Uniforming n-place Functions on Well Founded Trees

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Abstract. In this paper the Erdős-Rado theorem is generalized to the class of well founded trees. We define an equivalence relation on the class $\mathrm{ds}(\infty)^{<\aleph_0}$ (finite sequences of decreasing sequences of ordinals) with $\aleph_0$ equivalence classes, and for $n < \omega$ a notion of n-end-uniformity for a colouring of $\mathrm{ds}(\infty)^{<\aleph_0}$ with μ colours. We then show that for every ordinal α, $n < \omega$ and cardinal μ there is an ordinal λ so that for any colouring c of $T = \mathrm{ds}(\lambda)^{<\aleph_0}$ with μ colours, T contains S isomorphic to $\mathrm{ds}(\alpha)$ so that $c{\restriction}S^{<\aleph_0}$ is n-end uniform. For c with domain T^n this is equivalent to finding $S \subseteq T$ isomorphic to $\mathrm{ds}(\alpha)$ so that $c{\restriction}S^n$ depends only on the equivalence class of the defined relation, so in particular $T \to (\mathrm{ds}(\alpha))^n_{\mu,\aleph_0}$. We also draw a conclusion on colourings of n-tuples from a scattered linear order.

0. Introduction

This paper deals with a Ramsey-type theorem for scaterred order types. We dedicate this section to some general background. A Ramsey-type theorem begins with a target element φ and a fixed number of colors, μ. The statement asserts that there exists another element ψ (of the same type) so that for every coloring of ψ by μ colors, one can find a monochromatic φ-copy included in ψ.
The simplest example is the class of infinite cardinals, and coloring functions defined on singletons. For instance, $\mu^+ \to (\mu^+)^1_\mu$ holds for every infinite cardinal μ. It means that for any coloring $c : \mu^+ \to \mu$ there exists a copy of μ^+ (namely, a subset of μ^+ whose cardinality is μ^+) which is monochromatic under c.
This simple version works for order types as well. Given any order type θ (this is the target), and a fixed number of colors μ, one can find an order type ψ so that $\psi \to (\theta)^1_\mu$ (i.e., for every coloring $c : \psi \to \mu$ there exits a monochromatic copy of θ in ψ).
We concentrate, throughout the paper, in the interesting class of scaterred order types. Let us start with the following:

Definition 0.1. Scaterred order types.

(1) η is the order type of the set of rational numbers $(\mathbb{Q}, <)$
(2) For two order types φ, ψ we say that $\varphi \leq \psi$ iff there is an order preserving embedding of φ into ψ

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(3) An order type φ is scattered when $\neg(\varphi \leq \eta)$

The investigation of scaterred order types goes back to Hausdorff. This definition is a "negative" one. Hausdorff proved in [**3**] that the class of scaterred order types is characterized by a simple "positive" closure property. This class is the smallest class which contains $0, 1$ and is closed under well ordered and reverse well ordered sums. In fact, as a consequence of Hausdorff's proof we get that every linear ordrer is a dense sum of scattered ordered types (see as well [**5**]).

We shall use the followin notation:

Notation 0.2. The Erdös-Rado arrows.

(1) $\psi \to (\varphi)^{\ell}_{\mu}$ means that for every set S such that $\mathrm{otp}(S, <) = \psi$ and each coloring $c : [S]^{\ell} \to \mu$, there is an ordinal $i < \mu$ and a subset $T \subseteq S$ so that $\mathrm{otp}(T, <) = \varphi$ and $c \restriction [T]^{\ell} = \{i\}$
(2) $\psi \nrightarrow (\varphi)^{\ell}_{\mu}$ means that the statment $\psi \to (\varphi)^{\ell}_{\mu}$ does not hold

It is easy to show that if $\ell = 1$ (i.e., the colorings are defined on singletons) and μ is finite, then $\psi \to (\varphi)^{\ell}_{\mu}$ holds in the class of scattered order types. Trying to generalize it, we encounter with two problems. First, infinite amount of colors poses a limitation (in the case of scattered order types), even when using just $\aleph_0$ colors. Second, dealing with ℓ-tuples with $\ell > 1$ becomes much more complicated. For the first problem, $\psi \nrightarrow (\varphi)^1_{\omega}$ is exemplified by $\varphi = 1 + (\omega^* + \omega) + (\omega^* + \omega)^2 + \ldots$ (recall that if $\theta = \mathrm{otp}(S, <)$ then θ^* is $\mathrm{otp}(S, >)$). For the second problem, $\psi \nrightarrow (\omega^* + \omega)^2_2$, so we fail even when trying to use pairs.
Nevertheless, one can still prove positive results for infinitely many colors and ℓ-tuples, even when dealing with scattered order types. Aiming to these results, we need again a bit of notation:

Notation 0.3. Square brackets.

(1) $\psi \to [\varphi]^{\ell}_{\mu}$ means that for every set S such that $\mathrm{otp}(S, <) = \psi$ and each coloring $c : [S]^{\ell} \to \mu$, there is an ordinal $i < \mu$ and a subset $T \subseteq S$ so that $\mathrm{otp}(T, <) = \varphi$ and $i \notin c \restriction [T]^{\ell}$
(2) $\psi \to [\varphi]^{\ell}_{\lambda,\mu}$ means that for every set S such that $\mathrm{otp}(S, <) = \psi$ and each coloring $c : [S]^{\ell} \to \lambda$, there is a subset $X \subseteq \lambda$, $|X| = \mu$ and a subset $T \subseteq \{x \in S : c(x) \in X\}$ such that $\mathrm{otp}(T, <) = \varphi$

The former property in the above definition is a property of omitting a color, the latter property is the main concern of this paper. Notice that if $\psi \to [\varphi]^{\ell}_{\lambda,\mu}$ and $\kappa \leq \mu$, then $\psi \to [\varphi]^{\ell}_{\lambda,\kappa}$. Consequently, we may succeed even with infinite number of colors and colorings of ℓ-tuples, if we decrease κ. In particular, $\psi \to [\varphi]^{\ell}_{\lambda,1}$ is equivalent to $\psi \to (\varphi)^{\ell}_{\lambda}$.
In the general case (with no restriction to scattered order types) we can get both positive and negative results. For example, $\psi \to [\varphi]^{\ell}_{\mu,2}$ was proved by Shelah in [**6**], for every infinite μ and any natural number ℓ. On the other hand, it is consistent to have an order type θ of cardinality $\aleph_1$, such that $\psi \nrightarrow [\theta]^2_{\aleph_1}$ as shown by Hajnal and Komjáth in [**2**].
Under these considerations, we seek for ZFC theorems in the class of scattered order types. It was proved in [**4**] that $\psi \to [\varphi]^1_{\mu,\aleph_0}$ for such types. We generalize

it, to yield the relation $\psi \to [\varphi]^{\ell}_{\mu,\aleph_0}$ for every $\ell \in \omega$. Notice that $\psi \nrightarrow (\varphi)^1_{\aleph_0}$, so the subscript $\mu, \aleph_0$ is well motivated.

1. Some Definitions and Notation

This paper is a natural continuation of [**4**] in which Shelah and Komjáth prove that for any scattered order type φ and cardinal μ there exists a scattered order type ψ such that $\psi \to [\varphi]^1_{\mu,\aleph_0}$. This was proved by a theorem on colourings of well founded trees. By Hausdorff's characterization (see [**3**] and [**5**] and the introduction above) every scattered order type can be embedded in a well founded tree, so we can deduce a natural generalization of their theorem to the n-ary case, i.e for every scattered order type φ, $n < \omega$, and cardinal μ there is a scattered order type ψ such that $\psi \to [\varphi]^n_{\mu,\aleph_0}$.
We start with a few definitions.

Definition 1.1. For an ordinal α we define $\mathrm{ds}(\alpha) = \{\eta : \eta$ a decreasing sequence of ordinals $< \alpha\}$. By $\mathrm{ds}(\infty)$ we mean the class of decreasing sequences of ordinals.

We say $T \subseteq \mathrm{ds}(\infty)$ is a tree when T is non-empty and closed under initial segments. T, S will denote trees. For $S \subseteq T \subseteq \mathrm{ds}(\infty)$ we say that S is a subtree of T if it is also a tree. We use the following notation:

Notation 1.2.
(1) For $\eta, \nu \in \mathrm{ds}(\infty)$ by $\eta \cap \nu$ we mean $\eta{\restriction}\ell$ where ℓ is maximal such that $\eta{\restriction}\ell = \nu{\restriction}\ell$.
(2) For $\eta \in \mathrm{ds}(\infty)$ and a tree $T \subset \mathrm{ds}(\infty)$ we define

$$\eta^\frown T = \{\rho : \rho \trianglelefteq \eta \vee (\exists \nu \in T)(\rho = \eta^\frown \nu)\}$$

Note that for $\eta \in \mathrm{ds}(\infty \backslash \{\langle\rangle\})$ and $\{\langle\rangle\} \subsetneq T \subseteq \mathrm{ds}(\infty)$ if $\eta(\lg(\eta)-1) > \sup\{\rho(0) : \rho \in T\}$ then $\eta^\frown T \subseteq \mathrm{ds}(\infty)$.

Definition 1.3. We define the following four binary relations on $\mathrm{ds}(\infty)$:
(1) Let $<^1_{\ell x}$ be the two place relation on $\mathrm{ds}(\infty)$ defined by $\eta <^1_{\ell x} \nu$ iff one of the following: $(\exists \ell)(\eta(\ell) < \nu(\ell)$ and $\eta{\restriction}\ell = \nu{\restriction}\ell)$ <u>or</u> $\eta \triangleleft \nu$.
(2) Let $<^2_{\ell x}$ be the two place relation on $\mathrm{ds}(\infty)$ defined by $\eta <^2_{\ell x} \nu$ iff one of the following: $(\exists \ell)(\eta(\ell) < \nu(\ell)$ and $\eta{\restriction}\ell = \nu{\restriction}\ell)$ <u>or</u> $\nu \triangleleft \eta$.
(3) $<^*_{\ell x} = <^1_{\ell x} \cap <^2_{\ell x}$.
(4) Let $<^3$ be the two place relation on $\mathrm{ds}(\infty)$ defined by $\eta <^3 \nu$ iff one of the following holds: $\eta \triangleleft \nu$ or for the maximal ℓ such that $\eta{\restriction}\ell = \nu{\restriction}\ell$ if ℓ is even then $\eta(\ell) < \nu(\ell)$ and if ℓ is odd then $\eta(\ell) > \nu(\ell)$.

It is easily verified that $<^1_{\ell x}, <^2_{\ell x}$ and $<^3$ are complete orders of $\mathrm{ds}(\infty)$, and therefore $<^*_{\ell x}$ is a partial order. The following remark refers to their order types defined by $<^1_{\ell x}, <^2_{\ell x}$ and $<^3$ on $\mathrm{ds}(\infty)$ or $\mathrm{ds}(\alpha)$.

Observation 1.4.
(1) $<^1_{\ell x}, <^2_{\ell x}$ *are well orderings for* $\mathrm{ds}(\infty)$.
(2) $(\mathrm{ds}(\alpha), <^3)$ *is a scattered linear order type for every ordinal* α.
(3) *Every scattered linear order type can be embedded in* $(\mathrm{ds}(\alpha), <^3)$ *for some ordinal* α.

Proof.
(1) Let $\emptyset \neq A \subseteq \mathrm{ds}(\infty)$, we define by induction on $n < \omega$ an element a_n in the following manner $a_0 = \min\{\eta(0) : \eta \in A\}$, assume $a_0, \cdots, a_{n-1}$ have been chosen so that $\langle a_k : k < n\rangle \in \mathrm{ds}(\infty)$ and for every

$\eta \in A$ $\langle a_k : k < n\rangle \leq^2_{\ell x} \eta\restriction n$ (if $\lg(\eta) \leq n$ then $\eta\restriction n = \eta$). Now choose $a_n = \min\{\eta(n) : \eta \in A \wedge \eta\restriction n = \langle a_k : k < n\rangle\}$, if that set isn't empty. As the sequence derived in the above manner is a decreasing sequence of ordinals it is finite, say $a_0, \cdots a_{n-1}$ have been defined and a_n cannot be defined, we will show that $\bar{a} = \langle a_k : k < n\rangle$ is the minimal element of A with respect to $<^2_{\ell x}$. By the definition of the sequence there is an $\eta \in A$ so that $\eta\restriction n = \bar{a}$, if $\lg(\eta) > n$ then we could have defined a_n, so $\eta = \bar{a}$ and in particular $\bar{a} \in A$, and for every $\eta \in A\backslash\{\bar{a}\}$ we have $\bar{a} <^2_{\ell x} \eta$. Let $n_* = \min\{m : \bar{a}\restriction m \in A\}$ so $\bar{a}\restriction n_*$ is the $<^1_{\ell x}$- minimal element in A.

(2) The proof is by induction on α. Assume that $(\mathrm{ds}(\beta), <^3)$ is a scattered linear order type for every $\beta < \alpha$, and assume towards contradiction that $\mathbb{Q}$ can be embedded in $(\mathrm{ds}(\alpha), <^3)$, $q \mapsto \eta_q$. Let $C = \{\ell : (\exists p, q \in \mathbb{Q})(\eta_p(\ell) \neq \eta_q(\ell))\}$, $\ell = \min C$ and $\Gamma = \{\beta : (\exists q \in \mathbb{Q})(\eta_q(\ell) = \beta)\}$. Without loss of generality ℓ is even and for $\beta_0 = \min\Gamma$, $\beta_1 = \min\Gamma\backslash\{\beta_0\}$ there are $q_0 < q_1 \in \mathbb{Q}$ so that $\eta_{q_i}(\ell) = \beta_i$, $i = 0, 1$. Now $(q_0, q_1) = B_0 \cup B_1$ where $B_i = \{p \in (q_0, q_1) : \eta_p(\ell) = \beta_i\}$. For some $i \in \{0, 1\}$ the set B_i contains an interval of $\mathbb{Q}$ and is embedded in $(\eta_{q_i}\restriction(\ell+1)^\frown\mathrm{ds}(\beta_i), <^3)$ but this would imply that $\mathbb{Q}$ can be embedded in $(\mathrm{ds}(\beta_i), <^3)$ which is a contradiction to the induction hypothesis.

(3) By Hausdorff's characterization it is enough to show for ordinals α and β that both $A_{\alpha,\beta} = (\mathrm{ds}(\alpha), <^3) \times \beta$ and $A_{\alpha,\beta^*} = (\mathrm{ds}(\alpha), <^3) \times \beta^*$ can be embedded in $(\mathrm{ds}(\alpha + \beta \cdot 2 + 1), <^3)$. The embedding is given as follows, for $(\eta, \gamma) \in A_{\alpha,\beta}$ we have $(\eta, \gamma) \mapsto \langle \alpha + \beta + \gamma + 1, \alpha + \beta\rangle^\frown\eta$, and for $(\eta, \gamma) \in A_{\alpha,\beta^*}$ we have $(\eta, \gamma) \mapsto \langle \alpha + \beta \cdot 2, \alpha + \beta + \gamma\rangle^\frown\eta$.

□

Definition 1.5. For trees $T_1, T_2 \subset \mathrm{ds}(\infty)$, $f : T_1 \to T_2$ is an embedding of T_1 into T_2 if f preserves level, $\triangleleft$ and $<^1_{\ell x}$ (or equivalently, $<^2_{\ell x}, <^*_{\ell x}$ or $<^3$).

Observation 1.6. *For trees $T_1, T_2 \subset \mathrm{ds}(\infty)$, if $f : T_1 \to T_2$ preserves level and $\triangleleft$ then in order to determine whether f is an embedding it is enough to check for $\eta \in T_1$ and ordinals $\gamma_1 < \gamma_2$ such that $\nu_i = \eta^\frown\langle\gamma_i\rangle \in T_1$ ($i = 1, 2$) that $f(\nu_1) <^*_{\ell x} f(\nu_2)$.*

As $T \subseteq \mathrm{ds}(\infty)$ is well founded, i.e there are no infinite branches, it is natural to define a rank function. in the following definition $\mathrm{rk}_{T,\mu}$ isn't the standard rank function but for $\mu = 1$ we get a similar definition to the usual definition of a rank on a well founded tree.

Definition 1.7. For a tree $T \subset \mathrm{ds}(\infty)$ and cardinal μ define $\mathrm{rk}_{T,\mu}(\eta) : \mathrm{ds}(\infty) \to \{-1\} \cup \mathrm{Ord}$ by induction on α as follows:

(a) $\mathrm{rk}_{T,\mu}(\eta) \geq 0$ iff $\eta \in T$.
(b) $\mathrm{rk}_{T,\mu}(\eta) \geq \alpha + 1$ iff $\mu \leq |\{\gamma : \eta^\frown\langle\gamma\rangle \in T \wedge \mathrm{rk}_{T,\mu}(\eta^\frown\langle\gamma\rangle) \geq \alpha\}|$.
(c) $\mathrm{rk}_{T,\mu}(\eta) \geq \delta$ limit iff $(\forall\alpha < \delta)(\mathrm{rk}_{T,\mu}(\eta) \geq \alpha)$.

We say that $\mathrm{rk}_{T,\mu}(\eta) = \alpha$ iff $\mathrm{rk}_{T,\mu}(\eta) \geq \alpha$ but $\mathrm{rk}_{T,\mu}(\eta) \not\geq \alpha + 1$.
Denote $\mathrm{rk}_{T,\mu}(T) = \mathrm{rk}_{T,\mu}(\langle\rangle)$, and $\mathrm{rk}_T(\eta) = \mathrm{rk}_{T,1}(\eta)$.

Definition 1.8. For a tree $T \subset \mathrm{ds}(\infty)$, $\eta \in T$ and cardinals μ, λ we define the reduced rank $\mathrm{rk}^\lambda_{T,\mu}(\eta) = \min\{\lambda, \mathrm{rk}_{T,\mu}(\eta)\}$.

We first note a few properties of the rank function.

OBSERVATION 1.9. *For $\eta \in T \subset \mathrm{ds}(\infty)$ and an ordinal α we have:*

(1) *For cardinals $\mu \leq \mu'$ we have $\mathrm{rk}_{T,\mu}(\eta) \geq \mathrm{rk}_{T,\mu'}(\eta)$, and in particular $\mathrm{rk}_T(\eta) \geq \mathrm{rk}_{T,\mu}(\eta)$*
(2) $\mathrm{rk}_T(\eta) = \cup\{\mathrm{rk}_T(\eta\frown\langle\gamma\rangle) + 1 : \eta\frown\langle\gamma\rangle \in T\}$.
(3) $\mathrm{rk}_{\mathrm{ds}(\alpha)}(\langle\rangle) = \alpha$.
(4) *If $\mathrm{rk}_{T,\mu}(\eta) \geq \alpha$, $\mu \geq \alpha$ then we can embed $\eta\frown\mathrm{ds}(\alpha)$ into T, so that $\rho \mapsto \rho$ for $\rho \trianglelefteq \eta$.*

PROOF. 3 The proof is by induction on α.

For $\alpha = 0$ this is obvious. Assume correctness for every $\beta < \alpha$. $\mathrm{ds}(\alpha) = \bigcup_{\beta<\alpha} \{\langle\beta\rangle\frown\nu : \nu \in \mathrm{ds}(\beta)\}$. For every $\beta < \alpha, \nu \in \mathrm{ds}(\beta)$ we have $\mathrm{rk}_{\mathrm{ds}(\alpha)}(\langle\beta\rangle\frown\nu) = \mathrm{rk}_{\mathrm{ds}(\beta)}(\nu)$, therefore (the last equality is due to the induction hypothesis):

$$\begin{aligned}\cup\{\mathrm{rk}_{\mathrm{ds}(\alpha)}(\langle\beta\rangle\frown\nu) + 1 : \nu \in \mathrm{ds}(\beta)\} &= \cup\{\mathrm{rk}_{\mathrm{ds}(\beta)}(\nu) + 1 : \nu \in \mathrm{ds}(\beta)\} \\ &= \mathrm{rk}(\mathrm{ds}(\beta)) \\ &= \beta\end{aligned}$$

We therefore have $\mathrm{rk}(\mathrm{ds}(\alpha)) = \cup\{\beta + 1 : \beta < \alpha\} = \alpha$

4 The proof is by induction on α.

For $\alpha = 0$ there is nothing to prove.

Assume correctness for every $\beta < \alpha$, and $\mathrm{rk}_{T,\mu}(\eta) \geq \alpha$, $\alpha \leq \mu$. For $\beta < \alpha$ let $C_\beta = \{\gamma : \mathrm{rk}_{T,\mu}(\eta\frown\langle\gamma\rangle) \geq \beta\}$, so $|C_\beta| \geq \mu$ and $C_\beta \subseteq C_{\beta'}$ for $\beta' < \beta < \alpha$. By induction on $\beta < \alpha$ we can choose an increasing sequence of ordinals γ_β such that $\gamma_\beta = \min \Gamma_\beta$ where $\Gamma_\beta = \{\gamma \in C_\beta : (\forall\beta' < \beta)(\gamma > \gamma_{\beta'})\}$. Assume towards contradiction that Γ_β is empty, and let $C'_\beta = \langle\gamma_{\beta'} : \beta' < \beta\rangle \cap C_\beta$. For every $\gamma \in C_\beta \backslash C'_\beta$ (and there is such γ as $|C_\beta| \geq \mu$ whereas $|C'_\beta| \leq |\beta| < \mu$) as $\gamma \notin \Gamma_\beta$ then there is $\beta' < \beta$ such that $\gamma < \gamma_{\beta'}$, assume β' is minimal with this property, but that contradicts the choice of $\gamma_{\beta'}$.

By the induction hypothesis for every $\beta < \alpha$ there is φ_β which embeds $(\eta\frown\langle\gamma_\beta\rangle)\frown\mathrm{ds}(\beta)$ in T so that $\varphi_\beta \restriction \{\rho : \rho \trianglelefteq \eta\frown\langle\gamma_\beta\rangle\} = Id$. We now define $\varphi_\alpha : \eta\frown\mathrm{ds}(\alpha) \to T$ in the following manner, if $\rho \trianglelefteq \eta$ then $\varphi_\alpha(\rho) = \rho$, else $\rho = \eta\frown\nu$ for some $\nu \in \mathrm{ds}(\alpha)$, so there is $\beta < \alpha$ such that $\nu = \langle\beta\rangle\frown\nu_1$ with $\nu_1 \in \mathrm{ds}(\beta)$, and we define

$$\varphi_\alpha(\rho) = \varphi_\beta(\eta\frown\langle\gamma_\beta\rangle\frown\nu_1).$$

φ_α obviously preserves level.

For $\rho_1 \triangleleft \rho_2$ in $\eta\frown\mathrm{ds}(\alpha)$ if $\rho_1 \trianglelefteq \eta$ then obviously $\varphi_\alpha(\rho_1) \triangleleft \varphi_\alpha(\rho_2)$, and otherwise for some $\beta < \alpha$ we have $\rho_i = \eta\frown\langle\beta\rangle\frown\nu_i$, $i \in \{1, 2\}$, $\nu_1 \triangleleft \nu_2 \in \mathrm{ds}(\beta)$, and as φ_β is an embedding we have:

$$\varphi_\alpha(\rho_1) = \varphi_\beta(\eta\frown\langle\gamma_\beta\rangle\frown\nu_1) \triangleleft \varphi_\beta(\eta\frown\langle\gamma_\beta\rangle\frown\nu_2) = \varphi_\alpha(\rho_2).$$

For $\rho \in \eta\frown\mathrm{ds}(\alpha)$, $\gamma_1 < \gamma_2$ ordinals such that for $i = 1, 2$ $\rho_i = \rho\frown\langle\gamma_i\rangle \in \eta\frown\mathrm{ds}(\alpha)$, necessarily $\eta \trianglelefteq \rho$ and there are $\beta_1 \leq \beta_2 < \alpha$, $\nu_i \in \mathrm{ds}(\beta_i)$ so that $\rho_i = \eta\frown\langle\beta_i\rangle\frown\nu_i$. If $\beta_1 = \beta_2 = \beta$ then $\nu_1 <^*_{\ell x} \nu_2$, and as φ_β is an embedding,

$$\varphi_\alpha(\rho_1) = \varphi_\beta(\eta\frown\langle\gamma_\beta\rangle\frown\nu_1) <^*_{\ell x} \varphi_\beta(\eta\frown\langle\gamma_\beta\rangle\frown\nu_2) = \varphi_\alpha(\rho_2)$$

On the other hand, if $\beta_1 \neq \beta_2$ then $\varphi_\alpha(\rho_i)(\lg(\eta)) = \gamma_{\beta_i}$, and as $\gamma_{\beta_1} < \gamma_{\beta_2}$, also in this case $\varphi_\alpha(\rho_1) <^*_{\ell x} \varphi_\alpha(\rho_2)$.

By Observation 1.6 φ_α is an embedding, and by definition $\varphi_\alpha \restriction \{\rho : \rho \trianglelefteq \eta\} = Id$.

$\square$

The following theorem was proved By Komjáth and Shelah in [**4**]:

Theorem 1.10. *Assume α is an ordinal and μ a cardinal. Set $\lambda = (|\alpha|^{\mu^{\aleph_0}})^+$, and let $F : \mathrm{ds}(\lambda^+) \to \mu$. Then there is an embedding $\varphi : \mathrm{ds}(\alpha) \to \mathrm{ds}(\lambda^+)$ and a function $c : \omega \to \mu$ such that for every $\eta \in \mathrm{ds}(\alpha)$ of length $n+1$*

$$F(\varphi(\eta)) = c(n).$$

In what follows we will generalize the above theorem, in the process we will use infinitary logics. For the readers' convenience we include the following definitions.

Definition 1.11.
(1) For infinite cardinals κ, λ, and a vocabulary τ consisting of a list of relation and function symbols and their 'arity' which is finite, the infinitary language $\mathbb{L}_{\kappa,\lambda}$ for τ is defined in a similar manner to first order logic. The first subscript, κ, indicates that formulas have $< \kappa$ free variables and that we can join together $< \kappa$ formulas by $\bigwedge$ or $\bigvee$, the second subscript, λ, indicates that we can put $< \lambda$ quantifiers together in a row.
(2) Given a structure $\mathfrak{B}$ for τ we say that $\mathfrak{A}$ is an $\mathbb{L}_{\kappa,\lambda}$-elementary submodel (or substructure), and denote $\mathfrak{A} \prec_{\kappa,\lambda} \mathfrak{B}$ or $\mathfrak{A} \prec_{\mathbb{L}_{\kappa,\lambda}} \mathfrak{B}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ in the regular manner, and for any $\mathbb{L}_{\kappa,\lambda}$ formula φ with γ free variables and $\bar{a} \in {}^{\gamma}|\mathfrak{A}|$ we have
$$\mathfrak{B} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{A} \models \varphi(\bar{a}).$$
The Tarski-Vaught condition for a substructure $\mathfrak{A}$ of $\mathfrak{B}$ to be an elementary submodel is that for any $\mathbb{L}_{\kappa,\lambda}$-formula φ with parameters $\bar{a} \subseteq \mathfrak{A}$ we have
$$\mathfrak{B} \models \exists \bar{x} \varphi(\bar{x}\bar{a}) \Rightarrow \mathfrak{A} \models \exists \bar{x} \varphi(\bar{x}\bar{a}).$$
(3) A set X is transitive if for every $x \in X$ we have $x \subseteq X$.
(4) For every set X there exists a minimal transitive set, which is denoted by $TC(X)$, such that $X \subseteq TC(X)$.
(5) For an infinite regular cardinal κ we define
$$\mathcal{H}(\kappa) = \{X : |TC(X)| < \kappa\}.$$

Remark 1.12. In this paper the main use of infinitary logic will be in the following manner:
(1) τ will consist of the two binary relations $\in$ and $<^*$, so $|\mathbb{L}_{\kappa^+,\kappa^+}(\tau)| = 2^\kappa$.
(2) If $\kappa' \le \kappa, \lambda' \le \lambda$ and $\mathfrak{A} \prec_{\kappa,\lambda} \mathfrak{B}$ then also $\mathfrak{A} \prec_{\kappa',\lambda'} \mathfrak{B}$.
(3) $\prec_{\kappa,\lambda}$ is a transitive relation.
(4) For an infinite cardinal μ let $\kappa = \mu^+$, $\lambda = 2^\mu$, so κ is regular and $\lambda^{<\kappa} = \lambda$. Recall that for a structure $\mathfrak{B}$ and $X \subseteq \|\mathfrak{B}\|$ such that $|X| + \tau \le \lambda \le \mathfrak{B}$ there is an elementary $\mathbb{L}_{\kappa,\kappa}$ submodel $\mathfrak{A}$ of $\mathfrak{B}$ of cardinality λ which includes X.
For further reference on this point see [**1**].
(5) If $\mathfrak{A} \prec_{\kappa,\kappa} \mathfrak{B}$ and x is definable in $\mathfrak{B}$ over $\mathfrak{A}$ (i.e with parameters in $\mathfrak{A}$) by an $\mathbb{L}_{\kappa,\kappa}$-formula, then it is also definable in $\mathfrak{A}$ by the same formula. In particular if $\mathfrak{A} \prec_{\kappa,\kappa} \mathfrak{B}$ and $X \subseteq |\mathfrak{A}|, |X| < \kappa$ then $X \in |\mathfrak{A}|$.

DEFINITION 1.13. We say that two finite sequence $\langle \eta_\ell : \ell < n\rangle, \langle \nu_\ell : \ell < n\rangle$ are similar when:

(a) $\lg(\eta_\ell) = \lg(\nu_\ell)$ for $\ell < n$.
(b) $\lg(\eta_\ell \cap \eta_m) = \lg(\nu_\ell \cap \nu_m)$ for $\ell, m < n$.
(c) $(\eta_\ell <^2_{\ell x} \eta_m) \equiv (\nu_\ell <^2_{\ell x} \nu_m)$ for $\ell, m < n$ (equivalently, we could use $<^1_{\ell x}$).

OBSERVATION 1.14. (1) *Similarity is an equivalence relation and the number of equivalence classes of finite sequences is* $\aleph_0$.

(2) $\langle \eta_1, \dots, \eta_k, \nu'\rangle$, $\langle \eta_1, \dots, \eta_k, \nu''\rangle$ *are similar if*
  (a) $\eta_1 <^2_{\ell x} \eta_2 <^2_{\ell x} \dots <^2_{\ell x} \eta_k$
  (b) $\eta_k <^2_{\ell x} \nu'$
  (c) $\eta_k <^2_{\ell x} \nu''$
  (d) $\lg(\nu') = \lg(\nu'')$
  (e) $\lg(\nu' \cap \eta_k) = \lg(\nu'' \cap \eta_k)$

PROOF. (1) Similarity is obviously an equivalence relation.
The equivalence class of a finite sequence of $\mathrm{ds}(\infty)$ is determined by its length n, the lengths $\langle n_i : i < n\rangle$ of its elements, the lengths $\langle n_{i,j} : i, j < n\rangle$ of their intersections, and a permutation of n (the order of the elements according to $<^1_{\ell x}$). Therefore for each $n < \omega$ there are $\aleph_0$ equivalence classes of sequences of length n, and so the number of equivalence classes of finite sequences of $\mathrm{ds}(\infty)$ is $\aleph_0$.

(2) We need to show that $\lg(\nu' \cap \eta_i) = \lg(\nu'' \cap \eta_i)$ for every $0 < i < k$.
$\eta_k <^2_{\ell x} \nu'$ and $\eta_k <^2_{\ell x} \nu''$. If $\nu' \triangleleft \eta_k$ then we also have $\lg(\nu'' \cap \eta_k) = \lg(\nu' \cap \eta_k) = \lg(\nu') = \lg(\nu'')$ so $\nu'' \triangleleft \eta_k$, and $\nu' = \nu''$. In this case obviously the required sequences are similar, so we can assume that there is ℓ such that $\eta_k{\restriction}\ell = \nu'{\restriction}\ell$ and $\nu'(\ell) > \eta_k(\ell)$. By the same reasoning as above we deduce that $\eta_k{\restriction}\ell = \nu''{\restriction}\ell$ and $\nu''(\ell) \neq \eta_k(\ell)$ so necessarily $\nu''(\ell) > \eta_k(\ell)$. □

The last term we will need before moving on to the main theorem is that of uniformity.

DEFINITION 1.15. Let $T \subseteq \mathrm{ds}(\infty)$ be a tree, $c : [T]^{<\aleph_0} \to C$. We identify $u \in [T]^{<\aleph_0}$ with the $<^2_{\ell x}$-increasing sequence listing it.

(1) We say T is c-uniform if for any similar u_1, u_2 in $[T]^{<\aleph_0}$ we have $c(u_1) = c(u_2)$.
(2) We say T is c-end-uniform (or end-uniform for c) when
if $\eta_1 <^2_{\ell x} \eta_2 <^2_{\ell x} \dots <^2_{\ell x} \eta_k <^2_{\ell x} \rho', \rho''$ are in T and $\lg(\rho') = \lg(\rho''), \lg(\eta_k \cap \rho') = \lg(\eta_k \cap \rho'')$ (equivalently $\langle \eta_1 \dots \eta_k, \rho'\rangle, \langle \eta_1 \dots \eta_k, \rho''\rangle$ are similar-see 1.4(3))
then $c(\langle \eta_1 \dots \eta_k, \rho'\rangle) = c(\langle \eta_1, \dots, \eta_k, \rho''\rangle)$.
(3) We say T is c-n-end-uniform (or n-end-uniform for c) when for $k < \omega$, $\eta_i, \rho'_j, \rho''_j \in \mathrm{ds}(\infty)$ $(0 < i \le k, 0 < j \le n)$ such that

$$\eta_1 <^2_{\ell x}< \eta_2 <^2_{\ell x} \dots <^2_{\ell x} \eta_k <^2_{\ell x} \rho'_1 <^2_{\ell x} \dots <^2_{\ell x} \rho'_n$$

$$\eta_1 <^2_{\ell x}< \eta_2 <^2_{\ell x} \dots <^2_{\ell x} \eta_k <^2_{\ell x} \rho''_1 <^2_{\ell x}< \dots < \rho''_n$$

if those two sequences are similar then

$$c(\langle \eta_1 \dots, \rho'_1 \dots\rangle) = c(\langle \eta_1 \dots \rho''_1 \dots\rangle).$$

2. Uniforming n-place functions on $T \subset \mathrm{ds}(\alpha)$

We are now ready for the main theorem of this paper.

Main Claim 2.1. *Given a tree $S \subseteq \mathrm{ds}(\infty)$ and a cardinal μ we can find a tree $T \subseteq \mathrm{ds}(\infty)$ such that*

$(*)_1$ *for every $c : [T]^{<\aleph_0} \to \mu$ there is $T' \subseteq T$ isomorphic to S such that $c \restriction T'$ is c-end-uniform.*

$(*)_2$ $|T| < \beth_{|S|^+}(|S| + \mu)$.

Proof. We assume that $|S|, \mu$ are infinite cardinals since one of our main goals is proving a statement of the form $x \to [y]^n_{\mu,\aleph_0}$, otherwise the bound on T has to be slightly adjusted.
For each $\eta \in S$ let

$$\begin{aligned} \alpha_\eta &= \alpha_S(\eta) = \mathrm{otp}(\{\nu \in S : \nu <^2_{\ell x} \eta\}, <^2_{\ell x}), \\ \mu_\eta &= \beth_{5\alpha_\eta+1}(|S| + \mu), \\ \lambda_\eta &= \beth_3(\mu_\eta)^+. \end{aligned}$$

Note that $\mu_{\langle\rangle}, \lambda_{\langle\rangle}$ are the maximal ones, and let $\chi >> \lambda_{<>}$, and $<^*_\chi$ be a well ordering of $\mathcal{H}(\chi)$ (see 1.11(5)). By definition, for every $\eta, \nu \in S$ such that $\eta <^2_{\ell x} \nu$ we have $\mu_\eta < \mu_\nu$, and $\lambda_\eta < \lambda_\nu$ in the following we examine the relation between μ_ν and λ_η for $\eta \neq \nu$.

Observation 2.2. *For $\eta <^2_{\ell x} \nu$ we have $\mu_\nu \geq \lambda^+_\eta$.*

Proof. Since $\alpha_\nu \geq \alpha_\mu + 1$ we have:

$$\begin{aligned} \mu_\nu &= \beth_{5\alpha_\nu+1}(|S| + \mu) \\ &\geq \beth_{5(\alpha_\eta+1)+1}(|S| + \mu) \\ &= \beth_5(\mu_\eta) \\ &\geq \beth_3(\mu_\eta)^{++} \\ &= \lambda^+_\eta \end{aligned}$$

□

Let $T := \mathrm{ds}(\lambda^+_{\langle\rangle})$, we will show that T is as required. Obviously T meets requirement $(*)_2$, and let $c : [T]^{<\aleph_0} \to \mu$. Because of the many details in the following construction we bring it as a separate lemma.

Lemma 2.3. *For $\eta \in S$ we can choose M_η, T^*_η and $\nu_{\eta,n} \in T$ for $n < \omega$ with the following properties:*

(1) *M_η is an $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$-elementary submodel of $\mathbf{B} = (\mathcal{H}(\chi), \in, <^*_\chi)$.*
(2) $\|M_\eta\| = 2^{\mu_\eta}$.
(3) $S, T, c \in M_\eta$.
(4) *$M_\rho, \nu_{\rho,n} \in M_\eta$ for $\rho <^*_{\ell x} \eta$, $n < \omega$.*
(5) *Properties of T^*_η:*
   (a) *$T^*_\eta = \nu_{\eta,\lg(\eta)}{}^\frown T'$ where T' is isomorphic to $\mathrm{ds}(2^{2^{\mu_\eta}})$.*
   (b) *If $\nu', \nu'' \in T^*_\eta$ and are of the same length then they realize the same $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$-type over M_η.*
(6) *Properties of the $\nu_{\eta,n}$:*
   (a) *$\nu_{\eta,n} \in T$ is of length n.*
   (b) $\nu_{\eta,\lg(\eta)} \in M_\eta$.
   (c) $\lg(\eta) = m < n \Rightarrow \nu_{\eta,n}(m) \notin M_\eta$.

(d) $\nu_{\eta,n} \in T^*_\eta$, *and for* $n \geq \lg(\eta)$ *has at least* μ_η *immediate successors in* T^*_η.

(7) *If* $\eta = \eta_1{}^\frown\langle\alpha\rangle$, *then*

(a) $M_\eta, T^*_\eta, \nu_{\eta,n} \in M_{\eta_1}$ *for* $n < \omega$.

(b) $\nu_{\eta_1,n}, \nu_{\eta,n}$ *realize the same* $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$*-type over* $\{M_\rho, \nu_{\rho,n} : n < \omega, \rho <^*_{\ell x} \eta\}$.

(c) $\nu_{\eta_1,n} = \nu_{\eta,n}$ *for* $n \leq \lg\ (\eta_1)$.

(d) $\nu_{\eta,n} <^*_{\ell x} \nu_{\eta_1,n}$ *for* $n = \lg(\eta)$.

(e) $\nu_{\eta,\lg(\eta)} = \nu_{\eta,\lg\eta_1}{}^\frown\langle\gamma\rangle$ *for some* γ.

(f) *If* $\eta' = \eta_1{}^\frown\langle\alpha'\rangle$ *with* $\alpha' < \alpha$ *then* $\nu_{\eta',\lg(\eta')} <^*_{\ell x} \nu_{\eta,\lg(\eta)}$.

Proof. We show a construction for such a choice by induction on $<^1_{\ell x}$, yes, $<^1_{\ell x}$ not $<^2_{\ell x}$.
As the induction is on $<^1_{\ell x}$ the base of the induction is the case $\eta = \langle\rangle$. First choose $M_{\langle\rangle} \prec_{\mathbb{L}_{\mu^+_{\langle\rangle},\mu^+_{\langle\rangle}}} \mathbf{B}$ of cardinality $2^{\mu_{\langle\rangle}}$, so that $S, T, c \in M_{\langle\rangle}$ (this can be done, see Remark 1.12). The number of $\mathbb{L}_{\mu^+_{\langle\rangle},\mu^+_{\langle\rangle}}$ formulas $\varphi(\bar{x}, \bar{a})$ where $\bar{a} \subseteq {}^{\mu^+_{\langle\rangle}>}M_{\langle\rangle}$ (sequences of length $< \mu^+_{\langle\rangle}$ in $M_{\langle\rangle}$) is $\leq (2^{\mu_{\langle\rangle}})^{\mu_{\langle\rangle}} = 2^{\mu_{\langle\rangle}}$ hence the number of $\mathbb{L}_{\mu^+_{\langle\rangle},\mu^+_{\langle\rangle}}$-types over $M_{\langle\rangle}$ is at most $\mu' = 2^{2^{\mu_{\langle\rangle}}}$, so we color $T = \mathrm{ds}(\lambda^+_{\langle\rangle})$ by $\leq \mu'$ colors, $c_{\langle\rangle} : T \to \mu'$, so that for $\rho \in T$ its color, $c_{\langle\rangle}(\rho)$, codes the $\mathbb{L}_{\mu^+_{\langle\rangle},\mu^+_{\langle\rangle}}$-type which ρ realizes in $\mathbf{B}$ over $M_{\langle\rangle}$. As

$$((\beth_2(\mu_{\langle\rangle}))^{\mu'^{\aleph_0}})^+ = \beth_3(\mu_{\langle\rangle})^+ = \lambda_{\langle\rangle}$$

by Theorem 1.10 there is an embedding of $\mathrm{ds}(\beth_2(\mu_{\langle\rangle}))$ in T, and define $T^*_{\langle\rangle}$ to be its image, so that types of sequences from $T^*_{\langle\rangle}$ depend only on their length. We choose representatives $\langle\nu_{\langle\rangle,n} : 0 < n < \omega\rangle$ from each level larger than 0 so that for $n > 0$ $\nu_{\langle\rangle,n}$ and has at least $\mu_{\langle\rangle}$ immediate successors in $T^*_{\langle\rangle}$ and satisfies 6(c). The latter can be done by cardinality considerations, $\|M_{\langle\rangle}\| = 2^{\mu_{\langle\rangle}}$, while the cardinality of levels in $T^*_{\eta_{\langle\rangle}}$ is $\beth_2(\mu_{\langle\rangle})$. We let $\nu_{\langle\rangle,0} = \langle\rangle$.
It is easily verified that for $\eta = \langle\rangle$ all the requirements of the construction are met.
We now show the induction step.
Assume $\eta = \eta_1{}^\frown\langle\alpha_1\rangle$, $\lg(\eta_1) = r$, and that we have defined for η_1 (and below by $<^1_{\ell x}$) and we define for η.

$\circledast_1$ Let $A_\eta = \{M_\rho, \nu_{\rho,n} : n < \omega, \rho <^*_{\ell x} \eta\}$.

For any $\rho <^*_{\ell x} \eta$ if $\rho = \eta_1{}^\frown\langle\alpha\rangle$ for some $\alpha < \alpha_1$ then from requirement (7)(a) of the construction for ρ we have $M_\rho \in M_{\eta_1}$, and also for all $n < \omega$ $\nu_{\rho,n} \in M_{\eta_1}$, else $\rho <^*_{\ell x} \eta_1$ therefore from requirement (4) of the construction for η_1 we have for all $n < \omega$ $\nu_{\rho,n} \in M_{\eta_1}$, and $M_\rho \in M_{\eta_1}$. So $A_\eta \subseteq M_{\eta_1}$, and $|A_\eta| \leq \mu_{\eta_1}$, so A_η is definable by an $\mathbb{L}_{\mu^+_{\eta_1},\mu^+_{\eta_1}}$-formula with parameters in M_{η_1}, so we have:

$\circledast_2$ $A_\eta \subseteq M_{\eta_1}, |A_\eta| \leq \mu_\eta \leq \mu_{\eta_1}$, therefore $A_\eta \in M_{\eta_1}$.

For every $n < \omega$ let

$\circledast_3$ $\varphi_n(x) = \varphi_{\mu_{\eta_1},n}(x) = \bigwedge($ the $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta} -$ type which $\nu_{\eta_1,n}$ realizes over $A_\eta)$

And let

$\circledast_4$ $T_\varphi = \{\rho \in T : \mathbf{B} \models \varphi_{\lg(\rho)}(\rho)\}$.

As the cardinality of the $\mathbb{L}_{\mu_\eta^+,\mu_\eta^+}$-type of any $\nu \in \mathbf{B}$ over A_η is at most 2^{μ_η} which is less than μ_{η_1}, for every $n < \omega$ we have that φ_n is an $\mathbb{L}_{\mu_{\eta_1}^+,\mu_{\eta_1}^+}$-formula and therefore T_φ is definable in $M_{\mu_{\eta_1}}$ by an $\mathbb{L}_{\mu_{\eta_1}^+,\mu_{\eta_1}^+}$-formula, namely

$$\rho \in T_\varphi \leftrightarrow \Big(\rho \in T \wedge \big(\bigvee_{n<\omega} (\lg(\rho) = n \wedge \varphi_n(\rho))\big)\Big)$$

So

$\circledast_5$ $T_\varphi \in M_{\eta_1}$ and for every $n < \omega$ we obviously have $\nu_{\eta_1,n} \in T_\varphi$.

Recall that for all $n < \omega$ $\nu_{\eta_1,n} \in T^*_{\eta_1}$, so for any $\rho \in T^*_{\eta_1}$ of length n, we have that ρ realizes the same $\mathbb{L}_{\mu_{\eta_1}^+,\mu_{\eta_1}^+}$-type over M_{η_1} as $\nu_{\eta_1,n}$ so in particular they realize the same $\mathbb{L}_{\mu_\eta^+,\mu_\eta^+}$-type over A_η, so $\rho \in T_\varphi$. For $m \geq n$ $\nu_{\eta_1,n}, \nu_{\eta_1,m}{\restriction}n$ are of the same length, so in particular $\varphi_m(x) \vdash \varphi_n(x{\restriction}n)$. If $\rho \in T_\varphi$, $\lg\rho = m$ so $\mathbf{B} \models \varphi_m(\rho)$ therefore $\mathbf{B} \models \varphi_n(\rho{\restriction}n)$ and therefore also $\rho{\restriction}n \in T_\varphi$. We summarize:

$\circledast_6$ T_φ is a subtree of T and $T^*_{\eta_1} \subseteq T_\varphi$.

The following point is a crucial one, we show that:

$\circledast_7$ $\mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta_1,n}) > \mu_{\eta_1}$ for every n such that $\lg(\eta_1) \leq n < \omega$.

Assume toward contradiction that $\mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta_1,m}) \leq \mu_{\eta_1}$ for some $\lg(\eta_1) \leq m < \omega$, and define for each n such that $m \leq n < \omega$:

$$\gamma_n = \mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta_1,n}) \text{ and } \gamma^*_n = \mathrm{rk}^{\mu_{\eta_1}}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta_1,n})$$

(see Definitions 1.7 and 1.8). We now prove by induction on $n \geq m$ that $\gamma_n \leq \mu_{\eta_1}$, i.e $\gamma_n = \gamma^*_n$. For $n = m$ this is our assumption, and assume that it is known for n. The following can be expressed by $\mathbb{L}_{\mu_{\eta_1}^+,\mu_{\eta_1}^+}$-formulas with parameters in M_{η_1} :

ψ_1 : 'x has $\mathrm{rk}^{\mu_{\eta_1}}_{T_\varphi,\mu_{\eta_1}}(x) = \gamma_n$'

ψ_2 : 'x has at least μ_{η_1} immediate successors y in T_φ with $\mathrm{rk}^{\mu_{\eta_1}}_{T_\varphi,\mu_{\eta_1}}(y) \geq \gamma^*_{n+1}$'

We have $\mathbf{B} \models \psi_1(\nu_{\eta_1,n})$, and since $T^*_{\eta_1} \subset T_\varphi$ (see $\circledast_6$) we also have $\mathbf{B} \models \psi_2(\nu_{\eta_1,n})$. By the induction hypothesis for η_1 we have $\nu_{\eta_1,n}, \nu_{\eta_1,n+1}{\restriction}n \in T^*_{\eta_1}$ and as they are the same length realize the same $\mathbb{L}_{\mu_{\eta_1}^+,\mu_{\eta_1}^+}$-type over M_{η_1}, so $\mathbf{B} \models \psi_1 \wedge \psi_2(\nu_{\eta_1,n+1}{\restriction}n)$, or in more detail, we have that $\mathrm{rk}^{\mu_{\eta_1}}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta_1,n+1}{\restriction}n) = \gamma_n$, i.e $\mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta_1,n+1}{\restriction}n) = \gamma_n$, and $\nu_{\eta_1,n+1}{\restriction}n$ has at least μ_{η_1} immediate successors in T_φ with reduced rank γ^*_{n+1}, so by the definition of rank (Definition 1.7) we have $\gamma_n > \gamma^*_{n+1}$. By the induction hypothesis $\gamma_n \leq \mu_{\eta_1}$, therefore also $\gamma^*_{n+1} = \gamma_{n+1}$. In particular we can deduce that $\gamma_{n+1} < \gamma_n$, so having carried out the induction we have an infinite decreasing sequence of ordinals which is a contradiction.

Recall that $\lg(\eta_1) = r$ so $\lg(\eta) = r + 1$,

$\circledast_8$ Define $\nu_{\eta,\ell} = \nu_{\eta_1,\ell}$ for $\ell \leq r$.

By 2.2 $\mu_{\eta_1} \geq \lambda^+_\eta$, by $\circledast_7$ $\mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta_1,r}) > \mu_{\eta_1}$ therefore $\mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta_1,r}) > \lambda^+_\eta$ so by definition there are $\nu \in \mathrm{Suc}_T(\nu_{\eta_1,r}) \cap T_\varphi$ satisfying $\mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu) \geq \lambda^+_\eta$, defining $\nu_{\eta,r+1}$ to be one such ν which is minimal with respect to $<^1_{\ell x}$ (this is equivalent to demanding that $\nu(r)$ is minimal) can be done by an $\mathbb{L}_{\mu_{\eta_1}^+,\mu_{\eta_1}^+}$ formula. We therefore conclude:

$\circledast_9$ We can choose $\nu_{\eta,r+1} \in \mathrm{Suc}_T(\nu_{\eta_1,r}) \cap T_\varphi \cap M_{\eta_1}$ such that
- (i) $\mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta,r+1}) \geq \lambda^+_\eta$.
- (ii) $\nu_{\eta,r+1}$ is minimal under (i) in $<^1_{\ell x}$.

As $\nu_{\eta,\lg(\eta)} \in M_{\eta_1} and\ \nu_{\eta_1,\lg(\eta)}(\lg(\eta_1)) \notin M_{\eta_1}$, we have:

$\circledast_{10}$ $\nu_{\eta,\lg(\eta)} <^1_{\ell x} \nu_{\eta_1,\lg(\eta)}$, notice that as they are the same length $<^1_{\ell x} \Rightarrow <^*_{\ell x}$.

Now for any $\rho = \eta_1 {}^\frown \langle \alpha \rangle \in S$ where $\alpha < \alpha_1$ we have that $\rho <^*_{\ell x} \eta$ and therefore $\nu_{\rho,r+1} \in A_\eta$ (see $\circledast_1$). $\nu_{\eta,\lg(\eta)}, \nu_{\eta_1,\lg(\eta)}$ realize the same $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$-type over A_η, and by requirement (7)(d) of the construction for ρ ($\lg(\rho) = \lg(\eta)$) we have $\nu_{\rho,\lg(\eta)} <^1_{\ell x} \nu_{\eta_1,\lg(\eta)}$ so also $\nu_{\rho,\lg(\eta)} <^1_{\ell x} \nu_{\eta,\lg(\eta)}$ and as above, as they are the same length $<^1_{\ell x} \Rightarrow <^*_{\ell x}$, and we therefore conclude that:

$\circledast_{11}$ If $\rho = \eta_1 {}^\frown \langle \alpha \rangle \in S$ where $\alpha < \alpha_1$ then $\nu_{\rho,\lg(\eta)} <^*_{\ell x} \nu_{\eta,\lg(\eta)}$.

Since $|\{S, t, c, \nu_{\eta_{\lg(\eta)}}\} \cup A_\eta| < 2^{\mu_\eta}$ by Remark 1.12 we can choose M_η so that

$\circledast_{12}$ $M_\eta \prec_{\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}} M_{\eta_1}$, and therefore also $M_\eta \prec_{\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}} \mathbf{B}$, of cardinality 2^{μ_η} and $\{S, t, c, \nu_{\eta_{\lg(\eta)}}\} \cup A_\eta \subseteq M_\eta$.

By the same remark we can conclude that

$\circledast_{13}$ $M_\eta \in M_{\eta_1}$.

Lastly we choose T^*_η and $\nu_{\eta,m}$ for $m > \lg(\eta)$.

We have already commented that $\mathrm{rk}_{T_\varphi,\mu_{\eta_1}}(\nu_{\eta,\lg(\eta)}) > \lambda^+_\eta$, so from Observation 1.9 we can embed $\nu_{\eta,\lg(\eta)} {}^\frown \mathrm{ds}(\lambda^+_\eta)$ into T_φ so that $\rho \mapsto \rho$ for $\rho \trianglelefteq \nu_{\eta,\lg(\eta)}$, and denote one such embedding by ψ, without loss of generality $\psi \in M_{\eta_1}$.

The number of $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$-types over M_η is at most $\mu' = 2^{2^{\mu_\eta}}$. We color $\mathrm{ds}(\lambda^+_\eta)$ in $\leq \mu'$ colors, the color of $\rho \in \mathrm{ds}(\lambda^+_\eta)$ is determined by the $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$-type which $\psi(\nu_{\eta,\lg(\eta)} {}^\frown \rho)$ realizes over M_η, call this coloring c_η. As $((\beth_2(\mu_\eta))^{\mu'^{\aleph_0}})^+ = \beth_3(\mu_\eta)^+ = \lambda_\eta$, we can use 1.10 to get an embedding θ of $\mathrm{ds}(\beth_2(\mu_\eta))$ into $\mathrm{ds}(\lambda^+_\eta)$ so that for $\rho \in \mathrm{ds}(\beth_2(\mu_\eta))$ the $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$-type that $\nu_{\eta,n+1} {}^\frown \theta(\rho)$ realizes over M_η depends only on its length. Since the set X of $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$-types over M_η is in M_{η_1} of cardinality at most $\mu' < \mu_{\eta_1}$ we have $X \subset M_{\eta_1}$, also $\mathrm{ds}(\lambda^+_\eta) \in M_{\eta_1}$ so $c_\eta \in M_{\eta_1}$ and therefore without loss of generality $\theta \in M_{\eta_1}$. We define

$\circledast_{14}$ $T^*_\eta = \nu_{\eta,\lg(\eta)} {}^\frown \theta\big(\mathrm{ds}(\beth_2(\mu_\eta))\big)$.

$T^*_\eta \in M_{\eta_1}$ and meets requirement (5) of the construction. We will now choose representatives $\langle \rho_m : 0 < m < \omega \rangle$ from each level of $\mathrm{ds}(\beth_2(\mu_\eta))$ so that $\nu_{\eta,n+1} {}^\frown \theta(\rho_m)$ has at least μ_η immediate successors in T^*_η and $\nu_{\eta,n+1} {}^\frown \theta(\rho_m)(\lg(\eta)) \notin M_{\eta_1}$, since the existence of such representatives in $\mathbf{B}$ can be expressed by an $\mathbb{L}_{\mu^+_{\eta_1},\mu^+_{\eta_1}}$-formula with parameters in M_{η_1} so without loss of generality $\rho_m \in M_{\eta_1}$ and define

$\circledast_{15}$ $\nu_{\eta,\lg(\eta)+m} = \nu_{\eta,n+1} {}^\frown \theta(\rho_m)$.

T^*_η is a subtree of T_φ therefore $\rho \in T^*_\eta$ realizes the same $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$type over A_η as $\nu_{\eta_1,\lg(\rho)}$. The $\nu_{\eta,n}$ for $n > \lg(\eta)$ were chosen to satisfy (6)(c)-(d) so in particular they are in T_φ, and therefore realize the same $\mathbb{L}_{\mu^+_\eta,\mu^+_\eta}$-type over A_η as $\nu_{\eta_1,n}$. By the induction hypothesis we have already constructed for η_1 so for all n we have $\lg(\nu_{\eta,n}) = \lg(\nu_{\eta_1,n}) = n$ so also (6)(a) is satisfied. Requirements (1)-(4) and (6)(b) of the construction are taken care of by $\circledast_{12}$. $\circledast_7$-$\circledast_{11}$, $\circledast_{13}$ and $\circledast_{15}$ guarantee requirement (7). □

All that is left in order to complete the proof of the claim is to show that $\{\nu_{\eta,\lg(\eta)} : \eta \in S\}$ is end-uniform with respect to c.

Let $\eta_1 <^2_{\ell x} \eta_2 <^2_{\ell x} \ldots <^2_{\ell x} \eta_k <^2_{\ell x} \rho', \rho''$, be as in 1.15(2); without loss of generality

$\rho' <^*_{\ell x} \rho''$. Let $t = \lg(\rho' \cap \rho'')$, $\mu' = \mu^+_{\rho'}$ and $A = \{\nu_{\rho,\lg\rho} : \rho <^*_{\ell x} \rho' \restriction (t+1)\}$.
We first show that for every $i \leq k$ $\eta_i <^*_{\ell x} \rho' \restriction (t+1)$ so that $\nu_{\eta_i.\lg(\eta_i)} \in A$. As $\eta_i <^2_{\ell x} \rho'$ and $\lg(\eta_i \cap \rho'') = \lg(\eta_i \cap \rho')$ so $\rho' \ntriangleleft \eta_i$, therefore there is ℓ_i such that $\eta_i \restriction \ell_i = \rho' \restriction \ell_i$ and $\eta_i(\ell_i) < \rho'(\ell_i)$, but then $\eta_i \restriction \ell_i = \rho'' \restriction \ell_i$ i.e $\rho' \restriction \ell_i = \rho'' \restriction \ell_i$ so $\ell_i \leq t$ (and $\eta_i(\ell_i) < \rho''(\ell_i)$) and $\eta_i <^*_{\ell x} \rho' \restriction (t+1)$.
We now prove by induction on $\ell \in [t, \lg(\rho')]$ that $\nu_{\rho' \restriction \ell, \lg\rho'}$ and $\nu_{\rho' \restriction t, \lg\rho'}$ realize the same $\mathbb{L}_{\mu',\mu'}$-type over A. For $\ell = t$ this is obvious. Let us assume correctness for ℓ and prove for $\ell+1$. For every $n < \omega$ by (7)(b) of the construction $\nu_{\rho' \restriction \ell, n}, \nu_{\rho' \restriction (\ell+1), n}$ realize the same $\mathbb{L}_{\mu^+_{\rho' \restriction (\ell+1)}, \mu^+_{\rho' \restriction (\ell+1)}}$-type over $\{M_\rho, \nu_{\rho,n} : \rho <^*_{\ell x} \rho' \restriction (\ell+1)\}$ and in particular over A, for if $\rho <^*_{\ell x} \rho' \restriction (t+1)$ then also $\rho <^*_{\ell x} \rho' \restriction (\ell+1)$. So $\nu_{\rho' \restriction \ell, \lg\rho'}, \nu_{\rho' \restriction (\ell+1), \lg\rho}$ realize the same $\mathbb{L}_{\mu^+_{\rho' \restriction (\ell+1)}, \mu^+_{\rho' \restriction (\ell+1)}}$-type so also the same $\mathbb{L}_{\mu',\mu'}$-type over A, and from the induction hypothesis $\nu_{\rho' \restriction t, \lg\rho'}$ and $\nu_{\rho' \restriction \ell, \lg\rho'}$ realize the same $\mathbb{L}_{\mu',\mu'}$-type over A. Similarly we show for ρ'', so $\nu_{\rho',\lg\rho'}$ and $\nu_{\rho'',\lg\rho''}$ realize the same $\mathbb{L}_{\mu^+_{\eta_1}, \mu^+_{\eta_1}}$-type over A.
From the above we can deduce that in particular

$$c(\langle \nu_{\eta_1,\lg(\eta_1)}, \ldots, \nu_{\eta_k,\lg(\eta_k)}, \nu_{\rho',\lg(\rho')} \rangle) = c(\langle \nu_{\eta_1,\lg(\eta_1)}, \ldots, \nu_{\eta_k,\lg(\eta_k)}, \nu_{\rho'',\lg(\rho'')} \rangle).$$

□

Conclusion 2.4. Given a tree $S \subseteq \mathrm{ds}(\infty)$ and $n(*) < \omega$ and μ we can find a tree $T \subseteq \mathrm{ds}(\infty)$ such that:

- $(*)_1$ For every $c : [T]^{<\aleph_0} \to \mu$ there is $S' \subseteq T$ isomorphic to S such that S' is $n(*)$-end-uniform for c.
- $(*)_2$ In particular, for every $c : [T]^{n(*)} \to \mu$ is $S' \subseteq T$ isomorphic to S such that $c \restriction S'$ depends only on the equivalence classes of the equivalence relation defined in 1.13.
- $(*)_3$ $|T| < \beth_{1,n(*)}(|S|, \mu)$ (see Definition 2.5 below).

Proof. Let S, μ be as above. Since for $|S|, \mu \geq \aleph_0$ we have that $\beth_{1,n(*)}(|S|, \mu^{\aleph_0}) = \beth_{1,n(*)}(|S|, \mu)$, replacing μ with $\mu^{\aleph_0}$ gives the same bound, and we can therefore assume that $\mu = \mu^{\aleph_0}$.
Let $\langle h_n : n < \omega \rangle$ be the equivalence classes of the similarity relationship on finite sequences of $\mathrm{ds}(\infty)$ (see 1.14(1)), and let $f : {}^\omega(\mu \cup \{-1\}) \to \mu$ be one-to-one and onto.
We construct by induction a sequence $\langle T_n : n < \omega \rangle$ so that $T_0 = S$, and for every $n > 0$:

- (a) $|T_n| < \beth_{1,n}(|S|, \mu)$
- (b) T_{n-1}, T_n, μ correspond to S, T, μ in Theorem 2.1.
- (c) For every $c : [T_n]^{<\aleph_0} \to \mu$ there is $S' \subseteq T_n$ isomorphic to S such that S' is n-end-uniform for c.

By Theorem 2.1 we can obviously construct such a sequence satisfying clauses (a), (b), We will show by induction on n that for this sequence also clause (c) holds. For $n = 1$ this is Theorem 2.1. Assume correctness for n and let $c : [T_{n+1}]^{<\aleph_0} \to \mu$. By ($b$) there is $T' \subseteq T_{n+1}$ isomorphic to T_n so that T' is end-uniform for c. Let $\varphi : T_n \to T'$ be an isomorphism and let $d : [T']^{<\aleph_0} \to {}^\omega(\mu \cup \{-1\})$ as follows: for $\bar\rho = \langle \rho_1 \ldots \rho_k \rangle$ where $\rho_1 <^2_{\ell x} \rho_2 <^2_{\ell x} \ldots <^2_{\ell x} \rho_k$ and $m < \omega$

$$d(\bar\rho)(m) = \begin{cases} c(\bar\rho^\frown\langle\eta\rangle) & \text{if } \bar\rho^\frown\langle\eta\rangle \in h_m \text{ for some } \eta \\ \text{-1} & \text{otherwise} \end{cases}$$

d is well defined as T' is end-uniform for c, and by defining $\varphi(\rho_1,\dots\rho_k)=(\varphi(\rho_1),\dots\varphi(\rho_k))$ for $\rho_1,\dots\rho_k\in T_n$ we have $f\circ d\circ\varphi:[T_n]^{<\aleph_0}\to\mu$, so by the induction hypothesis there is $T''\subseteq T_n$ isomorphic to S so that T'' is n-end-uniform for $f\circ d\circ\varphi$. We claim that $S'=\varphi(T'')$ is isomorphic to S and that S' is $n+1$-end-uniform for c. As T'' is isomorphic to S and φ is an isomorphism S' is obviously isomorphic to S. Let the following sequences in S' be similar,

$$\eta_1 <^2_{\ell x}< \eta_2 <^2_{\ell x} \dots <^2_{\ell x} \eta_k <^2_{\ell x} \rho'_1 <^2_{\ell x} \dots <^2_{\ell x} \rho'_{n+1}$$

$$\eta_1 <^2_{\ell x}< \eta_2 <^2_{\ell x} \dots <^2_{\ell x} \eta_k <^2_{\ell x} \rho''_1 <^2_{\ell x}< \dots < \rho''_{n+1}$$

So in T'' the following sequences are similar:

$$\varphi^{-1}(\eta_1\dots\rho'_1\dots\rho'_n)=(\varphi^{-1}(\eta_1)\varphi^{-1}(\rho'_1)\dots\varphi^{-1}(\rho'_n))$$

$$\varphi^{-1}(\eta_1\dots\rho''_1\dots\rho''_n)=(\varphi^{-1}(\eta_1)\varphi^{-1}(\rho''_1)\dots\varphi^{-1}(\rho''_n))$$

so $f\circ d\circ\varphi(\varphi^{-1}(\eta_1\dots\eta_k,\rho'_1\dots\rho'_n))=f\circ d\circ\varphi(\varphi^{-1}(\eta_1\dots\eta_k,\rho''_1\dots\rho''_n))$. Therefore we have $f(d(\eta_1\dots\eta_k,\rho'_1\dots\rho'_n))=f(d(\eta_1\dots\eta_k,\rho''_1\dots\rho''_n))$, and as f is one-to-one, $d(\eta_1\dots\eta_k,\rho'_1\dots\rho'_n)=d(\eta_1\dots\eta_k,\rho''_1\dots\rho''_n)$, and therefore $c(\eta_1\dots\eta_k,\rho'_1\dots\rho'_{n+1})=c(\eta_1\dots\eta_k,\rho''_1\dots\rho''_{n+1})$, and $(*)_1$-$(*)_3$ are easily verified. $\square$

Definition 2.5. For cardinals $\lambda\geq\aleph_0$ and μ define $\beth_{1,\alpha}(\lambda,\mu)$ by induction on α. $\beth_{1,0}(\lambda,\mu)=\beth_0(\lambda)=\lambda$, $\beth_{1,\alpha+1}(\lambda,\mu)=\beth_{\beth_{1,\alpha}(\lambda,\mu)^+}(\beth_{1,\alpha}(\lambda,\mu)+\mu)$, and for a limit ordinal α $\beth_{1,\alpha}(\lambda,\mu)=\sum\limits_{\beta<\alpha}\beth_{1,\beta}(\lambda,\mu)$.

We end with a conclusion for scattered order types.

Conclusion 2.6. For a scattered order type φ, a cardinal μ and $n<\omega$, there is a scattered order type ψ so that $\psi\to[\varphi]^n_{\mu,\aleph_0}$.

Proof. Given a scattered order type φ, a cardinal μ and $n<\omega$ by Observation 1.4(3) we can embed φ in $(\mathrm{ds}(\alpha),<^3)$ for some ordinal α. By Conclusion 2.4$(*)_2$ above there is an ordinal λ and a tree $T\subset\mathrm{ds}(\lambda)$ so that for every coloring $c:T^n\to\mu$ there is a subtree $S\subseteq T$ isomorphic to $\mathrm{ds}(\alpha)$ so that $c{\restriction}S$ depends only on the equivalence class of similarity. Noting the above Observation, as $(T,<^3)$ is a scattered order, and as there are only $\aleph_0$ equivalence classes, we are done. $\square$

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