

# The stationary set splitting game\*

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## Abstract

The *stationary set splitting game* is a game of perfect information of length  $\omega_1$  between two players, *unsplit* and *split*, in which *unsplit* chooses stationarily many countable ordinals and *split* tries to continuously divide them into two stationary pieces. We show that it is possible in ZFC to force a winning strategy for either player, or for neither. This gives a new counterexample to  $\Sigma_2^2$  maximality with a predicate for the nonstationary ideal on  $\omega_1$ , and an example of a consistently undetermined game of length  $\omega_1$  with payoff definable in the second-order monadic logic of order. We also show that the determinacy of the game is consistent with Martin's Axiom but not Martin's Maximum.

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The *stationary set splitting game* ( $\mathcal{SG}$ ) is a game of perfect information of length  $\omega_1$  between two players, *unsplit* and *split*. In each round  $\alpha$ , *unsplit* either accepts or rejects  $\alpha$ . If *unsplit* accepts  $\alpha$ , then *split* puts  $\alpha$  into one of two sets  $A$  and  $B$ . If *unsplit* rejects  $\alpha$  then *split* does nothing. After all  $\omega_1$  many rounds have been played, *split* wins if *unsplit* has not accepted stationarily often, or if both of  $A$  and  $B$  are stationary.

In this note we prove that it is possible to force a winning strategy for either player in  $\mathcal{SG}$ , or for neither, and we also show that the determinacy of  $\mathcal{SG}$  is consistent with Martin's Axiom but not Martin's Maximum [4]. We also present two guessing principles,  $\mathcal{C}_s$  (*club for split*) and  $\mathcal{D}_u$  (*diamond for unsplit*), which imply the existence of winning strategies for *split* and *unsplit*, respectively (and are therefore incompatible; see Theorems 1.5 and 1.8). These principles may be of independent interest.

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# 1 Winning strategies

## 1.1 Strategies for *split*

A collection  $\mathcal{X}$  of countable sets is *stationary* if for every function  $F: [\bigcup \mathcal{X}]^{<\omega} \rightarrow \bigcup \mathcal{X}$  there is an element of  $\mathcal{X}$  closed under  $F$ . A set  $\mathcal{X}$  of countable sets is *projective stationary* [2] if for every stationary  $S \subset \omega_1$  the set of  $X \in \mathcal{X}$  with  $X \cap \omega_1 \in S$  is stationary. We note that a partial order  $P$  is said to be *proper* if forcing with  $P$  preserves the stationarity (in the sense above) of stationary sets from the ground model (see [11]).

The following statement holds in fine structural models such as  $L$ . It is a strengthening of the principle  $(+)$  used in [8]. Justin Moore has pointed out to us that his Mapping Reflection Principle [9] implies the failure of  $(+)$ . We note also that in the statement of  $(+)$ , “projective stationary” can be replaced with “club” without strengthening the statement. We do not know if that is the case for  $\mathcal{C}+$ .

**1.1 Definition.** Let  $\mathcal{C}+$  be the statement that there exists a projective stationary set  $\mathcal{X}$  consisting of countable elementary substructures of  $H(\aleph_2)$  such that for all  $X, Y$  in  $\mathcal{X}$  with  $X \cap \omega_1 = Y \cap \omega_1$ , either every for every club  $C \subset \omega_1$  in  $X$  there is a club  $D \subset \omega_1$  in  $Y$  with  $D \cap X \subset C \cap X$ , or for every for every club  $D \subset \omega_1$  in  $Y$  there is a club  $C \subset \omega_1$  in  $X$  with  $C \cap X \subset D \cap X$ .

Given a partial run of  $\mathcal{SG}$  of length  $\alpha$ , we let  $E_\alpha$  be the set of  $\beta < \alpha$  accepted by *unsplit*, and we let  $A_\alpha, B_\alpha$  be the partition of  $E_\alpha$  chosen by *split*.

**Theorem 1.2.** *If  $\mathcal{C}+$  holds then split has a winning strategy in  $\mathcal{SG}$ .*

*Proof.* Let  $\mathcal{X}$  be a set of countable elementary submodels of  $H(\aleph_2)$  witnessing  $\mathcal{C}+$ , and for each  $\alpha < \omega_1$  let  $\mathcal{X}_\alpha$  be the set of  $X \in \mathcal{X}$  with  $X \cap \omega_1 = \alpha$ . Let  $Z$  be the set of  $\alpha < \omega_1$  such that  $\mathcal{X}_\alpha$  is nonempty (since  $\mathcal{X}$  is projective stationary, this set contains a club).

Play for *split* as follows. In round  $\alpha \in Z$ , if *unsplit* accepts  $\alpha$ , let  $\mathcal{Y}_\alpha$  be the set of all  $X \in \mathcal{X}_\alpha$  such that  $X$  contains a stationary subset of  $\omega_1$ ,  $E_X$ , such that  $E_X \cap \alpha = E_\alpha$ . If  $\mathcal{Y}_\alpha = \emptyset$ , put  $\alpha \in A_{\alpha+1}$ . Otherwise, since every club subset of  $\omega_1$  in every member of  $\mathcal{Y}_\alpha$  intersects  $E_\alpha$ , there cannot be two club subsets of  $\omega_1$  in  $\bigcup \mathcal{Y}_\alpha$ , one disjoint from  $A_\alpha$  and one disjoint from  $B_\alpha$ , since some club subset of  $\omega_1$  in  $\bigcup \mathcal{Y}_\alpha$  would be contained in both of these clubs. If any member of  $\mathcal{Y}_\alpha$  contains a club subset of  $\omega_1$  disjoint from  $A_\alpha$ , then put  $\alpha$  in  $A_{\alpha+1}$ , and if any member of  $\mathcal{Y}_\alpha$  contains a club subset of  $\omega_1$  disjoint from  $B_\alpha$ , then put  $\alpha$  in  $B_{\alpha+1}$ . If neither case holds, put  $\alpha \in A_{\alpha+1}$ .

Let  $E$  be the play by *unsplit* in a run of  $\mathcal{SG}$  where *split* has played by this strategy, and let  $A$  and  $B$  be the corresponding play by *split*. Let  $C$  be a club subset of  $\omega_1$  and supposing that  $E$  is stationary, fix  $X \in \mathcal{X}$  containing  $E$ ,  $A$ ,  $B$  and  $C$  with  $X \cap \omega_1 \in E \cap C$ . Then if  $A \cap C \cap X \cap \omega_1 = \emptyset$ , then  $X \cap \omega_1 \in A \cap C$ , and if  $B \cap C \cap X \cap \omega_1 = \emptyset$ , then  $X \cap \omega_1 \in B \cap C$ , which shows that  $C$  does not witness that *unsplit* won this run of the game.  $\square$

The following fact, in conjunction with Theorem 1.2, shows that Martin's Axiom is consistent with the existence of a winning strategy for *split*.

**Theorem 1.3.** *The statement  $\mathcal{C}+$  is preserved by forcing with c.c.c. partial orders.*

*Proof.* Let  $P$  be a c.c.c. forcing and let  $\mathcal{X}$  witness  $\mathcal{C}+$ . Let  $\gamma$  be a regular cardinal greater than  $\aleph_2$  and  $2^{|P|}$ . Let  $G \subset P$  be a  $V$ -generic filter, and let

$$\mathcal{X}[G] = \{X[G] \cap H(\aleph_2)^{V[G]} : X \prec H(\gamma)^V, X \cap H(\aleph_2)^V \in \mathcal{X}\}.$$

Since every club subset of  $\omega_1$  in  $V[G]$  contains one in  $V$ , in order to show that  $\mathcal{X}[G]$  witnesses  $\mathcal{C}+$  in  $V[G]$ , it suffices to show that  $\mathcal{X}[G]$  is projective stationary there. Fix a  $P$ -name  $\rho$  for a function from  $[H(\aleph_2)^{V[G]}]^{<\omega}$  to  $H(\aleph_2)^{V[G]}$ . For any countable  $X \prec H(\gamma)$  with  $X \cap H(\aleph_2) \in \mathcal{X}$  and  $\rho \in X$ ,  $X[G] \cap H(\aleph_2)^{V[G]}$  is in  $\mathcal{X}[G]$  and closed under the realization of  $\rho$ . Fix a  $P$ -name  $\tau$  for a stationary subset of  $\omega_1$  and a condition  $p \in P$ . Let  $S$  be the set of countable ordinals forced to be in  $\tau$  by some condition below  $p$ . Then exist a countable  $X \prec H(\gamma)$  with  $X \cap H(\aleph_2) \in \mathcal{X}$ ,  $X \cap \omega_1 \in S$  and  $\rho \in X$  and a condition  $q$  below  $p$  forcing that  $X[\dot{G}] \cap \omega_1$  (where  $\dot{G}$  is the name for the generic filter) is in the realization of  $\tau$ . By genericity, then,  $\mathcal{X}[G]$  is projective stationary.  $\square$

We do not know how to force  $\mathcal{C}+$ , however, and use a different principle to force the existence of a winning strategy for *split*.

**1.4 Definition.** Let  $\mathcal{C}_s$  be the statement that there exist  $c_\alpha$  ( $\alpha < \omega_1$  limit) such that each  $c_\alpha$  is a sequence  $\langle a_\beta^\alpha : \beta < \gamma_\alpha \rangle$  (for some countable  $\gamma_\alpha$ ) of cofinal subsets of  $\alpha$  of ordertype  $\omega$  and

- for all limit  $\alpha < \omega_1$  and all  $\beta < \beta' < \gamma_\alpha$ ,  $a_{\beta'}^\alpha \setminus a_\beta^\alpha$  is finite;
- for every club  $C \subset \omega_1$  and every stationary  $E \subset \omega_1$  there exists an  $a_\beta^\alpha$  with  $\alpha \in E$  such that  $a_\beta^\alpha \setminus C$  is finite and  $a_\beta^\alpha \cap E$  is infinite.

The principle  $\mathcal{C}_s$  also holds in fine structural models such as  $L$ . The winning strategy for *split* given by  $\mathcal{C}_s$  is very similar to the one given by  $\mathcal{C}+$ .

**Theorem 1.5.** *If  $\mathcal{C}_s$  holds then *split* has a winning strategy in  $\mathcal{S}\mathcal{G}$ .*

*Proof.* Let  $a_\beta^\alpha$  ( $\alpha < \omega_1$  limit,  $\beta < \gamma_\alpha$ ) witness  $\mathcal{C}_s$ . Play for *split* as follows. In round  $\alpha$ ,  $\alpha$  a limit, if *unsplit* has accepted  $\alpha$  and if some  $a_\beta^\alpha$  intersects  $A_\alpha$  infinitely and  $B_\alpha$  finitely, then put  $\alpha$  in  $B_{\alpha+1}$ . If some  $a_\beta^\alpha$  intersects  $B_\alpha$  infinitely and  $A_\alpha$  finitely, then put  $\alpha$  in  $A_{\alpha+1}$ . Since the  $a_\beta^\alpha$ 's ( $\beta < \gamma_\alpha$ ) are  $\subset$ -decreasing mod finite, both cases cannot occur. If neither case occurs, put  $\alpha$  in  $A_{\alpha+1}$ .

Let  $E$  be the play by *unsplit* in a run of  $\mathcal{S}\mathcal{G}$  where *split* has played by this strategy, and let  $A$  and  $B$  be the corresponding play by *split*. Let  $C$  be a club subset of  $\omega_1$  and supposing that  $E$  is stationary, fix  $a_\beta^\alpha$  with  $\alpha \in E$  such that  $a_\beta^\alpha \setminus C$  is finite and  $a_\beta^\alpha \cap E$  is infinite. Then if  $A \cap a_\beta^\alpha$  is finite, then  $\alpha \in A \cap C$ , and if  $B \cap a_\beta^\alpha$  is finite, then  $\alpha \in B \cap C$ , which shows that  $C$  does not witness that *unsplit* won this run of the game.  $\square$

A partial order  $P$  is said to be *strategically*  $\omega$ -closed if there exists a function  $f: P^{<\omega} \rightarrow \mathcal{P}(P)$  such that whenever  $\langle p_i : i \leq n \rangle$  is a finite descending sequence in  $P$ ,  $f(\langle p_i : i \leq n \rangle)$  is a dense subset below  $p_n$  and, whenever  $\langle p_i : i < \omega \rangle$  is a descending sequence in  $P$  such that for each  $n$  there exists a  $j$  with

$$p_j \in f(\langle p_i : i \leq n \rangle),$$

the sequence has a lower bound in  $P$ . It is easy to see that strategic  $\omega$ -closure is equal to the property that for every countable  $X \prec H((2^{|P|})^+)$  and every  $(X, P)$ -generic filter  $g$  contained in  $X$  there is a condition in  $P$  extending  $g$ .

Let us say that a set  $a$  captures a pair  $E, C$  if  $a \setminus C$  is finite and  $a \cap E$  is infinite. Given  $A \subset \omega_1$ , let  $\mathbb{C}(A)$  be the partial order which adds a club subset of  $A$  by initial segments. We force  $\mathcal{C}_s$  by first adding a potential  $\mathcal{C}_s$ -sequence by initial segments, and then iterating to kill off every counterexample.

We refer the reader to [11] for background on countable support iterations of proper forcing.

**Theorem 1.6.** *Suppose that CH and  $2^{\aleph_1} = \aleph_2$  hold. Let  $\bar{P} = \langle P_\eta, \dot{Q}_\eta : \eta < \omega_2 \rangle$  be a countable support iteration such that  $P_0$  is the partial order consisting of sequences  $\langle c_\alpha : \alpha < \delta \text{ limit} \rangle$ , for some countable ordinal  $\delta$ , such that each  $c_\alpha$  is a sequence  $\langle a_\beta^\alpha : \beta < \gamma_\alpha \rangle$  (for some countable ordinal  $\gamma_\alpha$ ) of cofinal subsets of  $\alpha$  of ordertype  $\omega$ , decreasing by mod-finite inclusion (and  $P_0$  is ordered by extension). Suppose that the remainder of  $\bar{P}$  satisfies the following conditions.*

- For each nonzero  $\eta < \omega_2$  there is a  $P_\eta$ -name  $\tau_\eta$  for a subset of  $\omega_1$  such that if  $(\tau_\eta)_{G_\eta}$  (where  $G_\eta$  is the restriction of the generic filter to  $P_\eta$ ) is stationary in the  $P_\eta$  extension and there exists a club  $C \subset \omega_1$  in this extension such that no  $a_\beta^\alpha$  with  $\alpha \in \tau_{G_\eta}$  captures the pair  $\tau_{G_\eta}, C$ , then  $\dot{Q}_\eta$  is  $\mathbb{C}(\omega_1 \setminus (\tau_\eta)_{G_\eta})$  (and otherwise,  $\dot{Q}_\eta$  is  $\mathbb{C}(\omega_1)$ ).
- For every pair  $E, C$  of subsets of  $\omega_1$  in any  $P_\eta$ -extension ( $\eta < \omega_2$ ), if  $E$  is stationary in this extension and  $C$  is club and no  $a_\beta^\alpha$  with  $\alpha \in E$  captures  $E, C$ , then there is a  $\rho \in [\eta, \omega_2)$  such that if  $E$  is stationary in the  $P_\rho$  extension, then  $\dot{Q}_\rho$  is  $\mathbb{C}(\omega_1 \setminus E)$ .

Then  $\bar{P}$  is strategically  $\omega$ -closed, and  $\mathcal{C}_s$  holds in the  $\bar{P}$ -extension. Furthermore, in the  $\bar{P}$  extension,  $\diamond(S)$  holds for every stationary  $S \subset \omega_1$ .

*Proof.* Let  $X$  be a countable elementary submodel of  $H((2^{|\bar{P}|})^+)$  with  $\bar{P} \in X$ , let  $g$  be an  $X$ -generic filter contained in  $\bar{P} \cap X$ . Let  $\gamma_{X \cap \omega_1}$  be the ordertype of  $X \cap \omega_2$ , and for each  $\beta < \gamma_{X \cap \omega_1}$ , let  $\eta_\beta$  be the  $\beta$ th member of  $X \cap \omega_2$ . For each  $\beta < \gamma_{X \cap \omega_1}$ , let  $a_\beta^{X \cap \omega_1}$  be a cofinal subset of  $X \cap \omega_1$  of ordertype  $\omega$  such that, letting  $g_\eta$  denote the restriction of  $g$  to  $P_\eta$ ,

- for all  $\beta' < \beta < \gamma_{X \cap \omega_1}$ ,  $a_\beta^{X \cap \omega_1} \setminus a_{\beta'}^{X \cap \omega_1}$  is finite;
- $a_\beta^\alpha$  is eventually contained in every club subset of  $\omega_1$  in  $X[g_{\eta_\beta}]$  and intersects infinitely every stationary subset of  $\omega_1$  in every  $X[g_{\eta_{\beta'}}]$ ,  $\beta' \in [\beta, \gamma_{X \cap \omega_1})$ .

It remains to see that we can extend  $g$  to a condition whose first coordinate is given by adding  $c_{X \cap \omega_1} = \langle a_\beta^\alpha : \beta < \gamma_{X \cap \omega_1} \rangle$  to the union of the first coordinates of the elements of  $g$ , and whose  $\eta$ th coordinate, for each nonzero  $\eta \in X \cap \omega_2$  is the condition given by the union of  $\{X \cap \omega_1\}$  and the set of realizations of the  $\eta$ th coordinates of the members of  $g$ . We do this by induction on  $\eta$ , letting  $g'_\eta$  be our extended condition in  $P_\eta$ .

For each  $\eta \in \omega_2 \cap X$ , there is a  $P_\eta$ -name  $\sigma \in X$  for a club subset of  $\omega_1$  such that if, in the  $P_\eta$ -extension  $(\tau_\eta)_{G_\eta}$  is stationary and there exists a club  $C$  such that  $\tau_{G_\eta}, C$  is not captured by any  $a_\beta^\alpha$  with  $\alpha \in (\tau_\eta)_{G_\eta}$ , then  $\sigma_{G_\eta}$  is such a  $C$ . However, the realizations of  $\tau_\eta$  and  $\sigma$  by  $g$  are captured by  $a_{o.t.(\eta \cap \omega_2)}^{X \cap \omega_1}$ , so  $g'_\eta$  forces that  $\tau_{G_\eta}, \sigma_{G_\eta}$  is captured by  $a_{o.t.(\eta \cap \omega_2)}^{X \cap \omega_1}$ . It follows that  $g'_\eta$  forces that either  $\tilde{Q}_\eta$  is  $\mathbb{C}(\omega_1)$ , or  $X \cap \omega_1$  is not in  $\tau_{G_\eta}$ . In either case, the union of the members of  $g \cap \tilde{Q}_\eta$  be can extended to a condition in  $\tilde{Q}_\eta$  by adding  $\{X \cap \omega_1\}$ .

To see that  $\diamond(S)$  holds for every stationary  $S \subset \omega_1$  in the  $\bar{P}$  extension, fix such an  $S$  in the  $P_\alpha$  extension for some  $\alpha < \omega_2$ . Since  $\bar{P}$  is  $(\omega, \infty)$  distributive, there exists in this extension a set  $\langle e_\beta^\delta : \delta, \beta < \omega_1 \rangle$  such that for every  $\delta < \omega_1$  and every  $x \subset \delta$  there are uncountably many  $\beta$  such that  $e_\beta^\delta = x$ . Then, letting  $T \in \mathcal{P}(\omega_1)^{V[G_\alpha]}$  be the set such that the realization of  $\tilde{Q}_\alpha$  is  $\mathbb{C}(T)$ ,  $\tilde{Q}_\alpha$  adds a  $\diamond$  sequence  $\langle b_\delta : \delta \in S \rangle$  defined by letting  $b_\delta$  be  $e_\beta^\delta$ , where the  $\beta$ th element of  $T$  above  $\beta$  is the first element of the generic club for  $\tilde{Q}_\alpha$  above  $\delta$ . To see that this is a  $\diamond$  sequence, note that since  $S$  is stationary in the  $\bar{P}$  extension, there are stationarily many elementary submodels  $X$  of any sufficiently large  $H(\theta)^{V[G]}$  in this extension with  $X \cap \omega_1 \in S$ . Then  $X \cap (G/G_\alpha)$  is a  $(X \cap V[G_\alpha], \bar{P}/P_\alpha)$ -generic filter which can be extended to a condition in  $\bar{P}/P_\alpha$  by adding  $X \cap \omega_1$  to each coordinate, and extended again to make any element of  $T \setminus ((X \cap \omega_1) + 1)$  the least element of the generic club for  $\tilde{Q}_\alpha$  above  $X \cap \omega_1$ . That  $\langle b_\beta : \beta \in S \rangle$  is a  $\diamond$  sequence then follows by genericity.  $\square$

Section 2 shows that proper forcing does not always preserve the existence of a winning strategy for *split*.

## 1.2 A strategy for *unsplit*

In this section we show that it is consistent for *unsplit* to have a winning strategy in  $SG$ . We do this via the following guessing principle.

**1.7 Definition.** Let  $\mathcal{D}_u$  be the statement that there exists a diamond sequence  $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$  such that for every  $E \subset \omega_1$  there is a club  $C \subset \omega_1$  such that either

$$\forall \alpha \in C ((E \cap \alpha = \sigma_\alpha) \Rightarrow \alpha \in E)$$

or

$$\forall \alpha \in C ((E \cap \alpha = \sigma_\alpha) \Rightarrow \alpha \notin E).$$

**Theorem 1.8.** *If  $\mathcal{D}_u$  holds then unsplit has a winning strategy in  $SG$ .*

*Proof.* Let  $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$  witness  $\mathcal{D}_u$ . Play for *unsplit* by accepting  $\alpha$  if and only if  $\sigma_\alpha = A_\alpha$ . At the end of the game, the set of  $\alpha$  such that  $\sigma_\alpha = A_\alpha$  is stationary, and there is a club  $C$  such that either for all  $\alpha$  in  $C$ , if  $\sigma_\alpha = A_\alpha$ , then  $\alpha$  is in  $A$ , or for all  $\alpha$  in  $C$ , if  $\sigma_\alpha = A_\alpha$ , then  $\alpha$  is in  $B$ . In either case, *split* has lost.  $\square$

Our iteration to force  $\mathcal{D}_u$  employs the same strategy as the iteration for  $\mathcal{C}_s$ . We first force to add a  $\diamond$ -sequence  $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$  by initial segments, and we then iterate to make this sequence witness  $\mathcal{D}_u$ , iteratively forcing a club through the set of  $\alpha < \omega_1$  such that  $\sigma_\alpha \neq E \cap \alpha$  or  $\alpha \in E$  for each  $E \subset \omega_1$  such that the sets  $\{\alpha \in E \mid \sigma_\alpha = E \cap \alpha\}$  and  $\{\alpha \in \omega_1 \setminus E \mid \sigma_\alpha = E \cap \alpha\}$  are both stationary.

More specifically, we have the following. Given a sequence  $\Sigma = \langle \sigma_\alpha : \alpha < \omega_1 \rangle$  such that each  $\sigma_\alpha$  is a subset of  $\alpha$ , and given  $E \subset \omega_1$ , let  $A(\Sigma, E)$  be the set of  $\alpha \in E$  such that  $\sigma_\alpha = E \cap \alpha$ , and let  $B(\Sigma, E)$  be the set of  $\alpha \in \omega_1 \setminus E$  such that  $\sigma_\alpha = E \cap \alpha$ .

**Theorem 1.9.** *Suppose that  $CH + 2^{\aleph_1} = \aleph_2$  holds, and let  $\bar{P}$  be a countable support iteration  $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$  such that  $P_0$  is the partial order consisting of sequences  $\langle \sigma_\beta : \beta < \gamma \rangle$ , for some countable ordinal  $\gamma$ , such that each  $\sigma_\beta$  is a subset of  $\beta$ , ordered by extension. Let  $\Sigma$  be the sequence added by  $P_0$  and suppose that the remainder of  $\bar{P}$  satisfies the following conditions.*

- *Each  $\dot{Q}_\alpha$  is either  $\mathbb{C}(\omega_1)$  or  $\mathbb{C}(\omega_1 \setminus B(\Sigma, E))$  for some  $E \subset \omega_1$  such that  $A(\Sigma, E)$  and  $B(\Sigma, E)$  are both stationary.*
- *For every  $E \subset \omega_1$  in any  $P_\alpha$ -extension ( $\alpha < \omega_2$ ) there is a  $\gamma \in [\alpha, \omega_2)$  such that if  $A(\Sigma, E)$  and  $B(\Sigma, E)$  are both stationary in the  $P_\gamma$  extension, then  $\dot{Q}_\gamma$  is  $\mathbb{C}(\omega_1 \setminus B(\Sigma, E))$ .*

*Then  $\bar{P}$  is strategically  $\omega$ -closed, and in the  $\bar{P}$ -extension,  $\mathcal{D}_u$  holds. Furthermore, in the  $\bar{P}$  extension,  $\diamond(S)$  holds for every stationary  $S \subset \omega_1$ .*

*Proof.* The iteration  $\bar{P}$  is clearly strategically  $\omega$ -closed, since for any countable  $X \prec H((2^{|\bar{P}|})^+)$  and any  $(X, \bar{P})$ -generic filter  $g$  contained in  $X$ , one can extend  $g$  to a condition by making  $\sigma_{X \cap \omega_1}$  unequal to the realization by  $g$  of any name in  $X$  for a subset of  $\omega_1$ , and adding  $X \cap \omega_1$  to all the clubs being added by the  $\dot{Q}_\alpha$ 's,  $\alpha \in X \cap \omega_2$ . It is clear also that in the  $\bar{P}$ -extension there is no  $E \subset \omega_1$  such that  $A(\Sigma, E)$  and  $B(\Sigma, E)$  are both stationary.

To see that at least one of  $A(\Sigma, E)$  and  $B(\Sigma, E)$  is stationary for each  $E \subset \omega_1$ , we first note the following.

**Claim 1.** *Suppose that  $E \subset \omega_1$  is a member of the  $P_\alpha$  extension, for some  $\alpha < \omega_2$ , and  $A(\Sigma, E)$  is stationary in this extension. Then  $A(\Sigma, E)$  remains stationary in the  $\bar{P}$  extension.*

Note that  $A(\Sigma, E)$  has countable intersection with  $B(\Sigma, F)$ , for every  $F \subset \omega_1$ . Fix  $X \prec H(((2^{|\bar{P}|})^+)^V)^{V[G_\alpha]}$  (where  $G_\alpha$  is the restriction of the generic

filter  $G$  to  $P_\alpha$ ) with  $X \cap \omega_1 \in A(\Sigma, E)$  and  $A(\Sigma, E) \in X$ . Then any  $(X, \bar{P}/P_\alpha)$ -generic filter contained in  $X$  can be extended to a condition by adding  $X \cap \omega_1$  to the clubs being added at every stage of  $\bar{P}$  after the first.

Similar reasoning shows the following two facts, which complete the proof that  $\Sigma$  witnesses  $\mathcal{D}_u$  in the  $\bar{P}$  extension.

**Claim 2.** *Suppose that  $E \subset \omega_1$  is a member of the  $P_\alpha$  extension, for some  $\alpha < \omega_2$ , and not a member of the  $P_\gamma$  extension, for any  $\gamma < \alpha$ . Then  $A(\Sigma, E) \cup B(\Sigma, E)$  is stationary in the  $P_\alpha$  extension.*

To see Claim 2, let  $\tau$  be a  $P_\alpha$ -name for a subset of  $\omega_1$  which is forced to be unequal to any such subset in any  $P_\gamma$  extension, for any  $\gamma < \alpha$ . Fix  $X \prec H((2^{|\bar{P}|})^+)^V$  with  $\tau \in X$ . Let  $g$  be an  $(X, P_\alpha)$ -generic filter, and note that the realization of  $\tau \upharpoonright (X \cap \omega_1)$  by  $g$  is different from the realizations of  $\rho \upharpoonright (X \cap \omega_1)$  by  $g$  for any  $P_\gamma$ -name  $\rho \in X$  for a subset of  $\omega_1$ , for any  $\gamma \in X \cap \alpha$ . It follows that adding the realization of  $\tau \upharpoonright (X \cap \omega_1)$  by  $g$  to the union of the first coordinate projection of  $g$  gives a condition in  $P_0$  forcing that  $X \cap \omega_1$  is not in any  $\Sigma(B, \rho_{G_\gamma})$ , for any for any  $P_\gamma$ -name  $\rho \in X$  for a subset of  $\omega_1$ , for any  $\gamma \in X \cap \alpha$ . Therefore, we can add  $X \cap \omega_1$  to the clubs being added in every other stage of  $\bar{P}$  in  $X \cap \alpha$ , and get a condition extending every condition in  $g$ .

**Claim 3.** *Suppose that  $E \subset \omega_1$  is a member of the  $P_\alpha$  extension, for some  $\alpha < \omega_2$ , and  $A(\Sigma, E)$  is nonstationary in this extension. Then  $B(\Sigma, E)$  remains stationary in the  $\bar{P}$  extension.*

This is similar to the previous claims, noting that every subsequent stage of  $\bar{P}$  forces a club though the complement of a set with countable intersection with  $B(\Sigma, E)$ .

The proof that  $\diamond(S)$  holds for every stationary  $S \subset \omega_1$  in the  $\bar{P}$  extension is (literally) the same as in the proof of Theorem 1.6.  $\square$

Note that that the iterations  $\bar{P}$  in Theorems 1.6 and 1.9 are strategically  $\omega$ -closed.

### 1.3 $\Sigma_2^2$ maximality

The statements that *split* and *unsplit* have winning strategies in  $\mathcal{SG}$  are each  $\Sigma_2^2$  in a predicate for  $NS_{\omega_1}$ , and they are obviously not consistent with each other. Woodin (see [6]) has shown that if there is a proper class of measurable Woodin cardinals, then there exists in a forcing extension a transitive class model of ZFC satisfying all  $\Sigma_2^2$  sentences  $\phi$  such that  $\phi + \text{CH}$  can be forced over the ground model. The results here show that this result cannot be extended to include a predicate for  $NS_{\omega_1}$ . This was known already, in that  $\diamond^*$  (in the sense of [7]) and “the restriction of  $NS_{\omega_1}$  to some stationary set is  $\aleph_1$  dense” were both known to be consistent with  $\diamond$  (the second of these is due to Woodin, uses large cardinals and is unpublished, though a related proof, also due to Woodin, appears in [3]). Our example is simpler and doesn’t use large cardinals; it also gives (we believe, for the first time) a counterexample consisting of two sentences each consistent with “ $\diamond(S)$  holds for every stationary set  $S \subset \omega_1$ .”

## 1.4 A determined variation

There are many natural variations of  $\mathcal{SG}$ . We show that one such variation is determined.

**Theorem 1.10.** *Let  $\mathcal{G}$  be the following game of length  $\omega_1$ . In round  $\alpha$ , player I puts  $\alpha$  into one of two sets  $E_0$  and  $E_1$ , and player II puts  $\alpha$  into one of two sets  $A_0$  and  $A_1$ . After all  $\omega_1$  rounds have been played, II wins if one of the following pairs of set are both stationary.*

- $E_0 \cap A_0$  and  $E_0 \cap A_1$
- $E_1 \cap A_0$  and  $E_1 \cap A_1$

*Then II has a winning strategy in  $\mathcal{G}$ .*

*Proof.* Let  $B_{00}, B_{01}, B_{10}$  and  $B_{11}$  be pairwise disjoint stationary subsets of  $\omega_1$ . In round  $\alpha$ , if  $\alpha$  is in  $B_{ij}$ , let II put  $\alpha$  in  $A_i$  if I put  $\alpha$  in  $E_0$  and in  $A_j$  otherwise. Then after all  $\omega_1$  many rounds have been played, suppose that  $A_i \cap E_0$  is nonstationary. Then  $B_{i0}$  and  $B_{i1}$  are both contained in  $E_1$  modulo  $NS_{\omega_1}$ , which means that  $E_1 \cap A_0$  and  $E_1 \cap A_1$  are both stationary. Similarly, if  $A_i \cap E_1$  is nonstationary then  $B_{0i}$  and  $B_{1i}$  are both contained in  $E_0$  modulo  $NS_{\omega_1}$ , which means that  $E_0 \cap A_0$  and  $E_0 \cap A_1$  are both stationary.  $\square$

## 2 Indeterminacy from forcing axioms

The axiom  $\text{PFA}^{+2}$  says that whenever  $P$  is a proper partial order,  $D_\alpha$  ( $\alpha < \omega_1$ ) are dense subsets of  $P$  and  $\sigma_1, \sigma_2$  are  $P$ -names for stationary subsets of  $\omega_1$ , there is a filter  $G \subset P$  such that  $G \cap D_\alpha \neq \emptyset$  for each  $\alpha < \omega_1$ , and such that  $\{\alpha < \omega_1 \mid \exists p \in G \ p \Vdash \check{\alpha} \in \sigma_i\}$  is stationary for each  $i \in \{1, 2\}$ . Theorems 1.6 and 1.9 together show that  $\text{PFA}^{+2}$  implies the indeterminacy of  $\mathcal{SG}$ . Furthermore, a straightforward argument shows that the following statement implies the nonexistence of a winning strategy for *unsplit* in  $\mathcal{SG}$ , where  $\text{Add}(1, \omega_1)$  is the partial order that adds a subset of  $\omega_1$  by initial segments : for any pair  $\sigma_1, \sigma_2$  of  $\text{Add}(1, \omega_1)$ -names for stationary subsets of  $\omega_1$ , there is a filter  $G \subset \text{Add}(1, \omega_1)$  realizing both  $\sigma_1$  and  $\sigma_2$  as stationary sets. This statement is trivially subsumed by  $\text{PFA}^{+2}$ , but also holds in the collapse of a sufficiently large cardinal to be  $\omega_2$ , and thus is consistent with CH.

The axiom Martin's Maximum [4] says that whenever  $P$  is a partial order such that forcing with  $P$  preserves stationary subsets of  $\omega_1$  and  $D_\alpha$  ( $\alpha < \omega_1$ ) are dense subsets of  $P$ , there is a filter  $G \subset P$  such that  $G \cap D_\alpha \neq \emptyset$  for each  $\alpha < \omega_1$ .

**Theorem 2.1.** *Martin's Maximum implies that  $\mathcal{SG}$  is undetermined.*

*Proof.* Fix a strategy  $\Sigma$  for *unsplit* in  $\mathcal{SG}$ , and let  $E, A$ , and  $B$  be the result of a generic run of  $\mathcal{SG}$  where *unsplit* plays by  $\Sigma$  (the partial order consists of countable partial plays where *unsplit* plays by  $\Sigma$ , ordered by extension). If the

complement of  $E$  has stationary intersection with every stationary subset of  $\omega_1$  in the ground model, one can force to kill the stationarity of  $E$  in such a way that the induced two step forcing preserves stationary subsets of  $\omega_1$  and produces a run of  $\mathcal{SG}$  where *unsplit* plays by  $\Sigma$  and loses. If the complement of  $E$  does not have stationary intersection with some stationary  $F \subset \omega_1$  in the ground model, then there is a partial run of the game  $p$  and a name  $\tau$  for a club such that  $p$  forces that  $E$  will contain  $F \cap \tau_G$ . Then there exists in the ground model a run of  $\mathcal{SG}$  extending  $p$  in which *unsplit* plays by  $\Sigma$  and loses: *split* picks a pair of disjoint stationary subsets  $F_0, F_1$  of  $F$ , and plays so that

- for every  $\alpha < \omega_1$ , some initial segment of the play forces some ordinal greater than  $\alpha$  to be in  $\tau$ ,
- whenever *unsplit* accepts  $\alpha \in F$ , *split* puts  $\alpha$  in  $A$  if  $\alpha \in F_0$  and puts  $\alpha \in B$  if  $\alpha \in F_1$ .

Now fix a strategy  $\Sigma$  for *split* in  $\mathcal{SG}$ , and generically add a regressive function  $f$  on  $\omega_1$  by initial segments. Let  $E^\alpha = f^{-1}(\alpha)$  and let  $A^\alpha, B^\alpha$  be the responses given by  $\Sigma$  to a play of  $E^\alpha$  by *unsplit*. Note that each  $E^\alpha$  will be stationary.

Suppose that there exist an  $\alpha < \omega_1$  and stationary sets  $S, T$  in the ground model such that  $(S \cap E^\alpha) \setminus A^\alpha$  and  $(T \cap E^\alpha) \setminus B^\alpha$  are both nonstationary. Then there is a condition  $p$  in our forcing (i.e., a regressive function on some countable ordinal) such that  $p$  forces that  $(S \cap E^\alpha) \subset A^\alpha$  and  $(T \cap E^\alpha) \subset B^\alpha$ , modulo nonstationarity (and so in particular  $S$  and  $T$  have nonstationary intersection). Let  $\tau$  be a name for a club disjoint from  $(S \cap E^\alpha) \setminus A^\alpha$  and  $(T \cap E^\alpha) \setminus B^\alpha$ . Extend  $p$  to a filter  $f$  (identified with the corresponding function) realizing  $\tau$  as a club subset of  $\omega_1$ , at successor stages extending to add a new element to the realization of  $\tau$ , and at limit stages (when for some  $\beta < \omega$ ,  $f \upharpoonright \beta$  has been decided and  $f(\beta)$  has not, and  $\beta$  is forced by  $f \upharpoonright \beta$  to be a limit member of the realization of  $\tau$ ) extending so that  $f(\beta) = \alpha$  if and only if  $\beta \in S$ . Then the run of  $\mathcal{SG}$  corresponding to  $f^{-1}(\alpha)$  is winning for *unsplit*, since the corresponding set  $B^\alpha$  is nonstationary.

If there exist no such  $\alpha, S, T$ , there is a function  $h$  on  $\omega_1$  such that each  $h(\alpha) \in \{A^\alpha, B^\alpha\}$  and the forcing to shoot a club through the set of  $\beta$  such that  $f(\beta) = \alpha \Rightarrow \beta \in h(\alpha)$  preserves stationary subsets of the ground model. Then Martin's Maximum applied to the corresponding two step forcing produces a run of  $\mathcal{SG}$  (the run for any  $f^{-1}(\alpha)$  which is stationary) where *split* plays by  $\Sigma$  and loses.  $\square$

Theorem 2.1 leads to the following question.

**2.2 Question.** Does the Proper Forcing Axiom imply that  $\mathcal{SG}$  is not determined?

The following question is also interesting. The consistency of the  $\aleph_1$ -density of  $NS_{\omega_1}$  (relative to the consistency of  $AD^{L(\mathbb{R})}$ ) is shown in [13].

**2.3 Question.** Does the  $\aleph_1$ -density of  $NS_{\omega_1}$  decide the determinacy of  $\mathcal{SG}$ ?

### 3 MLO games

The second-order Monadic Logic of Order (MLO) is an extension of first-order logic with logical constants  $=$ ,  $\in$  and  $\subset$  and a binary symbol  $<$  as the only non-logical constant, allowing quantification over subsets of the domain. Every ordinal is a model for MLO, interpreting  $<$  as  $\in$ .

Given an ordinal  $\alpha$ , an MLO game of length  $\alpha$  is determined by an MLO formula  $\phi$  with two free variables for subsets of the domain. In such a game, two players each build a subset of  $\alpha$ , and the winner is determined by whether these two sets satisfy the formula in  $\alpha$ .

Büchi and Landweber [1] proved the determinacy of all MLO games of length  $\omega$ . Recently, Shomrat [12] extended this result to games of length less than  $\omega^\omega$ , and Rabinovich [10] extended it further to all MLO games of countable length. The stationary set splitting game is an example of an MLO game of length  $\omega_1$  whose determinacy is independent of ZFC.

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