ON LONG INCREASING CHAINS MODULO FLAT IDEALS SH908

Saharon Shelah

The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
Jerusalem 91904, Israel

Department of Mathematics
Hill Center-Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-8019 USA

Abstract. We prove that e.g. in ${}^{(\omega_3)}(\omega_3)$ there is no sequence of length ω_4 increasing modulo the ideal of countable sets.

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§0 Introduction

We hope sometime to prove: e.g.

0.1 Conjecture. For every $\mu > \theta$ in ${}^{(\theta^{+3})}\mu$ there is no increasing sequence of length μ^+ modulo $[\theta^{+3}]^{\le \theta}$.

Let $\kappa = \mathrm{cf}(\kappa) > \aleph_0$. If $\mu = \kappa$, then $\mathrm{Length}({}^\kappa\mu, <_{J^{\mathrm{bd}}_\kappa}) = \sup\{|\mathscr{F} : \mathscr{F} \subseteq {}^\kappa\kappa$ linearly ordered by $<_{J^{\mathrm{bd}}_\kappa}\}$ can be (forced to be) large. But for $\mu >$ Length$({}^\kappa\mu,$ $<_{J^{\mathrm{bd}}_\kappa})$ this implies pcf results (see [Sh 410], [Sh 589]).

However, e.g. for the ideal $\mathscr{I} = [\omega_3]^{\le \aleph_0}$ it is harder to get long increasing sequence, as above for "high μ", this leads to pcf results so let $\bar\lambda = \langle \lambda_i : i < \omega_3\rangle \in {}^{\omega_3}\mathrm{Reg}, (\pi\bar\lambda, <_{\mathscr{I}})$ has cofinality $> \sup\{\lambda_i; i < \omega_3\}$ which are much stronger than known consistency results. Even for $I = [\omega_1]^{\le\aleph_0}$ we do not know, for $I = [\beth_\omega]^{\le\aleph_0}$ we know ([Sh 460]), so even $[\aleph_\omega]^{\le\aleph_0}$ would be interesting good news

The following may be a first step and anyhow stand by itself.

§1 Increasing sequences modulo countable

1.1 Claim. *Assume $\partial^+ < \kappa \le \theta$ and $\mathscr{J} = [\kappa]^{<\partial}$, an ideal and $\mathrm{cf}([\theta]^\partial, \subseteq) \le \theta$. Then there is no $<_{\mathscr{J}}$-increasing sequence $\langle f_\alpha : \alpha < \theta^+\rangle$ of functions from ${}^\kappa\theta$ (or even ${}^\kappa\gamma(*)$ for some $\gamma(*) < \theta^+$).*

Remark. E.g. $\partial = \aleph_1, \kappa = \theta = \aleph_3$ was discussed above.

Proof. It is enough to treat the version with $\gamma(*)$. So toward contradiction assume $\langle f_\alpha : \alpha < \theta^+\rangle$ is a counterexample.

Let $\mathscr{S} \subseteq [\gamma(*)]^\partial$ be cofinal of cardinality $\le \theta$ exists as $|\gamma(*)| = \theta$ and $\mathrm{cf}([\theta]^\partial, \subseteq) \le \theta$. Now for every $s \in \mathscr{S}$ and $\beta < \kappa$ let $I_\beta = I(\beta) := [\beta, \beta + \partial)$ and we define

- $(*)_0$ for $\zeta < \theta^+$ let $f^{s,\beta}_\zeta \in {}^{I(\beta)}(\gamma(*) + 1)$ be defined as $f^s_\zeta \restriction I(\beta)$ where $f^s_\zeta \in {}^\kappa(\gamma(*) + 1)$ is defined by $f^s_\zeta(i) = \min(s \cup \{\gamma(*)\} \backslash f_\zeta(i))$.

Now

- $(*)_1$ for $s \in \mathscr{S}$ we have
  - (a) $\zeta < \theta^+ \Rightarrow f^{s,\beta}_\zeta \in {}^{I(\beta)}(s \cup \{\gamma(*)\})$
  - (b) $\zeta < \xi < \theta^+ \Rightarrow f^{s,\beta}_\zeta \le f^{s,\beta}_\xi \mod \mathscr{J} \restriction I_\beta$.

For $s \in \mathscr{S}$, let

- $(*)_2$ $B_s = \{\beta < \kappa :$ for every $\zeta < \theta^+$ there is $\xi \in (\zeta, \theta^+)$ such that $\neg(f^{s,\beta}_\zeta = f^{s,\beta}_\xi \mod \mathscr{J} \restriction I_\beta)\}$.

For $s \in \mathscr{S}$ and $\beta < \kappa$ clearly we can choose C^s_β such that

- $(*)_3$
  - (a) C^s_β is a club of θ^+
  - (b) if $\beta \in B_s$ and $\xi \in C^s_\beta$ then $\zeta < \xi \Rightarrow \neg(f^{s,\beta}_\zeta = f^{s,\beta}_\xi \mod \mathscr{J} \restriction I_\beta)$
  - (c) if $\beta \in \kappa \backslash B_s$ then $\theta^+ > \xi \ge \zeta \ge \mathrm{Min}(C^s_\beta) \Rightarrow (f^{s,\beta}_\zeta = f^{s,\beta}_\xi \mod \mathscr{J} \restriction I_\beta)$.

So

- $(*)_4$ $C := \cap\{C^s_\beta : s \in \mathscr{S}$ and $\beta < \kappa\}$ is a club of θ^+ as $|\mathscr{S}| \le \theta, \kappa \le \theta$.

Now choose

$(*)_5$ $\alpha_\varepsilon \in C$ for $\varepsilon < \partial^+$ increasing with ε

hence

$(*)_6$ $u_{\varepsilon,\zeta} := \{i < \kappa : f_{\alpha_\varepsilon}(i) \geq f_{\alpha_\zeta}(i)\} \in \mathscr{J}$ for $\varepsilon < \zeta < \partial^+$.

As we are assuming $\partial^+ < \kappa$ we can find $\beta(*)$ such that

$(*)_7$ (a) $\beta(*) < \kappa$
(b) $I_{\beta(*)} = [\beta(*), \beta(*) + \partial]$ is disjoint to $\cup\{u_{\varepsilon,\zeta} : \varepsilon < \zeta < \partial^+\}$ hence
(c) the sequence $\langle f_{\alpha_\varepsilon}(i) : \varepsilon < \partial^+\rangle$ is increasing for each $i \in I_{\beta(*)}$.

As $|I_{\beta(*)}| = \partial$ and $\mathscr{S} \subseteq [\gamma(*)]^{\leq \partial}$ is cofinal (for the partial order $\subseteq$), we can find s such that

$(*)_8$ $s \in \mathscr{S}$ and $s \supseteq \{f_{\alpha_0}(i), f_{\alpha_1}(i) : i \in I_{\beta(*)}\}$

hence by $(*)_6 + (*)_7 + (*)_8$ we have

$(*)_9$ $f^s_{\alpha_0}(i) = f_{\alpha_0}(i) < f_{\alpha_1}(i) = f^s_{\alpha_1}(i)$ for every $i \in I_{\beta(*)}$.

As $\alpha_0 < \alpha_1$ are from C and $I_{\beta(*)} \in \mathscr{J}^+$, recalling $(*)_2 + (*)_3 + (*)_4$, clearly

$(*)_{10}$ $\beta(*) \in B_s$,

recalling that $\alpha_\varepsilon \in C \subseteq C^s_{\beta(*)}$ for $\varepsilon < \partial^+$ and α_ε is increasing with ε, clearly (by $(*)_{10}$)

$(*)_{11}$ for every $\varepsilon < \partial^+$ there is $i_\varepsilon \in I_{\beta(*)}$ such that
(α) $f^s_{\alpha_\varepsilon}(i_\varepsilon) < f^s_{\alpha_{\varepsilon+1}}(i_\varepsilon)$
hence there is $j_\varepsilon \in s$ such that
(β) $f^s_{\alpha_\varepsilon}(i_\varepsilon) \leq j_\varepsilon < f^s_{\alpha_{\varepsilon+1}}(i_\varepsilon)$ hence
(γ) $f_{\alpha_\varepsilon}(i_\varepsilon) \leq j_\varepsilon < f_{\alpha_{\varepsilon+1}}(i_\varepsilon)$.

But $|I_{\beta(*)}| + |s| = \partial < \partial^+$ hence for some pair $(j_*, i_*) \in s \times I_{\beta(*)}$ we have

$(*)_{12}$ $\partial^+ = \sup(\mathscr{U})$ where $\mathscr{U} := \{\varepsilon < \theta^+ : j_\varepsilon = j_* \text{ and } i_\varepsilon = i_*\}$.

Hence we can find $\varepsilon_1 < \varepsilon_2$ from $\mathscr{U}$, but $\langle f_{\alpha_\varepsilon}(i_*) : \varepsilon < \theta^+\rangle$ is increasing by $(*)_7(c)$, i.e. the choice of $\beta(*)$, so $f_{\alpha_{\varepsilon_1}}(i_*) < f_{\alpha_{\varepsilon_1+1}}(i_*) \leq f_{\alpha_{\varepsilon_2}}(i_*) < f_{\alpha_{\varepsilon_2+1}}(i_*)$ but by $(*)_{11}(\gamma)$+$(*)_{12}$ the ordinal j_* belongs to $[f_{\alpha_{\varepsilon_1}}(i_*), f_{\alpha_{\varepsilon_1+1}}(i_*))$ and to $[f_{\alpha_{\varepsilon_2}}(i_*), f_{\alpha_{\varepsilon_2+1}}(i_*))$, which are disjoint intervals, contradiction. $\square_{1.1}$

Similarly

1.2 Claim. *There is no $<_{\mathscr{J}}$-increasing sequence of functions from κ to $\gamma(*)$ of length λ when*

$\circledast$ (a) $\mathscr{J}$ *is an ideal on* κ

(b) $I_\beta \in [\kappa]^\partial$ *for* $\beta < \kappa$

(c) $I_\beta \notin \mathscr{J}$ *for* $\beta < \kappa$

(d) $\mathrm{cf}([\theta]^\partial, \subseteq) < \lambda$ *where* $\theta = |\gamma(*)| + \kappa$

(e) *if* $u_\varepsilon \in \mathscr{J}$ *for* $\varepsilon < \partial^+$ *then* I_β *is disjoint to* $\bigcup\limits_{\varepsilon<\partial^+} u_\varepsilon$ *for some* $\beta < \kappa$.

Proof. Without loss of generality λ is the successor of $\mathrm{cf}([\theta]^\partial, \subseteq)$ hence is regular. The proof is similar to the proof of 1.1. $\square_{1.2}$

We may wonder could we have used same one "+", i.e.

1.3 Question: Is it consitent that ${}^\theta\theta$ contains $<_{\mathscr{J}}$-increasing sequence of length θ^+ when $\theta = \kappa^+, I = [\theta]^{<\kappa}$?

REFERENCES.

[Sh 410] Saharon Shelah. More on Cardinal Arithmetic. *Archive for Mathematical Logic*, **32**:399–428, 1993. arxiv:math.LO/0406550.

[Sh 589] Saharon Shelah. Applications of PCF theory. *Journal of Symbolic Logic*, **65**:1624–1674, 2000. arxiv:math.LO/9804155.

[Sh 460] Saharon Shelah. The Generalized Continuum Hypothesis revisited. *Israel Journal of Mathematics*, **116**:285–321, 2000. arxiv:math.LO/9809200.