THE FIRST ALMOST FREE WHITEHEAD GROUP
SH914

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ABSTRACT. Assume G.C.H. and κ is the first uncountable cardinal such that there is a non-free κ-free abelian Whitehead group of cardinality κ. We prove that if all κ-free Abelian group of cardinality κ are Whitehead then κ is necessarily an inaccessible cardinal.

§ 0. INTRODUCTION

For the non-specialist reader, note that we deal exclusively with abelian groups, we call such G Whitehead <u>when</u>: if $\mathbb{Z} \subseteq H$ and $H/\mathbb{Z} \cong G$ then $\mathbb{Z}$ is a direct summand of H. Why this property is worthwhile and generally on the subject see the book of Eklof-Mekler [EM02]. Recall G is a free abelian group if it is the direct sum of copies of $(\mathbb{Z}, +)$. All free abelian groups are Whitehead and every Whitehead group is "somewhat free" ($\aleph_1$-free, see below). Possibly (i.e. consistently with ZFC) every Whitehead group is free and possibly not (in fact, it seems that almost any behaviour is possible).

Recall that G.C.H., the generalized continuum hypothesis, says that $2^\lambda = \lambda^+$ for every infinite λ and "λ is an inaccessible cardinal" means that: it is strong limit (i.e. $\mu < \lambda \Rightarrow 2^\mu < \lambda$) and regular, i.e. $\lambda > \sum\limits_{i<\kappa} \lambda_i$: when $\kappa < \lambda$ and $i < \kappa \Rightarrow \lambda_i < \lambda$.

It is well known that inaccessible cardinals are large, e.g. it is not provable in ZFC, the usual axioms of set theory that such cardinals exists.

Also being "κ-free of cardinality κ" is a central notion where we say an abelian group G is κ-free when the pure closure inside G of any subgroup generated by $< \kappa$ elements is free. This explains the interest in the result, which restricts the behaviour. That is, assume κ is the first cardinal λ such that there is a non-free λ-free abelian Whitehead group of cardinality λ. The conclusion is, assuming GCH and κ not strongly inaccessible, that not all such abelian groups are Whitehead.

Set theorists may recall that in [Sh:667, §1] it is proved that if μ is strong limit singular, $\lambda = \mu^+ = 2^\mu$ and $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\mu)\}$ is stationary, <u>then</u> though $\diamondsuit_S$ may (= consistently) fail, still we can prove a relative of $\diamondsuit_S$ sufficient for constructing abelian groups G of cardinality λ related to satisfying $\text{Hom}(G, \mathbb{Z}) \neq \{0\}$. We prove here in 1.5 somewhat more and use it in the proof of the Theorem 1.1. This claim is complementary to [Sh:587] which shows that consistently there is such regular (in fact strongly inaccessible) λ.

Date: September 25, 2011.

1991 *Mathematics Subject Classification.* [2010] Primary 03E75; Secondary: 03C60, 20K20.

Key words and phrases. Abelian group, Whitehead group, almost free, stationary sets, λ-sets.

Research supported by German-Israeli Foundation for Scientific Research/Development Grant No. I-706-54.6/2001. I would like to thank Alice Leonhardt for the beautiful typing. First Typed - 05/Nov/17.

{5n.0A}

Convention 0.1. Saying group we mean abelian group.

In [Sh:587] we answer

{5n.0D}

Question 0.2. (Göbel) Assuming GCH can there be a regular κ such that:

$\boxdot_\kappa$ (a) $\kappa = \text{cf}(\kappa) > \aleph_0$ and there are κ-free not free groups of cardinality κ
(b) every κ-free group of cardinality κ is a Whitehead group.

Moreover by [Sh:587] it is consistent that: G.C.H. + for some strongly inaccessible κ we have $\circledast_\kappa$ where the statement $\circledast_\kappa$ is defined by:

$\circledast_\kappa$ (a) κ is regular uncountable and there are κ-free non-free (abelian) groups of cardinality κ
(b) every κ-free (abelian) group of cardinality κ is a Whitehead group
(c) every Whitehead group of cardinality $< \kappa$ is free.

[For clause (c), the following sufficient condition is used there: for every regular uncountable $\lambda < \kappa$ we have $\diamondsuit^*_\lambda$. The proof starts with κ weakly compact and adds no new sequence of length $< \kappa$, so e.g. starting there with **L** this holds.]

A natural question is whether

{5n.0H}

Question 0.3. Assume G.C.H., is it consistent that there is an accessible κ such that $\circledast_\kappa$ holds? In other words there is cardinal κ satisfying the following and the first such κ is accessible:

$(*)$ there is a κ-free non-free Whitehead group of cardinality κ.

Now Theorem 1.1 says that no. But another natural question is:

{5n.0K}

Question 0.4. Assume G.C.H. Can there be a κ such that $\boxdot_\kappa$ but is the first such κ accessible?

This is problem F4 of [EM02] and it remains open.

{0z.11}

Definition 0.5. 1) An abelian group G is free if G is the direct sum of copies of $\mathbb{Z}$.
2) For a cardinality $\kappa \geq \aleph_1$, an abelian group G is κ-free <u>when</u> every subgroup of G of cardinality $< \kappa$ is free.
3) $\bar{G} = \langle G_\alpha : \alpha < \lambda \rangle$ is a filtration of the abelian group G of cardinality λ if G_α is a subgroup of G of cardinality $< \lambda$, increasing continuous with α and $G = \cup\{G_\alpha : \alpha < \lambda\}$.
4) G, an abelian group of cardinality κ, is strongly κ-free <u>when</u> it is κ-free and for every subgroup G' of G of cardinality $< \kappa$ there is a subgroup G'' of cardinality $< \kappa$ such that $G' \subseteq G''$ and G/G'' is κ-free.

We thank the referees for many helpful comments, in particular for making the work more self-cointained.

§ 1. The First almost free non-free Whitehead

Theorem 1.1. *(G.C.H.) Let κ be the first $\lambda > \aleph_0$ such that there is a λ-free abelian Whitehead group, not free, of cardinality λ (and we assume there is such λ). If κ is not (strongly) inaccessible, then there is a non-Whitehead group G of cardinality κ which is κ-free (and necessarily non-free).* {5n.1}

Proof. Stage A: Let G exemplify the choice of κ.

Necessarily κ is regular, by the singular compactness, see e.g. [EM02, Ch.IV,3.5]. Let $\bar{G} = \langle G_\alpha : \alpha < \kappa \rangle$ be a filtration of G so as G is κ-free necessarily every G_α is free. Without loss of generality each G_α is a pure subgroup of G. Let $S := \Gamma(\bar{G}) = \{\alpha < \kappa : \alpha$ is a limit ordinal and G/G_α is not κ-free$\}$. Now recall G is not free hence $\Gamma(\bar{G})$ is stationary.

Stage B: G is strongly κ-free.

Toward contradiction assume that not. Without loss of generality κ is a successor cardinal (why? by the theorem's assumption and κ being regular but also as for κ a limit cardinal, κ-free implies strongly κ-free).

By our present assumption toward contradiction, for some $\alpha < \kappa$ for every $\beta \in [\alpha, \kappa)$, the abelian group, G/G_β is not κ-free. Hence without loss of generality $\alpha < \lambda \Rightarrow G_{\alpha+1}/G_\alpha$ is not free and if $G_{\alpha+1}/G_\alpha$ is uncountable then for some κ_α, it is κ_α-free not free. By the theorem assumption (and as countable Whitehead groups are free), $\alpha < \kappa \Rightarrow G_{\alpha+1}/G_\alpha$ is not a Whitehead group, i.e. $\Gamma(\bar{G}) = \kappa$. But κ is a successor cardinal, so let $\kappa = \mu^+$; also $2^\mu < 2^\kappa$ as we are assuming GCH so the weak diamond holds for κ (see [DvSh:65]) so by the previous sentence and e.g. [EM02, 1.10,pg.369], we know that G is not a Whitehead group, contradiction to the assumption on G.

Stage C: Hence without loss of generality

$(*)_1$ if α is a non-limit ordinal then G/G_α is κ-free

and obviously without loss of generality

$(*)_2$ (a) if $\alpha \in S$ then $G_{\alpha+1}/G_\alpha$ is not free
  (b) if $\alpha < \kappa$ then G_α is a pure subgroup of G and is free hence G/G_α is torsion free.

Also we can choose $\bar{H} = \langle H_\alpha : \alpha \in S \rangle$ such that

$(*)_3$ (a) H_α is a subgroup of G
  (b) $H_\alpha/(H_\alpha \cap G_\alpha)$ is not free
  (c) under (a) + (b) the rank of $H_\alpha/(H_\alpha \cap G_\alpha)$ is minimal call it θ_α hence θ_α is $< \aleph_0$ or is regular uncountable $< \kappa$
  (d) the cardinality of H_α is $\leq \theta_\alpha + \aleph_0$, in fact equality holds.

Note that $|H_\alpha|$ may be $< |G_\alpha|$ so it is unreasonable to ask for $G_\alpha \subseteq H_\alpha$.

$(*)_4$ without loss of generality
  (a) $H_\alpha \subseteq G_{\alpha+1}$
  (b) $G_\alpha/(H_\alpha \cap G_\alpha)$ is free
  (c) $G_\alpha + H_\alpha$ is a pure subgroup of $G_{\alpha+1}$.

[Why? For clause (a) we can restrict $\bar{G}$ to a club. For clause (c) let H''_α be the pure closure of $H_\alpha + G_\alpha$ inside $G_{\alpha+1}$ so $H''_\alpha/G_\alpha, (H_\alpha + G_\alpha)/G_\alpha, H_\alpha/(H_\alpha \cap G_\alpha)$ has the same rank, which is θ_α. As $G_{\alpha+1}/G_\alpha$ is torsion free, also H''_α/G_α is torsion free. Hence there is $H'_\alpha \subseteq H''_\alpha$ of cardinality $\le \theta_\alpha + \aleph_0$ such that $G_\alpha + H'_\alpha$ is a pure subgroup of H''_α hence of $G_{\alpha+1}$ and $H'_\alpha + G_\alpha = H''_\alpha$ and e.g. the rank of $H'_\alpha/(H'_\alpha \cap G_\alpha)$ is the same as the rank of H''_α/G_α which is θ_α, so replacing H_α by H'_α also clause (c) holds.

For clause (b) note that G_α is free; so there is $G'_\alpha \subseteq G_\alpha$ of cardinality $\theta_\alpha + \aleph_0$ such that G_α/G'_α is free and $H_\alpha \cap G_\alpha \subseteq G'_\alpha$ and replace H_α by $H^*_\alpha = H_\alpha + G'_\alpha$ noting that $H^*_\alpha + G_\alpha = H_\alpha + G_\alpha$.]

By the hypothesis on κ, if K is a λ-free not free group of cardinality λ and $\aleph_0 < \lambda < \kappa$ then K is not a Whitehead group so clearly (recalling that Whitehead groups which are countable are free):

$(*)_5$ $H_\alpha/(G_\alpha \cap H_\alpha)$ is not Whitehead for $\alpha \in S$.

Hence by [Sh:44] or see [EM02, Ch.VI,1.13] we know that

$(*)_6$ $\neg\diamondsuit_S$.

Recall κ is regular uncountable so toward contradiction assume $\kappa = \mu^+$. Let $\sigma = \mathrm{cf}(\mu)$. But, [Sh:108] or [EM02, Ch.VI,1.13], GCH implies that $\diamondsuit_{S'}$ for every stationary $S' \subseteq \kappa \backslash S^\kappa_\sigma$ where $S^\kappa_\sigma := \{\delta < \kappa : \mathrm{cf}(\delta) = \sigma\}$.

Hence we know that

$(*)_7$ for some club E of κ we have $S \cap E \subseteq S^\kappa_\sigma$ so without loss of generality $S \subseteq S^\kappa_\sigma$, i.e., $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \sigma$.

Also without loss of generality (from "strongly κ-free")

$(*)_8$ $\delta \in S \Rightarrow \theta_\delta + \aleph_0 \ge \sigma$.

[Why? For completeness we elaborate. Let $S^1 = \{\delta \in S : \theta_\delta + \aleph_0 < \sigma\}$; first assume S^1 is stationary. For each $\delta \in S^1$ necessarily $H_\delta \cap G_\delta$ has cardinality $\le \theta_\delta + \aleph_0 < \sigma = \mathrm{cf}(\delta)$ hence for some $\alpha_\delta < \delta, H_\delta \cap G_\delta \subseteq G_{\alpha_\delta}$. By Fodor lemma for some $\alpha(*) < \kappa$ the set $S^2 := \{\delta \in S^1 : \alpha_\delta = \alpha(*)\}$ is stationary. As $\sigma = \mathrm{cf}(\mu), \kappa = \mu^+$, clearly $\mu^{<\sigma} = \mu$ recalling GCH hence $\{H_\delta \cap G_{\alpha(*)} : \delta \in S^2\}$ has cardinality $\le \mu$ hence $S^3 = \{\delta \in S^2 : H_\delta \cap G_{\alpha(*)} = H_*, \theta_\delta = \theta_*\}$ is a stationary subset of κ for some θ_* and $H_* \subseteq H_{\alpha(*)}$. Let α_ε be the ε-th member of S^3 and $H^\varepsilon = H_* + \Sigma\{H_{\alpha_\zeta} : \zeta < \varepsilon\}$ for $\varepsilon \le (\theta_* + \aleph_0)^+$. Clearly for $\varepsilon = (\theta_* + \aleph_0)^+, H^\varepsilon$ is not free and has cardinality $\le \sigma = \mathrm{cf}(\mu) < \lambda$, contradiction to "$G$ is κ-free". So necessarily S^1 is not stationary, and as we can restrict $\bar{G}$ to a club, without loss of generality $S^1 = \emptyset$ so $(*)_8$ holds indeed.]

Stage D: $\alpha \in S \Rightarrow \theta_\alpha < \mu$.
Why? Otherwise for some $\alpha \in S, \theta_\alpha = \mu$ so θ_α is infinite hence θ_α is regular and by $(*)_3$ uncountable and there is a θ_α-free not free group of cardinality θ_α hence μ is regular and there is a μ-free not free abelian group of cardinality $\mu < \kappa$, i.e. $H_\alpha/(G_\alpha \cap H_\alpha)$, hence this group is not Whitehead. So there is a sequence $\langle H^*_\varepsilon : \varepsilon \le \mu + 1\rangle$ purely increasing continuous sequence of free groups such that $\varepsilon < \mu$ implies $H^*_{\mu+1}/H^*_\varepsilon$ is free but $H^*_{\mu+1}/H^*_\mu$ is not free and is isomorphic to $H_\alpha/(G_\alpha \cap H_\alpha)$.

[Why? Let $\langle y_\varepsilon : \varepsilon < \mu\rangle$ be a free basis of G_α, so $G_\alpha = \oplus\{\mathbb{Z}y_\varepsilon : \varepsilon < \mu\}$. As $\{y_\varepsilon : \varepsilon < \mu\} \subseteq G_\alpha$ and $\mathrm{cf}(\alpha) = \mu$, there is an increasing continuous sequence $\langle \gamma_\varepsilon : \varepsilon < \mu\rangle$ of ordinals $< \alpha$ with limit α such that $y_\varepsilon \in G_{\gamma_{\varepsilon+1}}$.

Let $H^*_\varepsilon = \oplus\{\mathbb{Z}y_\zeta : \zeta < \varepsilon\}$ for $\varepsilon < \mu, H^*_\mu = G_\alpha, H^*_{\mu+1} = G_\alpha + H_\alpha$, they are as required. E.g. why is $H^*_{\mu+1}/H_\varepsilon$ free? as $G_{\alpha+1}/G_{\alpha_\varepsilon+1}$ is free by $(*)_1$, also G_α/H^*_ε is free as $\{y_\zeta + H^*_\varepsilon : \zeta \in [\varepsilon,\mu)\}$ is a free basis but $H^*_\varepsilon \subseteq G_{\alpha_{\varepsilon+1}} \subseteq G_\alpha$ hence $G_{\alpha_\varepsilon+1}/H^*_\varepsilon$ is free so together $G_{\alpha+1}/H^*_\varepsilon$ is free which implies that its subgroup $H^*_{\mu+1}/H_\varepsilon$ is free as promised.]

We can find H_* and $\langle H_\eta, h_\eta : \eta \in {}^{\mu\geq}2\rangle$ such that:

- (a) $H_* = \oplus\{\mathbb{Z}x_t : t \in I\}$ and $|I| = \mu$
- (b) I is the disjoint union of $\{I_\eta : \eta \in {}^{\mu>}2\}$
- (c) $|I_\eta| = \mathrm{rk}(H^*_{\ell g(\eta)+1}/H^*_{\ell g(\eta)})$ for $\eta \in {}^{\mu>}2$
- (d) $H_\eta = \oplus\{\mathbb{Z}x_t : t \in \cup\{I_{\eta\restriction\varepsilon} : \varepsilon < \ell g(\eta)\}\} \subseteq H_*$ for $\eta \in {}^{\mu>}2$
- (e) h_η is an isomorphism from $H_{\ell g(\eta)}$ onto H_η
- (f) $h_{\eta\restriction\varepsilon} \subseteq h_\eta$ if $\varepsilon < \ell g(\eta), \eta \in {}^{\mu>}2$.

Now we can find $\langle H^+_\eta, h^+_\eta : \eta \in {}^\mu 2\rangle$ such that

- (g) $H_\eta \subseteq H^+_\eta$
- (h) h^+_η is an isomorphism from $H^*_{\mu+1}$ onto H^+_η extending h_η.

Without loss of generality

- (i) $H^+_\eta \cap H_* = H_\eta$

so there is

- (j) H^*_η extends H^+_η and H_* such that $H^+_\eta \cup H_*$ generates H^*_η and H^*_η/H_* is isomorphic to H^*_η/H_η.

Lastly, without loss of generality

- (k) the sets $\langle H^*_\eta \backslash H_* : \eta \in {}^\mu 2\rangle$ are pairwise disjoint,

so there is an abelian group H such that

- (l) $H^*_\eta \subseteq H$ for $\eta \in {}^\mu 2$ and
- (m) $H/H_* = \oplus\{H^*_\eta/H_* : \eta \in {}^\mu 2\}$
- (n) H has cardinality $2^\mu = \kappa$.

Next we note

- (o) H is κ-free.

[Why? Note that $\cup\{H^+_\eta : \eta \in {}^\mu 2\}$ include $\{x_t : t \in I\}$ hence the subgroup it generates includes H. Let $H' \subseteq H$ be a sub-group of cardinality $< \kappa$ so $\leq \mu$, hence there are $\eta_i \in {}^\mu 2$ for $i < \mu$ such that $H' \subseteq \Sigma\{H^+_{\eta_i} : i < \mu\}$ and $j < i \Rightarrow \eta_i \neq \eta_j$. We can choose $\zeta(i) < \mu$ by induction on $i < \mu$ such that $\langle\{\eta_i\restriction\varepsilon : \varepsilon \in [\zeta(i),\mu)\} : i < \mu\rangle$ is a sequence of pairwise disjoint sets.

For each i let $H^{**}_i \subseteq H^+_{\eta_i}$ be such that $H^+_{\eta_i} = H^{**}_i \oplus H_{\eta_i\restriction\zeta(i)}$. Let us define H'_i for $i \leq \mu$ by: $H'_0 \subseteq H$ is generated by $\{x_t : t \in I$ but $t \notin \cup\{I_{\eta_i\restriction\varepsilon} : i < \mu$ and $\varepsilon \in [\zeta(i),\mu)\}\}$ and H'_i is $\oplus\{H^{**}_j : j < i\} \oplus H'_0$. Clearly $H' \subseteq H'_\mu \subseteq H, \langle H'_i : i \leq \mu\rangle$

is increasing continuous, H'_0 is free and $H'_{i+1}/H'_i \cong H^{**}_i$ is free. It follows that H'_μ is free hence $H' \subseteq H'_\mu$ is free. So clause (o) holds indeed.]

Let $\langle \eta_\alpha : \alpha < \kappa \rangle$ list ${}^\mu 2$, let $H^{**}_\alpha \subseteq H$ be $\Sigma\{H^+_{\eta_\beta} : \beta < \alpha\} + H_*, \bar H^{**} = \langle H^{**}_\alpha : \alpha < \kappa\rangle$ is a filtration of H, and $\Gamma(\bar H^{**}) = \kappa$ and $H^{**}_{\alpha+1}/H^{**}_{\alpha+1}$ is isomorphic to $H^*_{\mu+1}/H^*_\mu$ so is not free and is not Whitehead. Hence, see [EM02, Ch.XII,1.10,pg.369], H is not Whitehead. So H is κ-free (see clause (o) of cardinality κ (see clause (o) and not Whitehead, and so not free), the desired conclusion of the theorem. So indeed without loss of generality the stage desired conclusion, $\alpha \in S \Rightarrow \theta_\alpha < \mu$ holds.]

Stage E: μ is singular.

Why? Because by earlier stages $\mathrm{cf}(\mu) = \sigma$ and $\alpha \in S$ implies $\mathrm{cf}(\alpha) \le \theta_\alpha < \mu$ and $\alpha \in S \Rightarrow \sigma = \mathrm{cf}(\alpha)$, so necessarily μ is singular.

Stage F: Let $\sigma = \mathrm{cf}(\mu)$ so σ is regular $< \mu$; also choose $\theta = \mathrm{cf}(\theta) < \mu$ such that $S^*_1 := \{\delta \in S : \theta_\delta = \theta\}$ is stationary. Note that $\mu = \mu^{<\sigma}$ as G.C.H. holds recalling $\sigma = \mathrm{cf}(\delta)$ and $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \sigma$ by $(*)_7$. We shall now use [Sh:521, §3] and its notation, see 1.2, 1.3, 1.5 below.

By the Theorem 1.2 below we can find a κ-witness $\mathbf{x}$, so it consists of $n \ge 1, \mathbf{S}, \langle B_\eta : \eta \in \mathbf{S}_c\rangle, \langle s^\ell_\eta : \eta \in \mathbf{S}_f, \ell < n\rangle$ as there, i.e. as in [Sh:521, 3.6] with (λ, κ^+, S) there standing for $(\kappa, \aleph_1, \mathbf{S})$ here, such that

$(*)$ $\langle \alpha \rangle \in \mathbf{S} \Leftrightarrow \alpha \in S^*_1$, this means $W(<>, \mathbf{S}) = S^*_1$.

Clearly continuing to use the notation there, $\alpha \in S^*_1 \Rightarrow \lambda(\langle \alpha \rangle, \mathbf{S}) \le \mu$ hence being regular it is $< \mu$, and so without loss of generality constant (in fact it is θ by the proof). Now we apply the claim 1.5 below with λ there standing for κ here.

Why clause (d) of the assumption of 1.5 holds? If $\theta > \aleph_0$, by [Sh:521, 1.2], the group $G_{\mathbf{x}(<\alpha>)}$, derived from $\mathbf{x}(\langle\alpha\rangle)$ is a $\lambda(<\infty, \mathbf{S}_\mathbf{x})$-free non-free abelian group of cardinality $\lambda(\langle \alpha \rangle, \mathbf{S}_\mathbf{x}) = \theta$, so is not Whitehead by the theorem assumption on κ; note that $G_{\mathbf{x}(<\alpha>)}$ was derived in some way from $H_\alpha/(H_\alpha \cap G_\alpha)$, but it is not necessarily equal to it. If $\theta \le \aleph_0$ then $G_{\mathbf{x}(<\alpha>)}$ is a non-free (abelian) countable group hence is not Whitehead.

So the assumption of 1.5 says that there is a strongly κ-free abelian group G of cardinality κ by the theorem's assumption which is not Whitehead, so we are done. $\square_{1.1}$

Recall (by [Sh:161, §5] and see [Sh:521, §3]; or see [EM02]).

{1n.4} **Theorem 1.2.** *For any $\lambda > \aleph_0$ the following conditions are equivalent:*

- (a) *there is a λ-free not free abelian group*
- (b) $\mathrm{PT}(\lambda, \aleph_1)$ *which means that: there is a family $\mathscr{P}$ of countable sets of cardinality λ with no transversal (i.e. a one-to-one choice function) but any subfamily of cardinality $< \lambda$ has a transversal*
- (c) *there is a λ-witness $\mathbf{x}$ as in* [Sh:521, 3.6,3.7]*; so $\mathbf{x}$ consists of*
  - (α) *a natural number n*
  - (β) *a so-called λ-set $\mathbf{S} \subseteq \{\eta \in {}^{n\ge}\lambda : \eta$ decreasing$\}$ closed under initial segments, (see* [Sh:521, 3.1]*)*
  - (γ) *disjoint λ-system $\bar B = \langle B_\eta : \eta \in \mathbf{S}_c\rangle$, see* [Sh:521, 3.4]
  - (δ) *$\bar s = \langle s^\ell_\eta : \eta \in \mathbf{S}_f, \ell < n\rangle$, etc.*

(d) *Without loss of generality* $\cup\{B_\eta : \eta \in \mathbf{S}_c\} = \lambda$, *so* $\mathbf{S} = \mathbf{S}_{\mathbf{x}}$, *etc.*

(e) *Let* $\langle a^\ell_{\eta,m} : m < \omega\rangle$ *list* $s^{\ell,\mathbf{x}}_\eta$ *with no repetition for* $\eta \in \mathbf{S}^{\mathbf{x}}_f, \ell < n$ *and* $a^{\ell(1)}_{\eta(1),m(1)} = a^{\ell(2)}_{\eta(2),m(2)}$ *implies* $\ell(1) = \ell(2), m(1) = m(2)$ *and* $m < m(1) \Rightarrow a^{\ell(1)}_{\eta(\ell),m} = a^{\ell(2)}_{\eta(2),m}$, *so* without loss of generality $a^\ell_{\eta,m} < a^\ell_{\eta,m+1}$ *for every relevant* η, ℓ, m; *also* $\delta = \sup\cup\{s^{0,\mathbf{x}}_\eta : \langle\delta\rangle \trianglelefteq \eta \in \mathbf{S}_f\}$ *when* $\langle\delta\rangle \in \mathbf{S}_f$ *and* $\eta \mapsto \lambda(\eta, \mathbf{S}_{\mathbf{x}}), \eta \mapsto W(\eta, S_{\mathbf{x}})$ *are well defined.*

Used but not named in [Sh:521]:

{1n.6}

Definition/Claim 1.3. *1) For a* λ*-witness* $\mathbf{x}$ *and* $\nu \in \mathbf{S}_{\mathbf{x}}$ *we define* $\mathbf{x}_\nu = \mathbf{x}(\nu)$ *by* $n_{\mathbf{x}(\nu)} = n_{\mathbf{x}} - \ell g(\nu), \mathbf{S}_{\mathbf{x}(\nu)} = \{\eta : \nu\hat{\ }\eta \in \mathbf{S}_{\mathbf{x}}\}, B^{\mathbf{x}(\nu)}_\eta = B^{\mathbf{x}}_{\eta\hat{\ }\nu}, s^{\ell,\mathbf{x}(\nu)}_\eta = s^{\ell g(\nu)+\ell,\mathbf{x}}_{\nu\hat{\ }\eta}$.
2) For a λ*-witness* $\mathbf{x}$ *let* $G_{\mathbf{x}}$ *be the abelian group* $G_{\{<\alpha>:\alpha\in W(<>,S_{\mathbf{x}})\}}$ *defined inside the proof of* [Sh:521, 1.2].

{a8}

Remark 1.4. We may use the following.
1) For a λ-witness $\mathbf{x}$ let

$$K_{\mathbf{x}} = \{I \subseteq \mathbf{S}_{\mathbf{x}} : I \quad \begin{array}{l}\text{is a set of pairwise } \triangleleft\text{-incomparable sequences such that}\\ \{\beta : \eta\hat{\ }\langle\beta\rangle \in I\} \text{ is an initial segment of}\\ W(\eta, S) \text{ for any } \eta \in \mathbf{S}\}.\end{array}$$

2) For $\mathbf{x}$ a λ-witness and $I \in K_{\mathbf{x}}$ let $Y[I]$ and G_I be defined as in the proof of [Sh:521, 1.2] before Fact A. We may write η instead of $I = \{\eta\}$.

{5n.2}

Claim 1.5. *Assume*

(a) μ *is strong limit singular,* $\lambda = \mu^+ = 2^\mu$ *and* $\sigma = \mathrm{cf}(\mu)$

(b) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \sigma\}$ *is stationary*

(c) $\mathbf{x}$ *is a* λ*-witness see 1.2 with* $W(\langle\rangle, \mathbf{S}) \subseteq S$

(d) *for each* $\alpha \in W(\langle\rangle, \mathbf{S})$, *the abelian group* $G_{\mathbf{x}(<\alpha>)}$ *is not Whitehead (where* $G_{\mathbf{x}(<\alpha>)}$ *is defined as inside the proof of* [Sh:521, 1.2]*).*

Then
1) There is a strongly λ*-free abelian group* G *of cardinality* λ *which is not Whitehead, in fact* $\Gamma(G) \subseteq S$.
2) There is a strongly λ*-free abelian group* G^* *of cardinality* λ *satisfying* $\mathrm{HOM}(G^*, \mathbb{Z}) = \{0\}$, *in fact* $\Gamma(G^*) \subseteq S$ *(in fact the same abelian group can serve).*

Remark 1.6. 1) We rely on [Sh:521, §3].
2) So in clause (d), $G_{\mathbf{x}(<\alpha>)}$ is the abelian group defined from $\mathbf{x}(\langle\alpha\rangle)$.
3) If you do not like clause (d) of 1.5, replace it by "λ is as in 1.1".

Proof. 1) Let $\mathbf{S} = \mathbf{S}_{\mathbf{x}}$, etc. Without loss of generality

$(*)_0$ $\cup\{B^{\mathbf{x}}_\eta : \eta \in \mathbf{S}_c\} = \lambda$.

Let $\mathscr{H}(\lambda) = \bigcup\limits_{\alpha<\lambda} M_\alpha$ where $M_\alpha \prec (\mathscr{H}(\lambda), \in)$ has cardinality μ, is increasing continuous with α such that $\mu + 1 \subseteq M_0$ and $\langle M_\beta : \beta \leq \alpha\rangle \in M_{\alpha+1}$.

Let $S_0 = \{\delta \in W(\langle\rangle, \mathbf{S}) : M_\delta \cap \lambda = \delta\}$, as $W(\langle\rangle, \mathbf{S})$ is stationary and $\{\delta < \lambda : M_\delta \cap \lambda = \delta\}$ is a club of λ; clearly also S_0 is a stationary subset of λ; now for each $\delta \in S_0$ as μ is strong limit (and so $\mu = \mu^{<\sigma}$) clearly

$(*)_1$ if $Y \subseteq M_\delta, |Y| < \mu$ and $\text{cf}(\delta) = \sigma$ <u>then</u> there is an increasing continuous sequence $\langle X_i : i < \sigma \rangle$ of subsets of Y with union Y such that $M_\delta \supseteq \{X_i : i < \sigma\}$.

As $\alpha \in W(\langle\rangle, \mathbf{S}) \Rightarrow \lambda(\langle \alpha \rangle, S) \leq \mu < \lambda$ clearly for some θ

$(*)_2$ θ is regular $< \mu$ the set $S_1 = \{\delta \in S_0 : \lambda(\langle \alpha \rangle, S) = \theta\}$ is stationary.

For each $\delta \in S_1$ clearly $\mathbf{x}(\langle \delta \rangle)$ is a θ-witness and let

$(*)_3$ (a) $X_\delta := \{(\rho, s) : \rho \in \mathbf{S}_f^{\mathbf{x}(<\delta>)}$ (i.e. is $\triangleleft$-maximal in $\mathbf{S}^{\mathbf{x}(<\delta>)}$) and s is a finite initial segment of $s^0_{\langle \delta \rangle \hat{} \rho}$ which is a set of members of $B_{\langle \delta \rangle}$ of order type $\omega\}$.

Clearly

$(*)_4$ $X_\delta \subseteq M_\delta$ has cardinality θ.

We can find $\bar{X}_\delta$ such that

$(*)_5$ $\bar{X}_\delta = \langle X_{\delta,i} : i < \sigma \rangle$ is as in $(*)_1$ above, i.e. is $\subseteq$-increasing continuous, so $X_{\delta,i} \in M_\delta, |X_{\delta,i}| \leq \theta$ and letting $X_{\delta,\sigma} := \cup\{X_{\delta,i} : i < \sigma\}$ we have $X_{\delta,\sigma} = X_\delta$

$(*)_6$ $Z_{\delta,i} := \{(\rho, |s|) : (\rho, s) \in X_{\delta,i}\}$ for $i \leq \sigma$.

We define equivalence relation E on S_1:

$(*)_7$ $\delta_1 E \delta_2$ iff
- (a) $\mathbf{S}^{\langle \delta_1 \rangle} = \mathbf{S}^{\langle \delta_2 \rangle}$ equivalently $\mathbf{S}_{\mathbf{x}(\langle \delta_1 \rangle)} = \mathbf{S}_{\mathbf{x}(\langle \delta_2 \rangle)}$
- (b) for each $i < \sigma$ we have $Z_{\delta_1,i} = Z_{\delta_2,i}$

Clearly E is actually an equivalence relation but $\theta < \mu$ and μ is strong limit hence E has $\leq 2^\theta < \mu < \lambda$ equivalence classes. So for some $\delta^* \in S_1, S_2 := \delta^*/E$ is a stationary subset of λ. Clearly there is $F \in M_0$ which is a one to one function with range $\subseteq \{\delta : \delta < \lambda, \delta = \mu\delta$ is $< \lambda$ but $> 0\}$ and with domain $\mathscr{H}(\lambda)$ hence for every $\delta \in S_2$ it maps M_δ onto $\text{Rang}(F) \cap \delta$ and without loss of generality if $\varrho_1 \triangleleft \varrho_2$ are from ${}^{\lambda >}\mathscr{H}(\lambda)$ then $F(\varrho_1) < F(\varrho_2)$ and $F(\langle X_{\delta,j} : j \leq i \rangle)$ is $> \sup\{s \cup \text{Rang}(\rho) : (\rho, s) \in X_{\delta,i}\}$.

Let $\alpha^*_{\delta,i} = F(\langle X_{\delta,j} : j \leq i \rangle)$ for $\delta \in S_2, i < \sigma$, so clearly $\langle \alpha^*_{\delta,i} : i < \sigma \rangle$ is increasing with limit δ by the choice of F and of S_2, as (see 1.2(2) - $\delta = \sup \cup \{s^{0,\mathbf{x}}_\eta : \langle \delta \rangle \trianglelefteq \eta \in \mathbf{S}_f\}$). By [Sh:667, 4.2], we can find $\langle (\nu_{\delta,1}, \nu_{\delta,2}) : \delta \in S_2 \rangle$ such that

$\circledast$ (a) $\nu_{\delta,\ell} \in {}^\sigma\mu$
- (b) $\nu_{\delta,1}(i) < \nu_{\delta,2}(i)$ for $i < \sigma$
- (c) if $\mathbf{c} : \lambda \to 2^\theta + \sigma$ then for stationarily many $\delta \in S_2$ we have
$$i < \sigma \wedge \varepsilon < \theta \Rightarrow \mathbf{c}(\alpha^*_{\delta,i} + \mu\varepsilon + \nu_{\delta,1}(i)) = \mathbf{c}(\alpha^*_{\delta,i} + \mu\varepsilon + \nu_{\delta,2}(i)).$$

Let $\mathbf{y} = \mathbf{x} \restriction \mathbf{S}'$ where $\mathbf{S}' = \{\langle\rangle\} \cup \{\rho \in \mathbf{S} : \rho \neq \langle\rangle$ and $\rho(0) \in S_2\}$.

Now at last we shall define the group. Essentially it will be similar to the group $G_{\mathbf{x}}$, see 1.3(2) so defined inside the proof of [Sh:521, 1.2] from the system $\mathbf{x}$, restricted to $\mathbf{S}'$ only, but whereas in $(*)^a_{I,\eta}$ before Fact A in the proof in [Sh:521, 1.2] we use "$2y_{\eta,m+1} = y_{\eta,m} + \Sigma\{x[a^\ell_{\eta,m}) : \ell < n$ and $a^\ell_{\eta,m} \in Y[I]\}$ here we replace $x[a^\ell_{\eta,m}]$ by the difference of two, related to $\circledast$; this may become clearer after reading the proof. The λ-freeness will be inherited from $\mathbf{S}$ being λ-free. The non-Whitehead comes from $\circledast$.

For $\delta \in S_1$ let $\mathbf{g}_\delta : \mathbf{S}_{\mathbf{x}(\langle \delta \rangle)} \to \theta$ be a one-to-one function, so by the choice of S_2 for some $\mathbf{g}$ we have $\delta \in S_2 \Rightarrow \mathbf{g}_\delta = \mathbf{g}$.

As in [Sh:521, §1] for each $\eta \in \mathbf{S}_f$ and $\ell < n$ recall, $\langle a^\ell_{\eta,m} : m < \omega \rangle$ lists $s^\ell_\eta \subseteq B_{\eta \restriction (\ell+1)}$, let $Y = \cup\{B_\nu : \nu \in \mathbf{S}_c\} \backslash B^{\mathbf{x}}_{\langle \lambda \rangle}$ and for $\eta \in \mathbf{S}_f, m < \omega$ let $\mathbf{i}_m(\eta)$ be the minimal i such that $(\eta \restriction [1,n), \{a^0_{\eta,\ell} : \ell \le m\}) \in X_{\eta(0),i}$. We define G as the abelian group generated by

$$\Xi = \{y_{\eta,m} : m < \omega \text{ and } \eta \in \mathbf{S}'_f\} \cup \{x[a] : a \in Y\} \cup \{z_\beta : \beta < \lambda\}$$

freely except for the equations

$(*)$ for $\eta \in \mathbf{S}'_f$ and $m < \omega$, so $\delta := \eta(0) \in S_2$ the equation (letting $i = \mathbf{i}_m(\eta)$)

$$\begin{aligned} 2y_{\eta,m+1} = y_{\eta,m} \quad & + \Sigma\{x[a^\ell_{\eta,m}] : 0 < \ell < n\} \\ & + z_{\alpha^*_{\delta,i} + \mu \mathbf{g}(\eta) + \nu_{\delta,2}(i)} \\ & - z_{\alpha^*_{\delta,i} + \mu \mathbf{g}(\eta) + \nu_{\delta,1}(i)} \end{aligned}$$

recalling $\mathbf{g} : \mathbf{S}_f \to \mu$ such that $\mathbf{g} \restriction \{\eta \in \mathbf{S}_f : \eta(0) = \delta\}$ is one to one, $\alpha^*_{\delta,i} = F(\langle X_{\delta,j} : j \le i \rangle)$.

For $\alpha \le \lambda$ let G_α be the subgroup of G generated by

$$\begin{aligned} \{y_{\eta,m} : \quad & m < \omega \text{ and } \eta \in \mathbf{S}'_f \text{ and } \eta(0) < \alpha\} \\ & \cup \{x[a] : a \in Y \cap \alpha \text{ and } a + 1 < \alpha\} \\ & \cup \{z_\beta : \beta < \alpha \text{ moreover } \beta + 1 < \alpha\}. \end{aligned}$$

Easily

$\oplus_1$ G_α is a pure subgroup of G, increasing continuous with α

$\oplus_2$ if $\delta \in S_2$ then $G_{\delta+1}/G_\delta$ is isomorphic to $G_{\mathbf{x}(\langle \delta \rangle)}$ which is not Whitehead.

[Why? By clause (d) of the assumption; now at first glance the set of generators and equations in the proof of [Sh:521, 1.2] and in this proof are different. But note that only $y_\eta, \langle \alpha \rangle \lhd \eta \in \mathbf{S}_f$ and $x[a], a \in \cup\{s^\ell_\eta : \ell \in [1,n), \langle \alpha \rangle \trianglelefteq \eta \in \mathbf{S}_f\}$ appear in the equation. Alternatively use Remark 1.4 and prove $G_{\alpha+1}/G_\alpha$ is isomorphic to $G_{\mathbf{x},\alpha+1}/G_{\mathbf{x},\alpha}$ in [Sh:521, 1.2] notation; again note that the z_α - here and $x[a], a \in \cup\{s^{0,\mathbf{x}}_\eta : \eta \in \mathbf{S}_f\}$ disappear.]

But recall

$\oplus_3$ G is strongly λ-free, moreover if $\alpha \in \lambda \backslash S_2$ and $\beta \in (\alpha, \lambda)$ then G_β/G_α is free.

[Why? As in the proof of Fact A inside the proof of [Sh:521, 1.2].]

$\oplus_4$ G is not Whitehead.

[Why? We choose $(H_\alpha, h_\alpha, g_\alpha)$ by induction on $\alpha \le \lambda$ such that

(a) H_α is an abelian group extending $\mathbb{Z}$

(b) h_α is a homomorphism from H_α onto G_α with kernel $\mathbb{Z}$

(c) g_α is a function from G_α to H_α inverting h_α (but in general not a homomorphism)
(d) H_α is increasing continuous with α
(e) h_α is increasing continuous with α
(f) g_α is increasing continuous with α
(g) if $\alpha = \delta + 1$ and $\delta \in S_2$ then there is no homomorphism g^* from G_α into H_α inverting h_α such that:
   $\odot$ $i < \sigma \wedge \varepsilon < \theta \Rightarrow g^*(z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,1}(i)}) - g_{\delta+1}(z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,1}}) = g^*(z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,2}(i)}) - g_{\delta+1}(z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,2}(i)})$

(note: the subtraction in $\mathbb{Z}$, the kernel of h_α)

For $\alpha = 0, \alpha$ limit and $\alpha = \beta + 1, \beta \notin S_2$ this is obvious. For $\alpha = \delta + 1, \delta \in S_2$ it is known that if instead of $\odot$ in clause (g) we know $g^* \restriction G_\delta$ this is possible. But $\odot$ gives all the necessary information. In more details let G'_δ be the subgroup of G_δ generated by $\{z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,2}(i)} - z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,1}(i)} : \varepsilon < \theta \text{ and } i < \sigma\}$.

Let $G'_{\delta+1}$ be the subgroup of $G_{\delta+1}$ generated by $G'_\delta \cup \{y_{\eta,m} : \langle\delta\rangle \trianglelefteq \eta \in \mathbf{S}'_f\}$. Clearly G'_δ is a pure subgroup of $G'_{\delta+1}$ and of G_δ and $G_{\delta+1} = G'_{\delta+1} \bigoplus_{G'_\delta} G_\delta$.

Let $H'_\delta = h_\delta^{-1}(G'_\delta)$, clearly $h_\delta \restriction H'_\delta$ is a homomorphism from H'_δ onto G'_δ with kernel $\mathbb{Z}$.

Clearly

$\oplus_{4.1}$ if $g', g'' \in \text{Hom}(G_\delta, H_{\delta+1})$ invert h_δ and both satisfies $\odot$ of clause (g) then $g' \restriction G'_\delta = g'' \restriction G'_\delta$.

So $|\mathscr{G}_\delta| \le 1$ where

$$\mathscr{G}_\delta = \{g \restriction G'_\delta : \; g \text{ is a homomorphism from } G_\delta \text{ to } H_\delta \text{ inverting } h_\alpha \text{ and satisfying } \odot \text{ in clause (g)}\}.$$

Let g^* be the unique member of $\mathscr{G}$ if $\mathscr{G}$ is non-empty and otherwise let it be any homomorphism from G'_δ into H'_δ inverting $h_\delta \restriction H'_\delta$, exist as G'_δ is free.

We now choose $(H'_{\delta+1}, h'_{\delta+1})$ such that

- $H'_\delta \subseteq H'_{\delta+1}$
- $h'_{\delta+1} \in \text{Hom}(H'_{\delta+1}, G'_{\delta+1})$
- $h'_{\delta+1}$ has kernel $\mathbb{Z}$
- $h'_{\delta+1}$ extends $h_\delta \restriction G'_\delta$
- g^* cannot be extended.

Next without loss of generality $H'_{\delta+1} \cap H_\delta = H'_\delta$, let $H_{\delta+1} = H'_{\delta+1} \bigoplus_{H'_\delta} H_\delta$, and let $h_{\delta+1} \in \text{Hom}(H_{\delta+1}, G_{\delta+1})$ extend $h_\delta, h'_{\delta+1}$. Let $g_{\delta+1} \supseteq g_\delta$ invert $h_{\delta+1}$ but is not necessarily a homomorphism. So we have chosen $(H_\alpha, h_\alpha, g_\alpha)$ such that: if $\mathscr{G}_\delta \neq \emptyset$ then we cannot find $g' \in \text{Hom}(G'_{\delta+1}, \mathbb{Z})$ inverting h_α such that $g' \restriction G'_\delta \in \mathscr{G}_\delta$. This suffices for carrying out the induction.

Having carried the induction, clearly $h = h_\lambda$ is a homomorphism from $H = H_\lambda$ onto $G_\lambda = G$ with kernel $\mathbb{Z}$. To show that G is not a Whitehead group it suffices to prove that h is not invertible as a homomorphism. But if $g \in \text{Hom}(G, H)$ inverts h then $x \in G \Rightarrow g(x) - g_\lambda(x) \in \mathbb{Z}$.

We define a function $\mathbf{f}$ with domain λ: for $\alpha < \lambda, \zeta < \mu$: $\mathbf{f}(\mu\alpha + \zeta) = \langle g(z_{\mu^2\alpha+\mu\varepsilon+\zeta}) - g_\lambda(z_{\mu^2\alpha+\mu\varepsilon+\zeta}) : \varepsilon < \theta\rangle$.

So for stationarily many $\delta \in S_2$ we have $i < \sigma \Rightarrow \mathbf{f}(\alpha^*_{\delta,i} + \nu_{\delta,1}(i)) = \mathbf{f}(\alpha^*_{\delta,i} + \nu_{\delta,2}(i))$. For any such δ we get a contradiction by clause (g) of the construction, so we have proved $\oplus_4$.

This finishes the proof of part (1), as $G = G_\lambda$ is as required.

2) For the proof of part (2) can use:

- $\boxtimes$ for regular uncountable λ, the following conditions are equivalent
  - (a) every λ-free abelian group of cardinality λ is Whitehead
  - (b) for every λ-free abelian group of cardinality λ we have $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$.

[Why? If (a) and G as in(b), let h be a pure embedding of $\mathbb{Z}$ into G, let $G' = G/\mathrm{Rang}(h)$ and use the definition of "G' is a Whitehead group". If G, H and $h \in \mathrm{Hom}(H, G)$ form a counterexample then we can find a purely increasing continuous sequence $\langle G_\alpha : \alpha \leq \lambda\rangle$ and $\langle x_\alpha : \alpha < \lambda\rangle$ such that: $\{x_\alpha : \alpha < \lambda\} = \{x \in G_\lambda : \mathbb{Z}x$ is a pure subgroup of $G_\lambda\}$ and for each α, there is a pure embedding h_α of H into $G_{\alpha+1}$ such that $h(1_{\mathbb{Z}}) = x_\alpha$ and $G_{\alpha+1} = G_\alpha \underset{\mathbb{Z}x_\alpha}{\oplus} h_\alpha(H)$.

Easily G_λ contradicts clause (b).]

We can also construct directly. $\square_{1.5}$

§ 2. Private Appendix
Assignments

1) The [EM02] conjecture: maybe can be contradicted in [Sh:832].
2) Assume G.C.H. and κ is first inaccessible but every stationary set reflects and more every κ-free abelian group is free. Prove the proofs in [Sh:587] apply.

§ 3. Private Appendix

{0n.21}

Discussion 3.1. Toward solving [**?**, Problem F4].

Step A: We start with $\mathbf{V} \models$ GCH + "diamond on every stationary set", Fsp $= \{\theta$: no almost free group in $\theta\}$ large enough.

Step B: Choose regular $\lambda > \theta$, force $S_1 \subseteq S^\lambda_\theta$ not reflecting, and force uniformization on S_1 let $\diamondsuit_S$ for every stationary $S \subseteq S_2 \leq \lambda, S_1, S_2$ stationary. Still $\neg(\exists\kappa)\boxdot_\kappa$.

Stage D: Let $\mu > \lambda$, force by initial segment $\bar\eta = \langle \eta_{\delta,i} : \delta \in S_2, i \in S\rangle$ such that

- (a) $S_3 \subseteq S^{\mu^+}_\lambda$ stationary
- (b) $\langle \eta_{\delta,i}\rangle$ is increasing of length θ
- (c) $\langle \sup \mathrm{Rang}(\eta_{\delta,i}) : \delta \in S\rangle$ increases with limit $\delta, \delta \in S_3, i < j \wedge i \in S_1 \wedge j \in S_2 \Rightarrow \sup(\mathrm{Rang}(\eta_{\delta,i}) < \eta_{\delta,j}(0)$
- (d) every initial segment of $\bar\eta$ is free
- (e) on λ, S_3 reflection as of old.

Alternative forcing: $S_3 = S^{\mu^+}_\lambda$, conditions apr? $< \mu =$ cf(μ).

Stage E: It is enough to prove that: if G is almost free in μ^+ then $\Gamma(G) \subseteq S_3$ moreover $\langle G_\alpha : \alpha < \mu^+\rangle$ a representation then for almost all $\alpha \in S_2, G_{\alpha+1}/G_\alpha$ is.

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