

ABSTRACT ELEMENTARY CLASSES NEAR \aleph_1 SH88R

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ABSTRACT. We prove in ZFC, that no $\psi \in \mathbb{L}_{\omega_1, \omega}[\mathbf{Q}]$ have unique model of uncountable cardinality, this confirms the Baldwin conjecture. But we analyze this in more general terms. We introduce and investigate a.e.c. and also versions of limit models, and prove some basic properties like representation by PC class, for any a.e.c. For PC_{\aleph_0} -representable a.e.c. we investigate the conclusion of having not too many non-isomorphic models in \aleph_1 and \aleph_2 , but have to assume $2^{\aleph_0} < 2^{\aleph_1}$ and even $2^{\aleph_1} < 2^{\aleph_2}$.

modified:2015-01-25

(88r) revision:2015-01-23

I would like to thank Alice Leonhardt for the beautiful typing.

This research was partially supported by the United States Israel Binational Science Foundation (BSF) and the NSF. First Typed - 04/May/18. Latest version - 2015/Jan/23; 2015/Jan/5.

0. INTRODUCTION

In [Sh:48], proving a conjecture of Baldwin, we show that (\mathbf{Q} here stands for the quantifier $\mathbf{Q}_{\geq \aleph_1}^{\text{car}}$, there are uncountably many)

(*)₁ no $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ has a unique uncountable model up to isomorphism

by showing that

(*)₂ categoricity (of $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$) in \aleph_1 implies the existence of a model of ψ of cardinality \aleph_2 (so ψ has ≥ 2 non-isomorphism models).

Unfortunately, both (*)₁ and (*)₂ were not proved in ZFC because diamond on \aleph_1 was assumed. In [Sh:87a] and [Sh:87b] this set theoretic assumption was weakened to $2^{\aleph_0} < 2^{\aleph_1}$; here we shall prove it in ZFC (see §3). However, for getting the conclusion from the weaker model theoretic assumption $\dot{I}(\aleph_1, \psi) < 2^{\aleph_1}$ as there, we still need $2^{\aleph_0} < 2^{\aleph_1}$.

The main result of [Sh:87a], [Sh:87b] was:

(*)₃ if $n > 0, 2^{\aleph_0} < 2^{\aleph_1} < \dots < 2^{\aleph_n}, \psi \in \mathbb{L}_{\omega_1, \omega}, 1 \leq \dot{I}(\aleph_\ell, \psi) < \mu_{\text{wd}}(\aleph_\ell)$ for $\ell \leq n, \ell \geq 1$ (where $\mu_{\text{wd}}(\aleph_\ell)$ is usually 2^{\aleph_ℓ} and always $> 2^{\aleph_{\ell-1}}$, see 0.6 below) then ψ has a model of cardinality \aleph_{n+1}

(*)₄ if $2^{\aleph_0} < 2^{\aleph_1} < \dots < 2^{\aleph_n} < 2^{\aleph_{n+1}} < \dots$ and $\psi \in \mathbb{L}_{\omega_1, \omega}, 1 \leq \dot{I}(\aleph_\ell, \psi) < \mu_{\text{wd}}(\aleph_\ell)$ for $\ell < \omega$ then ψ has a model in every infinite cardinal (and satisfies Los Conjecture), (note that (*)₃ for $n = 1$, assuming \diamond_{\aleph_1} was proved in [Sh:48]).

In (*)₄, it is proved that without loss of generality \mathfrak{K} is excellent; this means in particular that K is the class of atomic models of some countable first order T . The point is that an excellent class \mathfrak{K} is similar to the class of models of an \aleph_0 -stable first order T . In particular the set of relevant types, $\mathbf{S}_{\mathfrak{K}}(A, M)$ is defined as $\{p(x) : p(x) \text{ a complete type over } A \text{ in } M \text{ in the first order sense such that } p \upharpoonright B \text{ is isolated for every finite } B \subseteq A\}$. But we better restrict ourselves to “nice A ”, that is A which is the universe of some $N \prec M$ or $A = N_1 \cup N_2$ where N_0, N_1, N_2 are in stable amalgamation or $\cup\{N_u : u \in \mathcal{P} \subseteq \mathcal{P}(n)\}$ for some (so called) stable system $\langle N_u : u \in \mathcal{P} \rangle$; on stable such systems in the stable first order case see [Sh:c, XII, §5]. So types are quite like the first order case. In particular we say $M \in \mathfrak{K}$ is λ -full when: if $p \in \mathbf{S}_{\mathfrak{K}}(A, M)$, A as above, $|A| < \lambda$ implies p is realized in M ; this is the replacement of λ -saturated for that context.

Why in [Sh:87a] and [Sh:87b], ψ was assumed to be just in $\mathbb{L}_{\omega_1, \omega}$ and not more generally in $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$? Mainly because we feel that in [Sh:48], the logic $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ was incidental. We delay the search for the right context to this sequel. So here we are working in a.e.c., “abstract elementary class” (so no logic is present in the context) which are formally like elementary classes, i.e. $(\text{Mod}_T, \prec), T$ first order but note the absence of amalgamation, still they have closure under union of increasing chains. It is $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ where $\leq_{\mathfrak{K}}$ is the “abstract” notion of elementary submodel. So if \mathcal{L} is a fragment of $\mathbb{L}_{\infty, \omega}(\tau)$ (for a fixed vocabulary), $T \subseteq \mathcal{L}$ a theory included in \mathcal{L} , and we let $K = \{M : M \models T\}, M \leq_{\mathfrak{K}} N$ if and only if $M \prec_{\mathcal{L}} N$, we get such a class; if \mathcal{L} is countable then \mathfrak{K} has L.S.T. number \aleph_0 . So the class of models of $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ is not represented directly, but can be with

minor adaptation; see 3.19(2). Surprisingly (and by not so hard proof), every a.e.c. \mathfrak{K} can be represented as a pseudo elementary class if we allow omitting types, (see 1.10). We introduce a relative of saturated models (for stable first order T) and full models (for excellent classes, see [Sh:87a] and [Sh:87b]): limit models; really several variants of this notion. See Definition 3.3. The strongest and most important variant is “ $M \in K_\lambda$ superlimit” which means: M is universal (under $\leq_{\mathfrak{K}}$), $(\exists N)(M \leq_{\mathfrak{K}} N \wedge M \neq N)$ and if $M_i \cong M$ for $i < \delta \leq \|M\|$ and M_i is $\leq_{\mathfrak{K}}$ -increasing then $\bigcup_{i < \delta} M_i \cong M$. If we restrict ourselves to δ 's of cofinality κ we get (λ, κ) -superlimit. Such M exists for a first order T for some pairs λ, κ . In particular (see more in [Sh:868])

(*)₅ for every $\lambda \geq 2^{|T|} + \beth_\omega$, a superlimit model of T of cardinality λ exists if and only if T is superstable (by [Sh:868, 3.1]).

Moreover

(*)₆ “almost always”; for $\lambda \geq 2^{|T|} + \kappa, \kappa = \text{cf}(\kappa)$ (for simplicity) we have: a (λ, κ) -superlimit model exists iff T is stable in $\lambda \& \kappa \geq \kappa(T)$ or $\lambda = \lambda^{<\kappa}$.

But we can prove something under those circumstances: if K is categorical in λ or just have a superlimit model M^* in λ , but the λ -amalgamation property fails for M^* and $2^\lambda < 2^{\lambda^+}$ then $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ (see 3.9). With some reasonable restrictions on λ and K , we can prove e.g. $\dot{I}(\lambda, K) = \dot{I}(\lambda^+, K) = 1 \Rightarrow \dot{I}(\lambda^{++}, K) \geq 1$, (see 3.12, 3.14).

However, our long term main aim was to do the parallel of [Sh:87a] and [Sh:87b] in the present context, i.e., for an a.e.c. \mathfrak{K} and it is natural to assume \mathfrak{K} is PC_{\aleph_0} , here we prepare the ground.

Sections 4,5 present work toward this goal (§5 assuming $2^{\aleph_0} < 2^{\aleph_1}$; §4 without it). We should note that dealing with superlimit models rather than full ones make problems, as well as the fact that the class is not necessarily elementary in some reasonable logics. Because of the second we were driven to use formulas which hold “generically”, are “forced” instead of are satisfied, and “the type \bar{a} materialize” instead of realize and $\text{gtp}(\bar{a}, N, M)$ instead of $\text{tp}(\bar{a}, N, M)$. We also (necessarily) encounter the case “ $\mathbf{D}(N)$ of cardinality \aleph_1 for $N \in K_{\aleph_0}$ ”, see 5.2, 5.4(6). Because of the first, the scenario for getting a full model in \aleph_1 (which can be adapted to $(\aleph_1, \{\aleph_1\})$ -superlimit - see 5.18) does not seem to be enough for getting superlimit models in \aleph_1 (see 5.45).

We had felt that arriving at enough conclusions on the models of cardinality \aleph_1 to start dealing with models of cardinality \aleph_2 , will be a strong indication that we can complete the generalization of [Sh:87a] and [Sh:87b], so getting superlimits in \aleph_1 is the culmination of this paper and a natural stopping point. Trying to do the rest (of the parallel to [Sh:87a] and [Sh:87b]) was delayed.

Much remains to be done,

{88r-0.0}

0.1. **Problem.** 1) Prove (*)₃, (*)₄ in our context.

2) Parallel results in ZFC; e.g. prove (*)₃ for $n = 1, 2^{\aleph_0} = 2^{\aleph_1}$.

Note that if $2^{\aleph_0} = 2^{\aleph_1}$, assuming $1 \leq \dot{I}(\aleph_1, K) < 2^{\aleph_1}$ give really less model theoretic

consequences, as new phenomena arise (see §6). See §4 (and its concluding remarks).

3) Construct examples; e.g. (an a.e.c.) \mathfrak{K} (or $\psi \in \mathbb{L}_{\omega_1, \omega}$), categorical in $\aleph_0, \aleph_1, \dots, \aleph_n$ but not in \aleph_{n+1} .

4) If \mathfrak{K} is a PC_λ class, categorical in λ, λ^+ , does it necessarily have a model in λ^{++} ?

See the book's introduction [Sh:E53] on the progress on those problems in particular on [Sh:576], redone here in [Sh:46]. The direct motivation for [Sh:576] was that Grossberg asked me (Oct. 1994) some questions in this neighborhood (mainly 0.1(4)), in particular:

(*) assume $K = \text{Mod}(T)$, (i.e. K is the class of models of T), $T \subseteq L_{\omega_1, \omega}$, $|T| = \lambda$, $I(\lambda, K) = 1$ and $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$. Does it follow that $I(\lambda^{++}, K) > 0$?

We think of this as a test problem and much prefer a model theoretic to a set theoretic solution. This is closely related to 0.1(4) above and to 3.12 (where we assume categoricity in λ^+ , do not require $2^\lambda < 2^{\lambda^+}$ but take $\lambda = \aleph_0$ or some similar cases) and 5.30(4) (and see 5.2 and 4.8 on the assumptions) (there we require $2^\lambda < 2^{\lambda^+}$, $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$ and $\lambda = \aleph_0$).

Problem [Sh:576, 0.1] was stated a posteriori but is, I think, the real problem, it says:

(**) Can we have some (not necessarily much) classification theory for reasonable non-first order classes \mathfrak{K} of models, with no uses of even traces of compactness and only mild set theoretic assumptions?

This is a revised version of [Sh:88] which continues [Sh:87a], [Sh:87b] but do not use them. The paper [Sh:88] and the present chapter relies on [Sh:48] only when deducing results on $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbb{Q})$; it improves some of its early results and extends the context. The work on [Sh:88] was done in 1977, and a preprint was circulated. Before the paper had appeared, a user-friendly expository article of Makowsky [Mak85] represent, give background and explain the easy parts of the paper. In [Sh:88] the author have corrected and replaced some proofs and added mainly §6. See more in [Sh:F709].

We thank Rami Grossberg for lots of work in the early eighties on previous versions, i.e. [Sh:88], which improved this paper, and the writing up of an earlier version of §6 and Assaf Hasson on helpful comments in 2002 and Alex Usvyatsov for very careful reading, corrections and comments and Adi Jarden and Alon Sison on help in the final stages.

* * *

On history and background on $\mathbb{L}_{\omega_1, \omega}, \mathbb{L}_{\infty, \omega}$ and the quantifier \mathbf{Q} see [Kei71]. On (D, λ) -sequence-homogeneous (which 2.2 - 2.5 here generalized) see Keisler-Morley [KM67], this is defined in 2.3(5), and 2.5 is from there. Theorem 3.9 is similar to [Sh:87a, 2.7] and [Sh:87b, 6.3].

0.2. **Remark.** On non-splitting used here in 5.6 see [Sh:3], [Sh:c, Ch.I, Def.2.6, p.11] or [Sh:48].

We finish §0 by some necessary quotation.

By [Kei70] and [Mor70],

{88r-0.1}

0.3. **Claim.** 1) Assume that $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ has a model M in which $\{\text{tp}_\Delta(\bar{a}, \emptyset, M) : \bar{a} \in M\}$ is uncountable where $\Delta \subseteq \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ is countable, then ψ has 2^{\aleph_1} pairwise non-isomorphic models of cardinality \aleph_1 , in fact we can find models M_α of ψ of cardinality \aleph_1 for $\alpha < 2^{\aleph_1}$ such that $\{\text{tp}_\Delta(a; \emptyset, M_\alpha) : a \in M_\alpha\}$ are pairwise distinct where $\text{tp}_\Delta(\bar{a}, A, M) = \{\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \in \Delta \text{ and } M \models \varphi[\bar{a}, \bar{b}] \text{ and } \bar{b} \in {}^{\omega>}A\}$.
 2) If $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$, $\Delta \subseteq \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ is countable and $\{\text{tp}_\Delta(\bar{a}, \emptyset, M) : \bar{a} \in {}^{\omega>}M \text{ and } M \text{ is a model of } \psi\}$ is uncountable, then it has cardinality 2^{\aleph_0} .

Also note

{88r-0.9}

0.4. **Observation.** Assume (τ is a vocabulary and)

- (a) K is a family of τ -models of cardinality λ
- (b) $\mu > \lambda^\kappa$
- (c) $\{(M, \bar{a}) : M \in K \text{ and } \bar{a} \in {}^\kappa M\}$ has $\geq \mu$ members up to isomorphism.

Then K has $\geq \mu$ models up to isomorphisms (similarly for $= \mu$).

Proof. See [Sh:a, VIII,1.3] or just check by cardinal arithmetic. □_{0.4}

Further

{88r-0.2}

0.5. **Claim.** 1) Assume λ is regular uncountable, M_0 is a model with countable vocabulary and $T = \text{Th}_{\mathbb{L}}(M_0)$, $<$ a binary predicate from $\tau(T)$ and $(P^{M_0}, <^{M_0}) = (\lambda, <)$. Then every countable model M of T has an end extension, i.e., $M \prec N$ and $P^M \neq P^N$ and $a \in P^N \wedge b \in P^M \wedge a <^N b \Rightarrow a \in M$.

2) Moreover, we can further demand $(P^N, <^N)$ is non-well ordered and we can demand $|P^N| = \aleph_1$, $(P^N, <^N)$ is \aleph_1 -like (which means that it has cardinality \aleph_1 but every (proper) initial segment has cardinality $< \aleph_1$); and we can demand N is countable.

3) Moreover, we can add the demand that in $(P^N, <^N)$ there is a first element in $P^N \setminus P^M$ and we can add the demand: in $(P^N, <^N)$, there is no first element in $P^N \setminus P^M$.

Proof. 1),2) Keisler [Kei70].

3) By [Sh:43] and independently Schmerl [Sch76]. □_{0.5}

By Devlin-Shelah [DvSh:65], and [Sh:f, Ap,§1] (the so-called weak diamond).

{88r-0.wD}

0.6. **Theorem.** Assume that $2^\lambda < 2^{\lambda^+}$.

1) There is a normal ideal WDMId_{λ^+} on λ^+ and $\lambda^+ \notin \text{WDMId}_{\lambda^+}$, of course, (the members are called small set) such that: if $S \in (\text{WDMId}_{\lambda^+})^+$ (e.g., $S = \lambda^+$) and $\mathbf{c} : \lambda^+ > (\lambda^+) \rightarrow \{0, 1\}$, then there is $\bar{\ell} = \langle \ell_\alpha : \alpha < \lambda^+ \rangle \in {}^{\lambda^+}2$ such that for every $\eta \in {}^{\lambda^+}(\lambda^+)$ the set $\{\delta \in S : \mathbf{c}(\eta \upharpoonright \delta) = \ell_\alpha\}$ is stationary; we call $\bar{\ell}$ a weak diamond sequence (for the colouring

modified:2015-01-25

(88r) revision:2015-01-23

\mathbf{c} and the stationary set S).

2) $\mu_* = \mu_{\text{wd}}(\lambda^+)$, the cardinal defined by (*) below, is $> 2^\lambda$ (we do not say $\geq 2^{\lambda^+}$!)

(*) (α) if $\mu < \mu_*$ and \mathbf{c}_ε for $\varepsilon < \mu$ is as above then we can find $\bar{\ell}$ as in

part (1) for all the \mathbf{c}_ε 's simultaneously

(β) μ_* is maximal such that clause (α) holds.

3) $\mu_* = \mu_{\text{unif}}(\lambda^+, 2^\lambda)$ satisfies $\mu_*^{\aleph_0} = 2^{\lambda^+}$ and moreover $\lambda \geq \beth_\omega \Rightarrow \mu_* = 2^\lambda$ where $\mu_{\text{unif}}(\lambda^+, \chi)$ is the first cardinal μ such that we can find $\langle \mathbf{c}_\alpha : \alpha < \mu \rangle$ such that:

(a) \mathbf{c}_α is a function from ${}^{\lambda^+}(\lambda^+)$ to χ

(b) there is no $\rho \in {}^{\lambda^+}\chi$ such that for every $\alpha < \mu$ for some $\eta \in {}^{\lambda^+}(\lambda^+)$ the set $\{\delta < \lambda : \mathbf{c}_\alpha(\eta \upharpoonright \delta) \neq \rho(\delta)\}$ is stationary (so $\mu_{\text{wd}}(\lambda^+) = \mu_{\text{unif}}(\lambda^+, 2)$).

See more in [Sh:838, §0,§9] and hopefully in [Sh:E45].

The following are used in §2.

0.7. **Definition.** 1) For a regular uncountable cardinal λ let $\check{I}[\lambda] = \{S \subseteq \lambda : \text{some pair } (E, \bar{a}) \text{ witnesses } S \in \check{I}(\lambda), \text{ see below}\}$.

2) We say that (E, u) is a witness for $S \in \check{I}[\lambda]$ if:

(a) E is a club of the regular cardinal λ

(b) $u = \langle u_\alpha : \alpha < \lambda \rangle$, $a_\alpha \subseteq \alpha$ and $\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha$

(c) for every $\delta \in E \cap S$, u_δ is an unbounded subset of δ of order-type $< \delta$ (and δ is a limit ordinal).

By [Sh:420] and [Sh:E12]

0.8. **Claim.** Let λ be regular uncountable.

1) If $S \in \check{I}[\lambda]$ then we can find a witness (E, \bar{a}) for $S \in \check{I}[\lambda]$ such that:

(a) $\delta \in S \cap E \Rightarrow \text{otp}(a_\delta) = \text{cf}(\delta)$

(b) if $\alpha \notin S$ then $\text{otp}(a_\alpha) < \text{cf}(\delta)$ for some $\delta \in S \cap E$.

2) $S \in \check{I}[\lambda]$ iff there is a pair $(E, \bar{\mathcal{P}})$ such that:

(a) E is a club of the regular uncountable λ

(b) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$, where $\mathcal{P}_\alpha \subseteq \{u : u \subseteq \alpha\}$ has cardinality $< \lambda$

(c) if $\alpha < \beta < \lambda$ and $\alpha \in u \in \mathcal{P}_\beta$ then $u \cap \alpha \in \mathcal{P}_\alpha$

(d) if $\delta \in E \cap S$ then some $u \in \mathcal{P}_\delta$ is an unbounded subset of δ (and δ is a limit ordinal).

{88r-0.5}

{88r-0.6}

modified:2015-01-25

(88r) revision:2015-01-23

1. AXIOMS AND SIMPLE PROPERTIES FOR CLASSES OF MODELS

{88r-1.1}

1.1. **Context.** 1) Here in §1-§5, τ is a vocabulary, K will be a class of τ -models and $\leq_{\mathfrak{K}}$ a two-place relation on the models in K . We do not always strictly distinguish between K and $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$. We shall assume that $K, \leq_{\mathfrak{K}}$ are fixed; and usually we assume that \mathfrak{K} is an a.e.c. (abstract elementary class) which means that the following axioms hold.

2) For a logic \mathcal{L} let $M \prec_{\mathcal{L}} N$ mean M is an elementary submodel of N for the language $\mathcal{L}(\tau_M)$ and $\tau_M \subseteq \tau_N$, i.e., if $\varphi(\bar{x}) \in \mathcal{L}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ then $M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}]$; similarly $M \prec_L N$ for L a language, i.e. a set of formulas in some $\mathcal{L}(\tau_M)$. So $M \prec N$ in the usual sense means $M \prec_{\mathbb{L}} N$ as \mathbb{L} is first order logic and $M \subseteq N$ means M is a submodel of N .

{88r-1.2}

1.2. **Definition.** 1) We say \mathfrak{K} is a a.e.c. with L.S.T. number $\lambda(\mathfrak{K}) = \text{LST}(\mathfrak{K})$ if:

Ax 0: The holding of $M \in K, N \leq_{\mathfrak{K}} M$ depend on N, M only up to isomorphism, i.e. $[M \in K, M \cong N \Rightarrow N \in K]$ and $[\text{if } N \leq_{\mathfrak{K}} M \text{ and } f \text{ is an isomorphism from } M \text{ onto the } \tau\text{-model } M', f \upharpoonright N \text{ is an isomorphism from } N \text{ onto } N' \text{ then } N' \leq_{\mathfrak{K}} M']$.

Ax I: if $M \leq_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

Ax II: $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$ implies $M_0 \leq_{\mathfrak{K}} M_2$ and $M \leq_{\mathfrak{K}} M$ for $M \in K$.

Ax III: If λ is a regular cardinal, $M_i (i < \lambda)$ is a $\leq_{\mathfrak{K}}$ -increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{K}} M_j$) and continuous (i.e. for $\delta < \lambda, M_\delta = \bigcup_{i < \delta} M_i$) then $M_0 \leq_{\mathfrak{K}} \bigcup_{i < \lambda} M_i$.

Ax IV: If λ is a regular cardinal and M_i (for $i < \lambda$) is $\leq_{\mathfrak{K}}$ -increasing continuous and $M_i \leq_{\mathfrak{K}} N$ for $i < \lambda$ then $\bigcup_{i < \lambda} M_i \leq_{\mathfrak{K}} N$.

Ax V: If $N_0 \subseteq N_1 \leq_{\mathfrak{K}} M$ and $N_0 \leq_{\mathfrak{K}} M$ then $N_0 \leq_{\mathfrak{K}} N_1$.

Ax VI: If $A \subseteq N \in K$ and $|A| \leq \text{LST}(\mathfrak{K})$ then for some $M \leq_{\mathfrak{K}} N, A \subseteq |M|$ and $\|M\| \leq \text{LST}(\mathfrak{K})$ (and $\text{LST}(\mathfrak{K})$ is the minimal infinite cardinal satisfying this axiom which is $\geq |\tau|$; the $\geq |\tau|$ is for notational simplicity).

2) We say \mathfrak{K} is a weak ¹ a.e.c. if above we omit clause IV.

1.3. **Remark.** Note that AxV holds for $\prec_{\mathcal{L}}$ for any logic \mathcal{L} .

Notation: Let $K_\lambda = \{M \in K : \|M\| = \lambda\}$ and $K_{<\lambda} = \bigcup_{\mu < \lambda} K_\mu$ and $\mathfrak{K}_\lambda = (K_\lambda, \leq_{\mathfrak{K}} \upharpoonright K_\lambda)$ and similarly $\mathfrak{K}_{<\lambda}, K_{\leq\lambda}, \mathfrak{K}_{\geq\lambda}, K_{\geq\lambda}$. Recall \mathbb{L} is first order logic.

¹this is not really investigated here

1.4. **Definition.** The embedding $f : N \rightarrow M$ is called a $\leq_{\mathfrak{K}}$ -embedding if the range of f is the universe of a model $N' \leq_{\mathfrak{K}} M$ (so $f : N \rightarrow N'$ is an isomorphism onto).

{88r-1.4}

1.5. **Definition.** Let T_1 be a theory in $\mathcal{L}(\tau_1)$, Γ a set of types in $\mathcal{L}(\tau_1)$ for some logic \mathcal{L} , usually first order.

1) $\text{EC}(T_1, \Gamma) = \{M : M \text{ an } \tau_1\text{-model of } T_1 \text{ which omits every } p \in \Gamma\}$.

We implicitly use that τ_1 is reconstructible from T_1, Γ . A problem may arise only if some symbols from τ_1 are not mentioned in T_1 and in Γ , so we may write $\text{EC}(T_1, \Gamma, \tau_1)$, but usually we ignore this point.

2) For $\tau \subseteq \tau_1$ we let $\text{PC}(T_1, \Gamma, \tau) = \text{PC}_\tau(T_1, \Gamma) = \{M : M \text{ is a } \tau\text{-reduct of some } M_1 \in \text{EC}(T_1, \Gamma)\}$.

3) We say that K , a class of τ -models, is a PC_λ^μ or $\text{PC}_{\lambda, \mu}$ class when for some T_1, Γ_1, τ_1 we have $\tau \subseteq \tau_1, T_1$ a first order theory in the vocabulary τ_1, Γ_1 a set of types in $\mathbb{L}(\tau_1), K = \text{PC}_\tau(T_1, \Gamma_1)$ and $|T_1| \leq \lambda, |\Gamma_1| \leq \mu$.

4) We say \mathfrak{K} is PC_λ^μ or $\text{PC}_{\lambda, \mu}$ if for some $(T_1, \Gamma_1, \tau_1), (T_2, \Gamma_2, \tau_2)$ as in part (3) we have $K = \text{PC}(T_1, \Gamma_1, \tau)$ and $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ hence } M, N \in K\} = \text{PC}(T_2, \Gamma_2, \tau')$ where $\tau' = \tau \cup \{P\} \subseteq \tau_2, P$ a new one-place predicate, so $|\tau_\ell| \leq \lambda, |\Gamma_\ell| \leq \mu$ for $\ell = 1, 2$.

If $\mu = \lambda$ we may omit μ .

5) In (4) we may say “ \mathfrak{K} is (λ, μ) -presentable” and if $\lambda = \mu$ we may say “ \mathfrak{K} is λ -presentable”.

{88r-1.5}

1.6. **Example.** If $T \subseteq \mathbb{L}(\tau), \Gamma$ a set of types in $\mathbb{L}(\tau)$, then $K := \text{EC}(T, \Gamma), \leq_{\mathfrak{K}} := \prec_{\mathbb{L}}$ form an a.e.c. with LST-number $\leq |T| + |\tau| + \aleph_0$, that is, satisfy the Axioms from 1.2 (for $\text{LST}(\mathfrak{K}) := |\tau| + \aleph_0$).

{88r-1.6}

1.7. **Observation.** Let I be a directed set (i.e. partially ordered by \leq , such that any two elements have a common upper bound).

1) If M_t is defined for $t \in I$ and $t \leq s \in I$ implies $M_t \leq_{\mathfrak{K}} M_s$ then $\bigcup_{s \in I} M_s \in K$ and for

every $t \in I$ we have $M_t \leq_{\mathfrak{K}} \bigcup_{s \in I} M_s$.

2) If in addition $t \in I$ implies $M_t \leq_{\mathfrak{K}} N$ then $\bigcup_{s \in I} M_s \leq_{\mathfrak{K}} N$.

Proof. By induction on $|I|$ (simultaneously for (1) and (2)).

If I is finite, then I has a maximal element $t(0)$, hence $\bigcup_{t \in I} M_t = M_{t(0)}$, so there is nothing

to prove.

So suppose $|I| = \mu$ and we have proved the assertion when $|I| < \mu$. Let $\lambda = \text{cf}(\mu)$ so λ is a regular cardinal; hence we can find I_α (for $\alpha < \lambda$) such that $|I_\alpha| < |I|, \alpha < \beta < \lambda$

implies $I_\alpha \subseteq I_\beta \subseteq I$, $\bigcup_{\alpha < \lambda} I_\alpha = I$, for limit $\delta < \lambda$, $I_\delta = \bigcup_{\alpha < \delta} I_\alpha$ and each I_α is directed and non-empty; this is trivial when $\lambda > \aleph_0$ and obvious otherwise. Let $M^\alpha = \bigcup_{t \in I_\alpha} M_t$; so by the induction hypothesis on (1) we know that $t \in I_\alpha$ implies $M_t \leq_{\mathfrak{K}} M^\alpha$. If $\alpha < \beta$ then $t \in I_\alpha$ implies $t \in I_\beta$ hence $M_t \leq_{\mathfrak{K}} M^\beta$; hence by the induction hypothesis on (2) applied to $\langle M_t : t \in I_\alpha \rangle, M_\beta$ we have $M^\alpha = \bigcup_{t \in I_\alpha} M_t \leq_{\mathfrak{K}} M^\beta$. So by Ax III, applied to $\langle M^\alpha : \alpha < \lambda \rangle$ we have $M^\alpha \leq_{\mathfrak{K}} \bigcup_{\beta < \lambda} M^\beta = \bigcup_{t \in I} M_t$, and as $t \in I_\alpha$ implies $M_t \leq_{\mathfrak{K}} M^\alpha$, by Ax II, $t \in I$ implies $M_t \leq_{\mathfrak{K}} \bigcup_{s \in I} M_s$. So we have finished proving part (1) for the case $|I| = \mu$. To prove (2) in this case note that for each $\alpha < \lambda$, $\langle M_t : t \in I_\alpha \rangle$ is $\leq_{\mathfrak{K}}$ -directed and $t \in I_\alpha \Rightarrow M_t \leq_{\mathfrak{K}} N$, so clearly by the induction hypothesis for (2) we have $M^\alpha := \bigcup \{M_t : t \in I_\alpha\}$ is $\leq_{\mathfrak{K}} N$. So $\alpha < \lambda \Rightarrow M^\alpha \leq_{\mathfrak{K}} N$ and as proved above $\langle M^\alpha : \alpha < \lambda \rangle$ is $\leq_{\mathfrak{K}}$ -increasing and obviously it is continuous, hence by Ax IV, $\bigcup_{s \in I} M_s = \bigcup_{\alpha < \lambda} M^\alpha \leq_{\mathfrak{K}} N$. $\square_{1.7}$

1.8. Lemma. Let $\tau_1 = \tau \cup \{F_i^n : i < \text{LST}(\mathfrak{K}), n < \omega\}$, F_i^n an n -place function symbol (assuming, of course, $F_i^n \notin \tau$).

{88r-1.7}

Every model M (in K) can be expanded to an τ_1 -model M_1 such that:

- (A) $M_{\bar{a}} \leq_{\mathfrak{K}} M$ when $n < \omega$, $\bar{a} \in {}^n|M|$ and where $M_{\bar{a}}$ is the submodel of M with universe $\{F_i^n(\bar{a}) : i < \text{LST}(\mathfrak{K})\}$
- (B) if $\bar{a} \in {}^n|M|$ then $\|M_{\bar{a}}\| \leq \text{LST}(\mathfrak{K})$
- (C) if \bar{b} is a subsequence of a permutation of \bar{a} , then $M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}}$
- (D) for every $N_1 \subseteq M_1$ we have $N_1 \upharpoonright \tau \leq_{\mathfrak{K}} M$.

Proof. We define by induction on n , the values of $M_{\bar{a}}$ and of $F_i^n(\bar{a})$ for every $i < \text{LST}(\mathfrak{K})$, $\bar{a} \in {}^n|M|$ such that F_i^n is symmetric, i.e. preserved under permuting its variables. Arriving to n , for each $\bar{a} \in {}^nM$ by Ax VI there is an $M_{\bar{a}} \leq_{\mathfrak{K}} M$ such that $\|M_{\bar{a}}\| \leq \text{LST}(\mathfrak{K})$, $|M_{\bar{a}}|$ include $\cup \{M_{\bar{b}} : \bar{b} \text{ a subsequence of } \bar{a} \text{ of length } < n\} \cup \bar{a}$ and $M_{\bar{a}}$ does not depend on the order of \bar{a} . Let $|M_{\bar{a}}| = \{c_i : i < i_0 \leq \text{LST}(\mathfrak{K})\}$ and define $F_i^n(\bar{a}) = c_i$ for $i < i_0$ and c_0 for $i_0 \leq i < \text{LST}(\mathfrak{K})$.

Clearly our conditions are satisfied; in particular, if \bar{b} is a subsequence of \bar{a} , $M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}}$ by Ax V and clause (D) holds by 1.7 and Ax IV. $\square_{1.8}$

1.9. Remark. 1) This is the “main” place we use Ax V,VI; it seems that we use it rarely, e.g., in 2.12 which is not used later. It is clear that we can omit Ax V if we strengthen somewhat Ax VI for the proofs above.

{88r-1.7A}

2) Note that in 1.8, we do not require that $M_{\bar{a}}$ is closed under the functions $(F_i^n)^{M_1}$. By a different bookkeeping we can have it: renaming $\tau_{1,\varepsilon} = \tau \cup \{F_i^n : i < \text{LST}(\mathfrak{K}) \times \varepsilon, n < \omega\}$ for $\varepsilon \leq \omega$ and we choose a $\tau_{1,n}$ -expansion $M_{1,n}$ of M such that $m < n \Rightarrow M_{1,n} \upharpoonright \tau_{1,m} = M_{1,m}$. Let $M_{1,0} = M$, and if $M_{1,n}$ is defined, choose for every $\bar{a} \in {}^{\omega} (M_{1,n})$ a (non-empty) subset $A_{\bar{a}}^{1,n}$ of $M_{1,n}$ of cardinality $\leq \text{LST}(\mathfrak{K})$ such that $A_{\bar{a}}^{1,n}$ is closed under the functions of $M_{1,n}$ and $M \upharpoonright A_{\bar{a}}^{1,n} \leq_{\mathfrak{K}} M$, let $A_{\bar{a}}^{1,n} = \{c_{\bar{a},i} : i \in [\text{LST}(\mathfrak{K}) \times n, \text{LST}(\mathfrak{K}) \times (n+1))\}$ and define $M_{1,n+1}$ by letting $(F_i^m)^{M_{1,n+1}}(\bar{a}) = c_{\bar{a},i}$. Let $M_1 = M_{1,\omega}$ be the τ_ω -model with the universe of M such that $n < \omega \Rightarrow M_1 \upharpoonright \tau_{1,n} = M_{1,n}$.

3) Actually $M_{1,1}$ suffices if we expand it by making every term $\tau(\bar{x})$ equal to some function $F(\bar{x})$.

4) Alternatively demand for $n > 0$ that $F_i^n(\bar{a})$ is $F_i^{|\bar{a}|}(\bar{a} \upharpoonright u)$, $u = \{i < n : a_i \notin \{a_j : j < i\}\}$.

{88r-1.8}

1.10. **Lemma.** 1) \mathfrak{K} is $(\text{LST}(\mathfrak{K}), 2^{\text{LST}(\mathfrak{K})})$ -presentable.

2) There is a set Γ of types in $\mathbb{L}(\tau_1)$ in fact complete quantifier free (where τ_1 is from Lemma 1.8) such that $K = \text{PC}_\tau(\emptyset, \Gamma)$.

3) For the Γ from part (2), if $M_1 \subseteq N_1 \in \text{EC}(\emptyset, \Gamma)$ and M, N are the τ -reducts of M_1, N_1 respectively then $M \leq_{\mathfrak{K}} N$.

4) For the Γ from part (2), we have $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ so } N, M \in K\} = \{(M_1 \upharpoonright \tau, N_1 \upharpoonright \tau) : M_1 \subseteq N_1 \text{ are both from } \text{PC}_\Gamma(\emptyset, \Gamma)\}$.

Proof. 1) By part (2) the first half of “ \mathfrak{K} is $(\text{LST}(\mathfrak{K}), 2^{\text{LST}(\mathfrak{K})})$ -presentable holds”. The second part will be proved with part (4).

2) Let Γ_n be the set of complete quantifier free n -types $p(x_0, \dots, x_{n-1})$ in $\mathbb{L}(\tau_1)$ such that: if M_1 is a τ_1 -model, \bar{a} realizes p in M_1 and M is the τ -reduct of M_1 , then $M_{\bar{a}} \in K$ and $M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}}$ for any subsequence \bar{b} of any permutation of \bar{a} ; where $M_{\bar{c}}(\bar{c} \in {}^m |M_1|)$ is the submodel of M whose universe is $\{F_i^m(\bar{c}) : i < \text{LST}(\mathfrak{K})\}$. Clearly there are such submodels (when $K \neq \emptyset$).

Let Γ be the set of p which, for some n , are complete quantifier free n -types (in $\mathbb{L}(\tau_1)$) which do not belong to Γ_n . By 1.7(1) we have $\text{PC}_\tau(\emptyset, \Gamma) \subseteq K$ and by 1.8 $K \subseteq \text{PC}_\tau(\emptyset, \Gamma)$.

3) Similar to the proof of (2) using 1.7(2).

4) The inclusion \supseteq holds by part (3); so let us prove the other direction. Given $N \leq_{\mathfrak{K}} M$ we apply the proof of 1.8 to M , but demand further $\bar{a} \in {}^n N \Rightarrow M_{\bar{a}} \subseteq N$; simply add this demand to the choice of the $M_{\bar{a}}$'s (hence of the F_i^n 's). We still have a debt from part (1).

We let Γ'_n be the set of complete quantifier free n -types in $\tau'_1 := \tau_1 \cup \{P\}$ (P a new unary predicate), $p(x_0, \dots, x_{n-1})$ such that:

(*) if M_1 is an τ'_1 -model, \bar{a} realizes p in M_1 , M the τ -reduct of M_1 , then

(α) $M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}}$ for any subsequence \bar{b} of \bar{a} where $M_{\bar{c}}$ (for $\bar{c} \in |M_1|$) is the submodel of M whose universe is $\{(F_i^m)^{M_1}(\bar{c}) : i < \text{LST}(\mathfrak{K})\}$, where $m = \ell g(\bar{c})$ (and there are such models),

(β) $\bar{b} \subseteq P^{M_1} \Rightarrow M_{\bar{b}} \subseteq P^{M_1}$ for $\bar{b} \subseteq \bar{a}$.

We leave the rest to the reader (alternatively, use $\text{PC}_{\tau_1'}(T', \Gamma), T'$ saying “ P is closed under all the functions F_i^n ”). $\square_{1.10}$

By the proof of 1.10(4).

1.11. **Conclusion.** The τ_1 and Γ from 1.10 (so $|\tau_1| \leq \text{LST}(\mathfrak{K})$) satisfy: for any $M \in K$ and any τ_1 -expansion M_1 of M which is in $\text{EC}_{\tau_1}(\emptyset, \Gamma)$ {88r-1.9}

- (a) $N_1 \prec_{\mathbb{L}} M_1 \Rightarrow N_1 \subseteq M_1 \Rightarrow N_1 \upharpoonright \tau \leq_{\mathfrak{K}} M$
- (b) $N_1 \prec_{\mathbb{L}} N_2 \prec_{\mathbb{L}} M_1 \Rightarrow N_1 \subseteq N_2 \subseteq M_1 \Rightarrow N_1 \upharpoonright \tau \leq_{\mathfrak{K}} N_2 \upharpoonright \tau$
- (c) if $M \leq_{\mathfrak{K}} N$ then there is a τ_1 -expansion N_1 of N from $\text{EC}_{\tau_1}(\emptyset, \Gamma)$ which extends M_1 .

1.12. **Conclusion.** If for every $\alpha < (2^{\text{LST}(\mathfrak{K})})^+$, \mathfrak{K} has a model of cardinality $\geq \beth_{\alpha}$ then K has a model in every cardinality $\geq \text{LST}(\mathfrak{K})$. {88r-1.10}

Proof. Use 1.10 and the classical upper bound on value of the Hanf number for: first order theory and omitting any set of types, for languages of cardinality $\text{LST}(\mathfrak{K})$ (see, e.g., [Sh:c, VII,5.3,5.5]). $\square_{1.12}$

1.13. **Conclusion.** Assume that \mathfrak{K} is an a.e.c., $\mu = |\tau_{\mathfrak{K}}| + \text{LST}(\mathfrak{K})$ and for simplicity $\tau_{\mathfrak{K}} \subseteq \mu$ or just $\tau_{\mathfrak{K}} \subseteq \mathbf{L}_{\mu}$, recalling \mathbf{L} is the constructible universe of Göbel. If $\lambda > \mu$ and $\mathfrak{A} \prec (\mathcal{H}(\lambda, \in))$ and $\mu + 1 \subseteq \mathfrak{A}$ and $\mathfrak{K} \in \mathfrak{A}$ which means $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ has universe } \subseteq \mu\} \in \mathfrak{A}$ then: {88r-1a.1}

- (a) $M \in \mathfrak{K} \cap K \Rightarrow M \upharpoonright \mathfrak{A} \leq_{\mathfrak{K}} M$
- (b) if $M \leq_{\mathfrak{K}} N$ so both belongs to K and $M, N \in \mathfrak{A}$ then $M \upharpoonright \mathfrak{A} \leq_{\mathfrak{K}} N \upharpoonright \mathfrak{A}$
- (c) if $\mathfrak{A} \prec \mathfrak{B}$ and $[b <_{\mathfrak{B}} \mu \Rightarrow b \in \mathfrak{A}]$ and $\mathfrak{B} \models \text{“}M \in K\text{”}$ then $M[\mathfrak{B}] \in K$
- (d) similarly for $\mathfrak{B} \models \text{“}M \leq_{\mathfrak{K}} N\text{”}$
 where
 - (*)₁ if $M \in \mathfrak{A}$ then $M \upharpoonright \mathfrak{A}$ is the submodel of M with universe $|M| \cap |\mathfrak{A}|$
 - (*)₂ if $\mathfrak{B} \models \text{“}M \in \mathfrak{K}\text{”}$ then $M[\mathfrak{B}]$ is the following τ_K -model:
 - (a) it has universe $\{b \in \mathfrak{B} : \mathfrak{B} \models \text{“}b \text{ an element of the model } M\text{”}\}$
 - (b) for any m -place predicate Q of τ , $Q^M = \{\langle b_0, \dots, b_{m-1} \rangle : \mathfrak{B} \models \text{“}M \models Q[b_0, \dots, b_{m-1}]\text{”}\}$
 - (c) for any m -place function symbol G of τ , similarly.

Proof. Should be clear. $\square_{1.13}$

1.14. **Remark.** 1) Clearly $\{\mu : \mu \geq \text{LST}(\mathfrak{K}) \text{ and } K_\mu \neq \emptyset\}$ is an initial segment of the class of cardinals $\geq \text{LST}(\mathfrak{K})$.

2) For every cardinal $\kappa (\geq \aleph_0)$ and ordinal $\alpha < (2^\kappa)^+$ there is an a.e.c. \mathfrak{K} such that: $\text{LST}(\mathfrak{K}) = \kappa = |\tau_{\mathfrak{K}}|$ and \mathfrak{K} has a model of cardinality λ iff $\lambda \in [\kappa, \beth_\alpha(\kappa))$. This follows by [Sh:c, VII,§5,p.432] in particular [Sh:c, VII,5.5](6), because

(a) if a vocabulary of cardinality $\leq \kappa$ and $T \subseteq \mathbb{L}(\tau)$ and Γ a set of $(\mathbb{L}(\tau), < \omega)$ -types then $K = \{M : M \text{ a } \tau\text{-model of } T \text{ omitting every } \in \Gamma\}$ and $\leq_{\mathfrak{K}} = \prec \upharpoonright K$ form an a.e.c. (we can use Γ a set of quantifier free types, $T = \emptyset$), with $\text{LST}((\mathfrak{K}, \leq_{\mathfrak{K}}) \leq \kappa$

(b) if $\{c_i \neq c_j : i < j < \kappa\} \subseteq T$ then K above has no model of cardinality $< \kappa$.

3) More on such theorems see [Sh:394].

4) We can phrase 1.13 “for any \mathfrak{B} in appropriate $\text{EC}(T_1, \Gamma_1)$ ”, but the present formulation is the way we use it.

2. AMALGAMATION PROPERTIES AND HOMOGENEITY

{88r-2.0}

2.1. **Context.** \mathfrak{K} is an a.e.c.

The main theorem 2.9, the existence and uniqueness of the model-homogeneous models, is a generalization of Jonsson [Jón56], [Jón60] to the present context. The result on the upper bound $2^{2^{\aleph_0 + |\tau|}}$ for the number of D -sequence homogeneous universal-models of cardinality is of Keisler-Morley [KM67]. Earlier there were serious good reasons to concentrate on sequence-homogeneous models, but here we deal with the model-homogeneous case. From 2.14 to the end we consider what we can say when we omit smoothness, i.e. AxIV of Definition 1.2.

{88r-2.1}

- 2.2. **Definition.** 1) $\mathbb{D}(M) := \{N / \cong: N \leq_{\mathfrak{K}} M, \|N\| \leq \text{LST}(\mathfrak{K})\}$.
 2) $\mathbb{D}(\mathfrak{K}) := \{N / \cong: N \in K, \|N\| \leq \text{LST}(\mathfrak{K})\}$.
 3) $D(M) = \{\text{tp}_{\mathbb{L}(\tau_M)}(\bar{a}, \emptyset, M) : \bar{a} \in {}^{\omega}M\}$.

{88r-2.2}

2.3. **Definition.** Let $\lambda > \text{LST}(\mathfrak{K})$.

1) A model M is λ -model-homogeneous when: if $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} M, \|N_1\| < \lambda, f$ an $\leq_{\mathfrak{K}}$ -embedding of N_0 into M , then some $\leq_{\mathfrak{K}}$ -embedding $f' : N_1 \rightarrow M$ extends f .

1A) A model M is (\mathbb{D}, λ) -model-homogeneous if $\mathbb{D} = \mathbb{D}(M)$ and M is a λ -model homogeneous.

1B) Adding “above μ ” means in $\mathfrak{K}_{\geq \mu}$.

2) M is λ -strongly model-homogeneous if: for every $N \in K_{< \lambda}$ such that $N \leq_{\mathfrak{K}} M$ and a $\leq_{\mathfrak{K}}$ -embedding $f : N \rightarrow M$ there exists an automorphism g of M extending f .

3) M is λ -model universal homogeneous (for \mathfrak{K}) when: $\lambda > \text{LST}(\mathfrak{K})$, every² $N \in K_{\text{LST}(\mathfrak{K})}$ is $\leq_{\mathfrak{K}}$ -embeddable into M and for every $N_\ell \in K_{< \lambda}$ (for $\ell = 0, 1$) such that $N_0 \leq_{\mathfrak{K}} N_1$ and $\leq_{\mathfrak{K}}$ -embedding $f : N_0 \rightarrow M$ there exists a $\leq_{\mathfrak{K}}$ -embedding $g : N_1 \rightarrow M$ extending f (unlike (1), we do not demand that N_1 is $\leq_{\mathfrak{K}}$ -embeddable into M ; the universal is related to λ , it does not imply M is universal).

4) For each of the above three properties and the one below, if M has cardinality λ and has the λ -property then we may say for short that M has the property (i.e. omitting λ).

5) M is (D, λ) -sequence-homogeneous if:

(a) $D = D(M) = \{\text{tp}_{\mathbb{L}(\tau_M)}(\bar{a}, \emptyset, M) : \bar{a} \in |M|, \text{i.e., } \bar{a} \text{ a finite sequence from } M\}$ and

(b) if $a_i \in M$ for $i \leq \alpha < \lambda, b_j \in M$ for $j < \alpha$ and $\text{tp}_{\mathbb{L}(\tau_M)}(\langle a_i : i < \alpha \rangle, \emptyset, M) = \text{tp}_{\mathbb{L}(\tau_M)}(\langle b_i : i < \alpha \rangle, \emptyset, M)$, then for some $b_\alpha \in M, \text{tp}_{\mathbb{L}(\tau_M)}(\langle a_i : i \leq \alpha \rangle, \emptyset, M) = \text{tp}_{\mathbb{L}(\tau_M)}(\langle b_i : i \leq \alpha \rangle, \emptyset, M)$.

5A) In (5) we omit D when $D = \{\text{tp}_{\mathbb{L}(\tau_K)}(\bar{a}, \emptyset, N) : \bar{a} \in {}^n N \text{ where } n < \omega \text{ and } M \prec_{\mathbb{L}} N\}$.

6) We omit the “model/sequence”, when which one is clear from the context, i.e., if D is as in 2.2(3) = 2.3(5)(a), (D, λ) -homogeneous means (D, λ) -sequence-homogeneous: if \mathbb{D} is as in Definition 2.2(1), (\mathbb{D}, λ) -homogeneous means (\mathbb{D}, λ) -model-homogeneous, if not obvious

²in fact, $N \in K_{\leq \lambda}$ is O.K. by 2.5(2)

we mean the model version.

7) M is λ -universal when every $N \in K_\lambda$ can be $\leq_{\mathfrak{K}}$ -embedded into it. Similarly ($< \lambda$)-universal, ($\leq \lambda$)-universal.

{88r-2.3}

2.4. Claim. *Assume N is λ -model-homogeneous and $\mathbb{D}(M) \subseteq \mathbb{D}(N)$, (and $\text{LST}(\mathfrak{K}) < \lambda$, of course).*

1) *If $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M$, $\|M_0\| < \lambda$, $\|M_1\| \leq \lambda$ and f is a $\leq_{\mathfrak{K}}$ -embedding of M_0 into N , then we can extend f to a $\leq_{\mathfrak{K}}$ -embedding of M_1 into N .*

2) *If $M_1 \leq_{\mathfrak{K}} M$, $\|M_1\| \leq \lambda$ then there is a $\leq_{\mathfrak{K}}$ -embedding of M_1 into N .*

Proof. We prove by induction on $\mu \leq \lambda$ simultaneously that:

(i) $_{\mu}$ for every $M_1 \leq_{\mathfrak{K}} M$, $\|M_1\| \leq \mu$ (yes! not $< \mu$) there is a $\leq_{\mathfrak{K}}$ -embedding of M_1 into N

(ii) $_{\mu}$ if $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M$, $\|M_1\| \leq \mu$, $\|M_0\| < \lambda$ then any $\leq_{\mathfrak{K}}$ -embedding f of M_0 into N can be extended to a $\leq_{\mathfrak{K}}$ -embedding of M_1 into N .

Clearly (i) $_{\lambda}$ is part (2) and (ii) $_{\lambda}$ is part (1) so this is enough. \square

Proof of (i) $_{\mu}$. If $\mu \leq \text{LST}(\mathfrak{K})$, this follows by $\mathbb{D}(M) \subseteq \mathbb{D}(N)$.

If $\mu > \text{LST}(\mathfrak{K})$, then by 1.11 we can find $\bar{M}_1 = \langle M_1^\alpha : \alpha < \mu \rangle$ such that $M_1 = \bigcup_{\alpha < \mu} M_1^\alpha$ and

$\alpha < \mu \Rightarrow M_1^\alpha \leq_{\mathfrak{K}} M_1$ and M_1^α is $\leq_{\mathfrak{K}}$ -increasing continuous with α and $\alpha < \mu \Rightarrow \|M_1^\alpha\| < \mu$. We define by induction on α , a $\leq_{\mathfrak{K}}$ -embedding $f_\alpha : M_1^\alpha \rightarrow N$, such that for $\beta < \alpha$, f_α extend f_β . For $\alpha = 0$ we can define f_α by (i) $_{\chi(0)}$ which holds as by the induction hypothesis, where $\chi(\beta) := \|M_1^\beta\|$. We next define f_α for $\alpha = \gamma + 1$: by (ii) $_{\chi(\alpha)}$ which holds by the induction hypothesis there is a $\leq_{\mathfrak{K}}$ -embedding f_α of M_1^α into N extending f_γ .

Lastly, for limit α we let $f_\alpha = \bigcup_{\beta < \alpha} f_\beta$, it is a $\leq_{\mathfrak{K}}$ -embedding into N by 1.7. So we finish

the induction and $\bigcup_{\alpha < \mu} f_\alpha$ is as required. \square

Proof of (ii) $_{\mu}$. First, assume that $\mu = \lambda$ so we have proved (ii) $_{\theta}$ for $\theta < \lambda$ and $\|M_1\| = \lambda > \|M_0\|$, so $\text{LST}(\mathfrak{K}) < \mu = \lambda$ hence we can find $\langle M_1^\alpha : \alpha < \mu \rangle$ as in the proof of (i) $_{\mu}$ such that $M_1^0 = M_0$ and let $\chi(\beta) = \|M_1^\beta\|$. Now we define f_β by induction on $\beta \leq \mu$ such that f_β is a $\leq_{\mathfrak{K}}$ -embedding of M_1^β into N and f_β is increasing continuous in β and $f_0 = f$. We can do this as in the proof of (i) $_{\mu}$ by (ii) $_{\chi(\alpha)}$ for $\alpha < \mu$.

Second, assume $\|M_1\| < \lambda$. Let g be a $\leq_{\mathfrak{K}}$ -embedding of M_1 into N , it exists by (i) $_{\mu}$ which we have just proved. Let g be onto $N'_1 \leq_{\mathfrak{K}} N$, and let $g \upharpoonright M_0$ be onto $N'_0 \leq_{\mathfrak{K}} N'_1$, and let f be onto $N_0 \leq_{\mathfrak{K}} N$. So clearly $h : N'_0 \rightarrow N_0$ defined by $h(g(a)) = f(a)$ for $a \in |M_0|$, is an isomorphism from N'_0 onto N_0 . So $N_0, N'_0, N'_1 \leq_{\mathfrak{K}} N$. As $\|M_1\| < \lambda$ clearly $\|N'_1\| < \lambda$

so (by the assumption “ N is λ -model-homogeneous”, see Definition 2.3(1)) we can extend h to an isomorphism h' from N'_1 onto some $N_1 \leq_{\mathfrak{K}} N$, so $h' \circ g : M_1 \rightarrow N$ is as required. $\square_{2.4}$

{88r-2.4}

2.5. Conclusion. 1) If M, N are model-homogeneous, of the same cardinality ($> \text{LST}(\mathfrak{K})$) and $\mathbb{D}(M) = \mathbb{D}(N)$ then M, N are isomorphic. Moreover, if $M_0 \leq_{\mathfrak{K}} M$, $\|M_0\| < \|M\|$, then any $\leq_{\mathfrak{K}}$ -embedding of M_0 into N can be extended to an isomorphism from M onto N .

2) The number of model-homogeneous models from \mathfrak{K} of cardinality λ is $\leq 2^{2^{\text{LST}(\mathfrak{K})}}$; if in Definition 1.2, AxVI, in the definition of $\text{LST}(\mathfrak{K})$ we omit $|\tau| \leq \text{LST}(\mathfrak{K})$, the bound is $2^{2^{\text{LST}(\mathfrak{K})+|\tau(\mathfrak{K})|}}$.

3) If M is λ -model-homogeneous and $\mathbb{D}(M) = \mathbb{D}(\mathfrak{K})$ then M is $(\leq \lambda)$ -universal, i.e. every model N (in K) of cardinality $\leq \lambda$, has a $\leq_{\mathfrak{K}}$ -embedding into M . So if $\mathbb{D}(M) = \mathbb{D}(\mathfrak{K})$ then: M is λ -model universal homogeneous (see Definition 2.3(3)) iff M is a λ -model-homogeneous iff M is $(\lambda, \mathbb{D}(\mathfrak{K}))$ -homogeneous.

4) If M is λ -model-homogeneous then it is λ -universal for $\{N \in K_\lambda : \mathbb{D}(N) \subseteq \mathbb{D}(M)\}$.

5) If M is (D, λ) -sequence-homogeneous, $(\lambda > \text{LST}(\mathfrak{K}))$ then M is a λ -model homogeneous.

6) For $\lambda > \text{LST}(\mathfrak{K})$, M is λ -model universal homogeneous iff M is λ -model-homogeneous and $(\leq \text{LST}(\mathfrak{K}))$ -universal.

Proof. 1) Immediate by 2.4(1), using the standard hence and forth argument.

2) The number of models (in K) of power $\leq \text{LST}(\mathfrak{K})$ is, up to isomorphism, $\leq 2^{\text{LST}(\mathfrak{K})}$ (recalling that we are assuming $|\tau(\mathfrak{K})| \leq \text{LST}(\mathfrak{K})$). Hence the number of possible $\mathbb{D}(M)$ is $\leq 2^{2^{\text{LST}(\mathfrak{K})}}$. So by 2.5(1) we are done.

3),4),5) Immediate. $\square_{2.5}$

{88r-2.4A}

2.6. Remark. The results parallel to 2.5(1)-(4) for λ -sequence homogeneous models and $D(M)$ hold, too.

{88r-2.5}

2.7. Definition. 1) A model M has the (λ, μ) -amalgamation property (= am.p., in \mathfrak{K} , of course) if: for every M_1, M_2 such that $\|M_1\| = \lambda, \|M_2\| = \mu, M \leq_{\mathfrak{K}} M_1$ and $M \leq_{\mathfrak{K}} M_2$, there is a model N and $\leq_{\mathfrak{K}}$ -embeddings $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ such that $f_1 \upharpoonright |M| = f_2 \upharpoonright |M|$. Now the meaning of e.g. the $(\leq \lambda, < \mu)$ -amalgamation property should be clear. Always $\lambda, \mu \geq \text{LST}(\mathfrak{K})$ (and, of course, if we use $< \mu, \mu > \text{LST}(\mathfrak{K})$).

1A) In part (1) we add the adjective “disjoint” when $f_1(M_1) \cap f_2(M_2) = M$. Similarly in (2) below.

2) \mathfrak{K} has the (κ, λ, μ) -amalgamation property if every model M (in K) of cardinality κ has the (λ, μ) -amalgamation property. The (κ, λ) -amalgamation property for \mathfrak{K} means just the $(\kappa, \kappa, \lambda)$ -amalgamation property. The κ -amalgamation property for \mathfrak{K} is just the (κ, κ, κ) -amalgamation property.

- 3) \mathfrak{K} has the (λ, μ) -JEP (joint embedding property) if for any $M_1 \in K, M_2 \in K$ of cardinality λ, μ respectively there is $N \in K$ into which M_1 and M_2 are $\leq_{\mathfrak{K}}$ -embeddable.
- 4) The λ -JEP is the (λ, λ) -JEP.
- 5) The amalgamation property means the (κ, λ, μ) -amalgamation property for every $\lambda, \mu \geq \kappa (\geq \text{LST}(\mathfrak{K}))$.
- 6) The JEP means the (λ, μ) -JEP for every $\lambda, \mu \geq \text{LST}(\mathfrak{K})$.

2.8. **Remark.** Clearly in 2.7, parts (1), (2) first sentence, (3),(5), the roles of λ, μ are symmetric.

{88r-2.6}

- 2.9. **Theorem.** 1) If $\text{LST}(\mathfrak{K}) < \kappa \leq \lambda, \lambda = \lambda^{<\kappa}, K_\lambda \neq \emptyset$ and \mathfrak{K} has the $(< \kappa, \lambda)$ -amalgamation property then for every model M of cardinality λ , there is a κ -model-homogeneous model N of cardinality λ satisfying $M \leq_{\mathfrak{K}} N$. If $\kappa = \lambda$, alternatively the $(< \kappa, < \lambda)$ -amalgamation property suffices.
- 2) So in (1) if $\kappa = \lambda$, there is a universal, model-homogeneous model of cardinality λ , provided that for some $M \in K_{\leq \lambda}, \mathbb{D}(M) = \mathbb{D}(\mathfrak{K})$ or just \mathfrak{K} has the $\text{LST}(\mathfrak{K})$ -JEP.
- 3) If \mathfrak{K} has the amalgamation property and the $\text{LST}(\mathfrak{K})$ -JEP, then \mathfrak{K} has the JEP.

{88r-2.6A}

- 2.10. **Remark.** 1) The last assumption of 2.9(2) holds, e.g., if $(\leq \text{LST}(\mathfrak{K}), < 2^{\text{LST}(\mathfrak{K})})$ -JEP holds and $|\mathbb{D}(\mathfrak{K})| \leq \lambda$.
- 2) If for some $M \in K, \mathbb{D}(M) = \mathbb{D}(\mathfrak{K})$ then we can have such M of cardinality $\leq 2^{\text{LST}(\mathfrak{K})}$.
- 3) We can in 2.9 replace the assumption “ $(< \kappa, \lambda)$ -amalgamation property” by “ $(< \kappa, < \lambda)$ -amalgamation property” if, e.g., no $M \in K_{< \lambda}$ is maximal.

Proof. Immediate; in (1) note that if κ is singular then necessarily $\lambda > \kappa \& \lambda = \lambda^\kappa = \lambda^{<\kappa^+}$ so we can replace κ by κ^+ . $\square_{2.9}$

{88r-2.6B}

2.11. **Remark.** Also the corresponding converses hold.

{88r-2.7}

- 2.12. **Lemma.** 1) If $\text{LST}(\mathfrak{K}) \leq \kappa$ and \mathfrak{K} has the κ -amalgamation property then \mathfrak{K} has the (κ, κ^+) -amalgamation property and even the $(\kappa, \kappa^+, \kappa^+)$ -amalgamation property.
- 2) If $\kappa \leq \mu \leq \lambda$ and \mathfrak{K} has the (κ, μ) -amalgamation property and the (μ, λ) -amalgamation property then \mathfrak{K} has the (κ, λ) -amalgamation property. If \mathfrak{K} has the (κ, μ, μ) and the (μ, λ) -amalgamation property, then \mathfrak{K} has the (κ, λ, μ) -amalgamation property.
- 3) If $\lambda_i (i \leq \alpha)$ is increasing and continuous, $\text{LST}(\mathfrak{K}) \leq \lambda_0$ and for every $i < \alpha, \mathfrak{K}$ has the $(\lambda_i, \mu + \lambda_i, \lambda_{i+1})$ -amalgamation property then \mathfrak{K} has the $(\lambda_0, \mu + \lambda_0, \lambda_\alpha)$ -amalgamation property.
- 4) If $\kappa \leq \mu_1 \leq \mu$ and for every $M, \|M\| = \mu_1$, there is $N, M \leq_{\mathfrak{K}} N, \|N\| = \mu$, then the (κ, μ, λ) -amalgamation property (for \mathfrak{K}) implies the (κ, μ_1, λ) -amalgamation property (for

\mathfrak{K}).

5) Similarly with the disjoint amalgamation version.

Proof. Straightforward, e.g.

3) So assume $M_0 \in K_{\lambda_0}$, $M_0 \leq_{\mathfrak{K}} M_1 \in K_{\mu+\lambda_0}$ and $M_0 \leq_{\mathfrak{K}} M_2 \in K_{\lambda_\alpha}$ and for variety we prove for the disjoint amalgamation version (see part (5)). By e.g. 1.11 we can find an $\leq_{\mathfrak{K}}$ -increasing continuous sequence $\langle M_{2,i} : i \leq \alpha \rangle$ such that $M_{2,0} = M_0$, $M_{2,\alpha} = M_2$ and $M_{2,i} \in K_{\lambda_i}$ for $i \leq \alpha$.

Without loss of generality $M_1 \cap M_2 = M_0$. We now choose $M_{1,i}$ by induction on $i \leq \alpha$ such that:

- (*) (a) $\langle M_{1,j} : j \leq i \rangle$ is $\leq_{\mathfrak{K}}$ -increasing continuous
- (b) $M_{1,i} = M_1$ if $i = 0$
- (c) $M_{1,i} \in K_{\mu+\lambda_i}$
- (d) $M_{2,i} \leq_{\mathfrak{K}} M_{1,i}$
- (e) $M_{2,i} \cap M_{1,\alpha} = M_{1,i}$.

For $i = 0$ see clause (b), for i limit take union, for $i = j+1$ apply the disjoint $(\lambda_j, \mu+\lambda_j, \lambda_i)$ -amalgamation to $M_{2,j}, M_{1,j}, M_{2,j+1}$. For $i = \alpha$ we are done. $\square_{2.12}$

2.13. Conclusion. If $\text{LST}(\mathfrak{K}) \leq \chi_1 < \chi_2$ and \mathfrak{K} has the κ -amalgamation property whenever $\chi_1 \leq \kappa < \chi_2$ then \mathfrak{K} has the (κ, λ, μ) -amalgamation property whenever $\chi_1 \leq \kappa \leq \lambda \leq \chi_2$, $\kappa \leq \mu \leq \chi_2$ and $\kappa < \chi_2$. {88r-2.8}

* * *

It may be interesting to note that even waiving AX IV we can say something.

2.14. Context. For the remainder of this section \mathfrak{K} is just a weak a.e.c., i.e., Ax IV is not assumed. {88r-2.9}

2.15. Definition. Let $M \in K$ have cardinality λ , a regular uncountable cardinal $> \text{LST}(\mathfrak{K})$. We say M is smooth if there is a sequence $\langle M_i : i < \lambda \rangle$ with M_i being $\leq_{\mathfrak{K}}$ -increasing continuous, $M_i \leq_{\mathfrak{K}} M$ and $\|M_i\| < \lambda$ for $i < \lambda$ and $M = \bigcup_{i < \lambda} M_i$. {88r-2.10}

2.16. Remark. We can define S/\mathcal{D} -smooth, for S a subset of $\mathcal{P}(\lambda)$, \mathcal{D} a filter on $\mathcal{P}(\lambda)$, that is: $M \in K_\lambda$ is (S/\mathcal{D}) -smooth when for every one-to-one function f from $|M|$ onto λ the set $\{u \in \mathcal{P}(\lambda) : M \upharpoonright \{a : f(a) \in u\} \leq_{\mathfrak{K}} M\} \in \mathcal{D}$. Usually we demand that for every permutation f on λ , $\{u \subseteq \lambda : u \text{ is closed under } f\} \in \mathcal{D}$, and usually we demand that \mathcal{D} is a normal $\text{LST}(\mathfrak{K})^+$ -complete filter). {88r-2.10}

modified:2015-01-25

(88r) revision:2015-01-23

2.17. **Claim.** Assume that $\lambda = \lambda^{<\lambda} > |\tau_K|$, $\mathfrak{K}_{<\lambda}$ has no maximal member and \mathfrak{K} has $(<\lambda, <\lambda, <\lambda)$ -amalgamation property and $\text{LST}(\mathfrak{K}) < \lambda$ or at least assume in the $(<\lambda, <\lambda, <\lambda)$ -amalgamation demand that the resulting model has cardinality $< \lambda$. Then \mathfrak{K}_λ has a smooth model-homogeneous member.

Proof. Same proof. □_{2.17}

{88r-2.11}

2.18. **Lemma.** If $M, N \in K_\lambda$ ($\lambda > \text{LST}(\mathfrak{K})$) are smooth, model-homogeneous and $\mathbb{D}(M) = \mathbb{D}(N)$ then $M \cong N$.

Proof. By the hence and forth argument, left to the reader (the set of approximations is $\{f : f \text{ isomorphism from some } M' \leq_{\mathfrak{K}} M \text{ of cardinality } < \lambda \text{ onto some } N' \leq_{\mathfrak{K}} N\}$ but note that not for any increasing continuous sequence of approximations is the union an approximation). □_{2.18}

{88r-2.11A}

2.19. **Remark.** It is reasonable to consider

- (*) if $M \in K_\lambda$, ($\lambda > \text{LST}(\mathfrak{K})$) is smooth and model-homogeneous and $N \in K_\lambda$ is smooth, $\mathbb{D}(N) \subseteq \mathbb{D}(M)$ then N can be $\leq_{\mathfrak{K}}$ -embedded into M .

This can be proved in the context of universal classes (e.g. $AxFr_1$ from [Sh:300b]).

{88r-2.12}

2.20. **Fact.** 1) If $\mathfrak{K}_i = (K_i, <_i)$ is a (weak) a.e.c., i.e. with $\lambda_i = \text{LST}(K_i, \leq_i)$ where $\lambda_i \geq \aleph_0$ for $i < \alpha$, $i < \alpha \Rightarrow \tau_{K_i} = \tau$ and $K = \bigcap_{i < \alpha} K_i$ and \leq is defined by $M \leq N$ if and only if for

every $i < \alpha$, $M \leq_i N$ then $\mathfrak{K} = (K, \leq)$ is a [weak] a.e.c. with $\text{LST}(\mathfrak{K}) \leq \sum_{i < \alpha} \lambda_i$.

2) Concerning AxI-V, we can omit some of them in the assumption and still get the rest in the conclusion. But for AxVI we need in addition to assume $\text{AxV} + \text{AxIV}_\theta$ for at least one $\theta = \text{cf}(\theta) \leq \sum_{i < \alpha} \lambda_i$.

Proof. Easy. □_{2.20}

{88r-2.13}

2.21. **Example.** Consider the class K of norm spaces over the reals with $M \leq_{\mathfrak{K}} N$ iff $M \subseteq N$ and M is complete inside N . Now $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is a weak a.e.c. with $\text{LST}(\mathfrak{K}) = 2^{\aleph_0}$ and it is as required in 2.17.

3. LIMIT MODELS AND OTHER RESULTS

In this section we introduce various variants of limit models (the most important are the superlimit ones). We prove that if \mathfrak{K} has a superlimit model M^* of cardinality λ for which the λ -amalgamation property fails and $2^\lambda < 2^{\lambda^+}$ then $\dot{I}(\lambda, K) = 2^\lambda$ (see 3.9). We later prove that if $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ is categorical in \aleph_1 then it has model in \aleph_2 see 3.19(2). This finally solves Baldwin's problem (see §0). In fact we prove an essentially more general result on a.e.c. and λ (see 3.12, 3.14).

The reader can read 3.3(1),(1A),(1B) ignore the other definitions, and continue with 3.8(2),(5) and everything from 3.9 (interpreting all variants as superlimits).

You may wonder can we prove the parallel to Baldwin conjecture in λ^+ if $\lambda > \aleph_0$; it is

\otimes_λ if \mathfrak{K} is λ -presentable a.e.c. with $\text{LST}(\mathfrak{K}) = \lambda$, categorical in λ^+ then $K_{\lambda^{++}} \neq \emptyset$.

This is false when $\text{cf}(\lambda) > \aleph_0$.

3.1. **Context.** \mathfrak{K} is an a.e.c. {88r-3.0}

3.2. **Example.** Let λ be given and $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ be defined by {88r-3.0}

$$K = \{(A, <) : (A, <) \text{ a well order of order type } \leq \lambda^+\}$$

$$\leq_{\mathfrak{K}} = \{(M, N) : M, N \in K \text{ and } N \text{ is an end extension of } M\}.$$

Now

- (a) \mathfrak{K} is an abstract elementary class with $\text{LST}(\mathfrak{K}) = \lambda$ and \mathfrak{K} categorical in λ^+
- (b) if λ has cofinality $\geq \aleph_1$ then \mathfrak{K} is λ -presentable (see, e.g., [Sh:c, VII, §5] and history there); by clause (a) it is always $(\lambda, 2^\lambda)$ -presentable,
- (c) \mathfrak{K} has no model of cardinality $> \lambda^+$.

Note that if we are dealing with classes which are categorical (or just simple in some sense), we have a good chance to find limit models and they are useful in constructions. {88r-3.1}

3.3. **Definition.** Let λ be a cardinal $\geq \text{LST}(\mathfrak{K})$. For parts 3) - 7) but not 8), for simplifying the presentation we assume the axiom of global choice (alternatively, we restrict ourselves to models with universe an ordinal $< \lambda^+$).

1) $M \in K_\lambda$ is locally superlimit (for \mathfrak{K}) if:

- (a) for every $N \in K_\lambda$ such that $M \leq_{\mathfrak{K}} N$ there is $M' \in K_\lambda$ isomorphic to M such that $N \leq_{\mathfrak{K}} M'$ and $N \neq M'$
- (b) if $\delta < \lambda^+$ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$ -increasing sequence and $M_i \cong M$ for $i < \delta$ then $\bigcup_{i < \delta} M_i \cong M$.

1A) $M \in K_\lambda$ is globally superlimit if (a) +(b) and

- (c) M is universal in \mathfrak{K}_λ , i.e., any $N \in K_\lambda$ can be $\leq_{\mathfrak{K}}$ -embedded into M .

modified:2015-01-25

(88r) revision:2015-01-23

1B) Just superlimit means globally. Similarly with the other notions below we define the global version as adding clause (c) from (1A) and the default version is the global one. (Note that in the local version we can restrict our class to $\{N \in K_\lambda : M \text{ can be } \leq_{\mathfrak{R}}\text{-embedded into } N\}$ and get the global one).

2) For $\Theta \subseteq \{\mu : \aleph_0 \leq \mu < \lambda, \mu \text{ regular}\}$, $M \in K_\lambda$ is locally (λ, Θ) -superlimit if:

(a) as in part (1) above

(b) if $\langle M_i : i \leq \mu \rangle$ is $\leq_{\mathfrak{R}}$ -increasing, $M_i \cong M$ for $i < \mu$ and $\mu \in \Theta$ then $\cup\{M_i : i < \mu\} \cong M$.

2A) If Θ is a singleton, say $\Theta = \{\theta\}$, we may say that M is locally (λ, θ) -superlimit.

3) Let $S \subseteq \lambda^+$ be stationary. $M \in K_\lambda$ is called locally S -strongly limit or locally (λ, S) -strongly limit when for some function: $\mathbf{F} : K_\lambda \rightarrow K_\lambda$ we have:

(α) for $N \in K_\lambda$ we have $N \leq_{\mathfrak{R}} \mathbf{F}(N)$

(β) if $\delta \in S$ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathfrak{R}}$ -increasing continuous sequence³ in K_λ and $M_0 \cong M$ and $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$, then $M \cong \cup\{M_i : i < \delta\}$

(γ) if $M \leq_{\mathfrak{R}} M_1 \in K_\lambda$ then there is N such that $M_1 <_{\mathfrak{R}} N \in K_\lambda$.

4) Let $S \subseteq \lambda^+$ be stationary. $M \in K_\lambda$ is called locally S -limit or locally (λ, S) -limit if for some function $\mathbf{F} : K_\lambda \rightarrow K_\lambda$ we have:

(α) for every $N \in K_\lambda$ we have $N \leq_{\mathfrak{R}} \mathbf{F}(N)$

(β) if $\langle M_i : i < \lambda^+ \rangle$ is a $\leq_{\mathfrak{R}}$ -increasing continuous sequence of members of K_λ , $M_0 \cong M$, $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$ then for some closed unbounded⁴ subset C of λ^+ ,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

(γ) if $M \leq_{\mathfrak{R}} M_1 \in K_\lambda$ then there is N , $M_1 <_{\mathfrak{R}} N \in K_\lambda$.

5) We define “locally S -weakly limit”, “locally S -medium limit” like “locally S -limit”, “locally S -strongly limit” respectively by demanding that the domain of \mathbf{F} is the family of $\leq_{\mathfrak{R}}$ -increasing continuous sequence of members of $\mathfrak{K}_{<\lambda}$ of length $< \lambda$ and replacing “ $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$ ” by “ $M_{i+1} \leq_{\mathfrak{R}} \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \leq_{\mathfrak{R}} M_{i+2}$ ”. We replace “limit” by “limit⁻” if “ $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$ ”, “ $M_{i+1} \leq_{\mathfrak{R}} \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \leq_{\mathfrak{R}} M_{i+2}$ ” are replaced by “ $\mathbf{F}(M_i) \leq_{\mathfrak{R}} M_{i+1}$ ”, “ $M_i \leq_{\mathfrak{R}} \mathbf{F}(\langle M_j : j \leq i \rangle) \leq_{\mathfrak{R}} M_{i+1}$ ” respectively.

6) If $S = \lambda^+$ then we omit S (in parts (3), (4), (5)).

7) For $\Theta \subseteq \{\mu : \aleph_0 \leq \mu \leq \lambda \text{ and } \mu \text{ is regular}\}$, M is locally (λ, Θ) -strongly limit if M is locally $\{\delta < \lambda^+ : \text{cf}(\delta) \in \Theta\}$ -strongly limit. Similarly for the other notions (where $\Theta \subseteq \{\mu : \mu \text{ regular} \leq \lambda\}$). If we do not write λ we mean $\lambda = \|M\|$. Let locally (λ, θ) -strongly limit mean locally (λ, θ) -strongly limit.

8) We say that $M \in K_\lambda$ is invariantly strong limit when in part (3) we demand that \mathbf{F} is just a subset of $\{(M, N) / \cong : M \leq_{\mathfrak{R}} N \text{ are from } K_\lambda\}$ and in clause (b) of part (3) we

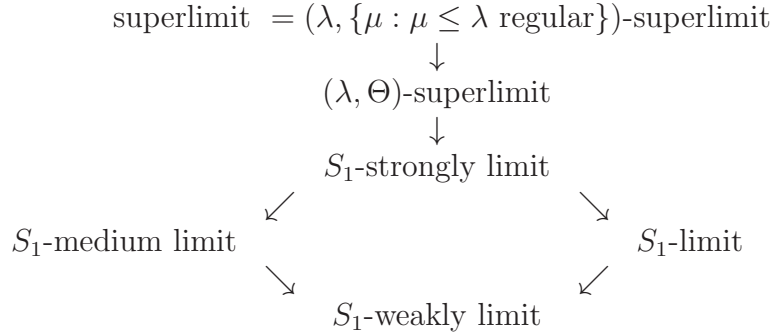
³no loss if we add $M_{i+1} \cong M$, so this simplifies the demand on \mathbf{F} , i.e., only $\mathbf{F}(M')$ for $M' \cong M$ are required

⁴we can use a filter as a parameter

replace “ $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$ ” by “ $(\exists N)(M_{i+1} \leq_{\mathfrak{R}} N \leq_{\mathfrak{R}} M_{i+2} \wedge ((M_{i+1}, N)/\cong) \in \mathbf{F})$ ” but abusing notation we still write $N = \mathbf{F}(M)$ instead $((M, N)/\cong) \in \mathbf{F}$. Similarly with the other notions, so if \mathbf{F} acts on suitable $\leq_{\mathfrak{R}}$ -increasing sequence of models then we use the isomorphic type of $\bar{M} \wedge \langle N \rangle$.

3.4. Obvious implication diagram. For Θ, S_1 as in 3.3(7) and $S_1 \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) \in \Theta\}$ is a stationary subset of λ^+ :

{88r-3.1A}



3.5. Lemma. 0) All the properties are preserved if S is replaced by a subset and if \mathfrak{R} has the λ -JEP, the local and global version in Definition 3.3 are equivalent.

{88r-3.2}

1) If $S_i \subseteq \lambda^+$ for $i < \lambda^+$, $S = \{\alpha < \lambda^+ : (\exists i < \alpha)\alpha \in S_i\}$ and $S_i \cap i = \emptyset$ for $i < \lambda$ then: M is S_i -strongly limit for each $i < \lambda$ if and only if M is S -strongly limit.

2) Suppose $\kappa \leq \lambda$ is regular and $S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \kappa\}$ is a stationary set and $M \in K_\lambda$ then the following are equivalent:

- (a) M is S -strongly limit
- (b) M is $(\lambda, \{\kappa\})$ -strongly limit
- (c) $M \in \mathfrak{K}_\lambda$ is $\leq_{\mathfrak{R}}$ -universal not $<_{\mathfrak{R}}$ -maximal and there is a function $\mathbf{F} : K_\lambda \rightarrow K_\lambda$ satisfying $(\forall N \in K_\lambda)[N \leq_{\mathfrak{R}} \mathbf{F}(N)]$ such that if $M_i \in K_\lambda$ for $i < \kappa$, $[i < j \Rightarrow M_i \leq_{\mathfrak{R}} M_j]$, $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$ and $M_0 \cong M$ then $\bigcup_{i < \kappa} M_i \cong M$.

2A) If $S \subseteq \lambda^+$, $\Theta = \{\text{cf}(\delta) : \delta \in S\}$ then M is S -strongly limit iff clause (c) in part (2) above holds for every $\kappa \in \Theta$.

3) In part (1) we can replace “strongly limit” by “limit”, “medium limit” and “weakly limit”.

4) Suppose $\kappa \leq \lambda$ is regular, $S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \kappa\}$ is a stationary set which belongs to $\bar{I}[\lambda]$ (see 0.7, 0.8 above) and $M \in K_\lambda$. The following are equivalent

- (a) M is S -medium limit in \mathfrak{K}_λ
- (b) $M \in K_\lambda$ is $\leq_{\mathfrak{R}}$ -universal not maximal and there is a function \mathbf{F} from $\bigcup_{\alpha < \kappa} K_\alpha \rightarrow \bigcup_{\alpha < \kappa} K_\alpha$ to K such that
 - (α) for any $\leq_{\mathfrak{R}}$ -increasing $\langle M_i : i \leq \alpha \rangle$ if $M_0 = M$, $\alpha < \kappa$, M_i is $\leq_{\mathfrak{R}}$ -increasing, $M_i \in K_\lambda$, then $M_\alpha \leq_{\mathfrak{R}} \mathbf{F}(\langle M_i : i \leq \alpha \rangle)$

modified:2015-01-25

(88r) revision:2015-01-23

(β) if $\langle M_i : i < \kappa \rangle$ is $\leq_{\mathfrak{R}}$ -increasing, $M_0 = M$, $M_i \in K_\lambda$ and for $i < \kappa$ we have $M_{i+1} \leq_{\mathfrak{R}} \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \leq_{\mathfrak{R}} M_{i+2}$ then $\bigcup_{i < \kappa} M_i \cong M$.

Proof. 0) Trivial.

1) Recall that in Definition 3.3(3), clause (b) we use \mathbf{F} only on M_{i+1} ; (see the proof of (2A) below, second part).

2) For (c) \Rightarrow (a) note that the demands on the sequence are “local”, $M_{i+1} \leq_{\mathfrak{R}} \mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$, (whereas in part (4) they are “global”).

2A) First assume that M is S -strongly limit and let \mathbf{F} witness it. Suppose $\kappa \in \Theta$, so we choose $\delta_\kappa \in S$ with $\text{cf}(\delta_\kappa) = \kappa$ and let $\langle \alpha_i : i < \kappa \rangle$ be increasing continuous with limit δ , $\alpha_0 = 0$, α_{i+1} a successor of a successor ordinal for each $i < \kappa$. We now define \mathbf{F}_κ as follows: to define $\mathbf{F}_\kappa(M)$ we define $\mathbf{F}_{\kappa,\alpha}$ for $\alpha \leq \delta$ by induction on $\alpha \leq \delta$. Let:

- (a) if $\alpha = 0$ then $\mathbf{F}_{\kappa,0}(M) = M$
- (b) if $\alpha = \beta + 1$ then $\mathbf{F}_{\kappa,\alpha}(M) = \mathbf{F}(\mathbf{F}_{\kappa,\beta}(M))$
- (c) if $\alpha \leq \delta$ a limit ordinal then $\mathbf{F}_{\kappa,\alpha}(M) = \cup\{\mathbf{F}_{\kappa,\beta}(M) : \beta < \alpha\}$.

Lastly, let $\mathbf{F}_\kappa(M)$ be $\mathbf{F}_{\kappa,\delta}(M)$.

Now suppose $\langle N_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous, $N_i \in K_\lambda$ and $\mathbf{F}_\kappa(N_{i+1}) \leq_{\mathfrak{R}} N_{i+2}$ for $i < \kappa$ and we should prove $N_\kappa \cong M$. Now we can find $\langle M_j : j < \lambda^+ \rangle$ such that it obeys \mathbf{F} and $M_{\alpha_i} = N_i$ for $i < \kappa$; so clearly we are done.

Second, assume that for each $\kappa \in \Theta$, clause (c) of 3.5(2) holds and let \mathbf{F}_κ exemplify this. Let $\langle \kappa_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ list Θ so $\varepsilon(*) < \lambda^+$ and define \mathbf{F} as follows. For any $M \in \mathfrak{K}$ choose $M_{[\varepsilon]}$ by induction on $\varepsilon \leq \varepsilon(*)$ as follows: $M_{[0]} = M$, $M_{[\varepsilon+1]} = \mathbf{F}_{\kappa_\varepsilon}(M_{[\varepsilon]})$ and for ε limit ordinal let $M_{[\varepsilon]} = \cup\{M_{[\zeta]} : \zeta < \varepsilon\}$. Lastly, let $\mathbf{F}[M] = M_{[\varepsilon(*)]}$. Now check.

3) No new point.

4) First note that (a) \Rightarrow (b) should be clear. Second, we prove that (b) \Rightarrow (a) so let \mathbf{F} witness that clause (b) holds. Let $E, \langle u_\alpha : \alpha < \lambda \rangle$ witness that $S \in \check{I}[\lambda]$, i.e.

- (*)₁ (a) E a club of λ
- (b) $u_\alpha \subseteq \alpha$ and $\text{otp}(u_\alpha) \leq \kappa$ for $\alpha < \lambda$
- (c) if $\alpha \in S \cap E$ then $\alpha = \sup(u_\alpha)$ and $\text{otp}(u_\alpha) = \kappa$
- (d) if $\alpha \in \lambda \setminus S \cap E$ then $\text{otp}(u_\alpha) < \kappa$
- (e) if $\alpha \in u_\beta$ then $u_\alpha = u_\beta \cap \alpha$.

We can add

- (*)₂ (f) if $\beta \in u_\alpha$ then β has the form $3\gamma + 1$.

Let $\langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ list E in increasing order and without loss of generality $\alpha_0 = 0$, $\alpha_{1+\varepsilon}$ is a limit ordinal (note that only the limit ordinals of S count).

To define \mathbf{F}' as required we shall deal with the requirement according to whether $\delta \in S$ is “easy”, i.e. $\delta \notin E$ so $\delta \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}]$ for some $\varepsilon < \lambda^+$ so after α_ε we can “take care of it”, or δ is “hard”, i.e. $\delta \in E$ so we use the $\alpha \in u_\delta$.

We choose $\langle e_\delta : \delta \in S \setminus E \rangle$ such that $\delta \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}] \cap S$ implies $e_\delta \subseteq \delta = \sup(e_\delta)$ and $\min(e_\delta) > \alpha_\varepsilon$, $\text{otp}(e_\delta) = \kappa$, e_δ is closed and $\alpha \in e_\delta \Rightarrow \alpha = \sup(e_\delta \cap \alpha) \vee (\alpha \in \{3\gamma + 2 : \gamma < \delta\})$. If $\delta \in S \cap E$ let e_δ be the closure of u_δ . Let $\langle \gamma_{\delta, \zeta} : \zeta < \kappa \rangle$ list e_δ in increasing order.

We now define a function \mathbf{F}' so let $\langle M_j : j \leq i + 1 \rangle$ be given and let $\alpha_\varepsilon \leq i < \alpha_{\varepsilon+1}$. We fix ε so $(\alpha_\varepsilon, \alpha_{\varepsilon+1})$ and now define $\mathbf{F}'(\langle M_j : j \leq i + 1 \rangle)$ by induction on $i \in [\alpha_\varepsilon, \alpha_{\varepsilon+1})$ assuming that if $\alpha_\varepsilon \leq j' + 1 < i + 1$ then $\mathbf{F}'(\langle M_j : j \leq j' + 1 \rangle) \leq_{\mathfrak{R}} M_{j'+2}$ and further there is $\bar{N}^{j'+1} = \langle N_{j'+1, \xi} : \xi < \alpha_{\varepsilon+1} \rangle$ such that the following holds:

- (*)₃ $\bar{N}^{j'+1}$ is $\leq_{\mathfrak{R}\lambda}$ -increasing continuous, $M_{j'+1} \leq_{\mathfrak{R}} N_{j'+1, 0}$ and $N_{j'+1, \xi} \leq_{\mathfrak{R}\lambda} M_{j'+2}$
- (*)₄ if $\delta \in (S \setminus E) \cap (\alpha_{\varepsilon+1} \setminus \alpha_\varepsilon)$, $j' + 1 = \gamma_{\delta, \zeta}$ (so necessarily $j' + 1 \in (\alpha_\varepsilon, \alpha_{\varepsilon+1})$, $j' + 1 \in \{3\gamma + 2 : \gamma < \lambda\}$, ζ is a successor ordinal) then let $\bar{N}_{\delta, j'}^* = \langle N_{\delta, j', \zeta'}^* : \zeta' \leq \zeta \rangle$ be the following sequence of length $\zeta + 1$, $N_{\delta, j', \zeta'}^*$ is $N_{\gamma_{\delta, \zeta'}, \zeta'}$ if ζ' is a successor ordinal and is $M_{\gamma_{\delta, \zeta'}}$ if ζ' is limit or zero, and we demand $\mathbf{F}'(\langle N_{\delta, j', \zeta'}^* : \zeta' \leq \zeta \rangle) \leq_{\mathfrak{R}} N_{j'+1, \zeta+1}$
- (*)₅ if $j'+1 \in u_\delta$ for some $\delta \in S \cap E$ hence $j'+1 \in \{3\gamma+1 : \gamma < \delta\}$ and $\zeta = \text{otp}(u_{j'+1}) < \kappa$ and f_ε is the one-to-one order preserving function from $\zeta + 1$ onto $\text{cl}(u_{j'+1} \cup \{j'+1\})$ and ζ' is a successor, then $\mathbf{F}'(\langle M_{\alpha_{f_\varepsilon(\zeta')}} : \zeta' \leq \zeta \rangle) \leq_{\mathfrak{R}} M_{\alpha_{\varepsilon+1}}$.

This implicitly defines \mathbf{F}' . Now \mathbf{F}' is as required: $M_i \cong M$ when $i < \lambda$, $\text{cf}(i) = \kappa$ by (*)₄ when $(\exists \varepsilon)(\alpha_\varepsilon < i < \alpha_{\varepsilon+1})$ and by (*)₅ when $(\exists \varepsilon)(i = \alpha_\varepsilon)$. $\square_{3.5}$

{88r-3.3}

3.6. Lemma. *Let T be a first order complete theory, K its class of models and $\leq_{\mathfrak{R}} = \prec_{\mathbb{L}}$.*

- 1) *If λ is regular, M a saturated model of T of cardinality λ , then M is $(\lambda, \{\lambda\})$ -superlimit.*
- 2) *If T is stable, and M a saturated model of T of cardinality λ then M is $(\lambda, \{\mu : \kappa(T) \leq \mu \leq \lambda \text{ and } \mu \text{ is regular}\})$ -superlimit (on $\kappa(T)$ -see [Sh:c, III, §3]). (Note that by [Sh:c] if λ is singular and T has a saturated model of cardinality λ then T is stable and $\text{cf}(\lambda) \geq \kappa(T)$).*
- 3) *If T is stable, λ singular $> \kappa(T)$, M a special model of T of cardinality λ , $S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \text{cf}(\lambda)\}$ is stationary and $S \in \check{I}[\lambda]$ (see above 0.7, 0.8) then M is (λ, S) -medium limit.*

3.7. Remark. See more in [Sh:868].

Proof. 1) Because if M_i is a λ -saturated model of T for $i < \delta$, $\text{cf}(\delta) \geq \lambda$, then $\bigcup_{i < \delta} M_i$ is λ -saturated. Remembering the uniqueness of a λ -saturated model of T of cardinality λ we finish.

2) Use [Sh:c, III, 3.11]: if M_i is a λ -saturated model of T , $\langle M_i : i < \delta \rangle$ increasing $\text{cf}(\delta) \geq \kappa(T)$ then $\bigcup_{i < \delta} M_i$ is λ -saturated.

3) Should be clear by now. $\square_{3.6}$

3.8. **Claim.** 1) If $M_\ell \in K_\lambda$ are S_ℓ -weakly limit and $S_0 \cap S_1$ is stationary, then $M_0 \cong M_1$, provided κ has (λ, λ) -JEP.

2) K has at most one locally weakly limit model of cardinality λ provided K has (λ, λ) -JEP.

3) If $M \in K_\lambda$ then $\{S \subseteq \lambda^+ : M \text{ is } S\text{-weakly limit or } S \text{ not stationary}\}$ is a normal ideal over λ^+ .

Instead “ S -weakly limit”, also “ S -medium limit”, “ S -limit”, “ S -strongly limit” can be used.

4) In Definition 3.3 without loss of generality $\mathbf{F}(N) \cong M$ or $\mathbf{F}(\bar{M}) \cong M$ according to the case (and we can add $N <_{\mathfrak{R}} \mathbf{F}(N)$, etc.)

5) If K is categorical in λ , then the $M \in K_\lambda$ is superlimit provided that $K_{\lambda^+} \neq \emptyset$ (or, what is equivalent, M has a proper $\leq_{\mathfrak{R}}$ -extension).

Proof. Easy.

1) E.g., let \mathbf{F}_ℓ witness that M_ℓ is S_ℓ -weakly limit. We can choose (M_α^0, M_α^1) by induction on α such that: $\langle M_\beta^\ell : \beta \leq \alpha \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous for $\ell = 0, 1$, $M_\alpha^0 \leq_{\mathfrak{R}} M_{\alpha+1}^1$, $M_\alpha^1 \leq_{\mathfrak{R}} M_{\alpha+1}^0$ and $\mathbf{F}_\ell(\langle M_\beta^\ell : \beta \leq \alpha+1 \rangle) \leq M_{\alpha+2}^\ell$. So for some club E_ℓ of λ^+ , $\delta \in S_\ell \cap E_\ell \Rightarrow M_\delta^\ell \cong M_\ell$ for $\ell = 0, 1$. But $S_0 \cap S_1$ is stationary hence there is a limit ordinal $\delta \in S_0 \cap S_1 \cap E_0 \cap E_1$, hence $M_0 \cong M_\delta^0 = M_\delta^1 \cong M_1$ as required. $\square_{3.8}$

{88r-3.5}

3.9. **Theorem.** If $2^\lambda < 2^{\lambda^+}$, $M \in K_\lambda$ superlimit, $S = \lambda^+$ or M is S -weakly limit, S is not small (see Definition 0.6) and M does not have the λ -amalgamation property (in \mathfrak{R}) then $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$, moreover there is no universal member in \mathfrak{R}_{λ^+} and $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow \dot{I}\dot{E}(\lambda^+, K) = 2^{\lambda^+}$, that is there are 2^{λ^+} models $M \in K_{\lambda^+}$ no one $\leq_{\mathfrak{R}}$ -embeddable into another.

{88r-3.5A}

3.10. **Remark.** 0) So in 3.9, if K is categorical in λ then it has λ -amalgamation.

1) We can define a superlimit for a family of models, i.e., when $\mathbf{N} = \{N_t : t \in I\} \subseteq \mathfrak{R}_\lambda$ is superlimit (i.e., if $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{R}}$ -increasing, $i < \delta \Rightarrow M_i \in \mathfrak{R}_\lambda$, δ a limit ordinal $< \lambda^+$, $M_\delta = \cup\{M_i : i < \delta\}$ then $\bigwedge_{i < \delta} \bigvee_{t \in I} M_i \cong N_t \Rightarrow \bigvee_{t \in I} M_\delta \cong N_t$ (and the other variants).

Of course, the family is $\subseteq K_\lambda$ and is not empty. Essentially everything generalizes but in 3.9 the hypothesis should be stronger: the family should satisfy that any member does not have the amalgamation property. E.g. $\mathbf{N} = \mathfrak{R}_\lambda$, (and we can reduce the general case to this by changing \mathfrak{R}). But this complicates the situation, and the gain is not clear, so we do not elaborate this.

2) We can many times (and in particular in 3.9) strengthen “there is no $\leq_{\mathfrak{R}}$ -universal $M \in K_{\lambda^+}$ ” to “there is no $M \in K_\mu$ into which every $N \in K_{\lambda^+}$ can be $\leq_{\mathfrak{R}}$ -embedded” for μ not too large. We need $\neg \text{Unif}(\lambda^+, S, 2, \mu)$, (see [Sh:f, AP, §1]).

Proof. Let \mathbf{F} be as in Definition 3.3(5) for M . We now choose by induction on $\alpha < \lambda^+$, models M_η for $\eta \in {}^\alpha 2$ such that:

- ⊗₁ (i) $M_\eta \in K_\lambda, M_{\langle \rangle} = M,$
- (ii) if $\beta < \alpha$ and $\eta \in {}^\alpha 2$ then $M_{\eta \upharpoonright \beta} \leq_{\mathfrak{K}} M_\eta$
- (iii) if $i + 2 \leq \alpha$ and $\eta \in {}^\alpha 2$, then $(\mathbf{F}(\langle M_{\eta \upharpoonright j} : j \leq i + 1 \rangle)) \leq_{\mathfrak{K}} M_{\eta \upharpoonright (i+2)}$
- (iv) if $\alpha = \beta + 1$ and β non-limit, $\eta \in {}^\alpha 2$, then $M_{\eta \upharpoonright \beta} \neq M_\eta$
- (v) if $\alpha < \lambda$ is a limit ordinal and $\eta \in {}^\alpha 2$ then:
 - (a) $M_\eta = \cup \{M_{\eta \upharpoonright \beta} : \beta < \ell g(\eta)\}$ and
 - (b) if M_η fails the λ -amalgamation property then $M_{\eta \hat{\ } \langle 0 \rangle}, M_{\eta \hat{\ } \langle 1 \rangle}$

cannot be amalgamated over M_η , i.e. for no N do we have:

$$M_\eta \leq_{\mathfrak{K}} N \in K \text{ and } M_{\eta \hat{\ } \langle 0 \rangle}, M_{\eta \hat{\ } \langle 1 \rangle} \text{ can be } \leq_{\mathfrak{K}}\text{-embedded}$$

into N over M_η .

For $\alpha = 0, \alpha$ limit, we have no problem, for $\alpha + 1, \alpha$ limit: if M_η fails the λ -amalgamation property - use its definition, otherwise let $M_{\eta \hat{\ } \langle 1 \rangle} = M_\eta = M_{\eta \hat{\ } \langle 0 \rangle}$; for $\alpha + 1, \alpha$ non-limit - use \mathbf{F} to guarantee clause (iii), and then for clause (iv) use clause (γ) of Definition 3.3(5), i.e., 3.3(4).

Let for $\eta \in {}^{\lambda^+} 2, M_\eta = \bigcup_{\alpha < \lambda^+} M_{\eta \upharpoonright \alpha}$. By changing names we can assume that

- ⊗₁ (vi) for $\eta \in {}^\alpha 2 (\alpha < \lambda^+)$ the universe of M_η is an ordinal $< \lambda^+$ (or even $\subseteq \lambda \times (1 + \ell g(\eta))$ and we could even demand equality).

So (by clause (iv)) for $\eta \in {}^{\lambda^+} 2, M_\eta$ has universe λ^+ .

First, why is there no universal member in \mathfrak{K}_{λ^+} ? If $N \in K_{\lambda^+}$ is universal (by $\leq_{\mathfrak{K}}$, of course), without loss of generality its universe is λ^+ . For $\eta \in {}^{\lambda^+} 2$ as $M_\eta \in K_{\lambda^+}$, there is a $\leq_{\mathfrak{K}}$ -embedding f_η of M_η into N . So f_η is a function from λ^+ to λ^+ . Let $\eta \in {}^{\lambda^+} 2$, by the choice of \mathbf{F} and of $\langle M_{\eta \upharpoonright \alpha} : \alpha < \lambda^+ \rangle$ there is a closed unbounded $C_\eta \subseteq \lambda^+$ such that $\alpha \in S \cap C_\eta \Rightarrow M_{\eta \upharpoonright \alpha} \cong M$, hence $M_{\eta \upharpoonright \alpha}$ fails the λ -amalgamation property. Without loss of generality for $\delta \in C_\eta, M_{\eta \upharpoonright \delta}$ has universe δ . Now by 0.6, if $\langle (f_\rho, C_\rho) : \rho \in {}^{\lambda^+} 2 \rangle$ satisfies that for each $\rho \in {}^{\lambda^+} 2, f_\rho : \lambda^+ \rightarrow \lambda^+$ and $C_\rho \subseteq \lambda^+$ is closed unbounded then for some $\eta \neq \nu \in {}^{\lambda^+} 2$ and $\delta \in C_\eta \cap S$ we have $\eta \upharpoonright \delta = \nu \upharpoonright \delta, \eta(\delta) \neq \nu(\delta)$ and $f_\eta \upharpoonright \delta = f_\nu \upharpoonright \delta$.

[Why? For every $\delta < \lambda^+, \rho \in {}^\delta 2$ and $f : \delta \rightarrow \lambda^+$ we define $\mathbf{c}(\rho, f) \in 2$ as follows: it is 1 iff there is $\nu \in {}^{\lambda^+} 2$ such that $\rho = \nu \upharpoonright \delta \& f = f_\nu \upharpoonright \delta \& \nu(\delta) = 0$ and is 0 otherwise. So some $\eta \in {}^{\lambda^+} 2$ is a weak diamond sequence for the colouring \mathbf{c} and the stationary set S . Now C_η, f_η are well defined and $S' = \{\delta \in S : \delta \text{ limit and } \eta(\delta) = \mathbf{c}(\eta \upharpoonright \delta, f \upharpoonright \delta)\}$ is a stationary subset of λ^+ , so we can choose $\delta \in S' \cap C_\eta$. If $\eta(\delta) = 0$, then $\mathbf{c}(\eta \upharpoonright \delta, f \upharpoonright \delta) = 0$ by the choice of S' but η witness that $\mathbf{c}(\eta \upharpoonright \delta, f \upharpoonright \delta)$ is 1, standing for ν there. If $\eta(\delta) = 1$ there is ν witnessing $\mathbf{c}(\eta \upharpoonright \delta, f_\nu \upharpoonright \delta) = 1$, in particular $\nu(\delta) = 0$, so $\eta, \nu, \eta \upharpoonright \delta$, are as required.]

Now as $\delta \in S \cap C_\eta \subseteq C_\eta$ it follows that $M_{\eta \upharpoonright \delta} \cong M$ hence $M_{\eta \upharpoonright \delta}$ fails the λ -amalgamation property. Also $M_{\eta \upharpoonright \delta}$ has universe δ as $\delta \in C_\eta$ and $M_{\eta \upharpoonright \delta} = M_{\nu \upharpoonright \delta}$ as $\eta \upharpoonright \delta = \nu \upharpoonright \delta$.

So $f_\eta \upharpoonright M_{\eta \upharpoonright \delta} = f_\eta \upharpoonright \delta = f_\nu \upharpoonright \delta = f_\nu \upharpoonright M_{\nu \upharpoonright \delta}$. So $f_\eta \upharpoonright M_{\eta \upharpoonright (\delta+1)}, f_\nu \upharpoonright M_{\nu \upharpoonright (\delta+1)}$ show that $M_{\eta \upharpoonright (\delta+1)}, M_{\nu \upharpoonright (\delta+1)}$, can be amalgamated over $M_{\eta \upharpoonright \delta}$ contradicting clause (v)(b) of the construction, i.e. of \otimes . So there is no $\leq_{\mathfrak{K}}$ -universal $N \in \mathfrak{K}_{\lambda^+}$.

It takes some more effort to get 2^{λ^+} pairwise non-isomorphic models (rather than just quite many).

Case A⁵: There is $M^* \in K_\lambda, M \leq_{\mathfrak{K}} M^*$ such that for every N satisfying $M^* \leq_{\mathfrak{K}} N \in K_\lambda$ there are $N^1, N^2 \in K_\lambda$ such that $N \leq_{\mathfrak{K}} N^1, N \leq_{\mathfrak{K}} N^2$ and N^2, N^1 cannot be $\leq_{\mathfrak{K}}$ -amalgamated over M^* (not just N). In this case we do not need “ M is S -weakly limit”.

We redefine $M_\eta, \eta \in {}^\alpha 2, \alpha < \lambda^+$ such that:

$$\otimes_2 \text{ (a) } \nu \triangleleft \eta \in {}^\alpha 2 \Rightarrow M_\nu \leq_{\mathfrak{K}} M_\eta \in K_\lambda;$$

$$\text{(b) if } \alpha = 0, M_{\langle \rangle} = M^*;$$

$$\text{(c) if } \alpha \text{ limit and } \eta \in {}^\alpha 2 \text{ then } M_\eta = \bigcup_{\beta < \alpha} M_{\eta \upharpoonright \beta};$$

$$\text{(d) if } \eta \in {}^\beta 2, \alpha = \beta + 1, \text{ use the assumption for } N = M_\eta, \text{ now}$$

obviously the (N^1, N^2) there satisfies $N^1 \neq N$ and $N^2 \neq N$, so we

can have $M_\eta <_{\mathfrak{K}} M_{\eta \hat{\ } \langle 1 \rangle} \in K_\lambda, M_\eta <_{\mathfrak{K}} M_{\eta \hat{\ } \langle 0 \rangle} \in K_\lambda$, such that

$M_{\eta \hat{\ } \langle 0 \rangle}, M_{\eta \hat{\ } \langle 1 \rangle}$ cannot be amalgamated over M^* .

Obviously, the models $M_\eta = \bigcup_{\alpha < \lambda^+} M_{\eta \upharpoonright \alpha}$, for $\eta \in {}^{\lambda^+} 2$ are pairwise non-isomorphic over M^*

and by 0.4 as $2^\lambda < 2^{\lambda^+}$ we finish proving $\dot{I}(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$.

Note also that for each $\eta \in {}^{\lambda^+} 2$ the set $\{\nu \in {}^{\lambda^+} 2 : M_\nu \text{ can be } \leq_{\mathfrak{K}}\text{-embedded into } M_\eta\}$ has cardinality $\leq |\{f : f \text{ a } \leq_{\mathfrak{K}}\text{-embedding of } M^* \text{ into } M_\eta\}| \leq 2^\lambda$. So if $(2^\lambda)^+ < 2^{\lambda^+}$, then by Hajnal free subset theorem ([Haj62]), there are 2^{λ^+} models $M_\eta \in K_{\lambda^+}(\eta \in {}^{\lambda^+} 2)$ no one $\leq_{\mathfrak{K}}$ -embeddable into another. Case B: Not Case A.

Now we return to the first construction, but we can add

(vii) if $\eta \in {}^{(\alpha+1)} 2$, then if $M_\eta \leq_{\mathfrak{K}} N^1, N^2$ both in K_λ , then N^1, N^2 can be $\leq_{\mathfrak{K}}$ -amalgamated over $M_{\eta \upharpoonright \alpha}$.

As $\{W \subseteq \lambda^+ : W \text{ is small}\}$ is a normal ideal (see 0.6), (and it is on a successor cardinal) it is well known that we can find λ^+ pairwise disjoint non-small $S_\zeta \subseteq S$ for $\zeta < \lambda^+$. We define a colouring (= function) \mathbf{c} :

⁵we can make it a separate claim

⊗₃ (a) $\mathbf{c}(\eta, \nu, f)$ will be defined iff for some limit ordinal $\delta < \lambda^+$, $\eta \in {}^{\delta}2$, $\nu \in {}^{\delta}2$

and f is a function from δ to λ^+

(b) $\mathbf{c}(\eta, \nu, f) = 1$ iff the triple (η, ν, f) belongs to the domain of \mathbf{c} (i.e., is

as in (a)) and M_η, M_ν have universe δ , f is a $\leq_{\bar{\kappa}}$ -embedding of M_η

into M_ν and for some $\rho, \nu^{\wedge} < 0 > \triangleleft \rho \in {}^{\lambda^+}2$ the function f can be

extended to a $\leq_{\bar{\kappa}}$ -embedding of $M_{\eta^{\wedge} < 0 >}$ into M_ρ

(c) $\mathbf{c}(\eta, \nu, f)$ is zero iff it is defined but is $\neq 1$.

For each ζ , as S_ζ is not small, by simple coding, for every $\zeta < \lambda^+$ there is $h_\zeta : S_\zeta \rightarrow \{0, 1\}$ such that:

(*) $_\zeta$ for every $\eta \in {}^{\lambda^+}2, \nu \in {}^{\lambda^+}2$ and $f : \lambda^+ \rightarrow \lambda^+$, for a stationary set of $\delta \in S_\zeta$

$$\mathbf{c}(\eta \upharpoonright \delta, \nu \upharpoonright \delta, f \upharpoonright \delta) = h_\zeta(\delta).$$

Now for every $W \subseteq \lambda^+$ we define $\eta_W \in {}^{\lambda^+}2$ as follows:

$\eta_W(\alpha)$ is $h_\zeta(\alpha)$, if $\zeta \in W$ and $\alpha \in S_\zeta$ (note that there is at most one ζ)

$\eta_W(\alpha)$ is zero if there is no such ζ .

Now we can show (chasing the definitions) that

⊗₄ if $W(1), W(2) \subseteq \lambda^+, W(1) \not\subseteq W(2)$, then $M_{\eta_{W(1)}}$ cannot be $\leq_{\bar{\kappa}}$ -embedded into $M_{\eta_{W(2)}}$.

This clearly suffices.

Why is ⊗₄ true? Suppose $W(1) \not\subseteq W(2)$, let $\zeta \in W(1) \setminus W(2)$ and toward contradiction let f be a $\leq_{\bar{\kappa}}$ -embedding of $M_{\eta_{W(1)}}$ into $M_{\eta_{W(2)}}$, so $E = \{\delta : M_{\eta_{W(1)} \upharpoonright \delta}, M_{\eta_{W(2)} \upharpoonright \delta} \text{ have universe } \delta \text{ and } f \upharpoonright \delta \text{ is a } \leq_{\bar{\kappa}}\text{-embedding of } M_{\eta_{W(1)} \upharpoonright \delta} \text{ into } M_{\eta_{W(2)} \upharpoonright \delta}\}$ is a club of λ^+ . Hence by the choice of \mathbf{c} and h_ζ there is $\delta \in E \cap S_\zeta$ such that

⊠ $\mathbf{c}(\eta_{W(1)} \upharpoonright \delta, \eta_{W(2)} \upharpoonright \delta, f \upharpoonright \delta) = h_\zeta(\delta)$ and $M_{\eta_{W(1)} \upharpoonright \delta}$ is not an amalgamation base.

Now the proof splits to two cases. Case 1: $h_\zeta(\delta) = 0$.

So $\eta_{W(1)}(\delta) = 0 = \eta_{W(2)}(\delta)$ and by clause (b) of ⊗₃ above, i.e., the definition of \mathbf{c} we have the objects $\eta_{W(1)}, \eta_{W(2)}, f \upharpoonright M_{\eta_{W(1)}^{\wedge} < 0 >} = f \upharpoonright M_{\eta_{W(1)} \upharpoonright (\delta+1)}$ witness that $\mathbf{c}(\eta_{W(1)} \upharpoonright \delta, \eta_{W(2)} \upharpoonright \delta, f \upharpoonright \delta) = 1$, contradiction. Case 2: $h_\zeta(\delta) = 1$.

So $\eta_{W(1)}(\delta) = 1, \eta_{W(2)}(\delta) = 0, \mathbf{c}(\eta_{W(1)} \upharpoonright \delta, \eta_{W(2)} \upharpoonright \delta, f \upharpoonright \delta) = 1$. By the definition of \mathbf{c} , we can find ν such that $(\eta_{W(2)} \upharpoonright \delta)^{\wedge} < 0 > \trianglelefteq \nu \in {}^{\lambda^+}2$ and a $\leq_{\bar{\kappa}}$ -embedding g of $M_{(\eta_{W(1)} \upharpoonright \delta)^{\wedge} < 0 >}$ into M_ν .

For some $\alpha \in (\delta, \lambda^+)$, f embeds $M_{\eta_{W(1)} \upharpoonright (\delta+1)} = M_{(\eta_{W(1)} \upharpoonright \delta)^{\wedge} < 1 >}$ into $M_{\eta_{W(2)} \upharpoonright \alpha}$ and g embeds $M_{(\eta_{W(1)} \upharpoonright \delta)^{\wedge} < 0 >}$ into $M_{\nu \upharpoonright \alpha}$.

As $\eta_{W(2)} \upharpoonright \delta^{\wedge} < 0 > \triangleleft \nu \upharpoonright \alpha$ and $\eta_{W(2)} \upharpoonright \delta^{\wedge} < 0 > \triangleleft \eta_{W(2)} \upharpoonright \alpha$ by clause (vii) above there are f_1, g_1 and $N \in K_\lambda$ such that

- (a) $M_{\eta_{W(2)} \upharpoonright \delta} \leq_{\mathfrak{K}} N$
- (b) f_1 is a $\leq_{\mathfrak{K}}$ -embedding of $M_{\eta_{W(2)} \upharpoonright \alpha}$ into N over $M_{\eta_{W(2)} \upharpoonright \delta}$
- (c) g_1 is a $\leq_{\mathfrak{K}}$ -embedding of $M_{\nu \upharpoonright \alpha}$ into N over $M_{\eta_{W(2)} \upharpoonright \delta}$.

So

- (b)* $f_1 \circ f$ is a $\leq_{\mathfrak{K}}$ -embedding of $M_{(\eta_{W(1)} \upharpoonright \delta)^{\prec 1 \succ}}$ into N
- (c)* $g_1 \circ g$ is a $\leq_{\mathfrak{K}}$ -embedding of $M_{(\eta_{W(1)} \upharpoonright \delta)^{\prec 0 \succ}}$ into N
- (d)* $f_1 \circ f, g_1 \circ g$ extend $f \upharpoonright \delta : M_{\eta_{W(1)} \upharpoonright \delta} \rightarrow N$ (both).

So together we get a contradiction to assumption $(*)_1(d)$. □_{3.9}

{88r-3.6}

3.11. Theorem. 1) Assume one of the following cases occurs:

- (a)₁ \mathfrak{K} is PC_{\aleph_0} (hence $\text{LST}(\mathfrak{K}) = \aleph_0$) and $1 \leq \dot{I}(\aleph_1, \mathfrak{K}) < 2^{\aleph_1}$
or

- (a)₂ \mathfrak{K} has models of arbitrarily large cardinality, $\text{LST}(\mathfrak{K}) = \aleph_0$ and $\dot{I}(\aleph_1, \mathfrak{K}) < 2^{\aleph_1}$.

Then there is an a.e.c. \mathfrak{K}_1 such that

- (A) $M \in K_1 \Rightarrow M \in K$ and $M \leq_{\mathfrak{K}_1} N \Rightarrow M \leq_{\mathfrak{K}} N$ and $\text{LST}(\mathfrak{K}_1) = \text{LST}(\mathfrak{K}) (= \aleph_0)$
- (B) if K has models of arbitrarily large cardinality then so does K_1
- (C) \mathfrak{K}_1 is PC_{\aleph_0}
- (D) $(K_1)_{\aleph_1} \neq \emptyset$
- (E) all models of K_1 are $\mathbb{L}_{\infty, \omega}$ -equivalent and $M \leq_{\mathfrak{K}_1} N \Leftrightarrow M \prec_{\mathbb{L}_{\infty, \omega}} N \& M \leq_{\mathfrak{K}} N$ and K_1 is categorical in \aleph_0 and $M_* \in (K_1)_{\aleph_0} \Rightarrow K_1 = \{N \in K : N \equiv_{\mathbb{L}_{\infty, \omega}(\tau_K)} M_*\}$
- (F) if \mathfrak{K} is categorical in \aleph_1 then $(K_1)_{\lambda} = K_{\lambda}$ for every $\lambda > \aleph_0$; moreover $\leq_{\mathfrak{K}_1} = \leq_{\mathfrak{K}} \upharpoonright (K_1)_{\geq \aleph_1}$.

2) If in (1) we add $\text{LST}(\mathfrak{K})$ names to formulas in $\mathbb{L}_{\infty, \omega}$ (i.e. to a set of representations up to equivalence) then we can assume each member of K is \aleph_0 -sequence-homogeneous. The vocabulary remains countable, in fact, for some countable first order theory T , the models of K are the atomic models of T (in the first order sense) and $\leq_{\mathfrak{K}}$ becomes \subseteq (being a submodel).

Proof. Like [Sh:48, 2.3,2.5] (using 2.20 here for $\alpha = 2$). E.g. why, if K is categorical in \aleph_1 then $\leq_{\mathfrak{K}_1} = \leq_{\mathfrak{K}} \upharpoonright (K_1)_{\geq \aleph_1}$? We have to prove that if $M \leq_{\mathfrak{K}} N$ are uncountable then $M \prec_{\mathbb{L}_{\infty, \omega}(\tau_K)} N$. But there is $M_* \in K_{\aleph_0}$ such that $K_1 = \{M' \in K : M' \equiv_{\mathbb{L}_{\infty, \omega}} M_*\}$ and $(K_1)_{\aleph_1} = K_{\aleph_1} \neq \emptyset$, so it suffices to prove $M \prec_{\mathbb{L}_{\omega_1, \omega}(T)} N$, so assume this is a counterexample so for some $\varphi(x, \bar{y}) \in \mathbb{L}_{\omega_1, \omega}(\tau)$ and $\bar{a} \in {}^{\ell g(\bar{y})} M, b \in N$ we have $N \models \varphi[b, \bar{a}]$ but for no $b' \in M$ do we have $N \models \varphi[b', \bar{a}]$ and without loss of generality the quantifier depth of $\varphi(x, \bar{y}), \gamma$ is minimal (for all such pairs (M, N)). Let $\Delta_{\gamma} = \{\psi(\bar{z}) \in \mathbb{L}_{\omega_1, \omega}(\tau_K) : \psi \text{ has quantifier depth } \leq \gamma\}$ hence $M' \leq_{\mathfrak{K}} N', M' \in K_{> \aleph_0} \Rightarrow M' \prec_{\Delta_{\gamma}} N'$. Also without loss of generality $\|M\| = \|N\| = \aleph_1$. Now choose $M_{\alpha} \in K_{\aleph_1}$ by induction on $\alpha < \omega_2$, which is $\leq_{\mathfrak{K}}$ -increasing continuous (hence $\prec_{\Delta_{\gamma}}$ increases) and for each α there is an isomorphism f_{α} from N onto

$M_{\alpha+1}$ mapping M onto M_α , recalling the categoricity. By Fodor lemma for some $\alpha < \beta$ we have $f_\alpha(\bar{a}) = f_\beta(\bar{a})$, so $f_\beta^{-1}(f_\alpha(b))$ contradict the choice of $\varphi(x, \bar{y}), b, \bar{a}$. $\square_{3.11}$

We arrive to the main theorem of this section.

{88r-3.7}

3.12. Theorem. *Suppose \mathfrak{K} and λ satisfy the following conditions:*

- (A) \mathfrak{K} has a superlimit member M^* of cardinality $\lambda, \lambda \geq \text{LST}(\mathfrak{K})$, (if K is categorical in λ , then by assumption (B) below there is such M^* ; really invariantly λ^+ -strongly limit suffice if (d) of (*) of 3.13(2) below holds, see Definition 3.3)
- (B) \mathfrak{K} is categorical in λ^+
- (C) (α) \mathfrak{K} is $\text{PC}_{\aleph_0}, \lambda = \aleph_0$ or
- (β) $\mathfrak{K} = \text{PC}_\lambda, \lambda = \beth_\delta, \text{cf}(\delta) = \aleph_0$ or
- (γ) $\lambda = \aleph_1, \mathfrak{K}$ is PC_{\aleph_0} or
- (δ) \mathfrak{K} is $\text{PC}_\mu, \lambda \geq \beth_{(2^\mu)^+}$; not useful for 3.12, still it too implies $(*)_{\lambda, \mu}$ in 3.13.

Then K has a model of cardinality λ^{++} .

{88r-3.7A}

3.13. Remark. 1) If $\lambda = \aleph_0$ we can wave hypothesis (A) by the previous theorem 3.11.

2) Hypothesis (C) can be replaced by (giving a stronger theorem):

- $(*)_{\lambda, \mu}$ (a) \mathfrak{K} is PC_μ and
- (b) any $\psi \in \mathbb{L}_{\mu^+, \omega}$ which has a model M of order-type $\lambda^+, |P^M| = \lambda$, has a non-well-ordered model N of cardinality λ
- (c) $\{M \in K_\lambda : M \cong M^*\}$ is PC_μ (among models in K_λ) and
- (d) for some \mathbf{F} witnessing “ M^* is invariantly λ -strongly limit”, that is the class $\{(M, \mathbf{F}(M)) : M \in K_\lambda\}$ is PC_μ (if M^* is superlimit this clause is not required as $\mathbf{F} =$ the identity on K_λ is O.K.)

3) It is well known, see e.g. [Sh:c, VII,§5] that hypothesis (C) implies $(*)_{\lambda, \mu}$ from part (2), see more [GrSh:259].

Proof. By 3.13(3) we can assume $(*)_{\lambda, \mu}$ from 3.13(2).

Stage a: It suffices to find $N_0 \leq_{\mathfrak{K}} N_1, \|N_0\| = \lambda^+, N_0 \neq N_1$.

Why? We define by induction on $\alpha < \lambda^{++}$ a model $N_\alpha \in K_{\lambda^+}$ such that $\beta < \alpha$ implies $N_\beta \leq_{\mathfrak{K}} N_\alpha$ and $N_\beta \neq N_\alpha$. Clearly N_0, N_1 are defined (without loss of generality $\|N_1\| = \lambda^+$ as $\lambda \geq \text{LST}(\mathfrak{K})$, also otherwise we already have the desired conclusion), for limit $\delta < \lambda^{++}$ the model $\bigcup_{\alpha < \delta} N_\alpha$ is as required. For $\alpha = \beta + 1$, by the λ^+ -categoricity, N_0 is isomorphic to N_β , say by f and we define $N_{\beta+1}$ such that f can be extended to an isomorphism from N_1 onto $N_{\beta+1}$, so clearly $N_{\beta+1}$ is as required. Now $\bigcup_{\alpha < \lambda^{++}} N_\alpha \in K_{\lambda^{++}}$ is as required. Hence

modified:2015-01-25

(88r) revision:2015-01-23

the following theorem completes the proof of 3.12 (use \mathbf{F} = the identity for the superlimit case). $\square_{3.12}$

We can find $N_0 \leq_{\mathfrak{K}} N_1, N_0 \neq N_1$ such that $N_0, N_1 \in K_{\lambda^+}^{\mathbf{F}}$, when the following clauses hold:

3.14. **Theorem.** *Suppose the following clauses:*

- (A) \mathfrak{K} has an invariantly λ -strongly limit member M^* of cardinality λ , as exemplified by $\mathbf{F} : K_{\lambda} \rightarrow K_{\lambda}$ and \mathfrak{K}_{λ} has the JEP (see Definition 3.3)
- (B) $\dot{I}(\lambda^+, K_{\lambda^+}) < 2^{\lambda^+}$ or even just $\dot{I}(\lambda^+, K_{\lambda^+}^{\mathbf{F}}) < 2^{\lambda^+}$ (or just $\dot{I}\dot{E}(\lambda^+, K_{\lambda^+}^{\mathbf{F}}) < 2^{\lambda^+}$ (see below))
- (C) \mathfrak{K} is a PC_{μ} class, as well as \mathbf{F} , i.e., K' is PC_{μ} where K' is a class closed under an isomorphism of $(\tau_{\mathfrak{K}} \cup \{P\})$ -models, P a unary predicate such that $K'_{\lambda} = \{(N, M) : N = \mathbf{F}(M)\}$
- (D) $\mu = \lambda = \aleph_0$ or $\mu = \lambda = \beth_{\delta}, \text{cf}(\delta) = \aleph_0$ or $\mu = \aleph_0, \lambda = \aleph_1$ or just $(*)_{\lambda, \mu}(c)$ from 3.13(2)
- (E) K categorical in λ or at least there is $\psi \in \mathbb{L}_{\omega_1, \omega}(\tau^+)$ such that $(M^* / \cong) = \{M \upharpoonright \tau_{\mathfrak{K}} : M \models \psi, \|M\| = \lambda\}$.

where

3.15. **Definition.** Assume $\mathbf{F} : K_{\lambda} \rightarrow K_{\lambda}$ satisfies $M \leq_{\mathfrak{K}} \mathbf{F}(M)$ for $M \in K_{\lambda}$ or more generally $\mathbf{F} \subseteq \{(M, N) : M \leq_{\mathfrak{K}} N \text{ are from } K_{\lambda}\}$ satisfies $(\forall M \in K_{\lambda})(\exists N)((M, N) \in \mathbf{F})$ or just $(\forall M \in K_{\lambda})(\exists N_0, N_1)[(N_0, N_1) \in \mathbf{F} \wedge M \leq_{\mathfrak{K}} N_0 \leq_{\mathfrak{K}} N_1]$. Then we let $K_{\lambda^+}^{\mathbf{F}} := \{\bigcup_{i < \lambda^+} M_i : M_i \in K_{\lambda}, \langle M_i : i < \lambda^+ \rangle \text{ is } \leq_{\mathfrak{K}}\text{-increasing continuous not eventually constant and } \mathbf{F}(M_{i+1}) \leq_{\mathfrak{K}} M_{i+2} \text{ or } (M_{i+1}, M_{i+2}) \in \mathbf{F}\} \text{ for } i < \lambda$.

3.16. **Remark.** 1) As the sequence in the definition of $K_{\lambda^+}^{\mathbf{F}}$ is $\leq_{\mathfrak{K}}$ -increasing and the sequence is not eventually constant (which follows if $(M, N) \in \mathbf{F} \Rightarrow M \neq N$), necessarily $K_{\lambda^+}^{\mathbf{F}} \subseteq \mathfrak{K}_{\lambda^+}$.

2) Theorem 3.14 is good for classes which are not exactly a.e.c., see, e.g., 3.19.

Considering $K_{\lambda^+}^{\mathbf{F}}$ we may note that the proofs of some earlier claims give more. In particular (before proving 3.14), similarly to 3.9:

3.17. **Claim.** *Assume that*

- (a) $2^{\lambda} < 2^{\lambda^+}$
- (b) \mathfrak{K} is an a.e.c. and $\text{LST}(\mathfrak{K}) \leq \lambda$
- (c) $M \in K_{\lambda}$ is S -weakly limit, S not small (see Definition 0.6)
- (d) M does not have the amalgamation property in \mathfrak{K} (= is an amalgamation base)
- (e) \mathbf{F} is as in 3.15.

{88r-3.8}

{88r-3.8.1}

{88r-3.8.3}

{88r-3.8.4}

modified: 2015-01-25

revision: 2015-01-23

(88r)

Then $\dot{I}(\lambda^+, K_{\lambda^+}^{\mathbf{F}}) = 2^{\lambda^+}$.

Proof. To avoid confusion rename \mathbf{F} of clause (e) as \mathbf{F}_1 , and choose \mathbf{F}_2 which exemplifies “ M is S -weakly limit”, i.e., as in Definition 3.3(5). Now we define \mathbf{F}' with the same domain as \mathbf{F}_2 by $\mathbf{F}'(\langle M_j : j \leq i \rangle) = \mathbf{F}_1(\mathbf{F}_2(\langle M_j : j \leq i \rangle))$, and continue as in the proof of 3.9 noting that \mathbf{F}' works as well there.

The sequence of models $\langle M_\eta : \eta \in \lambda^+ 2 \rangle$ we got there are from $K_{\lambda^+}^{\mathbf{F}_1}$ (so witness that $\dot{I}(\lambda^+, K_{\lambda^+}^{\mathbf{F}_1}) = 2^{\lambda^+}$) because:

- (*) if the sequence $\langle M_\alpha : \alpha < \lambda^+ \rangle, M_\alpha \in \mathfrak{K}_\lambda$ for $\alpha < \lambda^+$ is $\leq_{\mathfrak{K}}$ -increasing continuous and $\mathbf{F}'(\langle M_j : j \leq i+1 \rangle) \leq_{\mathfrak{K}} M_{i+2}$ then $\cup\{M_\alpha : \alpha < \lambda^+\} \in K_{\lambda^+}^{\mathbf{F}_1}$.

□_{3.17}

Also similarly to 3.11 we can prove:

3.18. Claim. *Assume \mathfrak{K} is a PC_{\aleph_0} and \mathbf{F} a PC_{\aleph_0} is as in 3.15. If $1 \leq \dot{I}(\aleph_1, K_{\aleph_1}^{\mathbf{F}}) < 2^{\aleph_1}$ then the conclusion of 3.11 above holds.*

{88r-3.8.

Proof of 3.14. (Hence of 3.12). The reader may do well to read it with \mathbf{F} = the identity in mind.

Stage b: We now try to find N_0, N_1 as mentioned in stage (a) above by approximations of cardinality λ . A triple will denote here (M, N, a) satisfying $M, N \cong M^*$ (see hypothesis (A)), $M \leq_{\mathfrak{K}} N$ and $a \in N \setminus M$. Let $<$ be the following partial order among this family of triples: $(M, N, a) < (M', N', a')$ if $a = a', N \leq_{\mathfrak{K}} N', M \leq_{\mathfrak{K}} M', M \neq M'$ and moreover $(\exists N'')[N \leq_{\mathfrak{K}} N'' \& \mathbf{F}(N'') \leq_{\mathfrak{K}} N']$ and $(\exists M'')[M \leq_{\mathfrak{K}} M'' \& \mathbf{F}(M'') \leq_{\mathfrak{K}} M']$. (It is tempting to omit a and require $M = M' \cap N$, but this apparently does not work as we do know if disjoint amalgamation \mathfrak{K}_{\aleph_0} exist).

We first note that there is at least one triple (as M^* has a proper elementary extension which is isomorphic to it, because it is a limit model by clause (A) of the assumption).

Stage c: We show that if there is no maximal triple, our conclusions follows.

We choose by induction on α a triple (M_α, N_α, a) increasing by $<$. For $\alpha = 0$ see the end of previous stage, for $\alpha = \beta + 1$, we can define (M_α, N_α, a) by the hypothesis of this stage. For limit $\delta < \lambda^+$, (M_δ, N_δ, a) will be $(\bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha, a)$ (notice $M_\delta \leq_{\mathfrak{K}} N_\delta$ by AxIV of 1.2 and M_δ, N_δ are isomorphic to M^* by the choice of \mathbf{F} and the definition of order on the family of triples). Now similarly $M = \bigcup_{\alpha < \lambda^+} M_\alpha \leq_{\mathfrak{K}} N = \bigcup_{\alpha < \lambda^+} N_\alpha$ are both from $\mathfrak{K}_{\lambda^+}^{\mathbf{F}}$ and the element a exemplifies $M \neq N$, so by Stage (a) we finish.

Recall

- ⊗ if (M, N, a) is a maximal triple then there is no triple (M', N', a) such that $M' \leq_{\mathfrak{R}} N', M <_{\mathfrak{R}} M', N \leq_{\mathfrak{R}} N', a \in N' \setminus M'$ and $(\exists M'')(M \leq_{\mathfrak{R}} M'' \leq_{\mathfrak{R}} \mathbf{F}(M'') \leq_{\mathfrak{R}} M')$ and $(\exists N'')(N \leq_{\mathfrak{R}} N'' \leq_{\mathfrak{R}} \mathbf{F}(N'') \leq_{\mathfrak{R}} N')$.

Stage d: There are $M_i \cong M^*$ for $i \leq \omega$ such that $[i < j \leq \omega \Rightarrow M_j <_{\mathfrak{R}} M_i], i < \omega \Rightarrow \mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_i$ and $|M_\omega| = \bigcap_{n < \omega} |M_n|$ and note that M_i is λ^+ -strongly limit.

This stage is dedicated to proving this statement. As M^* is superlimit (or just strongly limit), there is an $\leq_{\mathfrak{R}}$ -increasing continuous sequence $\langle M_i : i < \lambda^+ \rangle, M_i \cong M^*$ and $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$. (Note that this is true also for limit models as we can restrict ourselves to a club of i 's). So without loss of generality $\bigcup_{i < \lambda^+} M_i$ has universe λ^+ , M_0 has

universe λ .

Define a model \mathfrak{B} .

Its universe is λ^+ . Relations and Functions:

- (a) those of $\bigcup_{i < \lambda^+} M_i$
- (b) R -two place: aRi if and only if $a \in M_i$
- (c) P (monadic relation) $P = \lambda$ which is the universe of M_0
- (d) g , a two-place function such that for each $i, g(i, -)$ is an isomorphism from M_0 onto M_i
- (e) $<$ (two-place relation) - the usual ordering (on the ordinals $< \lambda^+$)
- (f) relations with parameter i witnessing $M_i \leq_{\mathfrak{R}} \bigcup_{j < \lambda^+} M_j$ (we can instead make functions witnessing $M \in K$ as in 1.10 (the strong version) and have: each M_i is closed under them))
- (g) relations with parameter i witnessing each $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{R}} M_{i+2}$ and $M_{i+1} \neq M_{i+2}$ (including $(M_{i+1}, \mathbf{F}(M_{i+1})) \in \mathbf{F}$)
- (h) if $\mu = \lambda$, also individual constant for each $a \in M_0$.

Let $\psi \in \mathbb{L}_{\mu^+, \omega}$ describe this, in particular for clauses (f), (g) use clause (C) of the assumptions. So ψ has a non-well ordered model $\mathfrak{B}^*, |P^{\mathfrak{B}^*}| = \lambda$ (by clause (D) of the assumption see 3.13(2)+(3)). So let

$$\mathfrak{B}^* \models "a_{n+1} < a_n" \text{ for } n < \omega.$$

Let for $a \in \mathfrak{B}^*, A_a = \{x \in \mathfrak{B}^* : \mathfrak{B}^* \models xRa\}$

$$M_a = (\mathfrak{B}^* \upharpoonright \tau_{\mathfrak{R}}) \upharpoonright A_a.$$

Easily $M_a \leq_{\mathfrak{R}} (\mathfrak{B}^* \upharpoonright \tau_{\mathfrak{R}})$ (use clause (f)) and $\|M_a\| = \lambda$. In fact M_a is superlimit or just isomorphic to M^* if $\mu = \lambda$, as ψ includes the diagram of $M_0 = M^*$, having names for

all members, and if $\mu < \lambda$ see assumption (E). So $M_{a_n} \leq_{\mathfrak{R}} \mathfrak{B}^* \upharpoonright \tau_{\mathfrak{R}}, M_{a_{n+1}} \subseteq M_{a_n}$ hence $M_{a_{n+1}} \leq_{\mathfrak{R}} M_{a_n}$ by Ax V. Let $M_n := M_{a_n}$. Let $I = \{b \in \mathfrak{B}^* : \bigwedge_{n < \omega} [\mathfrak{B}^* \models b < a_n]\}$.

Also as for $b \in I, M_b <_{\mathfrak{R}} \mathfrak{B}^* \upharpoonright \tau_{\mathfrak{R}}$ and $M_{b_1} <_{\mathfrak{R}} M_{b_2}$ for $b_1 <_{\mathfrak{B}^*} b_2$, by Ax IV clearly $M_\omega := (\mathfrak{B}^* \upharpoonright (\tau_{\mathfrak{R}})) \upharpoonright \bigcup_{b \in I} A_b$ satisfies $M_\omega \leq_{\mathfrak{R}} \mathfrak{B}^* \upharpoonright \tau_{\mathfrak{R}}$ hence $M_\omega \leq_{\mathfrak{R}} M_n$ for $n < \omega$. Obviously

$M_\omega \subseteq \bigcap_{n < \omega} M_n$ and equality holds as ψ guarantee

(*) for every $y \in \mathfrak{B}^*$ there is a minimal $x \in \mathfrak{B}^*$ such that $y \in M_x$.

As each M_b is isomorphic to M^* , of cardinality λ , also M_ω is. Stage e: Suppose that there

is a maximal triple, then we shall show $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ and moreover $\dot{I}(\lambda^+, K_{\lambda^+}^{\mathbf{F}}) = 2^{\lambda^+}$, and so we shall get a contradiction to assumption (B).

So there is a maximal triple (M^0, N^0, a) . Hence by the uniqueness of the limit model for each $M \in K_\lambda$ which is isomorphic to M^* hence to M^0 there are N, a satisfying $M \leq_{\mathfrak{R}} N \cong M^* \in K_\lambda, a \in N \setminus M$ such that: if $M <_{\mathfrak{R}} M' \leq_{\mathfrak{R}} N' \in \mathfrak{K}_\lambda, N <_{\mathfrak{R}} N', (\exists M'')(M \leq_{\mathfrak{R}} M'' \leq_{\mathfrak{R}} \mathbf{F}(M'') \leq_{\mathfrak{R}} M' \cong M^*)$ and $(\exists N'')(N \leq_{\mathfrak{R}} N'' \leq_{\mathfrak{R}} \mathbf{F}(N'') \leq_{\mathfrak{R}} N' \cong M^*)$ then $a \in M'$. (That is, in some sense a is algebraic over M). We can waive $(\exists N'')(N \leq_{\mathfrak{R}} N'' \leq_{\mathfrak{R}} \mathbf{F}(N'') \leq_{\mathfrak{R}} N' \cong M^*)$ as by the definition of strongly limit there is $N'_* \cong M^*$ such that $\mathbf{F}(N') \leq_{\mathfrak{R}} N'_*$. On the other hand by Stage d

(*)₁ for each $M \in K_\lambda$ isomorphic to M^* there are $M'_n (n < \omega)$ such that $M \leq_{\mathfrak{R}} M'_{n+1} <_{\mathfrak{R}} M'_n \in K_\lambda, M'_n \cong M^*$ and $\mathbf{F}(M'_{n+1}) \leq_{\mathfrak{R}} M'_n$ and $\bigcap_{n < \omega} M'_n = M$.

For notational simplicity: $M \in K_\lambda, |M|$ an ordinal $\Rightarrow |\mathbf{F}(M)|$ an ordinal.

Now for each $S \subseteq \lambda^+$ we define by induction on $\alpha \leq \lambda^+, M_\alpha^S$, increasing (by $<_{\mathfrak{R}}$) and continuous with universe an ordinal $< \lambda^+$ such that $M_\alpha^S \cong M^*$ and if $\beta + 2 \leq \alpha$ then $\mathbf{F}(M_{\beta+1}^S) \leq_{\mathfrak{R}} M_{\beta+1}^S$. Let $M_0^S = M^*$ and for limit $\delta < \lambda^+$ and let $M_\delta^S = \bigcup_{\alpha < \delta} M_\alpha^S$; by

the induction assumption and the choice of M^*, \mathbf{F} clearly M_δ^S is isomorphic to M^* . For $\alpha = \beta + 1, \beta$ successor let M_α^S be such that $\mathbf{F}(M_\beta^S) <_{\mathfrak{R}} M_\alpha^S \cong M^*$. So we are left with the case $\alpha = \delta + 1, \delta$ limit or zero.

Now if $\delta \in S$ hence $M_\delta^S \cong M^*$, choose $M_{\delta+1}, a_\delta^S$ such that $(M_{\delta+1}^S, M_\delta^S, a_\delta^S)$ is a maximal triple (possible as by the hypothesis of this case there is a maximal triple, and there is a unique strong limit model). If $\delta \notin S$ we choose $M_\delta^{S,n} \in K_\lambda$ for $n < \omega$ (not used) such that $M_\delta^S <_{\mathfrak{R}} M_\delta^{S,n+1} \leq_{\mathfrak{R}} M_\delta^{S,n}$ and $\mathbf{F}(M_\delta^{S,n+1}) \leq_{\mathfrak{R}} M_\delta^{S,n}$ for $n < \omega$ and $M_\delta^S = \bigcap_{n < \omega} M_\delta^{S,n}$ and

$M_\delta^{S,n} \cong M^*$; and let $M_{\delta+1}^S = M_\delta^{S,0}$ (again possible as $M_\delta \cong M^*$ and an (*)₁ above).

Lastly, let $M^S = \bigcup_{\alpha} M_\alpha^S$.

Now clearly it suffices to prove that if $S^0, S^1 \subseteq \lambda^+, S^1 \setminus S^0$, is stationary then $M^{S^1} \not\cong M^{S^0}$. So suppose f is a $\leq_{\mathfrak{R}}$ -embedding from M^{S^1} onto M^{S^0} or just into M^{S^0} . Then

$E^2 = \{\delta < \lambda^+ : M_\delta^{S^1}, M_\delta^{S^0}$ each has universe δ and $i < \lambda^+$ implies $[i < \delta \Leftrightarrow f(i) < \delta]\}$ is a closed unbounded subset of λ^+ , hence there is a limit ordinal $\delta \in (S^1 \setminus S^0) \cap E^2$. Let us look at $f(a_\delta^{S^1})$; as $\delta \in S^1$, $a_\delta^{S^1}$ is well defined, also $a_\delta^{S^1} \in M_{\delta+1}^{S^1} \setminus M_\delta^{S^1}$, as $\delta \in E^2$ it follows that $f(a_\delta^{S^1}) \not\prec \delta$ hence $f(a_\delta^{S^1})$ belongs to $M^{S^0} \setminus M_\delta^{S^0}$ but $M_\delta^{S^0} = \bigcap_{n < \omega} M_\delta^{S^0, n}$ (as $\delta \notin S^0$).

Hence for some n , $f(a_\delta^{S^1}) \notin M_\delta^{S^0, n}$. Let $\beta \in (\delta, \lambda^+)$ be large enough such that $f(M_{\delta+1}^{S^1}) \subseteq M_\beta^{S^0}$. But then $f(M_\delta^{S^1}) \leq_{\mathfrak{K}} M_\delta^{S^0, n} \leq_{\mathfrak{K}} M_\beta^{S^0}$ and $f(M_{\delta+1}^{S^1}) \leq_{\mathfrak{K}} M_\beta^{S^0}$ and $a_\delta^{S^1} \notin f^{-1}(M_\beta^{S^0, n})$. Now $(f(M_\delta^{S^1}), f(M_{\delta+1}^{S^1}), f(a_\delta^{S^1}))$ has the same properties as $(M_\delta^{S^1}, M_{\delta+1}^{S^1}, a_\delta^{S^1})$ because if f is an isomorphism from M' onto $M'' \in K_\lambda$ then we can extend f to an isomorphism from $\mathbf{F}(M')$ onto $\mathbf{F}(M'')$ (i.e., the “invariant”). But $(f(M_\delta^{S^1}), f(M_{\delta+1}^{S^1}), f(a_\delta^{S^1})) < (M_\delta^{S^0, n}, M_\beta^{S^0}, f(a_\delta^{S^1}))$, contradiction. So we are done. $\square_{3.12}$

{88r-3.9}

3.19. Conclusion. 1) If $\text{LST}(\mathfrak{K}) = \aleph_0$, K is PC_{\aleph_0} and $\dot{I}(\aleph_1, K) = 1$, then K has a model of cardinality \aleph_2 .
 2) If $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ (\mathbf{Q} is the quantifier “there are uncountably many”) has one and only one model of cardinality \aleph_1 up to isomorphism then ψ has a model in \aleph_2 .

Proof. 1) By 3.11 we get suitable \mathfrak{K}_1 (as in its conclusion) and by 3.12 the class \mathfrak{K}_1 has a model in \aleph_2 , hence \mathfrak{K} has a model in \aleph_2 .

2) We can replace ψ by a countable theory $T \subseteq \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$.

Let L be a fragment of $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau)$ in which T is included (e.g., L is the closure of $T \cup$ (the atomic formulas) under subformulas, \neg , \wedge , $(\exists x)$, $(\mathbf{Q}x)$; in particular L includes, of course, first order logic). By [Sh:48], without loss of generality T “says” that every formula $\varphi(x_0, \dots, x_{n-1})$ of L is equivalent to an atomic formula (i.e., $P(x_0, \dots, x_{n-1})$, P a predicate) and every type realized in model of T is isolated (i.e., every model is atomic), and T is complete in L . Let

$$\begin{aligned} K = \{M : M \text{ an atomic } \tau(T)\text{-model of } T \cap \mathbb{L} \text{ and if } M \models P[\bar{a}] \\ \text{and } (\forall \bar{x})[P(\bar{x}) \equiv \neg(\mathbf{Q}y)R(y, \bar{x})] \in T \\ \text{then } \{b : M \models R[b, \bar{a}]\} \text{ is countable}\} \end{aligned}$$

$M \leq_{\mathfrak{K}} N$ iff $M \leq^* N$ which means : (a) $M \prec_{\mathbb{L}} N$

(b) if $M \models P(\bar{a})$ and $\forall \bar{x}[P(\bar{x}) \equiv \neg \mathbf{Q}yR(y, \bar{x})] \in T$

then for no $b \in N \setminus M$ do we have $N \models R[b, \bar{a}]$.

So $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is categorical in \aleph_0 , is an a.e.c. and is PC_{\aleph_0} . Let \mathbf{F} be (see 3.3(8)) such that for $M \in K_{\aleph_0}$, $N = \mathbf{F}(M)$ iff: $M <^{**} N$ which says $M \leq_{\mathfrak{K}} N \in K_{\aleph_0}$ and if $\bar{a} \in M$, $M \models P[\bar{a}]$, $\forall \bar{x}[P(\bar{x}) \equiv \mathbf{Q}yR(y, \bar{x})] \in T$, then for some $b \in N \setminus M$ we have $N \models R[b, \bar{a}]$. So \mathbf{F} is invariant.

Note that every $M \in K_{\aleph_1}^{\mathbf{F}}$ is a model of ψ . So 3.14 gives that some $M \in K_{\aleph_1}^{\mathbf{F}}$ has a proper extension in $K_{\aleph_1}^{\mathbf{F}}$.

The rest should be easy, just as in stage (a) of the proof of 3.12. $\square_{3.19}$

3.20. Question. 1) Under the assumptions of 3.19(2), can we get $M \in K_{\aleph_2}$, such that: if $M \models P[\bar{a}]$, $\forall \bar{x}[P(\bar{x}) \equiv (\mathbf{Q}y)R(y, \bar{x})] \in T$ then $\{b \in M : M \models R[b, \bar{a}]\}$ has cardinality \aleph_2 ? Note that in the proof of 3.14 we show that no triple is maximal.

{88r-3.9A}

3.21. Remark. 1) We could have used multi-valued \mathbf{F} then in the proof above $N = \mathbf{F}(M)$ just means the demand there.

{88r-3.9.}

2) To answer 3.20, i.e., to prove the existence of $M \in K_{\aleph_2}$ as above we have to prove:

(*)₁ there are $N, N_i \in K_{\aleph_1}^{\mathbf{F}}$ for $i < \omega_1$ and $N \leq_{\aleph} N_i$ such that if $N \models P[\bar{a}]$ and the sentence $(\forall \bar{x})(P(\bar{x}) \equiv (\mathbf{Q}y)R(y, \bar{x}))$ belongs to T , then for some $i < \omega_1$ there is $b_* \in N_i \setminus N$ such that $N_i \models R[b, \bar{a}]$.

Clearly

(*)₂ the existence of N, N_i as in (*)₁ is equivalent to “ ψ^* has a model” for some $\psi^* \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ which is defined from T, \leq_{\aleph} .

Hence

(*)₃ it is enough to prove that for some forcing notion \mathbb{P} in $\mathbf{V}^{\mathbb{P}}$ there are N, N_i as in (*)₁.

There are some natural c.c.c. forcing notions tailor-made for this

(*)₄ consider the class of triples (M, N, a) such that $M \leq_{\aleph} N \in K_{\aleph_0}$, $\bar{a} \in {}^{\omega}N$, $\ell < \ell g(\bar{a}) \Rightarrow a_{\ell} \notin M$, order as in the proof of 3.14. By the same proof there is no maximal triple.

3) We can restrict ourselves in (*)₂ to

$$\{R(y, \bar{a}) : \bar{a} \in {}^{\ell g(\bar{x})}N \text{ and } \bar{a} \text{ realizes a type } p(\bar{x})\}.$$

Also we may demand $i < \omega_1 \Rightarrow N_i = N_0$ and we may try to force such a sequence of models (or pairs) and there is a natural forcing. By absoluteness it is enough to prove that it satisfies the c.c.c.

{88r-3.9B}

3.22. Problem. If \aleph is PC_{λ} , K categorical in λ and λ^+ , does it necessarily have a model in λ^{++} ?

3.23. Remark. The problem is proving (*) of 3.13.

3.24. **Question.** Assume $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau)$ is complete in $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau)$, is categorical in \aleph_1 , has an uncountable model M , $\bar{a} \in {}^n M$ and $\varphi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau)$ axiomatizes the $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau)$ -theory of (M, \bar{a}) . Is φ categorical in \aleph_1 ? {88r-3.9C}

3.25. **Question.** Can we weaken the demand on M^* in 3.14 to “ M^* is a λ^+ -limit model”? {88r-3.9D}

4. FORCING AND CATEGORICITY

The main aim in this section is, for \mathfrak{K} as in §1 with $\text{LST}(\mathfrak{K}) = \aleph_0$, to find what we can deduce from $1 \leq \dot{I}(\aleph_1, K) < 2^{\aleph_1}$, first without assuming $2^{\aleph_0} < 2^{\aleph_1}$.

We can build a model of cardinality \aleph_1 by an ω_1 -sequence of countable approximations. Among those, there are models which are the union of a quite generic $<_{\mathfrak{K}}$ -increasing sequence $\langle N_i : i < \omega_1 \rangle$ of countable models, so it is natural to look at them (e.g. if \mathfrak{K} is categorical in \aleph_1 , every model in K_{\aleph_1} is like that). We say on such models that they are quite generic. More exactly, we look at countable models and figure out properties of the quite generic models in \mathfrak{K}_{\aleph_1} . The main results are 4.13(a),(f). Note that the case $2^{\aleph_0} = 2^{\aleph_1}$, though in general making our work harder, can be utilized positively - see 4.11.

A central notion is (e.g.) “the type which $\bar{a} \in {}^{\omega}N_1$ materializes in (N_1, N_0) ”, $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$. This is as the name indicates, the type materialized in N_1^+ , which is N_1 expanded by $P^{N_1^+} = N_0$; it consists of the set of formulas forced (in the model theoretic sense started by Robinson) to satisfy; here forced is defined thinking on $(K_{\aleph_0}, \leq_{\aleph_0})$ so models in K_{\aleph_1} can be constructed as the union of quite generic $<_{\mathfrak{K}}$ -increasing ω_1 -sequence. As we would like to build models of cardinality \aleph_1 by such sequence, the “materialize” in (N_1, N_0) becomes realized in the (quite generic) $N \in K_{\aleph_1}$; but most of our work is in K_{\aleph_0} . This is also a way to express \mathbf{Q} speaking on countable models.

By the hypothesis 4.8 justified by §3, the $\mathbb{L}_{\infty, \omega}(\tau_{\mathfrak{K}})$ -theory of $M \in K$ is clear, in particular has elimination of quantifiers hence $M \leq_{\mathfrak{K}} N \Rightarrow M \prec_{\mathbb{L}_{\infty, \omega}} N$, but for $\bar{N} = \langle N_\alpha : \alpha < \omega_1 \rangle$ as above we would like to understand (N_β, N_α) for $\alpha < \beta$ (from the point of view of N, \bar{N} is not reconstructible, but its behaviour on a club is). Toward a parallel analysis of such pairs we again analyze them by $\langle L_\alpha^0 : \alpha < \omega_1 \rangle$ (similarly to [Mor70]).

4.1. *Convention.* We fix $\lambda > \text{LST}(\mathfrak{K})$ as well as the a.e.c. \mathfrak{K} .

{88r-4.0}

The main case below is here $\lambda = \aleph_1, \kappa = \aleph_0$. For $\lambda > \text{LST}(\mathfrak{K})$

{88r-4.1}

4.2. **Definition.** For $\lambda > \text{LST}(\mathfrak{K})$ and $N_* \in K_{<\lambda}$ and μ, κ satisfying $\lambda \geq \kappa \geq \aleph_0, \mu \geq \kappa$ and let

1) $\mathbb{L}_{\mu, \kappa}^0$ be first order logic enriched by conjunctions (and disjunctions) of length $< \mu$, homogeneous strings of existential quantifiers or of universal quantifiers of length $< \kappa$, and the cardinality quantifier \mathbf{Q} interpreted as $\exists^{\geq \lambda}$. But we apply those operations such that any formula has $< \kappa$ free variables, and the non-logical symbols are from $\tau(\mathfrak{K})$ so actually we should write $\mathbb{L}_{\mu, \kappa}^0(\tau_{\mathfrak{K}})$ but we may “forget” to say this when clear; the syntax does not depend on λ but we shall mention it in the definition of satisfaction.

2) For a logic \mathcal{L} and $A_i, A \subseteq N_*$ for $i < \alpha, \alpha < \lambda$ let $\mathcal{L}(N_*, A_i; A)_{i < \alpha}$ be the language, with the logic \mathcal{L} , and with the vocabulary $\tau_{N_*, \bar{A}, A}$ where $\bar{A} = \langle A_i : i < \alpha \rangle$ and $\tau_{N_*, \bar{A}, A}$ consists of $\tau(K)$, the predicates $x \in N_*$ and $x \in A_i$ for $i < \alpha$ and the individual constants c for $c \in A$. (If $A = \emptyset$, we may omit the A ; if we omit N_* then “ $x \in N_*$ ” is omitted, if the sequence of the A_i is omitted then the “ $x \in A_i$ ” are omitted, so $\mathcal{L}()$ means having the vocabulary

modified:2015-01-25

(88r) revision:2015-01-23

$\tau(K)$). So $\mathcal{L}(N_*, A_i; A)_{i < \alpha}$ formally should have been written $\mathcal{L}(\tau_{N_*, \bar{A}; A})$.

3) $\mathbb{L}_{\mu, \kappa}^1$ is defined as in part (1), but we have also variables (and quantification) over relations of cardinality $< \lambda$. Let $\mathbb{L}_{\mu, \kappa}^{-1}$ be as in part (1) but not allowing the cardinality quantifier \mathbf{Q} ; this is the classical logic $\mathbb{L}_{\mu, \kappa}$.

4) $(N, N_*, A_i; A)_{i < \alpha}$ is the model N expanded to a $\tau_{N_*, \bar{A}; A}$ -model by monadic predicates for N_* , $A_i (i < \alpha)$ and individual constants for every $c \in A$.

5) For “ $x \in N_*$ ”, “ $x \in A_i$ ” we use the predicates P, P_i respectively, so we may write $\mathcal{L}(\tau + P)$ instead $\mathcal{L}(N_*)$, but writing $\mathcal{L}(N_*)$ we fix the interpretation of P .

Let $\tau^{+\alpha} = \tau \cup \{P, P_\beta : \beta < \alpha\}$ and if $L = \mathcal{L}(\tau^{+0})$, i.e., for $\alpha = 0$ then $L(N)$ means L but we fix the interpretation of P as N , i.e., $|N|$, the set of elements of N .

Let $L(N_*, N_i)_{i \in u}$ where u a set of $< \kappa$ ordinals means the language L in the vocabulary $T \cup \{P, P_i : i \in u\}$ when we fix the interpretation of P as N_* and of $P_{\text{otp}(u \cap \alpha)}$ as N_α .

{88r-4.2}

4.3. Definition. 1) For $N_* \in K_{< \lambda}$ and $\varphi(x_0, \dots) \in \mathbb{L}_{\mu, \kappa}^1(N_*, \bar{A}; A)$ we define by induction on φ when $N_0 \Vdash_{\mathfrak{R}}^\lambda \varphi[a_0, \dots]$ where $N_* \leq_{\mathfrak{R}} N_0 \in K_{< \lambda}$, a_0, \dots are elements of N_0 or appropriate relations over it, depending on the kind of x_i . Pedantically we should write $(N_0, N_*, \bar{A}; A) \Vdash_{\mathfrak{R}}^\lambda \varphi[a_0, \dots]$; and we may do it when not clear from the context.

For φ atomic this means $N_0 \models \varphi[a_0, \dots]$. For $\varphi = \bigwedge_i \varphi_i$ this means

$$N_0 \Vdash_{\mathfrak{R}}^\lambda \varphi_i[a_0, \dots] \text{ for each } i.$$

For $\varphi = \exists \bar{x} \psi(\bar{x}, a_0, \dots)$ this means that for every N_1 satisfying $N_0 \leq_{\mathfrak{R}} N_1 \in K_{< \lambda}$ there is N_2 satisfying $N_1 \leq_{\mathfrak{R}} N_2 \in K_{< \lambda}$ and \bar{b} from N_2 of the appropriate length (and kind) such that $N_2 \Vdash_{\mathfrak{R}}^\lambda \psi[\bar{b}, a]$.

For $\varphi = \neg \psi$ this means that for no N_1 do we have $N_0 \leq_{\mathfrak{R}} N_1 \in K_{< \lambda}$ and $N_1 \Vdash_{\mathfrak{R}}^\lambda \psi[a_0, \dots]$.

For $\varphi(x_0, \dots) = (\mathbf{Q}y)\psi(y, x_0, \dots)$ this means that for every N_1 satisfying $N_0 \leq_{\mathfrak{R}} N_1 \in K_{< \lambda}$ there is N_2 satisfying $N_0 \leq_{\mathfrak{R}} N_2 \in K_{< \lambda}$ and $a \in N_2 \setminus N_1$ such that $N_2 \Vdash_{\mathfrak{R}}^\lambda \psi[a, a_0, \dots]$.

2) In part (1) if $\varphi \in \mathbb{L}_{\mu, \kappa}^1(N_*)$ we can omit the demand “ $N_* \leq_{\mathfrak{R}} N$ ” similarly below.

3) For a language $L \subseteq \mathbb{L}_{\mu, \kappa}^1(N_*, \bar{A}; A)$ and a model N satisfying $N_* \leq_{\mathfrak{R}} N \in K_{< \lambda}$ and a sequence $\bar{a} \in {}^\lambda N$ the L -generic type of \bar{a} in N is $\text{gtp}(\bar{a}; N_*, \bar{A}; A; N) = \{\varphi(\bar{x}) \in L : N \Vdash_{\mathfrak{R}}^\lambda \varphi[\bar{a}]\}$.

4) Let $\text{gtp}_L^\lambda(\bar{a}; N_*, \bar{A}; A; N)$ where $N_* \leq_{\mathfrak{R}} N \in K_\lambda$ and $L \subseteq \mathcal{L}(N_*, \bar{A}; A)$ be $\{\varphi(\bar{x}) : \varphi \in \mathcal{L}(N_*, \bar{A}; A) \text{ and for some } N' \in K_{< \lambda} \text{ we have } N \leq_{\mathfrak{R}} N' \leq_{\mathfrak{R}} N \text{ and } N' \Vdash_{\mathfrak{R}}^\lambda \varphi[\bar{a}]\}$; we may omit \bar{A}, A (and omit λ if clear from the context) and may write \mathcal{L} instead of $L = \mathcal{L}(N_*, \bar{A}; A)$; but note Definition 5.5.

5) We say “ \bar{a} materializes p (or φ)” if p (or $\{\varphi\}$) is a subset of the L -generic type of \bar{a} in N .

{88r-4.3}

4.4. Definition. Let $N_i (i < \lambda)$ be an increasing (by $\leq_{\mathfrak{R}}$) continuous sequence, $N = \bigcup_{i < \lambda} N_i$, $\|N_i\| < \lambda$ and $L^* \subseteq \bigcup_{\alpha < \kappa} \mathbb{L}_{\infty, \kappa}^1(\tau^{+\alpha})$.

- 1) N is L^* -generic, if for any formula $\varphi(x_0, \dots) \in L^* \cap \mathbb{L}_{\infty, \kappa}^1(\tau_{\mathfrak{R}})$ and $a_0, \dots \in N$ we have:
 $N \models \varphi[a_0, \dots] \Leftrightarrow$ for some $\alpha < \lambda$, $N_\alpha \Vdash_{\mathfrak{R}}^\lambda \varphi[a_0, \dots]$.
 2) The $\leq_{\mathfrak{R}}$ -presentation $\langle N_i : i < \lambda \rangle$ of N is L^* -generic when for any $\alpha < \lambda$ of cofinality $\geq \kappa$ and $\psi(x_0, \dots) \in L^*(N_\alpha, N_i)_{i \in I}$ satisfying $I \subseteq \alpha$, $|I| < \kappa$ and $a_0, \dots \in N$ we have:

$$N \models \psi[a_0, \dots] \Leftrightarrow \text{for some } \gamma < \lambda, N_\gamma \Vdash_{\mathfrak{R}}^\lambda \psi[a_0, \dots]$$

and for each $\beta \geq \alpha$, with cofinality $\geq \kappa$, N_β is almost $L^*(N_\alpha, N_i)_{i \in I}$ -generic (see part (5)).

3) N is strongly L^* -generic if it has an L^* -generic presentation (in this case, if λ is regular, then for any presentation $\langle N_i : i < \lambda \rangle$ of N there is a closed unbounded $E \subseteq \lambda$ such that $\langle N_i : i \in E \rangle$ is an L^* -generic presentation).

4) We say that $N \in K_{< \lambda}$ is pseudo L^* -generic if

(a) for every $\varphi(\bar{x}) = \exists \bar{y} \psi(\bar{x}, \bar{y}) \in L^*$, if $N \Vdash_{\mathfrak{R}}^\lambda \varphi(\bar{a})$ then for some \bar{b} , $N \Vdash_{\mathfrak{R}}^\lambda \psi(\bar{a}, \bar{b})$

(b) for every $\bar{a} \in N$, \bar{a} materializes in N some complete L^* -type.

5) We add “almost” to any of the above defined notions when: for $\Vdash_{\mathfrak{R}}^\lambda$, the inductive definitions of satisfaction works except possibly for \mathbf{Q} (e.g., $N \Vdash_{\mathfrak{R}}^\lambda \exists x \varphi(x, \dots)$ iff for some $a \in N$, $N \Vdash_{\mathfrak{R}}^\lambda \varphi(a, \dots)$).

4.5. **Remark.** 1) Notice we can choose $N_i = N_0 = N$, so $\|N\| < \lambda$. In particular almost (and pseudo) L^* -generic models of cardinality $< \lambda$ may well exist.

2) Here we concentrate on $\lambda = \aleph_1$ and fragments of $\mathbb{L}_{\infty, \omega}^0$ (mainly $\mathbb{L}_{\omega_1, \omega}^0$ and its countable fragments).

3) There are obvious implications, and forcing is preserved by isomorphism and replacing $N (\in K_{< \lambda})$ by N' , $N \leq_{\mathfrak{R}} N' \in K_{< \lambda}$.

There are obvious theorems on the existence of generic models, e.g.,

4.6. **Theorem.** 1) Assume $N_0 \in K_{< \lambda}$, $\lambda = \mu^+$, $\mu^{< \kappa} = \mu$, $L \subseteq \bigcup_{\alpha < \kappa} \mathbb{L}_{\infty, \kappa}(\tau^{+\alpha})$ and L is closed under subformulas and $|L| < \lambda$. Then there are $N_i (i < \lambda)$ such that $\langle N_i : i < \lambda \rangle$ is an L -generic representation of $N = \bigcup_{i < \lambda} N_i$, (hence N is strongly L -generic).

2) In part (1), $N \in K_\lambda$ if no $N', N_0 \leq_{\mathfrak{R}} N' \in K_{< \lambda}$ is $\leq_{\mathfrak{R}}$ -maximal.

Proof. Straightforward. □_{4.6}

4.7. **Remark.** 1) If $L = \bigcup_{i < \lambda} L_i$, $|L_i| < \lambda$, then we can get “ $\langle N_i : j < i < \lambda \rangle$ is an L_j -generic representation of N for each $j < \lambda$ ”.

2) When we speak on “complete L -type p ” we mean $p = p(x_0, \dots, x_{n-1})$ for some n .

From time to time we add some hypothesis and prove a series of claims; such that the hypothesis holds, at least without loss of generality, in the case we are interested in. We are mainly interested in the case $\dot{I}(\aleph_1, \aleph) < 2^{\aleph_1}$, etc., so by 3.11, 3.18 it is reasonable to make:

{88r-4.5}

4.8. **Hypothesis.** \aleph is $\text{PC}_{\aleph_0, \leq \aleph}$ refines $\mathbb{L}_{\infty, \omega}$ and \aleph is categorical in \aleph_0 and $1 \leq \dot{I}(\aleph_1, K)$ and $\dot{I}(\aleph_1, K_{\aleph_1}^{\mathbf{F}}) < 2^{\aleph_1}$ where $K_{\aleph_1}^{\mathbf{F}}$ is as in Definition 3.15 and is PC_{\aleph_0} or just $\mathbf{K}_{\aleph_1}^{\mathbf{F}} = \{M \upharpoonright \tau_{\aleph} : M \models \psi\}$ for some $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ (if \mathbf{F} is invariant, this follows).

{88r-4.5.3}

4.9. **Remark.** 0) We can add: every $M \in K_{\aleph_0}$ is atomic (model of $\text{Th}_{\mathbb{L}}(M)$).

1) Usually below we ignore the case $\dot{I}(\aleph_1, \aleph) < 2^{\aleph_0}$ as the proof is the same.

2) We can deal similarly with the case $1 \leq \dot{I}(\aleph_1, K') < 2^{\aleph_0}$ where $\aleph_{\aleph_1} \subseteq K'_{\aleph_1} \subseteq \{M \in \aleph_{\aleph_1} : M \text{ is strongly } L_*\text{-generic}\}$ and K' is PC_{\aleph_0} (or less: $\{M \upharpoonright \tau_{\aleph} : M \text{ a model of } \psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau^*)\}$).

3) Can we use \mathbf{F} a function with domain K_{\aleph_0} such that $M \leq_{\aleph} \mathbf{F}(M_0) \in K_{\aleph_0}$ for $M \in K_{\aleph_0}$ without the extra assumptions or even $\mathbf{F} : \{\bar{M} = \langle M_i : i \leq \alpha \rangle \text{ is } \leq_{\aleph_{\aleph_0}}\text{-increasing continuous}\} \rightarrow \aleph_{\aleph_0}$ such that $M_{\alpha} \leq_{\aleph} \mathbf{F}(M_i : i \leq \alpha)$? We cannot use the non-definability of well ordering (see 3.11(3)); (as in the proof of (f) of 4.13).

{88r-4.6}

4.10. **Claim.** 1) If $\bar{a} \in N \in K_{\aleph_0}$ and $\varphi(\bar{x}) \in \mathbb{L}_{\infty, \omega}^0(\tau^{+0})$ (so \bar{a} is a finite sequence) then $(N, N) \Vdash_{\aleph}^{\aleph_1} \varphi[\bar{a}]$ or $(N, N) \Vdash_{\aleph}^{\aleph_1} \neg\varphi[\bar{a}]$ (i.e. P is interpreted as N).

2) If $(N, N) \Vdash_{\aleph}^{\aleph_1} \exists \bar{x} \wedge p(\bar{x})$, where $p(\bar{x})$ is a not necessarily complete n -type ($n = \text{lg}(\bar{x})$) in L where $L \subseteq \mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ is countable, then for some complete n -type q in L extending p we have $(N, N) \Vdash_{\aleph}^{\aleph_1} \exists \bar{x} \wedge q(\bar{x})$.

Proof. 1) Suppose not, for each $S \subseteq \omega_1$, we define by induction on α , $N_{\alpha}^S \in K_{\aleph_0}$ ($\alpha < \omega_1$), increasing (by \leq_{\aleph}) and continuous, $N_0^S = N$ and for limit α , $N_{\alpha}^S = \bigcup_{\beta < \alpha} N_{\beta}^S$. For $\alpha = 2\beta + 1$

remember that $(N_{\beta}^S, \bar{a}) \cong (N, \bar{a})$ because $N = N_0 \leq_{\aleph} N_{\beta}^S$ hence $N_0 \prec_{\mathbb{L}_{\infty, \omega}} N_{\beta}^S \in K_{\aleph_0}$ hence $(N_{\beta}^S, \bar{a}) \equiv_{\mathbb{L}_{\infty, \omega}} (N, \bar{a})$ hence they are isomorphic. So $(N_{\beta}^S, N_{\beta}^S)$ forces ($\Vdash_{\aleph}^{\aleph_1}$) neither $\varphi[\bar{a}]$ nor $\neg\varphi[\bar{a}]$. So there are M_{ℓ} (for $\ell = 0, 1$) such that $N_{\beta}^S \leq_{\aleph} M_{\ell} \in K_{\aleph_0}$ and $(M_0, N_{\beta}^S) \Vdash_{\aleph}^{\aleph_1} \varphi[\bar{a}]$ but $(M_1, N_{\beta}^S) \Vdash_{\aleph}^{\aleph_1} \neg\varphi[\bar{a}]$. Now if $\beta \in S$ we let $N_{\alpha}^S = M_0$ and if $\beta \notin S$ we let $N_{\alpha}^S = M_1$.

Lastly, $M_{2\beta+2} = \mathbf{F}(M_{2\beta+1})$ recalling \mathbf{F} is from 4.8. Let $N^S = \bigcup_{\alpha < \omega_1} N_{\alpha}^S$. Now if $S(0) \setminus S(1)$

is stationary then $(N^{S(0)}, \bar{a}) \not\cong (N^{S(1)}, \bar{a})$. Why? Because if $f : N^{S(0)} \rightarrow N^{S(1)}$ is an isomorphism from $N^{S(0)}$ onto $N^{S(1)}$ mapping \bar{a} to \bar{a} then for some closed unbounded set $E \subseteq \omega_1$, we have: if $\alpha \in E$ then f maps $N_{\alpha}^{S(0)}$ onto $N_{\alpha}^{S(1)}$, so choose some $\alpha \in E \cap S(0) \setminus S(1)$ and choose $\beta \in E \setminus (\alpha + 1)$. Now $(N_{\alpha+1}^{S(0)}, N_{\alpha}^{S(0)}) \Vdash_{\aleph}^{\aleph_1} \varphi[\bar{a}]$, hence $(N_{\beta}^{S(0)}, N_{\alpha}^{S(0)}) \Vdash_{\aleph}^{\aleph_1} \varphi[\bar{a}]$, and similarly $(N_{\beta}^{S(1)}, N_{\alpha}^{S(1)}) \Vdash_{\aleph}^{\aleph_1} \neg\varphi[\bar{a}]$, but $f \upharpoonright N_{\beta}^{S(0)}$ is an isomorphism from $N_{\beta}^{S(0)}$ onto

$N_\beta^{S(1)}$ mapping $N_\alpha^{S(0)}$ onto $N_\alpha^{S(1)}$ and \bar{a} to itself and we get a contradiction. By 0.4, we get $\dot{I}(\aleph_1, K) = 2^{\aleph_1}$, contradiction.

2) Easy by 4.6 and part (1). In detail, if $N \leq_{\aleph} M_1 \in \mathfrak{K}_{\aleph_0}$ then by the definition of $\Vdash_{\aleph}^{\aleph_1}$ and the assumption we can find (M_2, \bar{a}) satisfying $M_1 \leq_{\aleph} M_2 \in \mathfrak{K}_{\aleph_0}$ and $\bar{a} \in M_2$ such that $(M_2, N) \Vdash_{\aleph}^{\aleph_1} \wedge p(\bar{a})$. As L is countable and the definition of $\Vdash_{\aleph}^{\aleph_1}$ without loss of generality for every formula $\varphi(\bar{x}) \in L$, $(M_2, N) \Vdash_{\aleph}^{\aleph_1} \varphi[\bar{a}]$ or $(M_2, N) \Vdash_{\aleph}^{\aleph_1} \neg\varphi[\bar{a}]$. (Why? Simply let $\langle \varphi_n(\bar{x}) : n < \omega \rangle$ list the formulas $\varphi(\bar{x}) \in L$ and choose $M_{2,n} \in \mathfrak{K}_{\aleph_0}$ by induction on n such that $M_{2,0} = M_2$, $M_{2,n} \leq_{\aleph} M_{2,n+1}$ such that $(M_{2,n+1}, N) \Vdash_{\aleph}^{\aleph_1} \varphi_n(\bar{x})$ or $(M_{2,n+1}, N) \Vdash_{\aleph}^{\aleph_1} \neg\varphi_n(\bar{x})$; now replace M_2 by $\cup\{M_{2,n} : n < \omega\}$). Recalling Definition 4.3(4), let $q = \text{gtp}_{L(N)}(\bar{a}, N, M_2)$, it is a complete $(L(N), n)$ -type. So clearly $(M_2, N) \Vdash_{\aleph}^{\aleph_1} (\exists \bar{x}) \wedge q(\bar{x})$. Now apply the proof of part (1) to the formula $(\exists \bar{x}) \wedge q(\bar{x})$ so we are done. $\square_{4.10}$

{88r-4.7}

4.11. Claim. For each countable $L \subseteq \mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ and $N \in K_{\aleph_0}$ the number of complete $L(N)$ -types p (with no parameters) such that $N \Vdash_{\aleph}^{\aleph_1} (\exists \bar{x}) \wedge p(\bar{x})$, is countable.

Proof. At first glance it seemed that 0.3 will imply this trivially. However, here we need the parameter N as an interpretation of the predicate P and if $2^{\aleph_0} = 2^{\aleph_1}$ there are too many choices. So we shall deal with “every N_α in some presentation”. Suppose the conclusion fails. First we choose by induction N_α (for $\alpha < \omega_1$) such that:

- ⊛ (i) $N_\alpha \in K_{\aleph_0}$ is \leq_{\aleph} -increasing and $\langle N_\alpha : \alpha < \omega_1 \rangle$ is L -generic
- (ii) for each $\beta < \alpha$, there is $a_\alpha^\beta \in N_{\alpha+1} \setminus N_\alpha$ materializing an $L(N_\beta)$ -type not materialized in N_α , (i.e. in (N_α, N_β) ; see Definition 4.3(2) on materialize), (possible by 4.10 and our assumption toward contradiction)
- (iii) $|N_\alpha| = \omega_\alpha$
- (iv) for $\alpha < \beta$, N_β is pseudo $L(N_\alpha)$ -generic and $\mathbf{F}(N_{2\beta+1}) \leq_{\aleph} N_{2\beta+2}$.

Now let $N = \cup\{N_\alpha : \alpha < \omega_1\}$ and we expand N by all relevant information: the order $<$ on the countable ordinals, $c(c \in N_0)$, enough “set theory”, “witness” for $N_\beta \leq_{\aleph} N_\alpha$ for $\beta < \alpha$ and the 2-place functions $F, F(\beta, \alpha) = a_\alpha^\beta$ and lastly witnesses of $\mathbf{F}(N_{2\beta+1}) \leq_{\aleph} N_{2\beta+2}$ recalling \mathbf{F} is quite definable by Definition 4.8 and names for all formulas in $L(N_\alpha)$ (with α as a parameter), i.e., the relations $R_{\varphi(\bar{x})} = \{\langle \alpha \rangle \hat{\ } \bar{a} : \alpha < \omega_1, \bar{a} \in {}^{\ell g(\bar{x})}N \text{ and for every } \beta < \omega_1 \text{ large enough } (N_\beta, N_\alpha) \Vdash_{\aleph}^{\aleph_1} \text{“}\varphi(\bar{a})\text{”}\}$ for $\varphi(\bar{x}) \in L$. Clearly for every $\alpha < \omega_1, \varphi(\bar{x}) \in L(N_\alpha)$ and $\bar{a} \in {}^{\ell g(\bar{x})}N$ we have $(N, N_\alpha) \models \varphi[\bar{a}]$ iff for every $\beta < \omega_1$ large enough we have $(N_\beta, N_\alpha) \Vdash_{\aleph}^{\aleph_1} \text{“}\varphi[\bar{a}]\text{”}$. We get a model \mathfrak{B} with countable vocabulary and $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ expressing all this. By 0.3(1) applied to the case $\Delta = L$, there are models \mathfrak{B}_i (for $i < 2^{\aleph_1}$) of cardinality \aleph_1 (note $N_0 \leq_{\aleph} \mathfrak{B} \upharpoonright \tau_{\aleph}$), so that the set of $L(N_0)$ -types realizes in N^i (the $\tau(K)$ -reduct of \mathfrak{B}_i) are distinct for distinct i 's. So $(N^i, c)_{c \in N_0}$ are pairwise non-isomorphic. If $2^{\aleph_0} < 2^{\aleph_1}$ we finish by 0.4.

So we can assume $2^{\aleph_0} = 2^{\aleph_1}$. In N , uncountably many complete $L(N_0)$ -n-types are realized hence by 0.3(2) the set $\{p : p \text{ a complete } L(N_0) - m\text{-type is realized in some } N', N_0 \leq_{\aleph} N' \in \mathfrak{K}_{\aleph_1} \text{ for some } m < \omega\}$ has cardinality continuum, hence by 4.10 the set of

complete $L(N_0)$ -types $p = p(x)$ such that $(N_0, N_0) \Vdash_{\aleph_1}^{\aleph_0} \exists \bar{x} \wedge p(\bar{x})$ has cardinality 2^{\aleph_0} . So we choose by induction on $\alpha < 2^{\aleph_0}$ a sequence $\langle N_i^\alpha, a_i^\alpha : i < \omega_1 \rangle$ such that:

- (a) $N_i^\alpha \in \mathfrak{K}_{\aleph_0}$
- (b) $N_{i_0}^\alpha \leq_{\aleph} N_i^\alpha$ for $i_0 < i < \omega_1$
- (c) $a_i^\alpha \in N_{i+1}^\alpha \setminus N_i^\alpha$ materialize a complete $L(N_i^\alpha)$ -type p_i^α
- (d) if $j < \omega_1$ is a limit ordinal then $N_j^\alpha = \cup \{N_i^\alpha : i < j\}$
- (e) $p_i^\alpha \notin \{\text{gtp}(\bar{a}; N_{j_1}^\beta; N_{j_2}^\beta) : j_1 < j_2 < \omega_1, \bar{a} \in {}^{\omega} (N_{j_2}^\beta) \text{ and } \beta < \alpha\}$ (see Definition 4.3(4))
- (f) $\mathbf{F}(N_{2\beta+1}) \leq_{\aleph} N_{2\beta+2}$.

As $\aleph_1 < 2^{\aleph_1} = 2^{\aleph_0}$ this is possible, i.e., in clause (e) we should find a type which is not in a set of $\leq \aleph_1 \times |\alpha| < 2^{\aleph_0}$ types, as the number of possibilities is 2^{\aleph_0} ; let $N_\alpha = \cup \{N_i^\alpha : i < \omega_1\}$ for $\alpha < 2^{\aleph_0}$, clearly $N_\alpha \in K_{\aleph_1}$. Now toward contradiction if $\beta < \alpha < 2^{\aleph_0}$ and $N_\alpha \cong N_\beta$ then there is an isomorphism f from N_α onto N_β ; necessarily f maps N_i^α onto N_i^β for a club of i . For any such $i, p_i^\alpha \in \text{gtp}_L(f(\bar{a}_i^\alpha); N_i^\beta; N_j^\beta)$ for j large enough, contradiction.

□_{4.11}

{88r-4.7A}

4.12. **Remark.** In the proof of 4.11(2), we can fix m and we can combine the two cases, when for $N \in K_{\aleph_1}^{\mathbf{F}}$ represent by $\langle N_\alpha : \alpha < \omega_1 \rangle$ we consider $\mathbf{P}_N = \{p : p \text{ a complete } L - m\text{-type such that for a club of } \alpha < \omega_1 \text{ for some } \beta \in (\alpha, \omega_1) \text{ and } \bar{a} \in {}^m(N_\beta) \text{ materialize } p \text{ in } (N_\beta, N_\alpha)\}$, can replace “club” by “stationarily many”. That is we can prove that $\{\mathbf{P}_N : N \in K_{\aleph_1}^{\mathbf{F}}\}$ has cardinality 2^{\aleph_1} .

{88r-4.8}

4.13. **Lemma.** 1) There are countable $L_\alpha^0 \subseteq \mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ for $\alpha < \omega_1$ increasing continuous in α , closed under finitary operations and subformulas such that, letting $L_{<\omega_1}^0 = \cup \{L_\alpha^0 : \alpha < \omega_1\}$ we have (some clauses do not mention the L_α^0 's):

- (a) for each $N \in K_{\aleph_0}$ and every complete $L_\alpha^0(N)$ -type $p(\bar{x})$ we have $N \Vdash_{\aleph_1}^{\aleph_0} (\exists \bar{x}) \wedge p(\bar{x}) \Rightarrow \wedge p \in L_{\alpha+1}^0(N)$.
Hence for every $\mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ -formula $\psi(\bar{x})$ there are formulas $\varphi_n(\bar{x}) \in L_{<\omega_1}^0$ for $n < \omega$ such that $(N, N) \Vdash_{\aleph_1}^{\aleph_0} (\forall \bar{x}) [\psi(\bar{x}) \equiv \bigvee_n \varphi_n(\bar{x})]$
- (b) for every $N_0 \leq_{\aleph} N_1 \in K_{\aleph_0}$ there is $N_2, N_1 \leq_{\aleph} N_2 \in K_{\aleph_0}$, such that for every $\bar{a} \in N_2$ and $\varphi(\bar{x}) \in \mathbb{L}_{\omega_1, \omega}^0(N_0)$, of course with $\ell g(\bar{a}) = \ell g(\bar{x}) < \omega$, we have $(N_2, N_0) \Vdash_{\aleph_1}^{\aleph_0} \varphi[\bar{a}]$ or $(N_2, N_0) \Vdash_{\aleph_1}^{\aleph_0} \neg \varphi[\bar{a}]$
- (c) If $N \leq_{\aleph} N_\ell \in K_{\aleph_0} (\ell = 1, 2), \bar{a}_\ell \in N_\ell$ and the $L_{<\omega_1}^0(N)$ -generic types of \bar{a}_ℓ in N_ℓ are equal (though they are not necessarily complete; i.e., for every $\varphi(\bar{x}) \in L_{<\omega_1}^0(N)$ we have $N_1 \Vdash_{\aleph_1}^{\aleph_0} \varphi(\bar{a}_1)$ iff $N_2 \Vdash_{\aleph_1}^{\aleph_0} \varphi(\bar{a}_2)$), then so are the $\mathbb{L}_{\infty, \omega}^0(N)$ -generic types. In fact, there is $M, N \leq_{\aleph} M$ and \leq_{\aleph} -embeddings $f_\ell : N_\ell \rightarrow M$ such that f_ℓ maps N onto itself and $f_1(\bar{a}_1) = f_2(\bar{a}_2)$ though we do not claim $f_1 \upharpoonright N = f_2 \upharpoonright N$. Also if

modified: 2015-01-25

(88r) revision: 2015-01-23

$N_1 = N_2$ then there is $M \in K_{\aleph_0}$ which \leq_{\aleph} -extends N_1 and an automorphism f of M mapping N onto itself and \bar{a}_1 to a_2 .

- (d) For each $N \in K_{\aleph_0}$ and complete $\mathbb{L}_{\omega_1, \omega}^0(N)$ -type $p(\bar{x})$, the class $K^1 := \{(N, M, \bar{a}) : M \in K_{\aleph_0}, N \leq_{\aleph} M \text{ and for some } M', M \leq_{\aleph} M' \in K_{\aleph_0} \text{ and } \bar{a} \text{ materialize } p \text{ in } (M; N)\}$ is a PC_{\aleph_0} -class.
- (e) for any complete $\mathbb{L}_{\omega_1, \omega}^{-1}(N)$ -type $p(\bar{x})$ for some complete $\mathbb{L}_{\omega_1, \omega}^0(N)$ -type q_p , if $N \leq_{\aleph} M \in K_{\aleph_0}, \bar{a} \in M$ and \bar{a} materialize p in (M, N) , then \bar{a} materialize q_p in (M, N) ; on $\mathbb{L}^0, \mathbb{L}^{-1}$ see Definition 4.2(1), (3)
- (f) the number of complete $\mathbb{L}_{\omega_1, \omega}^0(N)$ -types p which for some \bar{a}, M we have $\bar{a} \in {}^{\omega}M, M \in K_{\aleph_0}, N \leq_{\aleph} M$ and \bar{a} materialize in (M, N) is $\leq \aleph_1$
- (g) if in clause (f) we get that there are \aleph_1 such types then $\dot{I}(\aleph_1, K) \geq \aleph_1$
- (h) let $L_{\alpha}^{-1} := L_{\alpha}^0 \cap \mathbb{L}_{\omega_1, \omega}^{-1}(\tau^{+0})$ then the parallel clauses to (a)-(g) holds.

2) Clause (e) means that

- (i) assume further that $N_0 \leq_{\aleph} N_{\ell} \in K_{\aleph_0}$ for $\ell = 1, 2$ and $\bar{a}_{\ell} \in N_{\ell}$ and the $L_{<\omega_1}^{-1}(N)$ -type which \bar{a}_1 materializes in N_1 is equal to the $L_{<\omega_1}^{-1}(N)$ -type which \bar{a}_2 materializes in N_2 . Then we can find N_1^+, N_2^+ such that $N_{\ell} \leq_{\aleph} N_{\ell}^+ \in K_{\aleph_0}$ for $\ell = 1, 2$ and isomorphism f from N_1^+ onto N_2^+ mapping N onto itself and \bar{a}_1 to \bar{a}_2 .

4.14. **Remark.** 1) We cannot get rid of the case of \aleph_1 types (but see 5.23, 5.30) by the following variant of a well known example of Morley [Mor70] for $\dot{I}(\aleph_0, K) = \aleph_2$. For let $K = \{(A, E, <) : E \text{ an equivalence relation on } A, \text{ each } E\text{-equivalence class is countable, } x < y \Rightarrow xEy \text{ and on each } E\text{-equivalence class } < \text{ is a 1-transitive linear order, i.e. } xEy \Rightarrow (x/E, <, x) \cong (y/E, <, y)\} \text{ and } M \leq_{\aleph} N \text{ if } M \subseteq N \text{ and } [x \in M \wedge y \in N \wedge xEy \Rightarrow y \in M].$ By the analysis of such countable linear orders, each $(a/E^M, <)$ up to isomorphism is determined by $(\alpha, \ell) \in \omega_1 \times 2$. For appropriate \mathbf{F} , if $M = \mathbf{F}(N), a \in N$ and I is an interval of $(a/E^N, <^N)$ which is 1-transitive then for some $b \in M \setminus N, (b/E^M, <^M)$ is isomorphic to $(I, <^N)$. This is enough.

2) In clauses (c),(i) of 4.13 the mapping are not necessarily the identity on N . In clause (i) the assumption is apparently weaker (those by its conclusion the assumption of (c) holds).

3) Note that clause (f) of 4.13 does not follow from clause (a) as there may be \aleph_1 -Kurepa trees.

4) In clause (c) of 4.13 for the second sentence we can weaken the assumption: if $\varphi(\bar{x}) \in L_{<\omega_1}^0(N)$ and $(N_1; N) \not\vdash_{\aleph}^{\aleph_1} \varphi(\bar{a}_1)$ then $(N_2, N) \not\vdash_{\aleph}^{\aleph_1} \varphi(\bar{a}_2)$. This is enough to get the $M_{1,\alpha}, M_{2,\alpha}$ from the proof. (Why? For each $\alpha < \omega_1$, there are $M_{1,\alpha}$ such that $N_1 \leq_{\aleph} M_{1,\alpha} \in K_{\aleph_0}$ and a complete $L_{\alpha}^0 - \ell g(\bar{a}_i)$ -type $p_*(\bar{x})$ such that $(M_{1,\alpha}, N) \Vdash \wedge p_*(\bar{a}_1)$. But $\neg \wedge p_1(\bar{x}) \in L_{\alpha+1}$ and obviously $(N_1, N) \not\vdash \neg \wedge p_*(\bar{a}_1)$ hence $(N_2, N) \not\vdash_{\aleph}^{\aleph_1} \neg \wedge p_*(\bar{a}_2)$ hence there is $M_{2,\alpha}$ such that $N_2 \leq_{\aleph} M_{2,\alpha} \in K_{\aleph_0}$ and $(M_{2,\alpha}; N) \Vdash_{\aleph}^{\aleph_1} \wedge p_*[\bar{a}_2]$. Now continue as in the proof below).

{88r-4.8A

4.15. **Remark.** We can prove clause (b) and the last sentence in clause (c) of 4.13 directly not mentioning the L_α^0 -s.

Proof. Note that proving clause (e) we say “repeat the proof of clause (a),(b),(c),(d) for $L_{\omega,\omega}^{-1}$ ”.

Clause (a): We choose L_α^0 by induction on α using 4.11. The second phrase is proved by induction on the depth of the formula using 4.10.

Clause (b): By iterating ω times, it suffices to prove this for each $\bar{a} \in N_1$, so again by iterating ω times it suffices to prove this for a fix $\bar{a} \in N_1$. If the conclusion fails we can define by induction on $n < \omega$ for every $\eta \in {}^n 2$, a model M_η and $\varphi_\eta(\bar{x}) \in \mathbb{L}_{\omega_1,\omega}^0(N)$ such that:

- (i) $M_{\langle \rangle} = N_1$
- (ii) $M_\eta \leq_{\aleph} M_{\eta \hat{ } \langle \ell \rangle} \in K_{\aleph_0}$ for $\ell = 0, 1$
- (iii) $(M_\eta, N) \models_{\aleph}^{\aleph_1} \varphi_\eta(\bar{a})$
- (iv) $\varphi_{\eta \hat{ } \langle 1 \rangle}(\bar{x}) = \neg \varphi_{\eta \hat{ } \langle 0 \rangle}(\bar{x})$.

Now for $\eta \in {}^\omega 2$, let $M_\eta = \bigcup_{n < \omega} M_{\eta \upharpoonright n}$. Clearly for $\eta \in {}^\omega 2$ we have $M_\eta \models_{\aleph}^{\aleph_1} (\exists \bar{x}) [\bigwedge_{n < \omega} \varphi_{\eta \upharpoonright n}(\bar{x})]$ and, after slight work, we get contradiction to 4.11 + 4.10.

Clause (c): In general by clause (a) for each $\alpha < \omega_1$ we can find $M_\ell^\alpha \in K_{\aleph_1}$ for $\ell = 1, 2$ such that $N_\ell \leq_{\aleph} M_\ell^\alpha$ and $(M_1^\alpha, \bar{a}_1), (M_2^\alpha, \bar{a}_2)$ are $L_\alpha^0(N)$ -equivalent and without loss of generality each of N, N_ℓ, M_ℓ^α have universe an ordinal $< \omega_1$. Let $\mathfrak{A} = (\mathcal{H}(\aleph_2), N, N_1, N_2, \langle M_1^\alpha : \alpha < \omega_1 \rangle, \langle M_2^\alpha : \alpha < \omega_1 \rangle)$ let $\mathfrak{A}_1 \prec \mathfrak{A}$ be countable and recalling 0.5(3) find a non-well ordered countable model \mathfrak{A}_2 , which is an end extension of \mathfrak{A}_1 for $\omega_1^{\mathfrak{A}_1}$, hence $\omega^{\mathfrak{A}_2} = \omega$ so $N^{\mathfrak{A}_2} = N, N_\ell^{\mathfrak{A}_2} = N_\ell$ for $\ell = 1, 2$. For $x \in (\omega_1)^{\mathfrak{A}_2} \setminus \mathfrak{A}_1$ let $M_\ell^x = (M_\ell^x)^{\mathfrak{A}_2}$ so $N_\ell \leq_{\aleph} M_\ell^x \in K_{\aleph_0}$. Now there are x_n such that $\mathfrak{A}_2 \models$ “ $x_{n+1} < x_n$ are countable ordinals”; so using the hence and forth argument $(M_1^{x_0}, \bar{a}_1, N) \cong (M_2^{x_0}, \bar{a}_2, N)$.

[Why? Let $\mathcal{F}_n = \{(\bar{b}^1, \bar{b}^2) : \bar{b}^\ell \in {}^n(M_\ell^{x_0}) \text{ and } \mathfrak{A}_2 \models \text{gtp}_{L_{x_n}^0}(\bar{a}^1 \hat{ } \bar{b}^1, N; M_1^{x_0}) = \text{gtp}_{L_{x_n}^0}(\bar{a}^2 \hat{ } \bar{b}^2; N; M_2^{x_0})\}$.

Clearly $(\langle \rangle, \langle \rangle) \in \mathcal{F}_0$ and if $(\bar{b}^1, \bar{b}^2) \in \mathcal{F}_n, \ell \in \{1, 2\}$ and $b_n^\ell \in M_\ell^{x_0}$ then there is $b_n^{3-\ell} \in M_{3-\ell}^{x_0}$ such that $(\bar{b}^1 \hat{ } \langle b_n^1 \rangle, \bar{b}^2 \hat{ } \langle b_n^2 \rangle) \in \mathcal{F}_{n+1}$. As $M_1^{x_0}, M_2^{x_0}$ are countable we can find an isomorphism.]

But this is as required in the second phrase of (c).

We still have to prove the first phrase. For this we prove by induction on the ordinal α that

- \otimes_α^1 if for $\ell = 1, 2, \bar{a}_\ell \in {}^{\omega >}(N_\ell)$ materialize in (N_ℓ, N_*) a complete $L_{<\alpha}^0$ -type $p(\bar{x})$ not depending on ℓ and $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\omega}^0(N_*)$ has quantifier depth $< \alpha$ then: $\ell \in \{1, 2\} \Rightarrow (N_\ell, N_*) \models_{\aleph}^{\aleph_1} \varphi(\bar{a}_\ell)$ or $\ell \in \{1, 2\} \Rightarrow (N_\ell, N_*) \models_{\aleph}^{\aleph_1} \neg \varphi(\bar{a}_\ell)$.

For countable $N \leq_{\aleph} M$ and $\bar{a} \in {}^{\omega >}N$

- \odot_1 let $\mathbf{P}_\alpha(N, M, \bar{a}) = \{\text{gtp}_{L_{<\alpha}^0}(\bar{a}; N; M^\perp) : M \leq_{\aleph} M^+ \in K_{\aleph_0} \text{ and } \text{gtp}_{L_\alpha^0}(\bar{a}; N; M^+) \text{ is a complete } L_\alpha^0\text{-type}\}$.

Now

- ⊙₂ for $\beta < \alpha < \omega_1$, from $\text{gtp}_{L_\alpha^0}(\bar{a}; N; M)$ we can complete $\mathbf{P}_\beta(N, M, \bar{a})$
- ⊙₃ for $\alpha < \omega_1$, from $\mathbf{P}_\beta(N, M, \bar{a})$ we can compute $\text{gtp}_{L_\alpha^0}(\bar{a}; N; M)$
- ⊙₄ assume $N \leq_{\aleph} M$ are countable and $\bar{a} \in {}^\omega M$; for $\varphi(\bar{x}) \in L_{\omega_1, \omega}^0(N)$ of quantifier depth $< \alpha$ we have:
 $\varphi(\bar{x}) \in \text{gtp}_{\mathbb{L}_{\omega_1, \omega}^0(N)}(\bar{a}; N; M)$, iff for every $q(\bar{x}) \in \mathbf{P}_\alpha(N, M, \bar{a})$, $\varphi(\bar{x})$ belong to the type computed implicitly in \otimes_α , i.e. if $q(\bar{x}) = \text{gtp}_{L_{<\alpha}^0}(\bar{a}'; N'; M')$ then $(N', M') \Vdash_{\aleph}^{\aleph_1} \varphi(\bar{x})$.

Those three should be clear and gives the desired conclusion. Also the last sentence is easy.

Clause (d): Let $N_0 \leq_{\aleph} M_0 \in K_{\aleph_0}$ and $\bar{a}_0 \in M_0$ be such that $(M_0, N_0) \Vdash_{\aleph}^{\aleph_1} \bigwedge_{\varphi(\bar{x}) \in p} \varphi[\bar{a}_0]$,

(if it does not exist, the set of triples is empty). Let $K'' := \{(N, M, \bar{a}) : M \in K_{\aleph_0}, N \in K_{\aleph_0}, N \leq_{\aleph} M, \text{ and there are } M'' \in K_{\aleph_0}, M \leq_{\aleph} M'' \text{ and } \leq_{\aleph}\text{-embedding } f : M_0 \rightarrow M'', \text{ such that } f(N_0) = N, g(\bar{a}_0) = \bar{a}\}$. Clearly it is a PC_{\aleph_0} class. Also $M_0 \leq_{\aleph} M' \in K_{\aleph_0} \Rightarrow \text{gtp}_{\mathbb{L}_{\omega_1, \omega}^0(N_0)}(\bar{a}; N_0; M_0) = \text{gtp}_{\mathbb{L}_{\omega_1, \omega}^0(N_0)}(\bar{a}; N_0; M')$.

Now first if $(N, M, \bar{a}) \in K''$ let (M'', f) witness this so by applying clause (b) of 4.13 $\text{gtp}_{\mathbb{L}_{\omega_1, \omega}^0}(\bar{a}; N; M) \subseteq \text{gtp}_{\mathbb{L}_{\omega_1, \omega}^0}(\bar{a}; N; M'') = \text{gtp}_{\mathbb{L}_{\omega_1, \omega}^0}(\bar{a}; N; f(M_0)) = \text{gtp}_{\mathbb{L}_{\omega_1, \omega}^0}(a_0; N_0; M_0) = p$ so $(N, M, \bar{a}) \in K^1$.

Second, if $(N, M, \bar{a}) \in K^1$ let f_0 be an isomorphism from M_0 onto M . Let (M_1, f_1) be such that $N_0 \leq_{\aleph} M_1 \in K_{\aleph_0}$, $f_1 \supseteq f_0$ is an isomorphism from M_1 onto M and $\bar{a}_1 = f_1^{-1}(\bar{a})$ hence $p = \text{gtp}_{\mathbb{L}_{\omega_1, \omega}^0}(\bar{a}_1; N_0; M_1)$ and we apply clause (c) of 4.13 with $N_0, M_0, \bar{a}_0, M_1, \bar{a}_1$ here standing for $N, M_1, \bar{a}_1, M_2, \bar{a}_2$ there and can finish easily.

Clause (e): We can define $\langle L_\alpha^{-1} : \alpha < \omega_1 \rangle$ satisfying the parallel of Clause (a) and repeat the proofs of clauses (b),(c) and we are done.

Clause (f): Suppose this fails.

The proof splits to two cases.

Case A: $2^{\aleph_0} = 2^{\aleph_1}$.

We shall prove $\dot{I}(\aleph_1, K) \geq 2^{\aleph_0}$, thus, (as $2^{\aleph_0} = 2^{\aleph_1}$) contradicting Hypothesis 4.8. Let p_i (for $i < \omega_2$) be distinct complete $\mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ -types such that for each i , p_i is materialized in some pair $(M; N)$, so $N \leq_{\aleph} M \in K_{\aleph_0}$ (they exist by the assumption that (f) fails). For each $i < \omega_2$ we define $N_{i, \alpha}, \xi_{i, \alpha}$ (for $\alpha < \omega_1$) and $\bar{a}_{i, \alpha}$ such that:

- ⊠₁ (i) $N_{i, \alpha} \in K_{\aleph_0}$ has universe $\omega(1 + \alpha)$, $N_{0,0} = N$
- (ii) $\langle N_{i, \alpha} : \alpha < \omega_1 \rangle$ is \leq_{\aleph} -increasing continuous
- (iii) $\bar{a}_{i, \alpha} \in N_{i, \alpha+1}, \bar{a}_{i, \alpha}$ materialize p_i in $(N_{i, \alpha+1}, N_{i, \alpha})$
- (iv) for every $\alpha < \beta < \omega_1$ and $\bar{a} \in {}^\omega(N_{i, \beta})$, the sequence \bar{a} materialize

in $(N_{i, \beta}, N_{i, \alpha})$ a complete $\mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ -type

- (v) $\xi_{i,\alpha} < \omega_1$ is strictly increasing continuous in α
 (vi) for $\alpha < \beta$, $N_{i,\beta}$ is pseudo $L_{\beta}^0(N_{i,\alpha})$ -generic, see 4.4(4) and take care

of \mathbf{Q} , i.e., if $\gamma < \beta$, $p(y, \bar{x})$ a complete L_{γ}^0 -type and

$$(N_{i,\beta}, N_{i,\alpha}) \Vdash_{\mathfrak{K}}^{\aleph_1} (\mathbf{Q}y) \wedge p(y, \bar{a})$$

then for some $b \in N_{i,\beta+1} \setminus N_{i,\beta}$ we have $(N_{i,\beta+1}, N_{i,\alpha}) \Vdash_{\mathfrak{K}}^{\aleph_1} \wedge p(b, \bar{a})$

- (vii) if $\alpha < \beta$ and $\bar{a}, \bar{b} \in N_{\beta-1}$ materialize different $\mathbb{L}_{\omega_1, \omega}^0(N_{i,\alpha})$ -types in

$N_{i,\beta}$, then \bar{a}, \bar{b} realize different $(\mathbb{L}_{\omega_1, \omega}(\tau^{+0}) \cap L_{\xi_{i,\beta+1}}^{-1})(N_{\alpha})$ -types in

$N_{i,\beta}$

- (viii) $N_i = \cup\{N_{i,\alpha} : \alpha < \omega_1\}$

- (ix) if $\alpha_\ell < \beta$ for $\ell = 1, 2$, $\gamma < \beta$, $n < \omega$ and $\bar{a}_1 \in {}^n(N_{i,\beta})$ then for some

$$\bar{a}_2 \in {}^n(N_{i,\beta}) \text{ we have } \text{gtp}_{L_{\gamma}^0}(\bar{a}_1; N_{i,\alpha_1}; N_{i,\beta}) =$$

$$\text{gtp}_{L_{\gamma}^0}(\bar{a}_2; N_{i,\alpha_2}; N_{i,\beta})$$

- (ix)⁺ moreover, if $n < \omega$, $\gamma_1 < \gamma_2 < \beta$, $\alpha_\ell < \beta$, $\bar{a}_\ell \in {}^n(N_{i,\beta})$ for $\ell = 1, 2$

$$\text{and } \text{gtp}_{L_{\gamma_2}^0}(\bar{a}_1; N_{i,\alpha_{-1}}; N_{i,\beta}) = \text{gtp}_{L_{\gamma_2}^0}(\bar{a}_2; N_{i,\alpha_2}; N_{i,\beta}) \text{ and}$$

$$b_1 \in N_{i,\beta} \text{ then for some } b_2 \in N_{i,\beta} \text{ we have } \text{gtp}_{L_{\gamma_1}^0}(\bar{a}_1 \hat{\ } \langle b_1 \rangle; N_{i,\alpha_1}; N_{i,\beta})$$

$$= \text{gtp}_{L_{\gamma_1}^0}(\bar{a} \hat{\ } \langle b_2 \rangle; N_{i,\alpha_2}; N_{i,\beta}).$$

This is possible by the earlier claims. By clause (e) of 4.13 clearly

\boxtimes_2 the pair (N_i, N_0) is $L_{<\omega_1}^{-1}(\tau^{+0})$ -homogeneous.

We could below use D_i a set of complete $L_{\delta(i)}^0$ -types, the only problem is that the countable (D_i, \aleph_0) -homogeneous models have to be redefined using “materialized” instead “realized”. As it is we need to use clause (e) to translate the results on $L_{\delta(i)}^0$ to $L_{\delta(i)}^{-1}$.

Let $\tau^* = \{\in, Q_1, Q_2\} \cup \{c_\ell : \ell < 5\}$, c_ℓ an individual constant and \mathfrak{A}_i^* be $(\mathcal{H}(\aleph_2), \in)$ expanded to a τ^* -model, by predicates for $K, \leq_{\mathfrak{K}}$ with $Q_1^{\mathfrak{A}_i^*} = K \cap \mathcal{H}(\aleph_2)$, $Q_2^{\mathfrak{A}_i^*} = \{(M, N) : M \leq_{\mathfrak{K}} N \text{ both in } \mathcal{H}(\aleph_2)\}$, $c_0^{\mathfrak{A}_i^*}, \dots, c_4^{\mathfrak{A}_i^*}$ being $\{\langle N_{i,\alpha} : \alpha < \omega_1 \rangle\}, \langle \xi_{i,\alpha} : \alpha < \omega_1 \rangle, \{\langle \bar{a}_{i,\alpha} : \alpha < \omega_1 \rangle\}, N_i$ and $\{i\}$ respectively.

Let \mathfrak{A}_i be a countable elementary submodel of \mathfrak{A}_i^* so $|\mathfrak{A}_i| \cap \omega_1$ is an ordinal $\delta(i) < \omega_1$. It is also clear that $c_3^{\mathfrak{A}_i}$ is $N_{i,\delta(i)}$ as $c_3^{\mathfrak{A}_i^*} = N_i$. As \mathfrak{A}_i is defined for $i < \omega_2$, for some unbounded $S \subseteq \omega_2$ and $\delta < \omega_1$, for every $i \in S$, $\delta(i) = \delta$ and for $i, j \in S$, some sequence from N_j materializes p_i in the pair $(N_j, N_{j,\delta(j)})$ iff $i = j$. For $i \in S$ let $D_i = \{p : p \text{ is a complete } L_{\delta(i)}^{-1}\text{-type materialized in } (N_{i,\delta(i)}, N_{i,0})\}$. Because of the $\xi_{i,\alpha}$'s choice and \boxtimes_2 the pair $(N_{i,\delta}, N_0)$

is (D_i, \aleph_0) -homogeneous and D_i is a countable set of complete L_{δ}^{-1} -types. Note that by the choice of $S, i \neq j (\in S) \Rightarrow D_i \neq D_j$.

Let $\Gamma = \{D : D \text{ a countable set of complete } L_{\delta}^{-1}\text{-types, such that for some model } \mathfrak{A} = \mathfrak{A}_D \text{ of } \bigcap_{i \in S} \text{Th}_{\mathbb{L}_{\omega, \omega}}(\mathfrak{A}_i), \text{ with } \{a : \mathfrak{A}_D \models \text{“a countable ordinal”} = \delta \text{ (and the usual order) we have } D = \{\{\varphi(\bar{x}) : \varphi(\bar{x}) \in L_{\delta}^{-1} \text{ and } \mathfrak{A}_D \models \text{“}(N; N_0) \models_{\aleph_1} \varphi[\bar{a}]”\} : \bar{a} \in N \text{ where } N = c_3^{\mathfrak{A}_D}\}\}$.

So $D_i \in \Gamma$ for $i < \omega_2$, hence Γ is uncountable.

By standard descriptive set theory Γ (is an analytic set hence) has cardinality continuum. So let $D(\zeta) \in \Gamma$ be distinct for $\zeta < 2^{\aleph_0}$. For each ζ , let $\mathfrak{A}_{D(\zeta)}^0$ be as in the definition of Γ . We define by induction on $\alpha < \omega_1, \mathfrak{A}_{D(\zeta)}^{\alpha}$ such that

- (α) $\mathfrak{A}_{D(\zeta)}^{\alpha}$ is countable
- (β) $\alpha < \beta \Rightarrow \mathfrak{A}_{D(\zeta)}^{\alpha} \prec_{\mathbb{L}_{\omega, \omega}} \mathfrak{A}_{D(\zeta)}^{\beta}$
- (γ) for limit α we have $\mathfrak{A}_{D(\zeta)}^{\alpha} = \bigcup_{\beta < \alpha} \mathfrak{A}_{D(\zeta)}^{\beta}$
- (δ) if $d \in \mathfrak{A}_{D(\zeta)}^{\alpha+1} \setminus \mathfrak{A}_{D(\zeta)}^{\alpha}, \mathfrak{A}_{D(\zeta)}^{\alpha+1} \models \text{“}d \text{ a countable ordinal”}$ then for $a \in \mathfrak{A}_{D(\zeta)}^{\alpha}$ we have $\mathfrak{A}_{D(\zeta)}^{\alpha+1} \models \text{“if } a \text{ is a countable ordinal then } a < d\text{”}$
- (ε) for $\alpha = 0$ in clause (δ) there is no minimal such d
- (ζ) for every α there is $d_{\zeta, \alpha} \in \mathfrak{A}_{D(\zeta)}^{\alpha+1} \setminus \mathfrak{A}_{D(\zeta)}^{\alpha}$ satisfying $\mathfrak{A}_{D(\zeta)}^{\alpha+1} \models \text{“}d_{\zeta, \alpha} \text{ a countable ordinal”}$ and for $\alpha \neq 0$ it is minimal.

without loss of generality

(*) $(\mathcal{H}(\aleph_1)^{\mathfrak{A}_{D(\zeta)}^0}, \in^{\mathfrak{A}_{D(\zeta)}^0})$ is equal to its Mostowski collapse (and $\mathbb{L}_{\omega_1, \omega}(N) \subseteq \mathcal{H}(\aleph_1)$).

(We could have fixed also $\text{otp}(\mathfrak{A}_i \cap \omega_2)$, hence ensure also $(\mathfrak{A}_{D(\zeta)}^0, \in^{\mathfrak{A}_{D(\zeta)}^0})$ is equal to its Mostowski collapse).

Let $M_{\zeta, \alpha}$ be the $d_{\zeta, \alpha}$ -th member of the ω_1 -sequence of models in $\mathfrak{A}_{D(\zeta)}^{\beta}$ for $\beta > \alpha$ (remember $c_0^{\mathfrak{A}_i} = \langle N_{i, \alpha} : \alpha < \omega_1 \rangle$). Let $M_{\zeta} = \bigcup_{\alpha < \omega_1} M_{\zeta, \alpha}$. By absoluteness from $\mathfrak{A}_{D(\zeta)}^{\beta}$ we have

$M_{\zeta, \alpha} \leq_{\aleph} M_{\zeta, \beta} \in K_{\aleph_0}$. Now

(*) $0 < \alpha < \beta, (M_{\zeta, \beta}, M_{\zeta, \alpha})$ is $(D(\zeta), \aleph_0)$ -homogeneous.

[Why? Assume $\mathfrak{A}_{D(\zeta)}^{\alpha} \models \text{“}d_1 < d_2 \text{ are countable ordinals } > \gamma\text{”}$ when $\gamma < \delta$. Now if $\bar{a}, \bar{b} \in \omega^{\langle N_{d_2}^{\mathfrak{A}_{D(\zeta)}^{\alpha}} \rangle}$ and $[\gamma < \delta \Rightarrow \text{gtp}_{L_{\gamma}^0}(\bar{a}; N_{d_1}^{\mathfrak{A}_{D(\zeta)}^{\alpha}}; N_{d_2}^{\mathfrak{A}_{D(\zeta)}^{\alpha}}) = \text{gtp}_{L_{\gamma}^0}(\bar{b}; N_{d_1}^{\mathfrak{A}_{D(\zeta)}^{\alpha}}; N_{d_2}^{\mathfrak{A}_{D(\zeta)}^{\alpha}})]$ then also $\mathfrak{A}_{D(\zeta)}^{\alpha}$ satisfies this but “ $\mathfrak{A}_{D(\zeta)}^{\alpha}$ thinks that the countable ordinals are well ordered” hence for some $d, \mathfrak{A}_{D(\zeta)}^{\alpha} \models \text{“}d \text{ is a countable ordinal } > \gamma\text{”}$ for each $\gamma < \delta$ and we have $\mathfrak{A}_{D(\zeta)}^{\alpha} \models \text{“gtp}_{L_d^0}(\bar{a}; N_{d_1}; N_{d_2}) = \text{gtp}_{L_d^0}(\bar{a}; N_{d_1}; N_{d_2})\text{”}$. Hence if $\mathfrak{A}_{D(\zeta)}^{\alpha} \models \text{“}d' < d\text{”}$ then for every $a \in N_{d_2}^{\mathfrak{A}_{D(\zeta)}^{\alpha}}$ for some $b \in N_{d_2}^{\mathfrak{A}_{D(\zeta)}^{\alpha}}$ we have

$$\mathfrak{A}_{D(\zeta)}^{\alpha} \models \text{“gtp}_{L_d^0}(\bar{a} \hat{\ } \langle a \rangle; N_{d_1}; N_{d_2}) = \text{gtp}_{L_d^0}(\bar{b} \hat{\ } \langle b \rangle; N_{d_1}; N_{d_2})\text{”}$$

hence $\text{gtp}_{L_\gamma^0}(\bar{a} \hat{\langle} a \rangle; N^{\mathfrak{A}_{D(\zeta)}^\alpha}; N_{d_2}^{\mathfrak{A}_{D(\zeta)}^\alpha}) = \text{gtp}(\bar{b} \hat{\langle} b \rangle; N_{d_1}^{\mathfrak{A}_{D(\zeta)}^\alpha}; N_{d_2}^{\mathfrak{A}_{D(\zeta)}^\alpha})$.

Also we can replace L_δ^0 by L_δ^{-1} . By clause (x) of \boxtimes_1 the set $\{\text{gtp}_{L_\delta^0}(\bar{a}; N_{d_1}^{\mathfrak{A}_{D(\zeta)}^\alpha}; N_{d_2}^{\mathfrak{A}_{D(\zeta)}^\alpha}) : a \in {}^{\omega >}(N_{d_2}^{\mathfrak{A}_{D(\zeta)}^\alpha})\}$ is D_i .

So $(N_{d_2}^{\mathfrak{A}_{D(\zeta)}^\alpha}, N_{d_2}^{\mathfrak{A}_{D(\zeta)}^\alpha})$ is (D_i, \aleph_0) -homogenous.

So from the isomorphism type of M_ζ we can compute $D(\zeta)$. So $\zeta \neq \xi \Rightarrow M_\zeta \not\cong M_\xi$. As $M_\zeta \in K_{\aleph_1}$ we finish. Case B: $2^{\aleph_0} < 2^{\aleph_1}$.

By 3.9, \mathfrak{K} has the \aleph_0 -amalgamation property. So clearly if $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$, $\bar{a} \in M$, then \bar{a} materializes in (M, N) a complete $\mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ -type. We would now like to use descriptive set theory.

We represent a complete $\mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ -type materialized in some (N, M) by a real, by representing the isomorphism type of some (N, M, \bar{a}) , $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$, $\bar{a} \in M$. The set of representatives is analytic recalling \mathfrak{K} is PC_{\aleph_0} , and the equivalence relation is Σ_1^1 . [As $(N_1, M_1, \bar{a}_1), (N_2, M_2, \bar{a}_2)$ represents the same type if and only if for some (N, M) , $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$, there are $\leq_{\mathfrak{K}}$ -embeddings $f_1 : M_1 \rightarrow M, f_2 : M_2 \rightarrow M$ such that $f_1(N_1) = f_2(N_2) = N$ and $f_1(\bar{a}_1) = f_2(\bar{a}_2)$.]

By Burgess [Bur78] (or see [Sh:202]) as there are $> \aleph_1$ equivalence classes, there is a perfect set of representation, pairwise representing different types.

From this we easily get that without loss of generality that their restriction to some L_α^0 are distinct, contradicting part (a).

Clause (g): Easy by the proof of clause (f), Case A above but much simpler as in 4.12.

Clause (h): As in the proof of clause (e).

2) Should be clear by now. □_{4.13}

4.16. **Remark.** 1) Note that in the proof of Clause (f) of 4.13, in Case (A) we get many types too but it was not clear whether we can make the N_ζ to be generic enough, to get the contradiction we got in Case (B) but this is not crucial here.

2) We may like to replace $\mathbb{L}_{\omega_1, \omega}^0$ by $\mathbb{L}_{\omega_1, \omega}^1$ in 4.10, 4.11 and 4.13 (except that, for our benefit, in 4.13(e), we may retain the definition of $L^1(N)$). We lose the ability to build L -generic models in K_{\aleph_1} (as the number of (even unary) relations on $N \in K_{\aleph_0}$ is 2^{\aleph_0} , which may be $> \aleph_1$). However, we can say “ \bar{a} materializes in $N \in K_{\aleph_0}$ the type $p = p(\bar{x})$ which is a complete type in $\mathbb{L}_{\omega_1, \omega}^1(N_n, N_{n-1}, \dots, N_0)$; where $N_0 \leq_{\mathfrak{K}} \dots \leq_{\mathfrak{K}} N_n \leq_{\mathfrak{K}} N, N_\ell$ countable”.

[Why? Let some N^1, \bar{a}^1 be as above, \bar{a}^1 materialize p in (N^1, N_n, \dots, N_0) then this holds for (N, \bar{a}) iff for some N', f we have $N \leq_{\mathfrak{K}} N' \in K_{\aleph_1}$ and f is an isomorphism from N^1 onto N' mapping \bar{a}^1 to \bar{a} and N_ℓ to N_ℓ for $\ell \leq n$. If there is no such pair (N^1, \bar{a}^1) this is trivial.]

We can get something on formulas.

This suffices for 4.10.

{88r-4.9}

4.17. **Concluding remarks for §4.** 0) We can get more information on the case $1 \leq \dot{I}(\aleph_1, K) < 2^{\aleph_1}$ (and the case $1 \leq \dot{I}(\aleph_1, K_{\aleph_1}^{\mathbf{F}}) < 2^{\aleph_1}$, etc.).

1) As in 3.9, there is no difficulty in getting the results of this section for the class of models of $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ because using (K, \leq_{\aleph}) from the proof of 3.19(2) in all constructions we get many non-isomorphic models for appropriate \mathbf{F} , as in 4.9(2).

2) For generic enough $N \in K_{\aleph_1}$ with \leq_{\aleph} -representation $\langle N_\alpha : \alpha < \omega_1 \rangle$, we have determined the N_α 's (by having that without loss of generality K is categorical in \aleph_0). In this section we have shown that for some club E of ω_1 , for all $\alpha < \beta$ from E the isomorphism type of (N_β, N_α) essentially ⁶ is unique. We can continue the analysis, e.g., deal with sequences $N_0 \leq_{\aleph} N_1 \leq_{\aleph} \dots \leq_{\aleph} N_k \in K_{\aleph_0}$ such that $N_{\ell+1}$ is pseudo $L_\alpha^0(N_\ell, N_{\ell-1}, \dots, N_0)$ -generic. We can prove by induction on k that for any countable $L \subseteq \mathbb{L}_{\omega_1, \omega}^0(\tau^{+k})$ for some α , any strong L -generic $N \in K_{\aleph_1}$ is L -determined. That is, for any $\langle N_\alpha : \alpha < \omega_1 \rangle, N_\alpha \leq_{\aleph} N$ countable \leq_{\aleph} -increasing continuous with union N , for some club E for all $\alpha_0 < \dots < \alpha_k$ from N the isomorphic type of $\langle N_{\alpha_k}, N_{\alpha_k}, \dots, N_{\alpha_0} \rangle$ is the same; i.e., determining for $\mathbb{L}_{\infty, \omega}(aa)$.

3) We can do the same for stronger logics, let us elaborate.

Let us define a logic \mathcal{L}^* . It has as variable

variables for elements x_1, x_2, \dots and

variables for filters $\mathcal{Y}_1, \mathcal{Y}_2, \dots$. The atomic formulas are:

(i) the usual ones

(ii) $x \in \text{Dom}(\mathcal{Y})$.

The logical operations are:

(a) \wedge conjunction, \neg negation

(b) $(\exists x)$ existential quantification where x is individual variable

(c) the quantifier aa acting on variables \mathcal{Y} so we can form $(aa \mathcal{Y})\varphi$

(d) the quantification $(\exists x \in \text{Dom}(\mathcal{Y}))\varphi$

(e) the quantification $(\exists^f x \in \text{Dom}(\mathcal{Y}))\varphi$.

It should be clear what are the free variables of a formula φ . The variable \mathcal{Y} vary on pairs (a countable set, a filter on the set). Now in $\exists x[\varphi, \mathcal{Y}]$, $(\exists x \in \text{Dom}(\mathcal{Y}))\varphi$, $(\exists^f x \in \text{Dom}(\mathcal{Y}))\varphi$, x is bounded but not \mathcal{Y} and in $aa\mathcal{Y}$, \mathcal{Y} is bounded. The satisfaction relation is defined as usual plus

(α) $M \models (\exists x \in \text{Dom}(\mathcal{Y}))\varphi(x, \mathcal{Y}, \bar{a})$ if and only if for some b from the domain of \mathcal{Y} , $M \models \varphi[b, \mathcal{Y}, \bar{a}]$

(β) $M \models \exists^f x \in \text{Dom}(\mathcal{Y})\varphi(x, \mathcal{Y}, \bar{a})$ if and only if $\{x \in \text{Dom}(\mathcal{Y}) : \models \varphi(x, \mathcal{Y}, \bar{a})\} \in \mathcal{Y}$

(γ) $M \models (aa \mathcal{Y}, \bar{a})\varphi(\mathcal{Y})$ if and only if there is a function \mathbf{F} from ${}^{\omega}>([M]^{<\aleph_1}) \rightarrow [M]^{<\aleph_1}$ such that: if $A_n \subseteq M$, $|A_n| \leq \aleph_0$, $A_n \subseteq A_{n+1}$ and $\mathbf{F}(A_0, \dots, A_n) \subseteq A_{n+1}$ then

⁶why only essentially? as the number of relevant complete types can be \aleph_1 ; we can get rid of this by shrinking \aleph

modified:2015-01-25

(88r) revision:2015-01-23

$M \models \varphi[\mathcal{Y}_{\langle A_n : n < \omega \rangle}, \bar{a}]$ where $\mathcal{Y}_{\langle A_n : n < \omega \rangle}$ is the filter on $\bigcup_{n < \omega} A_n$, generated by $\{\cup\{A_n : n < \omega\} \setminus A_\ell : \ell < \omega\}$.

4) We, of course, can define $\mathcal{L}_{\mu, \kappa}^*$ (extending $\mathbb{L}_{\mu, \ell}$). As we like to analyze models in \aleph_1 , it is most natural to deal with $\mathcal{L}_{\omega_1, \omega}^*$. We can prove that (if $1 \leq \dot{I}(\aleph_1, \aleph) < 2^{\aleph_1}$) the quantifier $aa \mathcal{Y}$ is determined on K_{\aleph_1} (i.e., for almost all $\mathcal{Y}, \varphi(\mathcal{Y})$ iff not for almost all $\mathcal{Y}, \neg\varphi(\mathcal{Y})$).

5) The logic from (3) strengthens the stationary logic $\mathbb{L}(aa)$, see [Sh:43], [BKM78]. Not so strongly: looking at PC_{\aleph_0} class for $\mathbb{L}_{\omega_1, \omega}(aa)$ (i.e., $\{M \upharpoonright \tau : M \text{ a model of } \psi \text{ of cardinal } \aleph_1\}$), we can assume that $\psi \vdash$ “ $<$ is an \aleph_1 -like order”. Now we can express $\varphi \in \mathcal{L}_{\omega_1, \omega}^*$, but the determinacy tells us more. Also we can continue to define higher variables \mathcal{Y} .

5. THERE IS A SUPERLIMIT MODEL IN \aleph_1

Here we make

{88r-5.0}

5.1. **Hypothesis.** Like 4.8, but also $2^{\aleph_0} < 2^{\aleph_1}$.

(Note that we can assume that K_{\aleph_0} is the class of atomic models of a first order complete countable theory).

This section is the deepest (of this paper = chapter). The main difficulties are proving the facts which are obvious in the context of [Sh:48]. So while it was easy to show that every $p \in \mathbf{D}^*(N)$ is definable over a finite set ($\mathbf{D}^*(N)$ is defined below), it was not clear to me how to prove that if you extend the type p to $q \in \mathbf{D}^*(M)$ where $N \leq_{\aleph} M \in K_{\aleph_0}$, by the same definition, then $q \models p$ (remember p, q are types materialized not realized, and at this point in the paper we still do not have the tools to replace the models by uncountable generic enough models). So we rather have to show that failure is a non-structure property, i.e., implies existence of many models.

Also symmetry of stable amalgamation becomes much more complicated. We prove existence of stable amalgamation by four stages (5.29,5.30(3),5.34,5.37). The symmetry is proved as a consequence of uniqueness of one sided amalgamation, (so it cannot be used in its proof). Originally the intention was the culmination of the section to be the existence of a superlimit models in \aleph_1 (5.45). This seems a natural stopping point as it seems reasonable to expect that the next step should be phrasing the induction on n , i.e., dealing with \aleph_n and $\mathcal{P}(n - \ell)$ -diagrams of models of power \aleph_ℓ as in [Sh:87a], [Sh:87b]; (so this is done in [Sh:705]). But less is needed in [Sh:600].

5.2. **Definition.** We define functions \mathbf{D}, \mathbf{D}^* with domain K_{\aleph_0} .

{88r-5.1}

1) For $N \in K_{\aleph_0}$ let $\mathbf{D}(N) = \{p : p \text{ is a complete } \mathbb{L}_{\omega_1, \omega}^0(N)\text{-type over } N \text{ such that for some } \bar{a} \in M \in K_{\aleph_0}, N \leq_{\aleph} M \text{ and } \bar{a} \text{ materializes } p \text{ in } (M, N)\}$, (i.e. the members of p have the form $\varphi(\bar{x}, \bar{a})$, (\bar{x} finite and fixed for p) \bar{a} a finite sequence from N and $\varphi \in \mathbb{L}_{\omega_1, \omega}^0(N)$).

2) For $N \in K_{\aleph_0}$ let $\mathbf{D}^*(N) = \{p : p \text{ a complete } \mathbb{L}_{\omega_1, \omega}^0(N; N)\text{-type such that for some } \bar{a} \in M \in K_{\aleph_0}, N \leq_{\aleph} M \text{ and } \bar{a} \text{ materializes } p \text{ in } (M, N; N)\}$.

3) For $p(\bar{x}, \bar{y}) \in \mathbf{D}(N)$ let $p(\bar{x}, \bar{y}) \upharpoonright \bar{x} \in \mathbf{D}(N)$ be defined naturally; i.e. if for some $M, N \leq_{\aleph} M \in K_{\aleph_0}$ and $\bar{a} \upharpoonright \bar{b} \in {}^{\ell g(\bar{x} \wedge \bar{y})} M$ materializing $p(\bar{x}, \bar{y})$ such that $\ell g(\bar{x}) = \ell g(\bar{a})$, the sequence \bar{a} materializes $p(\bar{x}, \bar{y}) \upharpoonright \bar{x} \in \mathbf{D}(N)$. Similarly for permuting the variables.

5.3. **Explanation.** 0) Recall that any formula in $\mathbb{L}_{\omega_1, \omega}^0(N)$ has finitely many free variables.

{88r-5.1A}

1) So for every finite $\bar{b} \in N$ and $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\omega_1, \omega}^0(N)$, if $p \in \mathbf{D}(N)$, then $\varphi(\bar{x}, \bar{b}) \in p$ or $\neg \varphi(\bar{x}, \bar{b}) \in p$.

2) But a formula from $p \in \mathbf{D}^*(N)$ may have all $c \in N$ as parameters whereas a formula from $p \in \mathbf{D}(N)$ can mention only finitely many members of N .

5.4. **Lemma.** 1) \aleph has the \aleph_0 -amalgamation property.

{88r-5.2}

modified:2015-01-25

(88r) revision:2015-01-23

(88r)

- 2) If $N_* \leq_{\aleph} N \in K_{\aleph_0}$, $A_i \subseteq N_*$ for $i \leq n$ then for every sentence $\psi \in \mathbb{L}_{\infty, \omega}^1(N_*, A_n, \dots, A_1; A_0)$ we have

$$N \Vdash_{\aleph}^{\aleph_1} \psi \text{ or } N \Vdash_{\aleph}^{\aleph_1} \neg\psi.$$

- 3) If $N \leq_{\aleph} M \in K_{\aleph_0}$, then every $\bar{a} \in M$ materializes in $(M, N; N)$ one and only one type from $\mathbf{D}^*(N)$ and also materializes in (M, N) one and only one type from $\mathbf{D}(N)$. Also for every $N \leq_{\aleph} M \in K_{\aleph_0}$ and $q \in \mathbf{D}^*(N)$ for some $M', M \leq_{\aleph} M' \in K_{\aleph_0}$ and some $\bar{b} \in M'$ materializes q in $(M; N)$.

- 4) For every $N \in K_{\aleph_0}$ and countable $L \subseteq \mathbb{L}_{\omega_1, \omega}^0(N; N)$ the number of complete $L(N; N)$ -types p such that $N \Vdash_{\aleph}^{\aleph_1} "(\exists \bar{x}) \wedge p"$ is countable; note that pedantically $L \subseteq \mathbb{L}_{\omega_1, \omega}(\tau^+ \cup \{c : c \in N\})$ and we restrict ourselves to models M such that $P^M = |N|, c^M = c$.

- 5) For $N \in K_{\aleph_0}$ there are countable $L_\alpha^0 \subseteq \mathbb{L}_{\omega_1, \omega}^0(N; N)$ for $\alpha < \omega_1$ increasing continuous in α , closed under finitary operations (and subformulas) such that:

(*) for each complete L_α^0 -type p we have

$$[N \Vdash_{\aleph}^{\aleph_1} \exists \bar{x} \wedge p \Rightarrow \wedge p \in L_{\alpha+1}^0].$$

Hence for every $\mathbb{L}_{\omega_1, \omega}^0(N; N)$ formula $\psi(\bar{x})$ for some $\varphi_n(\bar{x}) \in \bigcup_{\alpha < \omega} L_\alpha^0$ for $n < \omega$ for every $N \in K_{\aleph_0}$

$$(N, N) \Vdash_{\aleph}^{\aleph_1} (\forall \bar{x})[\psi(\bar{x}) \equiv \bigvee_{n < \omega} \varphi_n(\bar{x})].$$

- 6) For $N \in K_{\aleph_0}$ we have $|\mathbf{D}^*(N)| \leq \aleph_1$ and $|\mathbf{D}(N)| \leq \aleph_1$.

- 7) If $p \in \mathbf{D}^*(N)$ then there is q such that: if $N \leq_{\aleph} M \in K_\lambda, \bar{a} \in M$ materializes p in $(M; N)$ then the complete $\mathbb{L}_{\infty, \omega}^0(N)$ -type which \bar{a} realizes in M over N is q ; also q belongs to $\mathbf{D}(N)$ and is unique. Moreover, we can replace q by the complete $\mathbb{L}_{\omega_1, \omega}^{-1}(N)$ -type which \bar{a} materializes in M . Similarly for $\mathbf{D}(N), \mathbb{L}_{\infty, \omega}^0(N), \mathbb{L}_{\omega_1, \omega}^{-1}(N)$.

- 8) If $n < \omega$ and $\bar{b}, \bar{c} \in {}^n N$ realize the same $\mathbb{L}_{\omega_1, \omega}(\tau)$ -type in N then they materialize the same $\mathbb{L}_{\omega_1, \omega}^1(\tau^{+0})$ -type in (N, N) .

- 9) If f is an isomorphism from $N_1 \in K_{\aleph_0}$ onto $N_2 \in K_{\aleph_0}$ then f induces a one to one function from $\mathbf{D}(N_1)$ onto $\mathbf{D}(N_2)$ and from $\mathbf{D}^*(N_1)$ onto $\mathbf{D}^*(N_2)$.

Proof. 1) By 3.9.

2) By 1).

3) By 2) and 1).

4) Like the proof of 4.11 (just easier).

- 5) Like the proof of 4.13(a).
 6) Like the proof of 4.13(f) (recalling 0.4).
 7) Clear as in $p \in \mathbf{D}^*(N)$ we allow more formulas than for $q \in \mathbf{D}(N)$.
 8),9) Easy, too. □_{5.4}

We shall use from now on a variant of gtp (in Definition 4.3(4) we define $\text{gtp}_L(\bar{a}; N_*, \bar{A}; A; N)$.

5.5. Definition. 1) If $N_0 \leq_{\bar{\kappa}} N_1 \in K_{\aleph_0}$, $\bar{a} \in N_1$, $\text{gtp}(\bar{a}, N_0, N_1)$ is the $p \in \mathbf{D}(N_0)$ such that $(N_1, N_0) \models_{\bar{\kappa}}^{\aleph_1} \wedge p[\bar{a}]$. So \bar{a} materializes (but does not necessarily realize) $\text{gtp}(\bar{a}, N_0, N_1)$. We may omit N_1 when clear from context. We define $\text{gtp}^*(\bar{a}, N_0, N_1) \in \mathbf{D}^*(N_0)$ similarly.
 2) We say $p = \text{gtp}^*(\bar{b}, N_0, N_1)$ is definable over $\bar{a} \in N_0$ if $\text{gtp}(\bar{b}, N_0, N_1) = p^- := \{\varphi(\bar{x}, \bar{a}) \in p : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\omega_1, \omega}^0(N_0) \text{ and } \bar{a} \in {}^{\ell g(\bar{y})}(N_0) \subseteq {}^{\omega >}(N_0)\}$ is definable over \bar{a} (see Definition 5.7 below, note that $p \mapsto p^-$ is a one-to-one mapping from $\mathbf{D}^*(N_0)$ onto $\mathbf{D}(N_0)$ by 5.9(1) below). So stationarization is defined for $p \in \mathbf{D}^*(N_0)$, too, after we know 5.9(1).

5.6. Claim. 1) Each $p \in \mathbf{D}(N)$ does not $(\mathbb{L}_{\omega_1, \omega}^0(\tau^{+0}), \mathbb{L}_{\omega_1, \omega}(\tau))$ -split (see Definition 5.7 below; also see more below) over some finite subset C of N , hence p is definable over it. Moreover, letting \bar{c} list C there is a function g_p satisfying $g_p(\varphi(\bar{x}, \bar{y}))$ is $\psi_{p, \varphi}(\bar{y}, \bar{z}) \in \mathbb{L}_{\omega_1, \omega}(\tau)$ such that for each $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\omega_1, \omega}^0(N)$ and $\bar{a} \in N$ we have $[\varphi(\bar{x}, \bar{a}) \in p \Leftrightarrow N \models \psi_{p, \varphi}(\bar{a}, \bar{c})]$, (in particular, \mathbf{Q} is “not necessary”).

2) Every automorphism of N maps $\mathbf{D}(N)$ onto itself and each $p \in \mathbf{D}(N)$ has at most \aleph_0 possible images; we may also call them conjugates. So if g is an isomorphism from $N_0 \in K_{\aleph_0}$ onto $N_1 \in K_{\aleph_0}$ then $g(\mathbf{D}(N_0)) = \mathbf{D}(N_1)$.

3) If $N_0 \leq_{\bar{\kappa}} N_1 \leq_{\bar{\kappa}} N_2 \in K_{\aleph_0}$ and $\bar{a} \in N_1$ then $\text{gtp}(\bar{a}, N_0, N_1) = \text{gtp}(\bar{a}, N_0, N_2)$.

Before we prove 5.6:

5.7. Definition. Assume

- (a) N is a model
 - (b) Δ_1 is a set of formulas (possibly in a vocabulary $\not\subseteq \tau_N$) closed under negation
 - (c) Δ_2 is a set of formulas in the vocabulary $\tau = \tau_N$
 - (d) p is a (Δ_1, n) -type over N (i.e., each member has the form $\varphi(\bar{x}, \bar{a})$, \bar{a} from N , $\varphi(\bar{x}, \bar{y})$ from Δ_1 , $\bar{x} = \langle x_\ell : \ell < n \rangle$; no more is required (we may allow other formulas but they are irrelevant)
 - (e) $A \subseteq N$.
- 0) We say p is a complete Δ_1 -type over B when:
- (i) $B \subseteq N$
 - (ii) $\varphi(\bar{x}, \bar{b}) \in p \Rightarrow \bar{b} \subseteq A \wedge \varphi(\bar{x}, \bar{y}) \in \Delta_1$
 - (iii) if $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ and $\bar{b} \in {}^{\ell g(\bar{y})}A$ then $\varphi(\bar{x}, \bar{b}) \in p$ or $\neg \varphi(\bar{x}, \bar{b}) \in p$.

The default value here for Δ_1 is $\mathbb{L}_{\omega_1, \omega}(\tau_{\mathbb{R}})$.

1) We say that p does (Δ_1, Δ_2) -split over A when there are $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ and $\bar{b}, \bar{c} \in {}^{\ell g(\bar{y})}N$ such that

$$(\alpha) \quad \varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p$$

$$(\beta) \quad \bar{b}, \bar{c} \text{ realize the same } \Delta_2\text{-type over } A.$$

2) We say that p is (Δ_1, Δ_2) -definable over A when: for every formula $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ there is a formula $\psi(\bar{y}, \bar{z}) \in \Delta_2$ and $\bar{c} \in {}^{\ell g(\bar{z})}A$ such that

$$\varphi(\bar{x}, \bar{b}) \in p \Rightarrow N \models \psi[\bar{b}, \bar{c}]$$

$$\neg\varphi(\bar{x}, \bar{b}) \in p \Rightarrow N \models \neg\psi[\bar{b}, \bar{c}]$$

(in the case p is complete over B , $\bar{b} \subseteq B$ we get “iff”).

3) Above we may write Δ_2 instead of (Δ_1, Δ_2) when this holds for every Δ_1 (equivalently Δ_1 is $\{\varphi(\bar{x}, \bar{y}) : \varphi(\bar{x}, \bar{a}) \in p\}$).

{88r-5.4.2}

5.8. **Observation.** Assume

- (a), (b), (c), (d), (e) as in 5.7 and in addition
 (d)⁺ p is a complete (Δ_1, n) -type over N , i.e., if

$$\varphi(\bar{x}, \bar{y}) \in \Delta_1, \bar{d} \in {}^{\ell g(\bar{y})}N, \bar{x} = \langle x_\ell : \ell < n \rangle$$

then $\varphi(\bar{x}, \bar{d}) \in p$ or $\neg\varphi(\bar{x}, \bar{d}) \in d$. Then the following conditions are

equivalent:

- (α) p does not (Δ_1, Δ_2) -splits over A
 (β) there is a sequence of $\langle g_{\varphi(\bar{x}, \bar{y})} : \varphi(\bar{x}, \bar{y}) \in \Delta_1 \rangle$ of functions such that:
 (i) $g_{\varphi(\bar{x}, \bar{y})}$ is a function with domain including $\{\text{tp}_{\Delta_2}(\bar{b}, A, N) :$

$$\bar{b} \in {}^{\ell g(\bar{y})}N\}$$

(ii) the values of $g_{\varphi(\bar{x}, \bar{y})}$ are truth values

(iii) if $\varphi(\bar{x}, \bar{y}) \in \Delta_1, \bar{b} \in {}^{\ell g(\bar{y})}N$ and $q = \text{tp}_{\Delta_2}(\bar{b}, A, N)$ then:

$$\varphi(\bar{x}, \bar{b}) \in p \Rightarrow g_{\varphi(\bar{x}, \bar{y})}(q) = \text{true, and}$$

$$\neg\varphi(\bar{x}, \bar{b}) \in p \Rightarrow g_{\varphi(\bar{x}, \bar{y})}(q) = \text{false.}$$

Proof of 5.8. Reflect on the definitions.

□_{5.8}

Proof of 5.6. 1) Clearly the second sentence follows from the first, so we shall prove the first. Assume this fails. Let (M, \bar{a}) be such that $N \leq_{\mathfrak{R}} M \in K_{\aleph_0}$ the sequence $\bar{a} \in M$ materializes p and clearly for every $\bar{b} \in M$, $(M, N) \Vdash \wedge q[\bar{b}]$ for some $q(\bar{x}) \in \mathbf{D}(N)$ and let $\langle b_\ell^* : \ell < \omega \rangle$ list N . We choose by induction on n , $\langle C_\eta^0, C_\eta^1, f_\eta, \bar{a}_\eta^0, \bar{a}_\eta^1 : \eta \in {}^n 2 \rangle$ such that

- (a) C_η^ℓ is a finite subset of N for $\ell < 2, \eta \in {}^n 2$
- (b) f_η is an automorphism of N mapping C_η^0 onto C_η^1
- (c) $\{b_{\ell g(\eta)}^*\} \cup C_\eta^0 \cup C_\eta^1 \subseteq C_{\eta \hat{\ } < \ell >}^0 \cap C_{\eta \hat{\ } < \ell >}^1$ for $\ell = 0, 1$
- (d) $\bar{a}_\eta^0, \bar{a}_\eta^1 \in N$ realize in N the same $\mathbb{L}_{\omega_1, \omega}(\tau)$ -type over $C_\eta^0 \cup C_\eta^1 \cup \{b_{\ell g(\eta)}^*\}$ in (M, N) but $\bar{a} \hat{\ } \bar{a}_\eta^0, \bar{a} \hat{\ } \bar{a}_\eta^1$ do not materialize the same $\mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ in (M, N) (this exemplifies splitting), so $\varphi_\eta(\bar{x}, \bar{y}_\eta)$ belongs to the first, $\neg \varphi_\eta(\bar{x}, \bar{y}_\eta)$ belongs to the second (where $\ell g(\bar{x}) = \ell g(\bar{a}), \ell g(\bar{y}_\eta) = \ell g(\bar{a}_\eta^0)$)
- (e) $f_{\eta \hat{\ } < 0 >}(\bar{a}_\eta^0) = \bar{a}_\eta^1, f_{\eta \hat{\ } < 1 >}(\bar{a}_\eta^1) = \bar{a}_\eta^1$
- (f) $f_\eta \upharpoonright C_\eta^0 \subseteq f_{\eta \hat{\ } < \ell >}$ for $\ell = 0, 1$
- (g) $\bar{a}_\eta^0 \hat{\ } \bar{a}_\eta^1 \subseteq C_{\eta \hat{\ } < \ell >}^0 \cap C_{\eta \hat{\ } < \ell >}^1$.

For $n = 0$ let $C_\eta^0, C_\eta^1 = \emptyset, f_\eta = \text{id}_N$. Recall that K_{\aleph_0} is categorical in \aleph_0 and N is countable, hence if $n < \omega, \bar{b}', \bar{b}'' \in {}^n N$ realize the same $\mathbb{L}_{\omega_1, \omega}(\tau)$ -type over a finite subset B of N , then some automorphism of N over B maps \bar{b}' to \bar{b}'' by a theorem of Scott (see [Kei71]). If $(C_\eta^0, C_\eta^1, f_\eta)$ are defined and satisfies clauses (a), (b) we recall that by our assumption toward contradiction as $C_\eta^0 \cup C_\eta^1 \cup \{b_{\ell g(\eta)}^*\}$ is a finite subset of N , there are $\bar{a}_\eta^0, \bar{a}_\eta^1 \in {}^\omega N$ as required in clause (d) again. So clearly there are automorphisms $f_{\eta \hat{\ } < 0 >}, f_{\eta \hat{\ } < 1 >}$ extending $f_\eta \upharpoonright C_\eta^0$ such that $f_{\eta \hat{\ } < 0 >}(\bar{a}_\eta^0) = \bar{a}_\eta^1, f_{\eta \hat{\ } < 1 >}(\bar{a}_\eta^1) = \bar{a}_\eta^1$ as required in clause (e), (f).

Lastly, choose $C_{\eta \hat{\ } < \ell >}^0 = C_\eta^0 \cup C_\eta^1 \cup f_{\eta \hat{\ } < \ell >}^{-1}(C_\eta^0) \cup \{b_{\ell g(\eta)}^*, f_{\eta \hat{\ } < \ell >}^{-1}(b_{\ell g(\eta)}^*)\}$, $\bar{a}_{\eta \hat{\ } < \ell >}^0 \hat{\ } \bar{a}_{\eta \hat{\ } < \ell >}^1, f_{\eta \hat{\ } < \ell >}^{-1}(\bar{a}_{\eta \hat{\ } < \ell >}^0 \hat{\ } \bar{a}_{\eta \hat{\ } < \ell >}^1)$ and $C_{\eta \hat{\ } < \ell >}^1 = f_{\eta \hat{\ } < \ell >}(C_{\eta \hat{\ } < \ell >}^0)$.

Having carried the induction, for every $\eta \in {}^\omega 2$ clearly $f_\eta = \cup \{f_{\eta \upharpoonright n} \upharpoonright C_{\eta \upharpoonright n}^0 : n < \omega\}$ is an automorphism of N .

[Why? As $\langle f_{\eta \upharpoonright n} \upharpoonright C_{\eta \upharpoonright n}^0 : n < \omega \rangle$ is an increasing sequence of functions by clauses (b) + (c) + (f), the union f_η is a partial function; as in addition each f_η is an automorphism of N by clause (b), also f_η is a partial automorphism of N . Recalling $\langle b_\ell^* : \ell < n \rangle$ list N , clearly f_η have domain N by clause (c) and as $f_{\eta \upharpoonright n}(C_{\eta \upharpoonright n}^0) = C_{\eta \upharpoonright n}^1$ the union f_η has range N by clause (c).] Hence for some $M_\eta \in K_{\aleph_0}$ there is an isomorphism f_η^+ from M onto M_η extending f . Now for some $p_\eta \in \mathbf{D}(N)$, $f_\eta(\bar{a})$ materialize p_η in (M_η, N) . Choose a countable $L \subseteq \mathbb{L}_{\omega_1, \omega}^0(\tau^+)$ which include $\{\varphi_\eta(\bar{x}, \bar{y}_\eta) : \eta \in {}^\omega 2\}$. Easily if $\eta \hat{\ } \langle \ell \rangle \triangleleft \eta_\ell \in {}^\omega 2$ for $\ell = 0, 1$ then $\varphi(\bar{x}, \bar{a}_\eta^1) \in p_0, \neg \varphi(\bar{x}, \bar{a}_\eta^1) \in p_1$. So $\eta \neq \nu \in {}^\omega 2 \Rightarrow p_\eta \cap L \neq p_\nu \cap L$ by clauses (d) + (e), contradiction to 5.4(4) as we can use \leq_{\aleph_0} formulas to distinguish.

2) Follows.

3) Trivial. □_{5.6}

5.9. Claim. 1) Suppose $N_0 \leq_{\mathfrak{R}} N_1 \in K_{\aleph_0}$ and N_1 forces that \bar{a}, \bar{b} (in N_1) realize the same $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -type over N_0 , then N_1 forces that they realize the same $\mathbb{L}_{\omega_1, \omega}^0(N_0; N_0)$ -type; (the

{88r-5.5}

inverse is trivial).

1A) Suppose $N_0 \subseteq_{\mathfrak{K}} N_\ell \in K_{\aleph_0}$ and $\bar{a}_\ell \in {}^{\omega>}(N_\ell)$ for $\ell = 1, 2$ and $\text{gtp}(\bar{a}_1, N_0, N_1) = \text{gtp}(\bar{a}_2, N_0, N_1)$ then we can find (N_1^+, N_2^+, f) such that $N_1 \leq_{\mathfrak{K}} N_1^+ \in K_{\aleph_0}$, $N_2 \leq_{\mathfrak{K}} N_2^+ \in K_{\aleph_0}$ and f is an isomorphism from N_1^+ onto N_2^+ over N_0 mapping \bar{a}_1 to \bar{a}_2 .

2) If $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ and $\bar{a}, \bar{b} \in N_2$ (remember N_2 determines the complete $\mathbb{L}_{\omega_1, \omega}^0(N_1)$ -generic types of \bar{a}, \bar{b}) then from the $\mathbb{L}_{\omega_1, \omega}^0(N_1)$ -generic type of \bar{a} over N_1 we can compute the $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -generic type of \bar{a} over N_0 (hence if the $\mathbb{L}_{\omega_1, \omega}^0(N_1)$ -generic types of \bar{a}, \bar{b} over N_1 are equal, then so are the $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -generic types of \bar{a}, \bar{b} over N_0).

3) For every $N_a \in K_{\aleph_0}$ there is a one-to-one function f from $\mathbf{D}(N)$ onto $\mathbf{D}^*(N)$ such that: if $N \subseteq_{\mathfrak{K}} M \in K_{\aleph_0}$ and $\bar{a} \in {}^{\omega>}M$ then $f(\text{gtp}(\bar{a}, N, M)) = \text{gtp}_{\mathbb{L}_{\omega_1, \omega}(N; N)}(\bar{a}; N; N; M)$.

5.10. **Remark.** 1) So there is no essential difference between $\mathbf{D}(N)$ and $\mathbf{D}^*(N)$.

2) Recall that in a formula of $\mathbb{L}_{\omega_1, \omega}^0(N_0; N_0)$ all $c \in N_0$ may appear as individual constants.

Proof. 1) We shall prove there are N_2 such that $N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ and an automorphism of N_2 over N_0 taking \bar{a} to \bar{b} ; this clearly suffices; and we prove the existence of such N_2 , of course, by hence and forth arguments. We shall use 5.4(2) freely. So by renaming and symmetry, it suffices to prove that

(*) if $m < \omega$, $N_0 \leq_{\mathfrak{K}} N_0$ and $\bar{a}, \bar{b} \in {}^m(N_1)$ materialize the same $\mathbb{L}_{\infty, \omega}^0(N_0)$ -type over N_0 then for every $c \in N_1$, there are N_2 and $d \in N_2$ such that $\bar{a} \hat{<} c \hat{>}, \bar{b} \hat{<} d \hat{>}$ materialize the same $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -type over N_0 .

However, by the previous claim 5.4 for some $\bar{a}^* \in {}^{\omega>}(N_0)$ the $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -type over N_0 that $\bar{a} \hat{<} c \hat{>}$ materialize in (N_1, N_0) does not $\mathbb{L}_{\omega_1, \omega}^0(\tau^{+0})$ -split over \bar{a}^* . Now \bar{a}, \bar{b} materialize in (N_1, N_0) the same $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -type over N_0 hence $\bar{a}^* \hat{<} \bar{a}, \bar{a}^* \hat{<} \bar{b}$ materialize in (N_1, N_0) the same $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -type. Hence there is $N_2, N_1 \leq_{\mathfrak{K}} N_2 \in K_0$ and an automorphism f of N_2 mapping N_0 onto N_1 and mapping $\bar{a}^* \hat{<} \bar{a}$ to $\bar{a}^* \hat{<} \bar{b}$ (but possibly $f \upharpoonright N_0 \neq \text{id}_{N_0}$), this holds by the last sentence in 4.13(c). Let $d = f(c)$, hence if $\bar{a} \hat{<} c \hat{>}, \bar{b} \hat{<} d \hat{>}$ materialize the same $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -type in (N_2, N_0) then they materialize the same $\mathbb{L}_{\omega_1, \omega}^0(N_0)$ -type over N_0 in (N_2, N_0) .

1A) Similarly to part (1).

2) Clearly it suffices to prove the “hence ” part. By the assumption and proof of 5.9(1) there are N_3 satisfying $N_2 \leq_{\mathfrak{K}} N_3 \in K_{\aleph_0}$ and f an automorphism of N_3 over N_1 taking \bar{a} to \bar{b} . Now the conclusion follows.

3) Should be clear. □_{5.9}

5.11. **Definition.** 1) We say that \mathbf{D}_* is a \mathfrak{K} -diagram function when

(a) \mathbf{D}_* is a function with domain K_{\aleph_0} (later we shall lift it to K)

(b) $\mathbf{D}_*(N) \subseteq \mathbf{D}(N)$ and has at least one non-algebraic member for $N \in K_{\aleph_0}$

(c) if $N_1, N_2 \in K_{\aleph_0}$ and f is an isomorphism from N_1 onto N_2 then f maps $\mathbf{D}_*(N_1)$ onto $\mathbf{D}_*(N_2)$, this applies in particular to an automorphism of $N \in K_{\aleph_0}$.

1A) Such \mathbf{D}_* is called weakly good when:

(d) (α) $\mathbf{D}_*(N)$ is closed under subtypes, that is: if $p(\bar{x}) \in \mathbf{D}_*(N)$, $\bar{x} = \langle x_\ell : \ell < m \rangle$, π is a function from $\{0, \dots, m-1\}$ into $\{0, \dots, n-1\}$ then some (necessarily unique) $\bar{q}(\langle x_0, \dots, x_{n-1} \rangle) \in \mathbf{D}_*(N)$ is equal to $\{\varphi(\langle x_0, \dots, \bar{x}_{n-1} \rangle) : \varphi(x_{\pi(0)}, \dots, x_{\pi(m-1)}) \in p(\bar{x})\}$

(β) if $N \leq_{\aleph} M \in K_{\aleph_0}$, $\bar{a}_1, \bar{b}_1 \in {}^{\omega}N$, $\bar{a}_2 \in {}^{\ell g(\bar{a}_1)}M$ and $(M, \bar{a}_1) \cong (M, \bar{a}_2)$ and $\text{gtp}_{\mathbb{L}_{\omega_1, \omega}(\tau^+)}(\bar{a}_2; N; M) \in \mathbf{D}(N)$ then for some M^+, \bar{b}_2 we have $M \leq_{\aleph} M^+ \in K_{\aleph_0}$, $\bar{b}_2 \in {}^{\ell g(\bar{b}_1)}(M^+)$, $(M^+, \bar{a}_1 \hat{\ } \bar{b}_1) \cong (M^+, \bar{a}_2 \hat{\ } \bar{b}_2)$ and $\text{gtp}_{\mathbb{L}_{\omega_1, \omega}(\tau^+)}(\bar{a}_2 \hat{\ } \bar{b}_2; N; M^+) \in \mathbf{D}(N)$

(γ) if $N \leq_{\aleph} M \in K_{\aleph_0}$, $\bar{a} \in {}^{\omega}M$ and $\bar{b} \in {}^{\omega}N$ and $\text{gtp}_{\mathbb{L}_{\omega_1, \omega}(\tau^+)}(\bar{a}; N; M) \in \mathbf{D}(N)$ then $\text{gtp}_{\mathbb{L}_{\omega_1, \omega}(\tau^+)}(\bar{a} \hat{\ } \bar{b}; N; M) \in \mathbf{D}(N)$.

2) Such \mathbf{D}_* is called countable if $N \in K_{\aleph_0} \Rightarrow |\mathbf{D}_*(N)| \leq \aleph_0$.

3) Such \mathbf{D}_* is called good when it is weakly good (i.e., clause (d) holds) and

(e) $\mathbf{D}_*(N)$ has amalgamation (i.e., if $p_0(\bar{x}), p_1(\bar{x}, \bar{y}), p_2(\bar{x}, \bar{z}) \in \mathbf{D}_*(N)$ and $p_0 \subseteq p_1 \cap p_2$ then there is $q(\bar{x}, \bar{y}, \bar{z}) \in \mathbf{D}_*(N)$ which includes $p_1(\bar{x}, \bar{y}) \cup p_2(\bar{x}, \bar{z})$).

4) Such \mathbf{D}_* is called very good if it is good and:

(f) $N_0 \leq_{\aleph} N_1 \leq_{\aleph} N_2 \in K_{\aleph_0}$, $\bar{a}_0 \subseteq \bar{a}_1 \subseteq \bar{a}_2$ and $\bar{a}_\ell \subseteq N_\ell$ for $\ell = 0, 1, 2$ and $\text{gtp}(\bar{a}_{\ell+1}, N_\ell, N_{\ell+1})$ is definable over \bar{a}_ℓ and belongs to $\mathbf{D}_*(N_\ell)$ for $\ell = 0, 1$ then $\text{gtp}(\bar{a}_2, N_0, N_2)$ belongs to $\mathbf{D}_*(N_0)$ and is definable over \bar{a}_0 .

5.12. **Remark.** 1) Note that if \mathbf{D} is a weakly good \aleph -diagram function and $N \in K_{\aleph_0}$ and $p \in \mathbf{D}(N)$ then we can find (M, \bar{a}) such that $N \leq_{\aleph} M \in K_{\aleph_0}$, $\bar{a} \in {}^{\omega}M$, $p = \text{gtp}_{\mathbb{L}_{\omega_1, \omega}(\tau^+)}(\bar{a}; N; M)$ and for every $\bar{b} \in {}^{\omega}M$ the type $\text{gtp}_{\mathbb{L}_{\omega_1, \omega}(\tau^+)}(\bar{b}; N; M)$ belongs to $\mathbf{D}(N)$.

2) If moreover, \mathbf{D} is a good \aleph -diagram function then we can demand above that M is $(\mathbf{D}(N), \aleph_0)^*$ -homogeneous, see Definition 5.15(1) below.

3) On very good \mathbf{D} see 5.13(2).

4) The \mathbf{D}_α 's in 5.13 below are very good \aleph -diagrams and for us it suffices to have then the properties mentioned above, so we do not elaborate.

5.13. **Fact.** 1) There are $\mathbf{D}_\alpha, \mathbf{D}_\alpha^*$ for $\alpha < \omega_1$, functions with domain K_{\aleph_0} such that:

(a) for $N \in K_{\aleph_0}$, $\mathbf{D}_\alpha(N), \mathbf{D}_\alpha^*(N)$ is a countable subset of $\mathbf{D}(N), \mathbf{D}^*(N)$ respectively

(b) for each $N \in K_{\aleph_0}$, $\langle \mathbf{D}_\alpha(N) : \alpha < \omega_1 \rangle$ as well as $\langle \mathbf{D}_\alpha^*(N) : \alpha < \omega_1 \rangle$ are increasing continuous

(c) $\mathbf{D}(N) = \bigcup_{\alpha < \omega_1} \mathbf{D}_\alpha(N)$ and $\mathbf{D}^*(N) = \bigcup_{\alpha < \omega_1} \mathbf{D}_\alpha^*(N)$

{88r-5.6.}

{88r-5.6}

- (d) if $N_1, N_2 \in K_{\aleph_0}$, f is an isomorphism from N_1 onto N_2 then f maps $\mathbf{D}_\alpha(N_1)$ onto $\mathbf{D}_\alpha(N_2)$ and $\mathbf{D}_\alpha^*(N_1)$ onto $\mathbf{D}_\alpha^*(N_2)$ for $\alpha < \omega_1$
- (e) for every $\alpha < \omega_1$ and $N \in K_{\aleph_0}$ there is a $(\mathbf{D}_\alpha(N), \aleph_0)^*$ -homogeneous model (see below Definition 5.15(1)) (obviously it is unique up to isomorphism over N)
- (f) if $N_0 \leq_{\aleph} N_1 \leq_{\aleph} N_2 \in K_{\aleph_0}$, N_2 is $(\mathbf{D}_\alpha(N_1), \aleph_0)^*$ -homogeneous (see Definition 5.15(1) below) and N_1 is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$ -homogeneous or just $(\mathbf{D}_\beta(N_0), \aleph_0)^*$ -homogeneous for some $\beta \leq \alpha$ or just $\bar{b} \in {}^{\omega>}(N_1) \Rightarrow \text{gtp}_{\mathbb{L}_{\omega_1, \omega}(\tau^+)}(\bar{b}; N_0; N_1) \in \mathbf{D}(N_0)$ then N_2 is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$ -homogeneous
- (f)⁺ if $\langle \alpha_\varepsilon : \varepsilon \leq \zeta \rangle$ is increasing continuous sequence of countable ordinals, $\zeta > 0$ and $\langle N_\varepsilon : \varepsilon \leq \zeta \rangle$ is \leq_{\aleph} -increasing continuous, $N_\varepsilon \in \aleph_{\aleph_0}$, for every $\bar{a} \in N_{\varepsilon+1}$, $\text{gtp}(\bar{a}, N_\varepsilon, N_{\varepsilon+1}) \in \mathbf{D}_\alpha(N_\varepsilon)$ and for every $\xi < \zeta$ for some $\varepsilon \in [\xi, \zeta)$, $N_{\varepsilon+1}$ is $(\mathbf{D}_{\alpha_\varepsilon}(N_\varepsilon), \aleph_0)^*$ -homogeneous then N_ζ is $(\mathbf{D}_{\alpha_\zeta}(N_0), \aleph_0)^*$ -homogeneous
- (g) N_1 is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$ -homogeneous if and only if N_1 is $(\mathbf{D}_\alpha^*(N_0), \aleph_0)^*$ -homogeneous where $N_0 \leq_{\aleph} N_1 \in K_{\aleph_0}$
- (h) \mathbf{D}_α is a very good countable \aleph -diagram function.
- 2) If \mathbf{D} is very good then clauses (d),(e),(f),(f)⁺ hold for it (and also (g), defining \mathbf{D}^* as $f''(\mathbf{D}), f$ from 5.17(3).

{88r-5.6.4}

5.14. **Remark.** 1) We can add

- (h) if $\aleph, <^*$ are as derived from the $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ in the proof of 3.19(2) then we can add: if $N_0 \leq_{\aleph} N_1 \in K_{\aleph_0}$ and every $p \in \mathbf{D}_0(N_0)$ is materialized in N_1 then $N_0 <^* N_1$.
- 2) So our results apply to $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$, too.
- 3) So it follows that if $\langle N_i : i \leq \alpha \rangle$ is \leq_{\aleph} -increasing in K_{\aleph_0} , N_{i+1} is $(\mathbf{D}_{\beta_i}(N_0), \aleph_0)^*$ -homogeneous and $\langle \beta_i : i < \alpha \rangle$ is non-decreasing with supremum β then N_α is $(\mathbf{D}_\beta, \aleph_0)^*$ -homogeneous.
- 4) So by 5.13(1)(h) each \mathbf{D}_α is very good and countable.

Proof of 5.13. First, \mathbf{D} is a \aleph -diagram function by Definition 5.2 and 5.4(9). As $\mathbf{D}(N)$ has cardinality $\leq \aleph_1$ by 5.4(6) we can find a sequence $\langle \mathbf{D}_\alpha : \alpha < \omega_1 \rangle$ such that

- ⊗ (a) \mathbf{D}_α is a countable \aleph -diagram function
- (b) for every $N \in K_{\aleph_0}$ the sequence $\langle \mathbf{D}_\alpha(N) : \alpha < \omega_1 \rangle$ is increasing

continuous with union $\mathbf{D}(N)$.

Second, \mathbf{D} is very good (clause (f) of 5.11 holds obviously but to prove that it reflects to \mathbf{D}_α for a cub of $\alpha < \omega_1$ we need 5.23 below, no vicious circle; the other - easier).

Third, note that for each of the demands (d),(e),(f) from Definition 5.11, for a club of $\delta < \omega_1$, \mathbf{D}_δ satisfies it. So without loss of generality each \mathbf{D}_α is very good.

The parts on \mathbf{D}_α^* follow by 5.9, and see 5.17(1) below which does not rely on 5.13-5.16 (and see proof of 5.19). □_{5.13}

{88r-5.7}

5.15. **Definition.** Assume $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ and \mathbf{D}_* is a \mathfrak{K} -diagram.

1) We say that (N_1, N_0) or just N_1 is $(\mathbf{D}_*(N_0), \aleph_0)^*$ -homogeneous over N_0 (but we may omit the “over N_0 ”) if:

- (a) every $\bar{a} \in N_1$ materializes in (N_1, N_0) over N_0 some $p \in \mathbf{D}_*(N_0)$ and every $q \in \mathbf{D}_\alpha(N_0)$ is materialized in (N_0, N_1) by some $\bar{b} \in N_1$
- (b) if $\bar{a}, \bar{b} \in N_1$, \bar{a}, \bar{b} materialize in (N_1, N_0) the same type over N_0 and $c \in N_1$ then for some $d \in N_1$ sequence $\bar{a} \hat{<} c \hat{>}, \bar{b} \hat{<} d \hat{>}$ materialize in (N_1, N_0) the same type from $\mathbf{D}_*(N_0)$.

2) Similarly for $(\mathbf{D}_*(N_0), \aleph_0)^*$ -homogeneity, pedantically we have to say $(N_1, N_0; N_0)$ is $(\mathbf{D}_*(N), \aleph_0)^*$ -homogeneous, but normally say N_1 is.

{88r-5.7A}

5.16. **Remark.** 1) Now this is meaningful only for $N \leq_{\mathfrak{K}} M \in K_{\aleph_0}$, but later it becomes meaningful for any $N \leq_{\mathfrak{K}} M \in K$.

2) Uniqueness for such countable models hold in this context too.

Now by 5.9.

{88r-5.8}

5.17. **Conclusion.** If (N_1, N_0) is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$ -homogeneous then N_1 , i.e. $(N_1, N_0, c)_{c \in N_0}$ is $(\mathbf{D}_\alpha^*(N_0), \aleph_0)^*$ -homogeneous.

Proof. This is easy by 5.9(1) and clause (g) of 5.13. □_{5.17}

{88r-5.9}

5.18. **Lemma.** *There is $N^* \in K_{\aleph_1}$ such that $N^* = \bigcup_{\alpha < \omega_1} N_\alpha$ and $N_\alpha \in K_{\aleph_0}$ is $\leq_{\mathfrak{K}}$ -increasing continuous with α and $N_{\alpha+1}$ is $(\mathbf{D}_{\alpha+1}(N_\alpha), \aleph_0)^*$ -homogeneous for $\alpha < \omega_1$.*

Proof. Should be clear. □_{5.18}

{88r-5.10}

5.19. **Theorem.** *The $N^* \in K_{\aleph_1}$ from 5.18 is unique (even not depending on the choice of $\mathbf{D}_\alpha(N)$ ’s), is universal and is, $(\mathbb{D}(\mathfrak{K}), \aleph_1)$ -model-homogeneous hence model-homogeneous (for \mathfrak{K}).*

modified:2015-01-25

(88r) revision:2015-01-23

Proof. Uniqueness: For $\ell = 0, 1$ let $N_\alpha^\ell, \mathbf{D}_\alpha^\ell$ ($\alpha < \omega_1$) be as in 5.13, 5.18 and we should prove $\bigcup_{\alpha < \omega_1} N_\alpha^0 \cong \bigcup_{\alpha < \omega_1} N_\alpha^1$; because of clause (g) of 5.13 it does not matter if we use the \mathbf{D} or \mathbf{D}^* version. As \mathbf{D}_α^ℓ ($\alpha < \omega_1$) is increasing and continuous, $|\mathbf{D}_\alpha^\ell(N)| \leq \aleph_0$ and $\bigcup_{\alpha < \omega_1} \mathbf{D}_\alpha^\ell(N) =$

$\mathbf{D}(N)$ for every $N \in K_{\aleph_0}$ and the \mathbf{D}_α^ℓ 's commute with isomorphisms, clearly there is a closed unbounded $E \subseteq \omega_1$ consisting of limit ordinals, such that $\alpha \in E \Rightarrow \mathbf{D}_\alpha^0 = \mathbf{D}_\alpha^1$. Let $E = \{\alpha(i) : i < \omega_1\}$, $\alpha(i)$ increasing and continuous. Now we define by induction on $i < \omega_1$, an isomorphism f_i from $N_{\alpha(i)}^0$ on $N_{\alpha(i)}^1$, increasing with i . For $i = 0$ use the \aleph_0 -categoricity of K and for limit i let $f_i = \bigcup_{j < i} f_j$. Suppose f_i is defined, then by clause (d) of 5.13 the function f_i maps $\mathbf{D}_{\alpha(i+1)}^0(N_{\alpha(i)}^0)$ onto $\mathbf{D}_{\alpha(i+1)}^0(N_{\alpha(i)}^1)$ and by the choice of $E, \mathbf{D}_{\alpha(i+1)}^0 = \mathbf{D}_{\alpha(i+1)}^1$. By the assumption on the N_α^ℓ and clause (f)⁺ of 5.13, $N_{\alpha(i+1)}^\ell$ is $(\mathbf{D}_{\alpha(i+1)}^\ell(N_{\alpha(i)}^\ell), \aleph_0)^*$ -homogeneous. Summing up those facts and 5.13(e) we see that we can extend f_i to an isomorphism f_{i+1} from $N_{\alpha(i+1)}^0$ onto $N_{\alpha(i+1)}^1$.

Now $\bigcup_{i < \omega_1} f_i$ is the required isomorphism. Universality: Let $M \in K_{\aleph_1}$, so $M = \bigcup_{\alpha < \omega_1} M_\alpha, M_\alpha$

is \leq_{\aleph} -increasing continuous and $\|M_\alpha\| \leq \aleph_0$. We now define $f_\alpha, N_\alpha, \gamma_\alpha$ by induction on $\alpha < \omega_1$ such that: $\gamma_\alpha \in [\alpha, \omega_1)$ is increasing continuous with α , f_α is a \leq_{\aleph} -embedding of M_α into $N_\alpha \in K_{\aleph_0}$, N_α is \leq_{\aleph} -increasing continuous, f_α is increasing and continuous, and for $\beta < \alpha, N_{\beta+1}$ is $(\mathbf{D}_{\gamma_{\beta+1}}(N_\beta), \aleph_0)^*$ -homogeneous. For $\alpha = 0$ let $N_\alpha = M_\alpha$ and $f_\alpha = \text{id}_{N_\alpha}$. For α limit use unions. Let $\alpha = \beta + 1$, we use the \aleph_0 -amalgamation property (which holds by 3.9, 4.8). So there is a pair (f_α, N'_α) such that $N_\beta \leq_{\aleph} N'_\alpha \in K_{\aleph_0}$ and f_α is a \leq_{\aleph} -embedding of M_α into N'_α extending f_β . The set $\{\text{gtp}(\bar{a}, N_\beta, N'_\alpha), \bar{a} \in {}^{\omega_1}(N'_\alpha)\}$ is a countable subset of $\mathbf{D}(N_\beta)$ hence is $\subseteq \mathbf{D}_{\gamma_\alpha}(N_\beta)$ for some $\gamma \in (\gamma_\beta, \omega_1)$. By 5.13(1)(c) there is N_α which \leq_{\aleph} -extends N'_α and is $(\mathbf{D}_{\gamma_\alpha}(N'_\alpha), \aleph_0)^*$ -homogeneous; by 5.13(1)(f) we are done. So $f = \cup\{f_\alpha : \alpha < \omega_1\}$ embeds M into $N = \cup\{N_\alpha : \alpha < \omega_1\}$ which is isomorphic to N^* by the uniqueness. So the universality follows from the uniqueness. ($\mathbb{D}(\aleph), \aleph_1$)-Model-homogeneity: So let $\langle N_\alpha : \alpha < \omega_1 \rangle, \mathbf{D}_\alpha, N^*$ be as in 5.13, 5.18 and we

are given (M_0, M_1, M_0^+, f) such that $M_0 \leq_{\aleph} M_0^+ \in K_{\aleph_0}, M_1 \leq_{\aleph} N^*, f$ an isomorphism from M_0 onto M_1 . For some $\gamma < \omega_1$ we have $M_1 \leq_{\aleph} N_\gamma$.

Now $\{\text{gtp}(\bar{a}, M_0, M_0^+) : \bar{a} \in {}^{\omega_1}(M_0^+)\}$ is a countable subset of $\mathbf{D}(M_0)$ hence $\subseteq \mathbf{D}_{\gamma_0}(M_0)$ for some $\gamma_0 < \omega_1$; also $\{\text{gtp}(\bar{a}, M_1, N_\gamma) : \bar{a} \in {}^{\omega_1}(N_\gamma)\}$ is a countable subset of $\mathbf{D}(M_1)$ hence $\subseteq \mathbf{D}_{\gamma_1}(M_1)$ for some $\gamma_1 < \omega_1$. Let $\beta = \max\{\gamma, \gamma_0, \gamma_1\}$ and let $M_0^* \in K_{\aleph_0}$ be $(\mathbf{D}_\beta(M_0^+), \aleph_0)^*$ -homogeneous so $M_0^+ \leq_{\aleph} M_0^*$, exists by 5.13(1)(e), hence $M_0^* \in K_{\aleph_0}$ is $(\mathbf{D}_\beta(M_0), \aleph_0)^*$ -homogeneous by 5.13(1)(f) because $\beta \geq \gamma_0$. Now N_β is $(\mathbf{D}(N_\gamma), \aleph_0)^*$ -homogeneous by 5.13(1), so as $\beta \geq \gamma_1$ it follows that N_β is $(\mathbf{D}_\gamma(M_1), \aleph_0)^*$ -homogeneous.

By 5.13(1)(d),(e) we can extend f to an isomorphism g from M_0^* onto N_β , so $g \upharpoonright M_0^+$ is a \leq_{\aleph} -embedding of M_0^+ into N .

We can deduce “ N^* is a model-homogeneous directly; let $M_0, M_1 \leq_{\mathfrak{K}} N^*$ be countable, and f is isomorphic from M_0 onto M_1 . Let $\gamma < \omega_1$ be such that $M_0, M_1 \leq_{\mathfrak{K}} N_\gamma$. Let γ_ℓ be such that $\{\text{gtp}(\bar{a}, M_\ell, N_\gamma) : \bar{a} \in {}^\omega(N_\gamma)\} \subseteq \mathbf{D}_{\gamma_\ell}(M_\ell)$ for $\ell = 0, 1$ and let $\beta = \max\{\gamma, \gamma_0, \gamma_1\} + 1$. As above N_β is $(\mathbf{D}_\beta(M_\ell), \aleph_0)^*$ -homogeneous, and now we choose an automorphism f_α of N_α increasing with α and extended f for $\alpha \in [\beta, \omega_1)$ by induction. Now $\cup\{f_\alpha : \alpha \in (\beta, \omega_1)\}$ is an automorphism of N^* extending f . $\square_{5.19}$

5.20. **Definition.** 1) If $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ and $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 1, 2$ and they are definable in the same way (see Definition 5.7 (and 5.6), so in particular both do not split over the same finite subset of N_0), then we call p_1 the stationarization of p_0 over N_1 .
 2) For $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0, 1$ let $p_1 \models p_0$ mean that $N_0 \leq_{\mathfrak{K}} N_1$ and if $N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ and $\bar{a} \in N_2$ materializes p_1 then it materializes p_0 .

{88r-5.11

5.21. **Remark.** It is easy to justify the uniqueness implied by “the stationarization”.

{88r-5.11.

Observe

{88r-5.12

5.22. **Claim.** If $p_\ell = \text{gtp}(\bar{a}, N_\ell, N_2)$ for $\ell = 0, 1$ and $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$ then $p_1 \models p_0$.

Proof. Easy. $\square_{5.22}$

{88r-5.16

5.23. **Claim.** 1) Suppose $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} N_2 \in K_{\aleph_0}$, $\bar{a}_i \in N_i$, (for $i = 0, 1, 2$), $\bar{a}_0 \subseteq \bar{a}_1 \subseteq \bar{a}_2$, i.e. the ranges increase, $\text{gtp}(\bar{a}_1, N_0, N_1)$ is definable over \bar{a}_0 and $\text{gtp}(\bar{a}_2, N_1, N_2)$ is definable over \bar{a}_1 . Then $\text{gtp}(\bar{a}_2, N_0, N_2)$ is definable over \bar{a}_0 . Moreover, the definition depends only on the definitions mentioned previously.
 2) If $N_0 \leq_{\mathfrak{K}} N_1 \leq_{\mathfrak{K}} N_2$ and $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0, 1, 2$ and $p_{\ell+1}$ is the stationarization of p_ℓ over $N_{\ell+1}$ for $\ell = 0, 1$, then p_2 is the stationarization of p_0 over N_2 .

Proof. 1) So we have to prove that $\text{gtp}(\bar{a}_2, N_0, N_2)$ does not split over \bar{a}_0 . Let $n < \omega$ and $\bar{b}, \bar{c} \in {}^n N_0$ realize the same type in N_0 over \bar{a}_0 (in the logic $\mathbb{L}_{\omega_1, \omega}(\tau_{\mathfrak{K}})$, or even first order logic when every $N \in K_{\aleph_0}$ is atomic). Now also $\bar{b} \hat{\ } \bar{a}_1, \bar{c} \hat{\ } \bar{a}_1$ materialize the same $\mathbb{L}_{\omega_1, \omega}(N_0)$ -type in N_1 hence they realize the same $\mathbb{L}_{\omega_1, \omega}(\tau_{\mathfrak{K}})$ -type (recall 5.4(8)). Hence \bar{b}, \bar{c} realize the same $\mathbb{L}_{\omega_1, \omega}(\tau_{\mathfrak{K}})$ -type in N_1 over \bar{a}_1 in N_1 . But $\text{gtp}(\bar{a}_2, N_0, N_2)$ does not split over \bar{a}_1 , so by the previous sentence we get that $\bar{b} \hat{\ } \bar{a}_2, \bar{c} \hat{\ } \bar{a}_2$ materializes the same $\mathbb{L}_{\omega_1, \omega}(N_0)$ -type in N_2 .
 2) Easy. The “moreover” is proved similarly. $\square_{5.23}$

{88r-5.12

5.24. **Lemma.** Suppose $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$, $p_\ell \in \mathbf{D}(N_\ell)$ and p_1 is a stationarization of p_0 over N_1 , then $p_1 \models p_0$, i.e., every sequence materializing p_1 materializes p_0 in any N_2 such that $N_1 \leq_{\mathfrak{K}} N_2$.

modified:2015-01-25

(88r) revision:2015-01-23

- 5.25. **Remark.** 1) In [Sh:48], [Sh:87a], [Sh:87b] and [Sh:c] the parallel proof of the claims were totally trivial, but here we need to invoke $\dot{I}(\aleph_1, K) < 2^{\aleph_1}$.
 2) A particular case can be proved in the context of §4.

Proof. So suppose N_0, N_1, p_0, p_1 contradict the claim and let $\bar{a}^* \in N_0$ be such that p_0 is definable over \bar{a}^* so p_1 , too. By 5.13(e)+(f) there are $\delta < \omega_1$ and $N_2 \in K_{\aleph_0}$ satisfying $N_1 \leq_{\aleph} N_2$ such that N_2 is $(\mathbf{D}_\delta^*(N_\ell), \aleph_0)^*$ -homogeneous for $\ell = 0, 1$. We can find $p_2 \in \mathbf{D}(N_2)$ which is the stationarization of p_0, p_1 . It is enough to prove that $p_2 \models p_1$.

[Why? First, note that there is an automorphism f of N_2 which maps N_1 onto N_0 and $f(\bar{a}^*) = \bar{a}^*$ hence $f(p_2) = p_2, f(p_1) = p_0$ hence $p_2 \models p_0$. Now assume that $N_1 \leq_{\aleph} N_1^+ \in K_{\aleph_0}$ and $\bar{a}_1 \in N_1^+$ materializes p_1 clearly we can find N_2^+, \bar{a}_2 such that $N_2 \leq_{\aleph} N_2^+ \in K_{\aleph_0}$ and $\bar{a}_2 \in N_2^+$ which materializes p_2 , as we are assuming $p_2 \models p_1$ it also materializes p_1 hence there are N_3, f such that $N_1^+ \leq_{\aleph} N_3 \in K_{\aleph_0}$ and f is a \leq_{\aleph} -embedding of N_2^+ into N_3 over N_1 mapping \bar{a}_2 to \bar{a}_1 . But $p_2 \models p_0$ (see above) hence $f(\bar{a}_2) = \bar{a}_1$ materializes p_0 and p_1 , too.]

So without loss of generality for some δ

⊗ N_1 is $(\mathbf{D}_\delta^*(N_0), \aleph_0)^*$ -homogeneous over N_0 .

For $N \in K_{\aleph_0}, N_0 \leq_{\aleph} N$, let p_N be the stationarization of p over N ; so

⊠₁ if $N_0 \leq_{\aleph} N \in K_{\aleph_0}$ then p_N is definable over \bar{a}^* .

Without loss of generality the universes of N_0, N_1 are $\omega, \omega \times 2$ respectively.

Now we choose by induction on α a model $N_\alpha \in K_{\aleph_0}$ ($\alpha < \omega_1$), $|N_\alpha| = \omega(1 + \alpha)$, [$\beta < \alpha \Rightarrow N_\beta \leq_{\aleph} N_\alpha$]; N_0, N_1 are the ones mentioned in the claim and $\bar{a}_\alpha \in N_{\alpha+1}$ materializes the stationarization $p_\alpha \in \mathbf{D}_\delta^*(N_\alpha)$ of p_0 over N_α and for $\alpha < \beta, N_\beta$ is $(\mathbf{D}_\delta^*(N_\alpha), \aleph_0)$ -homogeneous (see 5.13(f),(f)⁺). Recalling that \aleph is categorical in \aleph_0 (and uniqueness over N_0 of $(\mathbf{D}_\delta(N_0), \aleph_0)^*$ -homogeneous models) we have $\alpha > \beta \Rightarrow (N_\alpha, N_\beta) \cong (N_1, N_0)$ so, recalling ⊗ clearly \bar{a}_α does not materialize p_{N_β} (in $N_{\alpha+1}$). Let $N = \cup\{N_\alpha : \alpha < \omega_1\}$. Let \mathfrak{B} be $(\mathcal{H}(\aleph_2), \in)$ expanded by $N, K \cap \mathcal{H}(\aleph_2), \leq_{\aleph} \upharpoonright \mathcal{H}(\aleph_2)$ and anything else which is necessary. Let \mathfrak{B}^- be a countable elementary submodel of \mathfrak{B} to which $\langle N_\alpha : \alpha < \omega_1 \rangle, N$ belong and let $\delta(*) = \mathfrak{B}^- \cap \omega_1$. For any stationary co-stationary $S \subseteq \omega_1$, let \mathfrak{B}_S be a model which is

- (α) \mathfrak{B}_S an elementary extension of \mathfrak{B}^-
- (β) \mathfrak{B}_S is an end extension of \mathfrak{B}^- for ω_1 , that is, if $\mathfrak{B}_S \models "s < t \text{ are countable ordinals}"$ and $t \in \mathfrak{B}^-$ then $s \in \mathfrak{B}^-$
- (γ) among the \mathfrak{B}_S -countable ordinals not in \mathfrak{B}^- there is no first one
- (δ) “the set of countable ordinals” of \mathfrak{B}_S is $I_S, I_S = \bigcup_{\alpha < \omega_1} I_\alpha^S$, even I_0^S is not well ordered,
 each I_α a countable initial segment of $I_S, \alpha < \beta \Rightarrow I_\alpha^S \subseteq I_\beta^S \wedge I_\alpha^S \neq I_\beta^S$
- (ε) $I_S \setminus I_\alpha^S$ has a first element if and only if $\alpha \in S$ and then we call it $s(\alpha)$.

In particular ω and finite sets are standard in \mathfrak{B}_S . For $s \in I_S, N_s[\mathfrak{B}_s] := N_s^{\mathfrak{B}_s}$ is defined naturally, and so is $N^S = N^{\mathfrak{B}_S}$; clearly $N_s^{\mathfrak{B}_s} \in K_{\aleph_0}$ is \leq_{\aleph} -increasing with $s \in I$ as those

definitions are Σ_1^1 (as \mathfrak{K} is PC_{\aleph_0}). Let $N_\alpha^S = \bigcup_{s \in I_\alpha} N_s^{\mathfrak{B}^S}$ and let $s+1$ be the successor of s in

I_S .

So

⊞ if $\mathfrak{B}_S \models$ “ $s < t$ are countable ordinals”, then $(N_t^{\mathfrak{B}^S}, N_s^{\mathfrak{B}^S})$ is $(\mathbf{D}_\delta^*(N_s^{\mathfrak{B}^S}), \aleph_0)^*$ -homogeneous and if $s \in I_\alpha$ then N_α^S is $(\mathbf{D}_\delta^*(N_1^{\mathfrak{B}^S}), \aleph_0)^*$ -homogeneous.

If $\alpha \in S$ then clearly the type $p = p_{N_\alpha^S}$ satisfies (using absoluteness from \mathfrak{B}_S because N_α^S is definable in \mathfrak{B}_S as $N_{s(\alpha)}^{\mathfrak{B}^S}$):

(a) p is materialized in N^S (i.e. in N_β^S for a club of $\beta \in S$)

but by the assumption toward contradiction

(b) for a closed unbounded $E \subseteq \omega_1$ for no $\beta \in E \cap S, \beta > \alpha^*$ and $\gamma \in (\beta, \omega_1)$ does a sequence from N^S materialize both $p = p_{N_\alpha^S}$ and its stationarization $p_{N_\beta^S}$ over N_β^S in N_γ^S (again remember $N_\alpha^S = N_{s(\alpha)}^{\mathfrak{B}^S}$ because $\alpha \in S$)

and similarly

(c) for a closed unbounded set of $\beta > \alpha, N_\beta^S$ is $(\mathbf{D}_\delta^*(N_\alpha^S), \aleph_0)^*$ -homogeneous.

We shall prove that every $\alpha < \omega_1$,

⊠ if $\alpha \notin S$ then α cannot satisfy the statement (c) above.

This is sufficient because if $S(1), S(2) \subseteq \omega_1$ are stationary co-stationary, f is an isomorphism from $N^{S(1)}$ onto $N^{S(2)}$ mapping \bar{a}^* to itself, then for a closed unbounded set $E \subseteq \omega_1$, for each $\alpha < \omega_1$ the function f maps $N_\alpha^{S(1)}$ onto $N_\alpha^{S(2)}$, hence the property above is preserved, hence $S(1) \cap E = S(2) \cap E$. But there is a sequence $\langle S_i : i < 2^{\aleph_1} \rangle$ of subsets of ω_1 such that for $i \neq j$ the set $S_i \setminus S_j$ is stationary. So by 0.4 we have $\dot{I}(\aleph_1, K) = 2^{\aleph_1}$, contradiction.

So suppose $\alpha \in \omega_1 \setminus S, p = p_{N_\alpha^S}$ and clause (c) above hold; but obviously (c) \Rightarrow (a), recalling $p_0 \in \mathbf{D}_\delta(N_0)$ hence $p_{N_\alpha^S} \in \mathbf{D}_\delta(N_\alpha^S)$ so let $\bar{a} \in N^S$ materialize p in N^S and we shall get a contradiction.

There are elements $0 = t(0) < t(1) < \dots < t(k)$ of I^S and $\bar{a}_0 \in N_0 = N_{t(0)}^{\mathfrak{B}^S}, \bar{a}_{\ell+1} \in N_{t(\ell+1)}^{\mathfrak{B}^S}$ such that $\bar{a} \subseteq \bar{a}_k, \bar{a}^* \subseteq \bar{a}_0, \bar{a}_\ell \subseteq \bar{a}_{\ell+1}$ and $\text{gtp}(\bar{a}_{\ell+1}, N_{t(\ell)}^{\mathfrak{B}^S}, N_{t(\ell+1)}^{\mathfrak{B}^S})$ is definable over \bar{a}_ℓ and if $t(\ell+1)$ is a successor (in I_S) then it is the successor of $t(\ell)$ and if limit in I^S then $\bar{a}_\ell = \bar{a}_{\ell+1}$.

[Why do they exist? Because of the sentence saying that for every \bar{a} we can find such $k, t(\ell) (\ell \leq k), \bar{a}_\ell (\ell \leq k)$ as above is satisfied by \mathfrak{B} and involve parameters which belong to \mathfrak{B}^- hence to \mathfrak{B}_S , etc., so \mathfrak{B}_S inherits it (and finiteness is absolute from \mathfrak{B}_S)]. It follows that $\text{gtp}(\bar{a}, N_{t(\ell)}^{\mathfrak{B}^S}, N_{t(k)}^{\mathfrak{B}^S})$ is definable over \bar{a}_ℓ for each $\ell < k$.

Clearly $t(0) = 0 \in I_\alpha$ but $t(k) \notin I_\alpha$ (otherwise $t(k) + 1 \in I_\alpha$ hence $\bar{a} \in N_{t(k)+1}^{\mathfrak{B}^S} \leq_{\mathfrak{K}} N_\alpha^S$, impossible as p is a non-algebraic type over $N_\alpha^{\mathfrak{B}^S}$). Hence for some ℓ we have $t(\ell) \in I_\alpha, t(\ell+1) \notin I_\alpha$. By the construction $t(\ell+1)$ is limit (in I^S) hence $\bar{a}_{\ell+1} = \bar{a}_\ell$. As $\alpha \notin S$ we can choose $t(*) \in I_S \setminus I_\alpha^S, t(*) < t(\ell+1)$. As we are assuming (toward contradiction) that α, p satisfy clause (c), for some $\beta \in S, s(\beta)$ is well defined and $s(\beta) > t(k)$ (on the definition

of $s(\gamma)$ for $\gamma \in S$ see clause (ε) above) and N_β^S is $(\mathbf{D}_\delta^*(N_\alpha^S), \aleph_0)^*$ -homogeneous. Now $N_{s(\beta)}^{\mathfrak{B}_S} = N_\beta^S, N_{t(\ell+1)}^{\mathfrak{B}_S}$ are isomorphic over $N_{t(*)}$ (being both $(\mathbf{D}_\delta^*(N_{t(*)}^{\mathfrak{B}_S}), \aleph_0)^*$ -homogeneous by the choice of \mathfrak{B}_S , see \boxplus above).

So as $N_\alpha^S \leq_{\aleph} N_{t(\ell+1)}^{\mathfrak{B}_S} \leq_{\aleph} N_{s(\beta)}^{\mathfrak{B}_S} = N_\beta^S$ and, as said above, N_β^S is $(\mathbf{D}_\delta^*(N_\alpha^S), \aleph_0)^*$ -homogeneous also $N_{t(\ell+1)}^{\mathfrak{B}_S}$ is $(\mathbf{D}_\delta^*(N_\alpha^S), \aleph_0)^*$ -homogeneous, too, hence $(N_{t(\ell+1)}^{\mathfrak{B}_S}, N_\alpha^S, \bar{a}^*) \cong (N_1, N_0, \bar{a}^*)$.

As, by \boxplus above, clearly $N_\alpha^S, N_{t(*)}^{\mathfrak{B}_S}$ are $(\mathbf{D}_\delta^*(N_{t(\ell+1)}^{\mathfrak{B}_S}), \aleph_0)^*$ -homogeneous there is an isomorphism f_0 from N_α^S onto $N_{t(*)}^{\mathfrak{B}_S}$ over $N_{t(\ell+1)}^{\mathfrak{B}_S}$. As $N_{t(\ell+1)}^{\mathfrak{B}_S}$ is $(\mathbf{D}_\delta^*(N_{t(*)}^{\mathfrak{B}_S}), \aleph_0)^*$ -homogeneous and $(\mathbf{D}_\delta^*(N_\alpha^S), \aleph_0)^*$ -homogeneous by the previous paragraph (where we use β) we can extend f_0 to an automorphism f_1 of $N_{t(\ell+1)}^{\mathfrak{B}_S}$. Let $\gamma \in S \cap E$ satisfy $s(\gamma) \geq t(k) + 1$. As $\text{gtp}(\bar{a}_k, N_{t(\ell+1)}^{\mathfrak{B}_S}, N_\gamma^S)$ is definable over $\bar{a}_\ell = \bar{a}_{\ell+1}$ and $\bar{a}_\ell = f_0(\bar{a}_\ell) = f_1(\bar{a}_\ell)$ (as $\bar{a}_\ell \in N_{t(\ell+1)}^{\mathfrak{B}_S}$) and $N_{\gamma+1}^S$ is $(\mathbf{D}_\delta^*(N_{t(\ell+1)}^{\mathfrak{B}_S}), \aleph_0)^*$ -homogeneous, we can extend f_1 to an automorphism f_2 of N_γ^S satisfying $f_2(\bar{a}_k) = \bar{a}_k$.

Notice that by the choice of $\langle \bar{a}_\ell : \ell \leq k \rangle$ and $\langle t(\ell) : \ell \leq k \rangle$ it follows that for any $m < k$, $\text{gtp}(\bar{a}_k, N_{t(m)}, N_{t(k)+1})$ does not split over \bar{a}_m hence is definable over it by 5.23, and recall that we know that $\bar{a}_\ell = \bar{a}_{\ell+1}$.

So there is in N^S a sequence materializing both $\text{gtp}(\bar{a}, N_\alpha^S, N_\gamma^S) = p_{N_\alpha^S}$ and its stationarization over $N_{t(\ell+1)}^S$: just $\bar{a}(\subseteq \bar{a}_k)$ (so use f_2).

This contradicts the assumption as $(N_1, N_0, \bar{a}^*) \cong (N_{t(\ell+1)}^{\mathfrak{B}_S}, N_\alpha^S, \bar{a}^*)$. $\square_{5.24}$

The following claim 5.26(5)-(9) and Definition 5.27 are closely related.

5.26. **Claim.** 1) If $\bar{a} \in N_0 \leq_{\aleph} N_1 \leq_{\aleph} N_2 \in K_{\aleph_0}, \bar{b} \in N_2, p_1 = \text{gtp}(\bar{b}, N_1, N_2)$ is definable over $\bar{a} \in N_0$, then $p_0 = \text{gtp}(\bar{b}, N_0, N_2)$ is definable in the same way over \bar{a} , hence $\text{gtp}(\bar{b}, N_1, N_2)$ is its stationarization.

2) For a fixed countable $M \in K_{\aleph_0}$ to have a common stationarization in $\mathbf{D}(N')$ for some N' satisfying $M \leq_{\aleph} N'$ or $N' \leq_{\aleph} M$ is an equivalence relation over $\{p: \text{for some } N \leq_{\aleph} M, p \in \mathbf{D}(N)\}$ (and we can choose the common stationarization in $\mathbf{D}(M)$ as a representative). So if $N_0 \leq_{\aleph} N_1 \leq_{\aleph} N_2 \in K_{\aleph_0}, p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0, 1, 2$ and p_1, p_2 are stationarizations of p_0 then $p_2 \models p_1$.

3) If $N_\alpha \in K_{\aleph_0}$ ($\alpha \leq \omega + 1$) is \leq_{\aleph} -increasing and continuous and $\bar{a} \in N_{\omega+1}$ then for some $n < \omega$, for every k we have: $n < k \leq \alpha \leq \omega$ implies $\text{gtp}(\bar{a}, N_\alpha, N_{\omega+1})$ is the stationarization of $\text{gtp}(\bar{a}, N_k, N_{\omega+1})$.

4) If $N \leq_{\aleph} M \in K, N \in K_{\aleph_0}, \bar{a} \in M$ then for all $M' \in K_{\aleph_0}$, satisfying $\bar{a} \in M', N \leq_{\aleph} M' \leq_{\aleph} M, \text{gtp}(\bar{a}, N, M')$ is the same, we call it $\text{gtp}(\bar{a}, N, M)$ (the new point is that M is not necessarily countable. This is compatible with Definition 5.27(c) being a special case).

5) Suppose $N_0 \leq_{\aleph} N_1$ (in K), $\bar{a} \in N_1$, then there is a countable $M \leq_{\aleph} N_0$, such that for every countable M' satisfying $M \leq_{\aleph} M' \leq_{\aleph} N_0$ we have $\text{gtp}(\bar{a}, M', N_1)$ is the stationarization of $\text{gtp}(\bar{a}, M, N_1)$. Moreover there is a finite $A \subseteq N_0$ such that any countable $M \leq_{\aleph} N_0$ which includes A is O.K. So $\text{gtp}(\bar{a}, N_0, N_1)$ from 5.27(c) is well defined and $\in \mathbf{D}(N_0)$ and is definable over some finite $A \subseteq N_0$.

{88r}5.13}

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revision:2015-01-23

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- 6) The parallel of Part (3) holds for $N_\alpha \in K$, too, and any limit ordinal instead of ω . That is if $\langle N_\alpha : \alpha \leq \delta + 1 \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous and $\bar{a} \in N_{\delta+1}$, then for some $\alpha < \delta$ and countable $M \leq_{\mathfrak{R}} N_\alpha$ we have: $M \leq_{\mathfrak{R}} M' \leq_{\mathfrak{R}} M_\delta \Rightarrow \text{gtp}(\bar{a}, M', M_\delta)$ is the stationarization of $\text{gtp}(\bar{a}, M, M_\delta)$; similarly for every $p \in \mathbf{D}(N_\delta)$.
- 7) If $N_0 \leq_{\mathfrak{R}} N_1 \leq_{\mathfrak{R}} N_2 \leq_{\mathfrak{R}} N_3 \leq_{\mathfrak{R}} N_4$ and $\bar{a} \in N_4$ and $\text{gtp}(\bar{a}, N_3, N_4)$ is the stationarization of $\text{gtp}(\bar{a}, N_0, N_4)$ then $\text{gtp}(\bar{a}, N_2, N_4)$ is the stationarization of $\text{gtp}(\bar{a}, N_1, N_3)$. Also if \bar{b}' satisfies $\text{Rang}(\bar{b}') \subseteq \text{Rang}(\bar{a})$ and $\text{gtp}(\bar{a}, N_2, N_4)$ is the stationarization of $\text{gtp}(\bar{a}, N_1, N_4)$ then this holds also for \bar{b} . We can replace $\text{gtp}(\bar{a}, N_3, N_4)$ by $p \in \mathbf{D}(N_4)$.
- 8) If $N_0 \leq_{\mathfrak{R}} N_1 \leq_{\mathfrak{R}} N_2 \in K_{\aleph_0}$ and $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0, 1, 2$ and $p_{\ell+1}$ is the stationarization of p_ℓ for $\ell = 0, 1$ then p_2 is the stationarization of p_0 .
- 9) If $\langle M_\alpha : \alpha \leq \delta + 1 \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous, δ a limit ordinal and $\bar{a} \in {}^\omega(M_{\delta+1})$ then
- (a) for some $\alpha < \delta$ we have $\text{gtp}(\bar{a}, M_\beta, M_{\delta+1})$ is the stationarization of $\text{gtp}(\bar{a}, M_\alpha, M_{\delta+1})$ whenever $\beta \in [\alpha, \delta)$
 - (b) if $\text{gtp}(\bar{a}, M_\alpha, M_{\delta+1})$ is the stationarization of $\text{gtp}(\bar{a}, M_0, M_{\delta+1})$ for every $\alpha < \delta$ then this holds for $\alpha = \delta$, too.
- 10) If $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous, δ a limit ordinal and $p_\delta \in \mathbf{D}(M_\delta)$, then for some $\alpha < \beta$ there is $p_\alpha \in \mathbf{D}(M_\alpha)$ such that p_δ is the stationarization of p_α .
- 11) Those definitions in 5.27 are compatible with the ones for countable models.
- 12) $\text{gtp}(\bar{a}, N, M)$ (where $\bar{a} \in M, N \leq_{\mathfrak{R}} M$ are both in K) is the stationarization over N of $\text{gtp}(\bar{a}, N', M)$ for every large enough countable $N' \leq_{\mathfrak{R}} N$, see 5.26(5).

Proof. 1) As we can replace N_2 by any N'_2 satisfying $N_2 \leq_{\mathfrak{R}} N'_2 \in K_{\aleph_0}$, without loss of generality for some α, N_2 is $(\mathbf{D}_\alpha^*(N_0), \aleph_0)^*$ -homogeneous and $(\mathbf{D}_\alpha^*(N_1), \aleph_0)^*$ -homogeneous. Let $p_2 \in \mathbf{D}(N_2)$ be the stationarization of p_1 over N_2 .

So by 5.24 we get $p_2 \models p_1$. On the other hand, clearly there is an isomorphism f_0 from N_0 onto N_1 such that $f_0(\bar{a}) = \bar{a}$; and by the assumption above on N_2 , f_0 can be extended to an automorphism f_1 of N_2 .

Note that f_1 maps $p_0 = \text{gtp}(\bar{b}, N_0, N_2)$ to $p'_0 := \text{gtp}(f_1(\bar{b}), f_1(N_0), N_2)$ and maps p_2 to itself as $f_0(\bar{a}) = \bar{a}$.

Now $p_1 \models p_0$ (by the choices of p_1, p_0) and $p_2 \models p_1$ by 5.9(1), together $p_2 \models p_0$. As $f_1(p_2) = p_2, f_1(p_0) = p'_0$ it follows that $p_2 \models p'_0$. As also $p_2 \models p_1$ and $p'_0, p_1 \in \mathbf{D}(N_1)$ it follows that $p'_0 = p_1$ hence p_1, p'_0 have the same definition over \bar{a} , but now also $p_0 \in \mathbf{D}(N_0), p'_0 \in \mathbf{D}(N_1)$ have the same definition over \bar{a} (using f_1), together also p_1, p_0 have the same definition over \bar{a} , which means that p_1 is the stationarization of p_0 over N_1 and we are done.

- 2) Trivial.
- 3) By part (1).
- 4) Easy.
- 5) By (3) and (4).
- 6)-12) Easy by now.

□_{5.26}

{88r-5.14}

5.27. **Definition.** By 5.26(5) the type $\text{gtp}(\bar{a}, M, N)$ can be reasonably defined when $M \leq_{\aleph} N$, $\bar{a} \in {}^{\omega}N$ and we can define $\mathbf{D}(N)$ and $\mathbf{D}_*(N)$, $\text{gtp}(\bar{a}, N, M)$ and stationarization for not necessarily countable N and $N \leq_{\aleph} M \in K$. Everything still holds, except that maybe some p 's are not materialized in any \leq_{\aleph} -extension of N .

More formally

- (a) if $N \leq_{\aleph} M$ and $N \in K_{\aleph_0}$ and $p \in \mathbf{D}(N)$ then the stationarization of p over M is $\cup\{q : N_1 \in K_{\aleph_0}$ satisfies $N_1 \leq_{\aleph} N_1 \leq_{\aleph} M$ and q is the stationarization of $p \in \mathbf{D}(N_1)\}$
- (b) if $M \in \aleph$ then $\mathbf{D}(M) = \{q: \text{for some countable } N \leq_{\aleph} M \text{ and } p \in \mathbf{D}(N) \text{ the type } q \text{ is the stationarization of } p \text{ over } M\}$, similarly for \mathbf{D}_* , a \aleph -diagram
- (c) if $N \leq_{\aleph} M$ and $\bar{a} \in {}^{\omega}M$ then $\text{gtp}(\bar{a}, N, M)$ is defined as $\cup\{\text{gtp}(\bar{a}, N', M') : N_0 \leq_{\aleph} N' \leq_{\aleph} M' \in K_{\aleph_0}, M' \leq_{\aleph} M, N' \leq_{\aleph} N\}$ for every countable large enough $N_0 \leq_{\aleph} N$; it is well defined and belongs to $\mathbf{D}(N)$ by 5.26(5) and we say \bar{a} materializes $\text{gtp}(\bar{a}, N, M)$ in M
- (d) if $N \in \aleph, N \leq_{\aleph} M$ and $p \in \mathbf{D}(N)$ is definable over the countable $N_0 \leq_{\aleph} N$ equivalently is the stationarization of some $p' \in \mathbf{D}(N_0)$, then the stationarization of p over M is the stationarization of p' over M , see clause (a), equivalently $\cup\{p_{M_0} : N_0 \leq_{\aleph} M_0 \leq_{\aleph} M, M_0 \text{ is countable}\}$ where p_{M_0} is the stationarization of $p' \in \mathbf{D}(N_0)$ over M_0 ; it belongs to $\mathbf{D}(N_0)$
- (e) if $p(\bar{x}, \bar{y}) \in \mathbf{D}(M)$ then $p(\bar{x}, \bar{y}) \upharpoonright \bar{x} \in \mathbf{D}(M)$ is naturally defined; 5.2(3) similarly for permuting the variables
- (f) for $N \leq_{\aleph} M$ we say that M is $(\mathbf{D}(N), \aleph_0)^*$ -homogeneous when for every $p(\bar{x}, \bar{y}) \in \mathbf{D}(N)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ materializing $p(\bar{x}, \bar{y}) \upharpoonright x$ in M there is $\bar{b} \in {}^{\ell g(\bar{y})}M$ such that $\bar{a} \hat{\ } \bar{b}$ materializes $p(\bar{x}, \bar{y})$ in M .

5.28. **Remark.** Claim 5.29 below strengthens 3.9, it is a step toward non-forking amalgamation.

{88r-5.15}

5.29. **Claim.** Suppose $N_0 \leq_{\aleph} N_1 \in K_{\aleph_0}, N_0 \leq_{\aleph} N_2 \in K_{\aleph_0}, \bar{a} \in N_1$. Then we can find $M, N_0 \leq_{\aleph} M \in K_{\aleph_0}$ and \leq_{\aleph} -embeddings f_ℓ of N_ℓ into M over N_0 (for $\ell = 1, 2$) such that $\text{gtp}(f_1(\bar{a}), f_2(N_2), M)$ is a stationarization of $p_0 = \text{gtp}(\bar{a}, N_0, N_1)$ (so $f_1(\bar{a}) \notin f_2(N_2)$).

Proof. Let $p_2 \in \mathbf{D}(N_2)$ be the stationarization of p_0 . Clearly we can find an $\alpha < \omega_1$ (in fact, a closed unbounded set of α 's) and N'_1, N'_2 from K_{\aleph_0} which are $(D_\alpha^*(N_0), \aleph_0)^*$ -homogeneous and $N_\ell \leq_{\aleph} N'_\ell$ (for $\ell = 1, 2$) and some $\bar{b} \in N'_2$ materializing p_2 . But by 5.24, \bar{b} materializes p_0 hence there is an isomorphism f from N'_1 onto N'_2 over N_0 satisfying $f(\bar{a}) = \bar{b}$, recalling 5.9(1A). Now let $M = N'_2, f_1 = f \upharpoonright N_1, f_2 = \text{id}$. □_{5.29}

{88r-5.17}

- 5.30. **Claim.** 1) For any $N_0 \leq_{\mathfrak{R}} N_1 \in K_{\aleph_1}$ so $N_0 \in K_{\leq \aleph_1}$, there is N_2 such that $N_1 \leq_{\mathfrak{R}} N_2 \in K_{\aleph_1}$ and N_2 is $(\mathbf{D}(N_0), \aleph_0)^*$ -homogeneous.
 2) Also 5.29 holds for $N_2 \in K_{\aleph_1}$ (but still $N_0, N_1 \in K_{\aleph_0}$).
 3) If $N_0 \leq_{\mathfrak{R}} N_1 \in K_{\aleph_0}$ and $N_0 \leq_{\mathfrak{R}} N_2 \in K_{\leq \aleph_1}$ then we can find $M \in K_{\leq \aleph_1}$ and $\leq_{\mathfrak{R}}$ -embeddings f_1, f_2 of N_1, N_2 into M over N_0 respectively such that $\text{gtp}(f_1(\bar{c}), f_2(N_2), M)$ is a stationarization of $\text{gtp}(\bar{c}, N_0, N_1)$ for every $\bar{c} \in N_1$, hence $f_1(N_1) \cap f_2(N_2) = N_0$.
 4) $K_{\aleph_2} \neq \emptyset$.

- 5.31. **Remark.** 1) Note that 5.30(3) is another step toward stable amalgamation.
 2) Note that 5.30(3) strengthen 5.30(2) hence 5.29.

Proof. 1) As we can iterate $\leq_{\mathfrak{R}}$ -increasing N_1 in K_{\aleph_1} , it is enough to prove that: if $p(\bar{x}, \bar{y}) \in \mathbf{D}(N_0)$ and $\bar{a} \in N_1$ materializes $p(\bar{x}, \bar{y}) \upharpoonright \bar{x}$ in (N_1, N_0) then for some $N_2 \in K_{\aleph_1}, N_1 \leq_{\mathfrak{R}} N_2$ and for some $\bar{b} \in N_2$ the sequence $\bar{a} \hat{\ } \bar{b}$ materializes $p(\bar{x}, \bar{y})$ in (N_2, N_0) . Let $M_0 \leq_{\mathfrak{R}} N_0$ be countable and $q \in \mathbf{D}(M_0)$ be such that $p(\bar{x}, \bar{y})$ a stationarization of q . Without loss of generality if N_0 is countable then $M_0 = N_0$. Note that the case $N_0 = M_0$ is easier. Choose $M_i (0 < i < \omega_1)$ such that $M_i \leq_{\mathfrak{R}} N_1, N_1 = \bigcup_{i < \omega_1} M_i, \langle M_i : i < \omega_1 \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous sequence of countable models, $M_0 \cup \bar{a} \subseteq M_1$. As $\langle M_i \cap N_0 : i < \omega_1 \rangle$ is an increasing continuous sequence of countable sets with union N_0 clearly for a club of $i < \omega_1, M_i \cap N_0 \leq_{\mathfrak{R}} N_0$ hence $M_i \cap N_0 \leq_{\mathfrak{R}} M_i$. So without loss of generality $i < \omega_1 \Rightarrow M_i \cap N_0 \leq_{\mathfrak{R}} N_0, M_i$. For every $\bar{c} \in N_1$ there is a countable $N_{0, \bar{c}}$ such that $M_0 \leq_{\mathfrak{R}} N_{0, \bar{c}} \leq_{\mathfrak{R}} N_0$ and: if $N_{0, \bar{c}} \leq_{\mathfrak{R}} N' \leq_{\mathfrak{R}} N_0$ and $N' \in K_{\aleph_0}$ then $\text{gtp}(\bar{c}, N', N_1)$ is the stationarization of $\text{gtp}(\bar{c}, N_{0, \bar{c}}, N_1)$. Without loss of generality $\bar{c} \in M_i \Rightarrow N_{0, \bar{c}} \subseteq M_i$ hence

(*) for every $\bar{c} \in M_i, \text{gtp}(\bar{c}, N_0, N_1)$ is a stationarization of $\text{gtp}(\bar{c}, N_0 \cap M_i, M_i)$.

We can find $M_1^* \in K_{\aleph_0}$ satisfying $M_1 \leq_{\mathfrak{R}} M_1^*$ and $\bar{b} \in M_1^*$ such that $q = \text{gtp}(\bar{a} \hat{\ } \bar{b}, M_0, M_1^*)$. We can find $\bar{a}_2, \bar{a}_1, \bar{a}_0$ such that $\bar{a}_0 \in M_1 \cap N_0, \bar{a}_1 \in M_1, \bar{a}_2 \in M_1^*, \bar{b} \subseteq \bar{a}_2, \bar{a} \subseteq \bar{a}_1$ and $\bar{a}_0 \trianglelefteq \bar{a}_1 \trianglelefteq \bar{a}_2$ and $\text{gtp}(\bar{a}_2, M_1, M_1^*), \text{gtp}(\bar{a}_1, M_1 \cap N_0, M_1)$ are definable over \bar{a}_1, \bar{a}_0 , respectively. Now we define $f_j, M_j^*, 1 \leq j < \omega_1$ by induction on i such that:

- (i) $\langle M_i^* : 1 \leq i \leq j \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous
- (ii) M_j^* is countable, M_1^* already given
- (iii) f_j is a $\leq_{\mathfrak{R}}$ -embedding of M_j into M_j^*
- (iv) f_1 is the identity on M_1
- (v) f_j is increasing continuous with j
- (vi) $\text{gtp}(\bar{a}_2, f_j(M_j), M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}_2, M_1, M_1^*)$ (so definable over \bar{a}_1).

For $j = 1$ we have it letting $f_j^* = \text{id}_{M_1}$.

For $j > 1$ successor, use 5.29 to define (M_j, f_j) such that $\text{gtp}(\bar{a}_2, f_j(M_j), M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}_2, f_{j-1}(M_{j-1}), M_{j-1}^*)$. So clauses (i)-(v) clearly holds. Clause (vi) follows

modified:2015-01-25

(88r) revision:2015-01-23

(88r)

by 5.26(8).

For j limit: let $M_j^* = \bigcup_{1 \leq i < j} M_i^*$ and $f_j = \cup\{f_i : 1 \leq i < j\}$, condition (vi) holds by 5.26(3).

By renaming without loss of generality $f_j = \text{id}_{M_j}$ for $j \in [1, \omega_1)$.

By (*) we get that $\text{gtp}(\bar{a}_1, N_0 \cap M_j, M_j^*) = \text{gtp}(\bar{a}_1, N_0 \cap M_j, M_j)$ is definable over \bar{a}_0 (as this holds for $j = 1$). Combining this and clause (vi), by 5.23(1) we get for every $j \geq 1$, that $\text{gtp}(\bar{a}_2, N_0 \cap M_j, M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}_2, N_0 \cap M_1, M_1^*)$. Hence by the choice of $\bar{a}_2, \bar{a}_1, a_0$ and 5.26(7), easily $\text{gtp}(\bar{a} \hat{\ } \bar{b}, N_0 \cap M_j, M_j^*)$ is the stationarization of $\text{gtp}(\bar{a} \hat{\ } \bar{b}, N_0 \cap M_1, M_1^*)$ hence of $\text{gtp}(\bar{a} \hat{\ } \bar{b}, M_0, M_1^*)$. Let $N_2 = \cup\{M_j^* : j \in [1, \omega_1)\}$, clearly $N_1 \leq_{\aleph} N_2 \in K_{\aleph_1}$.

So by 5.26(9), clause (c) and the first sentence in the proof, we finish.

2) Similar proof ⁷ (or use the proof of part (3)).

3) Without loss of generality $N_2 \cong N^*$ from 5.18 (as we can replace N_2 by an extension so use 5.19 and 5.26(7)).

Also (by 5.30(1)) there is $M, N_2 \leq_{\aleph} M \in K_{\aleph_1}$ such that M is $(\mathbf{D}(N_2), \aleph_0)^*$ -homogeneous. As N_1 is countable there is $\alpha < \omega_1$ such that for every $\bar{c} \in N_1$, $\text{gtp}(\bar{c}, N_0, N_1) \in \mathbf{D}_\alpha(N_0)$. Let $M = \bigcup_{i < \omega_1} M_i$ with $M_i \in K_{\aleph_0}$ being \leq_{\aleph} -increasing continuous. So for some i we have

$\alpha < i < \omega_1, M_i \cap N_2 \leq_{\aleph} M$ and (recalling 5.26(6)) for every $\bar{c} \in M_i$, $\text{gtp}(\bar{c}, N_2, M)$ is stationarization of $\text{gtp}(\bar{c}, N_2 \cap M_i, M_i)$ and M_i is $(\mathbf{D}_i(N_2 \cap M_i), \aleph_0)^*$ -homogeneous. Now we can find an isomorphism f_0 from N_0 onto $N_2 \cap M_i$ (as K is \aleph_0 -categorical) and extend it to an automorphism f_2 of N_2 (by 5.19-model homogeneity). Also there is N'_1 such that $N_1 \leq_{\aleph} N'_1 \in K_{\aleph_0}$ and N'_1 is $(\mathbf{D}_i(N_1), \aleph_0)^*$ -homogeneous, hence is $(\mathbf{D}_i(N_0), \aleph_0)^*$ -homogeneous (by the choice of α as $\alpha < i$ see 5.13(f)), hence there is an isomorphism f'_1 from N'_1 onto M_i extending f_0 . Now $f_0, f'_1 \upharpoonright N_1, f_2, M$ show that amalgamation as required exists (we just change names).

4) Immediate, use 1) or 2) or 3) ω_2 -times. □_{5.30}

{88r_{modified:2015-01-25} 5.18}

5.32. Definition. For any $\mathbf{D}_* = \mathbf{D}_\alpha$ for some $\alpha < \omega_1$ (or just any very good \aleph -diagram \mathbf{D}_* , see 5.11, i.e., satisfies the demands on each \mathbf{D}_α in 5.13) we define:

1) $M \leq_{\mathbf{D}_*} N$ if $M \leq_{\aleph} N$ and for every $\bar{a} \in N$

$$\text{gtp}(\bar{a}, M, N) \in \mathbf{D}_*(M).$$

2) $K_{\mathbf{D}_*}$ is the class of $M \in K$ which are the union of a family of countable submodels, which is directed by $\leq_{\mathbf{D}_*}$.

3) $\aleph_{\mathbf{D}_*} = (K_{\mathbf{D}_*} \leq_{\mathbf{D}_*})$, or pedantically $(K_{\mathbf{D}_*}, \leq_{\mathbf{D}_*} \upharpoonright K_{\mathbf{D}_*})$.

⁷here $N_1 \in K_{\aleph_1}$ is O.K.; similar to 2.12(1)

{88r-5.19}

5.33. **Claim.** Let \mathbf{D}_* be countable and as in 5.32.

1) The pair $(K_{\mathbf{D}_*}, \leq_{\mathbf{D}_*})$ is an \aleph_0 -presentable a.e.c., that is it satisfies all the axioms from 1.2(1) and is PC_{\aleph_0} .

2) Also for $(K_{\mathbf{D}_*}, \leq_{\mathbf{D}_*})$, we get $\mathbf{D}(N)$ countable and equal to $\mathbf{D}_*(N)$ for every countable $N \in K_{\mathbf{D}_*}$.

Proof. 1) Obviously $K_{\mathbf{D}_*}$ is a class of τ -models and $\leq_{\mathbf{D}_*}$ is a two-place relation on $K_{\mathbf{D}_*}$; also they are preserved by isomorphisms. About being PC_{\aleph_0} note that

\otimes_1 $M \in K_{\mathbf{D}_*}$ iff $M \in K$ and for some model \mathfrak{B} with universe $|M|$ and countable vocabulary, for every countable $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \mathfrak{B}$ we have $M \upharpoonright \mathfrak{B}_1 \leq_{\mathbf{D}_*} M \upharpoonright \mathfrak{B}_2$ iff there is a directed partial order and $\langle M_t : t \in I \rangle$ such that $M_t \in K_{\aleph_0}$ and $s <_I t \Rightarrow M_s \leq_{\mathfrak{K}} M_t$ and $\bar{a} \subseteq M_t \Rightarrow \text{gtp}(\bar{a}, M_s, M_t) \in \mathbf{D}_*(M_s)$

\otimes_2 similarly for $M \leq_{\mathbf{D}_*} N$.

Ax I: If $M \leq_{\mathbf{D}_*} N$ then $M \leq_{\mathfrak{K}} N$ hence $M \subseteq N$. Ax II: The transitivity of $\leq_{\mathbf{D}_*}$ holds

by 5.11(4), 5.23(1) + Definition 5.27 (works as \mathbf{D}_* is closed enough or use clause (f) of 5.13). The demand $M \leq_{\mathbf{D}_*} M$ is trivial ⁸. Ax III: Assume $\langle M_i : i < \lambda \rangle$ is $\leq_{\mathbf{D}_*}$ -

increasing continuous and $M = \cup\{M_i : i < \lambda\}$. As \mathfrak{K} is an a.e.c. clearly $M \in K$ and $i < \lambda \Rightarrow M_i \leq_{\mathfrak{K}} M$. Also for each $i < \lambda$ and $\bar{a} \in M$ for some $j \in (i, \lambda)$ we have $\bar{a} \in M_j$ hence $\text{gtp}(\bar{a}, M_i, M_j) \in \mathbf{D}_*(M_i)$ but recalling 5.26(7) it follows that $\text{gtp}(\bar{a}, M_i, M) = \text{gtp}(\bar{a}, M_i, M_j) \in \mathbf{D}_*(M_i)$. So $i < \lambda \Rightarrow M_i \leq_{\mathbf{D}_*} M$. By applying \otimes_1 to every M_i and coding we can easily show that $M \in K_{\mathbf{D}_*}$ thus finishing. Ax IV: Assume $\langle M_i : i < \lambda \rangle, M$ are

as above and $i < \lambda \Rightarrow M_i \leq_{\mathbf{D}_*} N$. To prove $M \leq_{\mathbf{D}_*} N$ note that as \mathfrak{K} is an a.e.c., we have $M \leq_{\mathfrak{K}} N$ and consider $\bar{a} \in N$. By 5.26(6) for some $i < \lambda$, $\text{gtp}(\bar{a}, M, N)$ is the stationarization of $\text{gtp}(\bar{a}, M_i, N)$ but the latter belongs to $\mathbf{D}_*(M_i)$ hence $\text{gtp}(\bar{a}, M, N) \in \mathbf{D}_*(M)$ as required. Ax V: By \otimes_2 this is translated to the case $N_0, N_1, M \in K_{\aleph_0}$ but then

it holds easily. Ax VI: By $\otimes_1 + \otimes_2 + \text{Ax VI}$ for \mathfrak{K} .

2) So we replace \mathfrak{K} by $\mathfrak{K}' = \mathfrak{K}_{\mathbf{D}_*}$ and easily all that we need for \mathbf{D} for \mathfrak{K}' is satisfied by \mathbf{D}_* (actually repeating the works in §5 till now on \mathfrak{K}' we get it) noting that

\otimes if $M_0 \leq_{\mathbf{D}_*} M_\ell \in K_{\aleph_0}$ for $\ell = 1$ and $\text{gtp}(\bar{a}_1, M_0, M_1) = \text{gtp}(\bar{a}_2, M_0, M_2)$ then there is a triple (M_1^+, M_2^+, f) such that $M_\ell \leq_{\mathbf{D}_*} M_\ell^+ \in K_{\aleph_0}$, M_ℓ^+ is $(\mathbf{D}(M_i), \aleph_0)^*$ -homogeneous for $i = 0, \ell$ and f is an isomorphism from M_1^+ onto M_2^+ over M_0 mapping \bar{a}_1 to a_2 .

This by:

⁸recall that $M \upharpoonright \mathfrak{B} = M \upharpoonright \{a \in M : a \in \mathfrak{B}\}$

- ⊗₁ if $M_0 \leq_{\mathbf{D}_*} M_1 \leq_{\mathbf{D}_*} M_2$ and $\bar{a} \in M_1$ then $\text{gtp}(\bar{a}, M_0, M_1) = \text{gtp}(\bar{a}, M_0, M_2) \in \mathbf{D}_*(M_0)$
- ⊗₂ if $M_0 \in K_{\aleph_0}$ then for some $M_1 \in K_{\aleph_0}$ we have $M_0 \leq_{\mathbf{D}_*} M_2$ and M_1 is $(\mathbf{D}_*(M_0), \aleph_0)^*$ -homogeneous
- ⊗₃ if $M_0 \leq_{\mathbf{D}_*} M_1 \leq_{\mathbf{D}_*} M_2$ and M_2 is $(\mathbf{d}_*(M_1), \aleph_0)^*$ -homogeneous then M_2 is $(\mathbf{D}_*(M_0), \aleph_0)^*$ -homogeneous
- ⊗₄ if $M_0 \leq_{\mathbf{D}_*} M_\ell \in K_{\aleph_0}$, $\text{gtp}(\bar{a}_1, M_0, M_1) = \text{gtp}(\bar{a}_2, M_0, M_2)$ then there is an isomorphism from M_1 onto M_2 over M_0 mapping \bar{a}_1 to \bar{a}_2 .

□_{5.33}

{88r-5.20}

5.34. **Claim.** Suppose $N_0 \leq_{\aleph} N_\ell \in K_{\aleph_0}$ ($\ell = 1, 2$) and $\bar{c} \in N_2$, then there is $M, N_0 \leq_{\aleph} M$ and \leq_{\aleph} -embeddings f_ℓ of N_ℓ into M over N_0 for $\ell = 1, 2$ such that

- (i) for every $\bar{a} \in N_1$, $\text{gtp}(f_1(\bar{a}), f_2(N_2), M)$ is a stationarization of $\text{gtp}(\bar{a}, N_0, N_1)$
- (ii) $\text{gtp}(f_2(\bar{c}), f_1(N_1), M)$ is a stationarization of $\text{gtp}(\bar{c}, N_0, N_2)$.

5.35. **Remark.** This is one more step toward stable amalgamation: in 5.29 we have gotten it for one $\bar{a} \in N_1$, in 5.30(3) for every $\bar{a} \in N_1$, which gives disjoint amalgamation.

Proof. Clearly we can for $\ell = 1, 2$ replace N_ℓ by any $N'_\ell, N_\ell \leq_{\aleph} N'_\ell \in K_{\aleph_0}$, and without loss of generality $N_0 = N_1 \cap N_2$. By 5.30(3) there is $N_3 \in K_{\aleph_0}$ such that $N_\ell \leq_{\aleph} N_3$ for $\ell < 3$ and $\bar{a} \in {}^\omega(N_1) \Rightarrow \text{gtp}(\bar{a}, N_2, N_3)$ is the stationarization of $\text{gtp}(\bar{a}, N_0, N_1)$. So we can assume that for some \mathbf{D}_α as in Definition 5.32 we have N_ℓ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$ -homogeneous for $\ell = 1, 2$. As in the proof of 5.24, we can find a countable linear order I , such that every element $s \in I$ has an immediate successor $s + 1$, 0 is first element and I^* has a subset isomorphic to the rationals (follow really) and models $M_s \in K_{\aleph_0}$, (for $s \in I$) such that $s < t \Rightarrow M_s \leq_{\aleph} M_t$ and M_t is $(\mathbf{D}_\alpha(M_s), \aleph_0)$ -homogeneous when $s <_I t$, etc. So by 5.26(3) for every initial segment J of I and $t \in I$ such that $J < t$, that is, $(\forall s \in J)(s <_I t)$, if J has no last element and $I \setminus J$ has no first element then M_t is $(\mathbf{D}_\alpha(M_J), \aleph_0)^*$ -homogeneous, where $M_J = \bigcup_{s \in J} M_s = \bigcap_{t \in I \setminus J} M_t$. We let $N_0^J = M_J, N_1^J = M_I$ and N_2^J be a $(\mathbf{D}_\alpha(N_0^J), \aleph_0)^*$ -

homogeneous model satisfying $N_0^J \leq_{\aleph} N_2^J$ and without loss of generality $N_1^J \cap N_2^J = N_0^J$. Also easily there is $N'_0 <_{\aleph} N_0$ such that $\text{gtp}(\bar{c}, N_0, N_1)$ is definable over some $\bar{c}_0 \subseteq N'_0$ and N_0 is $(\mathbf{D}_\alpha(N'_0), \aleph_0)$ -homogeneous. Clearly the triples $(N_0, N_1, N_2), (N_0^J, N_1^J, N_2^J)$ are isomorphic and let f_0^J, f_1^J, f_2^J be appropriate isomorphisms such that $f_0^J \subseteq f_1^J, f_2^J$ and without loss of generality $f_0^J(N'_0) = M_0$. Now by 5.30(3), there is $M^J \in K_{\aleph_0}$ satisfying $N_\ell^J \leq_{\aleph} M^J$ ($\ell = 0, 1, 2$) such that for every $\bar{a} \in N_1^J$, $\text{gtp}(\bar{a}, N_2^J, M^J)$ is the stationarization of $\text{gtp}(\bar{a}, N_0^J, N_1^J)$ and there are $N_3 \in K_{\aleph_0}, N_\ell \leq_{\aleph} N_3$ for $\ell = 0, 1, 2$ and an isomorphism $f_3^J \supseteq f_1^J \cup f_2^J$ from N_3 onto M^J .

Suppose our conclusion fails, then $\text{gtp}(f_2^J(\bar{c}), N_1^J, M^J)$ is not the stationarization of $\text{gtp}(f_2^J(\bar{c}), N_0^J, M^J)$. Moreover, as in the proof of 5.24, $t \in I \setminus J \Rightarrow M_t = N_1^J, M_t$ are

isomorphic over $N_0^J = M_J$, hence we can replace N_1^J by M_t for any $t \in I \setminus J$ so as we assume that our conclusion fails, $t \in I \setminus J \Rightarrow \text{gtp}(f_2^J(\bar{c}), M_t, M^J)$ is not a stationarization of $\text{gtp}(f_2^J(\bar{c}), N_0^J, M^J)$ and the latter is the stationarization of $\text{gtp}(f_2^J(\bar{c}), M_0, M^J)$. Let $p_J = \text{gtp}(f_2^J(\bar{c}), N_1^J, M^J) = \text{gtp}(\bar{c}, M_I, M^J)$; all this was done for any appropriate J . So it is easy to check that $J_1 \neq J_2 \Rightarrow p_{J_1} \neq p_{J_2}$, but as $I^* \subseteq I \& |I| = \aleph_0$, we have continuum many such J 's hence such p_J 's. If CH fails, we are done. Otherwise, note that moreover, we can ensure that for $J_1 \neq J_2$ as above there is an automorphism of M_I taking p_{J_1} to p_{J_2} , hence for some $\beta < \omega_1$, $\{p_J : J \text{ as above}\} \subseteq \mathbf{D}_\beta(M_I)$, i.e., $(f_1^{J_2}) \circ (f_1^{J_1})^{-1}$ maps one to the other, contradiction by clause (d) of 5.13. (Alternatively repeat the proof of 5.24. More elaborately by the way \mathbf{D}_α was chosen, Claim 5.30(3) holds for $\mathfrak{K}_{\mathbf{D}_*}$ hence without loss of generality M^J is $(\mathbf{D}_\alpha(N_1), \aleph_0)$ -homogeneous and so without loss of generality for some $t(*) \in I \setminus J$, $N_1^J = M_{t(*)}$, $N^J = M_{t(*)+1}$ and we get a contradiction as in the proof of 5.24 (i.e., the choice of $\langle \bar{a}_\ell : \ell \leq \ell(*) \rangle$ there⁹). $\square_{5.34}$

{88r-5.21

5.36. Definition. 1) \mathfrak{K} has the symmetry property when the following holds: if $N_0 \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} N_3$ ($\ell = 1, 2$) and for every $\bar{a} \in N_1$, $\text{gtp}(\bar{a}, N_2, N_3)$ is the stationarization of $\text{gtp}(\bar{a}, N_0, N_3)$, then for every $\bar{b} \in N_2$, $\text{gtp}(\bar{b}, N_1, N_3)$ is the stationarization of $\text{gtp}(\bar{b}, N_0, N_3)$. 2) If $N_0, N_1, N_2 \leq_{\mathfrak{K}} N_3$ satisfies the assumption and conclusion of part (1) we say that N_1, N_2 are in stable amalgamation over N_0 inside N_3 (or in two-sided stable amalgamation over N_0 inside N_3). If only the hypothesis of (1) holds we say they are in a one sided stable amalgamation over N_0 inside N_3 (then the order of (N_1, N_2) is important). 3) We say that \mathfrak{K} has unique [one sided] amalgamation when: if $N_0 \leq_{\mathfrak{K}} N_\ell \in K_{\aleph_0}$ for $\ell = 1, 2$ then N_1, N_2 has unique [one sided] stable amalgamation, see part (4). 4) We say N_1, N_2 have a unique [one sided] stable amalgamation over N_0 , where for notational simplicity $N_1 \cap N_2 = N_0$, provided that: if (*), i.e. clauses (a)-(d) below hold then (**) below holds, where:

- (*) (a) $N_1 \leq_{\mathfrak{K}} N_3, N_2 \leq_{\mathfrak{K}} N_3$ and (N_1, N_2) in [one sided] stable amalgamation inside N_3 over N_0 and $\|N_3\| \leq \|N_1\| + \|N_2\|$
- (b) $M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3$ for $\ell = 1, 2$ and (M_1, M_2) are in [one sided] stable amalgamation inside M_3 over M_0 (hence $M_1 \cap M_2 = M_0$)
- (c) f_ℓ is an isomorphism from N_ℓ onto M_ℓ for $\ell = 0, 1, 2$
- (d) $f_0 \subseteq f_1$ and $f_0 \subseteq f_2$
- (**) we can find $M'_3, M_3 \leq_{\mathfrak{K}} M'_3$ and f_3 , a $\leq_{\mathfrak{K}}$ -embedding of N_3 into M'_3 extending $f_1 \cup f_2$.

We at last get the existence of stable amalgamation (to which we earlier get approximations).

{88r-5.22

5.37. Claim. For any $N_0 \leq_{\mathfrak{K}} N_1, N_2$, all from K_{\aleph_0} , we can find $M, N_0 \leq_{\mathfrak{K}} M \in K_{\aleph_0}$ and $\leq_{\mathfrak{K}}$ -embeddings f_1, f_2 of N_1, N_2 respectively over N_0 into M such that $N_0, f_1(N_1), f_2(N_2)$ are in stable amalgamation.

⁹A third way is to use forcing and absoluteness to use the case CH fail

5.38. **Remark.** In the proof we could have “inverted the tables” and used \bar{c}_ζ in the ω_1 direction.

Proof. We define by induction on $\zeta < \omega_1$, $\langle M_\alpha^\zeta : \alpha < \omega_1 \rangle$ and \bar{c}_ζ such that:

- (i) $\langle M_\alpha^\zeta : \alpha < \omega_1 \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous and $M_\alpha^\zeta \in K_{\aleph_0}$
- (ii) for $\alpha < \zeta$, $M_\alpha^\zeta = M_\alpha^\alpha$ and $\xi < \zeta \& \alpha < \omega_1 \Rightarrow M_\alpha^\xi \leq_{\mathfrak{R}} M_\alpha^\zeta$
- (iii) for ζ limit, $M_\alpha^\zeta = \bigcup_{\xi < \zeta} M_\alpha^\xi$
- (iv) for $\zeta \leq \alpha < \omega_1$, ζ non-limit $M_{\alpha+1}^\zeta$ is $(\mathbf{D}_{\alpha+1}(M_\alpha^\zeta), \aleph_0)^*$ -homogeneous
- (v) for every $\bar{c} \in M_{\alpha+1}^\zeta$, $\text{gtp}(\bar{c}, M_\alpha^{\zeta+1}, M_{\alpha+1}^{\zeta+1})$ is a stationarization of $\text{gtp}(\bar{c}, M_\alpha^\zeta, M_{\alpha+1}^\zeta)$
- (vi) $\bar{c}_\zeta \in M_{\zeta+1}^{\zeta+1}$ and for $\zeta + 1 < \alpha < \omega_1$, $\text{gtp}(\bar{c}_\zeta, M_\alpha^\zeta, M_\alpha^{\zeta+1})$ is the stationarization of $\text{gtp}(\bar{c}_\zeta, M_{\zeta+1}^\zeta, M_{\zeta+1}^{\zeta+1})$
- (vii) for every $p \in \mathbf{D}(M_\alpha^\xi)$ for some ζ satisfying $\xi + \alpha < \zeta < \omega_1$ we have $\text{gtp}(\bar{c}_\zeta, M_{\zeta+1}^\xi, M_{\zeta+1}^{\xi+1})$ is a stationarization of p .

There is no problem doing this (by 5.34 and as in earlier constructions); in limit stages we use local character 5.26(3) and \mathbf{D}_α being closed under stationarization.

Now easily for a thin enough closed unbounded set of $E \subseteq \omega_1$, for every $\zeta \in E$ we have

(*) $_\zeta$ (a) M_ζ^ζ is $(\mathbf{D}_\zeta(M_\zeta^0), \aleph_0)^*$ -homogeneous

(b) for every $\bar{c} \in M_\zeta^\zeta$, $\text{gtp}(\bar{c}, \bigcup_{\alpha < \omega_1} M_\alpha^0, \bigcup_{\xi < \omega_1} M_\xi^\xi)$ is a stationarization of $\text{gtp}(\bar{c}, M_\zeta^0, M_\zeta^\zeta)$

(c) for every $\bar{c} \in M_{\zeta+1}^0$, $\text{gtp}(\bar{c}, M_\zeta^{\zeta+1}, M_{\zeta+1}^{\zeta+1})$ is a stationarization of $\text{gtp}(\bar{c}, M_\zeta^0, M_{\zeta+1}^0)$.

[Why? Clause (c) holds by clause (v) of the construction (as $\langle M_\varepsilon^\zeta : \varepsilon \leq \zeta \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous). Clause (b) holds as E is thin enough, i.e., is proved as in earlier constructions (i.e., see the proof of (*) in the proof of 5.30(1)). As for Clause (a), first note that by clauses (i),(ii),(iii) the sequence $\langle M_\varepsilon^\zeta : \varepsilon \leq \zeta \rangle$ is $\leq_{\mathfrak{R}}$ -increasing continuous. By clause (vi) we have $\varepsilon < \zeta \Rightarrow \text{gtp}(\bar{c}_\varepsilon, M_\varepsilon^\varepsilon, M_\varepsilon^{\varepsilon+1})$ does not fork over $M_\varepsilon^\varepsilon$ and clause (vii) of the construction we have: if $p \in \mathbf{D}_\zeta(M_\varepsilon^\zeta)$, $\varepsilon < \zeta$ then for some $\xi \in (\varepsilon, \zeta)$, $\text{gtp}(\bar{c}_\xi, M_\xi^\xi, M_{\xi+1}^\xi)$ is a non-forking extension of p . As E is thin enough we have $\bar{d} \in M_\zeta^\zeta \Rightarrow \text{gtp}(\bar{d}, M_0^\zeta, M_\zeta^\zeta) \in \mathbf{D}_\zeta(M_0^\zeta)$. Together it is easy to get clause (a), e.g., see 5.47.]

So as in the proof of 5.30(3) we can finish (choose $\zeta \in E$, f_0 an isomorphism from N_0 onto M_ζ^0 , $f_1 \supseteq f_0$ is an $\leq_{\mathfrak{R}}$ -embedding of N_1 into M_ζ^ζ and $f_2 \supseteq f_0$ a $\leq_{\mathfrak{R}}$ -embedding of N_2 into $M_{\zeta+1}^0$). $\square_{5.37}$

5.39. **Remark.** Note that in [Sh:600] we use only the results up to this point.

{88r-5.23}

5.40. **Theorem.** 1) Suppose in addition to the hypothesis of this section that $2^{\aleph_1} < 2^{\aleph_2}$ and the club ideal on \aleph_1 is not \aleph_2 -saturated and $\dot{I}(\aleph_2, K) < 2^{\aleph_2}$ or just $\dot{I}(\aleph_2, K(\aleph_1\text{-saturated})) < 2^{\aleph_2}$. Then \aleph has the symmetry property.

2) Assume $2^{\aleph_1} < 2^{\aleph_2}$ and $\dot{I}(\aleph_2, K(\aleph_1\text{-saturated})) < \mu_{\text{unif}}(\aleph_2, 2^{\aleph_1})$; this number is always $> 2^{\aleph_1}$, usually 2^{\aleph_2} , see 0.6. Then \aleph has the symmetry property and stable amalgamation in K_{\aleph_0} is unique (we know that it always exists and it follows by (1) + (2) that one sided amalgamation is unique).

{88r-5.23}

5.41. **Discussion.** 1) This certainly gives a desirable conclusion. However, part (2) is not used so we shall return to it in [Sh:838].

More elaborately, in [Sh:838, 4.1], in the “lean version of [Sh:838], see reading plan A in [Sh:838, §0], assuming the weak diamond ideal is not \aleph_2 -saturated we prove 5.40(2) hence we also prove a slight weaker version of 5.40(1), replacing “ $\dot{I}(\aleph_2, K)(\aleph_1\text{-saturated}) < 2^{\aleph_2}$ ” by $\dot{I}(\aleph_2, K(\aleph_1\text{-saturated})) < \mu_{\text{unif}}(\aleph_2, 2^{\aleph_1})$. !!

Better, in [Sh:838, 4.40] we prove 5.40(2) fully. Still, the proof given below of part (1) is not covered presently by [Sh:838] and it gives nicer reasons for non-isomorphisms (essentially different natural invariants). !!

2) As for part (1), we can avoid using it (except in 5.45 below). More fully, in [Sh:600, §3] dealing with \aleph as here by [Sh:600, 3.4] for every $\alpha < \omega_1$ we derive a good \aleph_0 -frame \mathfrak{s}_α with $\aleph^{\mathfrak{s}_\alpha} = \aleph_{\mathcal{D}_\alpha}$ (if we would have liked to derive a good \aleph_1 -frame we would need 5.40). !!

Then in [Sh:705] if \mathfrak{s} is successful (holds, e.g. if $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}$ and $\dot{I}(\aleph_2, \aleph^{\mathfrak{s}_\alpha}) < 2^{\aleph_2}$ and WDMId_{\aleph_1} is not \aleph_2 -saturated) then we derive the successor \mathfrak{s}_α^+ , a good \aleph_1 -frame with $K^{\mathfrak{s}_\alpha^+} \subseteq \{M \in K_{\aleph_1}^{\mathfrak{s}_\alpha} : M \text{ is } \aleph_1\text{-saturated for } K^{\mathfrak{s}_\alpha}\}$, and \mathfrak{s}_α^+ is even good⁺ (see Claim [Sh:705, 1.6](2) and Definition [Sh:705, 1.3]). This suffices for the main conclusions of [Sh:600, §9] and end of [Sh:705, §12]. !!

3) Still we may wonder is $\leq_{\mathfrak{s}_\alpha^+} = \leq_{\aleph} \upharpoonright \aleph_{\mathfrak{s}_\alpha^+}$? If \mathfrak{s}_α is good⁺ then the answer is yes (see [Sh:705, 1.6](1)). That is, the present theorem 5.40 is used in [Sh:705, §1] to prove \mathfrak{s} is “good⁺”, really this is proved in 5.45. In fact part (1) of 5.40 is enough to prove that $\mathfrak{s}_{\mathcal{D}_*}$ is good⁺, see [Sh:705, 1.5](1A). !!

3) The proof of 5.40(1) gives that if \aleph fails the symmetry property then $\dot{I}(\aleph_2, K) \geq 2^{\aleph_1}$ even if $2^{\aleph_1} = 2^{\aleph_2}$ and do not use $2^{\aleph_0} = 2^{\aleph_1}$ directly (but use earlier results of §5). The case “ \mathcal{D}_{\aleph_1} is \aleph_2 -saturated, $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}$, $\dot{I}(\aleph_2, \aleph_2) < \mu_{\text{unif}}(\aleph_2, 2^{\aleph_2})$ ” is covered in [Sh:838]. !!

Proof. 1) So in the first part toward contradiction we can assume that $K^4 \neq \emptyset$ where K^4 is the class of quadruple $\bar{N} = (N_0, N_1, N_2, N_3)$ such that N_1, N_2 are one sided stably amalgamated over N_0 inside N_3 but N_2, N_1 are not. Hence there is $\bar{c} \in N_2$ such that $\text{gtp}(\bar{c}, N_1, N_3)$ is not the stationarization of $\text{gtp}(\bar{c}, N_0, N_2) = \text{gtp}(\bar{c}, N_0, N_3)$. We define a two-place relation \leq on K^4 by $\bar{N}^1 \leq \bar{N}^2$ iff $N_0^1 = N_0^2, N_\ell^1 \leq_{\aleph} N_\ell^2$ for $\ell = 0, 1, 2$ and $\bar{a} \in N_1^1 \Rightarrow \text{gtp}(\bar{a}, N_2^2, N_3^2)$ is definable over some $\bar{b} \in N_0^1$. Easily this is a partial order and K^4

modified:2015-01-25

(88r) revision:2015-01-23

is closed under union of increasing countable sequences. Hence without loss of generality for some \mathbf{D}_*, \bar{N}^*

- (*) (a) $\mathbf{D}_* \in \{\mathbf{D}_\alpha : \alpha < \omega_1\}$
- (b) $\bar{N}^* \in K^4$
- (c) N_ℓ^* is $(\mathbf{D}_*(N_0^*), \aleph_0)^*$ -homogeneous over N_0^* for $\ell = 1, 2$
- (d) N_3^* is $(\mathbf{D}_*(N_\ell^*), \aleph_0)^*$ -homogeneous over N_ℓ^* for $\ell = 1, 2$

□

So we have proved

{88r-5.23.8}

5.42. Observation. To prove 5.40, we can assume that $\mathbf{D} = \mathbf{D}_\alpha$ for $\alpha < \omega_1$, i.e., \mathbf{D} is countable.

Continuation of the proof of 5.40. A problem is that we still have not proven the existence of a superlimit model of K of cardinality \aleph_1 though we have a candidate N^* from 5.18. So we use N^* , but to ensure we get it at limit ordinals (in the induction on $\alpha < \aleph_2$), we have to take a stationary $S_0 \subseteq \omega_1$ with $\omega_1 \setminus S_0$ not small, i.e., $\omega_1 \setminus S_0$ does not belong to the ideal WDmId_{\aleph_1} from Theorem 0.6 and “devote” it to ensure this, using 5.37. □_{5.40}

The point of using S_0 is as follows (this is supposed to help to understand the quotation from [Sh:838]):

{88r-5.23A}

5.43. Definition. 1) Let $K^{\text{qt}} = \{\bar{N} : \bar{N} = \langle N_\alpha : \alpha < \omega_1 \rangle$ be \leq_{\aleph} -increasing continuous, $N_\alpha \in K_{\aleph_0}$, $N_{\alpha+1}$ is $(\mathbf{D}_\alpha(N_\alpha), \aleph_0)^*$ -homogeneous}.

2) On K^{qt} we define a two-place relation $<_S^a$ (for $S \subseteq \omega_1$) as follows:

$\bar{N}^1 <_S^a \bar{N}^2$ if and only if for some closed unbounded $E \subseteq \omega_1$

(a) for every $\alpha \in C$ we have $N_\alpha^1 \leq_{\aleph} N_\alpha^2$ and $N_{\alpha+1}^1 \leq_{\aleph} N_{\alpha+1}^2$

(b) for every $\alpha < \beta$ from E we have $N_\beta^2 \cap \bigcup_{\alpha < \omega_1} N_\alpha^1 = N_\beta^1$ and N_β^1, N_α^2 are in one sided stable amalgamation over N_α^1 inside N_β^2 , i.e. if $\bar{a} \in N_\beta^1$ then $\text{gtp}(\bar{a}, N_\alpha^2, N_\beta^2)$ is the stationarization of $\text{gtp}(\bar{a}, N_\alpha^1, N_\beta^1)$

(c) if $\alpha \in S \cap C$ then $N_\alpha^2, N_{\alpha+1}^1$ are in stable amalgamation over N_α^1 inside $N_{\alpha+1}^2$.

{88r-5.23B}

5.44. Fact. 0) The two-place relation $<_S^a$ defined in 5.43 are partial orders on K^{qt} for $n < \omega$.

1) If $\bar{N}^n \leq_{S_0}^a \bar{N}^{n+1}$ and let E_n exemplify this (as in the Definition 5.43) and let $E_\omega = \bigcap_{n < \omega} E_n$, $E'_\omega = \{\alpha, \alpha + 1 : \alpha \in C_\omega\}$ and let $N_\alpha^\omega = \bigcup_{n < \omega} N_\beta^n$ when $\beta = \text{Min}[E'_\omega \setminus \alpha]$. Then $\langle N_\alpha^\omega : \alpha < \omega_1 \rangle \in K_{< \aleph_1}$ and $\bar{N}^n \leq_{S_0}^a \langle N_\alpha^\omega : \alpha < \omega_1 \rangle$ for $n < \omega$.

2) If $\langle \bar{N}^\varepsilon : \varepsilon < \omega_1 \rangle$ is $<_S^a$ -increasing and $N^\varepsilon = \cup\{N_\alpha^\varepsilon : \alpha < \omega_1\} \in K_{\aleph_1}$ is \leq_{\aleph} -increasing

continuous, the club $E_{\varepsilon, \zeta}$ witness $\bar{N}^\varepsilon \leq \bar{N}^\zeta$ for $\varepsilon < \zeta < \aleph_1$ and $\langle N_\alpha : \alpha < \omega_1 \rangle$ a $\leq_{\mathfrak{K}}$ -representation of N , and for a club of $\alpha < \aleph_1$, $N_\alpha = \cup\{N_\alpha^\varepsilon : \varepsilon < \alpha\}$, $N_{\alpha+1} = \cup\{N_{\alpha+1}^\varepsilon : \varepsilon < \alpha\}$ then $\varepsilon < \omega_1 \Rightarrow \bar{N}^\varepsilon \leq_{S_0}^a \bar{N}$.

Proof. Should be easy by now.

Continuation of the proof of 5.40: it is done as follows.

There is $\langle S_\varepsilon : \varepsilon < \omega_1 \rangle$ such that $S_\varepsilon \subseteq \omega_1$, $\zeta < \varepsilon \Rightarrow S_\zeta \cap S_\varepsilon$ countable and $S_0, S_{\varepsilon+1} \setminus S_\varepsilon \in (\mathcal{D}_{\omega_1})^+$, possible by an assumption.

Now for any $u \subseteq \omega_2$ we choose $N_\varepsilon^u, \bar{N}_\varepsilon^u$ by induction on $\varepsilon < \omega_2$ such that

- ⊗ (a) $\bar{N}_\varepsilon^u = \langle N_{\varepsilon, \alpha}^u : \alpha < \omega_1 \rangle \in K^{\text{qt}}$
- (b) $N_\varepsilon^u = \cup\{N_{\varepsilon, \alpha}^u : \alpha < \omega_1\} \in K_{\aleph_1}$
- (c) for $\zeta < \varepsilon$ we have $\bar{N}_\zeta^u <_{S_\zeta}^1 \bar{N}_\varepsilon^u$ when $\xi \notin [\zeta, \varepsilon) \cap u$ (we can use $S'_{[\zeta, \varepsilon)}$,

the compliment of the diagonal union of $\{\langle S_\xi : \xi \in [\zeta, \varepsilon) \cap u \rangle\}$

- (d) we can demand continuity as defined implicitly in Fact 5.44
- (e) for each $\varepsilon \in u$ for a club of $\alpha < \omega_1$ if $\alpha \in S_\varepsilon$ then $N_{\varepsilon+1, \alpha}^u, N_{\varepsilon, \alpha+1}^u$

are not in stable amalgamation over $N_{\varepsilon, \alpha}^u$ inside $N_{\varepsilon+1, \alpha+1}^u$

(though is in one side).

Lastly, let $N^u = \cup\{N_\varepsilon^u : \varepsilon < \omega_1\} \in K_{\aleph_2}$. Now we can prove that if $u, v \subseteq \omega_2$ and $N^u \approx N^v$ then for some club C of ω_2 , $u \cap C = v \cap C$. So we can easily get $\dot{I}(\aleph_2, \mathfrak{K}) = 2^{\aleph_2}$ and even $\dot{I}(\aleph_2, \mathfrak{K}(\aleph_1\text{-saturated})) = 2^{\aleph_2}$. □_{5.40}

5.45. Theorem. *Suppose \mathfrak{K} has the symmetry property (holds if the assumption of 5.40(1) hold). Then \mathfrak{K} has a superlimit model in \aleph_1 .*

Proof. We have a candidate N^* from 5.18. So let $\langle N_i : i < \delta \rangle$ be $\leq_{\mathfrak{K}}$ -increasing, $N_i \cong N^*$ and without loss of generality $\delta = \text{cf}(\delta)$. If $\delta = \omega_1$ this is very easy. If $\delta = \omega$, let $N_\omega = \bigcup_{i < \omega} N_i$ and for each $i \leq \omega$ let $\langle N_i^\alpha : \alpha < \omega_1 \rangle$ be $\leq_{\mathfrak{K}}$ -increasing continuous with union N_i and $N_i^\alpha \in K_{\aleph_0}$. Now by restricting ourselves to a club E of α 's and renaming it $E = \omega_1$, we get: for $i < j \leq \omega$, $N_i^\alpha = N_i \cap N_j^\alpha$, and

- ⊗₁ for any $\alpha < \beta < \omega_1$, $\bar{a} \in N_\omega^\alpha$ and $i < \omega$, the type $\text{gtp}(\bar{a}, N_i^\beta, N_\omega^\beta)$ is a stationarization of $\text{gtp}(\bar{a}, N_i^\alpha, N_\omega^\alpha)$.

To prove $N_\omega \cong N^*$ it is enough to prove:

- ⊗₂ if $\alpha < \omega_1$, $p \in \mathbf{D}(N_\omega^\alpha)$ then some $\bar{b} \subseteq N_\omega$ realizes p in N_ω .

modified:2015-01-25

(88r) revision:2015-01-23

{88r-5.24

By 5.26(3) there is $i < \omega$ such that p is the stationarization of $q = p \upharpoonright N_i^\alpha \in \mathbf{D}(N_i^\alpha)$. As $N_i \cong N^*$, there is $\bar{b} \subseteq N_i$ which realizes q and we can find $\beta \in (\alpha, \omega_1)$ such that $\bar{b} \subseteq N_i^\beta$. By \otimes_1 we have $N_\omega^\alpha, N_i^\beta$ is in one sided stable amalgamation over N_i^α inside N_ω^β (see 5.36(2)).

As we assume \mathfrak{K} has the symmetry property, also $N_i^\beta, N_\omega^\alpha$ is in stable amalgamation over N_i^α inside N_ω^β . In particular, as $\bar{b} \subseteq N_i^\beta$, we have $\text{gtp}(\bar{b}, N_\omega^\alpha, N_\omega^\beta)$ is the stationarization of $\text{gtp}(\bar{b}, N_i^\alpha, N_i^\beta)$ but the latter is $p \upharpoonright N_i^\alpha$ so by uniqueness of stationarization, $p = \text{gtp}(\bar{b}, N_\omega^\alpha, N_\omega^\beta)$ which is $\text{gtp}(\bar{b}, N_\omega^\alpha, N_\omega)$, so p is realized in N_ω as required. $\square_{5.45}$

We have implicitly proved

{88r-5.24.5}

5.46. Claim. Assume that $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\aleph_0}$ and $\bar{a}_\ell \in {}^{\omega>}(N_1)$ for $\ell = 1, 2$. Then $(*)_1 \Leftrightarrow (*)_2$ where for $\ell = 1, 2$

- $(*)_\ell$ there are $M_1, M_2, \bar{b}_1, \bar{b}_2$ such that
 - (a) $N_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2 \in K_{\aleph_1}$
 - (b) $\bar{a}_k \in {}^{\omega>}(M_k)$ for $k = 1, 2$
 - (c) $\text{gtp}(\bar{b}_{3-\ell}, N_0, M_1) = \text{gtp}(\bar{a}_{3-\ell}, N_0, N_1)$
 - (d) $\text{gtp}(\bar{b}_\ell, M_1, M_2)$ is the stationarization of $\text{gtp}(\bar{a}_\ell, N_0, N_1)$ from $\mathbf{D}(M_1)$
 - (e) $\text{gtp}(\bar{b}_1 \hat{\ } \bar{b}_2, N_0, M_2) = \text{gtp}(\bar{a}_1 \hat{\ } \bar{a}_2, N_0, N_1)$

Proof. We can deduce it from 5.34 (or immitate the proof of 5.24).

In detail by symmetry it is enough to assume $(*)_2$ and prove $(*)_1$. So let $M_1, M_2, \bar{b}_1, \bar{b}_2$ witness $(*)_2$.

By 5.37 we can find M'_2, f such that: $M_2 \leq_{\mathfrak{K}} M'_2 \in K_{\aleph_0}$, f is a $\leq_{\mathfrak{K}}$ -embedding of M_2 into M'_2 over N_0 such that $M_1, f(M_2)$ is in stable amalgamation over N_0 inside M'_2 . Now, as $f(M_2), M_1$ are in one sided stable amalgamation over N_0 inside M'_2 by the choice of $(M_1, M_2, \bar{b}_1, \bar{b}_2)$ we get $\text{gtp}(f(\bar{b}_2), M_1, M'_2) = \text{gtp}(\bar{b}_2, M_1, M'_2)$ hence $\text{gtp}(\bar{b}_1 \hat{\ } \bar{b}_2, N_0, M'_2) = \text{gtp}(\bar{b}_1 \hat{\ } f(\bar{b}_2), N_0, M'_2)$.

By the choice of $M'_2, f, \text{gtp}(\bar{b}_1, f(M_2), M'_2)$ is the stationarization of $\text{gtp}(\bar{b}_1, N_0, M_2) = \text{gtp}(\bar{a}_1, N_0, N_1)$. Now $(*)_1$ holds as exemplified by $(f(M_2), M'_2, f(\bar{b}_2), \bar{b}_1)$. $\square_{5.46}$

{88r-5.25}

5.47. Exercise. Assume $\alpha \leq \omega_1$ and

- (a) $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{K}}$ -increasing continuous, δ a limit ordinal
- (b) if $p \in \mathbf{D}(M_i)$ is realized in M_{i+1} then it $\in \mathbf{D}_\alpha(M_i)$ or just $p \upharpoonright M_0 \in \mathbf{D}(M_0)$
- (c) if $i < \delta, p \in \mathbf{D}_\alpha(M_i)$ then p is materialized in M_j for some $j \in (i, \delta)$.

Then M_δ is $(\mathbf{D}_\alpha(M_0), \aleph_0)^*$ -homogeneous.

Proof. Easy. \square

{88r-5.26}

5.48. **Discussion.** 1) Consider $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$, $|\tau_\psi| \leq \aleph_0$, $1 \leq \dot{I}(\aleph_1, \psi) < 2^{\aleph_0}$. We translate it to \mathfrak{K} and $<^{**}$ as earlier, see 3.19.

2) What if we waive categoricity in \aleph_0 ? The adoption of this was O.K. as we shrink \mathfrak{K} but not too much. But without shrinking probably we still can say something on the models in $\mathfrak{K}^* = \{M \in \mathfrak{K}_{\geq \aleph_0} : \text{if } N_0 \leq_{\mathfrak{K}} M, N_0 \in K_{\aleph_0} \text{ then for some } N_1, N_0 <^* N_1 \leq_{\mathfrak{K}} M\}$ as there are good enough approximations.

modified:2015-01-25

(88r) revision:2015-01-23

6. COUNTEREXAMPLES

In [Sh:48] the statement of Conclusion 3.9 was proved for the first time where K is the class of atomic models of a first order theory assuming Jensen's diamond \diamond_{\aleph_1} (taking $\lambda = \aleph_0$). In [Sh:87a] and [Sh:87b] the same theorem was proved using $2^{\aleph_0} < 2^{\aleph_1}$ only (using 0.6). Let us now concentrate on the case $\lambda = \aleph_0$. We asked whether the assumption $2^{\aleph_0} < 2^{\aleph_1}$ is necessary to get Conclusion 3.9. In this section we construct three classes of models K^1, K^2, K^3, K^4 failing amalgamation, i.e., failing the conclusion of 3.9, K^2, K^3, K^4 are a.e.c. with LST-number \aleph_0 while K^1 satisfy all the axioms needed in the proof of Conclusion 3.9 (but it is not an abstract elementary class - fails to satisfy AxIV,AxV).

K^2 is PC_{\aleph_0} and is axiomatizable in $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$.

K^3 is PC_{\aleph_0} and is axiomatizable in $\mathbb{L}(\mathbf{Q})$. Now the common phenomena to K^1, K^2, K^3, K^4 are that all of them satisfy the hypothesis of Conclusion 3.9, i.e., for $\ell = 1, 2, 3$ we have $\dot{I}(\aleph_0, K^\ell) = 1$ and the \aleph_0 -amalgamation property fails in K^ℓ , but assuming $\aleph_1 < 2^{\aleph_0}$ and MA_{\aleph_1} for $\ell = 1, 2, 3$ we have $\dot{I}(\aleph_1, K^\ell) = 1$.

{88r-6.1}

6.1. Definition. Let Y be an infinite set. A family \mathcal{P} of infinite subsets of Y is called independent if for every $\eta \in \omega^{>2}$ and pairwise distinct $X_0, X_1, \dots, X_{\ell g(\eta)-1}$ (notation: for $X \in \mathcal{P}$ denote $X^0 = X$ and $X^1 = Y \setminus X$) the following set $\bigcap_{k < \ell g(\eta)} X_k^{\eta[k]}$ is infinite.

{88r-6.2}

6.2. Definition. 1) The class of models K^0 is defined by

$$K^0 = \{M : M = \langle |M|, P^M, Q^M, R^M \rangle, |M| = P^M \cup Q^M, \\ P^M \cap Q^M = \emptyset, |P^M| = \aleph_0 \leq |Q^M| \text{ and} \\ R \subseteq P^M \times Q^M \}.$$

2) For $M \in K^0$, let $A_y^M = \{x \in P^M : xR^M y\}$ for every $y \in Q^M$.

3) Let K^1 be the class of $M \in K^0$ such that

(a) the family $\{A_y^M : y \in Q^M\}$ is independent, which means that if $m < n$ and y_0, \dots, y_{n-1} are pairwise distinct members of Q^M then the set $\{x \in P^M : xR^M y_\ell \equiv \ell < m \text{ for every } \ell < n\}$ is infinite

(b) for every disjoint finite subsets u, w of P^M we have $\|M\| = |A_{u,w}^M|$ where $A_{u,w}^M := \{y \in Q^M : a \in u \Rightarrow (aR^M y) \text{ and } b \in w \Rightarrow \neg(bR^M y)\}$.

4) The notion of (strict) substructure \leq_{\aleph^1} is defined by: for $M_1, M_2 \in K^1$, $M_1 \leq_{\aleph^1} M_2$ iff $M_1 \subseteq M_2$, $P^{M_1} = P^{M_2}$ and for any finite disjoint $u, w \subseteq P^{M_2}$ the set $A_{u,w}^{M_2} \setminus M_1$ is infinite when $M_1 \neq M_2$ (equivalently - non-empty).

5) $\aleph^1 = (K^1, \leq_{\aleph^1})$.

{88r-6.3}

6.3. **Lemma.** *The class $(K^1, <_{\aleph^1})$ satisfies*

0) *Ax 0.*

1) *Ax I.*

2) *Ax II.*

3) *Ax III.*

4) *Ax IV fails even for $\lambda = \aleph_0$; but if $\langle M_\alpha : \alpha \leq \delta \rangle$ is \leq_{\aleph} -increasing and $\|\bigcup_{\alpha < \delta} M_\alpha\| < \|M_\delta\|$*

then $\bigcup_{\alpha < \delta} M_\alpha <_{\aleph^1} M_\delta$.

5) *Ax V fails for countable models.*

6) *Ax VI holds with $\text{LST}(\aleph^1) = \aleph_0$, in fact it holds for every cardinal.*

7) *For every $M \in K^1$, $\|M\| \leq 2^{\aleph_0}$.*

Proof. 0), 1), 2) follows trivially from the definition.

3) To prove that $M = \bigcup_{i < \lambda} M_i \in K^1$, it is enough to verify that for every finite disjoint $u, w \subseteq P^M$, $|A_{u,w}^M| = \|M\|$. If $\langle M_i : i < \lambda \rangle$ is eventually constant we are done hence without loss of generality $\langle M_i : i < \lambda \rangle$ is $<_{\aleph^1}$ -increasing; from the definition of $<_{\aleph^1}$ it follows that for each i , M_{i+1} has a new $y = y_i$ as above, i.e., $y_i \in A_{u,w}^{M_{i+1}} \setminus M_i$ for every $i < \lambda$. Also for each i there are at least $\|M_i\|$ many members in $A_{u,w}^{M_i} \subseteq A_{u,w}^M$. Together there are at least $\|M\|$ members in $A_{u,w}^M$.

4) Let $\{M_n : n < \omega\} \subseteq K_{\aleph_0}^1$ be an $<_{\aleph^1}$ -increasing chain, let $M = \bigcup_{n < \omega} M_n$; by part 3) we have $M \in K_{\aleph_0}^1$. Since $|Q^M| = \aleph_0$ by Claim 6.5(a) below there exists $A \subseteq P^M \setminus \{A_y^M : y \in Q^M\}$ infinite such that $\{A_y : y \in Q^M\} \cup \{A\}$ is independent. Now define $N \in K^1$ by $P^N = P^M$, let $y_0 \notin M$, $Q^N = Q^M \cup \{y_0\}$ and finally let $R^N = R^M \cup \{\langle a, y_0 \rangle : a \in P^N \ \& \ a \in A\}$. Clearly for every $n < \omega$, $M_n \leq_{\aleph^1} N$ but N is not an \leq_{\aleph^1} -extension of $M = \bigcup_{n < \omega} M_n$ because the

second part in Definition 6.2(4) is violated.

5) Let $N_0 <_{\aleph^1} N \in K^1$ be given; as in 4) define $N_1 \subseteq N$, $|N_0| \subseteq |N_1|$ by adding a single element to Q^{N_0} (from the elements of $Q^N \setminus Q^{N_0}$) it is obvious that $N_0 \leq_{\aleph^1} N$, $N_1 \leq_{\aleph^1} N$ but $N_0 \not\leq_{\aleph^1} N_1$.

6) By closing the set under the second requirement in Definition 6.2(3).

7) Let $y_1 \neq y_2 \in Q^M$, we show that $A_{y_1}^M \neq A_{y_2}^M$; if $A_{y_1}^M \subseteq A_{y_2}^M$ then $A_{y_1}^M \cap (P^M \setminus A_{y_2}^M) = \emptyset$ contradiction to the requirement that $\{A_y : y \in Q\}$ is independent hence $|Q^M| \leq 2^{|P^M|} = 2^{\aleph_0}$ and as $|P^M| = \aleph_0$ we are done. □_{6.3}

6.4. **Theorem.** $\aleph^1 = (K^1, <_{\aleph^1})$ *satisfies the hypothesis of Conclusion 3.9. Namely*

1) $\dot{I}(\aleph_0, K^1) = 1$.

2) *Every $M \in K_{\aleph_0}^1$ has a proper \leq_{\aleph^1} -extension in $K_{\aleph_0}^1$.*

{88r-6.4}

modified:2015-01-25

(88r) revision:2015-01-23

(88r)

- 3) \aleph^1 is closed under chains of length $\leq \omega_1$.
 4) \aleph^1 fails the \aleph_0 -amalgamation property.

Proof. 1) Let $M_1, M_2 \in K_{\aleph_0}^1$, pick the following enumerations $|M_1| = \{a_n : n < \omega\}$ and $|M_2| = \{b_n : n < \omega\}$. It is enough to define an increasing sequence of finite partial isomorphisms $\langle f_n : n < \omega \rangle$ from M_1 to M_2 such that for every $k < \omega$ for some $n(k) < \omega$ satisfy $a_k \in \text{Dom}(f_{n(k)})$ and $b_k \in \text{Range}(f_{n(k)})$, (finally take $f = \bigcup_{n < \omega} f_n$ and this will be an isomorphism from M_1 onto M_2).

Define the sequence $\langle f_n : n < \omega \rangle$ by induction on $n < \omega$: let $f_0 = \emptyset$, if $n = 2m$ denote $k = \min\{k < \omega : a_k \notin \text{Dom}(f_n)\}$. Distinguish between the following two alternatives:

- (A) if $a_k \in P^{M_1}$ let $\{a'_0, \dots, a'_{j-1}\} = Q^{M_1} \cap \text{Dom}(f_n)$. Without loss of generality there exists $i \leq j-1$ such that for all $\ell < i, a_k R^{M_1} a'_\ell$ and for all $i \leq \ell \leq j-1, \neg a_k R a'_\ell$. By 6.2(1), P^{M_ℓ} is infinite, hence by clause (b) of 6.2(2) also Q^{M_ℓ} is infinite. Hence by clause (a) of 6.2(3) there are infinitely many $y \in P^{M_2}$ such that $y R^{M_2} f_n(a'_\ell)$ for all $\ell < i$ and for all $i \leq \ell < j-1, \neg y R^{M_2} f_n(a'_\ell)$. But $\text{Rang}(f_n)$ is finite. Hence there is such $y \in P^{M_2} \setminus \text{Rang}(f_n)$. Finally let $f_{n+1} = f_n \cup \{\langle a_k, y \rangle\}$
- (B) if $a_k \in Q^{M_1}$ let $\{a'_0, \dots, a'_{j-1}\} = P^{M_1} \cap \text{Dom}(f_n)$ and as before we may assume that there exists $i \leq j-1$ such that for all $\ell < i, a'_\ell R^{M_1} a_k$ and for all $i \leq \ell < j-1$ we have $\neg(a'_\ell) R^{M_1} a_k$. By the second requirement in Definition 6.2(3) there exists $y \in Q^{M_2} \setminus \text{Dom}(f_n)$ such that $(\forall \ell < i)[f_n(a'_\ell) R^{M_2} y]$ and $(\forall \ell)[i \leq \ell < j-1 \Rightarrow \neg f_n(a'_\ell) R^{M_2} y]$. Now define $f_{n+1} = f_n \cup \{\langle a_k, y \rangle\}$.

2) First we prove the following.

- 6.5. **Observation.** (a) Let P be a countable set. For every countable family \mathcal{P} of infinite subsets of P if \mathcal{P} is independent then there exists an infinite $A \subseteq P$ such that $\mathcal{P} \cup \{A\}$ is independent and $A \notin \mathcal{P}$, of course
- (b) if A, \mathcal{P} are as in (a) then for every infinite $B \subseteq P$ satisfying $|A \Delta B| < \aleph_0$ also $\mathcal{P} \cup \{B\}$ is independent (and $B \notin \mathcal{P}$)
- (c) moreover in clause (a) we can require in addition that: for any disjoint finite $u, w \subseteq P$ there exists $A \subseteq P$ as in (a) satisfying $u \subseteq A$ and $A \cap w = \emptyset$.

Proof of Claim 6.5. Clause (a): Let $\mathcal{P}^* = \{X \subseteq P : (\exists n < \omega)(\exists X_0 \in \mathcal{P}) \dots (\exists X_{n-1} \in \mathcal{P})(\exists k \leq n) [X \text{ or } P \setminus X \text{ is equal to } \bigcap \{X_i : i < k\} \cap \bigcap \{P \setminus X_i : k \leq i < n\}]\}$.

Clearly $|\mathcal{P}^*| = \aleph_0$ hence we can find a sequence $\langle A_n : n < \omega \rangle$ such that $\{A_n : n < \omega\} = \mathcal{P}^*$ and such that for every $k < \omega$ there exists $n > k$ satisfying $A_n = A_k$ hence for some $n > k, A_n = P \setminus A_k$. Let $P = \{a_n : n < \omega\}$ without repetition.

Now define $i(n) < \omega$ by induction on n . Let $i(0) = 0$.

If $n = k + 1$, let $i(n) = \text{Min}\{\ell < \omega : i(n-1) < \ell \text{ and}$

{88r-01-25
modified:2015-6.5}

$$a_\ell \in (A_k \setminus \{a_{i(0)}, \dots, a_{i(n-1)}\}).$$

It is easy to verify that the construction is possible. Directly from the construction it follows that $A = \{a_{i(n)} : n < \omega\}$ is a set as required.

Clause (b): Easy.

Clause (c): Let $u, w \subseteq P$ be finite disjoint and \mathcal{P} a countable family of subsets of P which is independent.

Let $A' \subseteq P$ be as proved in clause (a). According to (b) also $A = (A' \cup u) \setminus w$ satisfies: the family $\mathcal{P} \cup \{A\}$ is independent. \square

Return to the proof of Theorem 6.4(2). Let $\mathcal{P} = \{A_y^M \subseteq P^M : y \in Q^M\}$. Let $\langle s_n : n < \omega \rangle$ be an enumeration of $[P^M]^{<\aleph_0}$ with repetitions such that for every finite disjoint $u, w \subseteq P^M$ there exists $n < \omega$ such that $s_{2n} = u, s_{2n+1} = w$ and for each $k < \omega, s_{2k} \cap s_{2k+1} = \emptyset$.

It is enough to define $\{\mathcal{P}_n : n < \omega\}$ increasing chain of countable independent families of subsets of P^M such that $\mathcal{P}_0 = \mathcal{P}$ and for all $k < \omega$ and every finite disjoint $u, w \subseteq P^M, (\exists n < \omega)(\exists A \in \mathcal{P}_n \setminus \mathcal{P}_k)[u \subseteq A \wedge A \cap w = \emptyset]$ because $\bigcup_{n < \omega} \mathcal{P}_n$ enables us to define

$N \in K_{\aleph_0}^1$ such that $M \leq_{\aleph_1} N$ as required. Assume \mathcal{P}_n is defined; apply Claim 6.5(c) on $P = P^M$ and \mathcal{P}_n when substituting $u = s_{2n}, w = s_{2n+1}$ let $A \subseteq P$ be supplied by the Claim and define $\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{A\}$. It is easy to check that $\{\mathcal{P}_n : n < \omega\}$ satisfies our requirements.

3) This is a special case of Ax III which we checked in Lemma 6.3(3).

4) Let $M \in K_{\aleph_0}^1$ and we shall find $M_\ell \in K_{\aleph_0}^1 (\ell = 0, 1), M \leq_{\aleph_1} M_\ell$, which cannot be amalgamated over M . By part (2) we can find a model M_1 such that $M <_{\aleph_1} M_1 \in K_{\aleph_0}^1$ and choose $y \in Q^{M_1} \setminus Q^M$. Define $M_2 \in K_{\aleph_0}^1$; its universe is $|M_1|, P^{M_2} = P^{M_1}, Q^{M_2} = Q^{M_1}$ and $R^{M_2} = \{(a, b) : aR^{M_1}b \& b \neq y \text{ or } a \in P^M \& b = y \& \neg(aRy)\}$. Clearly M_1, M_2 cannot be amalgamated over M (since the amalgamation must contain a set and its complement). $\square_{6.4}$

6.6. Theorem. *Assume MA_{\aleph_1} (hence $2^{\aleph_0} > \aleph_1$). The class $(K^1, <_{\aleph_1})$ is categorical in \aleph_1 .*

{88r-6.6}

Proof. Let $M, N \in K_{\aleph_1}^1$ and we shall prove that they are isomorphic. By repeated use of Lemma 6.3(6),(4) for AxVI we get (strictly) $<_{\aleph_1}$ -increasing continuous chains $\{M_\alpha : \alpha < \omega_1\}, \{N_\alpha : \alpha < \omega_1\} \subseteq K_{\aleph_0}^1$ such that $M = \bigcup_{\alpha < \omega_1} M_\alpha$ and $N = \bigcup_{\alpha < \omega_1} N_\alpha$, so for

$$\alpha < \beta, M_\alpha <_{\aleph_1} M_\beta, N_\alpha <_{\aleph_1} N_\beta.$$

Now define a forcing notion which supplies an isomorphism $g : M \rightarrow N$.

$$\mathbb{P} = \{f : f \text{ is a partial finite isomorphism from } M \text{ into } N \text{ satisfying} \\ (\forall \alpha < \omega_1)(\forall a \in \text{Dom}(f))[a \in M_\alpha \Leftrightarrow f(a) \in N_\alpha]\},$$

the order is inclusion. It is trivial to check that if $G \subseteq \mathbb{P}$ is a directed subset then $g = \bigcup G$ is a partial isomorphism from M to N , we show that $\text{Dom}(g) = |M|$ if G is generic enough. For every $a \in |M|$ define $\mathcal{J}_a = \{f \in \mathbb{P} : a \in \text{Dom}(f)\}$, and we shall show that for all $a \in |M|$ the set \mathcal{J}_a is dense. For $a \in M$ let $\alpha(a) = \text{Min}\{\alpha < \omega_1 : a \in M_\alpha\}$, clearly it is zero or a successor ordinal. Let $f \in \mathbb{P}$ be a given condition, it is enough to find $h \in \mathcal{J}_a$ such that $f \subseteq h$ and $a \in \text{Dom}(h)$. Let $A = \text{Dom}(f)$, let $B, C \subseteq A$ be disjoint sets such that $B \cup C = A$ and $B = \text{Dom}(f) \cap P^M, C = \text{Dom}(f) \cap Q^M$. Without loss of generality $a \notin B \cup C$. If $a \in P^M$ let $\varphi(x, \bar{c}) = \wedge \{\pm x R c : c \in C \text{ and } M \models \pm a R c\}$. From the definition of K^1 there exists $b \in P^N \setminus \text{Rang}(f)$ such that $N \models \varphi[b, f(\bar{c})]$. If $a \in Q^M$ let $\varphi(x, \bar{b}) = \wedge \{\pm b R x : b \in B, M \models \pm b R a\}$, we can find infinitely many $b \in Q^{N_{\alpha(a)}} \setminus \bigcup_{\beta < \alpha(a)} N_\beta$,

satisfying $\varphi(x, f(\bar{b}))$.

Why? This is as $\bigcup \{N_\beta : \beta < \alpha(a)\} <_{\aleph^1} N_{\alpha(a)}$ as C is finite without loss of generality $b \notin f(C)$.

Finally, let $h = f \cup \{ \langle a, b \rangle \}$.

The proof that $\text{Range}(g) = |N|$ is analogous to the proof that $\text{Dom}(g) = |M|$. In order to use MA we just have to show that R has the c.c.c. Let $\{f_\alpha : \alpha < \omega_1\} \subseteq R$ be given. It is enough to find $\alpha, \beta < \omega_1$ such that f_α, f_β have a common extension. Without loss of generality we may assume $|M| \cap |N| = \emptyset$. By the finitary Δ -system lemma there exists $S \subseteq \omega_1, |S| = \aleph_1$ such that $\{\text{Dom}(f_\alpha) \cup \text{Range}(f_\alpha) : \alpha \in S\}$ is a Δ -system with heart A . Let $B \subseteq |M|, C \subseteq |N|$ be such that $A = B \cup C$, now without loss of generality for every $\alpha \in S, f_\alpha$ maps B into C .

[Why? If not, $S_1 = \{\alpha \in S : \text{for some } b = b_\alpha \in B, f_\alpha(b_\alpha) \notin C\}$ is uncountable hence for some $b \in B, S_2 = \{\alpha \in S_1 : b_\alpha = b\}$ is uncountable; so $\langle f_\alpha(b) : \alpha \in S_2 \rangle$ is without repetitions hence is uncountable. But $\{f(b) : f \in \mathbb{P} \text{ and } b \in \text{Dom}(f) \cap B\}$ is countable because $f \in \mathbb{P} \& b \in \text{Dom}(f) \& \alpha < \omega_1 \Rightarrow [b \in M_\alpha \equiv f(b) \in N_\alpha]$. Similarly, f_α^{-1} maps C into B , so necessarily f_α maps B onto C ; but the number of possible functions from B to C is $|C|^{|B|} < \aleph_0$. Hence there exists $S_1 \subseteq S, |S_1| = \aleph_1$ such that for all $\alpha, \beta \in S_1, f_\alpha \upharpoonright B = f_\beta \upharpoonright B$ and $\text{Dom}(f_\alpha) \cap M_0 \subseteq B, \text{Rang}(f_\alpha) \cap N_0 \subseteq C$. As $P^{M_\alpha} = P^{M_0} \subseteq M_0, P^{N_\alpha} = P^{N_0} \subseteq N_0$ for every $\alpha \in S_1$ we have $P^M \cap \text{Dom}(f_\alpha) \subseteq B, P^N \cap \text{Range}(f_\alpha) \subseteq C$, therefore for all $\alpha, \beta \in S_1, f_\alpha \cup f_\beta \in \mathbb{P}$ and in particular there exists $\alpha \neq \beta < \omega_1$ such that $f_\alpha \cup f_\beta \in \mathbb{P}$.

□_{6.6}

In the terminology of [GrSh:174] Theorems 6.4 and 6.6 give us together:

6.7. **Conclusion.** Assuming $2^{\aleph_0} > \aleph_1$ and $\text{MA}_{\aleph_1}, \aleph^1$ is a nice category which has a universal object in \aleph_1 , moreover it is categorical in \aleph_1 .

6.8. **Definition.** 1) K^2 is the class of $M \in K^0$ (see Definition 6.2) satisfying:

- (a) $(\forall x \in Q^M)(\forall u \in [P^M]^{<\aleph_0})(\exists y \in Q)[A_x^M \Delta A_y^M = u]$
- (b) if $k < \omega$ and $y_0, \dots, y_{k-1} \in Q$ satisfies $|A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$ for $\ell < m < k$ then the set $\{A_{y_\ell}^M : \ell < k\}$ is an independent family of subsets of P^M
- (c) $Q(y) \wedge Q(z) \wedge (\forall x \in P)[xRy \leftrightarrow xRz] \rightarrow y = z,$
- (d) for every $k < \omega$ for some $y_0, \dots, y_k \in Q^M$ we have $\bigwedge_{\ell < m \leq k} |A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0.$

2) For $M_1, M_2 \in K^2$

$$M_1 \leq_{\aleph^2} M_2 \Leftrightarrow^{\text{df}} M_1 \subseteq M_2, P^{M_1} = P^{M_2}.$$

3) $\aleph^2 = (K^2, \leq_{\aleph^2}).$

4) K^3 is the class of models $M = (|M|, P^M, Q^M, R^M, E^M)$ such that

- (a) $(|M|, P^M, Q^M, R^M) \in K^1$
- (b) E^M is an equivalence relation on Q^M
- (c) E^M has infinitely many equivalence classes
- (d) each equivalence class of E^M is countable
- (e) if $u, w \subseteq P^M$ are finite disjoint and $y \in Q^M$ then for some $y' \in y/E^M$ we have $a \in u \Rightarrow aR^M y'$ and $b \in w \Rightarrow \neg(bR^M y')$.

5) We define \leq_{\aleph^3} : $M_1 \leq_{\aleph^3} M_2 \Leftrightarrow^{\text{df}} M_1 \subseteq M_2$ and $a \in M_1 \Rightarrow a/E^{M_2} = a/E^{M_1}.$

6) $\aleph^3 = (K^3, \leq_{\aleph^3}).$

If we like to have a class defined by a sentence from $\mathbb{L}_{\omega_1, \omega}$ (rather than $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$) we can use:

{88r-6.8A}

6.9. **Definition.** 1) \aleph^4 is defined as follows:

- (A) $\tau(\aleph^4) = \{P, Q, R\} \cup \{P_n : n < \omega\}$, R is two-place predicates, P, Q, P_n are unary predicates
- (B) $M \in K^4$ iff M is a $\tau(\aleph^4)$ -model such that $M \upharpoonright \{P, Q, R\} \in K^2$ and
 - (a) $\langle P_n^M : n < \omega \rangle$ is a partition of P^M
 - (b) P_n^M has exactly 2^n elements
 - (c) $(\forall x \in Q)(\forall u \in [P^M]^{<\aleph_0})(\exists y \in Q^M)[A_x^M \Delta A_y^M = u]$
 - (d) if $k < \omega$ and $y_0, \dots, y_{k-1} \in Q$ satisfies $|A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$ for $\ell < m < k$ then the set $\{A_{y_\ell}^M : \ell < k\}$ is an independent family of subsets of P^M ; moreover for any n large enough for any $\eta \in {}^k 2$ the set $P_n^M \cap \{A_{y_\ell}^M : \eta(\ell) = 1\} \setminus \cup \{A_{y_\ell}^M : \eta(\ell) = 0\}$ has exactly 2^{n-k} elements
 - (e) $Q^M(y) \wedge Q^M(z) \wedge (\forall x \in P^M)[xR^M y \leftrightarrow xR^M z] \rightarrow y = z,$
 - (f) for every $k < \omega$ for some $y_0, \dots, y_k \in Q^M$ we have $\bigwedge_{\ell < m \leq k} |A_{y_\ell} \Delta A_{y_m}| \geq \aleph_0$
- (C) $M \leq_{\aleph^4} N$ iff $M, N \in K^4$ and $M \subseteq N$ and $P^M = P^N.$

6.10. **Theorem.** 1) $(K^2, <_{\aleph^2})$ is an \aleph_0 -presentable abstract elementary class which is categorical in \aleph_0 .
 2) Also \aleph^3 and \aleph^4 are \aleph_0 -presentable a.e.c. categorical in \aleph_0 .

Proof. Similar to the proof for \aleph^1 . □_{6.10}

{88r-6.11}

6.11. **Theorem.** 1) $\aleph_{\aleph_1}^1$ has an axiomatization in $\mathbb{L}(\mathbf{Q})$ and \leq_{\aleph^1} is $<^{**}$ from the proof of 3.19 (this is $<^{**}$ from [Sh:87a] and [Sh:87b]).
 2) \aleph^2 has an axiomatization in $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ and \leq_{\aleph^2} is \leq^* from the proof of 3.19 (this is $<^*_{\omega_1, \omega}$ from [Sh:87a] and [Sh:87b]).
 3) \aleph^3 has an axiomatization in $\mathbb{L}(\mathbf{Q})$ and \leq_{\aleph^3} is $<^*$ from [Sh:87a] and [Sh:87b].
 4) \aleph^4 has an axiomatization in $\mathbb{L}_{\omega_1, \omega}$ and \leq_{\aleph^4} is just being a submodel.
 5) $(\forall \ell \in \{1, 2, 3, 4\})[K^\ell \text{ is } \text{PC}_{\aleph_0}]$.

Proof. Should be clear. □_{6.11}

{88r-6.11A}

6.12. **Theorem.** If MA_{\aleph_1} then K^ℓ is categorical in \aleph_1 for $\ell = 2, 3$.

Proof. Easy¹⁰. □

{88r-6.12}

6.13. **Conclusion.** Assuming MA_{\aleph_1} there exists an abstract elementary class, which is PC_{\aleph_0} , categorical in \aleph_0, \aleph_1 but without the \aleph_0 -amalgamation property.

¹⁰In the earlier version this was claimed also for $\ell = 4$, but, as Baldwin noted, this was wrong

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