

**NO LIMIT MODEL IN INACCESSIBLE
SH906**

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ABSTRACT. Our aim is to improve the negative results i.e., non-existence of limit models, and the failure of the generic pair property from [5] to inaccessible λ as promised there. In [5], the negative results were obtained only for non-strong limit cardinals.

0. INTRODUCTION

Let $\lambda = \lambda^{<\lambda} > \kappa$ be regular cardinals. A complete first order theory T may have (some variant of) (λ, κ) -limit model, which, if exists, is unique, see history in [5]. There we prove existence for the theory of linear order and non-existence for first order theories which are strongly independent and then just independent and even the parallel for $\kappa = 2$ (the so-called generic pair conjecture). Those non-existence results were for $\lambda = 2^\kappa$, here we deal with strongly inaccessible λ . In [6] there are existence results but for λ measurable, and we promise there the non-existence results for λ strongly inaccessible as complimentary results.

Let λ be strongly inaccessible ($> |T|$) such that $\lambda^+ = 2^\lambda$.

Here in §1 we prove that for strongly independent T (see Definition 0.2), a strong version of the generic pair conjecture (see Definition 0.7(2)) holds. We also prove the non-existence of (λ, κ) -limit models, a related property (for all versions of “limit model”).

In §2, we also prove this even for independent T . The use of $\lambda^+ = 2^\lambda$ is just to have a more transparent formulation of the conjecture. See more on the generic pair conjecture for dependent T in [4].

{0.4}

Notation 0.1. 1) \mathcal{D}_λ is the club filter on λ for λ regular uncountable.

2) $S_\kappa^\lambda = \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$.

3) For a limit ordinal δ let $\mathcal{P}^{\text{ub}}(\delta) = \{\mathcal{U} : \mathcal{U} \text{ is an unbounded subset of } \delta\}$.

4) T denotes a complete first order theory.

5) For a model M , $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M)$ and $\bar{d} \in {}^{\ell g(\bar{y})}M$, let $\varphi(M, \bar{d}) = \{\bar{c} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{c}, \bar{d}]\}$.

6) $\mathbf{S}^n(A, M) = \{\text{tp}(\bar{b}, A, N) : M \prec N \text{ and } \bar{b} \in {}^nN\}$ where $\text{tp}(\bar{b}, A, N) = \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M), \bar{a} \in {}^{\ell g(\bar{y})}A \text{ and } M \models \varphi[\bar{b}, \bar{a}]\}$.

7) $\mathbf{S}^n(M) = \mathbf{S}^n(M, M)$ and $\mathbf{S}^{<\omega}(M) = \cup\{\mathbf{S}^n(M) : n \in \omega\}$.

8) $\beth_\alpha(\lambda) = \lambda^+, \Sigma\{2^{\beth_\beta(\lambda)} : \beta < \alpha\}$ and $\beth_\alpha = \beth_\alpha(\aleph_0)$.

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Recall (as in [5, 2.3])

{0.7}

Definition 0.2. 1) T has the strong independence property (or is strongly independent) when: some $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ has it, where:

2) $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ has the strong independence property for T when for every $n < \omega$, model M of T and pairwise distinct $\bar{b}_0, \dots, \bar{b}_{2n-1} \in {}^{\ell g(\bar{y})}M$ for some $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have $\ell < 2n \Rightarrow M \models \varphi[\bar{a}, \bar{b}_\ell]^{\text{if } (\ell \text{ is even})}$.

Remark 0.3. 1) Elsewhere we use $\varphi(x, y)$, and the proofs are not affected.

2) Also we may restrict ourselves to $a_0, a_1, \dots, \in \psi(M, \bar{d})$ where $\psi \in \mathbb{L}(\tau_T)$ such that $\psi(M, \bar{d})$ is infinite, and we may restrict ourselves to \bar{b} 's realizing a fixed non-algebraic type $p \in \mathbf{S}^m(A, M)$ with M being $(|A|^+ + \aleph_0)$ -saturated. The results are not really affected.

{0q.37}

Question 0.4. 1) Assume $\lambda_2 = \lambda_2^{<\lambda_1} \geq \lambda_1 > |T|$, T a complete first order theory. When is the theory $T_{\lambda_1, \lambda_2}^*$ a dependent theory? where

(a) $T_{\lambda_1, \lambda_2}^* = \text{Th}(K_{\lambda_1, \lambda_2}^+)$ where

(b) $K_{\lambda_1, \lambda_2}^+ = \{(M, N) : M \text{ is a } \lambda_1\text{-saturated model of } T \text{ of cardinality } \lambda_2, N \text{ a } \lambda_2^+\text{-saturated elementary extension of } M\}$.

2) Similarly for other properties of $T_{\lambda_1, \lambda_2}^*$; note that this theory is complete.

3) When can we prove that $T_{\lambda_1, \lambda_2}^*$ does not depend on the cardinals at least for many pairs?

Remark 0.5. 1) Concerning failure of 0.4(1) see Kaplan-Shelah [3].

2) The solution of the generic pair conjecture answers positively 0.4(3).

3) It is known that in 0.4(1) if T extends PA or ZFC then in $T^* = \text{Th}(M, N)$ we can interpret the second order theory of λ_2 .

But may well be that as in Baldwin-Shelah [1]

{0q.44}

Question 0.6. Assume $|T| < \kappa \leq \lambda_1 \leq \lambda_2 = \lambda_2^{<\lambda_1}$, T a complete first order theory. For which T 's can we interpret in $M \in K_{\lambda_1, \lambda_2}^+$ a model of PA of cardinality $\geq \lambda_1$ by first order formula or just an $\mathbb{L}_{\infty, \kappa}(\tau_T)$ -formulas with parameters, the intention is that we assume λ_2 is large enough than λ_1 which is large enough than $|T|$; if $2^\kappa \geq \lambda_1$ this is trivial.

Recall (from ([5, 0.2])

{0.14}

Definition 0.7. 0) Let $\text{EC}_\lambda(T)$ be the class of models M of (the first order) T of cardinality λ . Let $\text{EC}_{\lambda, \kappa}(T)$ be the class of κ -saturated models $M \in \text{EC}_\lambda(T)$.

1) Assume $\lambda = \lambda^{<\lambda} > |T|$, $2^\lambda = \lambda^+$, $M_\alpha \in \text{EC}_\lambda(T)$ is \prec -increasing continuous for $\alpha < \lambda^+$ with $M = \cup\{M_\alpha : \alpha < \lambda^+\} \in \text{EC}_{\lambda^+}(T)$, and M is saturated. The generic pair property (for T, λ) says that for some club E of λ^+ for all pairs $\alpha < \beta$ of ordinals from E of cofinality λ , (M_β, M_α) has the same isomorphism type (we denote this property of T by $\text{Pr}_{\lambda, \lambda}^2(T)$).

2) The generic pair conjecture for $\lambda = \lambda^{<\lambda} > \aleph_0$ such that $2^\lambda = \lambda^+$ says that for any complete first order T of cardinality $< \lambda$, T is independent iff it has the generic pair property for λ .

3) Let $\mathbf{n}_{\lambda, \kappa}(T)$ be $\min\{|\{M_\delta / \cong : \delta \in E \text{ has cofinality } \kappa\}| : E \text{ a club of } \lambda^+\}$ for $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$ as above; clearly the choice of \bar{M} is immaterial.

{0.22}

Remark 0.8. 1) Note that to say $\mathbf{n}_{\lambda,\kappa}(T) = 1$ is a way to say that T has (some variant of) a (λ, κ) -limit model.

2) Recall that we conjecture that for $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa) > |T|, 2^\lambda = \lambda^+$ we have $\mathbf{n}_{\lambda,\kappa}(T) = 1 \Leftrightarrow \mathbf{n}_{\lambda,\kappa}(T) < 2^\lambda \Leftrightarrow T$ is dependent. The use of “ $\lambda^+ = 2^\lambda$ ” is just for clarity. See more in [5].

1. STRONGLY INDEPENDENT T

Context 1.1. 1) T is a fixed first order complete theory and $\mathfrak{C} = \mathfrak{C}_T$ a monster for it.

2) We let λ be a regular uncountable cardinal $> |T|$.

Here for λ strongly inaccessible and (complete first order) T with the strong independence property (of cardinality $< \lambda$) we prove the non-existence of (λ, κ) -limit models for $\kappa = \text{cf}(\kappa) < \lambda$ (in Theorem 1.9) and the generic pair conjecture for λ and T , in Theorem 1.10 (which shows non-isomorphism). Recall that the generic pair property speaks on the isomorphism type of pairs of models.

Definition 1.2 gives us a more constructive invariant of $(M, N)/\cong$. Unfortunately it seemed opaque how to manipulate it so we shall use a different version, the one from Definition 1.4. Naturally it concentrates on types in one formula $\varphi(\bar{y}, \bar{x})$ witnessing the strong independence property. But mainly gives the pair (M, N) an invariant $\langle \mathcal{P}_\delta : \delta < \lambda \rangle / \mathcal{D}_\lambda$ where $\mathcal{P}_\delta \subseteq \mathcal{P}(\mathcal{P}(\delta))$. Now always $|\mathcal{P}_\delta| \leq 2^{|\delta|}$ and it is easily computable from one $\mathcal{P} \subseteq \mathcal{P}(\delta)$, in fact from the invariant $\text{inv}_4(M, N)$ from Definition 1.2, but in our proofs its use is more transparent. It has monotonicity property and we can increase it.

We need different but similar version for the proof of non-existence of (λ, κ) -limit models.

{inv.7}

Definition 1.2. 1) Let \mathcal{E}_T^* be the following two-place relation on $\{(M, \mathbf{P}) : M \models T \text{ and } \mathbf{P} \subseteq \mathbf{S}^{<\omega}(M)\}$; let $(M_1, \mathbf{P}_1) \mathcal{E}_T^*(M_2, \mathbf{P}_2)$ iff there is an isomorphism h from M_1 onto M_2 mapping \mathbf{P}_1 onto \mathbf{P}_2 .

2) For models $M \prec N$ of T we define

$$(a) \text{ inv}_1(M, N) = \{p \in \mathbf{S}^{<\omega}(M) : p \text{ is realized in } N\}$$

$$(b) \text{ inv}_2(M, N) = (M, \text{inv}_1(M, N)) / \mathcal{E}_T^*.$$

3) If $M \prec N$ are models of T such that the universe of N is $\subseteq \lambda$, let, recalling \mathcal{D}_λ is the club filter on λ

(a) for any ordinal $\delta < \lambda$

$$\text{inv}_3(\delta, M, N) = (M \upharpoonright \delta, \{p \in \mathbf{S}^{<\omega}(N \upharpoonright \delta) : p \text{ is realized by some sequence from } N \upharpoonright \delta\}) / \mathcal{E}_T^*$$

$$(b) \text{ inv}_4(M, N) = \langle \text{inv}_3(\delta, M, N) : \delta < \lambda \rangle / \mathcal{D}_\lambda.$$

4) If $M \prec N$ are models of T of cardinality λ then $\text{inv}_4(M, N)$ is $\text{inv}_4(f(M), f(N))$ for every one-to-one function f from N into λ (equivalently some f , see below)

{inv.8}

Observation 1.3. 0) In Definition 1.2(3) for a club of δ 's below λ we have $M \upharpoonright \delta \prec M$ and $N \upharpoonright \delta \prec N$.

1) Concerning Definition 1.2(3), if $M \prec N$ are models of T of cardinality λ and f_1, f_2 are one-to-one functions from N into λ then $\text{inv}_4(f_1(M), f_1(N)) = \text{inv}_4(f_2(M), f_2(N))$.

2) Definitions 1.2(3), 1.2(4) are compatible and in 1.2(4), “some f such that f is a one-to-one function from N to λ ” is equivalent to “every f such that...”

{inv.10}

Definition 1.4. Assume $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ and $N_1 \prec N_2$ are models of T of cardinality λ .

1) For one-to-one mapping f from N_2 to λ and $\delta < \lambda$ we define

$\text{inv}_5^\varphi(\delta, f, N_1, N_2) = \{\mathcal{P} \subseteq \mathcal{P}(\delta) : \text{there are } \bar{a}_\gamma \in {}^{\ell g(\bar{x})}N_2 \text{ for } \gamma < \delta \text{ such that}$
 $f(\bar{a}_\gamma) \subseteq \delta \text{ and for every } \mathcal{U} \subseteq \delta \text{ the following are equivalent :}$
 (i) $\mathcal{U} \in \mathcal{P}$
 (ii) for some $\bar{b} \in {}^{\ell g(\bar{y})}N_1$ we have $\gamma < \delta \Rightarrow N_2 \models \varphi[\bar{a}_\gamma, \bar{b}]^{\text{if}(\gamma \in \mathcal{U})}\}$.

2) We let $\text{inv}_6^\varphi(N_1, N_2)$ be $\langle \text{inv}_5^\varphi(\delta, f, N_1, N_2) : \delta < \lambda \rangle / \mathcal{D}_\lambda$ for some (equivalently every) f as above.

Claim 1.5. 1) In Definition 1.4 we have $\text{inv}_6^\varphi(N_1, N_2)$ is well defined.

{inv.11}

2) In Definition 1.4, for $\delta, \lambda, N_1, N_2, \varphi(\bar{x}, \bar{y})$ as there

- (a) the set $\text{inv}_5^\varphi(\delta, f, N_1, N_2)$ has cardinality at most $2^{|\delta|}$
- (b) if π is a one-to-one function from $f(N_2)$ into λ mapping $f(N_2) \cap \delta$ onto $\pi(f(N_2)) \cap \delta$ then $\text{inv}_5^\varphi(\delta, \pi \circ f, N_1, N_2) = \text{inv}_5^\varphi(\delta, f, N_1, N_2)$.

Proof. Easy.

□_{1.5}

{inv.12}

Definition 1.6. 1) For $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$, a model N of T with universe λ , δ an ordinal $< \lambda$ and $\kappa < \lambda$ let

$\text{inv}_{7,\kappa}^\varphi(\delta, N) = \{\mathcal{P} \subseteq \mathcal{P}(\delta) : \text{we can find } \bar{a}_\gamma^i \in {}^{\ell g(\bar{x})}\delta \text{ for } \gamma < \delta, i < \kappa \text{ such that}$
 the following conditions on $\mathcal{U} \subseteq \delta$
 unbounded in δ are equivalent :
 (i) $\mathcal{U} \in \mathcal{P}$
 (ii) for some $\bar{b} \in {}^{\ell g(\bar{y})}N$ we have :
 for every $i < \kappa$ large enough for every
 $\gamma < \delta$ we have $N \models \varphi[\bar{a}_\gamma^i, \bar{b}]^{\text{if}(\gamma \in \mathcal{U})}\}$.

2) For $\varphi = \varphi(\bar{y}, \bar{x}) \in \mathbb{L}(\tau_T)$ and a model N of T of cardinality λ let $\text{inv}_{8,\kappa}^\varphi(N) = \langle \text{inv}_7^\varphi(\delta, N') : \delta < \lambda \rangle / \mathcal{D}_\lambda$ for every, equivalently some model N' isomorphic to N with universe λ .

{inv.13}

Observation 1.7. 1) $\text{inv}_{8,\kappa}^\varphi(N)$ is well defined for $N \in \text{EC}_\lambda(T)$ when $|T| + \kappa < \lambda$.

2) In Definition 1.6(1) we have $|\text{inv}_{7,\kappa}^\varphi(\delta, N)| \leq 2^{|\delta|}$.

Proof. Easy.

□_{1.7}

{inv.21}

Claim 1.8. Assume $\lambda > |T|$ is regular, $\varphi = \varphi(\bar{x}, \bar{y})$ and

- (a) $\langle N_i : i < \kappa \rangle$ is a \leftarrow -increasing sequence
- (b) $N_i \in \text{EC}_\lambda(T)$
- (c) $N = \cup\{N_i : i < \kappa\}$
- (d) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ where $\mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha)$
- (e) f is a one-to-one function from N onto λ
- (f) there are $\bar{b}_{\delta,u} \in {}^{\ell g(\bar{y})}(N_0)$ for $\delta \in S, \mathcal{U} \in \mathcal{P}_\delta$ such that for every $j < \kappa$ for a club of δ 's below λ there are $\bar{a}_\gamma \in N_{j+1} \cap f^{-1}(\delta)$ for $\gamma < \delta$ satisfying
 - (α) for every $\bar{c} \in {}^{\ell g(\bar{x})}(N_i)$ there is $\mathcal{U} \in \mathcal{P}_\delta$ such that $\gamma < \delta \Rightarrow N \models \varphi[\bar{a}_\gamma, \bar{c}_\gamma]^{\text{if}(\gamma \in \mathcal{U})}$
 - (β) for every $\mathcal{U} \in \mathcal{P}_\delta$ we have $\gamma < \delta \Rightarrow N \models \varphi[\bar{a}_\gamma, \bar{b}_{\delta,\mathcal{U}}]^{\text{if}(\gamma \in \mathcal{U})}$.

Then $\{\delta < \lambda : \mathcal{P}_\delta \in \text{inv}_{7,\kappa}^\varphi(\delta, f(N))\} \in \mathcal{D}_\lambda$.

Proof. Straight. □_{1.8}

{inv.14}

Now we come to the main two results of this section.

Claim 1.9. *For some club E of λ^+ , if $\delta_1 \neq \delta_2$ belong to $E \cap S_\kappa^{\lambda^+}$ then $M_{\delta_1}, M_{\delta_2}$ are not isomorphic, moreover $\text{inv}_{8,\kappa}^\varphi(M_{\delta_1}) \neq \text{inv}_{8,\kappa}^\varphi(M_{\delta_2})$ when:*

- ⊠ (a) T has the strong independence property (see Definition 0.2)
- (b) $\lambda = \lambda^{<\lambda}$ regular uncountable, $\lambda > |T|, \lambda > \kappa = \text{cf}(\kappa)$ and $\lambda^+ = 2^\lambda$
- (c) M is a saturated model of T of cardinality λ^+
- (d) $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is \prec -increasing continuous sequence with union M , each of cardinality λ .

{inv.24}

Theorem 1.10. *Assume ⊠ of 1.9.*

1) *For some club E of λ^+ , if $\delta_1 < \delta_2 < \delta_3$ are from E and $\delta_\ell \in S_\kappa^{\lambda^+}$ for $\ell = 1, 2, 3$ then $(M_{\delta_1}, M_{\delta_2}) \not\cong (M_{\delta_1}, M_{\delta_3})$, moreover $\text{inv}_6^\varphi(M_{\delta_1}, M_{\delta_2}) \neq \text{inv}_6^\varphi(M_{\delta_1}, M_{\delta_3})$ for some φ .*

2) *If $M \prec N_0$ are models of T of cardinality λ , then for some elementary extension $N_1 \in \text{EC}_\lambda(T)$ of N_0 we have $N_1 \prec N_2 \in \text{EC}_\lambda(T) \Rightarrow (M, N_1) \cong (M, N_2)$.*

Discussion 1.11. We shall below start with $M \in \text{EC}_\lambda(T)$ and a sequence $\langle b_i : i < \lambda \rangle$ of distinct members such that $\langle \varphi(\bar{b}_i, \bar{y}) : i < \lambda \rangle$ are independent, and like to find $N, \langle \bar{a}_i : i < \lambda \rangle$ such that $M \prec N \in \text{EC}_\lambda(T)$ and the $\langle \bar{b}_i : i < \lambda \rangle$ has a real affect on the relevant φ -invariant, in the case of 1.10(1) this is $\text{inv}_6^\varphi(M, N)$: for a stationary set of δ 's below λ it adds something to the δ -th component in a specific representation, i.e. assuming $f : N \rightarrow \lambda$ is a one-to-one function and we deal with $\langle \text{inv}_5^\varphi(\delta, f, M, N) : \delta < \lambda \rangle$; we have freedom about $\varphi(\bar{b}_i, \bar{a}_\alpha)$ and we can assume $b \in M \setminus \{\bar{b}_i : i < \lambda\} \Rightarrow N \models \neg \varphi[\bar{b}, \bar{a}_\alpha]$.

But the relevant \mathcal{P}_δ is influenced not just by say $\langle \bar{b}_i : i \in [\delta, 2^{|\delta|}] \rangle$ but also by later \bar{b}_i 's (and earlier b_i). To control this we use below $\langle \bar{a}_\alpha : \alpha < \lambda \rangle, S, E$ such that we deal with different $\delta \in S$ in an independent way; this is the reason for choosing the C_α 's.

Proof. Proof of 1.9 By [5, §2] without loss of generality λ is strongly inaccessible. Choose $\theta \in \text{Reg} \cap \lambda \setminus \{\aleph_0\}$, will be needed when we generalize the proof in §2.

Let $\langle \mathcal{U}_i : i < \theta \rangle$ be a \subseteq -increasing sequence of subsets of λ such that $\mathcal{U}_i^- = \mu \cap \mathcal{U}_i \setminus \cup \{\mathcal{U}_j : j < i\}$ has cardinality μ for each $i < \kappa$ and strong limit $\mu < \lambda$. Let $\varphi(\bar{x}, y) \in \mathbb{L}(\tau_T)$ have the strong independence property, see Definition 0.2.

Let $S_* = \{\mu : \mu = \beth_{\alpha+\omega} \text{ for some } \alpha < \lambda\}$. Let $E_*, \zeta, \langle C_\alpha : \alpha < \lambda \rangle$ be such that:

- ⊗₁ (a) $C_\alpha \subseteq \alpha \cap S_*$
- (b) $\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta$
- (c) $\text{otp}(C_\alpha) \leq \theta$
- (d) E_* is the club $\{\delta < \lambda : \delta = \beth_\delta\}$ of λ
- (e) $C_\alpha \subseteq E_*$ and $\text{otp}(C_\alpha) = \theta$ iff $\alpha \in E_* \cap S_\theta^\lambda$
- (f) if $\alpha \in S := E_* \cap S_\theta^\lambda$ then $\alpha = \sup(C_\alpha)$.

We shall prove that

⊗₂ if \square_2 below holds, then there is a pair (β, h) such that \odot_2 holds where:

- \square_2 (a) $\alpha < \lambda^+, i < \kappa$
 (b) f is a one-to-one function from M_α onto \mathcal{U}_i
 (c) $E \subseteq E_*$ is a club of λ such that $\delta \in E \Rightarrow f(M_\alpha) \upharpoonright \delta \prec f(M_\alpha)$
 (d) $\bar{\mathcal{P}} = \langle \mathcal{P}_\delta : \delta \in S \rangle$
 (e) $\mathcal{P}_\delta \subseteq \mathcal{P}(\delta)$ and $\emptyset \in \mathcal{P}_\delta$ and $\mathcal{P}_\delta \subseteq \bigcup_{\ell \leq 2} \mathcal{P}_\delta^{*,\ell}$ where
- (α) $\mathcal{P}_\delta^{*,0} = \{A \subseteq \delta : \sup(A) = \delta \text{ and } A \subseteq \cup\{[\mu, 2^\mu) : \mu \in C_\delta]\}$,
 (β) $\mathcal{P}_\delta^{*,1} = \cup\{\mathcal{P}_\mu^{*,0} : \mu \in S \cap \delta\}$,
 (γ) $\mathcal{P}_\delta^{*,2} = \{A \subseteq \delta : \text{for some } \mu \in \lambda \setminus (\delta + 1) \text{ we have } A \subseteq \cup\{[\partial, 2^\partial) : \partial \in C_\mu \cap \delta]\}$
- (f) if $\delta_1 < \delta_2$ are from S then
 (α) $[A \in \mathcal{P}_{\delta_1} \Rightarrow A \in \mathcal{P}_{\delta_2}^{*,1} \subseteq \mathcal{P}_{\delta_2}]$
 (β) $[A \in \mathcal{P}_{\delta_2} \Rightarrow A \cap \delta_1 \in \mathcal{P}_{\delta_1}^{*,2} \subseteq \mathcal{P}_{\delta_1}]$,
 (γ) for any $\delta \in S$ the family $\mathcal{P}_\delta^{*,1} \cup \mathcal{P}_\delta^{*,2}$ is a set of bounded subsets of δ ; (this follows)
- (g) $\bar{b}_{\delta, \mathcal{U}} \in M_\alpha$ for $\delta \in E, \mathcal{U} \in \mathcal{P}_\delta$ are such that
 $\bar{b}_{\delta_1, \mathcal{U}_1} = \bar{b}_{\delta_2, \mathcal{U}_2} \wedge \mathcal{U}_1 \in \mathcal{P}_{\delta_1} \wedge \mathcal{U}_2 \in \mathcal{P}_{\delta_2} \Rightarrow \delta_1 = \delta_2 \wedge \mathcal{U}_1 = \mathcal{U}_2$
- \odot_2 (α) $\beta \in (\alpha, \lambda^+)$
 (β) h is a one-to-one mapping from M_β onto \mathcal{U}_{i+1} extending f
 (γ) for a club of $\delta \in E$ there are $\bar{a}_\alpha \subseteq (\mathcal{U}_{i+1} \cap \delta)$ for $\alpha < \delta$ such that the following conditions on $\mathcal{U} \subseteq \delta$ are equivalent:
- (i) $\mathcal{U} \in \mathcal{P}_\delta$
 (ii) for some $\bar{b} \in {}^{\ell g(\bar{y})} M_\alpha$ we have: for every $\gamma < \delta$,
 $f(M_\beta) \models \varphi[\bar{a}_\gamma, \bar{b}]$ iff $\gamma \in \mathcal{U}$
 (iii) clause (ii) holds for $\bar{b} = \bar{b}_{\delta, \mathcal{U}}$.

[Why? Every $\delta \in E$ is a strong limit cardinal and $|\delta| = |\delta \cap \mathcal{U}_i| = |\delta \cap \mathcal{U}_{i+1} \setminus \mathcal{U}_i|$. For each $\delta \in E$ let $\langle \mathcal{U}_{\delta, \varepsilon} : \varepsilon < |\mathcal{P}_\delta| \leq 2^{|\delta|} \rangle$ list \mathcal{P}_δ and let $\bar{b}_{\delta, \varepsilon} := \bar{b}_{\delta, \mathcal{U}_{\delta, \varepsilon}}$.

Let

$$\Gamma = \{\varphi(\bar{x}_\gamma, \bar{b}_{\delta, \varepsilon})^{\text{if } (\gamma \in \mathcal{U}_{\delta, \varepsilon})} : \delta \in E \text{ and } \varepsilon < |\mathcal{P}_\delta|\} \\ \cup \{\neg \varphi(\bar{x}_\gamma, \bar{b}) : \gamma < \lambda, \bar{b} \in {}^{\ell g(\bar{y})} (M_\alpha) \text{ and for no } \delta \in E, \varepsilon < |\mathcal{P}_\delta| \text{ do we have } \bar{b} = \bar{b}_{\delta, \varepsilon}\}.$$

As $\varphi(\bar{x}, \bar{y})$ has the strong independence property and for each $\delta \in E$ the sequence $\langle \bar{b}_{\delta, \varepsilon} : \varepsilon \text{ satisfies } \gamma \in \mathcal{U}_{\delta, \varepsilon} \text{ and } \varepsilon < |\mathcal{P}_\delta| \rangle$ is with no repetitions, clearly Γ is finitely satisfiable in M_α , but M is λ^+ -saturated, $M_\alpha \prec M$ and $|\Gamma| = \lambda$ hence we can find $\bar{a}_\gamma \in M$ for $\gamma < \lambda$ such that the assignment $\bar{x}_\gamma \mapsto \bar{a}_\gamma$ ($\gamma < \lambda$) satisfies Γ in M . Lastly, choose $\beta \in (\alpha, \lambda^+)$ such that $\{\bar{a}_\gamma : \gamma < \lambda\} \subseteq M_\beta$ and let h be a one-to-one mapping from M_β onto \mathcal{U}_{i+1} extending f and let $E^* = \{\delta \in E : h(\bar{a}_\gamma) \in \mathcal{U}_{i+1} \cap \delta \text{ iff } \gamma < \delta \text{ for every } \gamma < \lambda\}$.

Now check.]

Next we can choose \bar{f} such that

- ⊗₃ (a) $\bar{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$
 (b) f_α is a one-to-one function from M_α into λ
 (c) if $\alpha \in C_\beta, \beta < \lambda$ then f_α is onto $\mathcal{U}_{\text{otp}(C_\beta \cap i)}$ and $f_\alpha \subseteq f_\beta$
- ⊗₄ for every $\alpha < \lambda^+$ there is $\bar{\mathcal{P}}^\alpha = \langle \mathcal{P}_\varepsilon^\alpha : \varepsilon < \lambda \rangle$ such that
 (i) $\mathcal{P}_\varepsilon^\alpha \subseteq \mathcal{P}(\varepsilon)$ are as in $\square_2(e)$ above
 (ii) for every $\beta \leq \alpha$, for a club of δ 's below λ we have $\mathcal{P}_\delta^\alpha \notin \text{inv}_{7,\kappa}^\varphi(\delta, f_\beta(M_\beta))$.

[Why? For every $\beta \leq \alpha$ and $\delta \in (\kappa, \lambda)$ we have $\text{inv}_{7,\kappa}^\varphi(\delta, f_\beta(N_\beta))$ is a subset of $\mathcal{P}(\mathcal{P}(\delta))$ of cardinality $\leq 2^{|\delta|}$. As the number of β 's is $\leq \lambda$, by diagonalization we can do this: let $\alpha + 1 = \bigcup_{\varepsilon < \lambda} u_\varepsilon$ and $u_\varepsilon \in [\alpha + 1]^{<\lambda}$ increasing continuous for $\varepsilon < \lambda$; moreover, $|u_\varepsilon| \leq |\varepsilon|$. By induction on $\varepsilon \in (\kappa, \lambda) \cap S$ choose $\mathcal{P}_\varepsilon^\alpha \subseteq \bigcup_{\ell < 3} \mathcal{P}_\alpha^{*,\ell}$ which includes $\bigcup \{ \mathcal{P}_\zeta^\alpha : \zeta \in u_\varepsilon \cap S \} \cup \mathcal{P}_\alpha^{*,2}$ and satisfies $\mathcal{P}_\alpha^{*,0} \cap \mathcal{P}_\varepsilon^\alpha \in \mathcal{P}(\mathcal{P}_\delta^{*,0}) \setminus \bigcup \{ \text{inv}_{7,\kappa}^\varphi(\delta, f_\beta(N_\beta)) \cap \mathcal{P}_\delta^{*,0} : \beta \in u_\varepsilon \}$.]

Now choose pairwise distinct $\bar{b}_{\delta,\mathcal{U}} \in {}^{\ell g(\bar{y})}(M_0)$ for $\delta \in E, \mathcal{U} \in \mathcal{P}_\delta^{*,0}$

- ⊗₅ for every $\alpha_* \leq \alpha < \lambda^+$ for some $\beta \in (\alpha, \lambda^+)$ and $\bar{a}_\gamma \in {}^{\ell g(\bar{x})}M_\beta$ for $\gamma < \lambda$ the condition in clause (γ) of \odot_2 holds with $\bar{\mathcal{P}}^{\alpha_*}$ here standing for $\bar{\mathcal{P}}$ there and the $\bar{b}_{\delta,\mathcal{U}}$ chosen above.

[Why? By \otimes_2 .]

- ⊗₆ let $E = \{ \delta < \lambda^+ : \delta \text{ is a limit ordinal such that for every } \alpha < \delta \text{ there is } \beta < \delta \text{ as in } \otimes_5 \}$.

Clearly E is a club of λ^+ .

- ⊗₇ if $\delta_1 < \delta_2$ are from $E \cap S_\kappa^{\lambda^+}$ then $M_{\delta_1}, M_{\delta_2}$ are not isomorphic.

[Why? We consider $\bar{\mathcal{P}}^{\delta_1}$ which is from \otimes_4 . On the one hand $\{ \varepsilon < \lambda : \mathcal{P}_\varepsilon^{\delta_1} \notin \text{inv}_{7,\kappa}^\varphi(\varepsilon, f_{\delta_2}(M_{\delta_2})) \}$ contains a club by $\otimes_4(ii)$.

On the other hand choose an increasing $\langle \alpha_i : i < \kappa \rangle$ with limit δ_2 satisfying $\alpha_0 = 0, \alpha_1 = \delta_1$ such that $(\delta_1, \alpha_{1+i}, \alpha_{1+i+1})$ are like $(\alpha_*, \alpha, \beta)$ in \otimes_5 for each $i < \kappa$. Now by 1.8, $\{ \varepsilon < \lambda : \mathcal{P}_\varepsilon^{\delta_1} \in \text{inv}_{7,\kappa}^\varphi(\varepsilon, f_{\delta_2}(M_{\delta_2})) \}$ contains a club. Hence by the last sentence and the end of the previous paragraph $M_{\delta_1} \not\cong M_{\delta_2}$ as required.]

So we are done. □_{1.9}

Proof. Proof of 1.10 Similar but easier (for λ regular not strong limit (but $2^\lambda > 2^{<\lambda}$) also easy), or see the proof of 2.8. □_{1.10}

2. INDEPENDENT T

We would like to do something similar to §1, but our control on the relevant family of subsets of μ is less tight.

{2d.1}

Context 2.1. T a complete first order theory, $\varphi(x, \bar{y})$ has the independence property (of course the existence of such φ follows from the strong independence property).

We continue [5, 2.1-2.12], but we do not rely on it.

{2d.4}

Definition 2.2. For a set I let

- (a) $\mathbb{B} = \mathbb{B}_I$ be the Boolean Algebra generated by $\langle e_t : t \in I \rangle$ freely,
- (b) \mathbb{B}_I^c is the completion of \mathbb{B}
- (c) for $J \subseteq I$ let $\mathbb{B}_{I,J}^c$ be the complete subalgebra of \mathbb{B}_I^c generated by $\{e_s : s \in J\}$
- (d) let $\text{uf}(\mathbb{B}_I^c)$ be the set of ultrafilters on I .

{2d.7}

Claim 2.3. Assume

- ⊗ (a) $M \models T$
- (b) $\bar{b}_t \in {}^{\ell g(\bar{y})}M$ for $t \in I$
- (c) $\langle \varphi(x, \bar{b}_t) : t \in I \rangle$ is an independent sequence of formulas.

Then there is a function F from ${}^{\ell g(\bar{y})}M$ to $\mathbb{B} = \mathbb{B}_I^c$ such that

- (α) $F(\bar{b}_t) = e_t$
- (β) for every ultrafilter D of \mathbb{B} there is $p = p_D = p_{F,D} \in \mathbf{S}_\varphi(M)$, in fact, a unique one, such that for every $\bar{b} \in {}^{\ell g(\bar{y})}M$ we have $\varphi(x, \bar{b}) \in p \Leftrightarrow F(\bar{b}) \in D$.

Remark 2.4. Note that the mapping $D \mapsto p_D$ is not necessarily one to one, but $D_1 \cap \{e_t : t \in I\} \neq D_2 \cap \{e_t : t \in I\} \Rightarrow p_{D_1} \neq p_{D_2}$.

Proof. $\mathcal{P}(M)$ is a Boolean algebra and $\{\varphi(M, \bar{b}_t) : t \in M\}$ generates freely a subalgebra of $\mathcal{P}(M)$ which we call \mathbb{B}' . So there is a homomorphism h from \mathbb{B}' into \mathbb{B} mapping $\varphi(M, \bar{b}_t)$ to e_t (moreover h is unique and is an isomorphism from \mathbb{B}' onto \mathbb{B}). So h is a homomorphism from $\mathbb{B}' \subseteq \mathcal{P}(M)$ into \mathbb{B}^c , which is a complete Boolean algebra hence there is a homomorphism h^+ from the Boolean algebra $\mathcal{P}(M)$ into \mathbb{B}^c extending h .

Lastly, define $F : {}^{\ell g(\bar{y})}M \rightarrow \mathbb{B}^c$ by $F(\bar{b}) = h^+(\varphi((M, \bar{a})))$. Now check. □_{2.3}

{2d.14}

Conclusion 2.5. Assume ⊗ from 2.3 and

- (a) $I = \lambda$ is regular uncountable
- (b) $|M| \subseteq \mathcal{U} \subseteq \lambda$
- (c) D_α is an ultrafilter of \mathbb{B}_I^c for $\alpha < \lambda$
- (d) $\mathcal{U} \setminus |M|$ is unbounded in λ .

Then we can find $\langle a_\alpha : \alpha < \lambda \rangle$ and N such that

- (α) $M \prec N$
- (β) $|N| \subseteq \mathcal{U}$
- (γ) $a_\alpha \in N$ for $\alpha < \lambda$
- (δ) a_α realizes $p_{D_\alpha} \in \mathbf{S}_\varphi(M)$.

{2d.21} *Proof.* Should be clear. □_{2.5}

Discussion 2.6. Note that compared to §1 instead $\bar{x}, y, \bar{a}_\alpha, b_\beta$ we have $x, \bar{y}, a_\alpha, \bar{b}_\beta$. Compared to §1, we have less control over $\{\text{tp}(a, M, N) : a \in N\}$. There, for the sequences \bar{b} of M which are not among $\{\bar{b}_\gamma : \gamma < \lambda\}$, we can demand $N \models \neg\varphi[\bar{a}_\gamma, \bar{b}]$ for $\gamma < \lambda$ so $\text{tp}_\varphi(\bar{a}_\gamma, M, N)$ can be clearly read. Here the complete Boolean Algebra \mathbb{B}_I^c is helping, a small price is that we need $\theta > \aleph_0$.

In order to try to keep track of what is going on we shall use only $\text{tp}(a_\gamma, M, N)$ of the form f_D for ultrafilter D on \mathbb{B}_I^c . Further, we better have, e.g. a nice function π from ${}^\lambda 2$ to $\text{uf}(\mathbb{B}_I^c)$ such that $(e_\alpha \in \pi(\eta)) \Leftrightarrow \eta(\alpha) = 1$.

A possible approach is: we define $\langle M_{\eta, u} : \eta \in \mathcal{T} \subseteq \text{des}(\lambda), u \in \mathcal{P}(n_\eta) \rangle$ as in [7, §3] and we define $D_\eta \in \text{uf}(\mathbb{B}^c \cap M)$ such that $\alpha \in M_\eta \cap \lambda \Rightarrow [e_\eta^{\eta(\alpha)} \in D_\eta]$ and $\bigcup_\eta D_\eta \in \text{uf}(\mathbb{B}^c)$.

{3e.7} We need some continuity so each “ $e \in D_\eta$ ” ($e \in \mathbb{B}^c$) depends on $\eta \upharpoonright u_e$ for some “small” $u_e \subseteq \lambda$.

Theorem 2.7. *In Theorem 1.9 it suffices to assume \boxtimes' which means clauses (b), (c), (d) of \boxtimes and*

(a)' T has the independence property.

{3e.14}

Theorem 2.8. *In Theorem 1.10 it suffices to assume \boxtimes' of 2.7.*

Proof. Proof of 2.7 Just combine the proofs of 1.9 from §1 and 2.8 below. □_{2.7}

Proof. Proof of 2.8 As in the proof of 1.9 we can assume λ is strongly inaccessible though the proof is just easier otherwise. We let

⊗₁ $E_* = \{\delta < \lambda : \delta = \beth_\delta\}$, a club of λ , choose a regular uncountable $\theta < \lambda$ and let

⊗₂ $S = \{\delta \in E_* : \text{cf}(\delta) = \theta\} = S_\theta^\lambda \cap E_*$ and let \bar{C} be as in ⊗₁ there.

Let \mathbb{D}_* be an ultrafilter of \mathbb{B}_λ^c such that $e_\alpha \notin \mathbb{D}_*$ for $\alpha < \lambda$.

Now for $\eta \in {}^\lambda 2$ we choose \mathbb{D}_η such that ¹

- ⊗₃ (a) \mathbb{D}_η is an ultrafilter of \mathbb{B}_λ^c
- (b) if $e \in \mathbb{D}_* \subseteq \mathbb{B}_\lambda^c$ belongs to $\mathbb{B}_{\lambda, \eta^{-1}\{0\}}^c$ (see 2.2, the closure of the subalgebra of \mathbb{B}_λ^c generated by $\{e_\alpha : \eta(\alpha) = 0\}$) then $e \in \mathbb{D}_\eta$.
- (c) if $\alpha < \lambda$ and $\eta(\alpha) = 1$ then $e_\alpha \in \mathbb{D}_\eta$.

So

- ⊗₄ (a) if $\eta \in {}^\lambda 2$ is constantly zero then $\mathbb{D}_\eta = \mathbb{D}_*$
- (b) if $e_\alpha \notin \mathbb{D}_*$ for $\alpha < \lambda$ then $-e_\alpha \in \mathbb{D}_*$
- (c) $e_\alpha \in \mathbb{D}_\eta \Leftrightarrow \eta(\alpha) = 1$ for $\alpha < \lambda, \eta \in {}^\lambda 2$.

Now let $\bar{\eta} = \langle \eta_\varepsilon : \varepsilon < \lambda \rangle$ be a sequence of members of ${}^\lambda 2$ with η_0 being constantly 0 and below we shall be interested mainly in the case $\alpha = \mu \in S$.

Define

¹letting π be the automorphism of \mathbb{B}_λ^c mapping e_α to $-e_\alpha$ for $\alpha \in \eta^{-1}\{0\}$ and to e_α for $\alpha \in \eta^{-1}\{1\}$ we can note that $\pi^{-1}(\mathbb{D}_*)$ is also an ultrafilter of \mathbb{B}_λ^c as required hence can add

(d) if $e \in \pi(\mathbb{D}_*)$ belongs to $\mathbb{B}_{\lambda, \eta^{-1}\{1\}}^c$ then $-e \in \mathbb{D}_\eta$.

⊗₅ for $e \in \mathbb{B}_\lambda^c$ and $\alpha \leq \lambda$ we let $Y_e^\alpha := \{\varepsilon < \alpha : e \in \mathbb{D}_{\eta_\varepsilon}\}$

⊗₆ $\mathcal{P}_{\bar{\eta}, \alpha} := \{\{\varepsilon < \alpha : e \in \mathbb{D}_{\eta_\varepsilon}\} : e \in \mathbb{B}_\lambda^c\}$.

Now what can we say on $\mathcal{P}_{\bar{\eta}, \mu}$ for $\mu \in S$? As we can consider $e \in \{e_\alpha : \alpha \in [\mu, 2^\mu]\}$, clearly

⊗₇ $\{\{\varepsilon < \mu : \eta_\varepsilon(\alpha) = 1\} : \alpha \in [\mu, 2^\mu]\} \subseteq \mathcal{P}_{\bar{\eta}, \mu} \subseteq \mathcal{P}(\mu)$.

This may be looked at as a lower bound of $\mathcal{P}_{\bar{\eta}, \mu}$. Naturally we try to get also an “upper bound” to $\mathcal{P}_{\bar{\eta}, \mu}$; now note

⊗₈ if $e \in \mathbb{B}_\lambda^c$ then $Y_{-e}^\mu = \mu \setminus Y_e^\mu$.

Also (by our knowledge of the completion of a free Boolean algebra) for every $e \in \mathbb{B}_\lambda^c$ we can choose u_e such that

⊕₁ (a) $u_e \subseteq \lambda$ is countable

(b) $e \in \mathbb{B}_{\lambda, u_e}^c$.

So by clause (b) of ⊗₄ clearly

⊕₂ if $e \in \mathbb{B}_\lambda^c, \varepsilon < \mu$ and $u_e \subseteq \eta_\varepsilon^{-1}\{0\}$ then $e \in \mathbb{D}_{\eta_\varepsilon} \Leftrightarrow e \in \mathbb{D}_*$

hence

⊕₃ if $e \in \mathbb{B}_\lambda^c \cap \mathbb{D}_*$ and $\mu \in S$ then $Y_e^\mu \supseteq \{\varepsilon < \mu : u_e \subseteq \eta_\varepsilon^{-1}\{0\}\}$.

Next

⊕₄ (a) let $\mathcal{D}_{\bar{\eta}, \mu}$ be the filter on μ generated by $\{\{\varepsilon < \mu : u \subseteq \eta_\varepsilon^{-1}\{0\}$
and $\varepsilon > \zeta\} : \zeta < \mu$ and $u \subseteq \mu$ is countable}

(b) let $\mathcal{I}_{\bar{\eta}, \mu}$ be the dual ideal

clearly

⊕₅ if $\emptyset \notin \mathcal{D}_{\bar{\eta}, \mu}$ then $\mathcal{D}_{\bar{\eta}, \mu}$ is a uniform \aleph_1 -complete filter on μ (recalling $\text{cf}(\mu) = \theta > \aleph_0$ as $\mu \in S$).

Now by ⊕₃ we have $e \in \mathbb{B}_\lambda^c \cap \mathbb{D}_* \Rightarrow Y_e^\mu \in \mathcal{D}_{\bar{\eta}, \mu}$ so recalling ⊗₈ we have $e \in \mathbb{B}_\lambda^c \setminus \mathbb{D}_* \Rightarrow Y_e^\mu = \emptyset \text{ mod } \mathcal{D}_{\bar{\eta}, \mu}$ hence

⊕₆ $\mathcal{P}_{\bar{\eta}, \mu} \subseteq \{X \subseteq \mu : X \in \mathcal{D}_{\bar{\eta}, \mu} \text{ or } \mu \setminus X \in \mathcal{D}_{\bar{\eta}, \mu}\}$.

Now

⊙₁ if $\mu \in S$ then we can find \bar{A}^ξ for $\xi < 2^{2^\mu}$ such that

(a) $\bar{A}^\xi = \langle A_\gamma^\xi : \gamma \in [\mu, 2^\mu] \rangle$

(b) A_γ^ξ is an unbounded subset of $A_\mu^* := \cup\{\chi, 2^\chi\} : \chi \in C_\mu\}$

(c) $\langle \mathcal{D}_\gamma^\xi \cup \mathcal{I}_\gamma^\xi : \gamma < 2^{2^\mu} \rangle$ is without repetition where: \mathcal{D}_γ^ξ is the \aleph_1 -complete filter of subsets of A_μ^* generated by $\{A_\beta^\xi \setminus \beta : \beta \in [\mu, 2^\mu] \text{ and } \beta < \mu\}$; let $\mathcal{I}_\gamma^\xi = \{A_\beta^* \setminus B : B \in \mathcal{D}_\gamma^\xi\}$, i.e. the dual ideal

(d) moreover if $\xi^1 \neq \xi^2$ are $< 2^{2^\mu}$, then

$$\{A_\gamma^{\xi^1} : \gamma \in [\mu, 2^{2^\mu}]\} \not\subseteq \mathcal{D}_\gamma^{\xi^2} \cup \mathcal{I}_\gamma^{\xi^2}$$

(e) for every $\mathcal{P} \subseteq \mathcal{P}(\delta)$ for at most one $\xi < 2^{2^\mu}$ we have

$$\{A_\gamma^\xi : \gamma \in [\mu, 2^\mu]\} \subseteq \mathcal{P} \subseteq \mathcal{D}_\gamma^\xi \cup \mathcal{I}_\gamma^\xi.$$

[Why \odot_1 holds? As $|A_\mu^*| = |\mu|$ is a strong limit cardinal of cofinality $\theta > \aleph_0$ clearly $\mu = |A_\mu^*| = |A_\mu^*|^{\aleph_0}$ hence by [2] there is a sequence $\langle B_\gamma : \gamma \in [\mu, 2^\mu] \rangle$ of subsets of A_μ^* such that any non-trivial Boolean combination of countably many of them has cardinality μ . Let $\langle U_\xi : \xi < 2^{2^\mu} \rangle$ be a sequence of pairwise distinct subsets of $[\mu, 2^\mu]$ each of cardinality $2^{|\mu|}$ no one included in another and let $\langle A_\gamma^\xi : \gamma \in [\mu, 2^{|\mu|}] \rangle$ list $\{B_\gamma : \gamma \in U_\xi\}$.
Now check.]

\odot_2 if $\gamma(*) < \lambda^+$ and $\bar{\mathcal{P}}^\gamma = \langle \mathbf{P}_\mu^\gamma : \mu \in S \rangle$, for $\gamma < \gamma(*)$ where $\mathbf{P}_\mu^\gamma \subseteq \mathcal{P}(\mathcal{P}(\mu))$ has cardinality $\leq 2^\mu$ for $\mu \in S, \gamma < \gamma(*)$ then we can find $\bar{\eta} = \langle \eta_\varepsilon : \varepsilon < \lambda \rangle$ with $\eta_\varepsilon \in {}^\lambda 2$ for $\varepsilon < \lambda$ such that for every $\gamma < \gamma(*)$ the set $\{\mu \in S : \text{for some } \mathcal{P} \in \mathbf{P}_\mu^\gamma \text{ we have } \mathcal{P} \subseteq \mathcal{D}_{\bar{\eta}, \mu} \cup \mathcal{I}_{\bar{\eta}, \mu} \text{ and } \mathcal{P} \text{ satisfies clause (e) of } \odot_1\}$ is not stationary.

[Why? Let $\langle u_\alpha : \alpha < \lambda \rangle$ be an increasing continuous sequence of subsets of $\gamma(*)$ with union $\gamma(*)$ such that $|u_\alpha| \leq |\alpha|$ for $\alpha < \lambda$. Now for each $\mu \in S$, the family $\cup \{\mathbf{P}_\mu^\gamma : \gamma \in u_\mu\}$ is a family of $\leq |u_\mu| \leq 2^\mu$ subsets of $\mathcal{P}(\mu)$.

Now by clause (e) of \odot_1 for each $\mu \in S, \gamma \in u_\mu, \mathcal{P} \in \mathbf{P}_\mu^\gamma$ let $\xi_{\mu, \gamma, \mathcal{P}} < 2^{2^\mu}$ be such that: if for some $\xi, \{A_\gamma^\xi : \gamma \in [\mu, 2^\mu]\} \subseteq \mathcal{P}_\gamma \subseteq \mathcal{D}_\gamma^\xi \cup \mathcal{I}_\gamma^\xi$ then $\xi_{\mu, \gamma}$ is the first such ξ . Choose $\xi(\mu) < 2^{2^\mu}$ which does not belong to $\{\xi_{\mu, \gamma, \mathcal{P}} : \gamma \in u_\mu \text{ and } \mathcal{P} \in \mathbf{P}_\mu^\gamma\}$.

Now for $\varepsilon < \lambda$ we define $\eta_\varepsilon \in {}^\lambda 2$ as follows: if $\varepsilon \in [\mu, 2^\mu]$ and $\mu \in S$ then $\eta_\varepsilon(i)$ is 1 if $i \in A_\varepsilon^{\xi(\mu)} (\subseteq A_\mu^* \subseteq \mu)$ and zero otherwise.

Now check.]

\odot_3 if $\langle M_\gamma : \gamma \in \gamma(*) \rangle$ is a \prec -increasing continuous and $M_\gamma \in \text{EC}_\lambda(T)$ and $\bar{b}_\alpha \in {}^{\ell g(\bar{y})}(M_0)$ for $\alpha < \lambda$ are such that $\langle \varphi_\alpha(x, \bar{b}_\alpha) : \alpha < \lambda \rangle$ is independent, then we can find N such that
(α) $M_{\gamma(*)} \prec N \in \text{EC}_\lambda(T)$
(β) if $N \prec N' \in \text{EC}_\lambda(T)$ and $\gamma < \gamma(*)$ then ${}^2 \text{inv}_6^\varphi(N_\gamma, N') \notin \{\text{inv}_6^\varphi(M_{\gamma_1}, M_{\gamma_2}) : \gamma_1 < \gamma_2 \leq \gamma(*)\}$.

[Why? Without loss of generality the universe of $M_{\gamma(*)}$ is $\mathcal{U}_1 \in [\lambda]^\lambda$ such that $\lambda \setminus \mathcal{U}_1$ has cardinality λ .

For $\gamma(1) < \gamma(2) \leq \gamma(*)$ let $\mathbf{P}_\delta^{\gamma(1), \gamma(2)} = \text{inv}_6^\varphi(\delta, \text{id}_{N_{\gamma(2)}}, N_{\gamma(1)}, N_{\gamma(2)})$, see Definition 1.4, clearly $\text{inv}_6^\varphi(N_{\gamma(1)}, N_{\gamma(2)}) = \langle \mathcal{P}_\delta^{\gamma(1), \gamma(2)} : \delta < \lambda \rangle / \mathcal{D}_\lambda$. So it is enough to find N and sequence $\langle a_\gamma : \gamma < \lambda \rangle$ of elements of N such that $M_{\gamma(*)} \prec N, |N| = \lambda$ and for each $\gamma(1) < \gamma(2) \leq \gamma(*)$, for every $\mu \in S$ except non-stationarily many, the family

$$\{\{\gamma < \mu : N \models \varphi[a, \bar{b}_\gamma]\} : \bar{b} \in {}^{\ell g(\bar{y})}(M_{\gamma(*)})\}$$

is not in $\mathbf{P}_\mu^{\gamma(1), \gamma(2)}$.

²really any pregiven set of $\leq \lambda$ “forbidden” inv_6^φ is O.K. and can make it work for $\text{inv}_6^\varphi(N_\gamma, N')$ for every $\gamma < \gamma(*)$.

We choose $\bar{\eta} = \langle \eta_\varepsilon : \varepsilon < \lambda \rangle$ as in \odot_2 ; so, recalling \otimes_3 clearly $\langle \mathbb{D}_{\eta_\varepsilon} : \varepsilon < \lambda \rangle$ is well defined. Now for each $\varepsilon < \alpha$ letting F be from 2.3 for the model $M_{\gamma(*)}$, let $p_\varepsilon \in \mathbf{S}_\varphi(M_{\gamma(*)})$ be such that for every $\bar{b} \in {}^{\ell g(\bar{y})}(M_0)$ we have $\varphi(x, \bar{b}) \in p_\varepsilon \Leftrightarrow F(\bar{b}) \in \mathbb{D}_{\eta_\varepsilon}$ so $\neg\varphi(x, \bar{b}) \in p_\varepsilon \Leftrightarrow F(\bar{b}) \notin \mathbb{D}_{\eta_\varepsilon}$.

So we can find an elementary extension N of $M_{\varepsilon(*)}$ and $a_\varepsilon \in N$ for $\varepsilon < \lambda$ such that a_ε realizes p_ε , and without loss of generality N has universe $\subseteq \lambda$ such that $\lambda \setminus |N|$ has cardinality λ . We can consider only N' such that ($N \prec N' \in \mathbf{EC}_\lambda(T)$ and) $|N'| \subseteq \lambda$. Now chasing our definitions and choices, it is clearly as required.]

\odot_4 if $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is as in \boxtimes' then for some club E of λ^+ , we have if $\alpha_1 < \alpha_2, \beta_1 < \beta_2$ are from E and $\alpha_2 \neq \beta_2$ then

$$(M_{\alpha_1}, M_{\alpha_2}) \not\cong (M_{\beta_1}, M_{\beta_2}).$$

[Why? For every $\beta < \lambda^+$ we apply \odot_3 to $\langle M_\alpha : \alpha \leq \beta \rangle$ and get N_β as there so $M_\beta \prec N_\beta \in \mathbf{EC}_\lambda(T)$. As $M = \cup\{M_\gamma : \gamma < \lambda^+\}$ is saturated, without loss of generality $N_\beta \prec M$ hence for some $\gamma_\beta < \lambda^+$ we have $N_\beta \prec M_{\gamma_\beta}$.

Let $E = \{\delta < \lambda^+ : \delta \text{ a limit ordinal such that } \beta < \delta \Rightarrow \gamma_\beta < \delta\}$.

So we are clearly done. □_{2.8}

3. PRIVATE APPENDIX

Definition 3.1. 1) We say that T has the PO-property when some $\varphi(\bar{x}, \bar{y})$ has it where:

2) We say $\varphi(\bar{x}, \bar{y})$ has the PO-property in T when: for every partial order I and for some model M of T and $\bar{a}_t \in {}^{\ell g(\bar{y})}M$ for $t \in I$ we have:

(a) if $s, t \in I$ has no common upper bound then $M \models \neg(\exists \bar{x})(\varphi(\bar{x}, \bar{a}_s) \wedge \varphi(\bar{x}, \bar{a}_t))$

(b) if $n < \omega$ and $t_0, \dots, t_{n-1} \in I$ has a common upper bound then $(\exists \bar{x}) \bigwedge_{\ell < n} \varphi(\bar{x}, \bar{a}_{t_\ell})$.

{3n.7}

Claim 3.2. In $\text{Th}(M, N)$ we can interpret by $\mathbb{L}_{|T|+, \omega}(\tau_T)$ -formulas with parameters, a Boolean algebra \mathbb{B} isomorphic to \mathbb{B}^* and can interpret qualification over ideals of \mathbb{B} when:

- ⊗ (a) $M \prec N$ are models of T
- (b) N is $\|M\|^+$ -saturated
- (c) \mathbb{B}^* is a complete Boolean Algebra of cardinality λ
- (d) $\langle \varphi(x, \bar{a}_\alpha) : \alpha < \lambda \rangle$ is an independent sequence of formulas in M
- (e) $\lambda = \lambda^{\aleph_0}$ (for simplicity).

Remark 3.3. Does this mean to imply interpretations of large models of PA?

Proof. Let $\mathbb{B} = \mathbb{B}_\lambda^c$ and let $\langle d_\alpha : \alpha < \lambda \rangle$ list the elements of \mathbb{B} , see 2.2. Let \mathbb{B}_M be the Boolean algebra of subsets of M definable by formulas with parameters, i.e. $\{\psi(M, \bar{b}) : \psi(x, \bar{y}) \in \mathbb{L}(\tau_T) \text{ and } \bar{b} \in {}^{\ell g(\bar{y})}M\}$ and let \mathbb{B}_M^- be the subalgebra generated by $\{\varphi(M, \bar{a}_\alpha) : \alpha < \lambda\}$. Let h be the unique homomorphism from \mathbb{B}_M^- into \mathbb{B}_λ^c mapping $\varphi(M, \bar{a}_\alpha)$ to d_α ; exists as $\langle \varphi(M, \bar{a}_\alpha) : \alpha < \lambda \rangle$ is an independent family of subsets of M . As \mathbb{B}_λ^c is a complete Boolean Algebra we can find an extension of h to a homomorphism from \mathbb{B}_M into (hence onto) \mathbb{B}_λ^c . \square

Moved from end of §1, pg.9:

Discussion: We may prefer to have an invariant which speaks only on one model.

Definition 3.4. Let $(\lambda$ be regular $\geq |T|$, $\varphi(x, \bar{y})$ as in xxx).

1) If $N \in \text{EC}_T(\lambda)$, $f : |N| \rightarrow \lambda$ and $\delta < \lambda$ and $\mathcal{P} \subseteq \mathcal{P}(\delta)$ we say (N, f, δ, κ) accept \mathcal{P} when: there is a witness $\langle a_{\eta, \varepsilon} : \eta \in \mathcal{T}, \varepsilon < \delta \rangle$ such that

- (a) $\mathcal{T} \in \mathbf{T}_\kappa$ where $\mathcal{T} \in \mathbf{T}_\kappa$ iff $\mathcal{T} \subseteq {}^{\omega} \kappa$ is non-empty, closed under initial segments, with no infinite branch
- (b) for every $\bar{b} \in {}^{\ell g(\bar{y})}N$ for some $\mathcal{U} \in \mathcal{P}$ for almost every $\eta \in \mathcal{T}$ we have $M \models \varphi[a_{\eta, \varepsilon}, \bar{b}]^{\text{if } (\varepsilon \in \mathcal{U})}$.

FILL

Moved from Definition 1.2, pg.3: 3) If N is a model of T with universe λ let

$$\text{inv}_3(M) = \langle \text{inv}_2(M, N) : M \prec N \text{ has universe an ordinal } < \lambda \rangle$$

$$\text{inv}_4(M) = \text{inv}_3(M) / \mathcal{D}_\lambda$$

recalling \mathcal{D}_λ the club filter.

4) If N is a model of cardinality λ let $\text{inv}_4(N)$ be $\text{inv}_4(N')$ for any equivalent some model N' with universe λ isomorphic to λ .

We first deal with the explicit caes (and use simplifying) cardinal arithmetic assumptions ³. Moved from pgs.10,11:

Remark 3.5. Assume

- (a) λ is strongly inaccessible of cofinality κ
- (b) $S \subseteq \{\mu < \lambda : \mu = \mu^{\aleph_0} \text{ is strong limit}\}$ is stationary.

Then we can find $\bar{\mathbb{D}}$ such that

- (α) $D = \langle \mathbb{D}_B : B \subseteq \lambda \rangle$
- (β) \mathbb{D}_B is an ultrafilter of \mathbb{B}_λ^c
- (γ)₁ if $B_\varepsilon^\ell \subseteq \lambda$ for $\varepsilon < \lambda$, $\ell = 1, 2$ and $B_{\varepsilon_1}^1 \neq B_{\varepsilon_2}^2$ for $\varepsilon_1, \varepsilon_2 < \lambda$ then for a club of $\mu \in S$ we have:
 - $\{\varepsilon < \mu : \{Y \subseteq \mu : \text{for some } e \in \mathbb{B}^\ell \text{ we have } (\forall \varepsilon < \mu)(\varepsilon \in Y \equiv e \in \mathbb{D}_{B_\varepsilon}^1)\}\} \neq \{y \subseteq \mu : \text{for some } e \in \mathbb{B}^c \text{ we have } (\forall \varepsilon < \mu)(\varepsilon \in y \equiv e \in \mathbb{D}_{B_\varepsilon}^2)\}$
- (γ)₂ if $\mathcal{P}_{\mu, \alpha} \subseteq \mathcal{P}(\mu)$ for $\alpha < 2^\mu$ then for some $\bar{\mathbb{D}} = \langle \mathbb{D}_\varepsilon : \varepsilon < \lambda \rangle$, for every $\mu \in S$ we have $\{\{\varepsilon < \mu : e \in \mathbb{D}_\varepsilon\} : e \in \mathbb{B}_\lambda^c\} \notin \{\mathcal{P}_{\mu, \alpha} : \alpha < 2^\mu\}$.

Proof. Choose \mathbb{D}_* such that

- \otimes_1 \mathbb{D}_* is an ultrafilter on \mathbb{B}_λ^c disjoint to $\{e_\alpha : \alpha < \lambda\}$.

Let $\bar{\pi}$ be such that

- \otimes_2 [??] (a) $\bar{\pi} = \langle \pi_{\mu_1, \mu_2} : \mu_1 < \mu_2 \text{ are from } S \rangle$
- (b) π_{μ_1, μ_2} is a mapping from $[\mu_2, 2^{\mu_2})$ to $[\mu_1, 2^{\mu_1})$
- (c) the π 's commute
- (d) if $(S \cap \mu)$ has not last inverse limit.

For each $\mu \in S$ we choose $\bar{\Lambda}_\mu$ such that

- \otimes'_2 ?? (a) $\bar{\Lambda}_\mu = \langle \Lambda_{\mu, \alpha} : \alpha \in [\mu, 2^\mu) \rangle$
- (b) $\Lambda_{\mu, \alpha} \subseteq \mu^2$
- (c) if $\mu_1 < \mu_2$ are from S and $\pi_{\mu_1, \mu_2}(\alpha_2) = \alpha_1$ then $\Lambda_{\mu_1, \alpha_1} \subseteq \{\nu \upharpoonright \mu_1 : \nu \in \Lambda_{\mu_2, \alpha_2}\}$
- \otimes'_2 let $\theta, S_*, E, S, \bar{c}$ be as in the proof of 1.9
- \otimes_3 if $\bar{\eta} = \langle \eta_\varepsilon : \varepsilon < \mu \rangle$, $\eta_\varepsilon \in {}^\lambda 2$ for $\varepsilon < \mu$ then let $\mathcal{D}_{\mu, \bar{\eta}}$ be the \aleph_1 -complete filter on μ generated by $\{\varepsilon < \mu : \eta_\varepsilon^{-1}\{0\} \supseteq u\} : u \subseteq \mu \text{ is countable}\}$.

Now we can choose $\bar{\eta}^*$ such that

- \boxplus_5 (a) $\bar{\eta}^* = \langle \eta_A^* : A \subseteq \lambda \rangle$
- (b) $\eta_A^* \in \mu^2$
- (c) if $\mu = \mu^{\aleph_0} < \lambda$ and $A_n \subseteq \lambda$, $\langle A_n \cap \mu : n < \omega \rangle$ are pairwise distinct then $\eta_{A_0}^{-1}\{1\} \not\subseteq \cup \{\eta_{A_n}^{-1}\{1\} : n < \omega\}$.

Hence

³if $2^\lambda > \lambda^+$ use the game formulation

⊗ if $A_\varepsilon \subseteq \lambda$ for $\varepsilon < \lambda$ and $B \in \mathcal{P}(\lambda) \setminus \{A_\varepsilon : \varepsilon < \lambda\}$ then $\eta_B^{-1}\{1\} \neq \emptyset, \lambda \bmod D_{\langle \eta_{A_\varepsilon} : \varepsilon < \lambda \rangle}$.

So $\langle \mathbb{D}_{\eta_B} : B \subseteq \lambda \rangle$ are as required in (γ) . □

Proof. Proof of $(\gamma)_2$ The proof is similar. FILL.

We choose $\bar{\Lambda}_\mu$ by induction on μ such that ... □

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