REASONABLE ULTRAFILTERS, AGAIN

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Abstract. We continue investigations of reasonable ultrafilters on uncountable cardinals defined in Shelah [12]. We introduce stronger properties of ultrafilters and we show that those properties may be handled in \( \lambda \)-support iterations of reasonably bounding forcing notions. We use this to show that consistently there are reasonable ultrafilters on an inaccessible cardinal \( \lambda \) with generating systems of size less than \( 2^\lambda \). We also show how ultrafilters generated by small systems can be killed by forcing notions which have enough reasonable completeness to be iterated with \( \lambda \)-supports.

0. Introduction

Reasonable ultrafilters were introduced in Shelah [12] in order to suggest a line of research that would repeat in some sense the beautiful theory created around the notion of \( P \)-points on \( \omega \). Most of the generalizations of \( P \)-points to uncountable cardinals in the literature go into the direction of normal ultrafilters and large cardinals (see, e.g., Gitik [3]), but one may be interested in the opposite direction. If one wants to keep away from normal ultrafilters on \( \lambda \), one may declare interest in ultrafilters which do not include some clubs and even demand that quotients by a closed unbounded subset of \( \lambda \) do not extend the club filter of \( \lambda \). Such ultrafilters are called weakly reasonable ultrafilters, see 1.1, 1.2. But if we are interested in generalizing \( P \)-points, we have to consider also properties that would correspond to any countable family of members of the ultrafilter has a pseudo-intersection in the ultrafilter. The choice of the right property in the declared context of very non-normal ultrafilters is not clear, and one of the goals of the present paper is to show that the very reasonable ultrafilters suggested in Shelah [12] (see Definition 1.3 here) are very reasonable indeed, that is we may prove interesting theorems on them.

In the first section we recall some of the concepts and results presented in Shelah [12] and we introduce strong properties of generating systems (super and strong reasonability, see Definitions 1.11, 1.12) and we show that there may exist super reasonable systems which generate ultrafilters (Propositions 1.15, 1.16).

In the next section we recall from [8] some properties of forcing notions relevant for \( \lambda \)-support iterations. We also improve in some sense a result of [8] and we show a preservation theorem for the nice double \( a \)-bounding property (Theorem 2.13).
Then in the third section we show that super reasonable families generating ultrafilters will be still at least strongly reasonable and will continue to generate ultrafilters after forcing with $\lambda$–support iterations of $A$–bounding forcing notions. Therefore, for an inaccessible cardinal $\lambda$, it is consistent that $2^\lambda = \lambda^{++}$ and there is a very reasonable ultrafilter generated by a system of size $\lambda^+$ (Corollary 3.4). It should be stressed that “generating an ultrafilter” has the specific meaning stated in Definition 1.3(3). In particular, “having a small generating system” does not imply “having small ultrafilter base”.

The fourth section shows that some technical inconveniences of the proofs from the third sections reflect the delicate nature of our concepts, not necessarily our lack of knowledge. We give an example of a nicely double $a$–bounding forcing notion which kills ultrafilters generated by systems from the ground model. Then we show that for an inaccessible cardinal $\lambda$, it is consistent that $2^\lambda = \lambda^{++}$ and there is no ultrafilter generated by a system of size $\lambda^+$ (see Corollary 3.4).

Studies of ultrafilters generated according to the schema introduced in [12] are also carried out in Roslanowski and Shelah [10].

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [5]). In forcing we keep the older convention that a stronger condition is the larger one.

(1) Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta \ldots$) and also by $i, j$ (with possible sub- and superscripts). Cardinal numbers will be called $\kappa, \lambda, \mu$ (with possible sub- and superscripts). $\lambda$ is always assumed to be regular, sometimes even strongly inaccessible.

By $\chi$ we will denote a sufficiently large regular cardinal; $\mathcal{H}(\chi)$ is the family of all sets hereditarily of size less than $\chi$. Moreover, we fix a well ordering $<^*_{\chi}$ of $\mathcal{H}(\chi)$.

(2) A sequence is a function with the domain being a set of ordinals. For two sequences $\eta, \nu$ we write $\nu <^* \eta$ whenever $\nu$ is a proper initial segment of $\eta$, and $\nu \leq \eta$ when either $\nu <^* \eta$ or $\nu = \eta$. The length of a sequence $\eta$ is the order type of its domain and it is denoted by $\text{lh}(\eta)$.

(3) We will consider several games of two players. One player will be called Generic or Complete or just $\text{COM}$, and we will refer to this player as “she”. Her opponent will be called Antigeneric or Incomplete or just $\text{INC}$ and will be referred to as “he”.

(4) For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\bar{\tau}, \bar{X}$). The canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$ is called $\bar{G}_\mathbb{P}$. The weakest element of $\mathbb{P}$ will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it). We will also assume that all forcing notions under consideration are atomless.

By “$\lambda$–support iterations” we mean iterations in which domains of conditions are of size $\leq \lambda$. However, we will pretend that conditions in a $\lambda$–support iteration $\bar{Q} = \langle \mathbb{P}_\zeta, \bar{Q}_\zeta : \zeta < \zeta^* \rangle$ are total functions on $\zeta^*$ and for $p \in \text{lim}(\bar{Q})$ and $\alpha \in \zeta^* \setminus \text{Dom}(p)$ we will let $p(\alpha) = \emptyset_{\mathbb{Q}_\alpha}$.

(5) For a filter $D$ on $\lambda$, the family of all $D$–positive subsets of $\lambda$ is called $D^+$. (So $A \in D^+$ if and only if $A \subseteq \lambda$ and $A \cap B \neq \emptyset$ for all $B \in D$.)

The club filter of $\lambda$ is denoted by $\mathcal{D}_\lambda$. 
1. More reasonable ultrafilters on $\lambda$

Here we recall some basic definitions and results from [12], and then we introduce even stronger properties of ultrafilters and/or generating systems. We also show that assumptions like $\mathcal{S}_{\lambda^+}$ imply the existence of such objects.

As explained in the introduction, we are interested in ultrafilters (on an uncountable cardinal $\lambda$) which are far from being normal. Weakly reasonable ultrafilters defined below do not contain some clubs even if we look at their quotients by a club.

Definition 1.1 ([12, Def. 1.4]). We say that a uniform ultrafilter $D$ on $\lambda$ is weakly reasonable if for every function $f \in \lambda^\lambda$ there is a club $C$ of $\lambda$ such that

$$\bigcup \{[\delta, \delta + f(\delta)) : \delta \in C\} \notin D.$$ 

Observation 1.2 ([12, Obs. 1.5]). Let $D$ be a uniform ultrafilter on $\lambda$. Then the following conditions are equivalent:

(A) $D$ is weakly reasonable,

(B) for every increasing continuous sequence $\langle \delta_\xi : \xi < \lambda \rangle \subseteq \lambda$ there is a club $C^*$ of $\lambda$ such that

$$\bigcup \{[\delta_\xi, \delta_{\xi+1}) : \xi \in C^*\} \notin D.$$ 

We want to investigate ultrafilters on $\lambda$ which are generated by systems defining “largeness in $\lambda$” by giving a condition based on “largeness in intervals below $\lambda$”. The family $Q^0_\lambda$ introduced below is a natural generalization of the approach used in [7, Sections 5, 6]. The directness of $G^*$ is an easy way to guarantee that $\text{fil}(G^*)$ is a filter, and $(<\lambda^+)$–directness has the flavour of $P$–pointness.

Definition 1.3 ([12, Def. 2.5]). (1) Let $Q^0_\lambda$ consist of all tuples

$$p = (C^p, (Z^p_\delta : \delta \in C^p), (d^p_\delta : \delta \in C^p))$$

such that

(i) $C^p$ is a club of $\lambda$ consisting of limit ordinals only, and for $\delta \in C^p$:

(ii) $Z^p_\delta = [\delta, \min \{C^p \setminus (\delta + 1)\}]$ and

(iii) $d^p_\delta \subseteq \mathcal{P}(Z^p_\delta)$ is a proper non-principal ultrafilter on $Z^p_\delta$.

(2) For $q \in Q^0_\lambda$ we let

$$\text{fil}(q) \overset{\text{def}}{=} \{A \subseteq \lambda : (\exists \delta < \lambda)(\forall \delta \in C^q \setminus \varepsilon)(A \cap Z^q_\delta \in d^q_\delta)\},$$

and for a set $G^* \subseteq Q^0_\lambda$ we let $\text{fil}(G^*) \overset{\text{def}}{=} \bigcup \{\text{fil}(p) : p \in G^*\}$. We also define a binary relation $\leq^0$ on $Q^0_\lambda$ by

$$p \leq^0 q \text{ if and only if } \text{fil}(p) \subseteq \text{fil}(q).$$

(3) We say that an ultrafilter $D$ on $\lambda$ is reasonable if it is weakly reasonable (see 1.1) and there is a directed (with respect to $\leq^0$) set $G^* \subseteq Q^0_\lambda$ such that $D = \text{fil}(G^*)$. The family $G^*$ may be called the generating system for $D$.

(4) An ultrafilter $D$ on $\lambda$ is said to be very reasonable if it is weakly reasonable and there is a $(<\lambda^+)$–directed (with respect to $\leq^0$) set $G^* \subseteq Q^0_\lambda$ such that $D = \text{fil}(G^*)$.

Definition 1.4. Suppose that

(a) $X$ is a non-empty set and $e$ is an ultrafilter on $X$,
(b) $d_x$ is an ultrafilter on a set $Z_x$ (for $x \in X$).

We let

$$
\bigoplus_{x \in X} d_x = \{ A \subseteq \bigcup_{x \in X} Z_x : \{ x \in X : Z_x \cap A \in d_x \} \in e \}.
$$

(Clearly, $\bigoplus_{x \in X} d_x$ is an ultrafilter on $\bigcup_{x \in X} Z_x$.)

**Proposition 1.5** ([12, Prop. 2.9]). Let $p,q \in Q^0_{\lambda}$. Then the following are equivalent:

(a) $p \leq^0 q$,

(b) there is $\varepsilon < \lambda$ such that

$$
(\forall \alpha \in C^\varepsilon) (\forall A \in d^p_\alpha) (\exists \beta \in C^p) (A \cap Z^p_\beta \in d^p_\beta),
$$

(c) there is $\varepsilon < \lambda$ such that

if $\alpha \in C^\varepsilon$, $\beta_0 = \sup (C^p \cap (\alpha + 1))$, $\beta_1 = \min (C^\varepsilon \cap (\alpha + 1))$, then there is an ultrafilter $e$ on $[\beta_0, \beta_1] \cap C^p$ such that

$$
d^e_\alpha = \{ A \in Z^p_\alpha : A \in \bigoplus \{ d^p_\beta : \beta \in [\beta_0, \beta_1] \cap C^p \} \}.
$$

**Observation 1.6** (Compare [12, Prop. 2.3(4)]). If $p \in Q^0_{\lambda}$, $A \subseteq \lambda$, then there is $q \in Q^0_{\lambda}$ such that $p \leq^0 q$ and either $A \in \text{fil}(q)$ or $\lambda \setminus A \in \text{fil}(q)$.

**Definition 1.7** ([12, Def. 2.10]). Let $p \in Q^0_{\lambda}$. Suppose that $X \subseteq [C^p]^{\lambda}$ and $C \subseteq C^p$ is a club of $\lambda$ such that

if $\alpha < \beta$ are successive elements of $C$,

then $|[\alpha, \beta) \cap X| = 1$.

(In this situation we say that $p$ is restrictable to $(X, C)$.) We define the restriction of $p$ to $(X, C)$ as an element $q = p|(X, C) \in Q^0_{\lambda}$ such that $C^q = C$, and if $\alpha < \beta$ are successive elements of $C$, $x \in [\alpha, \beta) \cap X$, then $Z^q_x = [\alpha, \beta)$ and $d^q_x = \{ A \subseteq Z^q_x : A \cap Z^p_x \in d^p_x \}$.

**Proposition 1.8** ([12, Prop. 2.11]).

1. If $G^* \subseteq Q^0_{\lambda}$ is $\leq^0$-directed and $|G^*| \leq \lambda$, then $G^*$ has a $\leq^0$-upper bound. (Hence, in particular, $\text{fil}(G^*)$ is not an ultrafilter.)

2. Assume that $G^* \subseteq Q^0_{\lambda}$ is $\leq^0$-directed and $\leq^0$-downward closed, $p \in G^*$, $X \subseteq [C^p]^{\lambda}$ and $C \subseteq C^p$ is a club of $\lambda$ such that $p$ is restrictable to $(X, C)$.

If $\bigcup_{x \in X} Z^p_x \in \text{fil}(G^*)$, then $p|(X, C) \in G^*$.

The following definition is used here to simplify our notation in 1.11 only. However, these concepts play a more central role in [10].

**Definition 1.9.** Let $Q^1_{\lambda}$ be the family of all sets $r$ such that

(a) members of $r$ are triples $(\alpha, Z, d)$ such that $\alpha < \lambda$, $Z \subseteq [\alpha, \lambda)$, $\Omega_0 \leq |Z| < \lambda$ and $d$ is a non-principal ultrafilter on $Z$, and

(b) $\forall \xi < \lambda (|\{ (\alpha, Z, d) \in r : \alpha = \xi \}| < \lambda)$, and $|r| = \lambda$.

For $r \in Q^1_{\lambda}$ we define

$\text{fil}^*(r) = \{ A \subseteq \lambda : (\exists \varepsilon < \lambda)(\forall (\alpha, Z, d) \in r) (\varepsilon \leq \alpha \Rightarrow A \cap Z \in d) \}$,

and we define a binary relation $\leq^*$ on $Q^1_{\lambda}$ by

$r_1 \leq^* r_2$ if and only if $(r_1, r_2 \in Q^1_{\lambda}$ and) $\text{fil}^*(r_1) \subseteq \text{fil}^*(r_2)$. 

1.10. \( \\forall \xi < \lambda \left( \{ (\alpha, Z, d) \in r : \alpha = \xi \} < 2 \right) \), and
\( \{ (\alpha_1, Z_1, d_1), (\alpha_2, Z_2, d_2) \in r : \alpha_1 < \alpha_2 \Rightarrow Z_1 \subseteq \alpha_2 \} \).

For \( p \in Q^*_\lambda \) we let \( \#(p) = \{ (\alpha, Z^\alpha, d^\alpha) : \alpha \in C^p \} \).

Observation 1.10. (1) If \( p \in Q^*_\lambda \) then \( \#(p) \in Q^*_\lambda \) is strongly disjoint and
\( \text{fil}(p) = \text{fil}^*(\#(p)). \) Also, if \( r \in Q^*_\lambda \) is strongly disjoint, then \( \text{fil}^*(r) = \text{fil}(p) \)
for some \( p \in Q^*_\lambda \).

Let \( r, s \in Q^*_\lambda \). Then \( r \leq^* s \) if and only if there is \( \varepsilon < \lambda \) such that
\( (\forall (\alpha, Z, d) \in s) (\forall A \in d) (\alpha > \varepsilon \Rightarrow (\exists (\alpha', Z', d') \in r) (A \cap Z' \subseteq d')) \).

The various definitions of super reasonable ultrafilters introduced in Definition 1.11 are motivated by the proof of “the Sacks forcing preserves \( P \)-points”. In that proof, a fusion sequence is constructed so that at a stage \( n < \omega \) of the construction one deals with finitely many nodes in a condition (the nodes that are declared to be kept). We would like to carry out this kind of argument, e.g., for forcing notions used in [9, B.8.3, B.8.5], but now we have to deal with \( < \lambda \) nodes in a tree, and the ultrafilter we try to preserve is not that complete. So what do we do? We deal with finitely many nodes at a time eventually taking care of everybody.

One can think that in the definition below the set \( I_\alpha \) is the set of nodes we have to keep and the finite sets \( u_{\alpha,i} \) are the nodes taken care of at a stage \( i \).

The technical aspects of 1.11 are motivated by the iteration theorems in [8] and [6]: our games here are tailored to fit the games played on trees of conditions in \( \lambda \)-support iterations, see Theorems 3.2, 3.3 later. As said earlier, the main goal is to have a property of \( G^* \) which implies the preservation of “fil\((G^*)\)” is an ultrafilter” by many forcing notions. We would also love to preserve that property itself, but we failed to achieve it. The “super reasonability” is what we need to preserve the ultrafilter (see 3.2), “strong reasonability” is what we can prove about \( G^* \) in the extension (see 3.3).

Definition 1.11. Let \( G^* \subseteq Q^*_\lambda \) and let \( \bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle \) be a sequence of cardinals, \( 2 \leq \mu_\alpha \leq \lambda \) for \( \alpha < \lambda \).

1. We define a game \( \mathcal{O}_\mu(G^*) \) between two players, COM and INC. A play of \( \mathcal{O}_\mu(G^*) \) lasts \( \lambda \) steps and at a stage \( \alpha < \lambda \) of the play the players choose \( I_\alpha, i_\alpha, \bar{u}_\alpha \) and \( (r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha) \) applying the following procedure.

- First, INC chooses a non-empty set \( I_\alpha \) of cardinality \( \mu_\alpha \), and an enumeration \( \bar{u}_\alpha = \langle u_{\alpha,i} : i < i_\alpha \rangle \) of \( [I_\alpha]^{< \omega} \) (so \( i_\alpha < \mu_\alpha \cdot \aleph_0 \)).
- Next the two players play a subgame of length \( i_\alpha \) in the \( i^\text{th} \) move of the subgame,
  (a) COM chooses \( r_{\alpha,i} \in G^* \), and then
  (b) INC chooses \( r'_{\alpha,i} \in G^* \) such that \( r_{\alpha,i} \leq^* r'_{\alpha,i} \), and finally
  (c) COM picks \( (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#(r'_{\alpha,i}) \) such that \( \beta_{\alpha,i} > \alpha \).
In the end of the play COM wins if and only if
\( (\exists \bar{j} = (j_\alpha : \alpha < \lambda) \in \prod_{\alpha < \lambda} I_\alpha \) we have
\[ \{ (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda, j_\alpha \in u_{\alpha,i} \text{ and } i < i_\alpha \} \leq^* \#(r) \].
A game $\mathcal{D}_\mu^\Box(G^*)$ is defined similarly to $\mathcal{D}_\nu^\Box(G^*)$ except that (E) is weakened to
\[(E)\] for every $j \in \prod_{\alpha < \lambda} I_\alpha$ the set $\bigcup\{Z_{\alpha,i} : \alpha < \lambda, i < i_\alpha\}$ belongs to $\text{fil}(G^*)$.

(2) We say that the family $G^*$ is $\mu$-super reasonable ($\mu$-super$^-$ reasonable, respectively) if
\[(i)\] $G^*$ is $(< \lambda^+)$-directed (with respect to $\leq^0$), and
\[(ii)\] if $s \in G^*$, $r \in Q_\lambda^0$ and for some $\alpha < \lambda$ we have $C^r = C^\alpha \setminus \alpha$ and $d^\prime_\beta = d^\prime_\alpha$ for $\beta \in C^\alpha$, then $r \in G^*$, and
\[(iii)\] INC has no winning strategy in the game $\mathcal{D}_\mu(G^*)$ ($\mathcal{D}_\nu(G^*)$, respectively).

(3) We say that a uniform ultrafilter $D$ on $\lambda$ is $\mu$-super reasonable ($\mu$-super$^-$ reasonable, respectively) if there is a $\mu$-super reasonable ($\mu$-super$^-$ reasonable, respectively) set $G^* \subseteq Q_\lambda^0$ such that $D = \text{fil}(G^*)$.

(4) If $\mu_\alpha = \lambda$ for all $\alpha < \lambda$, then we omit $\mu$ and say just super reasonable or super$^-$ reasonable (in reference to both ultrafilters on $\lambda$ and families $G^* \subseteq Q_\lambda^0$). Also in this case we may write $\mathcal{D}_\mu^\Box$ instead of $\mathcal{D}_\mu^\Box$.

**Definition 1.12.** Let $G^* \subseteq Q_\lambda^0$ be directed with respect to $\leq^0$ and let $\mu = \langle \mu_\alpha : \alpha < \lambda \rangle$ be a sequence of cardinals, $2 \leq \mu_\alpha \leq \lambda$ for $\alpha < \lambda$.

(1) A game $\mathcal{D}_\mu^\Box(G^*)$ between two players, COM and INC is defined as follows.
A play of $\mathcal{D}_\mu^\Box(G^*)$ lasts $\lambda$ steps and at a stage $\alpha < \lambda$ of the play the players choose $I_\alpha, i_\alpha, u_\alpha$ and $\langle r_{\alpha,i}, \delta_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha \rangle$ applying the following procedure.

- First, INC chooses a non-empty set $I_\alpha$ of cardinality $< \mu_\alpha$, and then
  COM chooses $i_\alpha < \lambda$ and a sequence $u_\alpha = \langle u_{\alpha,i} : i < i_\alpha \rangle$ of non-empty finite subsets of $I_\alpha$ such that $I_\alpha = \bigcup_{i < i_\alpha} u_{\alpha,i}$.

- Next the two players play a subgame of length $i_\alpha$. In the $i$th move of the subgame,
  (a) COM chooses $r_{\alpha,i} \in G^*$ and then
  (b) INC chooses $\delta_{\alpha,i} < \lambda$, and finally
  (c) COM picks $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#(r_{\alpha,i})$ such that $\beta_{\alpha,i}$ is above $\delta_{\alpha,i}$ and $\alpha$.

In the end of the play COM wins if and only if
\[(\star)\] there is $r \in G^*$ such that for every $j = \langle j_\alpha : \alpha < \lambda \rangle \in \prod_{\alpha < \lambda} I_\alpha$ we have

$$\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda, j_\alpha \in u_{\alpha,i} \text{ and } i < i_\alpha\} \subseteq \#(r).$$

A game $\mathcal{D}_\mu^\Box(G^*)$ is defined similarly to $\mathcal{D}_\nu^\Box(G^*)$ except that (E) is weakened to
\[(E)\] for every $j \in \prod_{\alpha < \lambda} I_\alpha$ the set $\bigcup\{Z_{\alpha,i} : \alpha < \lambda, i < i_\alpha\}$ belongs to $\text{fil}(G^*)$.

(2) If $G^* \subseteq Q_\lambda^0$ is $(< \lambda^+)$-directed (with respect to $\leq^0$) and INC has no winning strategy in the game $\mathcal{D}_\mu^\Box(G^*)$, then we say that $G^*$ is $\mu$-strongly reasonable. Also, $G^*$ is said to be $\mu$-strongly$^-$ reasonable if it is $(< \lambda^+)$-directed and INC has no winning strategy in the game $\mathcal{D}_\mu^\Box(G^*)$. 


(3) We say that a uniform ultrafilter $D$ on $\lambda$ is $\bar{\mu}$–strongly reasonable ($\bar{\mu}$–strongly reasonable, respectively) if there is a $\bar{\mu}$–strongly reasonable ($\bar{\mu}$–strongly reasonable, respectively) set $G^* \subseteq \mathcal{P}_\lambda^0$ such that $D = \text{fil}(G^*)$. If $\mu_\alpha = \lambda$ for all $\alpha < \lambda$, then we omit $\bar{\mu}$ and say just strongly reasonable or strongly reasonable.

**Observation 1.13.** Assume that $2 \leq \mu_\alpha \leq \kappa_\alpha \leq \lambda$ for $\alpha < \lambda$ and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$. Then for a family $G^* \subseteq \mathcal{P}_\lambda^0$ and/or a uniform ultrafilter $D$ on $\lambda$ the following implications hold.

\[ \bar{\kappa}–\text{super reasonable} \Rightarrow \bar{\mu}–\text{super reasonable} \Rightarrow \bar{\mu}–\text{strongly reasonable} \]

**Proposition 1.14.** Assume that $2 \leq \mu_\alpha \leq \lambda$ for $\alpha < \lambda$ and $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$. If a uniform ultrafilter $D$ on $\lambda$ is $\bar{\mu}$–strongly reasonable, then it is very reasonable.

**Proof.** Pick a $\bar{\mu}$–strongly reasonable family $G^* \subseteq \mathcal{P}_\lambda^0$ such that $D = \text{fil}(G^*)$. Then $G^*$ is $\langle \lambda^+ \rangle$–directed and the proof will be completed once we show that $D$ is weakly reasonable.

Let $f \in \lambda^\lambda$. We will argue that for some club $C = \{ \gamma_\alpha : \alpha < \gamma \} \subseteq \lambda$ we have $\bigcup \{ \langle \delta, \delta + f(\delta) : \delta \in C \rangle : \delta \notin D \}$, where $\gamma_\alpha$ are given by the arguments below.

We consider the following strategy $\text{st}(f)$ for INC in $\mathcal{D}_{\bar{\mu}}^\lambda(G^*)$. The strategy $\text{st}(f)$ instructs INC to construct on the side an increasing continuous sequence $\langle \gamma_\alpha : \alpha < \lambda \rangle \subseteq \lambda$ so that at a stage $\alpha < \lambda$ of the play, when

\[ \langle I_\xi, i_\xi, u_\xi, (\beta_\xi, i, Z_\xi, \delta_\xi, d_\xi) : i < i_\xi, \xi < \alpha \rangle \]

is the result of the play so far, then

- if $\alpha$ is limit, then $\gamma_\alpha = \sup(\gamma_\xi : \xi < \alpha)$,
- if $\alpha$ is not limit, then $\gamma_\alpha = \sup \left( \bigcup \{ Z_\xi, j < i \} \right) + 890$.

Now (at the stage $\alpha$) $\text{st}(f)$ instructs INC to choose $I_\alpha = \{0\}$ and then (after COM picks $i_\alpha, u_\alpha$) he is instructed to play in the subgame of this stage as follows. At stage $i < i_\alpha$, after COM has picked $r_{\alpha, i},$ INC lets

\[ \delta_{\alpha, i} = \gamma_\alpha + f(\gamma_\alpha) + \sup \left( \bigcup \{ Z_{\alpha, j} : j < i \} \right) + 890. \]

(After this COM chooses $\langle \beta_{\alpha, i}, Z_{\alpha, i}, d_{\alpha, i} \rangle \in \#(r_{\alpha, i})$ with $\beta_{\alpha, i} > \delta_{\alpha, i}$.)

The strategy $\text{st}(f)$ cannot be the winning one for INC, so there is a play

\[ \langle I_\alpha, i_\alpha, u_\alpha, (r_{\alpha, i}, \alpha, Z_{\alpha, i}, d_{\alpha, i}) : i < i_\alpha, \alpha < \lambda \rangle \]

of $\mathcal{D}_{\bar{\mu}}^\lambda(G^*)$ in which INC follows $\text{st}(f)$ but

\[ A^* \overset{\text{def}}{=} \bigcup \{ Z_{\alpha, i} : \alpha < \lambda, i < i_\alpha \} \in \text{fil}(G^*) = D \]

(note that necessarily $u_{\alpha, i} = I_\alpha = \{0\}$). It follows from the choice of $\gamma_\alpha, \delta_{\alpha, i}$ that for each $\alpha < \lambda$

\[ \{ \gamma_\alpha, \gamma_\alpha + f(\gamma_\alpha) \} \cap \bigcup \{ Z_\xi, \xi < \lambda, i < i_\xi \} = \emptyset, \]

and hence also $\bigcup \{ \gamma_\alpha, \gamma_\alpha + f(\gamma_\alpha) : \alpha < \lambda \} \cap A^* = \emptyset$. Consequently $\bigcup \{ \gamma_\alpha, \gamma_\alpha + f(\gamma_\alpha) : \alpha < \lambda \} \notin D$ and one can easily finish the proof.

**Proposition 1.15.** Assume $\lambda = \lambda^{<\lambda}$ and $\diamondsuit_{\lambda^+}$. There exists a sequence $\langle r_\xi : \xi < \lambda^+ \rangle \subseteq \mathcal{P}_\lambda^0$ such that
(i) $(\forall \xi < \zeta < \lambda^+) (r_\xi \leq^0 r_\zeta)$, and
(ii) the family

$$G^* \overset{\text{def}}{=} \{ r \in Q_\lambda^0 : (\exists \xi < \lambda^+) (r \leq^0 r_\xi) \}$$

is super reasonable and \( \text{fil}(G^*) \) is an ultrafilter on \( \lambda \).

Proof. The sequence \( (r_\xi : \xi < \lambda^+) \) will be constructed inductively. At successor stages we will use 1.6 to make sure that \( \text{fil}(G^*) \) is an ultrafilter. At limit stages we will use 1.8(1) to find upper bounds to the sequence constructed so far. Moreover, at (some) stages \( \xi \) of cofinality \( \lambda \) the element \( r_\xi \) will be chosen so that “it kills” a strategy for INC in \( \mathcal{D}(\mathbb{G}(\lambda)) \) predicted by the diamond sequence.

For \( \alpha < \lambda \) let \( X_\alpha^1 \) be the set of all legal plays of \( \mathcal{D}(\mathbb{G}(\lambda)) \) of the form

$$\langle \bigcup_{\alpha < \lambda} X_\alpha^1 \rangle \overset{(\circ)}{\sim} \langle I, i, u, \gamma, (r_{\gamma,i}, r'_{\gamma,i}) : i < i_\gamma \rangle : \gamma < \alpha \rangle$$

where each \( I_\gamma \) (for \( \gamma < \alpha \)) is an ordinal below \( \alpha \). Also let \( X^1 = \bigcup_{\alpha < \lambda} X_\alpha^1 \). Next, for \( \alpha < \lambda, 0 < I < \lambda \) and an enumeration \( \bar{u} = \langle u_j : j < i \rangle \) of \( [I]^{<\omega} \) let \( X_{\alpha,I,\bar{u}}^2 \) be the set of all legal plays of \( \mathcal{D}(\mathbb{G}(\lambda)) \) of the form

$$\langle \bigcup_{\alpha < \lambda} X_{\alpha,I,\bar{u}}^2 \rangle \overset{(\circ)}{\sim} \langle I, i, \bar{u}, (r_{j,i}, r'_{j,i}) : j < j^* \rangle \overset{\text{for}}{\sim} \langle r \rangle,$$

where \( \bar{u} \in X_{\alpha,I,\bar{u}}^2 \) and \( j^* < i \) (and \( (r_{j,i}, r'_{j,i}, \beta_j, Z_j, d_j) : j < j^* \rangle \overset{\text{for}}{\sim} \langle r \rangle \) is a legal partial play of the subgame of level \( \alpha \); in particular \( r_{j,i}, r'_{j,i}, r \in Q_\lambda^0 \)). Also let

$$X^2 = \bigcup \{ X_{\alpha,I,\bar{u}}^2 : \alpha < \lambda \text{ and } 0 < I < \lambda \text{ and } \bar{u} = \langle u_j : j < i \rangle \text{ is an enumeration of } [I]^{<\omega} \}.$$

Any strategy for INC in \( \mathcal{D}(\mathbb{G}(\lambda)) \) can be interpreted as a function \( \text{st} \) satisfying conditions \((\circ)^3\)–\((\circ)^5\).

Since \( |Q_\lambda^0| = 2^{2^{<\lambda}} = \lambda^+ \), we may pick a bijection \( \pi_0 : Q_\lambda^0 \overset{1:1}{\rightarrow} \lambda^+ \) and for \( \xi < \lambda^+ \) let \( X^1_\xi \) consist of all \( \bar{\sigma} \in X^1 \cup X^2 \) such that \( \pi_0(r) < \xi \) for all elements \( r \in Q_\lambda^0 \) involved in the representation of \( \bar{\sigma} \) as in \((\circ)^1, (\circ)^2\). We also let \( Y_\xi \) consist of all pairs \( (\bar{\sigma}, a) \) such that

- \( \bar{\sigma} \in X^1_\xi \) and \( a = \text{st}(\bar{\sigma}) \) for some strategy \( \text{st} \) of INC, and
- if \( \bar{\sigma} \in X^2 \) (and so \( a \in Q_\lambda^0 \)) then \( \pi_0(a) < \xi \).

Note that \( |X^1_\xi| \leq \lambda \) and \( |Y_\xi| \leq \lambda \) (for each \( \xi < \lambda^+ \)). Put \( \mathcal{Y} = \bigcup_{\xi < \lambda^+} Y_\xi \). Plainly \( |\mathcal{Y}| = \lambda^+ \) so we may fix a bijection \( \pi_1 : \lambda^+ \overset{\text{onto}}{\rightarrow} \mathcal{Y} \). Let

$$C = \{ \xi < \lambda^+ : \pi_1[\xi] = Y_\xi \}$$

it is a club of \( \lambda^+ \).

Let \( \langle A_\zeta : \zeta < \lambda^+ \rangle \) list all subsets of \( \lambda \) and let \( \langle B_\zeta : \zeta \in S^\lambda_{\lambda^+} \rangle \) be a diamond sequence on \( S^\lambda_{\lambda^+} = \{ \zeta < \lambda^+ : \text{cf}(\zeta) = \lambda \} \). By induction on \( \xi < \lambda^+ \) we choose a \( \leq^0 \)-increasing sequence \( \langle r_\zeta : \zeta < \lambda^+ \rangle \subseteq Q_\lambda^0 \) applying the following procedure. Assume \( \xi < \lambda^+ \) and we have constructed \( \langle r_\zeta : \zeta < \xi \rangle \).
CASE 0: $\xi = 0$.
We let $r_0$ be the $<^{*}\chi$-first member of $Q_\chi^0$.

CASE 1: $\xi = \chi + 1$.
Pick $r_\xi \in Q_\chi^0$ such that $r_\xi \leq^0 r_\xi$ and either $A_\xi \in \text{fil}(r_\xi)$ or $\lambda \setminus A_\xi \in \text{fil}(r_\xi)$ (remember Observation 1.6).

CASE 2: $\xi$ is a limit ordinal, cf($\xi) < \lambda$.
Pick $r_\xi \in Q_\chi^0$ such that $(\forall \zeta < \xi)(r_\zeta \leq^0 r_\xi)$ (exists by Proposition 1.8(1)).

CASE 3: $\xi$ is a limit ordinal, cf($\xi) = \lambda$.
Now we ask if

$$(\circ)^\xi_{\lambda, i} : \xi \in C \land (\forall \zeta < \xi)(\pi_0(r_\zeta) < \xi) \land \text{there is a strategy st for INC in} \, C^0(\xi_\lambda^0) \land \pi_1[B_\xi] = \text{st} \cap Y_\xi = \text{st}[A_\xi].$$

If the answer to $(\circ)^\xi_{\lambda, i}$ is negative, then we choose $r_\xi \in Q_\chi^0$ as in Case 2.

Suppose now that the answer to $(\circ)^\xi_{\lambda, i}$ is positive (so in particular $\xi \in C$) and st is a strategy for INC such that $\pi_1[B_\xi] = \text{st} \cap Y_\xi = \text{st}[A_\xi]$. Let $\xi = \langle \xi_\alpha : \alpha < \lambda \rangle$ be an increasing continuous sequence cofinal in $\xi$. Consider a play

$$\bar{\sigma} = \langle I_\alpha, i_\alpha, u_\alpha, (r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha) : \alpha < \lambda \rangle$$

of $C^0(Q^0_\lambda)$ in which INC follows the strategy st and COM proceeds as follows. When playing $C^0(Q^0_\lambda)$, at step $i < i_\alpha$ of the subgame of level $\alpha < \lambda$ (of $C^0(Q^0_\lambda)$), COM chooses $r_{\alpha,i} = r_{\xi_\alpha}$ and then, after INC determines $r'_{\alpha,i}$ by st, she picks the $<^{*}\chi$-first $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#(r'_{\alpha,i})$ satisfying:

$$(\circ)^0_{\xi, i} : (\forall \gamma \leq \alpha)(\forall A \in d_{\alpha,i})(\exists B \in C_{\gamma}(\alpha \land Z_{\gamma,i} \subseteq d_{\gamma}^* \land Z_{\gamma,j} \subseteq \beta_{\gamma,i})$$

and

$$(\circ)^0_{\xi, i} : (\forall \gamma < \alpha)(\forall j < i_\alpha)(Z_{\gamma,j} \subseteq \beta_{\gamma,i}) \land (\forall j < i_\alpha)(Z_{\gamma,j} \subseteq \beta_{\gamma,i}).$$

The above rules fully determine the play $\bar{\sigma}$ and it should be clear that $\bar{\sigma}|_{\alpha} \in X_\xi$ for each $\alpha < \lambda$. Note that $\bar{\sigma}$ depends on $B_\xi$ and $\xi$ only and not on st, provided it is as required by $(\circ)^0_{\xi, i}$).

By the demands $(\circ)^0_{\xi, i}$, we may choose an increasing continuous sequence $(\gamma_\alpha : \alpha < \lambda) \subseteq \lambda$ such that $\gamma_0 = 0$ and $\langle \forall \alpha < \lambda \rangle(\forall \gamma < i_\alpha)(Z_{\gamma,i} \subseteq [\gamma_\alpha, \gamma_{\alpha+1}))$. Now, for $\alpha < \lambda$ an ultrafilter $\mathcal{e}_\alpha$ on $i_\alpha$ such that

$$(\circ)^0_{\xi, \alpha, i} : (\forall j \in I_\alpha)(\{i < i_\alpha : j \in u_{\alpha,i}\} \subseteq \mathcal{e}_\alpha)$$

and let $d_{\alpha}$ be an ultrafilter on $[\gamma_\alpha, \gamma_{\alpha+1})$ such that

$$(\circ)^0_{\xi, \alpha} : \bigoplus \{d_{\alpha,i} : i < i_\alpha\} \subseteq d_{\alpha}.$$
Plainly, $G^*$ satisfies demands (i) and (ii) of 1.11(2) and fil($G^*$) is an ultrafilter on $\lambda$ (remember Case 1 of the construction). We should argue that INC has no winning strategy in $\mathcal{D}^+(G^*)$. To this end suppose that $st^\mathfrak{m}$ is a strategy of INC in $\mathcal{D}^+(G^*)$. Pick $\xi \in S_{\lambda^+} \cap C$ such that $(\forall \zeta < \xi)(\pi_0(r_\zeta) < \xi)$ and $\pi_1[B_\zeta] = st^\mathfrak{m} \cap \gamma_\zeta = st^\mathfrak{m}|\gamma_\zeta$. Then when choosing $r_\xi$ we gave a positive answer to $(\odot)^\mathfrak{m}_\xi$ and we constructed a play $\sigma$ of $\mathcal{D}^+(Q^0_\lambda)$. In that play, INC follows $st^\mathfrak{m}$ and COM chooses members of $G^*$, so it is a play of $\mathcal{D}^+(G^*)$. Now the condition $(\odot)^\mathfrak{m}_\xi$ means that $r_\xi$ witnesses that COM wins the play $\sigma$ and consequently $st^\mathfrak{m}$ is not a winning strategy for INC. \hfill $\Box$

**Proposition 1.16.** Let $Q^0_\lambda = (Q^0_\lambda, \leq^0)$. 

1. $Q^0_\lambda$ is a $(<\lambda^+)$-complete forcing notion of size $2^{<\lambda}$.
2. $\Vdash_{Q^0_\lambda}$ “$Q^0_\lambda$ is a super reasonable family and fil($Q^0_\lambda$) is an ultrafilter”.

**Proof.** (1) Should be clear; see also Proposition 1.8(1).

(2) By the completeness of $Q^0_\lambda$, forcing with it does not add new subsets of $\lambda$, and by 1.5

$$\Vdash_{Q^0_\lambda}$ “fil($Q^0_\lambda$) is a uniform ultrafilter on $\lambda$.”

It should also be clear that $G^*$ satisfies the demands of 1.11(2)(i+ii) (in $V^{Q^0_\lambda}$). Let us argue that

$$\Vdash_{Q^0_\lambda}$ “INC has no winning strategy in $\mathcal{D}^+(G^*)$”

and to this end suppose $p \in Q^0_\lambda$ and $st$ is a $Q^0_\lambda$-name such that

$$p \Vdash_{Q^0_\lambda}$ “$st$ is a strategy of INC in $\mathcal{D}^+(G^*)$”.

We are going to construct a condition $q \in Q^0_\lambda$ stronger than $p$ and a play $\bar{\sigma}$ of $\mathcal{D}^+(Q^0_\lambda)$ such that

$$q \Vdash_{Q^0_\lambda}$ “$\bar{\sigma}$ is a play of $\mathcal{D}^+(Q^0_\lambda)$ in which INC follows $st$ but COM wins”.

Let $X^1, X^2$ be defined as in the proof of 1.15 (see $(\odot)_2^1$, $(\odot)_2^1, i, \alpha$ there). We may assume that

$$p \Vdash_{Q^0_\lambda}$ “$st$ is a function satisfying $(\odot)^3$–$(\odot)^5$ of the proof of 1.15”.

By induction on $\alpha < \lambda$ we choose conditions $p_\alpha \in Q^0_\lambda$ and partial plays $\sigma _\alpha \in X^1_\alpha$ so that

1. $p \leq^0 p_\alpha \leq^0 p_\beta$ and $\sigma _\alpha < \sigma _\beta$ for $\alpha < \beta < \lambda$,
2. $p_\alpha \Vdash_{Q^0_\lambda}$ “$\sigma _\alpha$ is a partial play of $\mathcal{D}^+(Q^0_\lambda)$ in which INC uses $st$”,
3. If $\sigma _\alpha = (I_\gamma, i_\gamma, Z_\gamma, (\sigma _{\gamma, i_\gamma}) : i < i_\gamma)$, then for every $\gamma < \delta < \alpha$ and $j < i < i_\gamma$ we have

$$r^\prime_{\gamma, i} \leq^0 p_\alpha \quad \text{and} \quad Z_{\gamma, j} \subseteq \beta_{\gamma, i} \quad \text{and} \quad Z_{\gamma, j} \subseteq \beta_{\delta, 0}.$$

Suppose that $\alpha = \alpha^* + 1$ and we have determined $p_\alpha, \bar{\sigma} _\alpha$. Pick $p_\alpha^0 \geq^0 p_\alpha$ and $I_\alpha, i_\alpha, Z_\alpha$ such that $p_\alpha^0 \Vdash_{Q^0_\lambda}$ “$\bar{\sigma} _\alpha$ = $(I_\alpha, i_\alpha, Z_\alpha)$”. Now choose inductively $p_\alpha, r^\prime_{\alpha, i}, r^\prime_{\alpha, i_\alpha}$ and $(\beta_{\alpha, i}, Z_{\alpha, i}, d_{\alpha, i})$ for $i < i_\alpha$ so that for each $i < j < i_\alpha$ we have

1. $p_\alpha^0 = p_\alpha, p_{\alpha, i} \leq^0 p_\alpha^0, p_{\alpha, i} = r^\prime_{\alpha, i} \leq^0 p_\alpha^0$, and
2. $p_{\alpha, i} \Vdash_{Q^0_\lambda}$ “$\bar{\sigma} _\alpha$ is the answer by $st$ at stage $i$ of the subgame”,
3. $\beta_{\alpha, i}$ satisfies the demand in (1) and $(\beta_{\alpha, i}, Z_{\alpha, i}, d_{\alpha, i}) \in \#(r^\prime_{\alpha, i})$,
4. $(\forall A \in d_{\alpha, i})(\forall \gamma \leq \alpha^*)(\exists \delta \in C^p_{\gamma})(A \cap Z_{\alpha, i} \subseteq d^\delta_{\gamma})$. 


Then \( p_{n+1} \) is any \( \leq^\alpha \)-upper bound to \( \{ p_i : i < i_\alpha \} \).

The limit stages of the construction should be clear.

After the construction is carried out and we have \( \sigma_\lambda = \{ \sigma_\alpha : \alpha < \lambda \} \), we define \( r \in \mathcal{Q}_\lambda^0 \) like \( r^*_\epsilon \) in the proof of 1.15 (see \( \odot^\alpha_\chi^9 \) or \( \odot^\alpha_\chi^{10} \) there). Then \( r \) is \( \leq^\alpha \)-stronger then all \( p_\alpha \) (for \( \alpha < \lambda \)) and

\[
\models r \models \mathcal{Q}_\lambda^0 \quad \sigma_\lambda \text{ is a play of } \odot^\alpha(\mathcal{G}_{\mathcal{Q}_\lambda^0}) \text{ in which INC uses } \mathbf{st} \text{ but COM wins } ".
\]

(Note that the respective version of \( \odot^\alpha_\chi^{11} \) of the proof of 1.15 holds. By the completeness it continues to hold in \( \mathcal{V}_{\mathcal{Q}_\lambda^0} \).)

\[
\square
\]

2. More on reasonably complete forcing

**Definition 2.1.** Let \( \mathbb{P} \) be a forcing notion.

1. For a condition \( r \in \mathbb{P} \) let \( \mathcal{Q}_\lambda^0(\mathbb{P}, r) \) be the following game of two players, Complete and Incomplete:

   - The game lasts at most \( \lambda \) moves and during a play the players attempt construct a sequence \( \langle (p_i, q_i) : i < \lambda \rangle \) of pairs of conditions from \( \mathbb{P} \) in such a way that (\( \forall j < i < \lambda \)) \( r \models \mathcal{Q}_\lambda^0(\mathbb{P}, r) \).
   - At the stage \( i < \lambda \) of the game, first Incomplete chooses \( p_i \) and then Complete chooses \( q_i \).

   Complete wins if and only if for every \( i < \lambda \) there are legal moves for both players.

2. We say that the forcing notion \( \mathbb{P} \) is strategically \( (\leq \lambda) \)-complete if Complete has a winning strategy in the game \( \mathcal{Q}_\lambda^0(\mathbb{P}, p) \) for each condition \( p \in \mathbb{P} \).

3. Let \( N \prec (\mathcal{H}(\chi), \in, \prec^*) \) be a model such that \( (\leq \lambda) N \subseteq N \), \( |N| = \lambda \) and \( \mathbb{P} \in N \). We say that a condition \( p \in \mathbb{P} \) is \( (N, \mathbb{P}) \)-generic in the standard sense (or just: \( (N, \mathbb{P}) \)-generic) if for every \( \mathbb{P} \)-name \( \tau \in N \) for an ordinal we have \( p \models \tau \models \tau \in N \)."

4. \( \mathbb{P} \) is \( \lambda \)-proper in the standard sense (or just: \( \lambda \)-proper) if there is \( x \in \mathcal{H}(\chi) \) such that for every model \( N \prec (\mathcal{H}(\chi), \in, \prec^*) \)

   \[
   (\leq \lambda) N \subseteq N, \quad |N| = \lambda \quad \text{and} \quad \mathbb{P}, x \in N,
   \]

   and every condition \( p \in N \cap \mathbb{P} \) there is an \( (N, \mathbb{P}) \)-generic condition \( q \in \mathbb{P} \) stronger than \( p \).

**Theorem 2.2** (See Shelah [11, Ch. III, Thm 4.1], Abraham [1, §2] and Eisworth [2, §3]). Assume \( 2^\lambda = \lambda^+ \), \( \lambda < \lambda^\lambda = \lambda \). Let \( \bar{Q} = \langle \mathbb{P}_i, \mathcal{Q}_i : i < \lambda^+ \rangle \) be \( \lambda \)-support iteration such that for all \( i < \lambda^+ \) we have

- \( \mathbb{P}_i \) is \( \lambda \)-proper,
- \( \models_{\mathbb{P}_i} \ " |\mathcal{Q}_i| \leq \lambda^+ \" ")

Then

1. for every \( \delta < \lambda^++ \), \( \models_{\mathbb{P}_\delta} 2^\lambda = \lambda^+ \), and
2. the limit \( \mathbb{P}_{\lambda^+} \) satisfies the \( \lambda^++-cc \).

**Proposition 2.3** ([9, Prop. A.1.6]). Suppose \( \bar{Q} = \langle \mathbb{P}_i, \mathcal{Q}_i : i < \gamma \rangle \) is a \( \lambda \)-support iteration and, for each \( i < \gamma \), 

\[
\models_{\mathbb{P}_i} \ " \mathcal{Q}_i \text{ is strategically } (\leq \lambda) \text{-complete } ".
\]

Then, for each \( \varepsilon \leq \gamma \) and \( r \in \mathbb{P}_\varepsilon \), there is a winning strategy \( \mathbf{st}(\varepsilon, r) \) of Complete in the game \( \mathcal{Q}_\lambda^0(\mathbb{P}_\varepsilon, r) \) such that, whenever \( \varepsilon_0 < \varepsilon_1 \leq \gamma \) and \( r \in \mathbb{P}_{\varepsilon_1} \), we have:
(i) if \( \langle p_i, q_i : i < \lambda \rangle \) is a play of \( \mathcal{O}_0^\lambda(\mathbb{P}_{\varepsilon_0}, r|\varepsilon_0) \) in which Complete follows the strategy \( \text{st}(\varepsilon_0, r|\varepsilon_0) \), then \( \langle (p_i \upharpoonright r|\varepsilon_0, q_i \upharpoonright r|\varepsilon_0, \varepsilon_0) : i < \lambda \rangle \) is a play of \( \mathcal{O}_0^\lambda(\mathbb{P}_{\varepsilon_1}, r) \) in which Complete uses \( \text{st}(\varepsilon_1, r) \);

(ii) if \( \langle p_i, q_i : i < \lambda \rangle \) is a play of \( \mathcal{O}_0^\lambda(\mathbb{P}_{\varepsilon_0}, r) \) in which Complete plays according to the strategy \( \text{st}(\varepsilon_0, r) \), then \( \langle (p_i|\varepsilon_0, q_i|\varepsilon_0) : i < \lambda \rangle \) is a play of \( \mathcal{O}_0^\lambda(\mathbb{P}_{\varepsilon_0}, r|\varepsilon_0) \) in which Complete uses \( \text{st}(\varepsilon_0, r|\varepsilon_0) \);

(iii) if \( \langle p_i, q_i : i < i^* \rangle \) is a partial play of \( \mathcal{O}_0^\lambda(\mathbb{P}_{\varepsilon_0}, r) \) in which Complete uses \( \text{st}(\varepsilon_0, r) \) and \( p^* \in \mathbb{P}_{\varepsilon_0} \) is stronger than all \( p_i \varepsilon_0 \) (for \( i < i^* \)), then there is \( p^* \in \mathbb{P}_{\varepsilon_0} \) such that \( p^* = p^*|\varepsilon_0 \) and \( p^* \geq p_i \) for \( i < i^* \).

Definition 2.4 (Compare [8, Def. 2.2]).

1. Let \( \gamma \) be an ordinal, \( w \subseteq \gamma \). A standard \((w, 1)\)–tree is a pair \( T = (T, \text{rk}) \) such that
   - \( \text{rk} : T \rightarrow w \cup \{\gamma\} \),
   - if \( t \in T \) and \( \text{rk}(t) = \varepsilon \), then \( t \) is a sequence \( \langle (t) : \zeta \in w \cap \varepsilon \rangle \),
   - \( (T, <) \) is a tree with root \( \langle \rangle \) such that every chain in \( T \) has a \( < \)-upper bound in \( T \),
   - if \( t \in T \), then there is \( t' \in T \) such that \( t \leq t' \) and \( \text{rk}(t') = \gamma \).

We will keep the convention that \( T^*_0 \) is \( (T^*_0, \text{rk}^*_0) \).

2. Let \( \hat{Q} = (\bar{\mathbb{P}}, \bar{Q}_i : i < \gamma) \) be a \( \lambda \)-–support iteration. A standard tree of conditions in \( \hat{Q} \) is a system \( \hat{p} = \langle p_t : t \in T \rangle \) such that
   - \( (T, \text{rk}) \) is a standard \((w, 1)\)–tree for some \( w \subseteq \gamma \), and
   - \( p_t \in \mathbb{P}^{\text{rk}(t)} \) for \( t \in T \), and
   - if \( s, t \in T \), \( s \triangleleft t \), then \( p_s = p_t|\text{rk}(s) \).

3. Let \( \hat{p}^0, \hat{p}^1 \) be standard trees of conditions in \( \hat{Q} \), \( \hat{p}^0 = \langle p^i_t : t \in T \rangle \). We write \( \hat{p}^0 \leq \hat{p}^1 \) whenever for each \( t \in T \) we have \( p^0_t \leq p^1_t \).

Note that our standard trees and trees of conditions are a special case of that introduced in [9, Def. A.1.7] when \( \alpha = 1 \). Also, the rank function \( \text{rk} \) is essentially the function giving the level of a node, adjusted to have values in \( w \cup \{\gamma\} \) via the canonical increasing bijection.

Proposition 2.5 (See [9, Prop. A.1.9]). Assume that \( \hat{Q} = (\hat{\mathbb{P}}, \hat{Q}_i : i < \gamma) \) is a \( \lambda \)-–support iteration such that for all \( i < \gamma \) we have

\[ \models_{\hat{\mathbb{P}}_i} \text{ “}\hat{Q}_i \text{ is strategically (<}\lambda\text{-)–complete”}. \]

Suppose that \( \hat{p} = \langle p_t : t \in T \rangle \) is a standard tree of conditions in \( \hat{Q} \), \( |T| < \lambda \), and \( T \subseteq \mathbb{P} \) is open dense. Then there is a standard tree of conditions \( \hat{q} = \langle q_t : t \in T \rangle \) in \( \hat{Q} \) such that \( \hat{p} \leq \hat{q} \) and \( (\forall t \in T)(\text{rk}(t) = \gamma \Rightarrow q_t \in T) \), and such that conditions \( q_{t_0}, q_{t_1} \) are incompatible whenever \( t_0, t_1 \in T \), \( \text{rk}(t_0) = \text{rk}(t_1) \) but \( t_0 \neq t_1 \).

Definition 2.6 (See [8, Def. 3.1]). Let \( \hat{Q} \) be a forcing notion and let \( \hat{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle \) be a sequence of regular cardinals such that \( \mu_0 \leq \mu_\alpha \leq \lambda \) for all \( \alpha < \lambda \).

1. For a condition \( p \in \hat{Q} \) we define a reasonable \( \hat{A} \)-–completeness game \( \mathcal{O}_{\hat{\mu}}^{\hat{A}}(p, \hat{Q}) \) between two players, Generic and Antigeneric, as follows. A play of \( \mathcal{O}_{\hat{\mu}}^{\hat{A}}(p, \hat{Q}) \) lasts \( \lambda \) steps and during a play a sequence

\[ \langle I_\alpha, \langle q^0_t, q^1_t : t \in I_\alpha \rangle : \alpha < \lambda \rangle \]

is constructed. Suppose that the players have arrived to a stage \( \alpha < \lambda \) of the game. Now,

\[ (\mathcal{N})_\alpha \text{ first Generic chooses a non-empty set } I_\alpha \text{ of cardinality } < \mu_\alpha \text{ and a system } \langle p^i_t : t \in I_\alpha \rangle \text{ of conditions from } \hat{Q}, \]
Theorem 2.8 (See [8, Thm 3.2]). Assume that

(a) \( \lambda \) is a strongly inaccessible cardinal,
(b) \( \bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle \), each \( \mu_\alpha \) is a regular cardinal satisfying (for \( \alpha < \lambda \))
\[ \kappa_0 \leq \mu_\alpha \leq \lambda \quad \text{and} \quad \left( \forall f \in ^\alpha \mu_\alpha \right) \left( \prod_{\xi < \alpha} f(\xi) \right) < \mu_\alpha, \]
(c) \( \bar{\xi} = \langle \{ P_\xi, Q_\xi : \xi < \gamma \} \rangle \) is a \( \lambda \)-support iteration such that for every \( \xi < \gamma \),
\[ P_\xi \models " \bar{Q}_\xi \text{ is reasonably } A \text{-bounding over } \bar{\mu} \". \]

Then \( P_\gamma = \lim(\bar{Q}) \) is reasonableness \( A(\bar{Q}) \)-bounding over \( \bar{\mu} \) (and so \( P_\gamma \) is also \( \lambda \)-proper).

In [8, §3], in addition to \( A \)-reasonable completeness game we considered its variant called a \( \text{reasonably complete} \) game. In that variant, at stage \( \alpha < \lambda \) of the game the players played a subgame to construct a sequence \( \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle \) (corresponding to \( \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle \)). In the following definition we introduce
a further modification of that game. In the new game, the players will again play subgames, in some sense repeating several times the subgames from the a–reasonable completeness game.

**Definition 2.9.** Let $\mathbb{Q}$ be a forcing notion and let $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ be a sequence of cardinals such that $\kappa_0 \leq \mu_\alpha < \lambda$ for all $\alpha < \lambda$. Suppose also that $U$ is a normal filter on $\lambda$.

1. For a condition $p \in \mathbb{Q}$ we define a _reasonable double–a–completeness game_ $\mathcal{D}^{\text{rc}2a}_\mu(p, \mathbb{Q})$ between Generic and Antigeneric as follows. A play of $\mathcal{D}^{\text{rc}2a}_\mu(p, \mathbb{Q})$ lasts at most $\lambda$ steps and in the course of the play the players try to construct a sequence

   $$\langle \xi_\alpha, (p_\alpha^i, q_\alpha^i : \gamma < \mu_\alpha \cdot \xi_\alpha) : \alpha < \lambda \rangle.$$  

   (Here $\mu_\alpha$ is treated as an ordinal and $\mu_\alpha \cdot \xi_\alpha$ is the ordinal product of $\mu_\alpha$ and $\xi_\alpha$.) Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. First, Antigeneric picks a non-zero ordinal $\xi_\alpha < \lambda$. Then the two players start a subgame of length $\mu_\alpha \cdot \xi_\alpha$ alternately choosing the terms of the sequence $(p_\alpha^i, q_\alpha^i : \gamma < \mu_\alpha \cdot \xi_\alpha)$. At a stage $\gamma = \mu_\alpha \cdot i + j$ (where $i < \xi_\alpha$, $j < \mu_\alpha$) of the subgame, first Generic picks a condition $p_\alpha^i \in \mathbb{Q}$ stronger than all conditions $q_\alpha^i$ for $\delta < \gamma$ of the form $\delta = \mu_\alpha \cdot i' + j$ (where $i' < i$), and then Antigeneric answers with a condition $q_\alpha^i$ stronger than $p_\alpha^i$.

   At the end, Generic wins the play $(\mathbb{Q})$ of $\mathcal{D}^{\text{rc}2a}_\mu(p, \mathbb{Q})$ if and only if both players had always legal moves and

   $$(\star)_{2a}^\mathbb{Q} \text{ there is a condition } p^* \in \mathbb{Q} \text{ stronger than } p \text{ and such that }$$

   $$p^* \Vdash \mathbb{Q} \quad (\forall \alpha < \lambda)(\exists j < \mu_\alpha)(\{q_{\mu_\alpha \cdot i+j}^\alpha : i < \xi_\alpha \} \subseteq \mathcal{G}_\mathbb{Q})$$

2. Games $\mathcal{D}^{\text{rc}2b}_\mu(p, \mathbb{Q})$ (for $p \in \mathbb{Q}$) are defined similarly, we only replace condition $(\star)_{2a}^\mathbb{Q}$ by

   $$(\star)_{2b}^\mathbb{Q} \text{ there is a condition } p^* \in \mathbb{Q} \text{ stronger than } p \text{ and such that }$$

   $$p^* \Vdash \mathbb{Q} \quad \{ \alpha < \lambda : (\exists j < \mu_\alpha)(\{q_{\mu_\alpha \cdot i+j}^\alpha : i < \xi_\alpha \} \subseteq \mathcal{G}_\mathbb{Q}) \} \in U_{\mathbb{Q}}$$

   where $U_{\mathbb{Q}}$ is the (Q–name for the) normal filter generated by $U$ in $V^\mathbb{Q}$.

3. A strategy $\mathbf{st}$ for Generic in $\mathcal{D}^{\text{rc}2a}_\mu(p, \mathbb{Q})$ (or $\mathcal{D}^{\text{rc}2b}_\mu(p, \mathbb{Q})$) is said to be _nice_ if for every play $\langle \xi_\alpha, (p_\alpha^i, q_\alpha^i : \gamma < \mu_\alpha \cdot \xi_\alpha) : \alpha < \lambda \rangle$ in which she uses $\mathbf{st}$, for every $\alpha < \lambda$, the conditions in $(p_\alpha^i : \gamma < \mu_\alpha)$ are pairwise incompatible. (These are conditions played in the first “run” of the subgame. Note that then $p_\alpha^i, p_\alpha^j$ are incompatible whenever $\gamma \neq \gamma' \mod \mu_\alpha$.)

4. Let $x \in \{\mathbf{a}, \mathbf{b}\}$. A forcing notion $\mathbb{Q}$ is _nicely double $x$–bounded_ over $\bar{\mu}$ (and $U$ if $x = \mathbf{b}$) if

   (a) $\mathbb{Q}$ is strategically $<\lambda$–complete, and

   (b) Generic has a nice winning strategy in the game $\mathcal{D}^{\text{rc}2a}_\mu(p, \mathbb{Q})$ ($\mathcal{D}^{\text{rc}2b}_\mu(p, \mathbb{Q})$ if $x = \mathbf{b}$) for every $p \in \mathbb{Q}$.

**Remark 2.10.** (1) Reasonable double $x$–boundedness (for $x \in \{\mathbf{a}, \mathbf{b}\}$) is an iterative relative of reasonable $x$–boundedness introduced in [8, Definition 3.1, pp 206-207]. Technical differences in the definitions of suitable games are to achieve the preservation of the corresponding property in $\lambda$–support iterations (see Theorems 2.13, 2.14 below).
(2) The game $\mathcal{G}_{\mu,D}^{rc2b}(p,Q)$ is easier to win for Generic than $\mathcal{G}_{\mu,D}^{rc2a}(p,Q)$ (because the winning criterion is weaker). Therefore, if we are interested in $\lambda$-properness for $\lambda$-support iterations only, then 2.14 will cover a larger class of forcing notions than 2.13.

**Definition 2.11** (See [8, Def. 6.1]). Suppose that $\lambda$ is inaccessible and $\check{\kappa} = \langle \kappa_\alpha : \alpha < \lambda \rangle$ is a sequence of cardinals, $1 < \kappa_\alpha < \lambda$ for $\alpha < \lambda$. We define a forcing notion $\mathbb{P}_k$ as follows.

**A condition in** $\mathbb{P}_k$ **is a pair** $p = (f^p, C^p)$ **such that**

$$C^p \subseteq \lambda$$

is a club of $\lambda$ and $f^p \in \prod \{ \kappa_\alpha : \alpha \in \lambda \setminus C^p \}$.

The order $\leq_{\mathbb{P}_k}$ of $\mathbb{P}_k$ is given by:

$p \leq_{\mathbb{P}_k} q$ if and only if $C^p \subseteq C^q$ and $f^p \subseteq f^q$.

**Proposition 2.12.**

(1) Assume that $\check{\kappa}, \lambda$ are as in 2.11 above and let a sequence $\check{\mu} = (\mu_\alpha : \alpha < \lambda)$ be chosen so that $\prod_{\beta < \alpha} \kappa_\beta \leq \mu_\alpha < \lambda$ (for $\alpha < \lambda$).

Then the forcing notion $\mathbb{P}_k$ is nicely double $b$-bounding over $\check{\mu}, \mathcal{D}_\lambda$.

(2) If $\kappa_\alpha = \kappa$ for all $\alpha < \lambda$ and $\mu_\alpha \geq \kappa^\alpha$, then $\mathbb{P}_k$ is nicely double $a$-bounding over $\check{\mu}$.

**Proof.** (1) A natural modification of the proof of [8, Prop. 6.1] works here. Note that if $\delta = \langle \delta_\alpha : \alpha < \lambda \rangle$ is an increasing continuous sequence constructed as there during a play of $\mathcal{G}_{\mu,D}^{rc2b}(p,\mathbb{P}_k)$, then the set $B \overset{\text{def}}{=} \{ \alpha < \lambda : \prod_{\beta < \alpha} \kappa_\beta \leq \mu_\alpha \}$ is in the filter $\mathcal{D}_\lambda$. In the game, the stages $\alpha \in \lambda \setminus B$ are ignored and only those for $\alpha \in B$ are “active”. Also, at each stage $\alpha$ we may create $\mu_\alpha$ “not active” steps at each run of the subgame by picking an antichain of conditions incompatible with $p$.

(2) Similar; we get double $a$-bounding here as at each stage $\alpha < \lambda$ of the game we know that $\prod_{\beta < \alpha} \kappa_\beta = \kappa^\alpha \leq \mu_\alpha$ (so all steps are “active”). □

**Theorem 2.13.** Assume that

(a) $\lambda$ is a strongly inaccessible cardinal,

(b) $\check{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ is a sequence of cardinals below $\lambda$ such that $(\forall \alpha < \lambda)(\forall \delta_0 \leq \mu_\alpha = \mu_\alpha^{\delta_0 + 1})$,

(c) $\mathcal{Q} = (\mathbb{P}_\zeta, \mathcal{Q}_\zeta : \zeta < \zeta^\star)$ is a $\lambda$-support iteration such that for every $\zeta < \gamma$,

$$\|\mathbb{P}_\zeta \text{ nicely double } a \text{-bounding over } \check{\mu} \|$$

Then $\mathbb{P}_{\zeta^\star} = \lim(\mathcal{Q})$ is nicely double $a$-bounding over $\check{\mu}$ (and so $\mathbb{P}_{\zeta^\star}$ is also $\lambda$-proper).

**Proof.** Our arguments refine those presented in the proof of [8, Theorem 3.2, p. 217], but the differences in the games involved eliminate the use of trees of conditions. However, trees of conditions are implicitly present here too. The tree at level $\delta$ of the argument is indexed by

$$T_\delta = \bigcup_{\xi \in \omega_1 \cap \zeta} \prod_{\mu_\delta : \zeta \in w_\delta \cup \{ \zeta^\star \}}$$

and it is formed in part by conditions played in the game for various $t \in \prod_{\xi \in \omega_1} \mu_\delta = \{ t \in T_\delta : \text{rk}(t) = \zeta^\star \}$; note the coherence demand in $(\mathcal{E}_\gamma)$.
Let \( p \in \mathbb{P}_\ast \). We will describe a strategy \( \text{st} \) for Generic in the game \( \mathcal{G}_{\mu \ast}^{\text{cf}2\text{a}}(p, \mathbb{P}_\ast) \).

The strategy \( \text{st} \) instructs Generic to play the game \( \mathcal{G}_{\mu \ast}^{\text{cf}2\text{a}} \) on each relevant coordinate \( \zeta < \zeta^* \) using her winning strategy \( \text{st}_\zeta \). At stage \( \delta < \lambda \) Generic will be concerned with coordinates \( \zeta \in w_\delta \) for some set \( w_\delta \) of size \( \lambda \). If \( \zeta_\delta < \lambda \) is the ordinal put by Antigeneric in the play of \( \mathcal{G}_{\mu \ast}^{\text{cf}2\text{a}}(p, \mathbb{P}_\ast) \), then in the simulated plays on coordinates \( \zeta \in w_\delta \) Generic pretends that her opponent put \( \zeta_\delta = \mu_\delta \cdot \zeta_\delta \). The innings of the two players, Generic and Antigeneric, in the subgame of level \( \delta \) on a coordinate \( \zeta \) are

\[
\bar{\rho}_{\delta, \zeta} = \langle p^0_{\delta, \zeta} : \gamma < \mu_\delta \cdot \zeta^*_\delta \rangle \quad \text{and} \quad \bar{g}_{\delta, \zeta} = \langle g^0_{\delta, \zeta} : \gamma < \mu_\delta \cdot \zeta^*_\delta \rangle,
\]

respectively. Generic’s innings in the subgame of \( \mathcal{G}_{\mu \ast}^{\text{cf}2\text{a}}(p, \mathbb{P}_\ast) \) will be associated with sequences \( t \in \prod_{\zeta \in w_\delta} \mu_\delta = \{ t^j_\delta : j < \mu_\delta \} = \tau^\delta \). The innings of the two players will be \( p^0_\delta, q^0_\delta \) (for \( \varepsilon < \mu_\delta \cdot \zeta_\delta \)) and they will be related to what happens at coordinates \( \zeta \in w_\delta \) as follows. If \( t = t^j_\delta, \zeta \in w_\delta \) and \( \beta = (t)_\zeta < \mu_\delta \), then in the subgame of \( \mathcal{G}_{\mu \ast}^{\text{cf}2\text{a}}(p, \mathbb{P}_\ast) \) of level \( \delta \) at stages of the form \( \varepsilon = \mu_\delta \cdot i + j \) we will have \( p^0_\delta(\zeta) = p^0_{\delta, \zeta} \) and \( q^0_\delta(\zeta) = g^0_{\delta, \zeta} \), where \( \gamma = \mu_\delta \cdot \varepsilon + \beta \).

To keep track of what happens at coordinates \( \zeta \notin w_\delta \) Generic will use conditions \( r_\delta \).

Let us note that the construction of \( \text{st} \) presented in detail below would be somewhat simpler if we knew that all forcings \( Q_\zeta \) are \( (\lambda^\lambda, \lambda^\lambda) \)-complete (and not only strategically \( (\lambda^\lambda, \lambda^\lambda) \)-complete). Then \( \text{st}_\delta, r_\delta \) and \( p^0_\delta \) could be eliminated as their role is to make sure that some sequences of conditions (related to \( r_\delta \) and/or \( p_\delta^0 \)) have upper bounds. However, many natural forcing notions tend to have strategic completeness only (see [9, Part B]).

Let us formalize the ideas presented above. For each \( \zeta < \zeta^* \) pick a \( \mathbb{P}_\ast \)-name \( \text{st}_\zeta^0 \) such that

\[
\exists \mathbb{P}_\ast = \text{st}_\zeta^0 \text{ is a winning strategy for Complete in } \mathcal{G}_0^\lambda(Q_\zeta, \emptyset_{Q_\zeta}) \text{ such that } \\
\text{if Incomplete plays } \emptyset_{Q_\zeta} \text{ then Complete answers with } \emptyset_{Q_\zeta} \text{ as well.}
\]

In the course of a play of \( \mathcal{G}_{\mu \ast}^{\text{cf}2\text{a}}(p, \mathbb{P}_\ast) \), at a stage \( \delta < \lambda \), Generic will be instructed to construct on the side

\[
(\otimes)_\delta w_\delta, \bar{p}_\delta, \bar{q}_\delta, \text{st}_\zeta \text{ for } \zeta \in w_{\delta+1} \setminus w_\delta, \bar{p}_{\delta, \zeta}, \bar{g}_{\delta, \zeta}, p^0_{\zeta, \delta} \text{ (for } \varepsilon < \mu_\delta \cdot \zeta_\delta \), \text{ and } r^\delta_\delta, r^\delta_\delta.
\]

These objects will be chosen so that if

\[
\langle \zeta_\delta, (p^0_{\zeta, \delta}, q^0_{\zeta, \delta} : \gamma < \mu_\delta \cdot \zeta_\delta) : \delta < \lambda \rangle
\]

is a play of \( \mathcal{G}_{\mu \ast}^{\text{cf}2\text{a}}(p, \mathbb{P}_\ast) \) in which Generic follows \( \text{st} \), and the additional objects constructed at stage \( \delta < \lambda \) are listed in \( (\otimes)_\delta \), then the following conditions are satisfied (for each \( \delta < \lambda \)).

\[(\exists_1)\ r^\delta_\delta, r^\delta_\delta \in \mathbb{P}_\ast, r^0_0(0) = r_0(0) = p(0), w_\delta \subseteq \zeta^*, |w_\delta| = |\delta + 1|, \bigcup_{\alpha < \lambda} \text{Dom}(r_\alpha) = \bigcup_{\alpha < \lambda} w_\alpha, w_0 = \{ 0 \}, \text{ and if } \delta \text{ is limit then } w_\delta = \bigcup_{\alpha < \delta} w_\alpha.
\]

\[(\exists_2)\ \text{For each } \alpha < \delta < \lambda \text{ we have } (\forall \zeta \in w_{\alpha+1})(r_\alpha(\zeta) = r^\delta_\delta(\zeta) = r^\delta_\delta(\zeta)) \text{ and } \]

\[
p \leq r^\delta_\delta \leq r_\alpha' \leq r^\delta_\delta \leq r_\delta, \text{ and } p^0_{\zeta, \delta} \in \mathbb{P}_\ast \text{ (for } \varepsilon < \mu_\delta \cdot \zeta_\delta \).
\]

\[(\exists_3)\ \text{If } \zeta \notin \zeta^* \setminus w_\delta \text{, then } \]

\[
r_\delta \| r^\delta_\delta \text{ " is a legal partial play of } \mathcal{G}_0^\lambda(Q_\zeta, \emptyset_{Q_\zeta}) \text{ in which Complete follows } \text{st}_\zeta^0 "
\]
and if $\zeta \in w_{\beta+1} \setminus w_{\beta}$, then $st_{\zeta}$ is a $\mathbb{P}_{\zeta}$-name for a nice winning strategy for Generic in $\mathcal{O}_{\mu_{\omega}^{2a}}(Q_{\zeta}, \beta)$. (And $st_0$ is a nice winning strategy of Generic in $\mathcal{O}_{\mu_{\omega}^{2a}}(p(0), Q_0).)\\$$
(\mathbb{X})_4 \beta^* = (\delta^*_j : j < \mu_\beta)$ is an enumeration of $\prod_{\zeta \in w_{\beta}} \mu_\beta = w_{\beta} \mu_{\beta}$.\\$$(\mathbb{X})_5 \xi^*_\beta = \mu_\beta \cdot \xi_\beta$ (the ordinal product) and $\overline{p}_{\delta^*_\beta} = \langle q_{\delta^*_\beta}^\zeta : \gamma < \mu_\beta \cdot \xi_\beta \rangle$ and $\overline{q}_{\delta^*_\beta} = \langle \tilde{q}_{\delta^*_\beta}^\zeta : \gamma < \mu_\beta \cdot \xi_\beta \rangle$ are $\mathbb{P}_{\zeta}$-names for sequences of conditions in $Q_{\zeta}$ of length $\mu_\beta \cdot \xi_\beta$ (for $\zeta \in \bigcup_{\alpha < \lambda} w_{\alpha}$).\\$$(\mathbb{X})_6$ If $\zeta \in w_{\beta+1} \setminus w_\beta$, $\beta < \delta$ (or $\zeta = \beta = 0$), then $\parallel_{\mathbb{P}_\zeta} \langle \xi_\alpha, (p_{\alpha, \zeta}^\beta, \overline{q}_{\delta^*_\beta}^\zeta : \gamma < \mu_\alpha \cdot \xi_\alpha) : \alpha \leq \delta \rangle$ is a partial play of $\mathcal{O}_{\mu_{\omega}_{\zeta}}^{2a}(r_{\beta}(\xi), Q_{\zeta})$ in which Generic uses $st_{\zeta}$.\\$$(\mathbb{X})_7$ If $\varepsilon = \mu_\beta \cdot i + j$, $i < \xi_\beta$, $j < \mu_\beta$, then $\text{Dom}(p_{\delta^*_\beta}^\varepsilon) = \text{Dom}(p_{\delta^*_\beta}) = w_{\beta} \cup \text{Dom}(p) \cup \bigcup_{\alpha < \delta} \text{Dom}(r_\alpha) \cup \bigcup_{\varepsilon < \beta} \text{Dom}(q_{\delta^*_\beta}^\varepsilon)$, and for each $\zeta \in w_\beta \cup \{\zeta^*\}$ the condition $p_{\delta^*_\beta}^\varepsilon[\zeta]$ is an upper bound to $\langle \{p[\zeta] \cup \{r_\alpha[\zeta] : \alpha < \delta\} \cup \{q_{\delta^*_\beta}[\varepsilon'] : \varepsilon' = \mu_\beta \cdot i' + j' < \varepsilon \land i' < \xi_\beta \land j' < \mu_\beta \land t_{j'} \mid \zeta = t_{j'}^\delta \rangle \mid \zeta \rangle.$\\$$(\mathbb{X})_8$ If $j < \mu_\beta$, $i < \xi_\beta$, $\xi_\beta \in w_{\beta_1}$, $\langle t_{j'}^\delta \rangle_{\beta} = \beta$ and $\varepsilon = \mu_\beta \cdot i + j$, $\gamma = \mu_\beta \cdot \varepsilon + \beta$, then $p_{\delta^*_\beta}^\varepsilon[\zeta] = p_{\delta^*_\beta}^\gamma[\zeta]$ and $q_{\delta^*_\beta}[\varepsilon] \parallel_{\mathbb{P}_\zeta} q_{\delta^*_\beta}^\gamma[\zeta]$.\\$$(\mathbb{X})_9$ If $\varepsilon = \mu_\beta \cdot i + j$, $i < \xi_\beta$, $j < \mu_\beta$, $\zeta \in \zeta^* \setminus w_{\beta_1}$ and $t \in \prod_{\mu_\beta : \zeta \in w_\beta \cap \zeta}$, $t \leq t_{j'}$, then $p_{\delta^*_\beta}^\varepsilon[\zeta] \parallel_{\mathbb{P}_\zeta} \langle \langle \bigcup_{\{i : \zeta \in \zeta^* \setminus w_{\beta_1} \rangle} \bigcup_{\varepsilon < \beta} \{p[\zeta] \cup \{r_\alpha[\zeta] : \alpha < \delta\} \cup \{q_{\delta^*_\beta}[\varepsilon'] : \varepsilon' = \mu_\beta \cdot i' + j' < \varepsilon \land i' < \xi_\beta \land j' < \mu_\beta \land t_{j'} \mid \zeta = t_{j'}^\delta \rangle \mid \zeta \rangle.$\\$$(\mathbb{X})_{10}$ $\text{Dom}(r_{\delta_\beta}) = \text{Dom}(r_\beta) = \bigcup_{\mu_\beta : \zeta \in \zeta^* \setminus w_{\beta_1}} \text{Dom}(q_{\delta^*_\beta})$ and if $\zeta \in \zeta^* \setminus w_{\beta_1}$, $t \in \prod_{\mu_\beta : \zeta \in \zeta^* \setminus w_{\beta_1}} \text{Dom}(q_{\delta^*_\beta})$, $q \in \mathbb{P}_\zeta$, $q \geq r_{\delta_\beta}[\zeta]$ and $q \geq q_{\delta^*_\beta}[\zeta]$ whenever $\varepsilon = \mu_\beta \cdot i + j$, $i < \xi_\beta$, $j < \mu_\beta$ and $t \leq t_{j'}$, then $q \parallel_{\mathbb{P}_\zeta} \langle \bigcup_{\{i : \zeta \in \zeta^* \setminus w_{\beta_1} \rangle} \bigcup_{\varepsilon < \beta} \{p[\zeta] \cup \{r_\alpha[\zeta] : \alpha < \delta\} \cup \{q_{\delta^*_\beta}[\varepsilon'] : \varepsilon' = \mu_\beta \cdot i' + j' < \varepsilon \land i' < \xi_\beta \land j' < \mu_\beta \land t_{j'} \mid \zeta = t_{j'}^\delta \rangle \mid \zeta \rangle.$

Assume that the two players arrived to stage $\delta$ of $\mathcal{O}_{\mu_{\omega}^{2a}}(p, \mathbb{P}_{\zeta})$ and $\langle \xi_\alpha, \langle p_{\alpha, \zeta}^\delta, \overline{q}_{\delta^*_\beta}^\zeta : \varepsilon < \mu_\alpha \cdot \xi_\alpha : \alpha < \delta \rangle$ is the play constructed so far, and that Generic followed $st$ and determined objects listed in $(\mathbb{X})_9$ (for $\alpha < \delta$) with properties $(\mathbb{X})_1$–$(\mathbb{X})_{10}$.

Below, whenever we say "Generic chooses $x$ such that we mean Generic chooses the $<^\zeta$-first $x$ such that, etc.

First, Generic uses her favorite bookkeeping device to determine $w_{\beta_1}$ so that the demands of $(\mathbb{X})_1$ are satisfied (and that at the end we will have $\bigcup_{\alpha < \lambda} \text{Dom}(r_\alpha) =$
∪ w_α). If β < δ and ζ ∈ w_β, then we already have \( p_{α,ζ}, q_{α,ζ} \) for α < δ (see (3)) or we have not yet defined those objects when δ = δ_0 + 1 and ζ ∈ w_δ \( \setminus w_{δ_0} \).

So if δ = δ_0 + 1 and ζ ∈ w_δ \( \setminus w_{δ_0} \) then let \( p_{α,ζ} = \{ p_{α,ζ}^1 : γ < μ_α \cdot ξ^α \} \) and \( q_{α,ζ} = \{ q_{α,ζ}^1 : γ < μ_α \cdot ξ^α \} \) for α < δ be such that

\[
|p_{δ,ζ} \langle ξ^δ, p_{α,ζ}^1, q_{α,ζ}^1 : γ < μ_α \cdot ξ^α : α < δ \rangle| \text{ is a partial play of } \mathcal{P}_{μ_δ}^{2a}(r_{δ,ζ}(ζ), Q_ζ) \text{ in which Generic uses } s_ζ \text{ and }
\]

\[
q_{α,ζ}^1 = \tilde{q}_{α,ζ}^1 \text{ for all } α < δ, γ < μ_α \cdot ξ^α. 
\]

Condition (3) and our rule of taking “the first” determine the enumeration \( t^δ = (t^δ_j : j < μ_δ) \) of \( \prod_ζ μ_δ \). Now Antigeneric picks ξ_δ and the two players start a subgame of length μ_δ · ξ_δ. During the subgame Generic will simulate subgames of level δ at coordinates ζ ∈ w_δ pretending that Antigeneric played ξ_δ · μ_δ · ξ_δ there. Each step in the subgame of \( \mathcal{P}_{μ_δ}^{2a}(p, P_ζ) \) with correspond to μ_δ steps in the subgames of \( \mathcal{P}_{μ_δ}^{2a}(r_{δ,ζ}(ζ), Q_ζ) \) (when ζ ∈ w_{δ+1} \( \setminus w_δ, β < δ \)). So suppose that the two opponents have arrived at a stage \( ε = μ_δ \cdot i + j \) of the subgame, i < ξ_δ, j < μ_δ, and assume also that Generic (playing according to st) has already defined \( p_{δ,ζ}^j, q_{δ,ζ}^j \) for ζ ∈ w_δ, γ < μ_δ · ε and \( p_{ε,ζ}^j \) for ε < ε, so that the requirements of (3) at (3) are satisfied. Note that by (3) and (3)

\[
(α) \text{ if } ε > ε' = μ_δ \cdot i' + j', \text{ then } p_{ε,ζ}^j \upharpoonright |ζ| ≤ p_{ε',ζ}^j \upharpoonright |ζ| ≤ p_{ε,ζ}^j \upharpoonright |ζ|.
\]

For each ζ ∈ w_δ and β < (t^δ_j) ζ let \( p_{δ,ζ}^{β+ε} = q_{δ,ζ}^{β+ε} \) be \( P_ζ \)-names for conditions in \( Q_ζ \) such that (the relevant part of) (3) still holds. The same clause determines also \( p_{δ,ζ}^{β+ε} \) for β = (t^δ_j) ζ. Then the requirements in (3) and (3) essentially describe what \( p_{δ,ζ}^j \). Note that the “upper bound demands” in (3) can be satisfied because of (3) at (3) and (3) above. Next, Generic’s inning \( p_{δ,ζ}^j \) of \( \mathcal{P}_{μ_δ}^{2a}(p, P_ζ) \) is chosen so that \( \text{Dom}(p_{δ,ζ}^j) = \text{Dom}(p_{δ,ζ}^j) \) and clauses (3) at (3) hold. After this Antigeneric answers with a condition \( q_{δ,ζ}^j \) by \( p_{δ,ζ}^j \), and Generic picks for the construction on the side names \( q_{δ,ζ}^{β+ε} \) for ζ ∈ w_δ and β = (t^δ_j) ζ by the demand in (3). She also picks \( p_{δ,ζ}^{μ_δ \cdot ε+β} = q_{δ,ζ}^{μ_δ \cdot ε+β} \) for ζ ∈ w_δ and (t^δ_j) ζ < β < μ_δ so that (3) holds.

This completes the description of what happens during the μ_δ · ξ_δ steps of the subgame. After the subgame is over and the sequence \( (p_{δ,ζ}^j, q_{δ,ζ}^j : γ < μ_δ \cdot ξ_δ) \) is constructed, Generic chooses conditions \( r_{δ,ζ}^j, r_δ \) ∈ \( P_ζ \) by (3) at (3) and (3). (Note: since \( s_ζ \) are names for nice strategies, if ζ ∈ ζ \( \setminus w_δ, i_0 ≤ ξ_δ, j_0, j_1 < μ_δ, ε_0 = μ_δ \cdot i_0 + j_0, ε_1 = μ_δ \cdot i_1 + j_1, t_0, t_1 \in \prod_ζ \{ μ_δ : ζ ∈ w_δ \cap ζ \}, t_0 ≤ t_1^j_0, t_1 ≤ t_1^j_1, \) and \( t_0 ≠ t_1 \), then the conditions \( q_{δ,ζ}^{β_0} \upharpoonright |ζ|, q_{δ,ζ}^{β_1} \upharpoonright |ζ| \) are incompatible.)

This finishes the description of the strategy st.

Let us argue that st is a winning strategy for Generic. Suppose that

\[
(ξ_δ, (p_{δ,ζ}^j, q_{δ,ζ}^j : γ < μ_δ \cdot ξ_δ) : δ < λ)
\]

is a play of \( \mathcal{P}_{μ_δ}^{2a}(p, P_ζ) \) in which Generic followed st and she constructed the side objects listed in (3) at (3) (for δ < λ) so that demands (3) at (3) are satisfied. We define a condition \( r \) ∈ \( P_ζ \) as follows. Let Dom(r) = \( \bigcup_{δ < λ} \text{Dom}(r_δ) \). For ζ ∈ Dom(r) let \( r(ζ) \) be a \( P_ζ \)-name for a condition in \( Q_ζ \) such that
Suppose now that $δ < λ$ and $r' ≥ r.$ We are going to define $j < µ$ and a condition $r'' ≥ r'$ such that $(∀i < ξ δ)(q_{µ,i+j}^δ ≤ r'').$ To this end let $(ξ_α : α ≤ α^*)$ be the increasing enumeration of $w_δ ∪ \{ξ^*\}$. For $ζ ≤ ξ^*$ and $q ∈ P_ξ,$ let $st(ζ, q)$ be a winning strategy of Complete in $P_ζ^δ(P_ξ, q)$ with the coherence properties given in 2.3.

By induction on $α ≤ α^*$ we will choose conditions $r^*_α, r^*_α \in P_ζ^α$ and $(t)ζ_α < µδ$ such that

$(2)_{12}$ $r' |ζ_α ≤ r^*_α,$
$(2)_{13}$ if $i < ξ δ, j < µδ$ and $(t^j)ζ_α = (t)ζ_α$ for $β < α,$ then $q_{µ,i+j}^δ |ζ_α ≤ r^*_α,$
$(2)_{14}$ $(r^*_β − r' |ζ_β, ζ^*), r^*_ζ − r' |ζ_ζ, ζ^*: β < α)$ is a partial legal play of $P_ζ^δ(P_ζ^*, r')$ in which Complete uses her winning strategy $st(ζ^*, r').$

Suppose that $α ≤ α^*$ is a limit ordinal and we have already defined $(t)ζ_α < µδ$ and $r^*_β, r^*_ζ \in P_ζ^α$ for $β < α.$ Let $ζ = sup(ζ_β : β < α).$ It follows from $(2)_{14}$ that we may pick a condition $s ∈ P_ζ$ stronger than all $r^*_ζ$ for $β < α.$ Put $r^*_ζ = s − r' |ζ, ζ_α$ $∈ P_ζ.$ Then plainly $r' |ζ_α ≤ r^*_ζ$ and $q_{µ,i+j}^δ |ζ_α ≤ r^*_ζ |ζ_ζ$ whenever

$(2)_{15}^{i,j,α}$ $i < ζ δ, j < µδ$ and $(t^j)ζ_α = (t)ζ_α$ for all $β ≤ α.$

Now by induction on $ξ ≤ ζ_α$ we show that $q_{µ,i+j}^δ |ζ ≤ r^*_ζ |ξ$ whenever $(2)_{15}^{i,j,α}$ holds. For $ξ ≤ ζ$ we are already done, so assume $ξ ∈ [ζ, ζ_α)$ and we have shown that $q_{µ,i+j}^δ |ζ ≤ r^*_ζ |ξ$ whenever $(2)_{15}^{i,j,α}$ holds. It follows from $(2)_{10} + (2)_{9}$ that the condition $r^*_ζ |ξ$ forces in $P_ξ$ that

"the set
\[ \{p(ξ) \cup \{r_α(ξ) : α < δ\} \cup \{q^δ_ξ(ξ) : ε = µδ \cdot i + j \land i < ξ δ \land j < µδ \land (∀ β < α)((t^j)ζ_α = (t)ζ_α)) \} \]
has an upper bound in $Q_ξ"."

and therefore we may use $(2)_{16}$ to conclude that

$r^*_ζ |ξ \models " if $(2)_{15}^{i,j,α}$ holds, then $q_{µ,i+j}^δ |ζ_α ≤ r^*_ζ |ξ.$

The limit stages are trivial and we may claim that $q_{µ,i+j}^δ |ζ_α ≤ r^*_ζ$ whenever $(2)_{15}^{i,j,α}$ holds. Next, $r^*_ζ$ is determined by $(2)_{14}$.

Now suppose that $α = β + 1 ≤ α^*$ and we have already defined $r^*_β, r^*_ζ \in P_ζ^α$ and $(t)ζ_α : γ < β.$ It follows from $(2)_{11}$ that

$r^*_ζ \models " r(ζ_β) \models Q_ζ \ (\exists p < µδ) (\forall ε < ξ^*_ζ) (q^δ_ζ, ξ^*_ζ, p) ∈ G_ζ, ζ^* ".$

so we may pick $p = (t)ζ_α$ and a condition $s ∈ P_ζ, ζ_α$ such that $r^*_ζ ≤ s |ζ_β$ and

$s |ζ_β \models T_ζ − P_ζ \ (\forall ε < ξ^*_ζ) (q^δ_ζ, ξ^*_ζ, p) ≤ s(ζ_β).$

It follows from $(2)_{13} + (2)_{8}$ that then also $q_{µ,i+j}^δ |(ζ_β + 1) ≤ s$ whenever $i < ξ δ,$ $j < µδ$ and $(t^j)ζ_α = (t)ζ_α$ for $γ ≤ β.$ We let $r^*_ζ = s − r' |(ζ_β, ζ_α)$ and exactly like in the limit case we argue that $r' |ζ_α ≤ r^*_ζ$ and $q_{µ,i+j}^δ |ζ_α ≤ r^*_ζ$ whenever $i ≤ ξ δ,$ $j < µδ$ and $(t^j)ζ_α = (t)ζ_α$ for $γ ≤ β.$ Again, $r^*_ζ$ is determined by $(2)_{14}.$
After the induction is completed look at \( r'' = r_{\alpha^*}^* \) and \( j < \mu_\delta \) such that \( t^\delta_j = (l)_{\xi_\alpha^*} : \alpha < \alpha^* \).

\[ \textbf{Theorem 2.14.} \text{ Assume (a), (b) of 2.13. Suppose that } \mathcal{U} \text{ is a normal filter on } \lambda \text{ and } \]
\( (c) \mathcal{Q} = (\mathbb{P}_\mathcal{C}, \mathbb{Q}_\mathcal{C} : \zeta < \zeta^*) \text{ is a } \lambda \text{-support iteration such that for every } \zeta < \gamma, \]
\[ \vdash_{\mathbb{P}_\mathcal{C}} \textbf{” } \mathbb{Q}_\mathcal{C} \text{ is nicely double } \mathfrak{b} \text{-bounding over } \bar{\mu}, \mathcal{U}^{\mathcal{C}} \textbf{ “}. \]
Then \( \mathbb{P}_{\mathcal{C}} = \lim(\mathcal{Q}) \) is nicely double \( \mathfrak{b} \)-bounding over \( \bar{\mu}, \mathcal{U} \).

\[ \text{Proof.} \text{ The proof essentially repeats that of 2.13 with the following modifications in the arguments that } \textbf{st} \text{ is a winning strategy for Generic in } \otimes_{\mu, \mathcal{U}}(p, \mathbb{P}_{\mathcal{C}}). \]

We assume that \( (\xi_{\delta}, \langle p_{\delta}^j, q_{\delta}^j : \gamma < \mu_\delta \cdot \xi_{\delta} : \delta < \lambda \rangle) \) is a play in which Generic follows \textbf{st} and the objects listed in \( \otimes_{\delta < \lambda} \) were constructed on a side. A condition \( r \in \mathbb{P}_{\mathcal{C}} \) is chosen so that \( \text{Dom}(r) = \bigcup_{\delta < \lambda} \text{Dom}(r_{\delta}) = \bigcup_{\delta < \lambda} w_\delta \) and for each \( \zeta \in w_{\alpha + 1} \setminus w_\alpha, \alpha < \lambda, \) we have
\[ \vdash_{\mathbb{P}_{\mathcal{C}}} \textbf{” } r(\zeta) \geq r_\alpha(\zeta) \text{ and } r(\zeta) \vdash_{\mathbb{Q}_{\mathcal{C}}} \textbf{” } (\exists j < \mu_\delta)(\forall \varepsilon < \xi_{\delta}^*)((\bar{\mu}^{\delta_{\xi_{\delta}^*} + j} \in G_{\mathbb{Q}_{\mathcal{C}}})) \in U^{\mathcal{C} + 1} \textbf{ “}. \]

Then, for each \( \zeta \in \text{Dom}(r) \), we choose \( \mathbb{P}_{\mathcal{C} + 1} \)-names \( A_{\delta}^j \) for elements of \( \mathcal{U} \) such that
\[ \vdash_{\mathbb{P}_{\mathcal{C}}} \textbf{” } r(\zeta) \vdash_{\mathbb{Q}_{\mathcal{C}}} (\forall \delta \in \Delta)(\exists j < \mu_\delta)(\forall \varepsilon < \xi_{\delta}^*)((\bar{\mu}^{\delta_{\xi_{\delta}^*} + j} \in G_{\mathbb{Q}_{\mathcal{C}}})) \textbf{ “}. \]

Finally, we show that for each limit ordinal \( \delta < \lambda, \)
\[ r \vdash_{\mathbb{P}_{\mathcal{C}}} \textbf{” } (\forall \xi \in w_\delta)(\delta \in \Delta)(\exists j < \mu_\delta)(\forall \varepsilon < \xi_{\delta}^*)((\bar{\mu}^{\delta_{\xi_{\delta}^*} + j} \in G_{\mathbb{P}_{\mathcal{C}}})) \textbf{ “}. \]

For this we start with arbitrary condition \( r' \geq r \) such that
\[ r \vdash_{\mathbb{P}_{\mathcal{C}}} \textbf{” } (\forall \xi \in w_\delta)(\delta \in \Delta)(\exists j < \mu_\delta) \textbf{ “}. \]
and we repeat the arguments from the end of the proof of 2.13 to find \( j < \mu_\delta \) and \( r'' \geq r' \) such that \( (\forall \varepsilon < \xi_{\delta}^*)((\bar{\mu}^{\delta_{\xi_{\delta}^*} + j} \leq r'') \).

\[ \textbf{3. Reasonable Ultrafilters with Small Generating Systems} \]

Our aim here is to show that, consistently, there may exist a very reasonable ultrafilter on an inaccessible cardinal \( \lambda \) with generating system of size less than \( 2^\lambda \).

\[ \textbf{Lemma 3.1.} \text{ Assume that } G^* \subseteq \mathcal{Q}^\lambda_\mathcal{C} \text{ is directed (with respect to } \leq^0 \text{) and } \text{fil}(G^*) \text{ is an ultrafilter on } \lambda, r \in G^*. \text{ Let } \mathcal{P} \text{ be a forcing notion not adding bounded subsets of } \lambda, p \in \mathcal{P} \text{ and let } A \text{ be a } \mathcal{P} \text{-name for a subset of } \lambda \text{ such that } p \vdash_{\mathcal{P}} A \in (\text{fil}(G^*))^\mathcal{P}. \text{ Then } \]
\[ Y \overset{\text{def}}{=} \bigcup \{ Z_{\delta}^\mathcal{C} : \delta \in C^r \text{ and } p \not\vdash_{\mathcal{P}} \text{ “ } A \cap Z_{\delta}^\mathcal{C} \notin d_\delta \text{ “} \} \in \text{fil}(G^*). \]

\[ \text{Proof.} \text{ Assume towards contradiction that } Y \notin \text{fil}(G^*). \text{ Then we may find } s \in G^* \text{ such that } r \leq^0 s \text{ and } \lambda \setminus Y \in \text{fil}(s). \text{ Take } \varepsilon < \lambda \text{ such that } \]
\[ \text{if } \alpha \in C^n \setminus \varepsilon, \text{ then } Z_\alpha^\mathcal{C} \setminus Y \in d_\alpha^\mathcal{C} \text{ and } (\forall A \in d_\alpha^\mathcal{C})(\exists \beta \in C^r)(A \cap Z_{\beta}^\mathcal{C} \notin d_\beta^\mathcal{C}). \]
(Remember 1.5.) Now take a generic ultrafilter $G \subseteq P$ over $V$ such that $p \in G$ and work in $V[G]$. Since $A^G \in \text{fil}(s)^+$, we may pick $\alpha \in C^*$ such that $\varepsilon < \alpha$ and $A^G \cap Z_\alpha \in d^*_\varepsilon$. Then also $Z_\alpha \cap A^G \cap Y \in d^*_\varepsilon$ and thus we may find $\beta \in C^r$ such that $Z_\alpha \cap A^G \cap Z_\beta \cap Y \in d^*_\varepsilon$. In particular, $Z_\beta \cap Y \neq \emptyset$, so $p \Vdash A \cap Z_\beta \notin d^*_\varepsilon$, and thus $A^G \cap Z_\beta \notin d^*_\varepsilon$. Consequently $Z_\alpha \cap A^G \cap Z_\beta \cap Y \notin d^*_\varepsilon$ giving a contradiction. $$
abla$

**Theorem 3.2.** Assume that

(i) $\lambda$ is strongly inaccessible, $\bar{\mu} = (\mu_\alpha : \alpha < \lambda)$, each $\mu_\alpha$ is a regular cardinal, $\aleph_0 \leq \mu_\alpha \leq \lambda$ and $\forall f \in \alpha \mu_\alpha (|\prod_{\xi < \alpha} f(\xi)| < \mu_\alpha)$ for $\alpha < \lambda$;

(ii) $\bar{Q} = (\mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma)$ is a $\lambda$–support iteration such that for every $\xi < \gamma$,

$\Vdash_{\mathbb{P}_\xi} \bar{Q}_\xi$ is reasonably $A$–bounding over $\mu$;

(iii) $G^* \subseteq Q_\lambda^\gamma$ is a $\leq^0$–downward closed $\mu$–super reasonable family such that $\text{fil}(G^*)$ is an ultrafilter on $\lambda$.

Then

$\Vdash_{\mathbb{P}_\gamma} \text{fil}(G^*)$ is an ultrafilter on $\lambda$.

**Proof.** The proof is by induction on the length $\gamma$ of the iteration $\bar{Q}$. So we assume that (i)–(iii) hold and for each $\xi < \gamma$

(\circ) $\Vdash_{\mathbb{P}_\xi} \text{fil}(G^*)$ is an ultrafilter on $\lambda$.

Note that (by the strategic ($<\lambda$)–completeness of $\mathbb{P}_\gamma$) forcing with $\mathbb{P}_\gamma$ does not add bounded subsets of $\lambda$, and therefore $(Q_\lambda^\gamma)^V \subseteq (Q_\lambda^\gamma)^{V^\gamma}$.

**Claim 3.2.1.** Assume that

(a) $A$ is a $\mathbb{P}_\gamma$–name for a subset of $\lambda$ such that $\Vdash_{\mathbb{P}_\gamma} A \in (\text{fil}(G^*))^+$, 

(b) $w \in [\gamma]^{\omega}$ and $T$ is a finite standard $(w, 1)^\gamma$–tree, and

(c) $\bar{p} = (p_t : t \in T)$ is a (finite) tree of conditions in $\mathbb{Q}_\gamma$, and

(d) $r \in G^*$ and $X$ is the set of all $\alpha < C^r$ for which there is a tree of conditions

$\bar{q} = (q_t : t \in T)$ such that $\bar{q} \geq \bar{p}$ and

$(\forall t \in T)(\text{rk}(t) = \gamma \Rightarrow q_t \Vdash A \cap Z_\alpha \in d^*_\varepsilon)$.

Then $\bigcup \{Z_\alpha^r : \alpha \in X\} \in \text{fil}(G^*)$.

**Proof of the Claim.** Induction on $|w|$.

If $w = \emptyset$ and so $T = \{\emptyset\}$, then the assertion follows directly from Lemma 3.1 (with $p, \mathbb{P}$ there standing for $p_\gamma, \mathbb{P}_\gamma$ here).

Assume that $|w| = n + 1$, $\xi^* = \text{max}(w)$, $w' = w \setminus \{\xi^*\}$ and the claim is true for $w'$ (in place of $w$) and any $A, \bar{p}$. Let $P_{\xi^*, \gamma}$ be a $\mathbb{P}_\xi$–name for a forcing notion with the universe $P_{\xi^*, \gamma} = \{p[\xi^*, \gamma] : p \in \mathbb{P}_\gamma\}$ and the order relation $\leq_{\mathbb{P}_{\xi^* \gamma}}$, such that

if $G \subseteq P_{\xi^*, \gamma}$ is generic over $V$ and $f, g \in P_{\xi^*, \gamma}$, then $V[G] \models f \leq_{\mathbb{P}_{\xi^*, \gamma}} g$ if and only if $(\exists p \in G)(p \cup f \leq_{\mathbb{P}_\gamma} p \cup g)$.

Note that $P_{\xi^*, \gamma}$ is from $V$ but the relation $\leq_{\mathbb{P}_{\xi^*, \gamma}}$ is defined in $V[G]$ only. Also $\mathbb{P}_\gamma$ is isomorphic with a dense subset of the composition $P_{\xi^*, \gamma} \ast P_{\xi^*, \gamma}$.

We are going to define a $\mathbb{P}_{\xi^*, \gamma}$–name $\bar{Y}$ for a subset of $\lambda$. Suppose that $G \subseteq P_{\xi^*, \gamma}$ is generic over $V$ and work in $V[G]$. For $t \in T$ such that $\text{rk}(t) = \gamma$ let $X_t$ consist of all $\alpha < C^r$ for which there is $f \in P_{\xi^*, \gamma}$ such that

$p_t[\xi^*, \gamma] \leq_{\mathbb{P}_{\xi^*, \gamma}} f$ and $f \Vdash_{\mathbb{P}_{\xi^*, \gamma}} A \cap Z_\alpha^r \in d^*_\varepsilon$. 


Let \( Y_1 = \bigcup \{ Z^*_\alpha : \alpha \in X_t \} \) (for \( t \in T \) such that \( \text{rk}(t) = \gamma \)). It follows from Lemma 3.1 that each \( Y_1 \) belongs to \( \text{fil}(G^*) \) (remember that \( \Vdash_{\mathbb{P}_{\alpha}} \text{fil}(G^*) \) is an ultrafilter) by \((\circ)_{\xi^*}\). Hence
\[
Y^* \overset{\text{def}}{=} \bigcap \{ Y_1 : t \in T \& \text{rk}(t) = \gamma \} \in \text{fil}(G^*).
\]
Note that for each \( \alpha \in C^\gamma \), either \( Z^*_\alpha \cap Y^* = \emptyset \) or \( Z^*_\alpha \subseteq Y^* \).

Going back to \( \mathbf{V} \), let \( Y^*, Y_t, X_t \) be \( \mathbb{P}_\xi \)-names for the objects described as \( Y^*, Y_t, X_t \) above. Thus \( \Vdash_{\mathbb{P}_\xi} Y^* \in \text{fil}(G^*) \) and we may apply the inductive hypothesis to \( w', T' = \{ t(\xi^* : t \in T) \} \) and \( p' = \langle p_t : t' \in T' \rangle \subseteq \mathbb{P}_\xi \). Thus, if \( X^* \) is the set of all \( \alpha \in C^\gamma \) for which there is a tree of conditions \( q' = \langle q'_t : t' \in T' \rangle \subseteq \mathbb{P}_\xi \) such that \( q' \geq p' \) and
\[
(\forall t' \in T') (\text{rk}(t') = \xi^* \Rightarrow q'_t \Vdash_{\mathbb{P}_\xi} Y^* \cap Z^*_\alpha \in d^*_\alpha),
\]
then \( \bigcup \{ Z^*_\alpha : \alpha \in X^* \} \in \text{fil}(G^*) \).

Now suppose that \( \alpha \in X^* \) is witnessed by \( q' \) and let \( t' \in T \) be such that \( \text{rk}(t') = \xi^* \). Then \( q'_t \Vdash_{\mathbb{P}_\xi} \alpha \in X_t \) and hence \( q'_t \Vdash_{\mathbb{P}_\xi} \alpha \in X_t \) for all \( t \in T \) with \( \text{rk}(t) = \gamma \), so we have \( \mathbb{P}_\xi \)-names \( f'_t \) for elements of \( X_t \) such that
\[
q'_t \Vdash_{\mathbb{P}_\xi} \alpha \in X_t \cap Z^*_\alpha \subseteq \alpha \in X^*.
\]
Now use 2.5 (or just finite induction) to get a tree of conditions
\[
q'' = \langle q''_t : t' \in T' \rangle \subseteq \mathbb{P}_\xi
\]
and objects \( q''_t \) (for \( t' \in T' \), \( t \in T \), \( \text{rk}(t') = \xi^* \), \( \text{rk}(t) = \gamma \)) such that \( q' \leq q'' \) and
\[
\text{rk}(t) = \gamma \text{ for all } t \in T
\]
\[
q'_t = q''_t \text{ if } \text{rk}(t) \leq \xi^*, \text{ and}
\]
\[
q_t = q''_t \text{ if } \text{rk}(t) = \xi^* \text{ and } \text{rk}(t) = \gamma.
\]
It should be clear that \( \bar{q} = \langle q_t : t \in T \rangle \) is a tree of conditions in \( \mathbb{Q}, \bar{p} \leq \bar{q} \) and for every \( t \in T \) with \( \text{rk}(t) = \gamma \) we have \( q_t \Vdash_{\mathbb{P}_\gamma} Z^*_\alpha \in d^*_\alpha \). This shows that \( X^* \) is included in the set defined in the assumption (d), and hence \( \{ Z^*_\alpha : \alpha \in X \} \in \text{fil}(G^*) \).

Let \( \mathbb{A} \) be a \( \mathbb{P}_\gamma \)-name for a subset of \( \lambda \) such that \( \Vdash_{\mathbb{P}_\gamma} \mathbb{A} \in \text{fil}(G^*) \) and let \( p \in \mathbb{P}_\gamma \). We will find a condition \( p^* \geq p \) such that \( p^* \Vdash_{\mathbb{P}_\gamma} \mathbb{A} \in \text{fil}(G^*) \). It will be provided by the winning criterion \((\oplus)_{\mathbb{Q}}^{\text{tree}}\) of the game \( \mathbb{G}_{\mathbb{P}_\gamma}^{\text{tree}}(p, \mathbb{Q}) \) (see 2.7; remember \( \mathbb{P}_\gamma \) is reasonably \( A(\mathbb{Q}) \)-bounding over \( \mathbb{P}_\gamma \) by Theorem 2.8).

Let \( \mathbf{st} \) be a winning strategy of Generic in \( \mathbb{G}_{\mathbb{P}_\gamma}^{\text{tree}}(p, \mathbb{Q}) \), and for \( \varepsilon \leq \gamma \) and \( q \in \mathbb{P}_\varepsilon \) let us fix a winning strategy \( \mathbf{st}(\varepsilon, q) \) of Complete in \( \mathbb{G}_{\mathbb{P}_\varepsilon}^{\lambda}(p, q) \) so that the coherence demands (i)-(iii) of Proposition 2.3 are satisfied.

We are going to describe a strategy \( \mathbf{st}^{\oplus} \) of INC in the game \( \mathbb{G}_{\mathbb{P}_\gamma}^{\text{tree}}(G^*) \). In the course of a play of \( \mathbb{G}_{\mathbb{P}_\gamma}^{\text{tree}}(G^*) \), INC will construct on the side a play of \( \mathbb{G}_{\mathbb{P}_\gamma}^{\text{tree}}(p, \mathbb{Q}) \) in which Generic plays according to \( \mathbf{st} \). So suppose that INC and COM arrived to a stage \( \alpha < \lambda \) of a play of \( \mathbb{G}_{\mathbb{P}_\gamma}^{\text{tree}}(G^*) \), and they have constructed
\[
(\oplus)_{\gamma} \quad \langle I_{\gamma}, i_{\gamma}, \beta_{\gamma}, (r_{\gamma,i}, r'_{\gamma,i}, (\beta_{\gamma,i}, Z_{\gamma,i}, \psi_{\gamma,i}, \delta_{\gamma,i}) : i < i_{\gamma}) : \gamma < \alpha \rangle.
\]
Also, let us assume that INC (playing according to \( \mathbf{st}^{\oplus} \)) has written on the side a partial play
\[
(\oplus)_{\gamma} \quad \langle T_{\gamma}, p^\gamma, q^\gamma : \gamma < \alpha \rangle.
\]
of $\mathcal{O}^\alpha_{\text{tree}}(\mu, \bar{Q})$ (in which Generic plays according to $\mathbf{st}$). Let a standard tree $T_\alpha$ and a tree of conditions $\bar{p}^\alpha = \langle p^\alpha_i : t \in T_\alpha \rangle$ be given to Generic by the strategy $\mathbf{st}$ in answer to (\ref{eq:0}) (so $|T_\alpha| < \mu_\alpha$).

On the board of $\mathcal{O}^\alpha_{\mu}(G^*)$, the strategy $\mathbf{st}^{\text{BI}}$ instructs INC to play the set

$$I_\alpha \overset{\text{def}}{=} \{ t \in T_\alpha : \text{rk}_\alpha(t) = \gamma \}$$

and the $<^\chi$-first enumeration $u_\alpha = \langle u_{\alpha,i} : i < i_\alpha \rangle$ of $[I_\alpha]^\omega$ (so $i_\alpha < \mu_\alpha$). Now the two players start playing a subgame of length $i_\alpha$ to determine a sequence $\langle r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha \rangle$. During the subgame INC will construct on the side a sequence $\langle q^0_i, q^1_i : i < i_\alpha \rangle$ of trees of conditions in $P$ so that

\begin{equation}
\text{(\ref{eq:1})}_3 q^0_i = \langle q^0_{k,i} : t \in T_\alpha \rangle \quad (\text{for } t < 2, i < i_\alpha) \quad \text{and for each } t \in T_\alpha, \text{ the sequence}
\langle q^0_{k,i}, q^1_{k,i} : i < i_\alpha \rangle \text{ is a legal play of } \mathcal{O}^0_\alpha(P_{\text{rk}_\alpha(t)}, P^2_t) \text{ in which Complete uses her winning strategy } \mathbf{st}(\text{rk}_\alpha(t), P^2_t) \text{.}
\end{equation}

Suppose that COM and INC arrive at level $i < i_\alpha$ of the subgame (of $\mathcal{O}^\alpha_{\mu}(G^*)$) and

\begin{enumerate}
\item[(\ref{eq:1})] \quad (r_{\alpha,j}, r'_{\alpha,j}, (\beta_{\alpha,j}, Z_{\alpha,j}, d_{\alpha,j}) : j < i) \quad \text{and}
\item[(\ref{eq:2})] \quad \langle q^0_j, q^1_j : j < i \rangle
\end{enumerate}

have been determined and COM has chosen $r_{\alpha,i} \in G^*$. INC’s answer is given by $\mathbf{st}^{\text{BI}}$ as follows. First, INC takes the $<^\chi$-first tree of conditions $\bar{q}^\alpha$ in $\bar{Q}$ such that

\begin{enumerate}
\item[(\ref{eq:3})] \quad \text{there is a tree of conditions } q^\alpha \text{ of } \bar{Q} \text{ such that } q^\alpha \leq \bar{q}^\alpha \text{ and if } t \in u_{\alpha,i}, \text{ then } q^\alpha_t \models_{\forall, \bar{\pi}} A \cap Z^\alpha_{\alpha,i} \in d^\alpha_{\alpha,i}.
\end{enumerate}

Since $u_{\alpha,i}$ is finite, it follows from 3.2.1 that $\bigcup \{ Z^\alpha_{\alpha,i} : \beta \in X \} \in \text{fil}(G^*)$. Then INC picks also the club $C$ of $\lambda$ such that $C \subseteq C^{\alpha,i}$ and $r_{\alpha,i}$ is restrictable to $\langle X, C \rangle$ (see Definition 1.7) and $\min(C) = \min(X)$, and his inning at the stage $i$ of the subgame of $\mathcal{O}^\alpha_{\mu}(G^*)$ is $r'_{\alpha,i} = r_{\alpha,i} \upharpoonright \langle X, C \rangle$ (again, see Definition 1.7; note that $r'_{\alpha,i} \in G^*$ by 1.8).

After this COM answers with $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#(r'_{\alpha,i})$, and then INC chooses (for the construction on the side) the $<^\chi$-first tree of conditions $\bar{q}^\alpha$ in $\bar{Q}$ such that $\bar{q}^\alpha = \langle \bar{q}^\alpha_{k,i} : t \in T_\alpha \rangle$, and the strategies $\mathbf{st}(\text{rk}_\alpha(t), P^2_t)$ (for $t \in T_\alpha$); remember the coherence conditions of 2.3.

This completes the description of how INC plays in the subgame of stage $\alpha$. After the subgame is finished, INC determines the move $\bar{q}^\alpha$ of Antigeneric in the play of $\mathcal{O}^\alpha_{\text{tree}}(\mu, \bar{Q})$ which he is constructing on the side:

\begin{enumerate}
\item[(\ref{eq:4})] \quad $\bar{q}^\alpha$ is the $<^\chi$-first tree of conditions $\langle q^\alpha_t : t \in T_\alpha \rangle$ such that $q^\alpha_t \leq q^\alpha_t \leq \bar{q}^\alpha$ for all $i < i_\alpha$.
\end{enumerate}

(There is such a tree of conditions by (\ref{eq:3}); remember $i_\alpha < \mu_\alpha < \lambda$.)

This completes the description of the strategy $\mathbf{st}^{\text{BI}}$. Since $G^*$ is $\bar{\mu}$-super reasonable, $\mathbf{st}^{\text{BI}}$ cannot be a winning strategy, so there is a play...
\((\oplus)_0\) \(\langle I_\alpha, i_\alpha, u_\alpha, \langle r_{\alpha,i}, r'_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : i < i_\alpha \rangle : \alpha < \lambda \rangle\)
of \(\mathcal{D}_\mu^\Box(\mathcal{G}^*)\) in which INC follows \(\mathsf{st}^\Box\), but
\((\oplus)_9\) for some \(r \in \mathcal{G}^*\), for every \(\langle j_\alpha : \alpha < \lambda \rangle \in \prod_{\alpha < \lambda} I_\alpha\) we have
\[\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \& i < i_\alpha \& j_\alpha \in u_{\alpha,i}\} \leq^* \#(r)\]
Let \(\langle T_\alpha, \bar{p}_\alpha, \bar{q}_\alpha : \alpha < \lambda \rangle\) be the play of \(\mathcal{D}_\mu^\text{tree}\(\mathcal{V}_\alpha, \bar{q}_\alpha\)\) constructed on the side by INC (so this is a play in which Generic uses her winning strategy \(\mathsf{st}\)). Since Generic won that play, there is a condition \(p^* \in \mathbb{P}_\gamma\) stronger than \(p\) and such that for each \(\alpha < \lambda\) the set \(\{q_\alpha^*: t \in T_\alpha \& \text{rk}(t) = \gamma\}\) is pre-dense above \(p^*\). Note that if we show that
\((\oplus)_{10}\) it is forced in \(\mathbb{P}_\gamma\) that for every \(\bar{j} = \langle j_\alpha : \alpha < \lambda \rangle \in \prod_{\alpha < \lambda} I_\alpha\) we have
\[\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \& i < i_\alpha \& j_\alpha \in u_{\alpha,i}\} \leq^* \#(r)\]
then we will be able to conclude that \(p^* \vdash A \in \text{fil}(r)\) (remember \((\oplus)_6 + (\oplus)_7\) and 1.10), finishing the proof of the Theorem. So let us argue that \((\oplus)_{10}\) holds true.
It follows from the description of \(\mathsf{st}^\Box\) (see the description of \(X\) after \((\oplus)_6\)) that we may choose a continuous increasing sequence \(\langle \delta_{\alpha} : \alpha < \lambda \rangle \subseteq \lambda\) such that
\[(\forall \alpha < \lambda ) \ (\delta_\alpha \leq \delta_{\alpha,0} \leq \sup( \bigcup_{i < i_\alpha} Z_{\alpha,i} ) < \delta_{\alpha+1})\]
Now, we will say that \(\beta \in C^\circ\) is a sick case whenever there are \(\alpha_0 < \alpha_1 < \lambda\) and \(B \in d^\circ_{\alpha_1}\) such that \(Z^\beta_{\alpha} \subseteq [\delta_{\alpha_0}, \delta_{\alpha_1})\) and
\[\left(\forall \alpha \in [\alpha_0, \alpha_1) \right) \left( \exists \bar{t} \in I_\alpha \left( \forall i < i_\alpha \left( t \notin u_{\alpha,i} \ \text{or} \ \ B \cap Z_{\alpha,i} \neq d_{\alpha,i} \right) \right) \right)\]
Using 1.10(2) one can easily verify that the following two conditions are equivalent:
\((\oplus)_{11}^\text{me}\) there is \(\langle j_\alpha : \alpha < \lambda \rangle \in \prod_{\alpha < \lambda} I_\alpha\) such that
\[\{(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) : \alpha < \lambda \& i < i_\alpha \& j_\alpha \in u_{\alpha,i}\} \not\leq^* \#(r)\]
\((\oplus)_{11}^\text{wo}\) there are \(\lambda\) many sick cases of \(\beta \in C^\circ\).
Since the forcing with \(\mathbb{P}_\gamma\) does not add bounded subsets of \(\lambda\), being a sick case is absolute between \(\mathbb{V}\) and \(\mathbb{V}^\mathbb{P}_\gamma\). So we may conclude (from \((\oplus)_9\)) that \((\oplus)_{10}\) is true and thus the proof of Theorem 3.2 is complete. \(\Box\)

**Theorem 3.3.** Assume (i) and (ii) of 3.2 and
\((\alpha)\) \(\bar{k} = \langle k_\alpha : \alpha < \lambda \rangle\) is a sequence of regular cardinals such that for each \(\alpha < \lambda\):
\[\mu_\alpha \leq k_\alpha \leq \lambda \ \text{and} \ \ (\forall \mu < \mu_\alpha)(2^\mu < k_\alpha),\]
\((\beta)\) \(\mathcal{G}^* \subseteq \mathcal{Q}_\lambda^0\) \(\bar{k}\)-super reasonable.
Then \(\Vdash_{\mathbb{P}_\gamma}\) “ \(\mathcal{G}^*\) is \(\bar{k}\)-super reasonable “.

**Proof.** First of all note that the forcing notion \(\mathbb{P}_\gamma\) is reasonably\(^*\) \(A(Q)\)-bounding over \(\bar{\mu}\) and \(\lambda\)-proper (see 2.8). Therefore \(\Vdash_{\mathbb{P}_\gamma}\) “ \(([\mathcal{G}^*]^\lambda)^\mathbb{V}\) is cofinal in \([\mathcal{G}^*]^\lambda\)”, and consequently \(\Vdash_{\mathbb{P}_\gamma}\) “ \(\mathcal{G}^*\) is \((< \lambda)^\mu\)-directed (with respect to \(\leq^0\))”.
Suppose that \(\mathsf{st}^\Box\) is a \(\mathbb{P}_\gamma\)-name, \(p \in \mathbb{P}_\gamma\), and
\[\Vdash_{\mathbb{P}_\gamma}\) “ \(\mathsf{st}^\Box\) is a strategy of INC in \(\mathcal{D}_\mu^\Box(\mathcal{G}^*)\) such that all values given by it are from \(\mathbb{V}\) “.
We are going to find a condition \( p^* \geq p \) and a \( \mathbb{P}_\gamma \)-name \( g_\lambda \) such that
\[
p^* \forces_{\mathbb{P}_\gamma} \text{ " } g_\lambda \text{ is a play of } \mathcal{D}_\mathbb{P}^\alpha(G^*) \text{ in which INC uses } \text{st}_\beta \text{ but COM wins the play } \text{"} .
\]

The condition \( p^* \) will be provided by the winning criterion \((\oplus)_{\mathbb{A}}^{\text{tree}}\) of the game \( \mathcal{D}_\mathbb{P}^{\text{tree}}(p, \varnothing) \) (see Definition 2.7).

In the rest of the proof whenever we say “INC chooses/picks \( x \) such that” we mean “INC chooses/picks the \( \prec_\alpha \)-first \( x \) such that”. Let us fix

(i) a winning strategy \( \text{st} \) of Generic in \( \mathcal{D}_\mathbb{P}^{\text{tree}}(p, \varnothing) \),
(ii) winning strategies \( \text{st}(\varepsilon, q) \) of Complete in \( \mathcal{D}_\mathbb{P}^\alpha(\mathbb{P}_\varepsilon, q) \) (for \( \varepsilon \leq \gamma, q \in \mathbb{P}_\varepsilon \)) such that the coherence conditions of 2.3 are satisfied.

We are going to describe a strategy \( \text{st}_\beta^{\mathbb{V}} \) of INC in the game \( \mathcal{D}_\mathbb{P}^\beta(G^*) \). In the course of a play of \( \mathcal{D}_\mathbb{P}^\alpha(G^*) \), INC will simulate a play of \( \mathcal{D}_\mathbb{P}^{\text{tree}}(p, \varnothing) \) and he will consider names for partial plays of \( \mathcal{D}_\mathbb{P}^\beta(G^*) \) in which INC uses \( \text{st}_\beta \). Thus players INC/COM will appear in the play of \( \mathcal{D}_\mathbb{P}^\beta(G^*) \) in \( \mathbb{V} \) and in the play of \( \mathcal{D}_\mathbb{P}^\beta(G^*) \) in \( \mathbb{V}^{\mathbb{P}_\gamma} \). To avoid confusion we will refer to them as \( \text{COM}^\mathbb{V}, \text{INC}^\mathbb{V} \) for \( \mathcal{D}_\mathbb{P}^\beta(G^*) \) (in \( \mathbb{V} \)) and \( \text{COM}^{\mathbb{V}^{\mathbb{P}_\gamma}}, \text{INC}^{\mathbb{V}^{\mathbb{P}_\gamma}} \) for \( \mathcal{D}_\mathbb{P}^\beta(G^*) \) (in \( \mathbb{V}^{\mathbb{P}_\gamma} \)).

So suppose that \( \text{INC}^\mathbb{V} \) and \( \text{COM}^\mathbb{V} \) arrived at a stage \( \alpha < \lambda \) of the play of \( \mathcal{D}_\mathbb{P}^\beta(G^*) \) (in \( \mathbb{V} \)), and \( \text{INC}^{\mathbb{V}^{\mathbb{P}_\gamma}} \) (playing according to \( \text{st}_\beta^{\mathbb{V}} \)) has written on the side:

\((\oplus)_1^\mathbb{V}\) a partial play \( (T_\beta, \bar{p}^\beta, q^\beta : \beta < \alpha) \) of \( \mathcal{D}_\mathbb{P}^{\text{tree}}(p, \varnothing) \) in which Generic plays according to \( \text{st} \), and
\((\oplus)_2^\mathbb{V}\) a \( \mathbb{P}_\gamma \)-name \( g_\alpha = \langle \bar{I}_\beta, \bar{i}_\beta, \bar{u}_\beta, \bar{x}_\beta : \beta < \alpha \rangle \) of a partial play of \( \mathcal{D}_\mathbb{P}^\beta(G^*) \) (in \( \mathbb{V}^{\mathbb{P}_\gamma} \)) in which \( \text{INC}^{\mathbb{V}^{\mathbb{P}_\gamma}} \) uses the strategy \( \text{st}_\beta^{\mathbb{V}} \),
\((\oplus)_3^\mathbb{V}\) ordinals \( i_\beta < \mu_\beta \) such that \( q^\beta_t \forces_{\mathbb{P}_\beta} \bar{i}_\beta = i_\beta \) for every \( t \in T_\beta \) with \( \text{rk}_\alpha(t) = \gamma \) (for \( \beta < \alpha \)).

Note that \( I_\beta \) is a \( \mathbb{P}_\gamma \)-name for a set of size \( < \mu_\beta \) from \( \mathbb{V} \), \( \bar{u}_\beta \) is a \( \mathbb{P}_\gamma \)-name for an \( \bar{i}_\beta \)-sequence of finite subsets of \( I_\beta \) and \( \bar{x}_\beta \) is a \( \mathbb{P}_\gamma \)-name for the result of the subgame of length \( \bar{i}_\beta \) of level \( \beta \).

Let \( I_\alpha \) be a \( \mathbb{P}_\gamma \)-name for the answer by \( \text{st}_\beta^{\mathbb{V}} \) to the play \( g_\alpha \) of \( \mathcal{D}_\mathbb{P}^\beta(G^*) \) (in \( \mathbb{V}^{\mathbb{P}_\gamma} \)). Let \( T_\alpha \) and \( \bar{p}^\alpha = (q^\alpha_t : t \in T_\alpha) \) be given to Generic by the strategy \( \text{st} \) as an answer to \( (\oplus)_1^\mathbb{V} \). Let \( \bar{q}^\alpha = (q^\alpha_t : t \in T_\alpha) \) be a tree of conditions in \( \varnothing \) such that
\[(\oplus)_4^\mathbb{V} \bar{p}^\alpha \leq q^\alpha \text{ and } q^\alpha_t, q^\alpha_{t'} \text{ are incompatible for distinct } t, t' \in T_\alpha \text{ with } \text{rk}_\alpha(t_0) = \text{rk}_\alpha(t_1) = \gamma . \]
\[(\oplus)_5^\mathbb{V} \text{ for every } t \in T_\alpha \text{ with } \text{rk}_\alpha(t) = \gamma \text{ the condition } q^\alpha_t \text{ decides the value of } I_\alpha, \text{ say } q^\alpha_t \forces_{\mathbb{P}_\gamma} " I_\alpha = I_\alpha^t " . \]

(Note that \( \models_{\mathbb{P}_\gamma} I_\alpha \in \mathbb{V} \) by the choice of \( \text{st}_\beta^{\mathbb{V}} \); remember 2.5.)

In the play of \( \mathcal{D}_\mathbb{P}^\beta(G^*) \), the strategy \( \text{st}_\beta^{\mathbb{V}} \) instructs \( \text{INC}^{\mathbb{V}} \) to choose the set
\[ I_\alpha = \prod \{ I_\alpha^t : t \in T_\alpha \ \& \ \text{rk}_\alpha(t) = \gamma \} \]
and an enumeration \( \bar{u}_\alpha = \langle u_{\alpha,i} : i < i_\alpha \rangle \) of \( |I_\alpha|^\omega \). Note that \( |I_\alpha^t| < \mu_\alpha \) for all relevant \( t \in T_\alpha \) and \( |T_\alpha| < \mu_\alpha \), so by our assumptions on \( \mu_\alpha \) and \( \kappa_\alpha \) we know that \( |I_\alpha| < \kappa_\alpha \) (so also \( i_\alpha < \kappa_\alpha \)).
Then, in the play of $\exists_\alpha^R(G^*)$, INC$^{V^*_\gamma}$ pretends that COM$^{V^*_\gamma}$ played an ordinal $\bar{i}_\alpha \in [i_\alpha, \lambda)$ and $\bar{y}_\alpha = \langle y_{\alpha,i} : i < \bar{i}_\alpha \rangle$ such that

$$\Vdash_{P_{\alpha,i}} \bar{y}_\alpha \subseteq [\bar{i}_\alpha]^{<\omega}$$

and for each $t \in T_\alpha$ with $\text{rk}_t(t) = \gamma$ we have

$$q^t_\alpha \Vdash_{P_{\alpha,i}} \bar{i}_\alpha = i_\alpha \text{ and } y_{\alpha,i} = \{c(t) : c \in u_{\alpha,i} \} \text{ for } i < i_\alpha \, \ldots$$

Now, both in $\exists_\alpha^R(G^*)$ of $V^*_\gamma$ and in $\exists_\beta^R(G^*)$ of $V$ the two players start a subgame. The length of the subgame in $V^*_\gamma$ may be longer than $i_\alpha$, but we will restrict our attention to the first $i_\alpha$ steps of that subgame. In our active case we will have $\bar{i}_\alpha = i_\alpha$, see the choice of $i_\alpha$ above. When playing the subgame, INC$^{V^*_\gamma}$ will build a sequence $\langle q^0_{\alpha,i}, q^1_{\alpha,i} : i < i_\alpha \rangle$ of trees of conditions in $Q$ such that (in addition to demands stated later):

$$\left(\circlearrowright_0^\alpha \right)^{\alpha,0}_{\alpha,i} = \langle q^0_{\alpha,i}, q^1_{\alpha,i} : t \in T_\alpha \rangle, \text{ for } \ell < 2, j < i < i_\alpha, \text{ and }$$

$$\left(\circlearrowright_0^\alpha \right)^{\alpha,0}_{\alpha,i} \text{ for each } t \in T_\alpha, \text{ the sequence } \langle q^0_{\alpha,i}, q^1_{\alpha,i} : i < i_\alpha \rangle \text{ is a legal play of the game }$$

$$\exists_0^\alpha (P_{\alpha,i}(t), q^t_\alpha) \text{ in which Complete uses her winning strategy } st_\gamma (r_{\alpha,i}(t), q^t_\gamma).$$

He (as INC$^{V^*_\gamma}$) will also construct a name for a play of a subgame of $\exists_\alpha^R(G^*)$ of $V^*_\gamma$ for this stage.

Suppose that INC$^{V^*_\gamma}$ and COM$^{V^*_\gamma}$ have arrived to a stage $i < i_\alpha$ of the subgame and INC$^{V^*_\gamma}$ has determined on the side $q^0_{\alpha,i}$ for $j < i$, $\ell < 2$ and a $P_{\gamma}$-name $(s^\gamma_j : j < i)$ for a partial play of the subgame of $\exists_\alpha^R(G^*)$ of $V^*_\gamma$. Now COM$^{V^*_\gamma}$ chooses $r_{\alpha,i} \in G^*$ which INC$^{V^*_\gamma}$ passes to INC$^{V^*_\gamma}$ as an inning of COM$^{V^*_\gamma}$ at the $i$th step of the subgame of level $\alpha$ of $\exists_\alpha^R(G^*)$ in $V^*_\gamma$. There the strategy $st_\gamma$ gives INC$^{V^*_\gamma}$ an answer $\delta_{\alpha,i} = \lambda$.

Next, INC$^{V^*_\gamma}$ picks a tree of conditions $q^0_{\alpha,i} = \langle q^0_{t,i} : t \in T_\alpha \rangle$ in $Q$ such that

$$\left(\circlearrowright_0^\alpha \right)^{\alpha,0}_{\alpha,i} \text{ for every } t \in T_\alpha \text{ with } \text{rk}_t(t) = \gamma, \text{ the condition } q^0_{t,i} \text{ decides the value of } \delta_{\alpha,i},$$

say $q^0_{t,i} \Vdash_{P_{\alpha,i}} \delta_{\alpha,i} = \delta^t_{\alpha,i}$. Then INC$^{V^*_\gamma}$ lets

$$\delta^t_{\alpha,i} = \sup \left( \{ q^0_{t,i} : t \in T_\alpha \text{ & rk}_t(t) = \gamma \} \cup \bigcup_{\beta < \alpha, j < \alpha} \left( Z_{\beta,j} \cup \bigcup_{j < i} Z_{\alpha,j} \right) \right) + 890$$

and in the subgame of $\exists_\beta^R(G^*)$ (in $V$) he is instructed to put $r'_{\alpha,i}$ such that

$$C^{\alpha,i} = C^{\alpha,i} \setminus \delta^t_{\alpha,i} \text{ and } d^t_{\beta,i} = d^t_{\beta,i} \text{ for } \beta \in C^{\alpha,i}.$$  

(Note that $r'_{\alpha,i} \in G^*$ by 1.11(2)(ii).)

After this COM$^{V^*_\gamma}$ chooses $(\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i}) \in \#(r'_{\alpha,i})$, so $\beta_{\alpha,i} \in C^{\alpha,i}$, $\beta_{\alpha,i} = \delta_{\alpha,i}$ and $d_{\alpha,i} = d^t_{\beta,i}$. Next INC$^{V^*_\gamma}$ lets

- $q^0_{t,i} \vDash_{P_{\alpha,i}} \gamma = (r_{\alpha,i}, \delta_{\alpha,i}, (\beta_{\alpha,i}, Z_{\alpha,i}, d_{\alpha,i})).$
Then the subgame continues.

After all $i_{α}$ steps of the subgame are completed, INC$^V$ chooses a tree of conditions $\bar{q}_{α} = (q_{α}^i : t ∈ T_α)$ in $\bar{Q}$ such that $(∀i < i_{α})(q_{α}^i ≤ \bar{q}_{α})$ and he also lets $\bar{θ}_{α}$ be a $P_γ$-name for the result of the subgame of level $α$ of $\check{D}_α^{P}(G^*)$ in $V^{P_{γ}}$ such that $\bar{θ}_{α}↾i_{α} = (\bar{q}_{α}^i : i < i_{α})$. Note that all the objects described by $(\oplus)^{i+1}_{α}, (\oplus)^{i+1}_{j}$ are determined now.

This completes the description of the strategy $st^{P}$ of INC (i.e., INC$^V$) in $\check{D}_α^{P}(G^*)$. Since $G^*$ is $κ$–super reasonable, this strategy cannot be a winning one, so there is a play

$(\oplus)_{π} (I_{α}, i_{α}, u_{α}, (r_{α,i}, r_{α,i}', (β_{α,i}, Z_{α,i}, d_{α,i}) : i < i_{α}) : α < λ)$

of $\check{D}_α^{P}(G^*)$ in which INC follows $st^{P}$, but

$(\oplus)_{s}$ for some $r ∈ G^*$, for every $(j_{α} : α < λ) ∈ \prod_{α < λ} I_{α}$ we have

$\{(β_{α,i}, Z_{α,i}, d_{α,i}) : α < λ & i < i_{α} & j_{α} ∈ u_{α,i}\} ≤ ^* #(r)$. 

Exactly as in the proof of Theorem 3.2 we may argue that then also

$(\oplus)_{10}$ it is forced in $P_γ$ that

$$(\forall j \in \prod_{α < λ} I_{α})\{(β_{α,i}, Z_{α,i}, d_{α,i}) : α < λ & i < i_{α} & j_{α} ∈ u_{α,i}\} ≤ ^* #(r)).$$

(See $(\oplus)_{10}$ in the proof of 3.2.)

Let $⟨T_α, p^α, q^α : α < λ⟩$ be the play of $D^{treeA}_α(p, \bar{Q})$ constructed on the side by INC. Generic won that play, so there is a condition $p^* ∈ P_γ$ stronger than $p$ and such that for each $α < λ$ the set $\{q^α : t ∈ T_α & rk(t) = \gamma\}$ is pre-dense above $p^*$. Also, let $g_λ$ be the $P_γ$-name of a play of $D^{P}(G^*)$ (in $V^{P_{γ}}$) constructed on the side in the same run of $D^{P}(G^*)$ (see $(\oplus)_{2}$). We are going to argue that

$(\oplus)_{10}$ the condition $p^*$ forces (in $P_γ$) that

$$(\forall j \in \prod_{α < λ} I_{α})\{(β_{α,i}, Z_{α,i}, d_{α,i}) : α < λ & i < i_{α} & j_{α} ∈ u_{α,i}\} ≤ ^* #(r),$$

that is

$p^* \Vdash _{P_γ} " COM^{V_{P_γ}} wins the play g_λ as witnessed by r "$.

Suppose that $G ⊆ P_γ$ is generic over $V$, $p^* ∈ G$ and let us work in $V[G]$. For every $α < λ$ there is a unique $t = t(α) ∈ T_α$ such that $rk_α(t) = \gamma$ and $q^α ∈ G$, and thus $(I_{α})^G = I_{α}$, $(j_{α})^G = i_{α}$ and $(y_{α})^G = \gamma$. Suppose that $j = (j_{α} : α < λ) ∈ \prod_{α < λ} I^G_{α}$. For each $α < λ$ fix $j^*_{α} ∈ I_{α} = \prod_{t ∈ T_α & rk(t) = \gamma} I_{α}^t$ such that $j^*_{α}(t(α)) = j_{α}$. Note that if $j^*_{α} ∈ u_{α,i}, i < i_{α}$, then $j_{α} ∈ (y_{α,i})^G$ and therefore

$\{(β_{α,i}, Z_{α,i}, d_{α,i}) : α < λ & i < i_{α} & j_{α} ∈ (y_{α,i})^G\} ≤ ^*\{(β_{α,i}, Z_{α,i}, d_{α,i}) : α < λ & i < i_{α} & j_{α}^* ∈ u_{α,i}\} ≤ ^* #(r)$

(remember $(\oplus)_{9}$). Now $(\oplus)_{10}$ follows and the proof of the theorem is complete. □
Corollary 3.4. Assume that $\lambda$ is a strongly inaccessible cardinal. Then there is a forcing notion $P$ such that

- $\lambda$ is strongly inaccessible and $2^\lambda = \lambda^{++}$ and there is a strongly reasonable family $G^* \subseteq \mathcal{Q}^\lambda_\lambda$ such that $\text{fil}(G^*)$ is an ultrafilter on $\lambda$ and $|G^*| = \lambda^+$, in particular there is a very reasonable ultrafilter on $\lambda$.

Prove. We may start with a universe $V$ in which $\diamondsuit_{S_{\lambda^+}}$ holds (and $\lambda$ is strongly inaccessible). It follows from 1.15 that (in $V$) there is a $\leq^0_*$-increasing sequence $\langle r_\alpha : \alpha < \lambda \rangle \subseteq \mathcal{Q}^\lambda_\lambda$ such that $G^* \overset{\text{def}}{=} \{ r \in \mathcal{Q}^\lambda_\lambda : (\exists \alpha < \lambda^+)(r \leq^0 r_\alpha) \}$ is super reasonable and $\text{fil}(G^*)$ is an ultrafilter on $\lambda$.

Let $\mathcal{Q} = \langle P_\alpha, Q_\alpha : \alpha < \lambda^+ \rangle$ be a $\lambda$-support iteration of the forcing notion $\mathcal{Q}^\lambda_{\text{tree}}(K_1, \Sigma_1)$ defined in the proof of [9, Prop. B.8.5]. This forcing is reasonably $A$-bounding (by [8, Prop. 4.1, p. 221] and [9, Thm B.6.5]), so we may use Theorems 3.2 and 3.3 to conclude that $\text{fil}(G^*)$ is an ultrafilter on $\lambda$.

If one analyzes the proof of Theorem 3.3, one may notice that even $\text{fil}(G^*)$ is not an ultrafilter anymore.

If one analyzes the proof of Theorem 3.3, one may notice that even $\text{fil}(G^*)$ is not an ultrafilter anymore.

4. A feature, not a bug

One may wonder if Theorems 3.2, 3.3 could be improved by replacing the assumption that we are working with the iteration of reasonably $A$-bounding forcings by, say, just dealing with a nicely double $a$-bounding forcing. A result of that sort would be more natural and the fact that we had to refer to an iteration-specific property could be seen as some lack of knowledge. However, this is a feature, not a bug as nicely double $a$-bounding forcing notions may cause that $\text{fil}(G^*)$ is not an ultrafilter anymore.

In this section we assume that $\lambda$ is a strongly inaccessible cardinal.

Definition 4.1. (1) Let $P^*$ consist of all pairs $p = (\eta^p, C^p)$ such that $\eta^p : \lambda \rightarrow \{-1, 1\}$ and $C^p$ is a club of $\lambda$. A binary relation $\leq = \leq_{P^*}$ on $P^*$ is defined by letting $p \leq q$ if and only if

- $C^q \subseteq C^p$, $\eta^q|\min(C^p) = \eta^p|\min(C^p)$, and
- for every successive members $\alpha < \beta$ of $C^p$ we have

$$\forall \gamma \in [\alpha, \beta) \left( \eta^p(\gamma) = \frac{\eta^p(\alpha)}{\eta^p(\alpha)} \cdot \eta^p(\gamma) \right).$$

(2) For $p \in P^*$ and $\alpha \in C^p$ let

$$\text{pos}(p, \alpha) \overset{\text{def}}{=} \{ q \in P^* : p \leq q \}.$$

(3) For $p \in P^*$, $\alpha < \lambda$ and $\nu : \alpha \rightarrow \{-1, 1\}$ we define

$$\nu \ast_\alpha p = (\nu \overset{\sim}{\cdot} \eta^p|[\alpha, \lambda), C^p \setminus \alpha).$$

(Plainly, $\nu \ast_\alpha p \in P^*$.)
Proposition 4.3. Let $\mu = (\mu_\alpha : \alpha < \lambda)$, $\mu_\alpha = 2^{[\alpha]^{<\aleph_0}}$ (for $\alpha < \lambda$). Then $\mathbb{P}^*$ is a nicely double a-bounding over $\mu$ forcing notion. Also $|\mathbb{P}^*| = 2^\lambda$.

Proof. One easily verifies that the relation $\leq_{\mathbb{P}^*}$ is transitive and reflexive, also plainly $|\mathbb{P}^*| = 2^\lambda$.

Claim 4.3.1. $\mathbb{P}^*$ is $\langle \lambda \rangle$-complete.

Proof of the Claim. Suppose that $\delta < \lambda$ and $\langle p_\xi : \xi < \delta \rangle$ is a $\leq_{\mathbb{P}^*}$-increasing sequence of conditions in $\mathbb{P}^*$. Let $C = \bigcap C_{\mathcal{P}^\omega}$ (it is a club of $\lambda$) and let $\eta : \lambda \rightarrow \{-1,1\}$ be defined by

- if $\gamma < \min(C)$ and $\zeta = \min(\varepsilon < \delta : \gamma < \min(C_{\mathcal{P}^\omega}))$,
  then $\eta(\gamma) = \eta^\mathcal{P}(\gamma)$,
- if $\alpha < \beta$ are successive members of the club $C$, $\alpha \leq \gamma < \beta$ and $\zeta = \min(\varepsilon < \delta : \gamma < \min(C_{\mathcal{P}^\omega}) \setminus (\alpha + 1)))$, then $\eta(\gamma) = \eta^\mathcal{P}(\alpha) \cdot \eta^\mathcal{P}(\gamma)$.

Plainly, $\eta$ is well defined and $q = \eta(C) \in \mathbb{P}^*$. We claim that $(\forall \xi < \delta)(p_\xi \leq q)$. To this end suppose $\xi < \delta$. Clearly $C \subseteq C_{\mathcal{P}^\omega}$. Now, if $\gamma < \min(C_{\mathcal{P}^\omega})$, then $\eta(\gamma) = \eta^\mathcal{P}(\gamma)$ for some $\xi \leq \xi$ such that $\gamma < \min(C_{\mathcal{P}^\omega})$. Since $p_\xi \leq p_\xi$, we have $\eta^\mathcal{P}(\gamma) = \eta^\mathcal{P}(\gamma)$ and thus $\eta^\mathcal{P}(\gamma) = \eta(\gamma)$.

Next, suppose that $\alpha < \beta$ are successive members of $C_{\mathcal{P}^\omega}$ and $\alpha \leq \gamma < \beta$. If $\gamma < \min(C)$ and $\zeta = \min(\varepsilon < \delta : \gamma < \min(C_{\mathcal{P}^\omega}))$, then $\zeta > \xi$, $\eta(\alpha) = \eta^\mathcal{P}(\alpha)$ and

$$(*)^1 \quad \eta(\gamma) = \eta^\mathcal{P}(\gamma) = \frac{\eta^\mathcal{P}(\alpha)}{\eta(\alpha)} \cdot \eta^\mathcal{P}(\gamma) = \frac{\eta^\mathcal{P}(\alpha)}{\eta(\alpha)} \cdot \eta^\mathcal{P}(\gamma).$$

So assume $C \cap \beta \neq \emptyset$ and let $\alpha' < \beta'$ be successive members of $C$ such that $\alpha' \leq \alpha \leq \gamma < \beta'$. Let $\zeta = \min(\varepsilon < \delta : \gamma < \min(C_{\mathcal{P}^\omega}) \setminus (\alpha' + 1)))$. If $\alpha = \alpha'$, then $\zeta \leq \xi$ and

$$(*)^2 \quad \eta(\gamma) = \eta^\mathcal{P}(\alpha) \cdot \eta^\mathcal{P}(\gamma) = \eta^\mathcal{P}(\alpha) \cdot \frac{\eta^\mathcal{P}(\alpha)}{\eta(\alpha)} \cdot \eta^\mathcal{P}(\gamma) = \eta^\mathcal{P}(\alpha) \cdot \eta^\mathcal{P}(\gamma),$$

(as $\eta(\alpha) = \eta(\alpha') = 1$). If $\alpha' < \alpha$, then $\xi < \zeta$ and $\eta(\alpha) = \eta^\mathcal{P}(\alpha') \cdot \eta^\mathcal{P}(\alpha)$, and hence

$$(*)^3 \quad \eta(\gamma) = \eta^\mathcal{P}(\alpha') \cdot \eta^\mathcal{P}(\gamma) = \frac{\eta(\alpha)}{\eta^\mathcal{P}(\alpha') \cdot \eta^\mathcal{P}(\gamma)} \cdot \eta^\mathcal{P}(\gamma) = \frac{\eta(\alpha)}{\eta^\mathcal{P}(\alpha) \cdot \eta^\mathcal{P}(\gamma)} \cdot \eta^\mathcal{P}(\gamma).$$

Clearly $(*^1) - (*^3)$ are what we need to justify 4.1(1) and conclude $p_\xi \leq q$.

Claim 4.3.2. Let $p \in \mathbb{P}^*$. Then Generic has a nice winning strategy in the game $\mathcal{G}_{\mathcal{P}^\omega}^\mathbb{P}(p, \mathbb{P}^*)$ (see Definition 2.9).

Proof of the Claim. We will describe a strategy $\text{st}$ for Generic in $\mathcal{G}_{\mathcal{P}^\omega}^\mathbb{P}(p, \mathbb{P}^*)$. Whenever we say Generic chooses $x$ such that we mean Generic chooses the $\langle \lambda \rangle$-first $x$ such that (and likewise for other variants).

During a play of $\mathcal{G}_{\mathcal{P}^\omega}^\mathbb{P}(p, \mathbb{P}^*)$ Generic constructs on the side sequences $(p_\alpha : \alpha < \lambda)$ and $\delta = \langle \delta_\alpha : \alpha < \lambda \rangle$ so that for each $\alpha < \lambda$:

a) $\delta$ is a strictly increasing continuous sequence of ordinals below $\lambda$, $p_\alpha \in \mathbb{P}^*$ and $(\delta_\xi : \xi \leq \omega + \alpha) = C^{p_\alpha} \cap (\delta_{\omega + \alpha} + 1)$,

b) if $\beta < \alpha$, then $p_{\delta_\beta} \leq p_\alpha$ and $\eta^\mathcal{P}_{\beta} \cdot \eta^\mathcal{P}_{\delta_{\omega + \beta}} = \eta^\mathcal{P}_{\delta_{\omega + \beta}}$,

c) $\{\delta_\xi : \xi \leq \omega \} = \{\delta \in C^\omega : \text{otp} (\delta \cap C^\omega) \leq \omega\}$ and $p_0 = p$. 

Remark 4.2. $\mathbb{P}^*$ is a natural generalization of the forcing notion used by Goldstern and Shelah [4] to the context of uncountable cardinals.
(d) δ_{ω+α+1} and p_{α+1} are determined right after stage α of \( O_{\mu}^{2n}(p, \mathbb{P}^*) \).

So suppose that the two players have arrived to a stage \( α < \lambda \) of a play of \( O_{\mu}^{2n}(p, \mathbb{P}^*) \), and Generic has constructed on the side \( δ_{ω+β+1} \) and \( p_{β+1} \) for \( β < α \). If \( α = 0 \) or \( α \) is a limit ordinal, then conditions (a)–(e) and our rule of taking “the \( \prec \)–first” fully determine \( \{ δ_ξ : ξ ≤ ω + α \} \) and \( p_α \) (the suitable bounds exists essentially by 4.3.1).

Now Generic chooses an enumeration (without repetition) \( \bar{ρ} = \{ ρ_0^α : j < μ_α \} \) of pos(\( p_α, δ_{ω+α} \)) such that \( ρ_0^α = ρ_0^0 [δ_{ω+α} \). Antigeneric picks a non-zero ordinal \( ξ_α < \lambda \) and the two players start a subgame of length \( μ_α \cdot ξ_α \). In the course of the subgame, in addition to her innings \( p_0^{α} \), Generic will also choose ordinals \( ε_0^α = ε_γ < \lambda \) and sequences \( ϕ_0^{α} = ϕ_γ : ε_γ → \{-1, 1\} \). These objects will satisfy the following demands (letting \( q_0^α \) be the innings of Antigeneric):

\begin{enumerate}
  \item \( δ_{ω+α} < ε_γ < ε_γ ∈ C_0^0 \) and \( ϕ_γ [\{ δ_{ω+α}, ε_γ \} = \varphi_γ [\{ δ_{ω+α}, ε_γ \} \) for \( γ' < γ < μ_α \cdot ξ_α \),
  \item if \( γ = μ_α \cdot i + 2j, i < ξ_α \) and \( j < μ_α \), then
    \begin{enumerate}
      \item \( ρ_0^α < φ_γ < φ_γ+1, \ \varphi_γ = η^φ_γ [ε_γ, \ \text{and} \ \varphi_γ+1(δ) = -η^φ_γ+1(δ) \) for \( δ ∈ \{ δ_{ω+α}, ε_γ+1 \} \),
      \item \( p_0^α ≥ p_0^α * δ_{ω+α}, p_α, \ \min(C_0^0) > δ_{ω+α}, \ \text{and} \ (φ_γ | ε_γ) * ε_γ, q_0^α ≤ p_0^α \) for \( γ' < γ \),
      \item \( q_0^α ≤ φ_γ * ε_γ, p_0^α+1 ≤ φ_γ+1 * ε_γ+1, q_0^α \).
    \end{enumerate}
\end{enumerate}

So suppose that the two players have arrived to a stage \( γ = μ_α \cdot i + 2j \) \( (i < ξ_α, j < μ_α) \) of the subgame and \( p_0^α, q_0^α, φ_γ, ε_γ \) have been determined for \( γ' < γ \). Let \( φ = ρ_j^α ← ∪ \varphi_γ [\{ δ_{ω+α}, ε_γ \} \). It follows from (f) that the sequence \( ((φ | ε_γ) * ε_γ, q_0^α) : γ' < γ \) is \( ≤ψ_0 \)-increasing, so Generic may choose an upper bound \( p_0^α ∈ \mathbb{P}^* \) to it.

(Note that necessarily \( φ < η^φ_γ, \ \sup(φ_γ : γ' < γ) ≤ \min(C_0^0) \). She plays \( p_0^α \) in the subgame and Antigeneric answers with \( q_0^α ≥ p_0^α \). Now Generic lets \( ε_γ ∈ C_0^0 \) be such that \( | C_0^0 \cap ε_γ | = 1 \) and she puts \( φ_γ = η^φ_γ [ε_γ \) and she lets \( ψ = ε_γ → \{-1, 1\} \) be defined by \( ψ [δ_{ω+α} = ρ_j^α \) and \( ψ(δ) = -ϕ_γ(δ) \) for \( δ ∈ \{ δ_{ω+α}, ε_γ \} \). Then Generic plays \( p_{γ+1} = ψ * ε_γ, q_0^α \) as her inning at stage \( γ + 1 \) of the subgame and Antigeneric answers with \( q_0^α+1 ≥ p_0^α+1 \). Finally, Generic picks \( ε_γ+1 ∈ C_0^0+1 \) such that \( | C_0^0+1 \cap ε_γ+1 | = 1 \) and she takes \( φ_γ+1 : ε_γ+1 → \{-1, 1\} \) such that \( φ_γ < φ_γ+1 \) and \( φ_γ+1(δ) = -η^φ_γ+1(δ) \) for \( δ ∈ [ε_γ, ε_γ+1) \). Plainly, if \( i' < i, \ \gamma' = μ_α \cdot i' + 2j \) then \( q_0^α ≤ p_0^α \) and \( q_0^α+1 ≤ p_0^α+1 \) so both \( p_0^α \) and \( p_0^α+1 \) are legal innings in \( O_{\mu}^{2n}(p, \mathbb{P}^*) \). Also easily the demands in (e)–(f) are satisfied. Moreover, if \( j' < j < μ_α \), then the conditions \( p_0^α+j \) and \( p_0^α+j' \) are incompatible.

After the subgame is over, Generic lets

\( φ = ρ_0^α ← ∪ \{ φ_γ [\{ δ_{ω+α}, ε_γ \} : γ < μ_α \cdot ξ_α \}
\)

and she picks a \( ≤ψ_0 \)-upper bound \( p_{α+1}^0 \) to the increasing sequence

\( \{ φ_γ [ε_γ) * ε_γ, q_0^α : γ < μ_α \cdot ξ_α \} \).

Note that \( ε_γ ≤ \min(C_0^0+1) \) and \( φ_γ [ε_γ < η^φ_γ+1 \) for all \( γ < μ_α \cdot ξ_α \), so also \( ρ_0^α = η^φ_0 [δ_{ω+α} < η^φ_0^α+1 \). Also

\begin{enumerate}
  \item \( η^φ_0+1 [ε_γ) * ε_γ, q_0^α ≤ p_{α+1}^0 \) for all \( γ < μ_α \cdot ξ_α \).
\end{enumerate}
Let \( p_{\alpha + 1} \in \mathbb{P}^* \) be such that \( C^{p_{\alpha + 1}} = \{ \xi : \xi \leq \omega + \alpha \} \cup C^{p_{\alpha + 1}} \) and \( \eta^{p_{\alpha + 1}} = \eta^{p_{\alpha + 1}} \) (plainly \( p_\alpha \leq p_{\alpha + 1} \)) and let \( \delta_{\omega + \alpha + 1} = \min\{C^{p_{\alpha + 1}}\} \).

This finishes the description of the strategy \( \text{st} \). Let us argue that \( \text{st} \) is a winning strategy for Generic. To this end suppose that

\[
(\mathbb{H}) \quad (\xi_\alpha, \langle p^\alpha_\gamma, q^\alpha_\gamma : \gamma < \mu_\alpha : \xi_\alpha : \alpha < \lambda \rangle)
\]
is a result of a play of \( \mathcal{O}_{\mathbb{P}} 2^n(p, \mathbb{P}^*) \) in which Generic follows \( \text{st} \) and the objects constructed on the side are

\[(\mathbb{H}^*)_\alpha p^\alpha_{\alpha}, p^\alpha_{\alpha}, \delta_\xi, \langle \varepsilon^\alpha_\gamma, \varphi^\alpha_\gamma : \gamma < \mu_\alpha : \xi_\alpha : \alpha < \lambda \rangle, (p^\alpha_j : j < \mu_\alpha) \]

(and the demands in (a)–(g) are satisfied). Let \( C = \{ \delta_\xi : \xi < \lambda \} \) (so it is a club of \( \lambda \)) and \( \eta = \bigcup_{\alpha < \lambda} \eta^{p_\alpha} |_{\delta_\omega + \alpha} \) (clearly \( \eta : \lambda \rightarrow \{-1, 1\} \); remember (b)), and let \( p^* = (\eta, C) \). It is a condition in \( \mathbb{P}^* \) and it is stronger than all \( p_\alpha \) (for \( \alpha < \lambda \)) so also \( p^* \geq p \). Suppose that \( \alpha < \lambda \) and \( p'' \geq p' \). We will show that there is \( p'' \geq p' \) such that for some \( j < \mu_\alpha \), the condition \( p'' \) is stronger than all \( q^\alpha_{\alpha + i} \) \( j \) for all \( i < \xi_\alpha \). Without loss of generality, \( \min(C^{p''}) \geq \delta_{\omega + \alpha + 1} \). Let \( j' < \mu_\alpha \) be such that \( \eta^{p''} |_{\delta_{\omega + \alpha}} = p^{j'}_\alpha \).

We consider two cases now.

**Case 1:** \( \eta^{p''} |_{\delta_{\omega + \alpha}} = p^{j'}_\alpha = \eta^{p_\alpha} |_{\delta_{\omega + \alpha + 1}} \).

Then \( \eta^{p''} |_{\delta_{\omega + \alpha + 1}} = \eta^{p_\alpha} |_{\delta_{\omega + \alpha + 1}} \). Let \( j = 2 \cdot j' < \mu_\alpha \), and we will argue that \( q^\alpha_{\alpha + i + j} \leq p' \) for all \( i < \xi_\alpha \). So let \( i < \xi_\alpha \), \( \gamma = \mu_\alpha \cdot i + j \). By the choice of \( j' \) we know that \( \eta^{p''} |_{\delta_{\omega + \alpha}} = p^{j'}_\alpha = \eta^{p_\alpha} |_{\delta_{\omega + \alpha + 1}} \) and also

\[
\varphi^\alpha_\gamma |_{\delta_{\omega + \alpha}, \varepsilon^\alpha_\gamma} = \eta^{p_\alpha} |_{\delta_{\omega + \alpha}, \varepsilon^\alpha_\gamma} = \eta^{p''} |_{\delta_{\omega + \alpha}, \varepsilon^\alpha_\gamma}.
\]

Hence (by (f)(i)) \( \eta^{p''} |_{\varepsilon^\alpha_\gamma} = \eta^{p_\alpha} |_{\varepsilon^\alpha_\gamma} \) and now

\[
\begin{align*}
q^\alpha_\gamma & \leq (\eta^{p''} |_{\varepsilon^\alpha_\gamma}) *_{\varepsilon^\alpha_\gamma} q^\alpha_\gamma \leq (\eta^{p''} |_{\delta_{\omega + \alpha}}) *_{\delta_{\omega + \alpha + 1}} p^{j'}_\alpha = \\
(\eta^{p''} |_{\delta_{\omega + \alpha + 1}}) *_{\delta_{\omega + \alpha + 1}} p^{j'}_\alpha & = (\eta^{p''} |_{\delta_{\omega + \alpha + 1}}) *_{\delta_{\omega + \alpha + 1}} p^{j'}_\alpha \leq (\eta^{p''} |_{\delta_{\omega + \alpha + 1}}) *_{\delta_{\omega + \alpha + 1}} p'' = p'.
\end{align*}
\]

(for the second inequality remember (g)).

**Case 2:** \( \eta^{p''} |_{\delta_{\omega + \alpha}} = -\eta^{p''} |_{\delta_{\omega + \alpha}} = -\eta^{p_\alpha} |_{\delta_{\omega + \alpha}} \).

Then \( \eta^{p''} (\delta) = -\eta^{p''} (\delta) = -\eta^{p_\alpha} (\delta) \) for all \( \delta \in [\delta_{\omega + \alpha}, \delta_{\omega + \alpha + 1}] \). Let \( j = 2 \cdot j' + 1 \) and let us argue that \( q^\alpha_{\alpha + i + j} \leq p' \) for all \( i < \xi_\alpha \). So let \( i < \xi_\alpha \), \( \gamma = \mu_\alpha \cdot i + j \). Like in the previous case we show that \( \eta^{p''} |_{\varepsilon^\alpha_\gamma} = \eta^{p_\alpha} |_{\varepsilon^\alpha_\gamma} \) and then easily

\[
q^\alpha_\gamma \leq (\eta^{p''} |_{\varepsilon^\alpha_\gamma}) *_{\varepsilon^\alpha_\gamma} q^\alpha_\gamma \leq (\eta^{p''} |_{\delta_{\omega + \alpha + 1}}) *_{\delta_{\omega + \alpha + 1}} p^{j'}_\alpha \leq (\eta^{p''} |_{\delta_{\omega + \alpha + 1}}) *_{\delta_{\omega + \alpha + 1}} p'.
\]
Corollary 4.5. Assume \( \lambda \) is a strongly inaccessible cardinal. Then there is a forcing notion \( \mathbb{P} \) such that
\[
\vdash_{\mathbb{P}} " \lambda \text{ is strongly inaccessible and } 2^\lambda = \lambda^{++} \text{ and there is no very reasonable ultrafilter on } \lambda \text{ with a generating system of size } < 2^\lambda "
\]

Proof. We may start with the universe \( V \) in which \( 2^\lambda = \lambda^+ \).

Let \( \mathbb{Q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \lambda^{++} \rangle \) be a \( \lambda \)-support iteration of the forcing notion \( \mathbb{P}^* \) (see Definition 4.1). This forcing is nicely double \( a \)-bounding over \( \bar{\mu} \) (where \( \mu_\alpha = 2^{2^{\alpha+\aleph_0}} \); remember Proposition 4.3) and hence \( \mathbb{P}_{\lambda^{++}} \) is nicely double \( a \)-bounding over \( \bar{\mu} \) (by Theorem 2.13). Using Theorem 2.2 we conclude that \( \mathbb{P}_{\lambda^{++}} \) does not collapse any cardinals and forces that \( 2^\lambda = \lambda^{++} \). Proposition 4.4 implies that
\[
\vdash_{\mathbb{P}_{\lambda^{++}}} " \text{for no family } G^* \subseteq \mathbb{Q}_0^\mathbb{P} \text{ of size } < 2^\lambda, \text{fil}(G^*) \text{ is an ultrafilter on } \lambda ".
\]

Problem 4.6. (1) Is it consistent that for some uncountable regular cardinal \( \lambda \) we have that there is no super-reasonable ultrafilter on \( \lambda \)? Or even no very reasonable one?

(2) In particular, are there super-reasonable ultrafilters on \( \lambda \) in the model constructed for Corollary 4.5?

(3) Do we need the inaccessibility of \( \lambda \) for the assertions of Corollaries 3.4, 4.5 concerning ultrafilters on \( \lambda \)?

References


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