WINNING THE PRESSING DOWN GAME BUT NOT BANACH MAZUR

JAKOB KELLNER*, MATTI PAUNA, AND SAHARON SHELAH†

Abstract. Let S be the set of those $\alpha \in \omega_2$ that have cofinality ω_1. It is consistent relative to a measurable that the nonempty player wins the pressing down game of length ω_1, but not the Banach Mazur game of length $\omega+1$ (both games starting with S).

1. Introduction

We set $E^\kappa_\theta = \{\alpha \in \kappa : \text{cf}(\alpha) = \theta\}$. Let S be a stationary set. We investigate two games, each played by players called "empty" and "nonempty". Empty has the first move.

In the Banach Mazur game BM(S) of length $< \theta$, the players choose decreasing stationary subsets of S. Empty wins, if at some $\alpha < \theta$ the intersection of these sets is nonstationary. (Exact definitions are give in the next section.)

In the pressing down game PD(S), empty cannot choose a stationary subset of the moves so far, but only a regressive function. Nonempty chooses a homogeneous stationary subset.

So it is at least as hard for nonempty to win BM as to win PD.

In this paper, we show that BM can be really harder than PD. This follows easily from well known facts about precipitous ideals (cf. 2.4 for a more detailed explanation): Nonempty can never win $\text{BM}_{\leq\omega}(\omega_2)$, but it is consistent (relative to a measurable) that nonempty wins $\text{PD}_{<\omega_1}(\omega_2)$. The reason is the following: In BM, empty can first choose $E^{\omega_2}_{\omega}$, and empty always wins on this set. However in PD, it is enough for nonempty to win on $E^{\omega_2}_{\omega_1}$. In a certain way this is "cheating", since nonempty wins PD on $E^{\omega_2}_{\omega_1}$ but looses BM on the disjoint set $E^{\omega_2}_{\omega}$. So in a way the difference arises because empty has the first move in BM.

Therefore, a better question is: Can nonempty win PD(S) but loose BM(S) even if nonempty gets the first move,[1] e.g. on $S = E^{\omega_2}_{\omega_1}$?

This is indeed the case:

Theorem 1.1. *It is consistent relative to a measurable that for $\theta = \aleph_1$ and $S = E^{\theta^+}_\theta$, nonempty wins $PD_{<\omega_1}(S)$ but not $BM_{\leq\omega}(S)$, even if nonempty gets the first move.*

The same holds for $\theta = \aleph_n$ (for $n \in \omega$) etc.

Date: 2007-01-23.

2000 *Mathematics Subject Classification.* 03E35;03E55.

* supported by a European Union Marie Curie EIF fellowship, contract MEIF-CT-2006-024483.

† supported by the United States-Israel Binational Science Foundation (Grant no. 2002323), publication 896.

[1] Which is equivalent to: nonempty does not win $\text{BM}_{\leq\omega}(S')$ for any stationary $S' \subseteq S$.

Various aspects of these and related games have been studied for a long time.

Note that in this paper we consider the games on sets, i.e. a move is an element of the powerset of κ minus the (nonstationary) ideal. A popular (closely related but not always equivalent) variant are games on a Boolean algebra B: Moves are elements of B, in our case B would be the powerset of κ *modulo* the ideal.

Also note that in Banach Mazur games of length greater than ω, it is relevant which player moves first at limit stages (in our definition this is the empty player). Of course it is also important who moves first at stage 0 (in this paper again the empty player), but the difference here comes down to a simple density effect (cf. 2.1.4).

The Banach Mazur BM game has been investigated e.g. in [5] or [15]. It is closely related to the so-called "ideal game" and to precipitous ideals, cf. Theorem 2.3 and [7], [1], or [4]. BM is also related to the "cut & choose game" of [8].

The pressing down game is related to the Ehrenfeucht-Fraïssé game in model theory, cf. [13] or [3], and has applications in set theory as well [12].

Other related games have been studied e.g. in [9] or [14].

We thank Jouko Väänänen for asking about Theorem 1.1 and for pointing out Theorem 4.1.

2. Banach Mazur, pressing down, and precipitous ideals

Let κ and θ be regular, $\theta < \kappa$.

We set $E^\kappa_\theta = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \theta\}$. $\mathcal{E}^\kappa_\theta$ is the family of stationary subsets of E^κ_θ. Analogously for $E^\kappa_{>\theta}$ etc.

Instead of "the empty player has a winning strategy for the game G" we just say "empty wins G" (as opposed to: empty wins a specific run of the game).

$\mathcal{I}$ denotes a fine, normal ideal on κ (which implies $<\kappa$-completeness).

A set $S \subseteq \kappa$ is called $\mathcal{I}$-positive if $S \notin \mathcal{I}$.

Definition 2.1. Let κ be regular, and $S \subseteq \kappa$ an $\mathcal{I}$-positive set.

- $\mathrm{BM}_{<\zeta}(\mathcal{I}, S)$, the *Banach Mazur game* of length ζ starting with S, is played as follows:

  At stage 0, empty plays an $\mathcal{I}$-positive $S_0 \subseteq S$, nonempty plays $T_0 \subseteq S_0$. At stage $\alpha < \zeta$, empty plays an $\mathcal{I}$-positive $S_\alpha \subseteq \bigcap_{\beta<\alpha} S_\beta$ (if possible), and nonempty plays some $T_\alpha \subseteq S_\alpha$.

  Empty wins the run, if $\bigcap_{\beta<\alpha} S_\beta \in \mathcal{I}$ at any stage $\alpha < \zeta$. Otherwise nonempty wins.

  (For nonempty to win a run, it is not necessary that $\bigcap_{\beta<\zeta} S_\beta$ is $\mathcal{I}$-positive or even just nonempty.)
- $\mathrm{BM}_{\leq\omega}(\mathcal{I}, S)$ is $\mathrm{BM}_{<\omega+1}(\mathcal{I}, S)$. So empty wins the run iff $\bigcap_{n<\omega} S_n \in \mathcal{I}$.
- $\mathrm{PD}_{<\zeta}(\mathcal{I}, S)$, the *pressing down game* of length ζ starting with S, is played as follows:

  At stage $\alpha < \zeta$, empty plays a regressive function $f_\alpha : \kappa \to \kappa$, and nonempty plays some f_α-homogeneous $T_\alpha \subseteq \bigcap_{\beta<\alpha} T_\beta$.

  Empty wins the run, if $T_\alpha \in \mathcal{I}$ for any $\alpha < \zeta$. Otherwise, nonempty wins.
- $\mathrm{PD}_{\leq\omega}(\mathcal{I}, S)$ is $\mathrm{PD}_{<\omega+1}(\mathcal{I}, S)$.
- $\mathrm{BM}_{<\zeta}(S)$ is $\mathrm{BM}_{<\zeta}(\mathrm{NS}, S)$, and $\mathrm{PD}_{<\zeta}(S)$ is $\mathrm{PD}_{<\zeta}(\mathrm{NS}, S)$ (where NS denotes the nonstationary ideal).

The following is trivial:

Facts 2.1. (1) *Assume $S \subseteq T$.*
- *If empty wins $BM_{<\zeta}(\mathcal{I}, S)$, then empty wins $BM_{<\zeta}(\mathcal{I}, T)$.*
- *If nonempty wins $BM_{<\zeta}(\mathcal{I}, T)$, then nonempty wins $BM_{<\zeta}(\mathcal{I}, S)$.*
- *If empty wins $PD_{<\zeta}(\mathcal{I}, T)$, then empty wins $PD_{<\zeta}(\mathcal{I}, S)$.*
- *If nonempty wins $PD_{<\zeta}(\mathcal{I}, S)$, then nonempty wins $PD_{<\zeta}(\mathcal{I}, T)$.*

(2) *Assume $\mathcal{I} \subseteq \mathcal{J}$, and $\mathcal{J}$ also is fine and normal.*
- *If empty wins $PD_{<\zeta}(\mathcal{I}, S)$, then empty wins $PD_{<\zeta}(\mathcal{J}, S)$.*
- *If nonempty wins $PD_{<\zeta}(\mathcal{J}, S)$, then nonempty wins $PD_{<\zeta}(\mathcal{I}, S)$.*

(3) *In particular, if nonempty wins $PD_{<\zeta}(\mathcal{I}, S)$, then nonempty wins $PD_{<\zeta}(S)$.*

(4) *Let BM' be the variant of BM where nonempty gets the first move (at stage 0 only). The difference between BM and BM' is a simple density effect:*
- *Empty wins $BM'_{<\zeta}(\mathcal{I}, S)$ iff empty wins $BM_{<\zeta}(\mathcal{I}, S')$ for all positive $S' \subseteq S$ iff empty has a winning strategy for BM with S as first move.*
- *Empty wins $BM_{<\zeta}(\mathcal{I}, S)$ iff empty wins $BM'_{<\zeta}(\mathcal{I}, S')$ for some positive $S' \subseteq S$.*
- *Nonempty wins $BM'_{<\zeta}(\mathcal{I}, S)$ iff nonempty wins $BM_{<\zeta}(\mathcal{I}, S')$ for some positive $S' \subseteq S$.*

(For 3, use that $\mathcal{I}$ is normal, which implies $\mathrm{NS} \subseteq \mathcal{I}$.)

We will use the following definitions and facts concerning precipitous ideals, as introduced by Jech and Prikry [7]. We will usually refer to Jech's *Millennium Edition* [6] for details.

Definition 2.2. Let $\mathcal{I}$ be a normal ideal on κ.

- Let V be an inner model of W. $U \in W$ is called a normal V-ultrafilter if the following holds:
  - If $A \in U$, then $A \in V$ and A is a subset of κ.
  - $\emptyset \notin U$, and $\kappa \in U$.
  - If $A, B \in V$ are subsets of κ, $A \subseteq B$ and $A \in U$, then $B \in U$.
  - If $A \in V$ is a subset of κ, then either $A \in U$ or $\kappa \setminus A \in U$.
  - If $f \in V$ is a regressive function on $A \in U$, then f is constant on some $B \in U$.

  (Note that we do not require iterability or amenability.)
- A normal V-ultrafilter U is wellfounded, if the ultrapower of V modulo U is wellfounded. In this case the transitive collapse of the ultrapower is denoted by $\mathrm{Ult}_U(V)$.
- Let $P_{\mathcal{I}}$ be the family of $\mathcal{I}$-positive sets ordered by inclusion. Since $\mathcal{I}$ is normal, $P_{\mathcal{I}}$ forces that the generic filter G is a normal V-ultrafilter (cf. [6, 22.13]). $\mathcal{I}$ is called precipitous, if $P_{\mathcal{I}}$ forces that G is wellfounded.
- The ideal game on $\mathcal{I}$ is played just like $\mathrm{BM}_{\leq\omega}(\mathcal{I}, \kappa)$, but empty wins iff $\bigcap_{n\in\omega} S_n$ is empty (as opposed to "in $\mathcal{I}$").

So if empty wins the ideal game, then empty wins $\mathrm{BM}_{\leq\omega}(\mathcal{I}, \kappa)$. And if nonempty wins $\mathrm{BM}_{\leq\omega}(\mathcal{I}, \kappa)$, then nonempty wins the ideal game.

Theorem 2.3. *Let $\mathcal{I}$ be a normal ideal on κ.*

(1) *(Jech, cf* [6, 22.21]*) $\mathcal{I}$ is not precipitous iff empty wins the ideal game. So in this case empty also wins $BM_{\leq\omega}(\mathcal{I}, \kappa)$.*

(2) *(cf.* [1]*) If $E^\kappa_\omega\notin\mathcal{I}$, then nonempty cannot win the ideal game, and empty wins*[2] *$PD_{\le\omega}(\mathcal{I},E^\kappa_\omega)$ and therefore also $BM_{\le\omega}(\mathcal{I},\kappa)$.*
(3) *(Jech, Prikry* [4]*, cf* [6, 22.33]*) If $\mathcal{I}$ is precipitous, then κ is measurable in an inner model.*
(4) *(Laver, see* [1] *or* [6, 22.33]*) Assume that U is a normal ultrafilter on κ. Let $\aleph_1\le\theta<\kappa$ be regular and let $Q=Levy(\theta,<\kappa)$ be the Levy collapse (cf lemma 6.1). In $V[G_Q]$, let $\mathcal{F}$ be the filter generated by U and $\mathcal{I}$ the corresponding ideal. Then $\mathcal{I}$ is normal, and the family of $\mathcal{I}$-positive sets has a $<\theta$-closed dense subfamily.*

So in particular it is forced that nonempty wins $BM_{<\theta}(\mathcal{I},S)$ for all $\mathcal{I}$-positive sets S (nonempty just has to pick sets from the dense subfamily), and therefore that nonempty wins $PD_{<\theta}(S)$ (cf 2.1.3).
(5) *(Magidor* [4]*, penultimate paragraph) One can modify this forcing to get a $<\theta$-closed dense subset of $\mathcal{E}^{\theta^+}_\theta$.*

So in particular, $\mathcal{E}^{\theta^+}_\theta$ can be precipitous.

Mitchell [4] showed that the Levy$(\omega,<\kappa)$ gives a precipitous ideal on ω_1 (and with Magidor's extension, NS_{ω_1} can be made precipitous). So the ideal game is interesting on ω_1, but our games are not:

Corollary 2.4. (1) *Empty always wins $PD_{\le\omega}(S)$ (and $BM_{\le\omega}(S)$) for $S\subseteq\omega_1$.*
(2) *It is equiconsistent with a measurable that nonempty wins $BM_{<\theta}(E^{\theta^+}_\theta)$ for e.g. $\theta=\aleph_1$, $\theta=\aleph_2$, $\theta=\aleph^+_{\aleph_7}$ etc.*
(3) *The following is consistent relative to a measurable: Nonempty wins $PD_{<\theta}(\theta^+)$ but not $BM_{\le\omega}(\theta^+)$ for e.g. $\theta=\omega_1$.*

Proof. (1) is just 2.3.2, and (2) follows from 2.3.3–4.

(3) Let κ be measurable, and Levy-collapse κ to θ^+. According to 2.3.2, nonempty wins $\mathrm{PD}_{<\omega_1}(S)$ for all $S\in U$, in particular for $S=\theta^+$. However, empty wins $\mathrm{BM}_{\le\omega}(\theta^+)$ (by playing $E^{\theta^+}_\omega$). □

In the rest of the paper will deal with the proof of Theorem 1.1.

3. Overview of the proof

We assume that κ is measurable, and $\omega<\theta<\kappa$ regular.

Step 1. We construct models M satisfying:

$(*)$ κ is measurable and player empty wins $\mathrm{BM}_{\le\omega}(S)$ for every stationary S.

We present two constructions, showing that $(*)$ is true in $L[U]$ as well as compatible with larger cardinals:

(i) The inner model $L[U]$, Section 4:
Let D be a normal ultrafilter on κ, and set $U=D\cap L[D]$. Then in $L[U]$, (the dual ideal of) U is the only normal precipitous ideal on κ. In particular, $L[U]$ satisfies $(*)$.

[2]There is even a fixed sequence of winning moves for empty: For every $\alpha\in E^\kappa_\omega$ let $(\alpha_n)_{n\in\omega}$ be a normal sequence in α. As move n, empty plays the function that maps α to α_n. If β and β' are both in $\bigcap_{n\in\omega}T_n$, then $\beta_n=\beta'_n$ for all n and therefore $\beta=\beta'$.

(ii) Forcing (∗), Section 5:
(α) We construct a partial order $R(\kappa)$ forcing that empty wins $\mathrm{BM}_{\leq\omega}(\kappa)$. This $R(\kappa)$ does not preserve measurability of κ.
(β) We use $R(\kappa)$ to force (∗) while preserving e.g. supercompactness.

Step 2. Now we look at the Levy-collapse Q that collapses κ to θ^+.

In Section 6 we will see: If in $V[G_Q]$, nonempty wins $\mathrm{BM}_{\leq\omega}(\dot{S})$ for some $\dot{S} \in \mathcal{E}^\kappa_\theta$, then in V nonempty wins $\mathrm{BM}_{\leq\omega}(\tilde{S})$ for some $\tilde{S} \in \mathcal{E}^\kappa_{\geq\theta}$.

So if we start with V satisfying (∗) of Step 1, then Q forces:

- Nonempty wins $\mathrm{PD}_{<\theta}(E^\kappa_\theta)$ (by 2.3.4). Actually nonempty wins $\mathrm{PD}_{<\theta}(S)$ for all $S \in U$, and $E^\kappa_\theta = (E^\kappa_{\geq\theta})^V \in U$.
- Nonempty does not win $\mathrm{BM}_{\leq\omega}(\dot{S})$ for any stationary $\dot{S} \subseteq E^\kappa_\theta$. Equivalently: Nonempty does not win $\mathrm{BM}_{\leq\omega}(E^\kappa_\theta)$, even if nonempty gets the first move.

4. U is the only normal, precipitous ideal in $L[U]$

If $V = L$, then there are no normal, precipitous ideals (recall that a precipitous ideal implies a measurable in an inner model). Using Kunen's results on iterated ultrapowers, it is easy to relativize this to $L[U]$:

Theorem 4.1. *Assume $V = L[U]$, where U is a normal ultrafilter on κ. Then the dual ideal of U is the only normal, precipitous ideal on κ.*

In particular, NS_κ is nowhere precipitous, and empty wins $BM_{\leq\omega}(S)$ for any stationary $S \subseteq \kappa$.

Remark: Much deeper results by Gitik show that e.g.

($\star$) κ is measurable and either E^κ_λ or $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ is precipitous.

implies more than a measurable (in an inner model) [2, Sect. 5], so ($\star$) fails not only in $L[U]$ but also in any other universe without "larger inner-model-cardinals". However, it is not clear to us whether the same hold e.g. for

($\star'$) κ is measurable and $\mathrm{NS} \restriction S$ is precipitous for some S.

Back to the proof of Theorem 4.1.

If $\mathcal{I}$ is a normal, precipitous ideal, then $P_\mathcal{I}$ forces that the generic filter G is a normal, wellfounded V-ultrafilter (cf [6, 22.13]). So it is enough to show that in any forcing extension, U is the only normal wellfounded V-ultrafilter on κ. We will do this in Lemma 4.3.

If $U \in L[U]$ and $L[U]$ thinks that U is a normal ultrafilter on κ, then we call the pair $(L[U], U)$ a κ-model.

If D is a normal ultrafilter on κ, and $U = D \cap L[D]$, then $(L[U], U)$ is a κ-model.

We will use the following results of Kunen [10], cited as Theorem 19.14 and Lemma 19.16 in [6]:

Lemma 4.2. (1) *For every ordinal κ there is at most one κ-model.*

(2) *Assume $\kappa < \lambda$ are ordinals, $(L[U], U)$ is the κ-model and $(L[W], W)$ the λ-model. Then $(L[W], W)$ is an iterated ultrapower of $(L[U], U)$, in particular: There is an elementary embedding $i : L[U] \to L[W]$ definable in $L[U]$ such that $W = i(U)$.*

(3) *Assume that*
- *$(L[U], U)$ is the κ model,*
- *A is a set of ordinals of size at least κ^+,*
- *θ is a cardinal such that $A \cup \{U\} \subset L_\theta[U]$, and*

- *$X \subseteq \kappa$ is in $L[U]$.*

Then there is a formula φ, ordinals $\alpha_i < \kappa$ and $\gamma_i \in A$ such that in $L_\theta[U]$, X is defined by $\varphi(X, \alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_m, U)$.

(That means that in $L[U]$ there is exactly one y satisfying $\varphi(y, \alpha_1, \dots)$, and $y = X$.)

Lemma 4.3. *Assume $V = L[U]$, where U is a normal ultrafilter on κ. Let V' be a forcing extension of V, and $G \in V'$ a normal, wellfounded V-ultrafilter on κ. Then $G = U$.*

Proof. In V', let $j : V \to \mathrm{Ult}_G(V)$ be elementary. Set $\lambda = j(\kappa) > \kappa$ and $W = j[U]$. So $\mathrm{Ult}_G(V)$ is the λ-model $L[W]$.

In V, we can define a function $J : \mathrm{ON} \to \mathrm{ON}$ such that in V', $J(\alpha)$ is a cardinal greater than $(\alpha^\kappa)^{+V'}$. (After all, V' is just a forcing extension of V.) So $J(\alpha)$ is greater than both $i(\alpha)$ and $j(\alpha)$. In V, let $\mathcal{C}$ be the class of ordinals that are ω-limits of iterations of F, i.e. $\alpha \in \mathcal{C}$ if $\alpha = \sup(\alpha_0, F(\alpha_0), F(F(\alpha_0)), \dots)$. Then $i(\alpha) = j(\alpha) = \alpha$ and α is a cardinal in V'.

In V', pick a set A of κ^+ many members of $\mathcal{C}$, and $\theta \in \mathcal{C}$ such that and $A \cup \{U\} \subseteq L_\theta[U]$. Pick any $X \subseteq \kappa$. Then in $L[U]$, X is defined by

$$L_\theta[U] \vDash \varphi(X, \vec{\alpha}, \vec{\gamma}, U).$$

Let k be either i or j. Then by elementarity, in $L[W]$ $k(X)$ is the set Y such that

$$L_\theta[W] \vDash \varphi(Y, \vec{\alpha}, \vec{\gamma}, W),$$

since $W = k(U)$ and $k(\beta) = \beta$ for all $\beta \in \kappa \cup A \cup \{\theta\}$.

Therefore $i(X) = j(X) = Y$. So $X \in G$ iff $\kappa \in j(X) = i(X)$ iff $X \in U$, since both G and U are normal. □

5. Forcing empty to win

As in the last section, we construct a universe with in which empty wins $\mathrm{BM}_{\leq\omega}(S)$ for every stationary $S \subseteq \kappa$, this time using forcing. This shows that the assumption is also compatible with e.g. κ supercompact.

5.1. The basic forcing.

Assumption 5.1. κ is inaccessible, $2^\kappa = \kappa^+$, and $\lhd$ a wellordering of 2^κ (used for the bookkeeping).

We will define the $< \kappa$-support iteration $(P_\alpha, Q_\alpha)_{\alpha<\kappa^+}$ and show:

Lemma 5.2. *P_{κ^+} forces: Empty has a winning strategy for $BM_{\leq\omega}(\kappa)$ where empty's first move is κ. P_{κ^+} is κ^+-cc and has a dense subforcing P'_{κ^+} which is $< \kappa$-directed-closed and of size κ^+.*

We use two basic forcings in the iteration:

- If $S \subseteq \kappa$ is stationary, then $\mathrm{Cohen}(S)$ adds a Cohen subset of S. Conditions are functions $f : \zeta \to \{0,1\}$ with $\zeta < \kappa$ successor such that $\{\xi < \zeta : f(\xi) = 1\}$ is a subset of S. ζ is called height of f. $\mathrm{Cohen}(S)$ is ordered by inclusion.

  This forcing adds the generic set $S' = \{\zeta < \kappa : (\exists f \in G) f(\zeta) = 1\} \subset S$.

- If $\lambda \leq \kappa^+$, and $(S_i)_{i<\lambda}$ is a family of stationary sets, then $\text{Club}((S_i)_{i<\lambda})$ consists of $f : (\zeta \times u) \to \{0,1\}$, $\zeta < \kappa$ successor, $u \subseteq \lambda$, $|u| < \kappa$ such that $\{\xi < \zeta : f(\xi, i) = 1\}$ is a closed subset of S_i. ζ is called height of f, u domain of f. $\text{Club}((S_i)_{i<\lambda})$ is ordered by inclusion.

The following is well known:

Lemma 5.3. *Cohen(S) is $< \kappa$-closed and forces that the generic Cohen subset $S' \subseteq S$ is stationary.*

So Cohen(S) is a well-behaved forcing, adding a generic stationary subset of S. $\text{Club}((S_i)_{i<\lambda})$ adds unbounded closed subsets of each S_i. Other than that it is not clear why this forcing should e.g. preserve the regularity of κ (and it will generally not be σ-closed). However, we will shoot clubs only through complements of Cohen-generics we added previously, and this will simplify matters considerably.

The P_α will add more and more moves to our winning strategy.

Set $D = \{\delta < \kappa^+ : \delta \text{ limit}\}$ (for "destroy"), $M = (\kappa^+)^{<\omega}$ (for "moves"). Find a bijection of $i : M \to \kappa^+ \setminus D$ so that $s \preceq_M t$ implies $i(s) \leq i(t)$. We identify M with its image, i.e. $\kappa^+ = D \cup M$. So for $\alpha \in M$ there is a finite set $\alpha_0 < \alpha_1 \cdots < \alpha_m < \alpha$ of M-predecessors (in short: predecessors). For $\delta \in D$, we can look at all branches through $M \cap \delta$. Some of them will be "new", i.e. not in any $M \cap \gamma$ for $\gamma < D \cap \delta$. Let λ_δ be the number of these new branches, i.e. $0 \leq \lambda_\delta \leq 2^\kappa = \kappa^+$.

We define Q_α by induction on α, and assume that at stage α (i.e. after forcing with P_α) we have already defined a partial strategy. Work in $V[G_\alpha]$.

- $\alpha \in M$, with the predecessors $0 = \alpha_0 < \alpha_1 \cdots < \alpha_m < \alpha$. By induction we know that at stage α_m
  - we dealt with the sequence $x_{\alpha_m} = (\kappa, T_{\alpha_1}, S_{\alpha_1}, T_{\alpha_2}, \dots, S_{\alpha_{m-1}}, T_{\alpha_m})$, which is played to empty's partial strategy,
  - we defined Q_{α_m} to be Cohen(T_{α_m}), adding the generic set S_{α_m},
  - this S_{α_m} was added to the partial strategy as response to x_{α_m}.

  Now (using some simple bookkeeping) we pick a stationary $T_\alpha \subset S_{\alpha_m}$ such that the partial strategy is not already defined on $x_\alpha = x_{\alpha_m}{}^\frown(S_{\alpha_m}, T_\alpha)$, and set $Q_\alpha = \text{Cohen}(T_\alpha)$, and add the Q_α-generic $S_\alpha \in V[G_{\alpha+1}]$ to the partial strategy as response to x_α.
- $\alpha \in D$. In V, there are $0 \leq \lambda_\alpha \leq \kappa^+$ many new branches b_i. (All old branches have already been dealt with in the previous D-stages.) For each new branch $b_i = (\alpha_0^i < \alpha_1^i < \dots)$, we set $S^i = \bigcap_{n\in\omega} S_{\alpha_n^i}$, and we set $Q_\alpha = \text{Club}((\kappa \setminus S^i)_{i\in\lambda_\alpha})$.

So empty always responds to nonempty's move T with a Cohen subset of T, and the intersection of an ω-sequence of moves according to the strategy is made non-stationary.

We will show:

Lemma 5.4. *P_{κ^+} does not add any new countable sequences of ordinals, forces that κ is regular and that the Q_α-generic S_α (i.e. empty's move) is stationary for all $\alpha \in M$.*

We will prove this Lemma later. Then the rest follows easily:

Lemma 5.5. *P_{κ^+} forces that empty wins $BM_{\leq\omega}(\kappa)$, using κ as first move.*

Proof. At the final limit stage, P_{κ^+} does not add any new subsets of κ, nor any countable sequences of such subsets. So there are only κ^+ many names for countable sequences $x = (\kappa, T'_1, S'_1, T'_2, S'_2, T'_3, \dots)$. Our bookkeeping has to make sure that for every initial segment (if it consists of valid moves and uses the partial strategy so far) there has to be a response in the strategy.

Then $x \restriction 2n$ corresponds to an element of M for every n, and x defines a branch b through M. $b \in V$, since P_{κ^+} does not add new countable sequences of ordinals.

Let $\alpha \in D$ be minimal so that $x \restriction 2n < \alpha$ for all n. Then in the D-stage α, the stationarity of $\bigcap_{n\in\omega} S'_n$ was destroyed, i.e. empty wins the run x. □

We now define the dense subset of P_α:

Definition 5.6. $p \in P'_\alpha$ if $p \in P_\alpha$ and there are (in V) a successor ordinal $\epsilon(p) < \kappa$, $(f_\alpha)_{\alpha\in\mathrm{dom}(p)}$ and $(u_\alpha)_{\alpha\in\mathrm{dom}(p)\cap D}$ such that:

- If $\alpha \in M$, then $f_\alpha : \epsilon(p) \to \{0, 1\}$.
- If $\alpha \in D$, then $u_\alpha \subseteq \lambda_\alpha$, $|u_\alpha| < \kappa$, and $f_\alpha : \epsilon(p) \times u_\alpha \to \{0, 1\}$.
- Moreover, for $\alpha \in D$, u_α consists exactly of the new branches through $\mathrm{dom}(p) \cap \alpha \cap M$.
- $p \restriction \alpha \Vdash p(\alpha) = f_\alpha$.

So a $p \in P'_\alpha$ corresponds to a "rectangular" matrix with entries in $\{0, 1\}$. Of course only some of these matrices are conditions of P_α and therefore in P'_α.

Lemma 5.7. *(1) P'_α is ordered by extension. (I.e. if $p, q \in P'_\alpha$, then $q \le p$ iff q (as Matrix) extends p.)*
(2) $P'_\alpha \subseteq P_\alpha$ is a dense subset.
(3) P'_α is $< \kappa$-directed-closed, in particular P_α does not add any new sequences of length $< \kappa$ nor does it destroy stationarity of any subset of κ.

Proof. (1) should be clear.

(3) Assume all p_i are pairwise compatible. We construct a condition q by putting an additional row on top of $\bigcup p_i$ (and filling up at indices where new branches might have to be added). So we set

- $\mathrm{dom}(q) = \bigcup \mathrm{dom}(p_i)$.
- $\epsilon(q) = \bigcup \epsilon(p_i) + 1$.
- For $\alpha \in \mathrm{dom}(q) \cap M$, we put 0 on top, i.e. $q_\alpha(\epsilon(q) - 1) = 0$.
- For $\alpha \in \mathrm{dom}(q) \cap D$, and $i \in \bigcup \mathrm{dom}(p_i(\alpha))$, set $q_\alpha(\epsilon(q) - 1, i) = 1$.
- For $\alpha \in \mathrm{dom}(q) \cap D$, if i is a new branch through $M \cap \mathrm{dom}(q) \cap \alpha$ and not in $\bigcup \mathrm{dom}(p_i(\alpha))$, set $q_\alpha(\xi, i) = 0$ for all $\xi < \epsilon(q)$.

Why can we do that? If $\alpha \in M$, whether the bookkeeping says that $\epsilon(q)-1 \in T_\alpha$ or not, we can of course always choose to not put it into S_α (i.e. set $q_\alpha(\epsilon(q)-1) = 0$). Then for $\alpha \in D$, $\epsilon(q) - 1$ will definitely not be in the intersection along the branch i, so we can put it into the complement.

(2) By induction on α. Assume $p \in P_\alpha$.

$\alpha = \beta + 1$ is a successor. We know that P_β does not add any new $< \kappa$ sequences of ordinals, so we can strengthen $p \restriction \beta$ to a $q \in P'_\beta$ which decides $f = p(\beta) \in V$. Without loss of generality $\epsilon(q) \ge \mathrm{height}(f)$, and we can enlarge f up to $\epsilon(q)$ by adding values 0 (note that $\mathrm{height}(f) < \kappa$ is a successor, so we do not get problems with closedness when adding 0). And again, we also add values for the required "new branches" if necessary.

If α is a limit of cofinality $\ge \kappa$, then $p \in P_\beta$ for some $\beta < \alpha$, so there is nothing to do.

Let α be a limit of cofinality $< \kappa$, i.e. $(\alpha_i)_{i\in\lambda}$ is an increasing cofinal sequence in α, $\lambda < \kappa$. Using (2), define a sequence $p_i \in P'_{\alpha_i}$ such that $p_i < p_j \wedge p \restriction \alpha_i$ for all $j < i$, then use (3). $\Box$

How does the quotient forcing $P^\alpha_{\kappa^+}$ (i.e. P_{κ^+}/G_α) behave compared to P_{κ^+}?

- Assume $\alpha \in D$. In $V[G_\alpha]$, Q_α shoots a club through the complement of the (probably) stationary set $\bigcap_{i\in\omega} S^i$. In particular, Q_α cannot have a $< \kappa$-closed subset.
- Nevertheless, $P_\alpha * Q_\alpha$ has a $< \kappa$-closed subset (and preserves stationarity).
- So if we factor P_{κ^+} at some $\alpha \in D$, the remaining $P^\alpha_{\kappa^+}$ will look very different from P_{κ^+}.
- However, if we factor P_{κ^+} at $\alpha \in M$, $P^\alpha_{\kappa^+}$ will be more or less the same as $P^\alpha_{\kappa^+}$ (just with a slightly different bookkeeping).

In particular, we get:

Lemma 5.8. *If $\alpha \in M$, then the quotient $P^\alpha_{\kappa^+}$ will have a dense $< \kappa$-closed subset (and therefore it will not collapse stationary sets).*

(The proof is the same as for the last lemma.)

Note that for this result it was necessary to collapse the new branches as soon as they appear. If we wait with that, then (looking at the rest of the forcing from some stage $\alpha \in M$) we shoot clubs through stationary sets that already exist in the ground model, and things get more complicated.

Now we can easily prove lemma 5.4:

Proof of lemma 5.4. In stage $\alpha \in M$, nonempty's previous move S_{α_m} is still stationary (by induction), the bookkeeping chooses a stationary subset T_{α_m} of this move, and we add S_α as Cohen-generic subset of T_{α_m}. So according to lemma 5.3, S_α is stationary at stage $\alpha + 1$, i.e. in $V[G_{\alpha+1}]$. But since $\alpha + 1 \in M$, the rest of the forcing, $P^{\alpha+1}_{\kappa^+}$, is $< \kappa$-closed and does not destroy stationarity of S_α. $\Box$

5.2. **Preserving Measurability.** We can use the following theorem of Laver [11], generalizing an idea of Silver: If κ is supercompact, then there is a forcing extension in which κ is supercompact and every $< \kappa$-directed closed forcing preserves the supercompactness. Note that we can also get $2^\kappa = \kappa^+$ which such a forcing.

Corollary 5.9. *If κ is supercompact, we can force that κ remains supercompact and that empty wins $BM_{\leq\omega}(S)$ for all stationary $S \subseteq \kappa$.*

Remark: It is possible, but not obvious that we can also start with κ just measurable and preserve measurability. It is at least likely that it is enough to start with strong to get measurable. Much has been published on such constructions, starting with Silver's proof for violating GCH at a measurable (as outlined in [6, 21.4]).

6. The Levy collapse

We show that after collapsing κ to θ^+, nonempty still has no winning strategy in BM.

Assume that κ is inaccessible, $\theta < \kappa$ regular, and let $Q = \text{Levy}(\theta, < \kappa)$ be the Levy collapse of κ to θ^+: A condition $q \in Q$ is a function defined on a subset of $\kappa \times \theta$, such that $|\operatorname{dom}(q)| < \theta$ and $q(\alpha, \xi) < \alpha$ for $\alpha > 1, (\alpha, \xi) \in \operatorname{dom}(q)$ and $q(\alpha, \xi) = 0$ for $\alpha \in \{0, 1\}$.

Given $\alpha < \kappa$, define $Q_\alpha = \{q : \mathrm{dom}(q) \subseteq \alpha \times \theta\}$ and $\pi_\alpha : Q \to Q_\alpha$ by $q \mapsto q \restriction (\alpha \times \theta)$.

The following is well known (see e.g. [6, 15.22] for a proof):

Lemma 6.1.
- *Q is κ-cc and $< \theta$-closed.*
- *In particular, Q preserves stationarity of subsets of κ: If p forces that $\dot{C} \subseteq \kappa$ is club, then there is a $C' \subseteq \kappa$ club and a $q \leq p$ forcing that $C' \subseteq \dot{C}$.*
- *If $q \Vdash p \in G$, then $q \leq p$ (i.e. $\leq^*$ is the same as $\leq$).*

We will use the following simple consequence of Fodor's lemma (similar to a Δ-system lemma):

Lemma 6.2. *Assume that $p \in Q$ and $S \in \mathcal{E}^\kappa_{\geq\theta}$. If $\{q_\alpha \mid \alpha \in S\}$ is a sequence of conditions in Q, $q_\alpha < p$, then there is a $\beta < \kappa$, a $q \in Q_\beta$ and a stationary $S' \subseteq S$, such that $q \leq p$ and $\pi_\alpha(q_\alpha) = q$ for all $\alpha \in S'$.*

Proof. For $q \in Q$ set $\mathrm{dom}^\kappa(q) = \{\alpha \in \kappa : (\exists \zeta \in \theta)\, (\alpha, \zeta) \in \mathrm{dom}(q)\}$. For $\alpha \in S$ set $f(\alpha) = \sup(\mathrm{dom}^\kappa(q_\alpha) \cap \alpha)$. f is regressive, since $|\mathrm{dom}^\kappa(q_\alpha)| < \theta$ and $\mathrm{cf}(\alpha) \geq \theta$. By the pressing down lemma there is a $\beta < \kappa$ such that $T = f^{-1}(\beta) \subseteq S$ is stationary.

For $\alpha \in T$, set $h(\alpha) = \pi_{\beta+1}(q_\alpha)$. The range of h is of size at most $|\beta \times \theta|^{<\theta} < \kappa$. So there is a stationary $S' \subseteq T$ such that h is constant on S', say q. If $\alpha \in S'$, then $\sup(\mathrm{dom}^\kappa(q_\alpha)) \cap \alpha = \beta$, therefore $\pi_\alpha(q_\alpha) = \pi_{\beta+1}(q_\alpha) = q$.

Pick $\alpha \in S'$ such that $\alpha > \sup(\mathrm{dom}^\kappa(p))$. $q_\alpha \leq p$, so $q = \pi_\alpha(q_\alpha) \leq \pi_\alpha(p) = p$. □

Lemma 6.3. *Assume that*
- *κ is strongly inaccessible, $\theta < \kappa$ regular, $\mu \leq \theta$.*
- *$Q = Levy(\theta, < \kappa)$,*
- *$\dot{S}$ is a Q-name for an element of $\mathcal{E}^\kappa_\theta$,*
- *$\tilde{p} \in Q$ forces that $\dot{F}$ is a winning strategy of nonempty in $BM_{<\mu}(\dot{S})$.*

Then in V, nonempty wins $BM_{<\mu}(\tilde{S})$ for some $\tilde{S} \in E^\kappa_{\geq\theta}$.

If $\dot{S}$ is a standard name for $T \in (E^\kappa_{\geq\theta})^V$, then we can set $S = T$.

Proof. First assume that $\dot{S}$ is a standard name.

For a run of $\mathrm{BM}_{<\mu}(S)$, we let A_ε and B_ε denote the εth moves of empty and nonempty. We will construct by induction on $\varepsilon < \mu$ a strategy for empty, including not only the moves B_ε, but also Q-names $\dot{A}'_\varepsilon, \dot{B}'_\varepsilon$, and Q-conditions $p_\varepsilon, \langle p^\varepsilon_\alpha \mid \alpha \in B_\varepsilon\rangle$, such that the following holds:
- $p_\varepsilon \leq p_\xi$ and $p^\varepsilon_\alpha \leq p^\xi_\alpha$ for $\xi < \varepsilon$.
- p_ε forces that $(\dot{A}'_\xi, \dot{B}'_\xi)_{\xi\leq\varepsilon}$ is an initial segment of a run of $\mathrm{BM}_{<\mu}(\dot{S})$ in which nonempty uses the strategy $\dot{F}$.
- $p_\varepsilon \Vdash \dot{A}'_\varepsilon \subseteq A_\varepsilon$.
- For $\alpha \in B_\varepsilon$, $\pi_\alpha(p^\varepsilon_\alpha) = p_\varepsilon$ (in particular $p^\varepsilon_\alpha \leq p_\varepsilon$), and $p^\varepsilon_\alpha \Vdash$ "$\alpha \in \dot{B}'_\varepsilon$".

Assume that we have already constructed these objects for all $\xi < \varepsilon$.

In limit stages ε, we first have to make sure that $\bigcap_{\xi<\varepsilon} B_\xi$ is stationary (otherwise nonempty has already lost). Pick a q stronger than each p_ξ for $\xi < \varepsilon$. (This is possible since Q is $< \theta$-closed.) Then q forces that $\bigcap_{\xi<\varepsilon} B_\xi = \bigcap_{\xi<\varepsilon} A_\xi \supseteq \bigcap_{\xi<\varepsilon} \dot{A}'_\xi$ and that $(\dot{A}'_\xi, \dot{B}'_\xi)_{\xi\leq\varepsilon}$ is a valid initial segment of a run where nonempty uses the strategy, in particular $\bigcap_{\xi<\varepsilon} \dot{A}'_\xi$ is stationary.

So now ε can be a successor or a limit, and empty plays the stationary set $A_\varepsilon \subseteq \bigcap_{\xi<\varepsilon} B_\xi$. (That implies that p^ξ_α is defined for all $\alpha \in A_\varepsilon$ and $\xi < \varepsilon$.)

- Define the εth move of empty in $V[G_Q]$ to be
$$\dot{A}'_\varepsilon = \{\alpha \in A_\varepsilon : (\forall \xi < \varepsilon)\, p^\xi_\alpha \in G_Q\},$$
and pick $\tilde{p}_\varepsilon \leq p_\xi$ for $\xi < \varepsilon$ (for $\varepsilon = 0$, pick $\tilde{p}_0 = \tilde{p}$).
$\tilde{p}_\varepsilon$ forces that $\dot{A}'_\varepsilon \subseteq \bigcap_{\xi<\varepsilon} \dot{B}'_\xi$, since p^ξ_α forces that $\alpha \in \dot{B}'_\xi$. $\tilde{p}_\varepsilon$ also forces that $\dot{A}'_\varepsilon$ is stationary:
Otherwise there is a $C \subseteq \kappa$ club and a $q \leq \tilde{p}_\varepsilon$ forcing that $C \cap \dot{A}_\varepsilon$ is empty (cf 6.1). $q \in Q_\beta$ for some $\beta < \kappa$. Pick $\alpha \in (C \cap A_\varepsilon) \setminus (\beta+1)$. For $\xi < \varepsilon$, $\pi_\alpha(p^\xi_\alpha) = p_\xi \geq q$, and $q \in Q_\beta$, so q and p^ξ_α are compatible. Moreover, the conditions $(q \cup p^\xi_\alpha)_{\xi \in \varepsilon}$ are decreasing, so there is a common lower bound q' forcing that $p^\xi_\alpha \in G_Q$ for all ξ, i.e. that $\alpha \in \dot{A}'_\varepsilon$, a contradiction.
- Given $\dot{A}'_\varepsilon$, we define $\dot{B}'_\varepsilon$ as the response according to the strategy $\dot{F}$.
- Now we show how to obtain the next move of nonempty, B_ε, (in the ground model), as well as p^ε_α for $\alpha \in B_\varepsilon$. B_ε of course has to be a subset of the stationary set S defined by
$$S = \{\alpha \in A_\varepsilon \mid \tilde{p}_\varepsilon \not\Vdash \alpha \notin \dot{B}'_\varepsilon\}.$$
For each $\alpha \in S$, pick some $p^\varepsilon_\alpha \leq \tilde{p}_\varepsilon$ forcing that $\alpha \in \dot{B}'_\varepsilon$. By the definition of $\dot{A}'_\varepsilon$ and since $\tilde{p}_\varepsilon \Vdash \dot{B}'_\varepsilon \subseteq \dot{A}'_\varepsilon$, we get
$$p^\varepsilon_\alpha \Vdash (\forall \xi < \varepsilon)\, p^\xi_\alpha \in G_Q,$$
which means that for $\alpha \in S$ and $\xi < \varepsilon$, $p^\varepsilon_\alpha \leq p^\xi_\alpha$.
Now we apply lemma 6.2 (for $p = \tilde{p}_\varepsilon$). This gives us $S' \subseteq S$ and $q \leq \tilde{p}_\varepsilon$. We set $B_\varepsilon = S'$ and $p_\varepsilon = q$.

If $\dot{S}$ is not a standard name, set
$$S^0 = \{\alpha \in E^\kappa_{\geq\theta} : \tilde{p} \not\Vdash \alpha \notin \dot{S}\}$$
As above, for each $\alpha \in S_0$, pick a $\tilde{p}^{-1}_\alpha \leq \tilde{p}$ forcing that $\alpha \in \dot{S}$, and choose a stationary $\tilde{S} \subseteq S^0$ according to Lemma 6.2. Now repeat the proof, starting the sequence (p_ε) and (p^ε_α) already at $\varepsilon = -1$. □

References

1. F. Galvin, T. Jech, and M. Magidor, *An ideal game*, J. Symbolic Logic **43** (1978), no. 2, 284–292. MR MR0485391 (58 #5237)
2. Moti Gitik, *Some results on the nonstationary ideal*, Israel J. Math. **92** (1995), no. 1-3, 61–112. MR **MR1357746 (96k:**03108)
3. Tapani Hyttinen, Saharon Shelah, and Jouko Väänänen, *More on the Ehrenfeucht-Fraïssé game of length ω_1*, Fund. Math. **175** (2002), no. 1, 79–96. MR **MR1971240 (2004b:**03046)
4. T. Jech, M. Magidor, W. Mitchell, and K. Prikry, *Precipitous ideals*, J. Symbolic Logic **45** (1980), no. 1, 1–8. MR **MR560220 (81h:**03097)
5. Thomas Jech, *A game theoretic property of Boolean algebras*, Logic Colloquium '77 (Proc. Conf., Wrocław, 1977), Stud. Logic Foundations Math., vol. 96, North-Holland, Amsterdam, 1978, pp. 135–144. MR **MR519808 (80c:**90184)
6. ———, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR **MR1940513 (2004g:**03071)
7. Thomas Jech and Karel Prikry, *On ideals of sets and the power set operation*, Bull. Amer. Math. Soc. **82** (1976), no. 4, 593–595. MR MR0505504 (58 #21618)
8. Thomas J. Jech, *More game-theoretic properties of Boolean algebras*, Ann. Pure Appl. Logic **26** (1984), no. 1, 11–29. MR **MR739910 (85j:**03110)

896 revision:2007-02-17 modified:2007-02-17

9. ______, *Some properties of κ-complete ideals defined in terms of infinite games*, Ann. Pure Appl. Logic **26** (1984), no. 1, 31–45. MR **MR739911 (85h:**03057)
10. Kenneth Kunen, *Some applications of iterated ultrapowers in set theory*, Ann. Math. Logic **1** (1970), 179–227. MR MR0277346 (43 #3080)
11. Richard Laver, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, Israel J. Math. **29** (1978), no. 4, 385–388. MR MR0472529 (57 #12226)
12. Kecheng Liu and Saharon Shelah, *Cofinalities of elementary substructures of structures on $\aleph_\omega$*, Israel J. Math. **99** (1997), 189–205. MR **MR1469093 (98m:**03100)
13. Alan Mekler, Saharon Shelah, and Jouko Väänänen, *The Ehrenfeucht-Fraïssé-game of length ω_1*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 567–580. MR **MR1191613 (94a:**03058)
14. Saharon Shelah, *Large normal ideals concentrating on a fixed small cardinality*, Arch. Math. Logic **35** (1996), no. 5-6, 341–347. MR **MR1420262 (97m:**03078)
15. Boban Veličković, *Playful Boolean algebras*, Trans. Amer. Math. Soc. **296** (1986), no. 2, 727–740. MR **MR846604 (88a:**06017)

Kurt Gödel Research Center for Mathematical Logic, Universität Wien, Währinger Strasse 25, 1090 Wien, Austria
E-mail address: kellner@fsmat.at
URL: http://www.logic.univie.ac.at/~kellner

Department of Mathematics and Statistics, University of Helsinki, Gustaf Hällströmin katu 2b, FIN-00014, Finland
E-mail address: matti.pauna@helsinki.fi
URL: http://www.helsinki.fi/~pauna/

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA
E-mail address: shelah@math.huji.ac.il
URL: http://shelah.logic.at/