On long increasing chains modulo flat ideals

Saharon Shelah[*][1,2]

[1] The Hebrew University of Jerusalem, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel

[2] Department of Mathematics, Hill Center-Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

We prove that, e.g., in $^{(\omega_3)}(\omega_3)$ there is no sequence of length ω_4 increasing modulo the ideal of countable sets.

This note is concerned with the depth of the partial order of the functions in $^\kappa\gamma$ modulo the ideal of the form $\mathcal{I} = [\kappa]^{<\mu}$. Let us recall the following definitions.

Definition 1 For a partial order $(P, \sqsubset)$ we define

- $\mathrm{Depth}(P, \sqsubset) = \sup\{|\mathcal{F}| : \mathcal{F} \subseteq P \text{ is well–ordered by } \sqsubset \}$ *[the depth]*

- $\mathrm{cf}(P, \sqsubset) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq P \text{ is } \sqsubset\text{–cofinal which mean that for every } p \in P \text{ there is } q \in \mathcal{F} \text{ such that } p \sqsubseteq p \}$ *[the cofinality]*.

Our result (Theorem 4) states that under suitable assumptions the depth of the partial order $(^\kappa\gamma, <_{[\kappa]^{<\mu}})$ is at most $|\gamma|$. In particular, letting $\mu = \aleph_1$, $\kappa = |\gamma| = \aleph_3$ we obtain that in $^{(\omega_3)}(\omega_3)$ there is no sequence of length ω_4 increasing modulo the ideal of countable sets.

Let $\kappa = \mathrm{cf}(\kappa) > \aleph_0$. If $\mu = \kappa$, then $\mathrm{Depth}(^\kappa\kappa, <_{J_\kappa^{\mathrm{bd}}})$ can be (forced to be) large. But for $\mu > \mathrm{Depth}(^\kappa\mu, <_{J_\kappa^{\mathrm{bd}}})$ this implies pcf results (see [Sh 410], [Sh 589]).

However, e.g., for the ideal $\mathcal{I} = [\omega_3]^{\leq\aleph_0}$ it is harder to get long increasing sequence, as above for "high μ", this leads to pcf results e.g. if we assume that $\bar{\lambda} = \langle \lambda_i : i < \omega_3 \rangle \in {}^{\omega_3}\mathrm{Reg}$, and in $(\prod \bar{\lambda}, <_{\mathcal{I}})$ there is an increasing sequence moduo $\mathcal{I}$ of length say $> 2^{\aleph_3} + \sup\{\lambda_i : i < \omega_3\}$ are much stronger than known consistency results. Even for $I = [\omega_1]^{\leq\aleph_0}$ we do not know, for $I = [\beth_\omega]^{\leq\aleph_0}$ we know ([Sh 460]), so even $[\aleph_\omega]^{\leq\aleph_0}$ would be interesting good news.

We hope sometime to prove, e.g.,

Conjecture 2 *For every $\mu > \theta$, in $^{(\theta^{+3})}\mu$ there is no increasing sequence of length μ^+ modulo $[\theta^{+3}]^{\leq\theta}$.*

Problem 3 Is it consistent that $^\theta\theta$ contains $<_{\mathcal{I}}$-increasing sequence of length θ^+ when $\theta = \kappa^+$ and $\mathcal{I} = [\theta]^{<\kappa}$?

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [J]).

1. Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta, \ldots$ (with possible subscripts). Cardinal numbers will be called $\kappa, \lambda, \mu, \theta$.

2. For a set X and a cardinal θ, $[X]^\theta$ (or $[X]^{<\theta}$, respectively) stands for the family of subsets of X of size θ ($< \theta$, respectively).

[*] e-mail: shelah@math.huji.ac.il, URL: http://shelah.logic.at

Theorem 4 *Assume $\mu^+ < \kappa \leq \theta$ and $\mathcal{J} = [\kappa]^{<\mu}$ and $\mathrm{cf}([\theta]^\mu, \subseteq) \leq \theta$. Let $\gamma < \theta^+$. Then $\mathrm{Depth}(^\kappa\gamma, <_{\mathcal{J}}) \leq \theta$, i.e., there is no $<_{\mathcal{J}}$–increasing sequence $\langle f_\alpha : \alpha < \theta^+ \rangle$ of functions from $^\kappa\gamma$ modulo $\mathcal{J}$.*

P r o o f. Assume towards contradiction that there is a $<_{\mathcal{J}}$–increasing sequence $\langle f_\zeta : \zeta < \theta^+ \rangle \subseteq {}^\kappa\gamma$.

Let $\mathcal{S} \subseteq [\gamma]^\mu$ be cofinal of cardinality $\leq \theta$ (exists as $|\gamma| \leq \theta$ and $\mathrm{cf}([\theta]^\mu, \subseteq) \leq \theta$). For every $s \in \mathcal{S}$ and $\beta < \kappa$, $\zeta < \theta^+$ we let

- $I(\beta) := [\beta, \beta + \mu)$,

- $f_\zeta^s \in {}^\kappa(\gamma + 1)$ be defined by $f_\zeta^s(i) = \min(s \cup \{\gamma\} \setminus f_\zeta(i))$,

- $f_\zeta^{s,\beta} \in {}^{I(\beta)}(\gamma + 1)$ be defined as $f_\zeta^s \restriction I(\beta)$.

Now, for each $s \in \mathcal{S}$ we have

$(*)_1$ (a) for every $\zeta < \theta^+$, $f_\zeta^{s,\beta} : I(\beta) \longrightarrow s \cup \{\gamma\}$,

   (b) if $\zeta < \xi < \theta^+$, then $f_\zeta^{s,\beta} \leq f_\xi^{s,\beta} \mod [I(\beta)]^{<\mu}$.

For $s \in \mathcal{S}$ we define

$(*)_2$ $B_s = \{\beta < \kappa : (\forall \zeta < \theta^+)(\exists \xi > \zeta) \neg (f_\zeta^{s,\beta} = f_\xi^{s,\beta} \mod [I(\beta)]^{<\mu})\}$.

Plainly, we may choose a sequence $\langle C_\beta^s : \beta < \kappa, \ s \in \mathcal{S} \rangle$ such that

$(*)_3$ (a) C_β^s is a club of θ^+,

   (b) if $\beta \in B_s$ and $\xi, \zeta \in C_\beta^s$ are such that $\zeta < \xi$, then $\neg(f_\zeta^{s,\beta} = f_\xi^{s,\beta} \mod [I(\beta)]^{<\mu})$,

   (c) if $\beta \in \kappa \setminus B_s$, then $f_\zeta^{s,\beta} = f_\xi^{s,\beta} \mod [I(\beta)]^{<\mu}$ whenever $\min(C_\beta^s) \leq \zeta \leq \xi < \theta^+$.

Then, as $|\mathcal{S}| \leq \theta$ and $\kappa \leq \theta$, we have

$(*)_4$ the set $C := \bigcap\{C_\beta^s : s \in \mathcal{S} \text{ and } \beta < \kappa\}$ is a club of θ^+ .

Choose a sequence $\langle \alpha_\varepsilon : \varepsilon < \mu^+ \rangle \subseteq C$ increasing with ε. Then, for all $\varepsilon < \zeta < \mu^+$,

$(*)_5$ $u_{\varepsilon,\zeta} := \{i < \kappa : f_{\alpha_\varepsilon}(i) \geq f_{\alpha_\zeta}(i)\} \in \mathcal{J}$.

We have assumed that $\mu^+ < \kappa$, so we can find $\delta < \kappa$ such that

$(*)_6$ (a) $I(\delta) = [\delta, \delta + \mu)$ is disjoint from $\bigcup\{u_{\varepsilon,\zeta} : \varepsilon < \zeta < \mu^+\}$, and hence

   (b) the sequence $\langle f_{\alpha_\varepsilon}(i) : \varepsilon < \mu^+ \rangle$ is increasing for each $i \in I(\delta)$.

As $|I(\delta)| = \mu$ and $\mathcal{S} \subseteq [\gamma]^{\leq\mu}$ is cofinal (for the partial order $\subseteq$), we can find $s \in \mathcal{S}$ such that

$(*)_7$ $\{f_{\alpha_0}(i), f_{\alpha_1}(i) : i \in I(\delta)\} \subseteq s$.

It follows from $(*)_6 + (*)_7$ that for every $i \in I(\delta)$

$(*)_8$ $f_{\alpha_0}^s(i) = f_{\alpha_0}(i) < f_{\alpha_1}(i) = f_{\alpha_1}^s(i)$.

As $\alpha_0 < \alpha_1$ are from C and $I(\delta) \notin \mathcal{J}$, recalling $(*)_2 + (*)_3 + (*)_4$, clearly

$(*)_9$ $\delta \in B_s$.

Therefore, as $\alpha_\varepsilon \in C \subseteq C_\delta^s$ for $\varepsilon < \mu^+$ and α_ε is increasing with ε, we have

$(*)_{10}$ for every $\varepsilon < \mu^+$ there is $i_\varepsilon \in I(\delta)$ such that

   (α) $f_{\alpha_\varepsilon}^s(i_\varepsilon) < f_{\alpha_{\varepsilon+1}}^s(i_\varepsilon)$,

   and hence there is $j_\varepsilon \in s$ such that

(β) $f^s_{\alpha_\varepsilon}(i_\varepsilon) \leq j_\varepsilon < f^s_{\alpha_{\varepsilon+1}}(i_\varepsilon)$

and therefore

(γ) $f_{\alpha_\varepsilon}(i_\varepsilon) \leq j_\varepsilon < f_{\alpha_{\varepsilon+1}}(i_\varepsilon).$

But $|I(\delta)| + |s| = \mu < \mu^+$, so for some pair $(j_*, i_*) \in s \times I(\delta)$ we may choose $\varepsilon_1 < \varepsilon_2 < \mu^+$ such that

$(*)_{11}$ $j_{\varepsilon_1} = j_{\varepsilon_2} = j_*$ and $i_{\varepsilon_1} = i_{\varepsilon_2} = i_*.$

But the sequence $\langle f_{\alpha_\varepsilon}(i_*) : \varepsilon < \theta^+ \rangle$ is increasing by $(*)_6(b)$ (see the choice of δ), so

$$f_{\alpha_{\varepsilon_1}}(i_*) < f_{\alpha_{\varepsilon_1+1}}(i_*) \leq f_{\alpha_{\varepsilon_2}}(i_*) < f_{\alpha_{\varepsilon_2+1}}(i_*).$$

It follows from $(*)_{10}(\gamma) + (*)_{11}$ that the ordinal j_* belongs to $[f_{\alpha_{\varepsilon_1}}(i_*), f_{\alpha_{\varepsilon_1+1}}(i_*))$ and to $[f_{\alpha_{\varepsilon_2}}(i_*), f_{\alpha_{\varepsilon_2+1}}(i_*))$, which are disjoint intervals, a contradiction. $\square$

Similarly,

Theorem 5 *Assume that*

(a) *$\mathcal{J}$ is an ideal on κ,*

(b) *$I_\beta \in [\kappa]^\mu$, $I_\beta \notin \mathcal{J}$ for $\beta < \kappa$,*

(c) *$\theta = |\gamma| + \kappa$ and $\operatorname{cf}([\theta]^\mu, \subseteq) < \lambda$,*

(d) *if $u_\varepsilon \in \mathcal{J}$ for $\varepsilon < \mu^+$, then for some $\beta < \kappa$ the set I_β is disjoint from $\bigcup_{\varepsilon < \mu^+} u_\varepsilon$.*

Then there is no $<_\mathcal{J}$–increasing sequence of functions from κ to γ of length λ.

P r o o f. Without loss of generality λ is the successor of $\operatorname{cf}([\theta]^\mu, \subseteq)$ hence is regular. The proof is similar to the proof of Theorem 4. $\square$

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