THEORIES WITH EF-EQUIVALENT NON-ISOMORPHIC MODELS
SH897

SAHARON SHELAH

Abstract. Our “long term and large scale” aim is to characterize the first order theories \( T \) (at least the countable ones) such that: for every ordinal \( \alpha \) there are \( \lambda, M_1, M_2 \) such that \( M_1, M_2 \) are non-isomorphic models of \( T \) of cardinality \( \lambda \) which are \( \text{EF}^{+}_{\gamma,\theta,\lambda} \)-equivalent. We expect that as in the main gap ([Sh:c, ?]) we get a strong dichotomy, so in the non-structure side we have stronger, better examples, and in the structure side we have a parallel of [Sh:c, Ch.XIII]. We presently prove the consistency of the non-structure side for \( T \) which is \( \aleph_0 \)-independent (= not strongly dependent), even for \( \text{PC}(T_1, T) \).

Anotated Content

§0 Introduction

§1 Games, equivalences and the question

[We discuss what are the hopeful conjectures concerning versions of \( \text{EF} \)-equivalent non-isomorphic models for a given complete first order \( T \), i.e. how it fits classification. In particular we define when \( M_1, M_2 \) are \( \text{EF}_{\gamma,\lambda} \)-equivalent and when they are \( \text{EF}^{+}_{\gamma,\theta,\lambda} \)-equivalent and discuss those notions.]

§2 The properties of \( T \) and relevant indiscernibility

[We recall the definitions of “\( T \) strongly dependent”, “\( T \) is strongly dependent”, and prove the existence of models of such \( T \) suitable for proving non-structure theory.]

§3 Forcing \( \text{EF}^{+} \)-equivalent non-isomorphic models

[We force such an example.]

§4 Theories with order

[We prove in \( \text{ZFC} \), that for \( \lambda \) regular there are quite equivalent non-isomorphic models of cardinality \( \lambda^+ \).]

Date: December 17, 2012.
1991 Mathematics Subject Classification. Primary; Secondary:
Key words and phrases. Ehrenfuecht-Fraissé games, Isomorphism, Model theory, Classification theory (as in the J.).

The author would like to thank the Israel Science Foundation for partial support of this research (Grant No.242/03). I would like to thank Alice Leonhardt for the beautiful typing.
§ 0. Introduction

§ 0(A). Motivation.

We first give some an introduction for non-model theorists. A major theme in the author’s work in model theory is to find “main gap theorems”. This means, considering the family of elementary classes (e.g. the classes of the form \( \text{Mod}_T \) = the class of models of a (complete) first order theory \( T \), each such class is either very “simple” or is very complicated; expecting that we have much knowledge to gain on the “very simple” ones and even on approximations to them.

Of course, this depends on the criterion for “simple”. Essentially the main theorem of [Sh:c] does this for countable \( T \), with “complicated” interpreted as “\( \dot{I}(\lambda,T) \)”, the number of models in \( \text{Mod}_T \) of cardinality \( \lambda \) is maximal, i.e. \( 2^\lambda \), for every \( \lambda \).

See more e.g. in [Sh:E53]. Here we are interested with interpreting “complicated” as “for arbitrarily large cardinals, there are models \( M_1,M_2 \in \text{Mod}_T \) of cardinality \( \lambda \) which are “very similar” but not isomorphic”, where “very similar” is interpreted as a relative of the game of the following form. The isomorphism player constructs during the play, partial isomorphism of cardinality \( < \lambda \), in each move the anti-isomorphism player demands some elements to be in the domain or the range, the isomorphism player has to extend the partial isomorphism accordingly; in the play there are \( \alpha \) moves, \( \alpha < \lambda \); and the isomorphism player wins the play if he has a legal move in each stage (see Definition 1.5, 1.7).

In the present paper we try to deal with suggesting the “right” variant of the game, (see Definition 1.6), and give quite weak sufficient conditions for \( \text{Mod}_T \) being complicated.

\[ \star \star \star \]

Our aim is to prove (on \( \text{PC}(T_1,T) \), see Definition 0.3(3))

- if \( T \subseteq T_1 \) are complete first order theories such that \( T \) is not strongly stable, \( \alpha \) is an ordinal and \( \lambda > |T| \) (or at least for many such \( \lambda \)'s) \text{then}
  - \((*)\) there are \( M_1,M_2 \in \text{PC}(T_1,T) \) of cardinality \( \lambda \) which are \( \text{EF}_{\alpha,\lambda}^+ \)-equivalent for every \( \alpha < \lambda \) but not isomorphic (for the definition of \( \text{EF}_{\alpha,\lambda}^+ \), see Definition 1.7 below, it is a somewhat stronger relative of \( \text{EF}_{\alpha,\lambda} \)-equivalent).

§ 0(B). Related Works.

Concerning constructing non-isomorphic \( \text{EF}_{\alpha,\lambda}^+ \)-equivalent models \( M_1,M_2 \) (with no relation to \( T \)) we have intended to continue [Sh:836], or see more Havlin-Shelah [HvSh:866] and see history in Vaananen in [Vaa95]. Those works leave the case \( \lambda = \aleph_1 \) open; a recent construction [Sh:907] resolve this but whereas it applies to every regular uncountable \( \lambda \), it seems less amenable to generalizations.

By [Sh:c] we essentially know for \( T \) a countable complete first order when \text{there are} \( L_{\infty,\lambda}(\tau_T) \)-equivalent non-isomorphic models of \( T \) of cardinality \( \lambda \) for some \( \lambda \), see §4; this is exactly when \( T \) is superstable with NDOP, NOTOP; (see [Sh:220]).

On restricting ourselves to models of \( T \) for “\( \text{EF}_{\alpha,\lambda} \)-equivalent non-isomorphic”, Hyttinen-Tuuri [HT91] started, then Hyttinen-Shelah [HySh:474], [HySh:529], [HySh:602]. The notion “\( \text{EF}_{\alpha,\lambda}^+ \)-equivalent” is introduced here, in Definition 1.7.

By [HySh:474], if \( T \) is stable unsuperstable, complete first order theory, \( \lambda = \mu^+ \), \( \mu = \text{cf}(\mu) \geq |T| \), \text{then} there are \( \text{EF}_{\mu^+} \)-equivalent non-isomorphic models of \( T \) (even in \( \text{PC}(T_1,T) \)) of cardinality \( \lambda \). But by the variant \( \text{EF}_{\alpha,\lambda}^+ \)-equivalent, such
results are excluded; by it we define our choice test problem the version of being fat/lean, see Definition 0.1.

Why EF$^+$? See Discussion 1.6.

Concerning variants of strongly dependent theories see [Sh:783, §3],[Sh:863] (and maybe [Sh:F705]), most relevant is [Sh:863, §5], part (F). The best relative for us is “strongly$_4$ dependent”, a definition of it is given below but we delay the treatment to a subsequent paper, [Sh:F918]. There we also deal with the relevant logics and more.

We prove here that if $T$ is not strongly stable then $T$ is consistently fat. More specifically, for every $\mu = \mu^\lt \mu > |T|$ there is a $\mu$-complete class forcing notion $P$ such that in $V^P$ the theory $T$ is fat. The result holds even for $\text{PC}(T_1,T)$. This gives new cases even for $\text{PC}(T)$ by 0.2.

Also if $T$ is unstable or has the DOP or OTOP (see 0.7 below or [Sh:c]) then it is fat, i.e. already in $V$.

Of course, forcing the example is a drawback, but note that for proving there is no positive theory it is certainly enough. Hence it gives us an upper bound on the relevant dividing lines.

On Eherenfeucht-Mostowski models, see [Sh:e, Ch.III] or [Sh:c, Ch.VII] or [Sh:h], [Sh:e, Ch.III, §1]. I thank a referee for pointing out on earlier version that Hyttinen-Shelah [HySh:474] was forgotten hence as Definition 1.7 was not yet written, the main result 3.1 had not said anything new.

I also thank referees for many helpful remarks.

§ 0(C). Notations and Basic Definitions.

Definition 0.1. Let $T$ be a complete first order theory.

1) We say $T$ is fat when for every ordinal $\kappa$, for some (regular) cardinality $\lambda > \kappa$ there are non-isomorphic models $M_1,M_2$ of $T$ of cardinality $\lambda$ which are $\text{EF}_{\beta,\kappa,\kappa,\lambda}^+$ equivalent for every $\beta < \lambda$ (see Definition 1.7 below).

2) If $T$ is not fat, we say it is lean.

3) We say the pair $(T,T_1)$ is fat/lean when $(T_1$ is first order $\supseteq T$ and) $\text{PC}(T_1,T) := \{M \upharpoonright \tau T : M$ a model of $T_1\}$ is as above.

4) We say $(T,*)$ is fat when for every first order $T_1 \supseteq T$ the pair $(T,T_1)$ is fat. We say $(T,*)$ is lean otherwise.

Our claims (mainly 3.1) seem to make it clear that some stable $T$ has NDOT and NOTOP which falls under 3.1, but a referee asks for an example, see [Sh:863, §5(F)] for details.

Example 0.2. 1) There is a stable NDOP,NOTOP countable complete theory which is not strongly dependent; (moreover not is not strongly$_4$ stable), see [Sh:863, §5(G)].

2) $T = \text{Th}(\omega_1(\mathbb{Z}_2),E_\alpha)_{\alpha<\omega}$ is as above where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ as an additive group, $E_\alpha = \{(\eta,\nu) : \eta,\nu \in \omega_1(\mathbb{Z}_2) \text{ are such that } \eta|\omega\mu) = \nu|\omega\mu)$ where we interpret $\mathbb{Z}_2$ as the additive group $(\mathbb{Z}/2\mathbb{Z},+)$ and $\omega_1(\mathbb{Z}_2)$ as its $\omega_1$-th power as an abelian group.

* * *

The reader may look at the definitions below only when used.

Definition 0.3. 1) $\text{Mod}_T(\lambda) = \text{EC}_T(\lambda)$ is the class of models of $T$ of cardinality $\lambda$ and $\text{Mod}_T = \text{EC}_T$ is $\{\text{EC}_T(\lambda) : \lambda$ a cardinality\}.

}
2) \( PC_T(T) = \{ M \upharpoonright \tau : M \text{ a model of } T \} \) where \( T \) is a theory or a sentence, in whatever logic, in a vocabulary \( \tau_T \supseteq \tau \); if \( \tau = \tau_T \) we may omit \( \tau \).

3) If \( T \subseteq T_1 \) are complete first order theories then \( PC(T_1,T) = PC_T(T_1) \).

\( \{0z.11\} \)

**Notation 0.4.** 1) \( \ell g(\bar{a}) \) is the length of a sequence \( \bar{a} \).

2) \( \bar{a} \sqsubseteq \bar{b} \) means that \( \bar{a} \) is an initial segment of \( \bar{b} \).

3) \( \bar{a} \models \alpha \) is the unique initial segment of \( \bar{a} \) of length \( \alpha \) for \( \alpha \leq \ell g(\bar{a}) \).

\( \{0z.13\} \)

**Definition 0.5.** 1) For a regular uncountable cardinal \( \lambda \) let \( \bar{I}[\lambda] = \{ S \subseteq \lambda : \text{ some pair } (E,\bar{a}) \text{ witnesses } S \in \bar{I}(\lambda) \}, \) see below. 

2) We say that \( (E,u) \) is a witness for \( S \in \bar{I}[\lambda] \) when:

   \[
   (a) \ E \text{ is a club of the regular cardinal } \lambda \\
   (b) \ \bar{a} = (a_\alpha : \alpha < \lambda), a_\alpha \subseteq \alpha \text{ and } \beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha \\
   (c) \text{ for every } \delta \in E \cap S, u_\delta \text{ is an unbounded subset of } \delta \text{ of order-type } < \delta \text{ (and } \delta \text{ is a limit ordinal).}
   \]

\( \{0z.17\} \)

**Notation 0.6.** 1) For a model \( M, \bar{a} \in {}^\alpha M, B \subseteq M \) and \( \Delta \) a set of formulas, we are interested in formulas of the form \( \varphi(x,y), x = (x_i : i < \alpha), \) so \( \alpha \) may be infinite, but the formulas here are normally first order, so all but finitely many of the \( x_i \)'s are dummy variables.

1A) \( tp_\Delta(\bar{a},B,M) = \{ \varphi(x,\bar{a}) : \varphi(x,y) \in \Delta \text{ and } \bar{b} \in \ell g(\bar{a})A \text{ and } M \models \varphi[\bar{a},\bar{b}] \} \).

2) If \( \Delta_{qf} \) is the set of quantifier-free formulas in \( \mathbb{L}(\tau_M) \), we may write \( tp_{qf} \) instead of \( tp_\Delta \).

3) \( \bar{I}(\lambda,T) \) is the number of isomorphic types of models of \( T \) of cardinality \( \lambda \).

4) \( \bar{I}_b(\lambda,T) \) is the number of isomorphic types of \( M \upharpoonright \tau, M \) a model of \( T \) of cardinality \( \lambda \).

5) \( \bar{I}_E(\lambda,T) \) is the supremum of \( \{ |K| : K \subseteq PC_T(T) \text{ and } M \in K \Rightarrow |M| = \lambda \} \) no \( M \in K \) has an elementary embedding into any \( N \in K \setminus \{ M \} \), writing \( \bar{I}_E(\lambda,T) =^+ \chi \).

6) \( \bar{I}_E(\lambda,T) = \bar{I}_E(T)(\lambda,T) \).

\( \{0z.23\} \)

**Definition 0.7.** Let \( T \) be a first order complete theory.

1) \( T \) has OTOP when \( T \) is stable and for some \( n, m \) letting \( \bar{x} = (x_\ell : \ell < n), \bar{y} = (y_\ell : \ell < n), \bar{z} = (z_\ell : \ell < m) \), there are complete types \( p(\bar{x},\bar{y},\bar{z}) \) such that:

   \[
   (a) \ \langle a_\alpha : \alpha < \lambda \rangle \text{ is an indiscernible set} \\
   (b) \text{ for } \alpha \neq B < \lambda \text{ the type } (p(\bar{a}_\alpha, a_\beta, z)) \text{ is realized in } M \text{ iff } \alpha < B.
   \]

1A) \( T \) has the NOTOP when it is stable but fail the OTOP.

2) \( T \) has NDOP when \( T \) is stable and we can find \( \mathbb{I}^+ \)-saturated models \( M_\ell \) of \( T \) for \( \ell \leq 3 \) such that \( M_0 \prec M_1 \prec M_3 \) for \( \ell = 1, 2 \) and \( tp(M_1,M_2) \) does not fork over \( M_0, M_3 \) is \( |T|^+ \)-prime over \( M_1 \cup M_2 \) but not \( |T|^+ \)-minimal over it; equivalently for every \( \bar{c} \in \omega^{>}(M_3) \) the type \( tp(\bar{c},M_1 \cup M_2, M_3) \) is \( |T|^+ \)-isolated but there is no infinite \( I \subseteq M_3 \) which is indiscernible over \( M_1 \cup M_2 \).

2A) \( T \) has DOP when \( T \) is stable and fail to have the NDOP.

\( \{0z.25\} \)

**Definition 0.8.** 1) For a complete first order theory \( T \), we can say that \( \psi \) is a \( (\mu, \kappa, T) \)-candidate when:

   \[
   (a) \ \psi \in \mathbb{L}_{\mu,T}(\tau_\star) \text{ for some vocabulary } \tau_\star \supseteq \tau_T \text{ of cardinality } \leq \kappa
   \]
(b) $PC_{\tau(T)}(\psi) \subseteq EC(T)$

(c) for some $^1\Phi \in \mathcal{T}_{\kappa^{\omega^2}}$ satisfying $\tau_{\Phi} \supseteq \tau_{\psi}$ and $EM(\omega^2 \lambda, \Phi) \models \psi$ for every (equivalent some) $\lambda$ and $\Phi$ witness $T$ is not superstable.

Recall that by [Sh:c, Ch.VII]:

Claim 0.9. If a first order complete theory $T$ is not superstable, then for some $^1\Phi \in \mathcal{T}_2^{\omega^2}$ satisfying $\tau_{\Phi} \supseteq \tau_{\psi}$ and $EM(\omega^2 \lambda, \Phi)$ $|$ $\psi$ for every (equivalent some) $\lambda$ and $\Phi$ witness $T$ is not superstable.

Definition 0.10. 1) For any structure $I$ we say that $\langle \bar{a}_t: t \in I \rangle$ is indiscernible (in the model $\mathbb{C}$, over $A$, if $A = \emptyset$ we may omit it) when: $\check{\bar{a}}_t \in \ellg(\bar{a}_t)$, which is not necessarily finite depends only on the quantifier-free type of $t$ in $I$ and:

if $n < \omega$ and $\tilde{s} = \langle s_0, s_1, \ldots, s_{n-1} \rangle, \ell = \langle t_0, \ldots, t_{n-1} \rangle$ realize the same quantifier-free type in $I$ then $\check{\bar{a}}_t := \check{a}_{t_0} \hat{\ldots} \hat{a}_{t_{n-1}}$ and $\check{\bar{s}} = \check{a}_{s_0} \hat{\ldots} \hat{a}_{s_{n-1}}$ realizes the same type (over $A$) in $\mathbb{C}$.

2) We say that $\langle \bar{a}_u: u \in [I]^{<\aleph_0} \rangle$ is indiscernible (in $\mathbb{C}$, over $A$) similarly:

if $n < \omega, w_0, \ldots, w_{m-1} \subseteq \{0, \ldots, n-1\}$ and $\tilde{s} = \langle s_0: \ell < n \rangle, \ell = \langle t_0: \ell < n \rangle$ realize the same quantifier-free types in $I$ and $u_\ell = \langle s_k: k \in w_\ell \rangle, v_\ell = \langle t_k: k \in w_\ell \rangle$ then $\check{a}_{u_0} \hat{\ldots} \hat{a}_{u_{m-1}}, \check{a}_{v_0} \hat{\ldots} \hat{a}_{v_{m-1}}$ realize the same type in $\mathbb{C}$ (over $A$).

3) If $I$ is a linear order then we let $\text{incr}^{<\alpha}(I) = \text{incr}_\alpha(I) = \text{incr}(\alpha, I)$ be $\{\rho: \rho$ is an increasing sequence of length $\alpha$ of members of $I\}$; similarly $\text{incr}^{<\alpha\beta}(I) = \text{incr}^{<\alpha\beta}(I) = \cup\{\text{incr}_\beta(I): \beta < \alpha\}$. So instead $[I]^{<\aleph_0}$ we may use $\text{incr}^{<\omega}(I)$; clearly the difference is notational only.

---

$^1$ i.e. $\Phi$ proper for $K_\omega$, i.e. normal trees with $\omega + 1$ level, with linear order on the successor of each node of finite level, see Definition 2.2(7) or [Sh:c, Ch.VII]
§ 1. GAMES, EQUIVALENCES AND QUESTIONS

What is the meaning in using $E^+_\gamma,\lambda$? Consider for various $\gamma$’s the game $\overset{\gamma,\lambda}{\square}(M_1, M_2)$ where $M_1, M_2 \in \text{Mod}_T(\lambda)$, $T$ complete first order $\mathbb{L}(\tau)$-theory. During a play we can consider dependence relations on “short” sequences from $M_\ell$ (where $\ell \geq |\tau|+\omega$ is the default value), definable in a suitable sense. So if $T$ is a well understood unsuper- stable $T$ like $\text{Th}(\omega, E_n)_{n<\omega}$ with $E_n := \{(\eta, \nu) : \eta, \nu \in \omega \omega$ and $\eta \upharpoonright n = \nu \upharpoonright n\}$, then even for $\gamma = \omega + 2$ we have $E^+_{\gamma,\lambda}$-equivalence implies being isomorphic. This fits the thesis:

Thesis 1.1. The desirable dichotomy characterized, on the family of first order $T$, by the property “$M_1, M_2 \in \text{Mod}_T(\lambda)$ are long game $E^+$-equivalent iff they are isomorphic”, is quite similar to the one in [Sh:c, Ch.XIII]: the structure side is e.g.: $T$ is stable and every $M \in \text{Mod}_T$ is prime over some $\cup\{M_\eta : \eta \in I\}$, where $I$ is a subtree of $\kappa(T)>|M|$ and $\eta \vartriangleleft \nu \Rightarrow M_\eta \prec M_\nu \prec M, |M_\nu| \leq 2^{|\tau|}$ and $\eta \vartriangleleft \nu \in I \Rightarrow \text{tp}(M_\eta, \cup\{M_\rho : \rho \in I, \rho \upharpoonright (|(\ell (\eta (\nu) + 1)) \neq \eta \upharpoonright (|\ell (\nu (\nu) + 1))\})$ does not fork over $M_\nu$, i.e. $M = \langle M_\eta : \eta \in I \rangle$ is a non-forking tree of models with $\leq \kappa(T)$ many levels.

We think the right (variant of the) question is from 1.2. Probably a reasonable analog is the situation in [Sh:c, Ch.XII, XIII]: the original question was on the function $\lambda \mapsto I(\lambda, T)$, the number of non-isomorphic models; but the answer is more transparent for $\lambda \mapsto I E(\lambda, T)$.

If $\lambda = \mu^+, \mu = \mu^{[T]} = \text{cf}(\mu), T = \text{Th}(\omega, E_n)_{n<\omega}$ then by Hyttinen-Shelah [HySh:474, Th.4.4]: for $\gamma \geq \mu_\omega$ we get equivalence $\Rightarrow$ isomorphic, but not for $\gamma < \mu_\omega$; now 1.9 is parallel to that. This seems to indicate that $E^+_{\gamma,\lambda}$ is suitable for the questions we are asking: it uses the game $E^+$, which is more complicated but the length of the game is much “smaller” in the relevant results.

So the natural question concerning such equivalences is (see [Sh:c], [Sh:E53]):

Question 1.2. Classify first order complete $T$, or at least the countable ones by:

Version (A)_1: For every ordinal $\alpha$, there are a cardinal $\lambda$ and non-isomorphic $M_1, M_2 \in \text{Mod}_T(\lambda)$ which are $E^+_{\alpha,\lambda}$-equivalent (at least, e.g. in some $V^P, P$ is $(2^{|T|+|\alpha|})^+$-complete forcing notion).

Version (A)_0: Similar version for $E_{\alpha,\lambda}$.

Version (B)_1: For every cardinal $\kappa > |T|$ and vocabulary $\tau_1 \supseteq \tau_T$ and $\psi \in \mathbb{L}_{\kappa, \omega}(\tau_1)$ such that $PC_\tau(\psi) \subseteq \text{EC}_T$ has members of arbitrarily large cardinality we have

(a) $\Rightarrow$ (b) where

(a) for every cardinal $\mu$ in $PC_\tau(\psi) := \{M \mid \tau : M$ a model of $\psi\}$ there is a $\mu$-saturated member

(b) for every $\alpha$ for arbitrarily large $\lambda$ there are $M_1, M_2 \in PC_\tau(\psi)$ of cardinality $\lambda$ with non-isomorphic $\tau$-reducts which are $E^+_{\alpha,\lambda}$-equivalent.

Version (B)_0: Like (B)_1 for $E_{\alpha,\lambda}$.

Version (C)_1: Like (B)_1 using $\psi = \wedge T_1$ where $T_1$ is first order $\supseteq T$.

Version (C)_0: Like (B)_0 using $\psi = \wedge T_1$ where $T_1$ is a first order $\supseteq T$. 

Discussion 1.3. 1) For reasons to prefer version (B) over (C) - see [Sh:E53].
2) Now by the works quoted above, (see [HySh:529, 3.19] quoted in 4.1 below):
   $T$ satisfies $(A)_0$ iff $T$ is superstable NDOP, OTOP iff $(B)_0$. Of course if we change the order of the quantifier (to “for arbitrarily large some $\lambda$ for every $\alpha < \lambda$,...”) this is not so, but we believe solving $(A)_1$ and/or $(B)_1$ will eventually do much also for this.

So all this means

Conjecture 1.4. 1) For a complete (first order) $T$ the following are equivalent:
   \begin{enumerate}
   \item[$(a)$] for every ordinal $\alpha$ for some $\lambda$ there are non-isomorphic, $\text{EF}^+_\alpha,\lambda$-equivalent models $M_1, M_2 \in \text{EC}_T(N)$
   \item[$(b)$] for arbitrarily large $\lambda$ for every $\alpha < \lambda$ there are non-isomorphisms, $\text{EF}^+_\alpha,\lambda$-equivalent models $M_1, M_2 \in \text{EC}_T(\lambda)$
   \item[$(c)$] for every large enough regular $\lambda$ there are non-isomorphisms $M_1, M_2 \in \text{EC}_T(\lambda)$ which are $E_{\alpha,\lambda}$-equivalent for every $\alpha < \lambda$.
   \end{enumerate}

2) Similarly for “some $T_1 \supseteq T$, PC($T_1, T$) is lean.

We conjecture that proving that if we prove that a (countable) fat $T$ is close enough to superstable, will enable us to generalize proofs in [Sh:c, Ch.XII] only now the tree has $\leq \omega_1$ levels rather than $\omega$.

We can also return to the ordinals $\alpha \in (\lambda \omega, \lambda^+)$.

\* \* \*

Now we shall actually look at the games.

Definition 1.5. 1) "We say that $M_1, M_2$ are $\text{EF}_\alpha$-equivalent if $M_1, M_2$ are models (with same vocabulary) and $\alpha$ is an ordinal such that the isomorphism player has a winning strategy in the game $\mathcal{G}_\alpha^\alpha(M_1, M_2)$ defined below.

1A) Replacing $\alpha < \alpha$ means: for every $\beta < \alpha$; similarly below.

2) We say that $M_1, M_2$ are $\text{EF}_{\alpha,\mu}$-equivalent or $G_{\alpha,\mu}^\alpha$-equivalent when $M_1, M_2$ are models with the same vocabulary, $\alpha$ an ordinal, $\mu$ a cardinal such that the isomorphism player has a winning strategy in the game $\mathcal{G}_{\alpha,\mu}^\alpha(M_1, M_2)$ defined below.

3) For $M_1, M_2, \alpha, \mu$ as above and partial isomorphism $f$ from $M_1$ into $M_2$ we define the game $\mathcal{G}_{\mu}^\alpha(f, M_1, M_2)$ between the players ISO, the isomorphism player and AIS, the anti-isomorphism player as follows:

\begin{enumerate}
\item[$(a)$] a play lasts $\alpha$ moves
\item[$(b)$] after $\beta$ moves a partial isomorphism $f_\beta$ from $M_1$ into $M_2$ is chosen, increasing continuous with $\beta$
\item[$(c)$] in the $(\beta + 1)$-th move, the player AIS chooses $A_{\beta,1} \subseteq M_1, A_{\beta,2} \subseteq M_2$ such that $|A_{\beta,1}| + |A_{\beta,2}| < 1 + \mu$ and then the player ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that $A_{\beta,1} \subseteq \text{Dom}(f_{\beta+1})$ and $A_{\beta,2} \subseteq \text{Rang}(f_{\beta+1})$
\item[$(d)$] if $\beta = 0$, ISO chooses $f_0 = f$; if $\beta$ is a limit ordinal ISO chooses $f_\beta = \cup\{f_\gamma : \gamma < \beta\}$.
\end{enumerate}

The ISO player loses if he had no legal move for some $\beta < \alpha$, otherwise he wins the play.

4) If $f = \emptyset$ we may write $\mathcal{G}_{\mu}^\alpha(M_1, M_2)$. If $\mu$ is 1 we may omit it. We may write $\leq \mu$ instead of $\mu^+$.\]
Discussion 1.6. 1) Why do we need $\text{EF}^+$?

First, if we like a parallel of [Sh:c, Ch.XIII], i.e., a game in which set of small cardinality are chosen, say $|T|$ or $2^{[T]}$ or whatever rather than just $<\lambda = ||M||$, clearly $\text{EF}_{\alpha,\mu}$ cannot help.

2) Also, consider $\lambda = \mu^+, \mu = cf(\mu) > |T|$ and an ordinal $\alpha < \lambda$ and ask for which $T$: for any two models $M_1, M_2$ of $T$ of cardinality $\lambda$, $\text{EF}_{\alpha,\lambda}$-equivalence implies isomorphisms? (The $\text{EF}_{\alpha,\lambda}$-equivalence means that the isomorphism player wins in the game of length $\alpha$, in each step adding $\leq \mu$ elements to the domain and range of the partial isomorphism.)

Now we know (by earlier works, see 4.5) for countable $T$ that if $\alpha \in [\omega, \mu \times \omega]$ that the answer (for the pair $(\alpha, \lambda)$) is as in the main gap for $\check{\text{I}}E$ ($T$ superstable with NDOP and NOTOP). But for larger $\alpha < \lambda$ this is not so, e.g., for the prototypical stable unsuperstable $T$ for $\alpha = \mu \times (\omega + 2)$ we get yes, “it is low”.

3) Looking at the reason for this, i.e., why we need $\mu \times (\omega + 2)$, moves, not $(\omega + 2)$ moves we formulate $\text{EF}^+$. We think that with $\text{EF}^+_{\alpha, \theta, \mu, \lambda}$ for small $\alpha, \theta, \mu$ and just $\lambda = ||M||$ we get the desired dichotomy. In general, we expect the results will be robust under choosing such an exact game; and will resolve the case $\alpha \in (\mu \times (\omega, 2), \lambda)$ case above.

4) More specifically, the reason $\text{EF}_{\alpha, \lambda}$-equivalence does not imply isomorphisms for $M_1, M_2 \in \text{EC}_\lambda(T)$, even in the case $T = \text{Th}(\omega^\omega, E_\alpha)_{\alpha < \omega}$, is that: assume we fix a winning strategy $\textbf{st}$ for $\mathcal{G}_{\alpha, \lambda}(M_1, M_2)$, if we let $\{a_\alpha^0/E_1^M : \alpha < \lambda\}$ list $M_1/E_1^M$ and $R = \{(\alpha, \beta) : \text{in some short initial segment } x \text{ of a play of } \mathcal{G}_{\alpha, \lambda}(M_1, M_2) \text{ in which the player ISO uses the strategy } \textbf{st}, \text{ we have } f_\alpha^x(a_\alpha^0)E_1^M a_\alpha^1\}$, we have to find a function $h$ from $\lambda$ onto $\lambda$ whose graph is $\subseteq R$.

Now being in a winning position is enough to show the existence of such $h$, only when the game is long enough. For $\text{EF}^+_{\alpha, \theta}$ this is different.

5) Note: we use the case $k = 1$ from 1.7. If we shall have good structure theorems then even $k = 2$ is O.K. For $k = \kappa$ it expresses the logic in [Sh:c, Ch.XIII] when we add the game quantifier of appropriate length.

6) Of course, the case $k = 0$ is easier for ISO then the case $k = 2$ which is easier than $k = 1$, so the relevant implications hold.

Definition 1.7. 1) For $k \in \{0, 1, 2\}$ the models $M_1, M_2$ are $\text{EF}^+_{\alpha, \theta, \mu, \lambda}$-equivalent, but if $k = 1$ we may omit it, when the isomorphism player, ISO, has a winning strategy in the game $\mathcal{G}_{\alpha, \theta, \mu, \lambda}^k(M_1, M_2)$ defined below.

We always assume $\theta_0 \leq \theta \leq \mu$. If $\mu = \min\{||M_1||, ||M_2||\}$ then we may omit it. If also $\theta = (2||\tau(M_3)||^{+\kappa_0})^+$ we may omit it, too.

2) For an ordinal $\gamma$, cardinals $\theta \leq \mu$, vocabulary $\tau$ and $\tau$-models $M_1, M_2$ and partial isomorphism $f$ from $M_1$ to $M_2$, we define a game $\mathcal{G}^k = \mathcal{G}_{\gamma, \theta, \mu, \lambda}^k(f, M_1, M_2)$, between the player ISO (isomorphism) and AIS (anti-isomorphism).

A play last $\gamma$ moves; in the $\beta$-th move a partial isomorphism $f_\beta$ from $M_1$ to $M_2$ is chosen by ISO, extending $f_\alpha$ for $\alpha < \beta$ such that $f_0 = f$ and for limit $\beta$ we have $f_\beta = \bigcup\{f_\alpha : \alpha < \beta\}$ and for every $\beta < \alpha$ the set $\text{Dom}(f_{\beta+1}) \setminus \text{Dom}(f_\beta)$ has cardinality $< 1 + \mu$; let $f^\ell_\beta$ be $f_\beta$ if $\ell = 1, f^{\ell-1}_\beta$ if $\ell = 2$.

During a play, the player ISO loses if he has no legal move and he wins in the end of the play iff he always had a legal move.

In the $(\beta + 1)$-th move, the AIS player does one of the following cases:
Case 1: The AIS player chooses $A_\ell = A_\beta^\ell \subseteq M_\ell$ for $\ell = 1, 2$ such that $|A_1| + |A_2| < 1 + \mu$ and then ISO chooses $f_\beta$ as above such that $A_\ell \subseteq \text{Dom}(f_\beta^\ell)$ for $\ell = 1, 2$.

Case 2: First the AIS player chooses a pre-dependence relation $R_\ell$ on $^{a_\ell}(M_\ell)$ (see Definition 1.8 below) and $\mathcal{A}_\ell \subseteq ^\ell(M_\ell)$ of cardinality $\leq \lambda$ for $\ell = 1, 2$ such that:

(1) if $k = 0$ then $R_\ell = [^{a_\ell}(M_\ell)]^{<\lambda}$, so really an empty case

(b) if $k = 1, 2$ then $R_\ell$ is a 1-dependence relation (see 1.8(4)(b)(B) below)

(c) if $k = 1, 2$ and $\ell = 1, 2$ and $n < \omega$ and $\vec{a}_0, \ldots, \vec{a}_{n-1} \in ^\ell(M_\ell)$ then the truth value of $\{\vec{a}_0, \ldots, \vec{a}_{n-1}\} \in R_\ell$ depends just on the complete first order type which $(\vec{a}_0, \ldots, \vec{a}_{n-1})$ realizes on $\text{Dom}(f_\beta^\ell)$ inside the model $M_\ell$.

Second, the ISO does one of the following:

Subcase 2A: First, assume $k = 2$. The player ISO chooses $\langle (\vec{a}_k^1, \vec{a}_k^2) : \zeta < \lambda \rangle$ such that for $\ell = 1, 2$:

(a) for each $\zeta < \lambda$ for some $\epsilon < \theta$ we have $\vec{a}_\zeta^\ell \in ^\epsilon(M_\ell)$

(b) $\langle \vec{a}_\zeta^\ell : \zeta < \lambda \rangle$ is dependent for $R_\ell$

(c) each $\vec{a} \in \mathcal{A}_\ell$ does $R_\ell$-depend on $\{\vec{a}_\zeta^\ell : \zeta < \lambda\}$.

Then AIS chooses $\zeta < \lambda$ and ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that $f_\beta(\vec{a}_1^1) = \vec{a}_2^1$.

Second, assume $k = 1$. Then the ISO player chooses equivalence relations $E_\ell$ on $^{a_\ell}(M_\ell)$ which the dependence relation, i.e. $E_{\mathcal{R}_\ell}$, i.e. 1.8(6) and equality of length refine for $\ell = 1, 2$ and choose a function $h$ from the family of $E_1$-equivalence classes onto the family of $E_2$-equivalence classes which preserve cardinality up to $\lambda$; that is, if $h(\vec{a}_1/E_1) = \vec{a}_2/E_2$ then $\ell g(\vec{a}_1) = \ell g(\vec{a}_2)$ and $\min\{\text{dim}(\vec{a}_1/E_1), \lambda\} = \min\{\text{dim}(\vec{a}_2/E_2), \lambda\}$.

Then the AIS player chooses a pair $(\vec{a}_1, \vec{a}_2)$ such that $\vec{a}_\ell \in ^\ell(M_\ell)$ for $\ell = 1, 2$ such that $h(\vec{a}_1/E_1) = (\vec{a}_2/E_2)$ and ISO has to choose $f_{\beta+1} \supseteq f_\beta$ such that $f(\vec{a}_1) = \vec{a}_2$.

Subcase 2B: The player ISO chooses $f_{\beta+1} \supseteq f_\beta$ as required such that for some $n < \omega$ and $\vec{a}_1^\ell \in ^\ell(M_\ell)$ for $\ell \leq n$ we have: $\{\vec{a}_0^1, \ldots, \vec{a}_n^1\}$ is $R_1$-dependent if $\{f_\beta(\vec{a}_0^1), \ldots, f_\beta(\vec{a}_n^1)\}$ is not $R_2$-dependent.

Definition 1.8. 1) We say $R$ is a pre-dependence relation on $X$ when $R$ is a subset of $[X]^{<\omega}$.

2) For $X, R$ as above, we say $Y \subseteq X$ is $R$-independent when $[Y]^{<\omega} \cap R = \emptyset$; of course, an index set with repetitions is considered dependent.

3) We say $R$ or $(X, R)$ has character $\leq \kappa$ when for every $R$-independent $Y \subseteq X$ and $\{x\} \subseteq X$ for some $Z \subseteq [Y]^{<\kappa}$ the set $(Y \setminus Z) \cup \{x\}$ is $R$-independent.

4) We say that $R$ is a $k$-dependence relation on $X$ (if $k = 1$ we may omit it) when:

\[1a.15\]
(a) \( R \) is a subset of \([X]^{<\aleph_0}\)

(b) (α) if \( k = 0 \) then \( R = [X]^{<\aleph_0} \)

(β) if \( k = 1 \) then \( R \)-independence satisfies the exchange principle

(\text{so dimension is well defined, as for regular types}).

5) We say \( R \) is trivial when for every \( Y \subseteq X, Y \) is \( R \)-independent iff every \( Z \subseteq [Y]^{\leq 2} \) is \( R \)-independent.

6) For \( R \) as in (a), (b)(β) let \( E_R = \{ \{ x_1, x_2 \} : x_1 = x_2 \in X \text{ on } \{ \lambda_1 \}, \{ x_2 \} \in R \text{ or } \{ x_1, x_2 \} \in R \land \{ x_1 \} \notin R \land \{ x_2 \} \notin R \} \) is an equivalence relation on \( X \); pedantically we should write \( E_X, R \).

\{1a.19\}

Claim 1.9. \( M_1, M_2 \) are isomorphic when:

(a) \( M_1, M_2 \) are models of \( T \) of cardinality \( \lambda \)

(b) \( M_1, M_2 \) are \( EF_{\omega+2, \aleph_0, \aleph_0}^+ \)-equivalent

(c) \( T = \text{Th}(\omega, E_n)_{n<\omega} \) and \( E_n = \{ (\eta, \nu) : \eta \in \omega, \nu \in \omega, \text{ and } \eta \upharpoonright n = \nu \downharpoonright n \} \).

Proof. Step A: We choose a winning strategy \( \mathbf{w} \) of the isomorphism player in the game \( \mathcal{G}_{\omega+2, \aleph_0, \aleph_0, \lambda}(M_1, M_2) \).

Step B: By the choice of \( T \) for \( \ell = 1, 2 \) we can find \( \mathcal{T}_\ell, \bar{a}_\ell \) such that:

\( \exists \ell \)

(a) \( \mathcal{T}_\ell \) is a subtree of \( \omega^{>\lambda} \)

(b) \( \bar{a}_\ell = (a^\ell_\eta : \eta \in \mathcal{T}_\ell) \)

(c) \( a^\ell_\eta \in M_\ell \)

(d) if \( \eta \in \mathcal{T}_\ell \) and \( \ell g(\eta) = n \) then \( (a^\ell_\nu / E^{M_\ell}_{n+1} : \nu \in \text{ suc } \mathcal{T}_\ell(\eta)) \) list

\( \{ b / E^{M_\ell}_{n+1} : b \in M_\ell, b \in a^\ell_\eta / E^{M_\ell}_n \} \) without repetitions.

Let \( \mathcal{T}_{\ell,n} = \{ \eta \in \mathcal{T}_\ell : \ell g(\eta) = n \} \) and let \( \mathcal{T}_{\ell,\omega} = \{ \eta \in \omega : \eta \upharpoonright n \in \mathcal{T}_\ell \text{ for every } n < \omega \} \).

Lastly, let \( \bar{\mu}_\ell = (\mu^\ell_\eta : \eta \in \mathcal{T}_{\ell,\omega}) \), where

\( \mu^\ell_\eta = | \{ b \in M_\ell : b \in a^\ell_\eta / E^{M_\ell}_n \text{ for every } n < \omega \} | \).

Step C:

Clearly

\( \Box \) \( M_1, M_2 \) are isomorphic iff there is an isomorphism \( h \) from \( \mathcal{T}_1 \) onto \( \mathcal{T}_2 \) (i.e. \( h \) maps \( \mathcal{T}_{1,n} \) onto \( \mathcal{T}_{2,n} \), \( h \) preserves the length, \( \eta \downarrow \nu \) and \( \eta \not\downarrow \nu \) such that letting \( h_n = h \upharpoonright \mathcal{T}_{1,n} \) and \( h_\omega \) be the mapping from \( \mathcal{T}_{1,\omega} \) onto \( \mathcal{T}_{2,\omega} \) which \( h \) induces (so \( h_\omega(\eta) = \bigcup_{n<\omega} h_n(\eta \upharpoonright n) \)) we have \( \eta \in \mathcal{T}_{1,\omega} \Rightarrow \mu^1_\eta = \mu^2_{h_\omega(\eta)} \).

Step D:

By induction on \( n \) we choose \( h_n, x_n \) such that

\( \circ \)

(a) \( h_n \) is a one-to-one mapping from \( \mathcal{T}_{1,n} \) onto \( \mathcal{T}_{2,n} \)

(b) if \( m < n \) and \( \eta \in \mathcal{T}_{1,n} \) then \( h_m(\eta \upharpoonright m) = (h_n(\eta)) \upharpoonright m \)

(c) \( x_n = (x^n_\eta : \eta \in \mathcal{T}_{1,n}) \)
(d) \((\alpha)\) \(x^0_n\) is an initial segment of a play of the game \(\omega_+2,\kappa_0,\kappa_0,\lambda(M_1, M_2)\)

\[(\beta)\] in \(x^0_n\) only finitely many moves have been played (can specify), the last one is \(m(x^0_n)\)

\[(\gamma)\] in \(x^0_n\), the player ISO uses his winning strategy \(\textbf{st}\)

(e) if \(\eta_1 \in \mathcal{T}_{1,n}\) and \(\eta_2 = h_n(\eta_1)\), then for some \(b_1 \in \text{Dom}(f^x_n(\eta_1))\) we have

\[(\alpha)\] \(b_1 \in a^1_{\eta_1}/E^M_n\)

\[(\beta)\] \(f^x_n(b_1) \in a^2_{\eta_1(n)}/E^M_2\)

\[(f)\] if \(\nu \in \mathcal{T}_{1,n}\) then \(x^{\nu(\nu)}_\eta\) is an initial segment of \(x^\eta_n\).

Why can we carry the induction?

For \(n = 0\):

Note that \(h_0\) is uniquely determined. As for \(x^0_{>\epsilon}\), any \(x\) as in \(\oplus\)(d) is O.K., as long as at least one move was done (note that \(E^M_0\) has one and only one equivalence class.

For \(n = m + 1\): So \(h_m, x_m\) has been chosen.

Let \(\eta_1 \in \mathcal{T}_{1,m}\) and let \(\eta_2 = h_m(\eta_1)\) and

\[
\textbf{F}_{\eta_1} := \{ (\nu_1, \nu_2) : \nu_1 \in \text{suc}\mathcal{T}_{1}(\eta_1), \nu_2 \in \text{suc}\mathcal{T}_{1}(\eta_2) \}
\]

and there is \(x\) as in \(\oplus\)(d) such that \(x^m_n\) is an initial segment of \(x\) and for some \(b_1 \in \text{Dom}(f^x_m(\eta_1))\) we have

\[b_1 \in a^1_{\nu_1}/E^M_1\text{ and }f^x(b_1) \in a^2_{\nu_2}/E^M_2\}\]

Now

\(\circ\) to do the induction step, it suffices to prove that: if \(\eta_1 \in \mathcal{T}_{1,m}\) then there is a one-to-one function \(h_{m,\eta_1}\) from \(\text{suc}\mathcal{T}_{1}(\eta_1)\) onto \(\text{suc}\mathcal{T}_{1}(\eta_2)\) such that \(\nu \in \text{suc}\mathcal{T}_{1}(\eta_1)\) \(\Rightarrow\) \((\nu, h_{m,\eta_1}(\nu)) \in \textbf{F}_{\eta_1}\).

However by case 2 in Definition 1.7 this holds.

Stage E:

So we can find \(\langle h_n : n < \omega \rangle\), \(\langle x_n : n \in \mathcal{T} \rangle\) as in \(\circ\). Let \(h := \cup\{ h_n : n < \omega \}\), clearly it is an isomorphism from \(\mathcal{T}_{1}\) onto \(\mathcal{T}_{2}\) and \(h_\omega\) is well defined, see \(\oplus\) from Stage C.

So it is enough to check the sufficient condition for \(M_1 \cong M_2\), i.e. \(\eta \in \mathcal{T}_{1,\omega} \Rightarrow \mu_{1,\eta} = \mu_{2, h_{\eta}(\eta)}\). But if \(\eta \in \mathcal{T}_{1,\omega}\) then \(\langle x^\eta_{\eta(n)} : n < \omega \rangle\) is a sequence of initial segments of a play of \(\mathcal{G}\) with ISO using his winning strategy \(\textbf{st}\), increasing with \(n\), each with finitely many moves. So \(x^\eta_{\eta}\), defined as the limit \(\langle x^\eta_{\eta(n)} : n < \omega \rangle\), is an initial segment of the play \(\mathcal{G}\), with \(\omega\) moves and \(f^x_{m(x^\eta)} = \cup\{ f^x_{m(x^\eta(n))} : n < \omega \}\).

Clearly \(n < \omega \Rightarrow f^x_{m(x^\eta_n)} E^M_2 a^2_{h_{\eta}(\eta(n))}\). As we have one move left and can use case 2 in Definition 1.7(2) we are done.

\(\square\)

The following claim says that the game in 1.5, 1.7 when \(\lambda = \mu^+, \alpha < \lambda\) divisible enough are equivalent, i.e. the ISO player wins one iff he wins the other.
Claim 1.10. 1) $M_1, M_2$ are $\text{EF}^+_{\gamma, \theta, \mu, \lambda}$-equivalent when:

(a) $M_1, M_2$ are $\tau$-models

(b) $\lambda = \lambda_1^\tau, \lambda_1 \geq \mu$ and $\theta \leq \mu \leq \lambda$ and $\gamma \leq \mu$ and $\text{cf}(\mu) < \mu \Rightarrow \lambda_1 > \mu$ and $\lambda \in I[\lambda]$, see Definition 0.5

(c) $M_1, M_2$ are $\text{EF}_{\gamma(*)}^+_{\mu}$-equivalent where $\gamma(*) = \lambda_1 \times \gamma$ (see Definition 1.5(2))

(d) $\|M_\ell\| = \lambda = \lambda^{<\theta}$ for $\ell = 1, 2$

2) $M_1, M_2$ are $\text{EF}^+_{\gamma, \mu}$-equivalent when they are $\text{EF}_{\gamma, \theta, \mu, \lambda}$-equivalent.

3) $M_1, M_2$ are $\text{EF}^+_{\gamma_1, \theta_1, \mu_1, \lambda_1}$-equivalent when they are $\text{EF}^+_{\gamma_2, \theta_2, \mu_2, \lambda_2}$-equivalent and $\gamma_1 \leq \gamma_2, \theta_1 \leq \theta_2, \mu_1 \leq \mu_2, \lambda_1 \leq \lambda_2$.

Proof. 1) First, we do not save on $\gamma(*)$, say use $\lambda_1 \times \lambda_1 \times \gamma$.

Let $\text{st}$ be a winning strategy of the ISO player in the game $\mathcal{G}^\gamma(*)$. We try to use it as a winning strategy of the ISO player in the game $\mathcal{G}_{\gamma, \theta, \mu, \lambda}(M_1, M_0)$. Well, the $f_\alpha^\gamma$ may have too large a domain, so “on the side” in the $\beta$-th move ISO play $x_\beta$ for $\mathcal{G}_{\gamma, \theta, \mu, \lambda}$ and $A_1^\beta \subseteq \text{Dom}(f_\alpha^\gamma)$ of cardinality $< \mu$ (or $\leq \mu$ if $\mu > \text{cf}(\mu) \land \beta \geq \text{cf}(\mu)$) and he actually plays $f_\alpha^\gamma : A_1^\beta$, i.e. is an initial segment of a play of $\mathcal{G}_{\gamma, \mu}$ of length $\beta$ in which the ISO player uses the strategy $\text{st}$ such that $[\beta_1 < \beta \Rightarrow x_{\beta_1}$ is an initial segment of $x_\beta]$. The only problem is when $\beta = \alpha + 1$ and in Definition 1.7, Case 2 occurs, i.e. with the AIS player choosing $R_1^\beta, R_2^\beta$. We may for notational simplicity choose $\varepsilon < \theta$ and deal only with $A_\ell \cap \varepsilon(M_\ell)$ for $\ell = 1, 2$.

We can consider $x_\beta$ extending $x_\alpha$; if it is as required in subcase (2B) of Definition 1.7 we are done. Let

$$F_\beta^1 = \{(\bar{a}_1, \bar{a}_2) : \text{ for some } \varepsilon < \theta, \bar{a}_\ell \in \varepsilon(M_\ell) \text{ for } \ell = 1, 2 \text{ and there is a candidate } x_\beta \text{ for the } \beta \text{-th move such that } f_\alpha^\varepsilon(\bar{a}_1) = \bar{a}_2\}.$$ 

Let

$$F_\beta^2 = \{(\bar{a}_2, \bar{a}_1) : (\bar{a}_1, \bar{a}_2) \in F_\beta^1\}$$

$$\mathcal{A}_\beta^1 = \mathcal{A}_\ell$$

$$\mathcal{A}_\beta^2 = \{\bar{a} \in \mathcal{A}_\ell : \text{ the number of } \bar{b} \text{ such that } (\bar{a}, \bar{b}) \in F_\beta^\varepsilon \leq \lambda\}$$

$$\mathcal{A}_\beta^3 = \mathcal{A}_\beta^2 \cup \{\bar{a} : \text{ for some } \bar{b} \in \mathcal{A}_\beta^2 \text{ we have } (\bar{a}, \bar{b}) \in F_\beta^\varepsilon\}.$$ 

So $|\mathcal{A}_\beta^3| \leq \lambda$ by clause (d) of the assumption and let $\langle \bar{a}_\zeta^\varepsilon : \zeta < \lambda\rangle$ list $\mathcal{A}_\beta^3$ possibly with repetitions

(*) it is enough to take care of $\mathcal{A}_\beta^3 \cap \mathcal{A}_\ell$ for $\ell = 1, 2$. 

(897)
Remark to the definition of $S_2$, 3) Obvious. □

Claim 1.12. We can use a variant of Definition 1.7(2) as follows: we can in case
So we simulate two plays and use the composition of the $f$ and $R_2$
assume $\sigma$ x $\delta>\gamma$ and $a_2$ is $R_2$-dependent over $\{\bar{a}^\xi : \xi < \delta\}$
if $\xi < \delta_4$ and $\bar{a}^\xi \in \mathcal{A} \subseteq \mathcal{R}$ then $a^\xi$ does $R_2$-depend on $\{\bar{a}^\xi : \xi < \delta\} \cup \{\bar{b}^\xi : \xi \in [\delta, \delta_3]\}$.
If $S$ is not stationary we can easily finish (we start by playing $\omega$ moves in $\mathcal{Q}_\gamma$). So assume $S$ is stationary, hence for some regular $\sigma = \lambda_1$ the set $\mathcal{S}' = \{\delta \in S : d(\delta) = \sigma\}$ is stationary. By playing $\sigma + \omega$ moves (recalling $\lambda \in I[\lambda]$) we get a contradiction to the definition of $S$.
2), 3) Obvious.

Remark 1.11. In 1.10(1), to get the exact $\gamma(*)$, we combine partial isomorphisms.
So we simulate two plays and use the composition of the $f^{x_1}$’s from two plays where
in each ISO use a winning strategy $\mathcal{S}$.

Claim 1.12. We can use a variant of Definition 1.7(2) as follows: we can in case
2 make a $R_2$ dependence relation on $\kappa \times ^\theta>(M_2)$, but equivalently $C \times ^\theta>(M_2)$ for
a set $C$ of cardinality $\leq \kappa$, but
(a) it seems to help presently relevant only for $\kappa \leq 2^{\tau(M_1)} + \aleph_0$
(b) if $\kappa \leq 2^{<\theta}$ we get an equivalent game.

Remark 1.13. 1) We can replace $2^{<\theta}$ by a larger cardinal in clause (b) for “interesting” cases of $M_1, M_2$.
2) Anyhow we use only (b).

Proof. Clause (a) is obvious.

For clause (b), without loss of generality $\|M_t\| > 1$, now let $\langle \eta_\alpha : \alpha < \kappa \rangle$ be
a sequence of pairwise distinct members of $\kappa > 2$. now we define $F_t : ^\theta>(M_t) \rightarrow (2^{<\theta}) \times ^\theta>(M_t)$ as follows: for $\bar{a} \in ^\theta>(M_t)$ let
\[
i(\bar{a}) = \min \{i : 2i \geq \ell g(\bar{a}) \text{ or } 2i + 1 < \ell g(\bar{a}) \land a_{2i} \neq a_{2i+1}\}
\]
\[
\eta_\alpha = \langle T.V.(a_{2i(\alpha)+2+2j}) = a_{2i(\alpha)+2+2j+1}) : j \geq 0 \text{ and } 2i(\bar{a}) + 2 + 2j + 1 < \ell g(\bar{a}) \rangle
\]
where T.V. stands for “truth value”.
\[
\alpha(\bar{a}) = \text{Min}\{\alpha \leq \kappa : \text{if } \alpha < \kappa \text{ then } \eta_\alpha = \eta_\alpha\}.
\]
Finally, $F_t(\bar{a}) \in (\alpha(\bar{a}), \langle a_{2j} : j < i(\bar{a}) \rangle) \text{ if } i(\alpha) < \ell g(\bar{a}) \land \alpha(\bar{a}) < \kappa, \text{ and is } (0, \bar{a}) \text{ if otherwise}.$

Let
\[
R'_t := \{\mathcal{A} : \mathcal{A} \subseteq ^\theta>(M_t) \text{ and } \{F(\bar{a}) : \bar{a} \in \mathcal{A}\} \in \mathcal{R}
\text{ or for some } \bar{a}' \neq \bar{a}'' \in \mathcal{A} \text{ we have } F(\bar{a}') = \bar{a}''\}.
\]
Now check. □_{1.12}

Claim 1.14. \( M_1, M_2 \) are \( \text{EF}_{\gamma, \theta, \mu, \lambda}^+ \)-equivalent when:

(a) \( K \) a class of \( \tau_0 \)-structures and \( \Phi \in \Upsilon[K] \), see 2.2(8), used here for \( K = K_{\text{or}} \) the class of linear orders and \( K_{\text{oi}} \), see Definition 2.1

(b) the structures \( I_1, I_2 \in K \) are \( \text{EF}_{\gamma, \theta, \mu, \lambda}^+ \)-equivalent

(c) \( M_\ell = \text{EM}_\tau(I_\ell, \Phi) \) for \( \ell = 1, 2 \) for some \( \tau \subseteq \tau_\Phi \)

(d) \( \mu \geq \aleph_0 \) and \( |\tau_\Phi| < \theta \).

Proof. Let \( \text{St} \) be a winning strategy of the ISO player in the game \( \mathcal{G}_{\gamma, \theta, \mu, \lambda}(I_1, I_2) \). We define a strategy \( \text{st}^* \) of the ISO player in the game \( \mathcal{G}_{\gamma, \theta, \mu, \lambda}(M_1, M_2) \) as follows.

During a play of it after \( \beta \) moves a partial isomorphism \( f^* \) from \( M_1 \) to \( M_2 \) has been chosen, but the ISO player also simulates a play of \( \mathcal{G}_{\gamma, \theta, \mu, \lambda}(I_1, I_2) \) in which we call the function \( h_\alpha \), and in which he uses the winning strategy \( \text{st} \) and \( \Box f_\alpha \subseteq h_\alpha \) where \( h_\alpha \) is defined by

\[
\hat{h}_\alpha(a_{t_0}, \ldots, a_{t_{n-1}}) = \sigma^M_\ell(a_{b_a(t_0)}, \ldots, a_{b_a(t_{n-1})}) \quad \text{for } n < \omega, \sigma(x_0, \ldots, x_{n-1}) \text{ a term of } \tau_\Phi \text{ and } t_0, \ldots, t_{n-1} \in \text{Dom}(h_\alpha).
\]

Why can the player ISO carry this strategy \( \text{st}^* \)? Suppose we arrive to the \( \beta \)-th move. The point to check is Case 2 in Definition 1.7(2), so the AIS player has chosen \( R_1, R_2, \sigma_1, \sigma_2 \) as there.

Let \( \{\sigma_\zeta(\bar{x}_\zeta) : \zeta < 2^{<\theta}\} \) list \( \{\sigma(\bar{x}) : \sigma_i(\bar{x}) = (\sigma : i < \ell g(\sigma)), \ell g(\sigma) < \theta, \ell g(\bar{x}) < \theta \text{ and each } \sigma_i \text{ is a } \tau_\Phi \text{-term.} \]

Clearly \( \theta^>(M_\ell) = \{\sigma^M_\ell(i) : \zeta < 2^{<\theta} \text{ and } i \in \ell g(\bar{x}_\zeta)(I_\theta) \} \text{, so by clause (b) of 1.12, we can assume } \text{“} R_\ell \text{ is a dependence relation on } \{\zeta, \bar{t}_\zeta : \zeta < 2^{<\theta}, \bar{t}_\zeta \in \theta(I_\theta) \text{ and } \ell g(\bar{t}) = \ell g(\bar{t}_\zeta)\}. \)

That is

\[
R'_\ell = \{ u : \{\sigma^M_\zeta(\bar{t}) : (\zeta, \bar{t}) \in u \} \in \mathcal{R}_\ell \text{ or there are } (\zeta_1, \bar{t}_1) \neq (\zeta_2, \bar{t}_2) \text{ from } u \text{ such that } \sigma^M_\zeta_1(\bar{a}_{\bar{t}_1}) = \sigma^M_\zeta_2(\bar{a}_{\bar{t}_2}) \}.
\]

The rest should be clear. □_{1.14}
§ 2. The Properties of $T$ and Relevant Indiscernibility

In [Sh:c, Ch.VIII], [Sh:c, Ch.VI] we use as indiscernible sets trees with $\omega + 1$ levels, suitable for dealing with unsuperstable (complete first order) theories.

Here instead we use a linear order and family of $\omega$-sequences from it, suitable for our case. The letters “oi” stands for order $+$ increasing ($\omega$-sequences).

**Definition 2.1.** 1) $K_{oi}^\omega$ is the class of structures $J$ of the form $(J, Q, P <, F_n)_{n<\omega} = ([J], P^J, Q^J, <^J, F_n^J)$, where $J = [J]$ is a set of cardinality $\lambda, <^J$ a linear order on $Q^J \subseteq J, P^J = [J] \setminus Q^J, F_n^J$ a unary function, $P^J_n \supseteq P^J = J$ the identity and $a \in J \setminus Q^J \Rightarrow F_n^J(a) \in Q^J$ and $n \neq m \Rightarrow F_n^J(a) \neq F_m^J(a)$ and for simplicity $a \neq b \in P^M \Rightarrow \bigvee \bar{F}_n(a) \neq a_n(b)$; lastly, we add $n < m \Rightarrow F_n^J(a) <^J F_m^J(a)$ (there is a small price). We stipulate $F^J_\omega = \text{the identity on } [J]$ and $I^J = (Q^J, <^J)$.

\begin{enumerate}
1) $K_{oi} = \{ \lambda \in \text{a cardinal} \}$.
2) For a linear order $I$ and $\mathcal{G} \subseteq \text{inc}(\omega)I$ (see Definition 0.10(3)), we let $J = J_{I, \mathcal{G}}$ be the derived member of $K_{oi}$ which means: $|J| = |I| \cup \mathcal{G}, (Q^J, <^J) = I$, $F^J_n(\eta) = \eta(n)$ for $n < \omega, F^J_n(t) = t$ for $t \in I$.
3) $K_{oi}^\omega = \text{the class of linear order of cardinality } \lambda, K_{oi} = \bigcup \{ K_{oi}^\omega : \lambda \in \text{a cardinal} \}$. 

**Definition 2.2.** 1) For a vocabulary $\tau_1$ let $\Sigma_{\tau_1}$ be the class of functions $\Phi$ with domain $(tp_{\Phi}(\emptyset, J) : \emptyset \in \text{inc}(\omega)J, J \in K_{oi}^\omega)$, see 0.3 and if $q(x_0, \ldots, x_{n-1}) \in \text{Dom}(\Phi)$ then $\Phi(q)$ is a complete quantifier free $n$-type in $L(\tau_1)$ with the natural compatibility functions.

\begin{enumerate}
2) $\Sigma_{\tau_1}^\omega = \{ \Phi : \Phi \in \Sigma_{\tau_1} \}$ for some vocabulary $\tau_1$ of cardinality $\kappa$.
3) For $\Phi \in \Sigma_{\tau_1}^\omega$ let $\tau(\Phi) = \tau_{\Phi}$ be the vocabulary $\tau_1$ such that $\Phi \in \Sigma_{\tau_{\Phi}}^\omega$.
4) For $\Phi \in \Sigma_{\tau_1}^\omega, J \in K_{oi}^\omega$ let $EM(J, \Phi)$ be “the” $\tau_{\Phi}$-model $M_1$ generated by $\{a_t : t \in J\}$ such that: $n < \omega, t \in n \Rightarrow tp_{\Phi}(a_0, \ldots, a_{n-1}) = \Phi(tp_{\Phi}(\emptyset, a_0, \ldots, a_{n-1}), \emptyset, M_1)$.
5) If $\tau \subseteq \tau_{\Phi}$ then $EM_\tau(J, \Phi)$ is the $\tau$-reduct of $EM(J, \Phi)$.
6) Let $\Sigma_{\tau_1}^{\tau_{\Phi}}, \Sigma_{\tau_1}^{\tau_{\Phi}}$ and $EM(I, \Phi), EM_\tau(I, \Phi)$ be defined similarly for $J$ a linear order.
7) Let $\Sigma_{\tau_1, \text{tr}}^{\tau_{\Phi}}, \Sigma_{\tau_1, \text{tr}}^{\tau_{\Phi}}$ and $EM(I, \Phi), EM_\tau(I, \Phi)$ be defined similarly for $J \in K_{oi}^\omega$, i.e. trees with $\omega + 1$ levels (with a linear order on the successor of any member of level $\omega$).
8) We can above replace $K_{oi}$ by any class $K$ of $\tau_{\Phi}$-structures.

**Definition 2.3.** 1) A (complete first order) $T$ is $\aleph_0$-independent $\equiv$ not strongly dependent (this is from [Sh:783, §33], see [Sh:863, §1]) when: there is a sequence $\varphi = \{ \varphi_n(x,y_n) : n < \omega \}$, (may use finite $\bar{x}$, as usual does not matter by [Sh:863, 2.1]) of (first order) formulas such that $T$ is consist with $\Gamma_\lambda$ for some (equivalently $\lambda > \aleph_0$) where

$$\Gamma_\lambda = \{ \varphi_n(x_\eta, y_\eta)^{\text{Id}(\eta(n))} : \eta \in \omega, \alpha < \lambda, n < \omega \}.$$ 2) $T$ is strongly stable when it is stable and strongly dependent.

**Claim 2.4.** If $T$ is first order complete, $T_1 \supseteq T$ is first order complete, without loss of generality with Skolem functions and $T$ is not strongly dependent then we can find $\varphi = \{ \varphi_n(x,y_n) : n < \omega \}, y_n \subseteq y_{n+1}$ and $\varphi_n(x,y_n) \in L_{\{T\}}$ for $n < \omega$ such that

- for any $J \in K_{oi}$ we can find $M, \langle a_t : t \in J \rangle$ such that
- $(a)$ $M$ is the Skolem hull of $\{a_t : t \in J \}$

\begin{enumerate}
(b) $a_t \in \omega M$ for $t \in I^1$, $a_\eta = \langle a_\eta \rangle \in M_1$ for $\eta \in \mathcal{P}^I$

c) for $\eta \in \mathcal{P}^I$, $t \in Q^\mathcal{J}$ and $n < \omega$ we have $M \models \varphi_n[a_\eta, a_t]$ iff $F^\eta_n = t$; 
(pedantically we should write $\varphi_n(a_\eta, a_t \mid \ell g(\gamma_n))$

(d) $\langle a_t : t \in J \rangle$ is indiscernible in $M$ for the index model $J$

e) $M$ is a model of $T_1$

(f) in fact (not actually used, see 2.2) there is $\Phi \in \Upsilon_{|T_1|}$ depending on 
$T_1, \varphi$ only such that $M = EM(J, \Phi)$, in fact if $n < \omega, t = (t_\ell : \ell < n) \in J$
then $tp_{\varphi}(a_{t_0}, \ldots, a_{t_n-1}, \emptyset, M) = \Phi(tp_{\varphi}(t, \emptyset, J))$.

Proof. Let $I = (Q^J, <_J)$. By an assumption, i.e. 2.3 there is a sequence $\langle \varphi_n(x, y_n) \rangle : n < \omega$ as in Definition 2.3 and let $k_n = \ell g(\gamma_n)$.

Let $J$ be an infinite linear order. Easily we can find $M_1 \models T_1$ and a sequence 
$\langle a_t : t \in J \rangle$ with $a_t \in \omega(M_1)$ such that for every $\eta \in \omega I$, the set $\{ \varphi_n(x, a_t) \mid \ell g(\gamma_n) : t \in I, n < \omega \}$ is a type, i.e. finitely satisfiable in $M_1$.

Now by Ramsey theorem without loss of generality $\langle a_t : t \in I \rangle$ is an indiscernible sequence in $M_1$. Without loss of generality $M_1 \models \lambda^+ \text{-saturated}$, we then expand 
$M_1$ to $M_1^+$ by function $F_n^M$ ($n < \omega$), (of finite arity) such that for $t_0 <_J \ldots <_J t_n, n \in \omega I$ the element $F_n^M(a_{t_0}, a_{t_1}, \ldots, a_{t_n-1})$ or more exactly $F_n^M(a_{t_0}, k_0a_1, k_1, \ldots, a_{t_n-1} \upharpoonright \ell k_n-1)$ realizes in $M_1$ the type $\{ \varphi_n(x, a) \mid \ell g(\gamma_n) : t \in I, n < \omega \}$. 
Let $M_2^+$ be an expansion of $M_1^+$ by Skolem functions such that $|\tau_{M_2^+}| = |T_1|$, 
(natural, though not strictly required). Without loss of generality $\langle a_t : t \in I \rangle$ is an 
indiscernible sequence also in $M_2^+$.

Let $D$ be a non-principal ultrafilter on $\omega$ and in $M_2^+ = (M_2^+)^\omega / D$, we let $a_\eta' = 
\langle a_t : n < \omega \rangle / D$ for $t \in I$, and $a_\eta'' = \langle F_n^M(a_{t_0(0)}, a_{t_0(1)}, \ldots, a_{t_0(n-1)}) : n < \omega \rangle / D$ for 
$\eta \in \alpha < \omega I$ and $a_\eta' = a_\eta^\prime_{F^\eta_n(t), n < \omega}$ for $t \in \mathcal{P}^I$.

Let $M_2^+$ be the submodel of $M_2^+$ generated by $\langle a_t' : t \in J \rangle$ and $M$ be $\langle a_t' : t \in J, M \rangle$ as required. 
\[ \Box \]

Claim 2.5. Assume $J \in K_\omega$, and $M, \varphi, T_1, T$ as in 2.4 for $\ell = 1, 2$. A sufficient condition for $M_1 \models \tau_T \not\equiv M_2 \models \tau_T$ is:

(\#) if $f$ is a function from $J_1$ (i.e. its universe) into $\mathcal{M}_{\tau_{T_1}}(\mathcal{J}_2)$ (i.e. the free 
algebra generated by $\{ x_t : t \in J_2 \}$ in the vocabulary $\tau_{T_1, \mathcal{J}_2} = \{ F_n^\mathcal{J}_2 : n < \omega \}$ 
and $\alpha < |\mathcal{J}_2|$, $F_n^\mathcal{J}_2$ has arity $n$, see more in [Shc, Ch.III, §1] = [Sh:265]) we can find $t \in \mathcal{P}^{J_1}, n < \omega$, and $s_1, s_2 \in Q^{J_1}$ and $k, r^\ell_1(\ell = 1, 2$ and 
i < k), $m, r^\ell_2(\ell = 1, 2$ and 
i < k), $m$ such that:

(a) $F_n^\mathcal{J}_2(t) = s_1 \neq s_2$ 

(b) for $\ell \in \{ 1, 2 \}$ we have $f(s_\ell) = \sigma(r^\ell_1, \ldots, r^\ell_{k-1})$ so $k < \omega, r^\ell_k \in J_2$ for 
i < k and $\sigma$ is a $\tau_{\tau_{T_1, \mathcal{J}_2}}$-term not dependent on $\ell$

(\gamma) $\ell = 1, 2$ we have $f(s_\ell) = \sigma^*(r_0, \ldots, r_{m-1}),$ $\sigma^*$ is a $\tau_{\tau_{T_1, \mathcal{J}_2}}$-term and $r_0, \ldots, r_{m-1} \in J_2$

(d) the sequences

\[ r^1_i : i < k \}\upharpoonright {r^1_i : i < m} \]

\[ r^2_i : i < k \}\upharpoonright {r^2_i : i < m} \]

realize the same quantifier free type in $J_2$ (note: we should close by the 
$F_n^{\mathcal{J}_1}$'s, but no need to iterate as $F_n^{\mathcal{J}_2}|Q^{J_2}$ is the identity so quantifier
free type mean the truth value of the inequalities $F_{n_1}(r') \neq F_{n_2}(r')$ (including $F_\omega$ and the order between those terms).

Proof. Straight (or as in [Sh:Ch.III] = [Sh:E59]). \hfill \square_{2.5}

Remark 2.6. We could have replaced $Q^J$ by the disjoint union of $Q^J_n: n < \omega$, $<^J$ linearly order each $Q^J_n$ (and $\leq^J = \cup\{<^J_n: n < \omega\}$ and use $Q_n$ to index parameters for $\varphi(x, y_n)$). Does not matter at present.

But for our aim we can replace “not strongly stable” by a weaker demand, though this will not be carried here, we present it. Recall (from [Sh:863, \S5(G)], i.e. [Sh:863, 5.39=\text{dw5.35tex(2A)}]) the following, an equivalent definition to “a (complete first order theory) $T$ is strongly dependent”.

Definition 2.7. 1) A (complete first order) $T$ is not strongly$_4$ dependent if there is a sequence $\bar{\varphi} = \langle \varphi_n(x, y_n) : n < \omega \rangle$, (finite $\bar{x}$ of length $m < \omega$, as usual) (of first order) formulas from $L(\tau_T)$, an infinite linear order $I$, a sequence $\langle a_\eta : \eta \in \text{inc}_{<\omega}(I) \rangle$ indiscernible in $M$ with $\ell g(a_\eta) \subseteq \omega$ and letting $B = \cup\{a_\eta : \eta \in \text{inc}_{<\omega}(I)\}$ for some $m < \omega$ and $p \in S^m(B, M)$ for every $k < \omega$ there is $n < \omega$, satisfying: for no linear order $I^+$ extending $I$ and subset $I_0$ of $I^+$ with $\leq k$ members, do we have:

1A) In (1) without loss of generality $\bar{y}_n \subset \bar{y}_{n+1}$ for $n < \omega$.
2) $T$ is strongly$_4$ stable if it is stable and strongly dependent.

Remark 2.8. 1) We can write the condition in 2.7(1) without $I^+$ speaking on finite sets as done in (*) in the proof of 2.9 below.
2) In 2.7 by compactness we can get such $\langle a^*_\rho : \rho \in \text{inc}_{<\omega}(I') \rangle$ for any infinite linear order $I'$.

Next we deduce a consequence of being non-strongly$_4$-dependent, see Definition 2.7 helpful in proving non-structure results.

Claim 2.9. If $T$ is first order complete, $T_1 \supseteq T$ is first order complete, without loss of generality with Skolem functions and $T$ is not strongly$_4$ dependent as witnessed by $\bar{\varphi} = \langle \varphi_n(x, y_n) : n < \omega \rangle$, i.e. as in Definition 2.7(1A), then there is $\tau_1 \supseteq \tau_{T_1}, |\tau_1| = |T_1|$ and $\bar{\sigma}(\bar{z}_n) = \langle \sigma_{n, \ell}(\bar{z}_n) = \ell < \ell g(\bar{\sigma}_n)\rangle, \sigma_{n, \ell}$ is a $\tau_1$-term such that:

\begin{itemize}
  \item[\textcircled{3}] if $I, \mathcal{G}$ and $J = J_{1, \mathcal{G}}$ are as in Definition 2.1(2), then there are $M_1$ and $\langle a_t : t \in I \rangle$ and $\langle a_\eta : \eta \in \mathcal{G} \rangle$ such that:
    \begin{itemize}
      \item[(a)] $M_1$ is a $\tau_1$-model and is the Skolem hull of $\{a_t : t \in I\} \cup \{a_\eta : \eta \in \mathcal{G}\}$ (we write $a_\eta$ for $t \in \mathcal{G} \subseteq J$ for uniformity)
      \item[(b)] $\langle a_t : t \in J \rangle$ is indiscernible in $M_1$,
      \item[(c)] if $\eta \in \mathcal{G}$ and $k < \omega$ then for large enough $n(\star)$ we have:
        \begin{itemize}
          \item[(\star)] if $u \subseteq n(\star)$, $|u| \leq k$, then we can find $\bar{s}, \bar{t}$ and $n_* < n(\star)$ and $\bar{\sigma}$ such that
            \begin{itemize}
              \item[(i)] $\bar{s}, \bar{t}$ are sequences of members of $\{P^J_\eta(n) : n < n(\star)\}$
            \end{itemize}
        \end{itemize}
    \end{itemize}
\end{itemize}
(ii) $\ell g(\bar{s}) = \ell g(\bar{t}) \leq n(*)$

(iii) $s_i < t_j \iff t_i < t_j$ for $i, j < \ell g(\bar{s})$

(iv) if $i < \ell g(\bar{s}) = \ell g(\bar{t})$ then

$(\forall n \in u)(F^3(n) \leq \ell g(\bar{s}) \iff s_i \leq t_i)$

(v) $\sigma = (\sigma_i(y) : i < \ell g(y_n)), \sigma, \tau_1$-term

(vi) $M_1, \bar{y}_n \models \phi_{<m}(a_\eta, \ldots, \sigma^M_i(a_{t_0}, \ldots, \bar{a}_t, \ldots), \ldots)_{i < \ell g(y_n)} \equiv$

$(\bar{\eta}_n, \ldots, \sigma^M_i(a_{t_0}, \bar{a}_t, \ldots, \ldots))_{i < \ell g(y_n)}$

(\begin{align*}
(\delta) M_1 & \text{ is a model of } T_1 \text{ so } \tau_M \supseteq \tau_T.$
\end{align*})

Proof. Fix $I, \mathcal{S}$; without loss of generality $I$ is dense with neither first nor least element and is $\aleph$-homogeneous hence there are infinite increasing sequences of members of $I$.

Let $I, \langle \bar{x}, y_n \rangle : n < \omega, \langle \bar{a}_\eta : \eta \in \text{ incr}_{<\omega}(I) \rangle \in \mathcal{S}^n(\cup \{ \bar{a}_\eta : \eta \in \text{ incr}_{<\omega}(I) \})$ exemplify $T$ is not strongly $\omega$-dependent, i.e. be as in the Definition so $m = \ell g(\bar{x})$. For notational simplicity (and even without loss of generality by [Sh:863, \S 5]) assume $m = 1$.

Now in 2.7 we can add:

$(*)$ there is a sequence $\langle \langle n_k, m_k, I^c_k \rangle : k < \omega \rangle$ such that $k < n_k < m_k < m_{k+1}, I^c_k \subseteq I$ has $m_k$ members, for no $I^c_0 \subseteq I^c_k$ with at most $k$ members does $\otimes$ from 2.7 holds for $I^c_1, I^c_2 \in \text{ incr}_{<\omega}(I^c_k)$ and $\langle k, n_k \rangle$ here standing for $k, n$.

[Why? By compactness.]

Without loss of generality $I$ is the reduct to the vocabulary $\{<\}$, i.e. to just a linear order of an ordered field $\mathbb{F}$ and $t_q \in \mathbb{F}$ for $q \in \mathbb{Q}$ are such that $0 < t_q, (t_q)^2 < t_q$ for $q_1 < q_2$ (hence $n < \omega \Rightarrow n < n < n < n < n$). By easy manipulation without loss of generality $I^c_k = \{ t_i : i = 0, \ldots, m_k \}$.

Now for each $m < \omega$ and $\eta \in \text{ incr}_m(I)$ we can choose $c_0$ such that if $m = m_k$ then for some automorphism $h$ of $I$ mapping $I^c_k$ onto $\text{ Rang} (\eta)$, letting $\hat{h}$ be an automorphism of $M_1$ mapping $\bar{a}_\nu$ to $\bar{a}_{\hat{h}(\nu)}$ for $\nu \in \text{ incr}_{<\omega}(I)$, the element $c_0$ realizes $\hat{h}(\eta)$ and $\langle c_0 : \eta \in \text{ incr}_{<\omega}(I) \rangle$ is without repetitions.

Now without loss of generality $\langle c_\eta \rangle_{\bar{\eta} : \eta \in \text{ incr}_{<\omega}(I)}$ is an indiscernible sequence and $\bar{a}_l = c_{<\ell}$ be such that $M_0$ be a model of $T_1$ satisfying $\cup \{ \langle c_\eta \rangle_{\bar{\eta} : \eta \in \text{ incr}_{<\omega}(I)} \subseteq M_0 \mid \tau < \mathcal{C}$.

Without loss of generality $\langle c_\eta \rangle_{\bar{\eta} : \eta \in \text{ incr}_{<\omega}(I)}$ is indiscernible in $M_0$ and we can find an expansion $M_1$ of $M_0$ such that $|\tau_{M_1}| = |\tau_{T_1}|$ such that $\bar{a}_\eta = \langle F^\ell_1(\bar{q}) \bar{a}_n(0), \ldots, \bar{a}_n(n-1) \rangle : i < \ell g(\bar{a}_\eta), c_\eta = F^\ell_1(\bar{q}) \bar{a}_n(0), \ldots, \bar{a}_n(n-1) \rangle$

if $\eta \in \text{ incr}(I)$ and $M_1$ has Skolem functions.

By manipulating $I$ without loss of generality we can find $I_* \subseteq I$ of order type $\omega$.

So for some $H_n \in \tau_{I_*}$ for $n < \omega$,

$\otimes$ if $t_0 < t_1 < \ldots \text{ list } I_*$, for every $k < \omega$ large enough, for every $u \subseteq n(*)$

satisfying $|u| \leq k$ for every $n$ large enough $H_n^M(a_{t_0}, a_{t_1}, \ldots, a_{t_{n-1}})$ satisfies the demand (on the singleton $a_\eta$ from clause (\gamma) in the claim).

Let $D$ be a non-principal ultrafilter on $\omega$ such that $\{ m_k : k < \omega \} \in D$, let $M_2$ be isomorphic to $M^c_1/D$ over $M_1$, i.e. $M_1 \prec M_2$ and there is an isomorphism $I$ from $M_2$ onto $M^c_1/D$ extending the canonical embedding.

If $\eta$ is an increasing $\omega$-sequence of members of $I$, we let
\[ a^n_\eta = H^M_n(a_\eta(0), \ldots, a_\eta(n-1)) \in M_1 \]

and let

\[ a_\eta = j^{-1}(a^n_\eta, a^n_\eta, \ldots, a^n_\eta: n < \omega)/D \in M_2. \]

Let \( M'_2 \) be the Skolem hull of \( \{ \bar{a}_t: t \in I \} \cup \{ a_\eta: \eta \in S \} \) inside \( M_2 \). It is easy to check that it is as required. \( \Box \) \text{2.9}

Naturally it is helpful to have a sufficient condition for the non-isomorphism of two such models:

\textbf{Claim 2.10.} Assume \( J_\ell \subseteq K^{\alpha_\ell} \), and \( M_\ell, \bar{\varphi}, T_1, T \) as in 2.9 for \( \ell = 1, 2 \). A sufficient condition for \( M_1 \not\cong M_2 \) is

\begin{itemize}
  \item[(\ast)] if \( f \) is a function from \( J_1 \) (i.e. its universe) into \( \mathcal{H}_{|T_1|, \aleph_0}(J_2) \) (i.e. the free algebra generated by \( \{ x_t: t \in J_1 \} \) the vocabulary \( \tau_{|T_1|, \aleph_0} = \{ F^n_\alpha: n < \omega \) and \( \alpha < |T_1| \}, F^n_\alpha \) has arity \( n \)\), we can find \( t \in P^\mathcal{J}_1 \) and \( k_\ast < \omega \) such that for every \( n_\ast < \omega \) we can find \( s_1, s_2 \) such that:
    \begin{enumerate}
      \item[(a)] \( s_1, s_2 \in k_\ast (Q^{\mathcal{J}}) \) are increasing, \( s_1 = (F^n_\alpha(t): n < n_\ast) \) and \( n < k_\ast \Rightarrow s_2.n = s_1.n \) and \( s_1.n - 1 < I \) \( s_2.k_\ast \).
      \item[(b)] \( f(s_\ell) = \bar{\sigma}(r^0_\ell, \ldots, r_{k_\ast-1}^\ell) \) so \( k < \omega, r^\ell_\ell \in J_2 \) for \( i < k \) so \( \sigma \) is a \( \tau_{|T_1|, \aleph_0} \)-term not dependent on \( \ell \).
      \item[(c)] \( f(t) = \sigma^\ast(r_0, \ldots, r_{m-1}), \sigma^\ast \) is a \( \tau_{|T_1|, \aleph_0} \)-term and \( r_0, \ldots, r_{m-1} \in J_2 \).
      \item[(d)] the sequences\[ \langle r^1_i: i < k \rangle, \langle r_i: i < m \rangle \]
    \end{enumerate}
    realize the same quantifier free type in \( J_2 \) (note: we should close by the \( F^n_{\omega_1}(r') \neq F^n_{\omega_2}(r') \) (including \( F^n_\omega \)) and the order between those terms).
\end{itemize}

\textit{Proof.} As in [Sh:300, Ch.III] or better in [Sh:e, Ch.III] = [Sh:E59], called unembeddability. \( \Box \) 2.10
§ 3. Forcing EF+-equivalent Consistency non-isomorphic models

The following result is not optimal, but it is enough to prove necessary conditions on $T$ for being lean and even on $(T,+)$. As for unstable $T$, see below in §4. So our main result is

Claim 3.1. Assume $(\varphi,T,T_1,\Phi)$ is as in 2.4, $T$ stable and $\lambda = \lambda^{\aleph_0} \geq \aleph_1 + \# T_1$ and $\mu = \lambda^+ > \lambda$. Then for some $\lambda$-complete $\lambda^+$-c.c. forcing notion $\mathbb{Q}$ we have: $\text{IF}_{\mathbb{Q}}$ “there are models $M_1, M_2$ of $T$ of cardinality $\lambda^+$ such that $M_1 \upharpoonright \tau(T), M_2 \upharpoonright \tau(T)$ are EF$_{\alpha,\lambda,\lambda^+}$-equivalent for every $\alpha < \lambda$ but are not isomorphic”.

Remark 3.2. 1) It should be clear that we can improve it allowing $\alpha < \lambda^+$ and replacing forcing by e.g. $2^{\lambda}$ and $\lambda = \lambda^{\aleph_0}$, but we shall continue in [Sh:F918].

Proof. We define $\mathbb{Q}$ as follows:

(1) $p \in \mathbb{Q}$ iff $p$ consist of the following objects satisfying the following conditions:

(a) $u = u^p \in [\mu]^{\lambda}$ such that $\alpha + i \in u \wedge i < \lambda \Rightarrow \alpha \in u$
(b) $<p$ a linear order of $u$ such that
\[ \alpha, \beta \in u \wedge \alpha + \lambda \leq \beta \Rightarrow \alpha <p \beta \]
(c) for $\ell = 1, 2 \quad \mathcal{G}_\ell^p$ is a subset of $\{ \eta \in \omega^u : \eta(n) + \lambda \leq \eta(n+1) \text{ for } n < \omega \}$ such that $\eta \neq \nu \in \mathcal{G}_\ell^p \Rightarrow \text{Rang}(\eta) \cap \text{Rang}(\nu)$ is finite; note that in particular $\eta \in \mathcal{G}_\ell^p$ is without repetitions and is $<p$-increasing
(d) $\Lambda^p$ a set of $< \lambda$ increasing sequences of ordinals from $\{ \alpha \in u^p : \lambda|\alpha \}$ hence of length $< \lambda$
(e) $f^p_\ell = (f^p_\ell : \rho \in \Lambda^p)$ such that
\[ f^p_\ell \text{ is a partial automorphism of the linear order } (u^p, <p) \text{ such that } \alpha \in \text{Dom}(f^p_\ell) \Rightarrow \alpha + \lambda = f^p_\ell(\alpha) + \lambda \text{ and we let } f^p_\ell = f^p_\ell, f^p_\ell = (f^p_\ell)^{-1} \]
(f) if $\eta \in \mathcal{G}_\ell^p, \rho \in \Lambda^p, \ell \in \{1, 2\}$ then $\text{Rang}(\eta)$ is included in $\text{Dom}(f^p_\ell)$ or is almost disjoint to it (i.e. except finitely many “errors”)
(g) if $\rho \cup q \in \Lambda^p$ then $\rho \in \Lambda^p$ and $f^p_\ell \subseteq f^p_\ell$
(h) if $\rho \cup q \in \Lambda^p$ then $\rho \in \Lambda^p$ and $f^p_\ell \subseteq f^p_\ell$
(i) $f^p_< \subseteq$ is the empty function and if $\rho \in \Lambda^p$ has limit length then
\[ f^p_\ell = \cup\{ f^p_{\ell+1} : i < \ell \rho(\rho) \} \]
(j) if $\rho \in \Lambda^p$ has length $i + 1$ then $\text{Dom}(f^p_{\ell+1}) \subseteq \rho(i)$ for $\ell = 1, 2$
(k) if $\rho \in \Lambda^p$ and $\eta \in \omega(\text{Dom}(f^p_\ell))$ then $\eta \in \mathcal{G}_\ell^p \Leftrightarrow (f^p_\ell(\eta(n)) : n < \omega) \in \mathcal{G}_2^p$

(\ell) if $\rho_n \in \Lambda^p$ for $n < \omega$ and $\rho_n < \rho_{n+1}$ and $\lambda > \aleph_0$ then $\cup\{ \rho_n : n < \omega \} \in \Lambda$

$\mathbb{Q}_2$ We define the order $\leq = \leq_{\mathbb{Q}}$ on $\mathbb{Q}$ as follows: $p \leq q$ iff $(p, q) \in \mathbb{Q}$ and

(a) $u^p \subseteq u^q$
(b) $\leq^p = \leq^q \upharpoonright u^p$
(c) $\mathcal{G}_\ell^p \subseteq \mathcal{G}_\ell^q$ for $\ell = 1, 2$
(d) $\Lambda^p \subseteq \Lambda^q$

(e) if $p \in \Lambda^p$ then $f^p_\rho \subseteq f^q_\rho$

(f) if $\eta \in \mathcal{S}_\ell^p \setminus \mathcal{S}_\ell^q$ then $\text{Rang}(\eta) \cap \Lambda^p$ is finite

(g) if $p \in \Lambda^p$ and $f^p_\rho \neq f^q_\rho$ then $u^p \cap \text{sup Rang}(\rho) \subseteq \text{Dom}(f^\ell_{\rho,q})$ for $\ell = 1, 2$

(h) if $p \in \Lambda^p$ and $\ell \in \{1, 2\}, \alpha \in u^p \setminus \text{Dom}(f^\ell_{\rho,p})$ and $\alpha \in \text{Dom}(f^\ell_{\rho,q})$ then $f^\ell_{\rho,p}(\alpha) \notin u^p$

Having defined the forcing notion $\mathcal{Q}$ we start to investigate it.

$\circ_3 \mathcal{Q}$ is a partial order of cardinality $\mu^{\mathcal{S}_\ell^\Lambda} = \lambda^+$.

[Why? Obviously.]

$\circ_4 (i)$ if $\bar{\rho} = (p_i : i < \delta)$ is $\leq^\mathcal{Q}$-increasing, $\delta$ a limit ordinal $< \lambda$ of uncountable cofinality then $p_\delta := \cup\{p_i : i < \delta\}$ defined naturally is an upper bound of $\bar{\rho}$

(ii) if $\delta < \lambda$ is a limit ordinal of cofinality $\aleph_0$ and the sequence $\bar{\rho} = (p_i : i < \delta)$ is increasing (in $\mathcal{Q}$), then it has an upper bound.

Why (i)? Think or see (ii); why the case $\text{cf}(\delta) > \aleph_0$ is easier? Because of clause $\circ_1(i)$ and $\circ_1(\ell)$. Why (ii)? We define $p_\delta \in \mathcal{Q}$ as follows: $u^p_\delta = \cup\{u^p_i : i < \delta\}$, $\leq^\mathcal{P}_\delta = \cup\{\leq^\mathcal{P}_i : i < \delta\}, \Lambda^p_\delta = \cup\{\Lambda^p_i : i < \delta\} \cup \{\rho : \rho$ is an increasing sequence of ordinals from $u^p_\delta$, length a limit ordinal of cofinality $\aleph_0$ such that $\varepsilon < \ell \text{g}(\rho) \Rightarrow \rho \in \cup\{\Lambda^p_i : i < \delta\}\}$. Let $f^\mathcal{P}_\delta = (f^\mathcal{P}_\delta : \rho \in \Lambda^p_\delta)$ where: if $i < \delta$ and $\rho \in \Lambda^p_i \cup \{\Lambda^p_j : j < i\}$, then $f^\mathcal{P}_\delta = \cup\{f^\mathcal{P}_j : j \in [i, \delta]\}$ and if $\rho \in \Lambda^p_i \setminus \{\Lambda^p_j : j < i\}$ then $f^\mathcal{P}_\delta = \cup\{f^\mathcal{P}_\varepsilon : \varepsilon < \ell \text{g}(\rho)\}$ is well defined as $\varepsilon < \ell \text{g}(\rho) \Rightarrow \rho^\varepsilon(\varepsilon) \in \cup\{\Lambda^p_j : j < \delta\}$. Clearly clauses (a),(b),(d),(e),(f),(h),(i),(j),(k) from $\circ_1$ for $p_\delta \in \mathcal{Q}$ hold.

Lastly, let $\mathcal{S}^\mathcal{P}_\ell = \cup\{\mathcal{S}^\mathcal{P}_\delta : \alpha < \delta\}$ for $\ell = 1, 2$.

Note

$\circ_1$ if $p \in \Lambda^p \setminus \{\Lambda^p_\alpha : \alpha < \delta\}$ then $\text{Dom}(f^\mathcal{P}_\rho) = u^p \cap \text{sup Rang}(\rho) = \text{Rang}(f^\mathcal{P}_p)$ and for every $\alpha < \delta$ for some $\beta < \delta$ we have $f^\mathcal{P}_\rho | u^\alpha \subseteq f^\mathcal{P}_\rho | u^\alpha$ for some $i < \ell \text{g}(\rho)$.

[Why? Clearly, $\text{cf}(\ell \text{g}(\rho)) = \aleph_0$.]

Assume $\alpha < \delta$ and $i < \ell \text{g}(\rho)$. Clearly for some $\beta \in (\alpha, \delta)$ we have $p_\beta \in \Lambda^p_\beta$. Also the set $\{j < \ell \text{g}(\rho) : \rho(j) \in \Lambda^p_\beta\}$ is an initial segment of $\ell \text{g}(\rho)$ and cannot be $\ell \text{g}(\rho)$ because $\rho \notin \Lambda^p_\beta$ by clause $\circ_1(\ell)$. So for some $j < \ell \text{g}(\rho)$ we have $p_\beta \in \Lambda^p_\beta$ but by the choice of $\rho$ for some $\gamma < \delta$ we have $p(j) \in \Lambda^p_\gamma$, so necessarily $\beta < \gamma$. As $p_\alpha \leq_{\mathcal{Q}} p_\beta \leq_{\mathcal{Q}} p_\gamma$ by clause (g) of $\circ_2$, as $p_\beta \in \Lambda^p_\gamma \setminus \Lambda^p_\beta$ we know that $u^p_\gamma \cap \text{sup Rang}(\rho(i))$ is included in $\text{Dom}(f^\mathcal{P}_{\rho(i)})$ for $\ell = 1, 2$ by $p_\alpha \leq_{\mathcal{Q}} p_\beta$ hence $u^p_\alpha \cap \text{sup Rang}(\rho(i))$ is included in $\text{Dom}(f^\mathcal{P}_{\rho(i)})$ which $\subseteq \text{Dom}(f^\mathcal{P}_{\rho(i)})$ for $\ell = 1, 2$.

As this holds for any $\alpha < \delta$ and $i < \ell \text{g}(\rho)$ and $u^p \cap \text{sup Rang}(\rho(i)) = \cup\{u^p \cap \text{sup Rang}(\rho(i)) : \alpha < \delta\}$ it follows that for $\ell = 1, 2$ we have $\varepsilon \in u^p \cap \text{sup Rang}(\rho) \Rightarrow (\exists \beta < \delta) (\varepsilon \in u^\beta \cap \text{sup Rang}(\rho)) \Rightarrow (\exists \beta < \delta) (\varepsilon \in \text{Dom}(f^\mathcal{P}_{\rho(i)}) \Rightarrow \varepsilon \in \text{Dom}(f^\mathcal{P}_{\rho(i)})$ so are done.]

$\circ_2$ if $p \in \cup\{\Lambda^p_\alpha : \alpha < \delta\}$ then exactly one of the following occurs:
(a) there is a unique $\alpha = \alpha(\rho) < \delta$ such that $\rho \in \Lambda^{p_\delta},(\forall \beta)(\alpha \leq \beta < \delta \Rightarrow f^p_\rho = f^p_{\rho_\delta})$ and $(\forall \beta < \alpha)(\rho \in \Lambda^{p_\delta} \rightarrow f^p_\rho \neq f^p_{\rho_\delta})$

(b) Dom($f^p_\rho$) = $w^p \cap \sup \operatorname{Rang}(\rho) = \operatorname{Rang}(f^p_\rho)$ and $(\forall \alpha < \delta)(\exists \beta < \delta)(f^p_\rho[u^{p_\delta} \subseteq f^p_{\rho_\delta}]).$

[Why? Similarly to the proof of $\oplus_1.$]

To finish proving $p_\delta \in \mathcal{Q},$ i.e. verifying $\oplus_1$ holds, we have to check clauses (c),(g),(k).

Clause (c): Obvious by the choice of $\mathcal{G}^p_{\delta_1}.$

Clause (g):

So let $\eta \in \mathcal{G}^p_{\delta_1}, \rho \in \Lambda^p$ where $\ell \in \{1,2\}$ and we should prove that $\operatorname{Rang}(\eta) \subseteq \operatorname{Dom}(f^p_{\ell},\rho)$ or $\operatorname{Rang}(\eta) \cap \operatorname{Dom}(f^p_{\ell},\rho)$ is finite. For some $\alpha < \delta$ we have $\eta \in \mathcal{G}^p_{\delta_1}.$ If $\rho \in \{\Lambda^{p_\delta} : \beta < \delta\}$ then we apply $\oplus_2,$ now if clause (a) there holds so $\alpha = \alpha(\rho) < \delta$ is well defined and we use $p_\alpha \in \mathcal{Q}$ and if clause (b) there holds then trivially $\operatorname{Rang}(\eta) \subseteq w^p \subseteq \operatorname{Dom}(f^p_{\ell},\rho)$ so assume $\rho \in \Lambda^{p_\delta} \cap \{\Lambda^{p_\delta} : \beta < \delta\}.$ By $\oplus_1$ we finish as in the case $\oplus_2(b)$ holds.

Clause (k):

By the choice of $\mathcal{G}^p_{\delta_1}$ and the proof of clause (g).

Checking $p_{\alpha_0} \leq p_\delta$ where $\alpha < \delta$

We should check that the pair $(p_{\alpha_0},p_\delta)$ satisfies the demands in $\oplus_2$ which is straight. $\oplus_2.$

So we have proved $\oplus_4.$

$\oplus_5$ if $\alpha < \mu$ then $\mathcal{X}^1_{\alpha} := \{p \in \mathcal{Q} : \alpha \in w^p\}$ is dense and open as well as

$\mathcal{X}_{\alpha} = \{p \in \mathcal{Q} : \delta \in u^p, \lambda|\delta \text{ and } \operatorname{cf}(\delta) < \lambda \text{ then } \delta = \sup(\delta \cap u)\}.$

[Why? Straight. For the first, $\mathcal{X}^1_\alpha,$ given $p \in \mathcal{Q}$ we define $q \in \mathcal{Q}$ by

(a) $u^q$ is $u^p \cup \{\beta \leq \alpha : \beta + \lambda = \alpha + \lambda\},$ so clause $\oplus_1(a)$ holds

(b) $u^q$ is the following linear order on $u^q$

$\alpha_1 < \alpha_2$ if $\alpha_1 < \alpha_2 \lor \alpha_1 < \alpha_2 \land \{\alpha_1,\alpha_2\} \not\subseteq u^p \land \{\alpha_1,\alpha_2\} \subseteq u^q$

(c) $\mathcal{G}^q_1 = \mathcal{G}^p_{\delta_1}$ for $\ell = 1,2$

(d) $\Lambda^q = \Lambda^p$ and

(e) $f^p_\rho = f^p_\rho$ for $p \in \Lambda^q.$

Now check.

For the second, $\mathcal{X}_\alpha$ use the first and $\oplus_4.$]

$\oplus_6$ if $q \in \Lambda^* := \{p : p \text{ is an increasing sequence of ordinals } < \lambda^+ \text{ divisible by } \lambda \}$ of length $< \lambda$ then $\mathcal{X}^2_\alpha = \{p \in \mathcal{Q} : q \in \mathcal{Q}\}$ is dense open.

[Why? Let $p \in \mathcal{Q},$ by $\oplus_5 + \oplus_4$ there is $q \geq p$ (from $\mathcal{Q}$) such that $\operatorname{Rang}(q) \subseteq u^q.$ If $q \in \Lambda^q$ we are done, otherwise define $q'$ as follows: $u^{q'} = u^{q},<^{q'} = <^{q},\mathcal{G}^{q'} = \mathcal{G}^q,\Lambda^{q'} = \Lambda^q \cup \{q : \in \in \leq \leq \ell g(q) \}$ and if $i \leq \ell g(q),q \not\in i \not\in \Lambda^q$ then we let $f^p_{\rho,i} = f^p_{\rho,i} \cup \{f^p_{\rho,i} : p \in \Lambda^q \land \rho < q \not\in i\}.$ We should check all the clauses of $\oplus_1$ for “$q \in \mathcal{Q}$” and e.g. clause (k) there holds because $q$ satisfies clause ($\ell$). Then we should check all the clauses of $\oplus_2$ for “$q \leq \mathcal{Q} q'$.”]
\( \oplus_7 \) if \( p \) is as in \( \oplus_8 \) and \( \alpha < \lambda^+ \) and \( \ell \in \{1, 2\} \) then
\[
\mathcal{S}^3_{p, \alpha, \ell} = \{ p \in Q : \alpha \in \text{Dom}(f_{\rho, p}^\ell) \text{ so } q \in \Lambda^p, \alpha \in u^p \}
\]
is dense open.

[Why? By \( \oplus_5 + \oplus_6 \).

\( \oplus_8 \) if \( p \in Q \) and \( q \in \Lambda^p \) then for some \( q \) we have \( p \leq_q q \wedge f_{\rho, p}^{\ell} \neq f_{\rho, p}^{\ell} \wedge \{ \alpha + \lambda : \alpha \in u^p \} = \{ \alpha + \lambda : \alpha \in u^p \} \).

Why? For each \( \delta \in u \cap \sup \text{Rang}(\rho) \) divisible by \( \lambda \) let \( u_{\delta} = u \cap [\delta, \delta + \lambda) \). So
\[ g_{\delta} := f_{\rho, p}^{\ell} \upharpoonright u_{\delta} \] is a partial function from \( u_{\delta} \) into \( u_{\delta} \) and \( f_{\rho, p}^{\ell} = \bigcup \{ g_{\delta} : \delta \text{ as above} \} \).

Now, for \( \delta \) as above we can find \( f_{\delta} \) such that:

\begin{enumerate}
\item \( f_{\delta} \) is a one-to-one function.
\item \( g_{\delta} = f_{\rho, p}^{\ell} \upharpoonright u_{\delta} \subseteq f_{\delta} \).
\item \( f_{\delta} \) maps \( [\alpha_3, \alpha_2] \) onto \( u_{\delta} \).
\item \( f_{\delta} \) maps \( [\alpha_2, \alpha_1] \) onto \( u_{\delta} \).
\end{enumerate}

Now we can find a linear order \( <_{\delta} \) on \( u_{\delta} \cup [\alpha_3, \alpha_2] \) such that \( f_{\delta} \) is order preserving (as the class of linear orders has amalgamation).

Lastly, we define \( q \):

\begin{enumerate}
\item \( u^p = u^p \cup \{ [\alpha_3, \alpha_2] : \delta \text{ as above} \} \).
\item \( \langle q \rangle < \delta \) is defined by \( \alpha <_{\delta} \beta \) iff \( (\exists \delta)(\delta <_{\delta} \beta) \) or \( \alpha + \lambda < \beta \).
\item \( \Lambda^q = \Lambda^p \).
\item \( \mathcal{S}^3_q = \mathcal{S}^3_{\rho, p} \cup \{ (f_{\rho, p}^{\ell-\ell}(\eta(n)) : n < \omega) : \ell \in \{1, 2\}, p \in \Lambda^p \text{ and } \eta \in \mathcal{S}^3_{\rho, p, \ell} \}. \)
\end{enumerate}

Now we have to check \( q \in Q \), i.e. all the clauses of \( \oplus_1 \). This is straight; e.g. for clause (c), assume \( \eta \neq \nu \in \mathcal{S}^3_q \) and we have to prove that \( \text{Rang}(\eta) \cap \text{Rang}(\nu) \) is finite.

Now we have four cases: first \( \eta, \nu \in \mathcal{S}^3_{\rho, p} \), so use \( p \in Q \), clause \( \oplus_1(c) \) for \( \ell \). Second, \( \eta, \nu \in \mathcal{S}^3_q \setminus \mathcal{S}^3_{\rho, p} \), so \( \eta, \nu \) are images by \( f_{\rho, p}^{\ell} \) of members of \( \mathcal{S}^3_{\rho, p} \), as this function is one-to-one, this follows from \( p, \mathcal{S}^3_{\rho, p} \) satisfying clause \( \oplus_1(c) \).

Third, \( \eta \in \mathcal{S}^3_q \setminus \mathcal{S}^3_{\rho, p} \), \( \nu \in \mathcal{S}^3_q \setminus \mathcal{S}^3_{\rho, p} \), then \( \nu = (f_{\rho, p}^{\ell-\ell}(\nu(n)) : n < \omega) \) for some \( \nu \in \mathcal{S}^3_{\rho, p} \) satisfying \( \text{Rang}(\nu') \subseteq \mathcal{S}^3_{\rho, p} \), hence for some \( n_\ast < \omega \) we have \( n \in [n_\ast, \omega) \Rightarrow \nu'(n) \notin \text{Dom}(f_{\rho, p}^{\ell-\ell}) \Rightarrow \nu(n) \notin u^p \) but \( \text{Rang}(\eta) \subseteq u^p \) so we are done. Fourth, \( \eta \in \mathcal{S}^3_q \setminus \mathcal{S}^3_{\rho, p} \), \( \nu \in \mathcal{S}^3_q \), the proof is dual.

The proof of clause (g) is similar.

Also we have to check that \( p \leq_q q \), i.e. all the clauses of \( \oplus_2 \) for the pair \( (p, q) \). This is straight, clause (f) is proved as in the proof of \( \oplus_1(c) \) above and clause (h) holds by our choice of the \( f_{\delta} \)’s.

Now check that \( q \) is as required.]

Let
\[ \oplus_1 Q^+ = \{ p \in Q : \text{if } \ell \in \{1, 2\} \text{ and } p \in \Lambda^p \text{ then } \text{Dom}(f_{\rho, p}^{\ell}) = u^p \cap \sup \text{Rang}(\rho) \} \]
$\otimes_2 Q^+$ is a dense subset of $Q$, moreover $(\forall p \in Q)(\exists q \in Q^+) (p \leq q \land \{\alpha + \lambda : \alpha \in u^p\})$.

[Why? Let $p \in Q, \kappa = |\Lambda|, \delta = \kappa \times \kappa$ and $\{p_i : i < i_* < \lambda\}$ list $\Lambda^p$ each appearing unboundedly often. We choose $p_i$ by induction on $i \leq \delta$ such that

(a) $p_i \in Q$
(b) $j < i \Rightarrow p_i \leq Q p_j$
(c) $p_0 = p$
(d) $\Lambda^{p_i} = \Lambda^p$
(e) $f^{p_i}_{p_i+1} \neq f^{p_i}_{p_i}$
(f) $\{\alpha + \lambda : \alpha \in u^{p_i}\} = \{\alpha + \lambda : \alpha \in u^p\}$.

For $i = 0$ use clause (c) for $i$ limit use $\otimes_4$, for $i = j + 1$ use $\otimes_8$. Now $p_\delta$ is as required.]

$\otimes_3$ for $p \in Q$ and $\delta < \lambda^+$ divisible by $\lambda, p \upharpoonright \delta$ is naturally defined, belongs to $Q$ and $u^p \subseteq \delta \Rightarrow p|\delta = p$ and $p \upharpoonright \delta \leq Q p$, where $q = p \upharpoonright \delta$ be defined by:

(a) $u^q = u^p \cap \delta$
(b) $<^q = <^p| \delta$
(c) $\mathcal{S}^q_\ell = \{\eta \in \mathcal{S}^p_\ell : \operatorname{Rang}(\eta) \subseteq \delta\}$
(d) $\Lambda^q = \{\rho \in \Lambda^p : \operatorname{sup} \operatorname{Rang}(\rho) \leq \delta\}$
(e) $f^q = (f^p_\rho : \rho \in \Lambda^q)$ where $f^p_\rho = f^p_{\rho|\delta}$.

[Why? Check.]

$\otimes_4$ if $\delta < \lambda^+$ is divisible by $\lambda, p \in Q^+$ and $(p|\delta) \leq Q q \in Q^+$ but $u^q \subseteq \delta$ then $p, q$ are compatible in $Q$, moreover has a common upper bound $r = p + q$ such that $r|\delta = q \upharpoonright u^r = u^p \cup u^q$.

[Why? Note that if $\rho \in \Lambda^p \cap \Lambda^q$ then $\operatorname{sup} \operatorname{Rang}(\rho) \leq \delta$ by clause (i) + (j) of $\otimes_1$; also $\Lambda^p \cap \Lambda^q = \Lambda_{p|\delta}$. We define $r$ as follows:

(a) $u^r = u^p \cup u^q$
(b) $\leq r$ is defined by: for $\alpha, \beta \in u^r$ we have $\alpha <^r \beta$ if $\alpha + \lambda \leq \beta$ or $\alpha <^q \beta$ or $\alpha <^p \beta$
(c) $\mathcal{S}^r_\ell = \mathcal{S}^p_\ell \cup \mathcal{S}^q_\ell$ for $\ell = 1, 2$
(d) $\Lambda^r = \Lambda^p \cup \Lambda^q$
(e) $f^r = (f^p_\rho : \rho \in \Lambda^p)$ where $f^p_\rho$ is:

- $f^p_\rho$ when $\rho \in \Lambda^q$
- $f^p_\rho \cup \{f^p_{\rho|\delta} : i \leq \ell \theta(\rho) \text{ and } \rho|\delta \in \Lambda^r\}$ when $\rho \in \Lambda^p \setminus \Lambda^q$.

Why $r \in Q$? We should check all the clauses in $\otimes_1$, which are easy. E.g. in clause (c), $\eta \neq \nu \in \mathcal{S}^r_\ell \Rightarrow N_0 \geq |\operatorname{Rang}(\eta) \cap \operatorname{Rang}(\nu)|$, the only new case is $\eta \in \mathcal{S}^p_\ell \Leftrightarrow \nu \notin \mathcal{S}^q_\ell$ so without loss of generality $\eta \in \mathcal{S}^p_\ell \mathcal{S}^q_\ell \land \nu \in \{\mathcal{S}^r_\ell\},$ hence $\operatorname{sup}(\eta) > \delta$ hence $\operatorname{Rang}(\eta) \cap \delta$ is finite but $\operatorname{Rang}(\nu) \subseteq u^\delta \subseteq \delta$.

Also clauses (g) + (k) should be checked only when $f^p_\rho$ is new so necessarily $\rho \in \Lambda^p$ so $f^p_\rho = f^p_\rho \cup \{f^p_{\rho|\delta} : i \in \Lambda^r\}$, but recalling that any $\eta \in \mathcal{S}^r_\ell$ is an increasing $\omega$-sequence, clearly if $\operatorname{sup} \operatorname{Rang}(\eta) > \delta$ we use "$p$ satisfies clauses (g) + (k)" and if $\operatorname{sup} \operatorname{Rang}(\eta) \leq \delta$ we use "$\eta$ satisfies clauses (g) + (k) and (k)".
Theories with EF-Equivalent Non-Isomorphic Models

Why? Let $p_\alpha \in Q$ for $\alpha < \lambda^+$, so by $\oplus_{10}$ there are $q_\alpha$ such that $p_\alpha \leq Q q_\alpha \in Q^+$, now use the $\Delta$-symmetry lemma that is first $S^\lambda_\alpha = \{\delta < \lambda^+ : cf(\delta) = \lambda\}$; now $\delta \in S^\lambda_\alpha \Rightarrow p[\delta \in Q \land \sup(u^p|\delta) < \delta$ and $\lambda \geq |\{p \in Q : u^p = u\}|$ for any $u$. Hence for some stationary $S \subseteq S^\lambda_\alpha$ and $p_\alpha$ we have $\delta \in S \Rightarrow q_\delta = p_\alpha$ and $\delta_1 < \delta_2 \in S \Rightarrow \sup(u^{q_\delta_2}) < \delta_2$. So for any $\delta_2 < \delta_2$ from $S$ by $\oplus_4$ the condition $q_\delta_1, q_\delta_2$ are compatible.

\[ \exists \delta \leq \lambda \] define $J_\ell \in K_\ell^\alpha$, a $Q$-name as follows:

(a) $Q^{\exists \delta} = \mu$
(b) $\mathfrak{S}^J_\ell = \cup\{\mathfrak{S}^p : p \in G_Q\}$
(c) $<^J_\ell = \cup<^p : p \in G_Q\}$
(d) $F^J_\ell$ is a unary function, the identity on $\lambda^+$ and
(e) $\eta \in \mathfrak{S}^J_\ell \Rightarrow F^{J_\ell}(\eta) = \eta(n)$

\[ \oplus_2 \] for $\ell \in \{1, 2\}$ and $p \in Q$ let $J_\ell^p \in K_\alpha^\ell$ be defined as follows:

(a) $J_\ell^p \in$ has universe $u^p \subseteq \mathfrak{S}^p$
(b) $<^J_\ell = ^p$
(c) $Q^{J_\ell^p} = u^p$
(d) $F^{J_\ell^p}(\eta) = \eta(n)$

\[ \exists_3 \] (a) $\models_Q \exists J_\ell \in K_\alpha^\ell$
(b) $\models_Q \forall \delta \leq \lambda^+$ divisible by $\lambda$ the linear order $((\delta, \delta + \lambda), <^J_\ell$ $\{\delta, \delta + \lambda\}$) is a saturated linear order and $\alpha + \lambda \leq \beta < \lambda^+ \Rightarrow \alpha <^J_\ell \beta$
(c) $p \in Q \Rightarrow p \models_Q \exists J_\ell^p \subseteq J_\ell$ for $\ell = 1, 2$.

Why? Think

\[ \exists_4 \] if $\delta < \lambda^+$ is divisible by $\lambda$ then $\models Q \exists J_\ell | \delta \in K_\alpha$ where $J_\ell | \delta = (\{\delta \cup (P^J_\ell \cap -\delta), Q^{J_\ell} \cap \delta, P^J_\ell \cap \delta, F_\delta^{J_\ell} \cap \delta \cup (P^J_\ell \cap -\delta)\})_{n < \omega}$

\[ \oplus_5 \] $\models Q \exists EM_{(T)}(J_1, \Phi), EM_{(T)}(J_2, \Phi)$ are $EF^{+, \lambda, \lambda}_\ell$-equivalent

(so the games of length $< \lambda$, and the player INC chooses sets of cardinality $< \lambda^+$).

Why? To show the $EF^{+, \lambda, \lambda}_\ell$-equivalence, it suffices to show that $\models Q \exists J_1, J_2$ are $EF^{+, \lambda, \lambda}_\ell$-equivalent" by 1.14 as $\lambda \geq \aleph_1 \land |T_1|$. From $\oplus_6$, recall $\Lambda^* = \{\rho : \rho$ is an increasing sequence of ordinals $< \lambda^+$ divisible by $\lambda$ of length $< \lambda\}$, (is the same in $V$ and $V^Q)$. For $\rho \in \Lambda^*$ let $f_\rho = \cup\{f^p_{\rho} : p \in G, p \in P^\ell\}$ and by $\oplus_7(f) + (j)$and $\oplus_7(e)$ easily $\models Q \exists J_\ell, f^p_{\rho}$ a partial isomorphism from $J_\ell | \sup Rang(\rho)$ into $J_2 | \sup Rang(\rho)$, see Definition inside $\oplus_4$.  

(897)
Now $\models Q \ "\text{Dom}(f_\rho) = \sup \text{Rang}(\rho)\"$ as if $G \subseteq Q$ is generic over $V$, for any $\alpha < \sup \text{Rang}(\rho)$ for some $p \in G$ we have $\alpha \in u^p \land \rho \in \mathcal{P}^\ell \text{ by } \otimes_7$ and there is $q$ such that $p \leq q \in G, p \neq q$ by $\otimes_8$, so recalling $\otimes_2(g)$ we are done.

Similarly $\models Q \ "\text{Rang}(f_\rho) = \sup \text{Rang}(\rho)\"$.

Also $\rho \not\subseteq \rho \models Q \ \{p \subseteq \ell_q\}$. For the $\text{EF}^+\text{-version}$ we have to analyze dependence relations, which is straight as in the proof in 4.2. So $\langle f_\rho : \rho \in \Lambda^\ast \rangle$ exemplify the equivalence.]

$\text{EF}_6 \models Q \ "M_1 = EM_{\tau(T)}(J_1, \Phi), M_2 = EM_{\tau(T)}(J_2, \Phi) are not isomorphic".$

Why? Let $M_1^\tau = EM(J_1, \Phi)$ so $M_1^\tau \restriction \tau(T) = M_\ell$ for $\ell = 1, 2$, and assume toward contradiction that $p \in Q$, and $p \models Q \ "g is an isomorphism from $M_1$ onto $M_2\"$. For each $\delta \in S_\lambda^\lambda := \{\delta < \lambda^+ : cf(\delta) = \lambda\}$ by $\otimes_4$ we can find $p_\delta \in Q$ above $p$ and $g_\delta$ such that:

$\square_1 (\alpha)$ $p \leq p_\delta, \delta \in u^{P\delta}$
$\quad (\beta)$ $p_\delta \models \ "g_\delta is g \upharpoonright EM(J_\rho, \Phi)\"$
$\quad (c)$ $g_\delta$ is an isomorphism from $EM_{\tau(T)}(J_\rho, \Phi)$ onto $EM_{\tau(T)}(J_\rho, \Phi)$.

We can find stationary $S \subseteq S_\lambda^\lambda$ and $P^\ast$ such that

$\square_2 (\alpha)$ $p_\delta \upharpoonright \delta$, defined in $\square_3$ is $P^\ast$ for $\delta \in S$
$\quad (b)$ for $\delta_1, \delta_2 \in S$, $u^{P\delta_1}, u^{P\delta_2}$ has the same order type and the order preserving mapping $\pi_{\delta_1, \delta_2}$ from $u^{P\delta_2}$ onto $u^{P\delta_1}$ induce an isomorphism from $p_{\delta_2}$ onto $p_{\delta_1}$
$\quad (c)$ if $\delta_1 < \delta_2 \in S$ then $\sup(u^{P\delta_1}) < \delta_2$.

Now choose $\eta^\ast = \langle \delta^\ast_n : n < \omega \rangle$ such that

$\square_3 (\alpha)$ $\delta^\ast_0 \in \delta_0^\ast_{n+1}$
$\quad (b)$ $\delta^\ast_n = \sup(S \cap \delta^\ast_n)$ and $\delta^\ast_n \in S$
$\quad (c)$ let $\delta^\ast = \sup\{\delta^\ast_n : n < \omega\}$.

We define $q \in Q$ as follows

$\square_4 (\alpha)$ $u^q = \cup\{p_{\delta_2} : n < \omega\}$
$\quad (b)$ $<^q = \{(\alpha, \beta) : \alpha < ^{P\delta_2} \beta \text{ for some } n \text{ or } \alpha + \lambda < \beta \wedge \{\alpha, \beta\} \subseteq u^q, \text{equivalently for some } m < n, \alpha \in u^{P\delta^\ast_m} \setminus \delta^\ast_m \text{ and } \beta \in u^{P\delta^\ast} \setminus \delta^\ast_n\}$
$\quad (c)$ $\mathcal{S}_1^q = \cup \{\mathcal{S}_1^{P\delta^\ast} : n < \omega\} \cup \{\eta^\ast\}$
$\quad (d)$ $\mathcal{S}_2^q = \cup \{\mathcal{S}_2^{P\delta^\ast} : n < \omega\}$
$\quad (e)$ $\Lambda^q = \cup \{\Lambda^{P\delta^\ast} : n < \omega\}$
$\quad (f)$ $f^q_\rho = f^{P\delta^\ast}_\rho$ if $\rho \in \Lambda^{P\delta^\ast}$.

So there is a pair $(q_\ast, g^\ast)$ such that:

$\square_5 (\alpha)$ $q \leq q_\ast$
$\quad (b)$ $q_\ast \models Q \ "g^\ast = g \upharpoonright EM(J_1^\ast, \Phi)\"$
$\quad (c)$ $g^\ast$ is an isomorphism from $EM_{\tau(T)}(J_1^\ast, \Phi)$ onto $EM_{\tau(T)}(J_2^\ast, \Phi)$. 

So $g^+(a_{n'}) \in \text{EM}(J^*_q, \Phi)$ hence is of the form $\sigma^{M^*_2}(a_{t_0}, \ldots, a_{t_{n-1}})$ for some $t_0, \ldots, t_{n-1} \in J^*_q$ and a $\tau_\Phi$-term $\sigma(x_0, \ldots, x_{n-1})$.

Note that by the definition of $\leq_Q$ in $\oplus_2$:

$\Box_6$ if $\eta \in \mathcal{G}_2^q$ then $\text{Rang}(\eta) \cap u^q$ is bounded in $\delta^*$.

[Why? If $\eta \in \mathcal{G}_2^q$ this holds by our choice of $q$ and if $\eta \in \mathcal{G}_2^q \setminus \mathcal{G}_2^q$ then $\text{Rang}(\eta) \cap u^q$ is finite so as $u^q \subseteq \delta$ it follows that $\text{Rang}(\eta) \cap u^q$ is bounded in $\delta^*$.]

We can find $n(*) < \omega$ such that:

$\Box_7$ for each $k < n$ and $\ell < n$ we have

(a) if $t_\ell \in Q_{J^*_q}$, i.e. $t_\ell \in u^q \subseteq \lambda^+$ then $t_\ell \leq^q \delta^*_n(*)$ or $\delta^* \leq t_\ell$ (hence $\bigwedge_n \delta^*_n \leq^q t_\ell$)

(b) if $t_\ell \in P_{J^*_q}$, i.e. $t_\ell \in \mathcal{G}_2^q$ then $\{F^*_{j_\ell} (t_\ell) : n < \omega\}$ is disjoint to $[\delta^*_n(*), \delta^*) \cap u^q$.

Now using “$T$ is stable”, the rest is as in 2.5, 2.9. $\Box_{3.1} \{3c.12\}$

**Discussion 3.3.** (2012.11.23) 1) Can we do it in ZFC? It is natural to use $\langle W_\alpha : \alpha \in S_\mu^\omega \rangle$ be stationary pairwise almost disjoint, see [Sh:F980].

2) Instead of “not strongly stable” it suffices to assume “not strongly $^2$ stable”, see [Sh:863]. In [Sh:F930] even much less.
\section*{§ 4. Theories with order}

\{4d.2\}

**Claim 4.1.** If $\lambda = \mu^+$, $\text{cf}(\mu)$, $\lambda = \lambda^\kappa$, $\kappa = \text{cf}(\kappa) < \kappa(T)$ and $T$ is unstable then there are $\text{EF}_{\mu \times \kappa, \lambda^\kappa}$-equivalent non-isomorphic models of $T$ of cardinality $\lambda$.

The new point in 4.2 is the $\text{EF}^+$ rather than $\text{EF}$.

\{4d.7\}

**Claim 4.2.** Assume $\lambda = \lambda^{<\theta}$ and $\lambda$ is regular uncountable, $T \subseteq T_1$ are complete first order theories of cardinality $\lambda$.

1) If $T$ is unstable then there are models $M_1, M_2$ of $T_1$ of cardinality $\lambda^+$, $\text{EF}^+_{\lambda^+, \lambda^+}$-equivalent with non-isomorphic $\tau_T$-reducts.

2) Assume $\Phi \in \mathcal{T}_{\omega}^m$ is proper for linear orders, $\bar{\sigma} = (\sigma_i(x) : i < i(\ast))$ a sequence of terms from $\tau_\mathcal{B}, \bar{x}^i = (x^i_1 : i < i(\ast)), i(\ast) < \lambda, \varphi(\bar{x}^1, \bar{x}^2)$ is a formula in $\mathcal{L}(\tau_T)$, $\tau \subseteq \tau_T$ (any logic) and for every linear order $I$ letting $M = \text{EM}(I, \Phi), b_i = (\sigma^M_i(a) : i < i(\ast))$ we have $(M \upharpoonright \tau) \models \varphi[b_s, b_t]^{(i(\ast)=t)}$ for every $s, t \in I$. Then there are linear orders $I_1, I_2$ of cardinality $\lambda^+$ such that $M_1, M_2$ are $\text{EF}^+_{\lambda^+, \lambda^+}$-equivalent but not isomorphic where $M_\ell = \text{EM}_\ell(I_\ell, \Phi)$ for $\ell = 1, 2$.

3) If every $\text{EM}(I, \Phi)$ is a model of $T_1$ then in (2) the models $M_1, M_2$ are in $\text{PC}(T_1, T)$.

**Proof.** 1) Let $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}(\tau_T)$ order some infinite subset of $\mathcal{M}$ for some $M \models T$.

Let $\Phi$ be as in Definition 2.3, i.e., $[\text{Sh:c}, \text{Ch.VII, VIII}]$, i.e. proper for linear orders such that $\tau_{T_1} \subseteq \tau(\Phi), |\tau(\Phi)| = |T_1|$ and for every linear order $I, \text{EM}(I, \Phi)$ (we allow the skeleton to consist of $m$-tuples rather than elements) is a model of $T_1$ satisfying $\varphi[\bar{a}_s, \bar{a}_t]$ iff $s < t, t$. Now we can apply part (2) with $i(\ast) = m$.

2) We choose $I$ such that

(\ast) (a) $I$ is a linear order of cardinality $\lambda$ (yes, not $\lambda^+$)

(b) if $\alpha, \beta \in (1, \lambda)$ then $(I \times \alpha) + (I \times \beta)^* \cong I$ (equivalently every $\alpha, \beta \in [1, \lambda^+]$)

(c) $I$ is isomorphic to its inverse

(d) $I$ has cofinality $\lambda$.

For every $S \subseteq S^\lambda_\lambda = \{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\}$ we define $I_S = \sum_{\alpha < \lambda^+} I_{S, \alpha}$ where $I_{S, \alpha}$ is isomorphic to $I$ if $\alpha \in \lambda^+ \setminus S$ and isomorphic to the inverse of $I \times \omega$ otherwise. Now

(\ast)_2 if $S_1, S_2 \subseteq S^\lambda_\lambda$ then the models $\text{EM}(I_{S_2}, \Phi), \text{EM}(I_{S_1}, \Phi)$ are $\text{EF}^+_{\lambda, \theta, \lambda^+}$-equivalent.

[Why? Let $J_{\ell, \gamma} = \sum_{\alpha < \gamma} I_{S_1, \alpha}$. Let $\mathcal{F}_\gamma := \{f : \text{for some non-zero ordinal } \gamma < \lambda^+, f \in \mathcal{F}_\gamma \}$ and $[\gamma \in S_1 \leftrightarrow \gamma \in S_2]$ where $\mathcal{F}_\gamma := \{f \text{ an isomorphism from } \sum_{\alpha < \gamma} I_{S_1, \alpha} \text{ onto } \sum_{\alpha < \gamma} I_{S_2, \alpha}\}$.

Now

(\ast)_1 $\mathcal{F}_\gamma \neq \emptyset$ for $\gamma < \lambda^+$

(\ast)_2 if $f \in \mathcal{F}_\gamma$ and $[\gamma \in S_1 \equiv \gamma \in S_2]$ and $\gamma < \beta < \lambda$ then $f$ can be extended to some $g \in \mathcal{F}_\beta$]
\(\ast\)\(_3\) if \(\gamma < \lambda, X_\ell \subseteq I_\ell\) has cardinality \(< \lambda^+\) for \(\ell = 1, 2\) then for some successor \(\beta, \gamma < \beta < \lambda^+\) and \(X_\ell \subseteq I_\ell, \beta\) for \(\ell = 1, 2\)

\(\ast\)\(_4\) if \(\gamma_1 \in S_1 \iff \gamma_2 \in S_2\) for \(i < \delta, \delta\) a limit ordinal \(< \lambda\) and \(\langle \gamma_i : i < \delta \rangle\) is increasing then \(\gamma_\delta := \cup\{\gamma_i : i < \delta\}\) satisfies \(\gamma_\delta \in S_1 \equiv \gamma_\delta \in S_1\).

Lastly, we have to deal with case 2 in Definition 1.7(2) so assume

\(\ast\)\(_5\) \(f_\ast \in \mathcal{F}_\ast, [\gamma_\ast \in S_1 \equiv \gamma_\ast \in S_2]\) and \(R_\ell \subseteq \theta^>(M_\ell)\) for \(\ell = 1, 2\) are as there for \(f_\ast\).

This holds because the strategy is simple, e.g. with no memory. Now if \(f\) does not map the definition of \(R_1\) in \(M_1\) to the definition of \(R_2\) in \(M_2\) we can use subcase 2B there, so we assume this does not occur. Let \(\ell \in \{1, 2\}\).

\(\ast\)\(_6\) Let \(e_\ell = \{(s, t) : s, t \in \theta^>(I_\ell)\) and some automorphism of \(I_\ell\) over \(I_\ell, \gamma\), maps \(s\) to \(t\).

\(\ast\)\(_7\) Let \(Y_\ell\) be the set of \(e_\ell\)-equivalence classes.

Note

\(\circ\)\(_1\) for \(\ell \in \{1, 2\}, n < \omega\) and \(y_0, \ldots, y_n \in Y_\ell\) the following are equivalent:

(a) some \(\bar{a} \in y_n\) depend (by \(R_1\)) on \(y_0 \cup \ldots \cup y_{n-1}\)

(b) every \(\bar{a} \in y_n\) depend (by \(R_1\)) on \(y_0 \cup \ldots \cup y_{n-1}\).

So \(R_1\) induce a 1-dependence relation on \(Y_1\), so let \((y_i : i < i(\ast))\) be a maximal independent subset of \(Y_1\) such that \([i < i(\ast) \land \bar{a} \in y_i \Rightarrow \bar{a}\) does not depend on \(\cup\{y_j : j < i(\ast), j \neq i\}\).

So \(\circ\)\(_2\) it is enough to deal with one \(y_i\).

Now we can find \(i_\ell, \gamma \in y_i\) such that \(\text{Rang}(i_{\ell, \gamma}) \setminus I_{\ell, \gamma^*} \subseteq I_{\ell, \gamma^2} \setminus I_{\ell, \gamma+1}\) for each \(\gamma \in [\gamma_\ast, \lambda^+]\) as \(I_1\) has enough automorphisms

\(\circ\)\(_3\) if \(\{i_{\ell, \gamma} : \gamma \in [\gamma_\ast, \lambda^+]\}\) is not \(R_1\)-independent, then dim\((y_i)\) is finite, in fact 1 or 0.

So we choose \(\beta_\ast\) such that

\(\circ\)\(_4\) \(\gamma_\ast < \beta_\ast < \lambda^+\) and \(\beta_\ast \in S_1 \equiv \beta_\ast \in S_2\) and for every \(i < i(\ast)\), if dim\((X_\ell)\) is finite then \(y_i\) has a maximal \(R_1\)-independent set included in \(e(y_i)(J_1, \beta_\ast)\).

[Why possible? Because for any such \(\beta_\ast\) is an automorphism of \(I_2\) over \(J_1, \gamma_\ast\), mapping \(I_{\beta_\ast, +2}\) onto \(I_{\gamma_\ast, +2}\).]

Let \(e \in \mathcal{F}_\beta, e\) extend \(f\) and using it we can choose \(\langle (\bar{a}_1^\gamma, \bar{a}_2^\gamma) : \zeta < \zeta^* \rangle\) as required.

\(\circ\)\(_5\) if \(S_1, S_2 \subseteq S_\lambda^+\) and \(S_1 \setminus S_2\) is stationary, then \(\text{EM}_\ast(I_{S_1}, \Phi), \text{EM}_\ast(I_{S_2}, \Phi)\) are not isomorphic.

[Why? By the proof in [Sh:300, Ch.III,\S3] or [Sh:59,\S3] = [Sh:E59,\S3] only easier. In fact, immitating it we can represent the invariants from there. If \(k = 2\) we have to work somewhat more.]

2) As in [HySh:529].

3) Obvious.

\(\square\)\(_{4.2}\)
Conclusion 4.3. Assume $T$ is a (first order complete) theory.
1) If $T$ is unstable, then $(T, *)$ is fat.
2) If $T$ is unstable or stable with DOP, or stable with OTOP, then $T$ is fat.
3) For every $\mu$ there is a $\mu$-complete, class forcing $\mathbb{P}$ such that in $\mathbb{V}^\mathbb{P}$ we have: if $T$ is not strongly dependent or just not strongly stable then $T$ is fat, moreover $(T, *)$ is fat.

Proof. 1) By 4.2.
2) Similar, the only difference is that the formula defining the “order” is not first order and the length of the relevant sequences may be infinite but still $\leq |T|$ (see [Sh:c, Ch.XIII]).
3) By parts (1),(2) we should consider only stable - not strongly stable $T$. Choose a class $C$ of regular cardinals such that $\lambda \in C \Rightarrow (2^{<\lambda})^+ < \min(C \setminus \lambda^+)$ and $\min(C) > \mu$. We iterate with full support $\langle \mathbb{P}_\mu, \mathbb{Q}_\mu : \mu \in C \rangle$ with $\mathbb{Q}_\mu$ as in 3.1. □

Claim 4.4. Assume $T \subseteq T_1$, $\lambda = \lambda^\kappa$ is not necessary regular and $\kappa = \cf(\kappa) < \kappa(T)$, e.g. $T$ is unstable. Then there are $\EF_{\lambda^\kappa, \lambda^\lambda^+}$-equivalent non-isomorphic models from $\text{PC}(T_1, T)$ of cardinality $\lambda^+$.

Proof. As in [HySh:529], seeing the proof of 4.2. □

As said in the introduction by the old results (note: 4.5 is on elementary classes and 4.6 on small enough pseudo elementary classes).

Conclusion 4.5. (ZFC) For first order countable complete first order theory $T$ the following conditions are equivalent:

(A) $T$ is superstable with NDOM and NOTOP
(B) if $\lambda = \cf(\lambda) > |T|$ and $M_1, M_2 \in \text{Mod}_T(\lambda)$ are $L_{\infty, \lambda}(\tau_T)$-equivalent then $M_1, M_2$ are isomorphic
(C) if $\lambda = \cf(\lambda) > |T|$ and $M_1, M_2 \in \text{Mod}_T(\lambda^+)$ are $\EF_{\omega, \lambda}$-equivalent then $M_1, M_2$ are isomorphic
(D) for some regular $\lambda > |T|$, if $M_1, M_2 \in \text{Mod}_T(\lambda^+)$ are $\EF_{\omega, \lambda}$-equivalent then they are isomorphic.

Proof. Clause (A), clause (B)$_1$, clause (B)$_2$ are equivalent because: as proved in [Sh:c, Ch.XIII,Th.1.11], we have $(A) \Rightarrow (B)_1$ and $(B)_2$ and the inverse implication holds by [Sh:220]. Now by the definitions trivially $(B)_1 \Rightarrow (C) \Rightarrow (D)$.

Lastly, by [HySh:529], i.e. by 4.1 we have $\neg(A) \Rightarrow \neg(D)$, i.e. $(D) \Rightarrow (A)$ so we have the circle.

So 4.5 tells us what we know about Question (A)$_0$ of 1.2. Similarly concerning (B)$_0$ of Question 1.2. □

Conclusion 4.6. (ZFC) For first order countable complete first order theory $T$ and $\kappa \geq 2^{8\omega}$ the following conditions are equivalent:

(A) $T$ is unsuperstable
(B)$_{\kappa}$ for every $\lambda > \kappa \geq |T|$ and $(\kappa, T)$-candidate $\psi$ (see Definition 0.8), and ordinal $\alpha < \lambda$ satisfying $|\alpha|^+ = \lambda \Rightarrow |\alpha| \leq |\alpha| \times \omega$, there are $\EF_{\omega, \lambda}$-equivalent non-isomorphic models $M_1, M_2 \in PC_\tau(T)(\psi)$ of cardinality $\lambda$
(C) for some \( \lambda > \kappa \geq |T| \), for no \((\kappa,T)\)-candidate \( \psi \) is the class \( \text{PC}_{\tau(T)}(\psi) \) categorical in \( \lambda \).

Proof. First, assume \( T \) is superstable, so clause (A) holds. By the proofs of [Sh:c, Ch.VI,§4] there is a \((\kappa,T)\)-candidate \( \psi \), \( \text{PC}_{\tau(T)}(\psi) \) is the class of saturated models of \( T \), (in details, if \( n < \omega, a \in {}^nC \), \( \text{tp}(b,\emptyset,C) \) is stationary, \( q = \text{tp}(\langle a \rangle,\emptyset,C) \), \( p = p(x,y) = \text{tp}(\langle a \rangle,\emptyset,C) \) then let \( \psi_{p,q} \) be such that \( M \models \psi_{p,q} \) if for every \( b' \in {}^nM \) realizing the type \( q(y) \), the function \( c \mapsto F_{p,q}(c,b') \) is one-to-one and if \( k < \omega, c_0,\ldots,c_k \in M \) are pairwise distinct then \( \text{tp}_{L(\tau(T))}(F_{p,q}(c_k),\{F_{p,q}(c_0),\ldots,F_{p,q}(c_{k-1})\} \cup b',M) \) extends \( p(x,b') \) and does not fork over \( M \).

Lastly, \( \psi = \land\{\psi_{p,q} : p,q \text{ as above}\} \) so \( \in L_{\kappa^+,\omega} \). So in the present case also \((B)_{\kappa},(C)_{\kappa},(D)_{\kappa} \) holds.

Second, assume \( T \) is not superstable, so clause (A) holds and we shall prove the rest. Let \( \psi \) be a \((\kappa,T)\)-candidate.

By 0.9 and let there is \( \Phi \in T_{\kappa^+,\omega}^{\omega-1} \) witnessing this hence witnessing unsuperstability and now we can use Theorem 4.1 quoted above. \( \square_{4.6} \)
References


[Sh:F918] , *Theories with EF-Equivalent Non-Isomorphic Models II*.

[Sh:F930] , *Category of metric elementary classes with uncountable vocabulary*.

[Sh:F980] , *Creature forcing and large continuum*.


E-mail address: shelah@math.huji.ac.il

URL: http://shelah.logic.at