \alpha < \omega_1\rangle$ be $\le_{\mathfrak{K}}$-increasing continuous with union $N_i$ and $N^\alpha_i \in K_{\aleph_0}$. Now by restricting ourselves to a club $E$ of $\alpha$'s and renaming it $E = \omega_1$, we get: for $i < j \le \omega, N^\alpha_i = N_i \cap N^\alpha_j$, and

$\circledast_1$ for any $\alpha < \beta < \omega_1, \bar{a} \in N^\alpha_\omega$ and $i < \omega$, the type $\mathrm{gtp}(\bar{a}, N^\beta_i, N^\beta_\omega)$ is a stationarization of $\mathrm{gtp}(\bar{a}, N^\alpha_i, N^\alpha_\omega)$.

To prove $N_\omega \cong N^*$ it is enough to prove:

$\circledast_2$ if $\alpha < \omega_1, p \in \mathbf{D}(N^\alpha_\omega)$ then some $\bar{b} \subseteq N_\omega$ realizes $p$ in $N_\omega$.

By 5.26(3) there is $i < \omega$ such that $p$ is the stationarization of $q = p \restriction N_i^\alpha \in \mathbf{D}(N_i^\alpha)$. As $N_i \cong N^*$, there is $\bar b \subseteq N_i$ which realizes $q$ and we can find $\beta \in (\alpha, \omega_1)$ such that $\bar b \subseteq N_i^\beta$. By $\circledast_1$ we have $N_\omega^\alpha, N_i^\beta$ is in one sided stable amalgamation over $N_i^\alpha$ inside $N_\omega^\beta$ (see 5.36(2)).

As we assume $\mathfrak{K}$ has the symmetry property, also $N_i^\beta, N_\omega^\alpha$ is in stable amalgamation over $N_i^\alpha$ inside $N_\omega^\beta$. In particular, as $\bar b \subseteq N_i^\beta$, we have $\mathrm{gtp}(\bar b, N_\omega^\alpha, N_\omega^\beta)$ is the stationarization of $\mathrm{gtp}(\bar b, N_i^\alpha, N_i^\beta)$ but the latter is $p \restriction N_i^\alpha$ so by uniqueness of stationarization, $p = \mathrm{gtp}(\bar b, N_\omega^\alpha, N_\omega^\beta)$ which is $\mathrm{gtp}(\bar b, N_\omega^\alpha, N_\omega)$, so $p$ is realized in $N_\omega$ as required. $\square_{5.45}$

We have implicitly proved

{88r-5.24.5}

5.46. **Claim.** *Assume that $N_0 \le_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ and $\bar a_\ell \in {}^{\omega>}(N_1)$ for $\ell = 1, 2$. Then $(*)_1 \Leftrightarrow (*)_2$ where for $\ell = 1, 2$*

- $(*)_\ell$ *there are $M_1, M_2, \bar b_1, \bar b_2$ such that*
  - $(a)$ $N_0 \le_{\mathfrak{K}} M_1 \le_{\mathfrak{K}} M_2 \in K_{\aleph_1}$
  - $(b)$ $\bar a_k \in {}^{\omega>}(M_k)$ *for* $k = 1, 2$
  - $(c)$ $\mathrm{gtp}(\bar b_{3-\ell}, N_0, M_1) = \mathrm{gtp}(\bar a_{3-\ell}, N_0, N_1)$
  - $(d)$ $\mathrm{gtp}(\bar b_\ell, M_1, M_2)$ *is the stationarization of* $\mathrm{gtp}(\bar a_\ell, N_0, N_1)$ *from* $\mathbf{D}(M_1)$
  - $(e)$ $\mathrm{gtp}(\bar b_1 \hat{\ } \bar b_2, N_0, M_2) = \mathrm{gtp}(\bar a_1 \hat{\ } \bar a_2, N_0, N_1)$

*Proof.* We can deduce it from 5.34 (or immitate the proof of 5.24).

In detail by symmetry it is enough to assume $(*)_2$ and prove $(*)_1$. So let $M_1, M_2, \bar b_1, \bar b_2$ witness $(*)_2$.

By 5.37 we can find $M_2', f$ such that: $M_2 \le_{\mathfrak{K}} M_2' \in K_{\aleph_0}, f$ is a $\le_{\mathfrak{K}}$-embedding of $M_2$ into $M_2'$ over $N_0$ such that $M_1, f(M_2)$ is in stable amalgamation over $N_0$ inside $M_2'$. Now, as $f(M_2), M_1$ are in one sided stable amalgamation over $N_0$ inside $M_2'$ by the choice of $(M_1, M_2, \bar b_1, \bar b_2)$ we get $\mathrm{gtp}(f(\bar b_2), M_1, M_2') = \mathrm{gtp}(\bar b_2, M_1, M_2')$ hence $\mathrm{gtp}(\bar b_1 \hat{\ } \bar b_2, N_0, M_2') = \mathrm{gtp}(\bar b_1 \hat{\ } f(\bar b_2), N_0, M_2')$.

By the choice of $M_1^2, f$,$\mathrm{gtp}(\bar b_1, f(M_2), M_2')$ is the stationarization of $\mathrm{gtp}(\bar b_1, N_0, M_2) = \mathrm{gtp}(\bar a_1, N_0, N_1)$. Now $(*)_1$ holds as exemplified by $(f(M_2), M_2', f(\bar b_2), \bar b_1)$. $\square_{5.46}$

{88r-5.25}

5.47. **Exercise.** Assume $\alpha \le \omega_1$ and

- $(a)$ $\langle M_i : i \le \delta \rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, $\delta$ a limit ordinal
- $(b)$ if $p \in \mathbf{D}(M_i)$ is realized in $M_{i+1}$ then it $\in \mathbf{D}_\alpha(M_i)$ or just $p \restriction M_0 \in \mathbf{D}(M_0)$
- $(c)$ if $i < \delta, p \in \mathbf{D}_\alpha(M_i)$ then $p$ is materialized in $M_j$ for some $j \in (i, \delta)$.

Then $M_\delta$ is $(\mathbf{D}_\alpha(M_0), \aleph_0)^*$-homogeneous.

*Proof.* Easy. $\square$

{88r-5.26}

5.48. **Discussion.** 1) Consider $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q}), |\tau_\psi| \le \aleph_0, 1 \le \dot{I}(\aleph_1, \psi) < 2^{\aleph_0}$. We translate it to $\mathfrak{K}$ and $<^{**}$ as earlier, see 3.19.

2) What if we waive categoricity in $\aleph_0$? The adoption of this was O.K. as we shrink $\mathfrak{K}$ but not too much. But without shrinking probably we still can say something on the models in $\mathfrak{K}^* = \{M \in \mathfrak{K}_{\ge \aleph_0}$: if $N_0 \le_{\mathfrak{K}} M, N_0 \in K_{\aleph_0}$ then for some $N_1, N_0 <^* N_1 \le_{\mathfrak{K}} M\}$ as there are good enough approximations.

## 6. Counterexamples

In [Sh:48] the statement of Conclusion 3.9 was proved for the first time where $K$ is the class of atomic models of a first order theory assuming Jensen's diamond $\diamondsuit_{\aleph_1}$ (taking $\lambda = \aleph_0$). In [Sh:87a] and [Sh:87b] the same theorem was proved using $2^{\aleph_0} < 2^{\aleph_1}$ only (using 0.6). Let us now concentrate on the case $\lambda = \aleph_0$. We asked whether the assumption $2^{\aleph_0} < 2^{\aleph_1}$ is necessary to get Conclusion 3.9. In this section we construct three classes of models $K^1, K^2, K^3, K^4$ failing amalgamation, i.e., failing the conclusion of 3.9, $K^2, K^3, K^4$ are a.e.c. with LST-number $\aleph_0$ while $K^1$ satisfy all the axioms needed in the proof of Conclusion 3.9 (but it is not an abstract elementary class - fails to satisfy AxIV,AxV).

$K^2$ is $\mathrm{PC}_{\aleph_0}$ and is axiomatizable in $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$.

$K^3$ is $\mathrm{PC}_{\aleph_0}$ and is axiomatizable in $\mathbb{L}(\mathbf{Q})$. Now the common phenomena to $K^1, K^2, K^3, K^4$ are that all of them satisfy the hypothesis of Conclusion 3.9, i.e., for $\ell = 1, 2, 3$ we have $\dot{I}(\aleph_0, K^\ell) = 1$ and the $\aleph_0$-amalgamation property fails in $K^\ell$, but assuming $\aleph_1 < 2^{\aleph_0}$ and $\mathrm{MA}_{\aleph_1}$ for $\ell = 1, 2, 3$ we have $\dot{I}(\aleph_1, K^\ell) = 1$.

{88r-6.1}

6.1. **Definition.** Let $Y$ be an infinite set. A family $\mathcal{P}$ of infinite subsets of $Y$ is called independent if for every $\eta \in {}^{\omega>}2$ and pairwise distinct $X_0, X_1, \ldots, X_{\ell g(\eta)-1}$ (notation: for $X \in \mathcal{P}$ denote $X^0 = X$ and $X^1 = Y \backslash X$) the following set $\bigcap\limits_{k<\ell g(\eta)} X_k^{\eta[k]}$ is infinite.

{88r-6.2}

6.2. **Definition.** 1) The class of models $K^0$ is defined by

$$\begin{aligned} K^0 = \{M : &M = \langle |M|, P^M, Q^M, R^M\rangle, |M| = P^M \cup Q^M, \\ &P^M \cap Q^M = \emptyset, |P^M| = \aleph_0 \le |Q^M| \text{ and} \\ &R \subseteq P^M \times Q^M\}. \end{aligned}$$

2) For $M \in K^0$, let $A_y^M = \{x \in P^M : xR^M y\}$ for every $y \in Q^M$.

3) Let $K^1$ be the class of $M \in K^0$ such that

- $(a)$ the family $\{A_y^M : y \in Q^M\}$ is independent, which means that if $m < n$ and $y_0, \ldots, y_{n-1}$ are pairwise distinct members of $Q^M$ then the set $\{x \in P^M : xR^M y_\ell \equiv \ell < m$ for every $\ell < n\}$ is infinite
- $(b)$ for every disjoint finite subsets $u, w$ of $P^M$ we have $\|M\| = |A_{u,w}^M|$ where $A_{u,w}^M := \{y \in Q^M : a \in u \Rightarrow (aR^M y)$ and $b \in w \Rightarrow \neg(bR^M y)\}$.

4) The notion of (strict) substructure $\le_{\mathfrak{K}^1}$ is defined by: for $M_1, M_2 \in K^1, M_1 \le_{\mathfrak{K}^1} M_2$ <u>iff</u> $M_1 \subseteq M_2, P^{M_1} = P^{M_2}$ and for any finite disjoint $u, w \subseteq P^{M_2}$ the set $A_{u,w}^{M_2} \backslash M_1$ is infinite when $M_1 \ne M_2$ (equivalently - non-empty).

5) $\mathfrak{K}^1 = (K^1, \le_{\mathfrak{K}^1})$.

{88r-6.3}
6.3. **Lemma.** *The class* $(K^1, <_{\mathfrak{K}^1})$ *satisfies*
*0) Ax 0.*
*1) Ax I.*
*2) Ax II.*
*3) Ax III.*
*4) Ax IV fails even for* $\lambda = \aleph_0$*; but if* $\langle M_\alpha : \alpha \le \delta\rangle$ *is* $\le_{\mathfrak{K}}$*-increasing and* $\|\bigcup_{\alpha<\delta} M_\alpha\| < \|M_\delta\|$ *then* $\bigcup_{\alpha<\delta} M_\alpha <_{\mathfrak{K}^1} M_\delta$.
*5) Ax V fails for countable models.*
*6) Ax VI holds with* $\mathrm{LST}(\mathfrak{K}^1) = \aleph_0$*, in fact it holds for every cardinal.*
*7) For every* $M \in K^1, \|M\| \le 2^{\aleph_0}$.

*Proof.* 0), 1), 2) follows trivially from the definition.
3) To prove that $M = \bigcup_{i<\lambda} M_i \in K^1$, it is enough to verify that for every finite disjoint $u, w \subseteq P^M, |A^M_{u,w}| = \|M\|$. If $\langle M_i : i < \lambda\rangle$ is eventually constant we are done hence without loss of generality $\langle M_i : i < \lambda\rangle$ is $<_{\mathfrak{K}^1}$-increasing; from the definition of $<_{\mathfrak{K}^1}$ it follows that for each $i, M_{i+1}$ has a new $y = y_i$ as above, i.e., $y_i \in A^{M_{i+1}}_{u,w} \backslash M_i$ for every $i < \lambda$. Also for each $i$ there are at least $\|M_i\|$ many members in $A^{M_i}_{u,w} \subseteq A^M_{u,w}$. Together there are at least $\|M\|$ members in $A^M_{u,w}$.
4) Let $\{M_n : n < \omega\} \subseteq K^1_{\aleph_0}$ be an $<_{\mathfrak{K}^1}$-increasing chain, let $M = \bigcup_{n<\omega} M_n$; by part 3) we have $M \in K^1_{\aleph_0}$. Since $|Q^M| = \aleph_0$ by Claim 6.5(a) below there exists $A \subseteq P^M \backslash \{A^M_y : y \in Q^M\}$ infinite such that $\{A_y : y \in Q^M\} \cup \{A\}$ is independent. Now define $N \in K^1$ by $P^N = P^M$, let $y_0 \notin M, Q^N = Q^M \cup \{y_0\}$ and finally let $R^N = R^M \cup \{\langle a, y_0\rangle : a \in P^N \& a \in A\}$. Clearly for every $n < \omega, M_n \le_{\mathfrak{K}^1} N$ but $N$ is not an $\le_{\mathfrak{K}^1}$-extension of $M = \bigcup_{n<\omega} M_n$ because the second part in Definition 6.2(4) is violated.
5) Let $N_0 <_{\mathfrak{K}^1} N \in K^1$ be given; as in 4) define $N_1 \subseteq N, |N_0| \subseteq |N_1|$ by adding a single element to $Q^{N_0}$ (from the elements of $Q^N \backslash Q^{N_0}$) it is obvious that $N_0 \le_{\mathfrak{K}^1} N, N_1 \le_{\mathfrak{K}^1} N$ but $N_0 \nleq_{\mathfrak{K}^1} N_1$.
6) By closing the set under the second requirement in Definition 6.2(3).
7) Let $y_1 \ne y_2 \in Q^M$, we show that $A^M_{y_1} \ne A^M_{y_2}$; if $A^M_{y_1} \subseteq A^M_{y_2}$ then $A^M_{y_1} \cap (P^M \backslash A^M_{y_2}) = \emptyset$ contradiction to the requirement that $\{A_y : y \in Q\}$ is independent hence $|Q^M| \le 2^{|P^M|} = 2^{\aleph_0}$ and as $|P^M| = \aleph_0$ we are done. $\square_{6.3}$

6.4. **Theorem.** $\mathfrak{K}^1 = (K^1, <_{\mathfrak{K}^1})$ *satisfies the hypothesis of Conclusion 3.9. Namely* {88r-6.4}
*1)* $\dot{I}(\aleph_0, K^1) = 1$.
*2) Every* $M \in K^1_{\aleph_0}$ *has a proper* $\le_{\mathfrak{K}^1}$*-extension in* $K^1_{\aleph_0}$.

*3)* $\mathfrak{K}^1$ *is closed under chains of length* $\leq \omega_1$.
*4)* $\mathfrak{K}^1$ *fails the* $\aleph_0$*-amalgamation property.*

*Proof.* 1) Let $M_1, M_2 \in K^1_{\aleph_0}$, pick the following enumerations $|M_1| = \{a_n : n < \omega\}$ and $|M_2| = \{b_n : n < \omega\}$. It is enough to define an increasing sequence of finite partial isomorphisms $\langle f_n : n < \omega \rangle$ from $M_1$ to $M_2$ such that for every $k < \omega$ for some $n(k) < \omega$ satisfy $a_k \in \mathrm{Dom}(f_{n(k)})$ and $b_k \in \mathrm{Range}(f_{n(k)})$, (finally take $f = \bigcup_{n<\omega} f_n$ and this will be an isomorphism from $M_1$ onto $M_2$).

Define the sequence $\langle f_n : n < \omega \rangle$ by induction on $n < \omega$: let $f_0 = \emptyset$, if $n = 2m$ denote $k = \min\{k < \omega : a_k \notin \mathrm{Dom}(f_n)\}$. Distinguish between the following two alternatives:

- $(A)$ if $a_k \in P^{M_1}$ let $\{a'_0, \ldots, a'_{j-1}\} = Q^{M_1} \cap \mathrm{Dom}(f_n)$. Without loss of generality there exists $i \leq j-1$ such that for all $\ell < i, a_k R^{M_1} a'_\ell$ and for all $i \leq \ell \leq j-1, \neg a_k R a'_\ell$. By 6.2(1), $P^{M_\ell}$ is infinite, hence by clause (b) of 6.2(2) also $Q^{M_\ell}$ is infinite. Hence by clause (a) of 6.2(3) there are infinitely many $y \in P^{M_2}$ such that $y R^{M_2} f_n(a'_\ell)$ for all $\ell < i$ and for all $i \leq \ell < j-1, \neg y R^{M_2} f_n(a'_\ell)$. But $\mathrm{Rang}(f_n)$ is finite. Hence there is such $y \in P^{M_2} \setminus \mathrm{Rang}(f_n)$. Finally let $f_{n+1} = f_n \cup \{\langle a_k, y \rangle\}$
- $(B)$ if $a_k \in Q^{M_1}$ let $\{a'_0, \ldots, a'_{j-1}\} = P^{M_1} \cap \mathrm{Dom}(f_n)$ and as before we may assume that there exists $i \leq j-1$ such that for all $\ell < i, a'_\ell R^{M_1} a_k$ and for all $i \leq \ell < j-1$ we have $\neg(a'_\ell) R^{M_1} a_k$. By the second requirement in Definition 6.2(3) there exists $y \in Q^{M_2} \setminus \mathrm{Dom}(f_n)$ such that $(\forall \ell < i)[f_n(a'_\ell) R^{M_2} y]$ and $(\forall \ell)[i \leq \ell < j-1 \Rightarrow \neg f_n(a'_\ell) R^{M_2} y]$. Now define $f_{n+1} = f_n \cup \{\langle a_k, y \rangle\}$.

2) First we prove the following.

{88r-6.5} 6.5. **Observation.** $(a)$ Let $P$ be a countable set. For every countable family $\mathcal{P}$ of infinite subsets of $P$ <u>if</u> $\mathcal{P}$ is independent <u>then</u> there exists an infinite $A \subseteq P$ such that $\mathcal{P} \cup \{A\}$ is independent and $A \notin \mathcal{P}$, of course

- $(b)$ if $A, \mathcal{P}$ are as in (a) <u>then</u> for every infinite $B \subseteq P$ satisfying $|A \Delta B| < \aleph_0$ also $\mathcal{P} \cup \{B\}$ is independent (and $B \notin \mathcal{P}$)
- $(c)$ moreover in clause (a) we can require in addition that: for any disjoint finite $u, w \subseteq P$ there exists $A \subseteq P$ as in (a) satisfying $u \subseteq A$ and $A \cap w = \emptyset$.

*Proof of Claim 6.5.* <u>Clause (*a*)</u>: Let $\mathcal{P}^* = \{X \subseteq P : (\exists n < \omega)(\exists X_0 \in \mathcal{P}) \ldots (\exists X_{n-1} \in \mathcal{P})(\exists k \leq n)\ [X \text{ or } P \setminus X \text{ is equal to } \cap\{X_i : i < k\} \cap \cap\{P \setminus X_i : k \leq i < n\}\}$.

Clearly $|\mathcal{P}^*| = \aleph_0$ hence we can find a sequence $\langle A_n : n < \omega \rangle$ such that $\{A_n : n < \omega\} = \mathcal{P}^*$ and such that for every $k < \omega$ there exists $n > k$ satisfying $A_n = A_k$ hence for some $n > k, A_n = P \setminus A_k$. Let $P = \{a_n : n < \omega\}$ without repetition.

Now define $i(n) < \omega$ by induction on $n$. Let $i(0) = 0$.

If $n = k+1$, let $i(n) = \mathrm{Min}\{\ell < \omega : i(n-1) < \ell$ and

$$a_\ell \in (A_k \backslash \{a_{i(0)}, \dots, a_{i(n-1)})\}.$$

It is easy to verify that the construction is possible. Directly from the construction it follows that $A = \{a_{i(n)} : n < \omega\}$ is a set as required.

Clause $(b)$: Easy.

Clause $(c)$: Let $u, w \subseteq P$ be finite disjoint and $\mathcal{P}$ a countable family of subsets of $P$ which is independent.

Let $A' \subseteq P$ be as proved in clause (a). According to (b) also $A = (A' \cup u) \backslash w$ satisfies: the family $\mathcal{P} \cup \{A\}$ is independent. $\square$

*Return to the proof of Theorem 6.4(2).* Let $\mathcal{P} = \{A^M_y \subseteq P^M : y \in Q^M\}$. Let $\langle s_n : n < \omega\rangle$ be an enumeration of $[P^M]^{<\aleph_0}$ with repetitions such that for every finite disjoint $u, w \subseteq P^M$ there exists $n < \omega$ such that $s_{2n} = u, s_{2n+1} = w$ and for each $k < \omega, s_{2k} \cap s_{2k+1} = \emptyset$.

It is enough to define $\{\mathcal{P}_n : n < \omega\}$ increasing chain of countable independent families of subsets of $P^M$ such that $\mathcal{P}_0 = \mathcal{P}$ and for all $k < \omega$ and every finite disjoint $u, w \subseteq P^M, (\exists n < \omega)(\exists A \in \mathcal{P}_n \backslash \mathcal{P}_k)[u \subseteq A \wedge A \cap w = \emptyset]$ because $\bigcup\limits_{n<\omega} \mathcal{P}_n$ enables us to define $N \in K^1_{\aleph_0}$ such that $M \leq_{\mathfrak{K}^1} N$ as required. Assume $\mathcal{P}_n$ is defined; apply Claim 6.5(c) on $P = P^M$ and $\mathcal{P}_n$ when substituting $u = s_{2n}, w = s_{2n+1}$ let $A \subseteq P$ be supplied by the Claim and define $\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{A\}$. It is easy to check that $\{\mathcal{P}_n : n < \omega\}$ satisfies our requirements.

3) This is a special case of Ax III which we checked in Lemma 6.3(3).

4) Let $M \in K^1_{\aleph_0}$ and we shall find $M_\ell \in K^1_{\aleph_0} (\ell = 0, 1), M \leq_{\mathfrak{K}^1} M_\ell$, which cannot be amalgamated over $M$. By part (2) we can find a model $M_1$ such that $M <_{\mathfrak{K}^1} M_1 \in K^1_{\aleph_0}$ and choose $y \in Q^{M_1} \backslash Q^M$. Define $M_2 \in K^1_{\aleph_0}$; its universe is $|M_1|, P^{M_2} = P^{M_1}, Q^{M_2} = Q^{M_1}$ and $R^{M_2} = \{(a, b) : aR^{M_1}b \& b \neq y$ or $a \in P^M \& b = y \& \neg(aRy)\}$. Clearly $M_1, M_2$ cannot be amalgamated over $M$ (since the amalgamation must contain a set and its complement). $\square_{6.4}$

6.6. **Theorem.** *Assume* $\mathrm{MA}_{\aleph_1}$ *(hence* $2^{\aleph_0} > \aleph_1$*). The class* $(K^1, <_{\mathfrak{K}^1})$ *is categorical in* $\aleph_1$. {88r-6.6}

*Proof.* Let $M, N \in K^1_{\aleph_1}$ and we shall prove that they are isomorphic. By repeated use of Lemma 6.3(6),(4) for AxVI we get (strictly) $<_{\mathfrak{K}^1}$-increasing continuous chains $\{M_\alpha : \alpha < \omega_1\}, \{N_\alpha : \alpha < \omega_1\} \subseteq K^1_{\aleph_0}$ such that $M = \bigcup\limits_{\alpha<\omega_1} M_\alpha$ and $N = \bigcup\limits_{\alpha<\omega_1} N_\alpha$, so for $\alpha < \beta, M_\alpha <_{\mathfrak{K}^1} M_\beta, N_\alpha <_{\mathfrak{K}^1} N_\beta$.

Now define a forcing notion which supplies an isomorphism $g : M \to N$.

$$\mathbb{P} = \{f : f \text{ is a partial finite isomorphism from } M \text{ into } N \text{ satisfying}$$
$$(\forall \alpha < \omega_1)(\forall a \in \text{Dom}(f))[a \in M_\alpha \Leftrightarrow f(a) \in N_\alpha]\},$$

the order is inclusion. It is trivial to check that if $G \subseteq \mathbb{P}$ is a directed subset then $g = \cup G$ is a partial isomorphism from $M$ to $N$, we show that $\text{Dom}(g) = |M|$ if $G$ is generic enough. For every $a \in |M|$ define $\mathcal{J}_a = \{f \in \mathbb{P} : a \in \text{Dom}(f)\}$, and we shall show that for all $a \in |M|$ the set $\mathcal{J}_a$ is dense. For $a \in M$ let $\alpha(a) = \text{Min}\{\alpha < \omega_1 : a \in M_\alpha\}$, clearly it is zero or a successor ordinal. Let $f \in \mathbb{P}$ be a given condition, it is enough to find $h \in \mathcal{J}_a$ such that $f \subseteq h$ and $a \in \text{Dom}(h)$. Let $A = \text{Dom}(f)$, let $B, C \subseteq A$ be disjoint sets such that $B \cup C = A$ and $B = \text{Dom}(f) \cap P^M, C = \text{Dom}(f) \cap Q^M$. Without loss of generality $a \notin B \cup C$. If $a \in P^M$ let $\varphi(x, \bar{c}) = \wedge\{\pm xRc : c \in C \text{ and } M \models \pm aRc\}$. From the definition of $K^1$ there exists $b \in P^N \setminus \text{Rang}(f)$ such that $N \models \varphi[b, f(\bar{c})]$. If $a \in Q^M$ let $\varphi(x, \bar{b}) = \wedge\{\pm bRx : b \in B, M \models \pm bRa\}$, we can find infinitely many $b \in Q^{N_{\alpha(a)}} \setminus \bigcup\limits_{\beta < \alpha(a)} N_\beta$, satisfying $\varphi(x, f(\bar{b}))$.
Why? This is as $\cup\{N_\beta : \beta < \alpha(a)\} <_{\mathfrak{K}^1} N_{\alpha(a)}$ as $C$ is finite without loss of generality $b \notin f(C)$.
Finally, let $h = f \cup \{\langle a, b\rangle\}$.

The proof that $\text{Range}(g) = |N|$ is analogous to the proof that $\text{Dom}(g) = |M|$. In order to use MA we just have to show that $R$ has the c.c.c. Let $\{f_\alpha : \alpha < \omega_1\} \subseteq R$ be given. It is enough to find $\alpha, \beta < \omega_1$ such that $f_\alpha, f_\beta$ have a common extension. Without loss of generality we may assume $|M| \cap |N| = \emptyset$. By the finitary $\Delta$-system lemma there exists $S \subseteq \omega_1, |S| = \aleph_1$ such that $\{\text{Dom}(f_\alpha) \cup \text{Range}(f_\alpha) : \alpha \in S\}$ is a $\Delta$-system with heart $A$. Let $B \subseteq |M|, C \subseteq |N|$ be such that $A = B \cup C$, now without loss of generality for every $\alpha \in S, f_\alpha$ maps $B$ into $C$.
[Why? If not, $S_1 = \{\alpha \in S$: for some $b = b_\alpha \in B, f_\alpha(b_\alpha) \notin C\}$ is uncountable hence for some $b \in B, S_2 = \{\alpha \in S_1 : b_\alpha = b\}$ is uncountable; so $\langle f_\alpha(b) : \alpha \in S_2\rangle$ is without repetitions hence is uncountable. But $\{f(b) : f \in \mathbb{P} \text{ and } b \in \text{Dom}(f) \cap B\}$ is countable because $f \in \mathbb{P} \& b \in \text{Dom}(f) \& \alpha < \omega_1 \Rightarrow [b \in M_\alpha \equiv f(b) \in N_\alpha]$. Similarly, $f_\alpha^{-1}$ maps $C$ into $B$, so necessarily $f_\alpha$ maps $B$ onto $C$; but the number of possible functions from $B$ to $C$ is $|C|^{|B|} < \aleph_0$. Hence there exists $S_1 \subseteq S, |S_1| = \aleph_1$ such that for all $\alpha, \beta \in S_1, f_\alpha \restriction B = f_\beta \restriction B$ and $\text{Dom}(f_\alpha) \cap M_0 \subseteq B$, $\text{Rang}(f_\alpha) \cap N_0 \subseteq C$. As $P^{M_\alpha} = P^{M_0} \subseteq M_0, P^{N_\alpha} = P^{N_0} \subseteq N_0$ for every $\alpha \in S_1$ we have $P^M \cap \text{Dom}(f_\alpha) \subseteq B, P^N \cap \text{Range}(f_\alpha) \subseteq C$, therefore for all $\alpha, \beta \in S_1, f_\alpha \cup f_\beta \in \mathbb{P}$ and in particular there exists $\alpha \neq \beta < \omega_1$ such that $f_\alpha \cup f_\beta \in \mathbb{P}$. $\square_{6.6}$

In the terminology of [GrSh:174] Theorems 6.4 and 6.6 give us together:


6.7. **Conclusion.** Assuming $2^{\aleph_0} > \aleph_1$ and $\text{MA}_{\aleph_1}, \mathfrak{K}^1$ is a nice category which has a universal object in $\aleph_1$, moreover it is categorical in $\aleph_1$.


6.8. **Definition.** 1) $K^2$ is the class of $M \in K^0$ (see Definition 6.2) satisfying:

(a) $(\forall x \in Q^M)(\forall u \in [P^M]^{<\aleph_0})(\exists y \in Q)[A^M_x \Delta A^M_y = u]$

(b) if $k < \omega$ and $y_0, \dots, y_{k-1} \in Q$ satisfies $|A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$ for $\ell < m < k$ <u>then</u> the set $\{A^M_{y_\ell} : \ell < k\}$ is an independent family of subsets of $P^M$

(c) $Q(y) \wedge Q(z) \wedge (\forall x \in P)[xRy \leftrightarrow xRz] \rightarrow y = z$,

(d) for every $k < \omega$ for some $y_0, \dots, y_k \in Q^M$ we have $\bigwedge\limits_{\ell<m\leq k} |A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$.

2) For $M_1, M_2 \in K^2$

$$M_1 \leq_{\mathfrak{K}^2} M_2 \Leftrightarrow^{\mathrm{df}} M_1 \subseteq M_2, P^{M_1} = P^{M_2}.$$

3) $\mathfrak{K}^2 = (K^2, \leq_{\mathfrak{K}^2})$.

4) $K^3$ is the class of models $M = (|M|, P^M, Q^M, R^M, E^M)$ such that

(a) $(|M|, P^M, Q^M, R^M) \in K^1$

(b) $E^M$ is an equivalence relation on $Q^M$

(c) $E^M$ has infinitely many equivalence classes

(d) each equivalence class of $E^M$ is countable

(e) if $u, w \subseteq P^M$ are finite disjoint and $y \in Q^M$ then for some $y' \in y/E^M$ we have $a \in u \Rightarrow aR^M y'$ and $b \in w \Rightarrow \neg(bR^M y')$.

5) We define $\leq_{\mathfrak{K}^3}$: $M_1 \leq_{\mathfrak{K}^3} M_2 \Leftrightarrow^{\mathrm{df}} M_1 \subseteq M_2$ and $a \in M_1 \Rightarrow a/E^{M_2} = a/E^{M_1}$.

6) $\mathfrak{K}^3 = (K^3, \leq_{\mathfrak{K}^3})$.

If we like to have a class defined by a sentence from $\mathbb{L}_{\omega_1,\omega}$ (rather than $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$) we can use:

{88r-6.8A

6.9. **Definition.** 1) $\mathfrak{K}^4$ is defined as follows:

(A) $\tau(\mathfrak{K}^4) = \{P, Q, R\} \cup \{P_n : n < \omega\}$, $R$ is two-place predicates, $P, Q, P_n$ are unary predicates

(B) $M \in K^4$ <u>iff</u> $M$ is a $\tau(\mathfrak{K}^4)$-model such that $M \restriction \{P, Q, R\} \in K^2$ and

(a) $\langle P^M_n : n < \omega\rangle$ is a partition of $P^M$

(b) $P^M_n$ has exactly $2^n$ elements

(c) $(\forall x \in Q)(\forall u \in [P^M]^{<\aleph_0})(\exists y \in Q^M)[A^M_x \Delta A^M_y = u]$

(d) if $k < \omega$ and $y_0, \dots, y_{k-1} \in Q$ satisfies $|A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$ for $\ell < m < k$ <u>then</u> the set $\{A^M_{y_\ell} : \ell < k\}$ is an independent family of subsets of $P^M$; moreover for any $n$ large enough for any $\eta \in {}^k 2$ the set $P^M_n \cap \cap\{A^M_{y_\ell} : \eta(\ell) = 1\} \setminus \cup\{A^M_{y_\ell} : \eta(\ell) = 0\}$ has exactly $2^{n-k}$ elements

(e) $Q^M(y) \wedge Q^M(z) \wedge (\forall x \in P^M)[xR^M y \leftrightarrow xR^M z] \rightarrow y = z$,

(f) for every $k < \omega$ for some $y_0, \dots, y_k \in Q^M$ we have $\bigwedge\limits_{\ell<m\leq k} |A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$

(C) $M \leq_{\mathfrak{K}^4} N$ iff $M, N \in K^4$ and $M \subseteq N$ and $P^M = P^N$.

{88r-6.9}

6.10. **Theorem.** *1)* $(K^2, <_{\mathfrak{K}^2})$ *is an* $\aleph_0$*-presentable abstract elementary class which is categorical in* $\aleph_0$.
*2) Also* $\mathfrak{K}^3$ *and* $\mathfrak{K}^4$ *are* $\aleph_0$*-presentable a.e.c. categorical in* $\aleph_0$.

*Proof.* Similar to the proof for $\mathfrak{K}^1$. $\square_{6.10}$

{88r-6.11}

6.11. **Theorem.** *1)* $\mathfrak{K}^1_{\aleph_1}$ *has an axiomatization in* $\mathbb{L}(\mathbf{Q})$ *and* $\leq_{\mathfrak{K}^1}$ *is* $<^{**}$ *from the proof of 3.19 (this is* $<^{**}$ *from* [Sh:87a] *and* [Sh:87b]*).*
*2)* $\mathfrak{K}^2$ *has an axiomatization in* $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ *and* $\leq_{\mathfrak{K}^2}$ *is* $\leq^*$ *from the proof of 3.19 (this is* $<^*_{\omega_1,\omega}$ *from* [Sh:87a] *and* [Sh:87b]*).*
*3)* $\mathfrak{K}^3$ *has an axiomatization in* $\mathbb{L}(\mathbf{Q})$ *and* $\leq_{\mathfrak{K}^3}$ *is* $<^*$ *from* [Sh:87a] *and* [Sh:87b].
*4)* $\mathfrak{K}^4$ *has an axiomatization in* $\mathbb{L}_{\omega_1,\omega}$ *and* $\leq_{\mathfrak{K}^4}$ *is just being a submodel.*
*5)* $(\forall \ell \in \{1,2,3,4\})[K^\ell$ *is* $\mathrm{PC}_{\aleph_0}]$.

*Proof.* Should be clear. $\square_{6.11}$

{88r-6.11A}

6.12. **Theorem.** *If* $\mathrm{MA}_{\aleph_1}$ <u>*then*</u> $K^\ell$ *is categorical in* $\aleph_1$ *for* $\ell = 2, 3$.

*Proof.* Easy[10]. $\square$

{88r-6.12}

6.13. **Conclusion.** Assuming $\mathrm{MA}_{\aleph_1}$ there exists an abstract elementary class, which is $\mathrm{PC}_{\aleph_0}$, categorical in $\aleph_0, \aleph_1$ but without the $\aleph_0$-amalgamation property.

[10]In the earlier version this was claimed also for $\ell = 4$, but, as Baldwin noted, this was wrong

REFERENCES.

## References


[BKM78] Jon Barwise, Matt Kaufmann, and Michael Makkai, *Stationary logic*, Annals of Mathematical Logic **13** (1978), 171–224.

[Bur78] John P. Burgess, *Equivalences generated by families of borel sets*, Proceedings of the AMS **69** (1978), 323–326.

[Haj62] Andras Hajnal, *Proof of a conjecture of S.Ruziewicz*, Fundamenta Mathematicae **50** (1961/1962), 123–128.

[Jón56] Bjarni Jónsson, *Universal relational systems*, Mathematica Scandinavica **4** (1956), 193–208.

[Jón60] ______, *Homogeneous universal relational systems*, Mathematica Scandinavica **8** (1960), 137–142.

[Kei70] H. Jerome Keisler, *Logic with the quantifier "there exist uncountably many"*, Annals of Mathematical Logic **1** (1970), 1–93.

[Kei71] ______, *Model theory for infinitary logic. Logic with countable conjunctions and finite quantifiers*, Studies in Logic and the Foundations of Mathematics, vol. 62, North–Holland Publishing Co., Amsterdam–London, 1971.

[KM67] H. Jerome Keisler and Michael D. Morley, *On the number of homogeneous models of a given power*, Israel Journal of Mathematics **5** (1967), 73–78.

[Mak85] Johann A. Makowsky, *Abstract embedding relations*, Model-Theoretic Logics (J. Barwise and S. Feferman, eds.), Springer-Verlag, 1985, pp. 747–791.

[Mor70] Michael D. Morley, *The number of countable models*, Journal of Symbolic Logic **35** (1970), 14–18.

[Sch76] James H. Schmerl, *On $\kappa$-like structures which embed stationary and closed unbounded subsets*, Annals of Mathematical Logic **10** (1976), 289–314.

[Sh:a] Saharon Shelah, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam-New York, xvi+544 pp, $62.25, 1978.

[Sh:c] ______, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, xxxiv+705 pp, 1990.

[Sh:f] ______, *Proper and improper forcing*, Perspectives in Mathematical Logic, Springer, 1998.

[Sh:3] ______, *Finite diagrams stable in power*, Annals of Mathematical Logic **2** (1970), 69–118.

[Sh:E12] ______, *Analytical Guide and Corrections to* [**?**]., arxiv:math.LO/9906022.

[Sh:43] ______, *Generalized quantifiers and compact logic*, Transactions of the American Mathematical Society **204** (1975), 342–364.

[Sh:E45] ______, *Basic non-structure for a.e.c.*

[Sh:46] ______, *Colouring without triangles and partition relation*, Israel Journal of Mathematics **20** (1975), 1–12.

[Sh:48] ______, *Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$*, Israel Journal of Mathematics **20** (1975), 127–148.

[Sh:E53] ______, *Introduction and Annotated Contents*, arxiv:0903.3428.

[DvSh:65] Keith J. Devlin and Saharon Shelah, *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel Journal of Mathematics **29** (1978), 239–247.

[Sh:87a] Saharon Shelah, *Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel Journal of Mathematics **46** (1983), 212–240.

[Sh:87b] ______, *Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel Journal of Mathematics **46** (1983), 241–273.

[Sh:88] ______, *Classification of nonelementary classes. II. Abstract elementary classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Mathematics, vol. 1292, Springer, Berlin, 1987, Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T., pp. 419–497.

[GrSh:174] Rami Grossberg and Saharon Shelah, *On universal locally finite groups*, Israel Journal of Mathematics **44** (1983), 289–302.
[Sh:202] Saharon Shelah, *On co-$\kappa$-Souslin relations*, Israel Journal of Mathematics **47** (1984), 139–153.
[GrSh:259] Rami Grossberg and Saharon Shelah, *On Hanf numbers of the infinitary order property*, Mathematica Japonica **submitted**, arxiv:math.LO/9809196.
[Sh:300b] Saharon Shelah, *Universal Classes: Axiomatic Framework [Sh:h]*, Chapter V (B).
[Sh:394] ———, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), 261–294, arxiv:math.LO/9809197.
[Sh:420] ———, *Advances in Cardinal Arithmetic*, Finite and Infinite Combinatorics in Sets and Logic, Kluwer Academic Publishers, 1993, N.W. Sauer et al (eds.). arxiv:0708.1979, pp. 355–383.
[Sh:576] ———, *Categoricity of an abstract elementary class in two successive cardinals*, Israel Journal of Mathematics **126** (2001), 29–128, arxiv:math.LO/9805146.
[Sh:600] ———, *Categoricity in abstract elementary classes: going up inductively*, arxiv:math.LO/0011215.
[Sh:705] ———, *Toward classification theory of good $\lambda$ frames and abstract elementary classes*, arxiv:math.LO/0404272.
[Sh:F709] ———, *Good* $\lambda$-frames*.
[Sh:838] ———, *Non-structure in $\lambda^{++}$ using instances of WGCH*, Chapter VII, in series Studies in Logic, volume 20, College Publications. arxiv:0808.3020.
[Sh:868] ———, *When first order T has limit models*, Colloquium Mathematicum **126** (2012), 187–204, arxiv:math.LO/0603651.

The Hebrew University of Jerusalem, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel University of Wisconsin, Madison, Wisconsin, Institute of Advanced Studies, Jerusalem, Israel Department of Mathematics,

Hill Center-Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA