

# TIE-POINTS AND FIXED-POINTS IN $\mathbb{N}^*$

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ABSTRACT. A point  $x$  is a (bow) tie-point of a space  $X$  if  $X \setminus \{x\}$  can be partitioned into (relatively) clopen sets each with  $x$  in its closure. Tie-points have appeared in the construction of non-trivial autohomeomorphisms of  $\beta\mathbb{N} \setminus \mathbb{N}$  (e.g. [10, 8]) and in the recent study of (precisely) 2-to-1 maps on  $\beta\mathbb{N} \setminus \mathbb{N}$ . In these cases the tie-points have been the unique fixed point of an involution on  $\beta\mathbb{N} \setminus \mathbb{N}$ . This paper is motivated by the search for 2-to-1 maps and obtaining tie-points of strikingly differing characteristics.

## 1. INTRODUCTION

A point  $x$  is a tie-point of a space  $X$  if there are closed sets  $A, B$  of  $X$  such that  $\{x\} = A \cap B$  and  $x$  is an adherent point of each of  $A$  and  $B$ . We picture (and denote) this as  $X = A \bowtie_x B$  where  $A, B$  are the closed sets which have a unique common accumulation point  $x$  and say that  $x$  is a tie-point as witnessed by  $A, B$ . Let  $A \equiv_x B$  mean that there is a homeomorphism from  $A$  to  $B$  with  $x$  as a fixed point. If  $X = A \bowtie_x B$  and  $A \equiv_x B$ , then there is an involution  $F$  of  $X$  (i.e.  $F^2 = \text{id}$ ) such that  $\{x\} = \text{fix}(F)$ . In this case we will say that  $x$  is a symmetric tie-point of  $X$ .

An autohomeomorphism  $F$  of  $\beta\mathbb{N} \setminus \mathbb{N}$  (or  $\mathbb{N}^*$ ) is said to be *trivial* if there is a bijection  $f$  between cofinite subsets of  $\mathbb{N}$  such that  $F = \beta f \upharpoonright \beta\mathbb{N} \setminus \mathbb{N}$ . If  $F$  is a trivial autohomeomorphism, then  $\text{fix}(F)$  is clopen; so of course  $\beta\mathbb{N} \setminus \mathbb{N}$  will have no symmetric tie-points in this case if all autohomeomorphisms are trivial.

If  $A$  and  $B$  are arbitrary compact spaces, and if  $x \in A$  and  $y \in B$  are accumulation points, then let  $A \bowtie_{x=y} B$  denote the quotient space of

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$A \oplus B$  obtained by identifying  $x$  and  $y$  and let  $xy$  denote the collapsed point. Clearly the point  $xy$  is a tie-point of this space.

We came to the study of tie-points via the following observation.

**Proposition 1.1.** *If  $x, y$  are symmetric tie-points of  $\beta\mathbb{N} \setminus \mathbb{N}$  as witnessed by  $A, B$  and  $A', B'$  respectively, then there is a 2-to-1 mapping from  $\beta\mathbb{N} \setminus \mathbb{N}$  onto the space  $A \underset{x=y}{\times} B'$ .*

The proposition holds more generally if  $x$  and  $y$  are fixed points of involutions  $F, F'$  respectively. That is, replace  $A$  by the quotient space of  $\beta\mathbb{N} \setminus \mathbb{N}$  obtained by collapsing all sets  $\{z, F(z)\}$  to single points and similarly replace  $B'$  by the quotient space induced by  $F'$ . It is an open problem to determine if 2-to-1 continuous images of  $\beta\mathbb{N} \setminus \mathbb{N}$  are homeomorphic to  $\beta\mathbb{N} \setminus \mathbb{N}$  [5]. It is known to be true if CH [3] or PFA [2] holds.

There are many interesting questions that arise naturally when considering the concept of tie-points in  $\beta\mathbb{N} \setminus \mathbb{N}$ . Given a closed set  $A \subset \beta\mathbb{N} \setminus \mathbb{N}$ , let  $\mathcal{I}_A = \{a \subset \mathbb{N} : a^* \subset A\}$ . Given an ideal  $\mathcal{I}$  of subsets of  $\mathbb{N}$ , let  $\mathcal{I}^\perp = \{b \subset \mathbb{N} : (\forall a \in \mathcal{I}) a \cap b =^* \emptyset\}$  and  $\mathcal{I}^+ = \{d \subset \mathbb{N} : (\forall a \in \mathcal{I}) d \setminus a \notin \mathcal{I}^\perp\}$ . If  $\mathcal{J} \subset [\mathbb{N}]^\omega$ , let  $\mathcal{J}^\downarrow = \bigcup_{J \in \mathcal{J}} \mathcal{P}(J)$ . Say that  $\mathcal{J} \subset \mathcal{I}$  is unbounded in  $\mathcal{I}$  if for each  $a \in \mathcal{I}$ , there is a  $b \in \mathcal{J}$  such that  $b \setminus a$  is infinite.

**Definition 1.1.** If  $\mathcal{I}$  is an ideal of subsets of  $\mathbb{N}$ , set  $\text{cf}(\mathcal{I})$  to be the cofinality of  $\mathcal{I}$ ;  $\mathfrak{b}(\mathcal{I})$  is the minimum cardinality of an unbounded family in  $\mathcal{I}$ ;  $\delta(\mathcal{I})$  is the minimum cardinality of a subset  $\mathcal{J}$  of  $\mathcal{I}$  such that  $\mathcal{J}^\downarrow$  is dense in  $\mathcal{I}$ .

If  $\beta\mathbb{N} \setminus \mathbb{N} = A \underset{x}{\times} B$ , then  $\mathcal{I}_B = \mathcal{I}_A^\perp$  and  $x$  is the unique ultrafilter on  $\mathbb{N}$  extending  $\mathcal{I}_A^+ \cap \mathcal{I}_B^+$ . The character of  $x$  in  $\beta\mathbb{N} \setminus \mathbb{N}$  is equal to the maximum of  $\text{cf}(\mathcal{I}_A)$  and  $\text{cf}(\mathcal{I}_B)$ .

**Definition 1.2.** Say that a tie-point  $x$  has (i)  $\mathfrak{b}$ -type; (ii)  $\delta$ -type; respectively (iii)  $\mathfrak{b}\delta$ -type,  $(\kappa, \lambda)$  if  $\beta\mathbb{N} \setminus \mathbb{N} = A \underset{x}{\times} B$  and  $(\kappa, \lambda)$  equals: (i)  $(\mathfrak{b}(\mathcal{I}_A), \mathfrak{b}(\mathcal{I}_B))$  (ii)  $(\delta(\mathcal{I}_A), \delta(\mathcal{I}_B))$ ; and (iii) each of  $(\mathfrak{b}(\mathcal{I}_A), \mathfrak{b}(\mathcal{I}_B))$  and  $(\delta(\mathcal{I}_A), \delta(\mathcal{I}_B))$ . We will adopt the convention to put the smaller of the pair  $(\kappa, \lambda)$  in the first coordinate.

Again, it is interesting to note that if  $x$  is a tie-point of  $\mathfrak{b}$ -type  $(\kappa, \lambda)$ , then it is uniquely determined (in  $\beta\mathbb{N} \setminus \mathbb{N}$ ) by  $\lambda$  many subsets of  $\mathbb{N}$  since  $x$  will be the unique point extending the family  $((\mathcal{J}_A)^\downarrow)^+ \cap ((\mathcal{J}_B)^\downarrow)^+$  where  $\mathcal{J}_A$  and  $\mathcal{J}_B$  are unbounded subfamilies of  $\mathcal{I}_A$  and  $\mathcal{I}_B$ .

**Question 1.1.** Can there be a tie-point in  $\beta\mathbb{N} \setminus \mathbb{N}$  with  $\delta$ -type  $(\kappa, \lambda)$  with  $\kappa \leq \lambda$  less than the character of the point?

**Question 1.2.** Can  $\beta\mathbb{N} \setminus \mathbb{N}$  have tie-points of  $\delta$ -type  $(\omega_1, \omega_1)$  and  $(\omega_2, \omega_2)$ ?

**Proposition 1.2.** *If  $\beta\mathbb{N} \setminus \mathbb{N}$  has symmetric tie-points of  $\delta$ -type  $(\kappa, \kappa)$  and  $(\lambda, \lambda)$ , but no tie-points of  $\delta$ -type  $(\kappa, \lambda)$ , then  $\beta\mathbb{N} \setminus \mathbb{N}$  has a 2-to-1 image which is not homeomorphic to  $\beta\mathbb{N} \setminus \mathbb{N}$ .*

One could say that a tie-point  $x$  was *radioactive* in  $X$  (i.e.  $\bowtie_x$ ) if  $X \setminus \{x\}$  can be similarly split into 3 (or more) relatively clopen sets accumulating to  $x$ . This is equivalent to  $X = A \bowtie_x B$  such that  $x$  is a tie-point in either  $A$  or  $B$ .

Each point of character  $\omega_1$  in  $\beta\mathbb{N} \setminus \mathbb{N}$  is a radioactive point (in particular is a tie-point). P-points of character  $\omega_1$  are symmetric tie-points of  $\mathfrak{b}\delta$ -type  $(\omega_1, \omega_1)$ , while points of character  $\omega_1$  which are not P-points will have  $\mathfrak{b}$ -type  $(\omega, \omega_1)$  and  $\delta$ -type  $(\omega_1, \omega_1)$ . If there is a tie-point of  $\mathfrak{b}$ -type  $(\kappa, \lambda)$ , then of course there are  $(\kappa, \lambda)$ -gaps. If there is a tie-point of  $\delta$ -type  $(\kappa, \lambda)$ , then  $\mathfrak{p} \leq \kappa$ .

**Proposition 1.3.** *If  $\beta\mathbb{N} \setminus \mathbb{N} = A \bowtie_x B$ , then  $\mathfrak{p} \leq \delta(\mathcal{I}_A)$ .*

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*Proof.* If  $\mathcal{J} \subset \mathcal{I}_A$  has cardinality less than  $\mathfrak{p}$ , there is, by Solovay's Lemma (and Bell's Theorem) an infinite set  $C \subset \mathbb{N}$  such that  $C$  and  $\mathbb{N} \setminus C$  each meet every infinite set of the form  $J \setminus (\bigcup \mathcal{J}')$  where  $\{J\} \cup \mathcal{J}' \in [\mathcal{J}]^{<\omega}$ . We may assume that  $C \not\subset x$ , hence there are  $a \in \mathcal{I}_A$  and  $b \in \mathcal{I}_B$  such that  $C \subset a \cup b$ . However no finite union from  $\mathcal{J}$  covers  $a$  showing that  $\mathcal{J}^\downarrow$  can not be dense in  $\mathcal{I}_A$ .  $\square$

Although it does not seem to be completely trivial, it can be shown that PFA implies there are no tie-points (the hardest case to eliminate is those of  $\mathfrak{b}$ -type  $(\omega_1, \omega_1)$ ).

**Question 1.3.** Does  $\mathfrak{p} > \omega_1$  imply there are no tie-points of  $\mathfrak{b}$ -type  $(\omega_1, \omega_1)$ ?

Analogous to tie-points, we also define a tie-set: say that  $K \subset \beta\mathbb{N} \setminus \mathbb{N}$  is a tie-set if  $\beta\mathbb{N} \setminus \mathbb{N} = A \bowtie_K B$  and  $K = A \cap B$ ,  $A = \overline{A \setminus K}$ , and  $B = \overline{B \setminus K}$ . Say that  $K$  is a symmetric tie-set if there is an involution  $F$  such that  $K = \text{fix}(F)$  and  $F[A] = B$ .

**Question 1.4.** If  $F$  is an involution on  $\beta\mathbb{N} \setminus \mathbb{N}$  such that  $K = \text{fix}(F)$  has empty interior, is  $K$  a (symmetric) tie-set?

**Question 1.5.** Is there some natural restriction on which compact spaces can (or can not) be homeomorphic to the fixed point set of some involution of  $\beta\mathbb{N} \setminus \mathbb{N}$ ?

Again, we note a possible application to 2-to-1 maps.

**Proposition 1.4.** *Assume that  $F$  is an involution of  $\beta\mathbb{N} \setminus \mathbb{N}$  with  $K = \text{fix}(F) \neq \emptyset$ . Further assume that  $K$  has a symmetric tie-point  $x$  (i.e.  $K = A \underset{x}{\bowtie} B$ ), then  $\beta\mathbb{N} \setminus \mathbb{N}$  has a 2-to-1 continuous image which has a symmetric tie-point (and possibly  $\beta\mathbb{N} \setminus \mathbb{N}$  does not have such a tie-point).*

**Question 1.6.** If  $F$  is an involution of  $\mathbb{N}^*$ , is the quotient space  $\mathbb{N}^*/F$  (in which each  $\{x, F(x)\}$  is collapsed to a single point) a homeomorphic copy of  $\beta\mathbb{N} \setminus \mathbb{N}$ ?

**Proposition 1.5 (CH).** *If  $F$  is an involution of  $\beta\mathbb{N} \setminus \mathbb{N}$ , then the quotient space  $\mathbb{N}^*/F$  is homeomorphic to  $\beta\mathbb{N} \setminus \mathbb{N}$ .*

*Proof.* If  $\text{fix}(F)$  is empty, then  $\mathbb{N}^*/F$  is a 2-to-1 image of  $\beta\mathbb{N} \setminus \mathbb{N}$ , and so is a copy of  $\beta\mathbb{N} \setminus \mathbb{N}$ . If  $\text{fix}(F)$  is not empty, then consider two copies,  $(\mathbb{N}_1^*, F_1)$  and  $(\mathbb{N}_2^*, F_2)$ , of  $(\mathbb{N}^*, F)$ . The quotient space of  $\mathbb{N}_1^*/F_1 \oplus \mathbb{N}_2^*/F_2$  obtained by identifying the two homeomorphic sets  $\text{fix}(F_1)$  and  $\text{fix}(F_2)$  will be a 2-to-1-image of  $\mathbb{N}^*$ , hence again a copy of  $\mathbb{N}^*$ . Since  $\mathbb{N}_1^* \setminus \text{fix}(F_1)$  and  $\mathbb{N}_2^* \setminus \text{fix}(F_2)$  are disjoint and homeomorphic, it follows easily that  $\text{fix}(F)$  must be a P-set in  $\mathbb{N}^*$ . It is trivial to verify that a regular closed set of  $\mathbb{N}^*$  with a P-set boundary will be (in a model of CH) a copy of  $\mathbb{N}^*$ . Therefore the copy of  $\mathbb{N}_1^*/F_1$  in this final quotient space is a copy of  $\mathbb{N}^*$ .  $\square$

## 2. A SPECTRUM OF TIE-SETS

We adapt a method from [1] to produce a model in which there are tie-sets of specified  $\mathfrak{bd}$ -types. We further arrange that these tie-sets will themselves have tie-points but unfortunately we are not able to make the tie-sets symmetric. In the next section we make some progress in involving involutions.

**Theorem 2.1.** *Assume GCH and that  $\Lambda$  is a set of regular uncountable cardinals such that for each  $\lambda \in \Lambda$ ,  $T_\lambda$  is a  $<\lambda$ -closed  $\lambda^+$ -Souslin tree. There is a forcing extension in which there is a tie-set  $K$  (of  $\mathfrak{bd}$ -type  $(\mathfrak{c}, \mathfrak{c})$ ) and for each  $\lambda \in \Lambda$ , there is a tie-set  $K_\lambda$  of  $\mathfrak{bd}$ -type  $(\lambda^+, \lambda^+)$  such that  $K \cap K_\lambda$  is a single point which is a tie-point of  $K_\lambda$ . Furthermore, for  $\mu \leq \lambda < \mathfrak{c}$ , if  $\mu \neq \lambda$  or  $\lambda \notin \Lambda$ , then there is no tie-set of  $\mathfrak{bd}$ -type  $(\mu, \lambda)$ .*

We will assume that our Souslin trees are well-pruned and are ever  $\omega$ -ary branching. That is, if  $T_\lambda$  is a  $\lambda^+$ -Souslin tree, we assume that for each  $t \in T$ ,  $t$  has exactly  $\omega$  immediate successors denoted  $\{t \smallfrown \ell : \ell \in \omega\}$  and that  $\{s \in T_\lambda : t < s\}$  has cardinality  $\lambda^+$  (and so has successors on every level). A poset is  $<\kappa$ -closed if every directed subset

of cardinality less than  $\kappa$  has a lower bound. A poset is  $<\kappa$ -distributive if the intersection of any family of fewer than  $\kappa$  dense open subsets is again dense. For a cardinal  $\mu$ , let  $\mu^-$  be the minimum cardinal such that  $(\mu^-)^+ \geq \mu$  (i.e. the predecessor if  $\mu$  is a successor).

The main idea of the construction is nicely illustrated by the following.

**Proposition 2.2.** *Assume that  $\beta\mathbb{N} \setminus \mathbb{N}$  has no tie-sets of  $\mathfrak{bd}$ -type  $(\kappa_1, \kappa_2)$  for some  $\kappa_1 \leq \kappa_2 < \mathfrak{c}$ . Also assume that  $\lambda^+ < \mathfrak{c}$  is such that  $\lambda^+$  is distinct from one of  $\kappa_1, \kappa_2$  and that  $T_\lambda$  is a  $\lambda^+$ -Souslin tree and  $\{(a_t, x_t, b_t) : t \in T_\lambda\} \subset ([\mathbb{N}]^\omega)^3$  satisfy that, for  $t < s \in T_\lambda$ :*

- (1)  $\{a_t, x_t, b_t\}$  is a partition of  $\mathbb{N}$ ,
- (2)  $x_{t \smallfrown j} \cap x_{t \smallfrown \ell} = \emptyset$  for  $j < \ell$ ,
- (3)  $x_s \subset^* x_t$ ,  $a_t \subset^* a_s$ , and  $b_t \subset^* b_s$ ,
- (4) for each  $\ell \in \omega$ ,  $x_{t \smallfrown \ell+1} \subset^* a_{t \smallfrown \ell}$  and  $x_{t \smallfrown \ell+2} \subset^* b_{t \smallfrown \ell}$ ,

then if  $\rho \in [T_\lambda]^{\lambda^+}$  is a generic branch (i.e.  $\rho(\alpha)$  is an element of the  $\alpha$ -th level of  $T_\lambda$  for each  $\alpha \in \lambda^+$ ), then  $K_\rho = \bigcap_{\alpha \in \lambda^+} x_{\rho(\alpha)}^*$  is a tie-set of  $\beta\mathbb{N} \setminus \mathbb{N}$  of  $\mathfrak{bd}$ -type  $(\lambda^+, \lambda^+)$ , and there is no tie-set of  $\mathfrak{bd}$ -type  $(\kappa_1, \kappa_2)$ .

- (5) Assume further that  $\{(a_\xi, x_\xi, b_\xi) : \xi \in \mathfrak{c}\}$  is a family of partitions of  $\mathbb{N}$  such that  $\{x_\xi : \xi \in \mathfrak{c}\}$  is a mod finite descending family of subsets of  $\mathbb{N}$  such that for each  $Y \subset \mathbb{N}$ , there is a maximal antichain  $A_Y \subset T_\lambda$  and some  $\xi \in \mathfrak{c}$  such that for each  $t \in A_Y$ ,  $x_t \cap x_\xi$  is a proper subset of either  $Y$  or  $\mathbb{N} \setminus Y$ , then  $K = \bigcap_{\xi \in \mathfrak{c}} x_\xi^*$  meets  $K_\rho$  in a single point  $z_\lambda$ .
- (6) If we assume further that for each  $\xi < \eta < \mathfrak{c}$ ,  $a_\xi \subset^* a_\eta$  and  $b_\xi \subset^* b_\eta$ , and for each  $t \in T_\lambda$ ,  $\eta$  may be chosen so that  $x_t$  meets each of  $(a_\eta \setminus a_\xi)$  and  $(b_\eta \setminus b_\xi)$ , then  $z_\lambda$  is a tie-point of  $K_\rho$ .

*Proof.* To show that  $K_\rho$  is a tie-set it is sufficient to show that  $K_\rho \subset \overline{\bigcup_{\alpha \in \lambda^+} a_\alpha^*} \cap \overline{\bigcup_{\alpha \in \lambda^+} b_\alpha^*}$ . Since  $T_\lambda$  is a  $\lambda^+$ -Souslin tree, no new subset of  $\lambda$  is added when forcing with  $T_\lambda$ . Of course we use that  $\rho$  is  $T_\lambda$  is generic, so assume that  $Y \subset \mathbb{N}$  and that some  $t \in T_\lambda$  forces that  $Y^* \cap K_\rho$  is not empty. We must show that there is some  $t < s$  such that  $s$  forces that  $a_s \cap Y$  and  $b_s \cap Y$  are both infinite. However, we know that  $x_{t \smallfrown \ell} \cap Y$  is infinite for each  $\ell \in \omega$  since  $t \smallfrown \ell \Vdash_{T_\lambda} "K_\rho \subset x_{t \smallfrown \ell}^*"$ . Therefore, by condition 4, for each  $\ell \in \omega$ ,  $Y \cap a_{t \smallfrown \ell}$  and  $Y \cap b_{t \smallfrown \ell}$  are both infinite.

Now let  $\kappa_1, \kappa_2$  be regular cardinals at least one of which is distinct from  $\lambda^+$ . Recall that forcing with  $T_\lambda$  preserves cardinals. Assume that in  $V[\rho]$ ,  $K \subset \mathbb{N}^*$  and  $\mathbb{N}^* = C \underset{K}{\times} D$  with  $\mathfrak{b}(\mathcal{I}_C) = \delta(\mathcal{I}_C) = \kappa_1$  and  $\mathfrak{b}(\mathcal{I}_D) = \delta(\mathcal{I}_D) = \kappa_2$ . In  $V$ , let  $\{c_\gamma : \gamma \in \kappa_1\}$  be  $T_\lambda$ -names for the increasing cofinal sequence in  $\mathcal{I}_C$  and let  $\{d_\xi : \xi \in \kappa_2\}$  be  $T_\lambda$ -names for the increasing cofinal sequence in  $\mathcal{I}_D$ . Again using the fact that  $T_\lambda$  adds

no new subsets of  $\mathbb{N}$  and the fact that every dense open subset of  $T_\lambda$  will contain an entire level of  $T_\lambda$ , we may choose ordinals  $\{\alpha_\gamma : \gamma \in \kappa_1\}$  and  $\{\beta_\xi : \xi \in \kappa_2\}$  such that each  $t \in T_\lambda$ , if  $t$  is on level  $\alpha_\gamma$  it will force a value on  $c_\gamma$  and if  $t$  is on level  $\beta_\xi$  it will force a value on  $d_\xi$ . If  $\kappa_1 < \lambda^+$ , then  $\sup\{\alpha_\gamma : \gamma \in \kappa_1\} < \lambda^+$ , hence there are  $t \in T_\lambda$  which force a value on each  $c_\gamma$ . If  $\lambda^+ < \kappa_2$ , then there is some  $\beta < \lambda^+$ , such that  $\{\xi \in \kappa_2 : \beta_\xi \leq \beta\}$  has cardinality  $\kappa_2$ . Therefore there is some  $t \in T_\lambda$  such that  $t$  forces a value on  $d_\xi$  for a cofinal set of  $\xi \in \kappa_2$ . Of course, if neither  $\kappa_1$  nor  $\kappa_2$  is equal to  $\lambda^+$ , then we have a condition that decided cofinal families of each of  $\mathcal{I}_C$  and  $\mathcal{I}_D$ . This implies that  $\mathbb{N}^*$  already has tie-sets of  $\mathfrak{bd}$ -type  $(\kappa_1, \kappa_2)$ .

If  $\kappa_1 < \kappa_2 = \lambda^+$ , then fix  $t \in T_\lambda$  deciding  $\mathfrak{C} = \{c_\gamma : \gamma \in \kappa_1\}$ , and let  $\mathfrak{D} = \{d \subset \mathbb{N} : (\exists s > t) s \Vdash_{T_\lambda} "d^* \subset D"\}$ . It follows easily that  $\mathfrak{D} = \mathfrak{C}^\perp$ . But also, since forcing with  $T_\lambda$  can not raise  $\mathfrak{b}(\mathfrak{D})$  and can not lower  $\delta(\mathfrak{D})$ , we again have that there are tie-sets of  $\mathfrak{bd}$ -type in  $V$ .

The case  $\kappa_1 = \lambda^+ < \kappa_2$  is similar.

Now assume we have the family  $\{(a_\xi, x_\xi, b_\xi) : \xi \in \mathfrak{c}\}$  as in (5) and (6) and set  $K = \bigcap_\xi x_\xi^*$ ,  $A = \{K\} \cup \bigcup\{a_\xi^* : \xi \in \mathfrak{c}\}$ , and  $B = \{K\} \cup \bigcup\{b_\xi^* : \xi \in \mathfrak{c}\}$ . It is routine to see that (5) ensures that the family  $\{x_\xi \cap x_{\rho(\alpha)} : \xi \in \mathfrak{c} \text{ and } \alpha \in \lambda^+\}$  generates an ultrafilter when  $\rho$  meets each maximal antichain  $A_Y$  ( $Y \subset \mathbb{N}$ ). Condition (6) clearly ensures that  $A \setminus K$  and  $B \setminus K$  each meet  $(x_\xi \cap x_{\rho(\alpha)})^*$  for each  $\xi \in \mathfrak{c}$  and  $\alpha \in \lambda^+$ . Thus  $A \cap K_\rho$  and  $B \cap K_\rho$  witness that  $z_\lambda$  is a tie-point of  $K_\rho$ .  $\square$

Let  $\theta$  be a regular cardinal greater than  $\lambda^+$  for all  $\lambda \in \Lambda$ . We will need the following well-known Easton lemma (see [4, p234]).

**Lemma 2.3.** *Let  $\mu$  be a regular cardinal and assume that  $P_1$  is a poset satisfying the  $\mu$ -cc. Then any  $<\mu$ -closed poset  $P_2$  remains  $<\mu$ -distributive after forcing with  $P_1$ . Furthermore any  $<\mu$ -distributive poset remains  $<\mu$ -distributive after forcing with a poset of cardinality less than  $\mu$ .*

*Proof.* Recall that a poset  $P$  is  $<\mu$ -distributive if forcing with it does not add, for any  $\gamma < \mu$ , any new  $\gamma$ -sequences of ordinals. Since  $P_2$  is  $<\mu$ -closed, forcing with  $P_2$  does not add any new antichains to  $P_1$ . Therefore it follows that forcing with  $P_2$  preserves that  $P_1$  has the  $\mu$ -cc and that for every  $\gamma < \mu$ , each  $\gamma$ -sequence of ordinals in the forcing extension by  $P_2 \times P_1$  is really just a  $P_1$ -name. Since forcing with  $P_1 \times P_2$  is the same as  $P_2 \times P_1$ , this shows that in the extension by  $P_1$ , there are no new  $P_2$ -names of  $\gamma$ -sequences of ordinals.

Now suppose that  $P_2$  is  $\mu$ -distributive and that  $P_1$  has cardinality less than  $\mu$ . Let  $\dot{D}$  be a  $P_1$ -name of a dense open subset of  $P_2$ . For each  $p \in P_1$ , let  $D_p \subset P_2$  be the set of all  $q$  such that some extension of  $p$  forces that  $q \in \dot{D}$ . Since  $p$  forces that  $\dot{D}$  is dense and that  $\dot{D} \subset D_p$ , it follows that  $D_p$  is dense (and open). Since  $P_2$  is  $\mu$ -distributive,  $\bigcap_{p \in P_1} D_p$  is dense and is clearly going to be a subset of  $\dot{D}$ . Repeating this argument for at most  $\mu$  many  $P_1$ -names of dense open subsets of  $P_2$  completes the proof.  $\square$

We recall the definition of Easton supported product of posets (see [4, p233]).

**Definition 2.1.** If  $\Lambda$  is a set of cardinals and  $\{P_\lambda : \lambda \in \Lambda\}$  is a set of posets, then we will use  $\prod_{\lambda \in \Lambda} P_\lambda$  to denote the collection of partial functions  $p$  such that

- (1)  $\text{dom}(p) \subset \Lambda$ ,
- (2)  $|\text{dom}(p) \cap \mu| < \mu$  for all regular cardinals  $\mu$ ,
- (3)  $p(\lambda) \in P_\lambda$  for all  $\lambda \in \text{dom}(p)$ .

This collection is a poset when ordered by  $q < p$  if  $\text{dom}(q) \supset \text{dom}(p)$  and  $q(\lambda) \leq p(\lambda)$  for all  $\lambda \in \text{dom}(p)$ .

**Lemma 2.4.** For each cardinal  $\mu$ ,  $\prod_{\lambda \in \Lambda \setminus \mu^+} T_\lambda$  is  $< \mu^+$ -closed and, if  $\mu$  is regular,  $\prod_{\lambda \in \Lambda \cap \mu} T_\lambda$  has cardinality at most  $2^{< \mu} \leq \min(\Lambda \setminus \mu)$ . chain.condition

**Lemma 2.5.** If  $P$  is ccc and  $G \subset P \times \prod_{\lambda \in \Lambda} T_\lambda$  is generic, then in  $V[G]$ , for any  $\mu$  and any family  $\mathcal{A} \subset [\mathbb{N}]^\omega$  with  $|\mathcal{A}| = \mu$ : chains

- (1) if  $\mu \leq \omega$ , then  $\mathcal{A}$  is a member of  $V[G \cap P]$ ;
- (2) if  $\mu = \lambda^+$ ,  $\lambda \in \Lambda$ , then there is an  $\mathcal{A}' \subset \mathcal{A}$  of cardinality  $\lambda^+$  such that  $\mathcal{A}'$  is a member of  $V[G \cap (P \times T_\lambda)]$ ;
- (3) if  $\mu^- \notin \Lambda$ , then there is an  $\mathcal{A}' \subset \mathcal{A}$  of cardinality  $\mu$  which is a member of  $V[G \cap P]$ .

**Corollary 2.6.** If  $P$  is ccc and  $G \subset P \times \prod_{\lambda \in \Lambda} T_\lambda$  is generic, then for any  $\kappa \leq \mu < \mathfrak{c}$  such that either  $\kappa \neq \mu$  or  $\kappa \notin \{\lambda^+ : \lambda \in \Lambda\}$ , if there is a tie-set of  $\mathfrak{bd}$ -type  $(\kappa, \mu)$  in  $V[G]$ , then there is such a tie-set in  $V[G \cap P]$ . omit

*Proof.* Assume that  $\beta\mathbb{N} \setminus \mathbb{N} = A \boxtimes_K B$  in  $V[G]$  with  $\mu = \mathfrak{b}(A)$  and  $\lambda = \mathfrak{b}(B)$ . Let  $\mathcal{J}_A \subset \mathcal{I}_A$  be an increasing mod finite chain, of order type  $\mu$ , which is dense in  $\mathcal{I}_A$ . Similarly let  $\mathcal{J}_B \subset \mathcal{I}_B$  be such a chain of order type  $\lambda$ . By Lemma 2.5,  $\mathcal{J}_A$  and  $\mathcal{J}_B$  are subsets of  $[\mathbb{N}]^\omega \cap V[G \cap P] = [\mathbb{N}]^\omega$ . Choose, if possible  $\mu_1 \in \Lambda$  such that  $\mu_1^+ = \mu$  and  $\lambda_1 \in \Lambda$  such that  $\lambda_1^+ = \lambda$ . Also by Lemma 2.5, we can, by passing to a subcollection, assume that  $\mathcal{J}_A \in V[G \cap (P \times T_{\mu_1})]$  (if there is no  $\mu_1$ , then let  $T_{\mu_1}$  denote

the trivial order). Similarly, we may assume that  $\mathcal{J}_B \in V[G \cap (P \times T_{\lambda_1})]$ . Fix a condition  $q \in G \subset (P \times \prod_{\lambda \in \Lambda} T_\lambda)$  which forces that  $(\mathcal{J}_A)^\perp$  is a  $\subset$ -dense subset of  $\mathcal{I}_A$ , that  $(\mathcal{J}_B)^\perp$  is a  $\subset$ -dense subset of  $\mathcal{I}_B$ , and that  $(\mathcal{I}_A)^\perp = \mathcal{I}_B$ .

Working in the model  $V[G \cap P]$  then, there is a family  $\{\dot{a}_\alpha : \alpha \in \mu\}$  of  $T_{\mu_1}$ -names for the members of  $\mathcal{J}_A$ ; and a family  $\{\dot{b}_\beta : \beta \in \lambda_1\}$  of  $T_{\lambda_1}$ -names for the members of  $\mathcal{J}_B$ . Of course if  $\mu = \lambda$  and  $T_{\mu_1}$  is the trivial order, then  $\mathcal{J}_A$  and  $\mathcal{J}_B$  are already in  $V[G \cap P]$  and we have our tie-set in  $V[G \cap P]$ .

Otherwise, we assume that  $\mu_1 < \lambda_1$ . Set  $\mathcal{A}$  to be the set of all  $a \in \mathbb{N}$  such that there is some  $q(\mu_1) \leq t \in T_{\mu_1}$  and  $\alpha \in \mu$  such that  $t \Vdash_{T_{\mu_1}} "a = \dot{a}_\alpha"$ . Similarly let  $\mathcal{B}$  be the set of all  $b \in \mathbb{N}$  such that there is some  $q(\lambda_1) \leq s \in T_{\lambda_1}$  and  $\beta \in \lambda$  such that  $s \Vdash_{T_{\lambda_1}} "b = \dot{b}_\beta"$ . It follows from the construction that, in  $V[G]$ , for any  $(a', b') \in \mathcal{J}_A \times \mathcal{J}_B$ , there is an  $(a, b) \in \mathcal{A} \times \mathcal{B}$  such that  $a' \subset^* a$  and  $b' \subset^* b$ . Therefore the ideal generated by  $\mathcal{A} \cup \mathcal{B}$  is certainly dense. It remains only to show that  $\mathcal{B} \subset (\mathcal{A})^\perp$ . Consider any  $(a, b) \in \mathcal{A} \times \mathcal{B}$ , and choose  $(q(\mu_1), q(\lambda_1)) \leq (t, s) \in T_{\mu_1} \times T_{\lambda_1}$  such that  $t \Vdash_{T_{\mu_1}} "a \in \mathcal{J}_A"$  and  $s \Vdash_{T_{\lambda_1}} "b \in \mathcal{J}_B"$ . It follows that for any condition  $\bar{q} \leq q$  with  $\bar{q} \in (P \times \prod_{\lambda \in \Lambda} T_\lambda)$ ,  $\bar{q}(\mu_1) = t$ ,  $\bar{q}(\lambda_1) = s$ , we have that

$$\bar{q} \Vdash_{(P \times \prod_{\lambda \in \Lambda} T_\lambda)} "a \in \mathcal{J}_A \text{ and } b \in \mathcal{J}_B" .$$

It is routine now to check that, in  $V[G \cap P]$ ,  $\mathcal{A}$  and  $\mathcal{B}$  generate ideals that witness that  $\bigcap \{(\mathbb{N} \setminus (a \cup b))^* : (a, b) \in \mathcal{A} \times \mathcal{B}\}$  is a tie-set of  $\mathfrak{bd}$ -type  $(\mu, \lambda)$ .  $\square$

Let  $T$  be the rooted tree  $\{\emptyset\} \cup \bigcup_{\lambda \in \Lambda} T_\lambda$  and we will force an embedding of  $T$  into  $\mathcal{P}(\mathbb{N}) \bmod \text{finite}$ . In fact, we force a structure  $\{(a_t, x_t, b_t) : t \in T\}$  satisfying the conditions (1)-(4) of Proposition 2.2.

**Definition 2.2.** The poset  $Q_0$  is defined as the set of elements  $q = (n^q, T^q, f^q)$  where  $n^q \in \mathbb{N}$ ,  $T^q \in [T]^{<\omega}$ , and  $f^q : n^q \times T^q \rightarrow \{0, 1, 2\}$ . The idea is that  $x_t$  will be  $\bigcup_{q \in G} \{j \in n^q : f^q(j, t) = 0\}$ ,  $a_t$  will be  $\bigcup_{q \in G} \{j \in n^q : f^q(j, t) = 1\}$  and  $b_t = \mathbb{N} \setminus (a_t \cup x_t)$ . We set  $q < p$  if  $n^q \geq n^p$ ,  $T^q \supset T^p$ ,  $f^q \supset f^p$  and for  $t, s \in T^p$  and  $i \in [n^p, n^q]$

- (1) if  $t < s$  and  $f^q(i, t) \in \{1, 2\}$ , then  $f^q(i, s) = f^q(i, t)$ ;
- (2) if  $t < s$  and  $f^q(i, s) = 0$ , then  $f^q(i, t) = 0$ ;
- (3) if  $t \perp s$ , then  $f^q(i, t) + f^q(i, s) > 0$ .
- (4) if  $j \in \{1, 2\}$  and  $\{t^\frown \ell, t^\frown(\ell+j)\} \subset T^p$  and  $f^q(i, t^\frown(\ell+j)) = 0$ , then  $f^q(i, t^\frown \ell) = j$ .

The next lemma is very routine but we record it for reference.



**Lemma 2.7.** *The poset  $Q_0$  is ccc and if  $G \subset Q_0$  is generic, the family  $\mathcal{X}_T = \{(a_t, x_t, b_t) : t \in T\}$  satisfies the conditions of Proposition 2.2.* Q0

We will need some other combinatorial properties of the family  $\mathcal{X}_T$ .

**Definition 2.3.** For any  $\tilde{T} \in [T]^{<\omega}$ , we define the following ( $Q_0$ -names).

- (1) for  $i \in \mathbb{N}$ ,  $[i]_{\tilde{T}} = \{j \in \mathbb{N} : (\forall t \in \tilde{T}) i \in x_t \text{ iff } j \in x_t\}$ ,
- (2) the collection  $\text{fin}(\tilde{T})$  is the set of  $[i]_{\tilde{T}}$  which are finite.

We abuse notation and let  $\text{fin}(\tilde{T}) \subset n$  abbreviate  $\text{fin}(\tilde{T}) \subset \mathcal{P}(n)$ .

**Lemma 2.8.** *For each  $q \in Q_0$  and each  $\tilde{T} \subset T^q$ ,  $\text{fin}(\tilde{T}) \subset n^q$  and for  $i \geq n_q$ ,  $[i]_{\tilde{T}}$  is infinite.*

**Definition 2.4.** A sequence  $\mathcal{S}_W = \{(a_\xi, x_\xi, b_\xi) : \xi \in W\}$  is a tower of  $T$ -splitters if for  $\xi < \eta \in W$  and  $t \in T$ :

- (1)  $\{a_\xi, x_\xi, b_\xi\}$  is a partition of  $\mathbb{N}$ ,
- (2)  $a_\xi \subset^* a_\eta$ ,  $b_\xi \subset^* b_\eta$ ,
- (3)  $x_t \cap x_\xi$  is infinite.

**Definition 2.5.** If  $\mathcal{S}_W$  is a tower of  $T$ -splitters and  $Y$  is a subset  $\mathbb{N}$ , then the poset  $Q(\mathcal{S}_W, Y)$  is defined as follows. Let  $E_Y$  be the (possibly empty) set of minimal elements of  $T$  such that there is some finite  $H \subset W$  such that  $x_t \cap Y \cap \bigcap_{\xi \in H} x_\xi$  is finite. Let  $D_Y = E_Y^\perp = \{t \in T : (\forall s \in E_Y) t \perp s\}$ . A condition  $q \in Q(\mathcal{S}_W, Y)$  is a tuple  $(n^q, a^q, x^q, b^q, T^q, H^q)$  where

- (1)  $n^q \in \mathbb{N}$  and  $\{a^q, x^q, b^q\}$  is a partition of  $n^q$ ,
- (2)  $T^q \in [T]^{<\omega}$  and  $H^q \in [W]^{<\omega}$ ,
- (3)  $(a_\xi \setminus a_\eta)$ ,  $(b_\xi \setminus b_\eta)$ , and  $(x_\eta \setminus x_\xi)$  are all contained in  $n^q$  for  $\xi < \eta \in H^q$ .

We define  $q < p$  to mean  $n^p \leq n^q$ ,  $T^p \subset T^q$ ,  $H^p \subset H^q$ , and

- (4) for  $t \in T^p \cap D_Y$ ,  $x_t \cap (x^q \setminus x^p) \subset Y$ ,
- (5)  $x^q \setminus x^p \subset \bigcap_{\xi \in H^p} x_\xi$ ,
- (6)  $a^q \setminus a^p$  is disjoint from  $b_{\max(H^p)}$ ,
- (7)  $b^q \setminus b^p$  is disjoint from  $a_{\max(H^p)}$ .

**Lemma 2.9.** *If  $W \subset \gamma$ ,  $\mathcal{S}_W$  is a tower of  $T$ -splitters, and if  $G$  is  $Q(\mathcal{S}_W, Y)$ -generic, then  $\mathcal{S}_W \cup \{(a_\gamma, x_\gamma, b_\gamma)\}$  is also a tower of  $T$ -splitters where  $a_\gamma = \bigcup \{a_q : q \in G\}$ ,  $x_\gamma = \bigcup \{x_q : q \in G\}$ , and  $b_\gamma = \bigcup \{b_q : q \in G\}$ . In addition, for each  $t \in D_Y$ ,  $x_t \cap x_\xi \subset^* Y$  (and  $x_t \cap x_\xi \subset^* \mathbb{N} \setminus Y$  for  $t \in E_Y$ ).*

**Lemma 2.10.** *If  $W$  does not have cofinality  $\omega_1$ , then  $Q(\mathcal{S}_W, Y)$  is  $\sigma$ -centered.*

As usual with  $(\omega_1, \omega_1)$ -gaps,  $Q(\mathcal{S}_W, Y)$  may not (in general) be ccc if  $W$  has a cofinal  $\omega_1$  sequence.

Let  $0 \notin C \subset \theta$  be cofinal and assume that if  $C \cap \gamma$  is cofinal in  $\gamma$  and  $\text{cf}(\gamma) = \omega_1$ , then  $\gamma \in C$ .

poset

**Definition 2.6.** Fix any well-ordering  $\prec$  of  $H(\theta)$ . We define a finite support iteration sequence  $\{P_\gamma, \dot{Q}_\gamma : \gamma \in \theta\} \subset H(\theta)$ . We abuse notation and use  $Q_0$  rather than  $\dot{Q}_0$  from definition 2.2. If  $\gamma \notin C$ , then let  $\dot{Q}_\gamma$  be the  $\prec$ -least among the list of  $P_\gamma$ -names of ccc posets in  $H(\theta) \setminus \{\dot{Q}_\xi : \xi \in \gamma\}$ . If  $\gamma \in C$ , then let  $\dot{Y}_\gamma$  be the  $\prec$ -least  $P_\gamma$ -name of a subset  $\mathbb{N}$  which is in  $H(\theta) \setminus \{\dot{Y}_\xi : \xi \in C \cap \gamma\}$ . Set  $\dot{Q}_\gamma$  to be the  $P_\gamma$  name of  $Q(\mathcal{S}_{C \cap \gamma}, \dot{Y}_\gamma)$  adding the partition  $\{\dot{a}_\gamma, \dot{x}_\gamma, \dot{b}_\gamma\}$  and, where  $\mathcal{S}_{C \cap \gamma}$  is the  $P_\gamma$ -name of the  $T$ -splitting tower  $\{(a_\xi, x_\xi, b_\xi) : \xi \in C \cap \gamma\}$ .

We view the members of  $P_\theta$  as functions  $p$  with finite domain (or support) denoted  $\text{dom}(p)$ .

The main difficulty to the proof of Theorem 2.1 is to prove that the iteration  $P_\theta$  is ccc. Of course, since it is a finite support iteration, this can be proven by induction at successor ordinals.

main2ccc

**Lemma 2.11.** *For each  $\gamma \in C$  such that  $C \cap \gamma$  has cofinality  $\omega_1$ ,  $P_{\gamma+1}$  is ccc.*

*Proof.* We proceed by induction. For each  $\alpha$ , define  $p \in P_\alpha^*$  if  $p \in P_\alpha$  and there is an  $n \in \mathbb{N}$  such that

- (1) for each  $\beta \in \text{dom}(p) \cap C$ , with  $H^\beta = \text{dom}(p) \cap C \cap \beta$ , there are subsets  $a^\beta, x^\beta, b^\beta$  of  $n$  and  $T^\beta \in [T]^{<\omega}$  such that  $p \upharpoonright \beta \Vdash_{P_\beta}$  “ $p(\beta) = (n, a^\beta, x^\beta, b^\beta, T^\beta, H^\beta)$ ”

Assume that  $P_\beta^*$  is dense in  $P_\beta$  and let  $p \in P_{\beta+1}$ . To show that  $P_{\beta+1}^*$  is dense in  $P_{\beta+1}$  we must find some  $p^* \leq p$  in  $P_{\beta+1}^*$ . If  $\beta \notin C$  and  $p^* \in P_\beta^*$  is below  $p \upharpoonright \beta$ , then  $p^* \cup \{(\beta, p(\beta))\}$  is the desired element of  $P_{\beta+1}^*$ . Now assume that  $\beta \in C$  and assume that  $p \upharpoonright \beta \in P_\beta^*$  and that  $p \upharpoonright \beta$  forces that  $p(\beta)$  is the tuple  $(n_0, a, x, b, \tilde{T}, \tilde{H})$ . By an easy density argument, we may assume that  $\tilde{H} \subset \text{dom}(p)$ . Let  $n^*$  be the integer witnessing that  $p \upharpoonright \beta \in P_\beta^*$ . Let  $\zeta$  be the maximum element of  $\text{dom}(p) \cap C \cap \beta$  and let  $p \upharpoonright \zeta \Vdash_{P_\zeta}$  “ $p(\zeta) = (n^*, a^\zeta, x^\zeta, b^\zeta, T^\zeta, H^\zeta)$ ” as per the definition of  $P_{\zeta+1}^*$ . Notice that since  $\tilde{H} \subset H^\zeta$  we have that

$$p \upharpoonright \beta \Vdash_{P_\beta} “(n^*, a^*, x, b^*, T^\zeta \cup \tilde{T}, H^\zeta \cup \{\zeta\}) \leq p(\beta)”$$

where  $a^* = a \cup ([n_0, n^*] \setminus b^\zeta)$  and  $b^* = b \cup ([n_0, n^*] \cap b^\zeta)$ . Defining  $p^* \in P_{\beta+1}$  by  $p^* \upharpoonright \beta = p \upharpoonright \beta$  and  $p^*(\beta) = (n^*, a^*, x, b^*, T^\zeta \cup \tilde{T}, H^\zeta \cup \{\zeta\})$

completes the proof that  $P_{\beta+1}^*$  is dense in  $P_{\beta+1}$ , and by induction, that this holds for  $\beta = \gamma$ .

Now assume that  $\{p_\alpha : \alpha \in \omega_1\} \subset P_{\gamma+1}^*$ . By passing to a subcollection, we may assume that

- (1) the collection  $\{T^{p_\alpha(\gamma)} : \alpha \in \omega_1\}$  forms a  $\Delta$ -system with root  $T^*$ ;
- (2) the collection  $\{\text{dom}(p_\alpha) : \alpha \in \omega_1\}$  also forms a  $\Delta$ -system with root  $R$ ;
- (3) there is a tuple  $(n^*, a^*, x^*, b^*)$  so that for all  $\alpha \in \omega_1$ ,  $a^{p_\alpha(\gamma)} = a^*$ ,  $x^{p_\alpha(\gamma)} = x^*$ , and  $b^{p_\alpha(\gamma)} = b^*$ .

Since  $C \cap \gamma$  has a cofinal sequence of order type  $\omega_1$ , there is a  $\delta \in \gamma$  such that  $R \subset \delta$  and, we may assume,  $(\text{dom}(p_\alpha) \setminus \delta) \subset \min(\text{dom}(p_\beta) \setminus \delta)$  for  $\alpha < \beta < \omega_1$ . Since  $P_\delta$  is ccc, there is a pair  $\alpha < \beta < \omega_1$  such that  $p_\alpha \upharpoonright \delta$  is compatible with  $p_\beta \upharpoonright \delta$ . Define  $q \in P_{\gamma+1}$  by

- (1)  $q \upharpoonright \delta$  is any element of  $P_\delta$  which is below each of  $p_\alpha \upharpoonright \delta$  and  $p_\beta \upharpoonright \delta$ ,
- (2) if  $\delta \leq \xi \in \gamma \cap \text{dom}(p_\alpha)$ , then  $q(\xi) = p_\alpha(\xi)$ ,
- (3) if  $\delta \leq \xi \in \text{dom}(p_\beta) \setminus C$ , then  $q(\xi) = p_\beta(\xi)$ ,
- (4) if  $\delta \leq \xi \in \text{dom}(p_\beta) \cap C$ , then

$$q(\xi) = (n^*, a^{p_\beta(\xi)}, x^{p_\beta(\xi)}, b^{p_\beta(\xi)}, T^{p_\beta(\xi)}, H^{p_\beta(\xi)} \cup H^{p_\alpha(\gamma)}).$$

The main non-trivial fact about  $q$  is that it is in  $P_{\gamma+1}$  which depends on the fact that, by induction on  $\eta \in C \cap \gamma$ ,  $q \upharpoonright \eta$  forces that

$$(a_\eta \setminus a_\xi) \cup (b_\eta \setminus b_\xi) \cup (x_\eta \setminus x_\xi) \subset n^* \text{ for } \xi \in C \cap \eta.$$

It now follows trivially that  $q$  is below each of  $p_\alpha$  and  $p_\beta$ . □

*Proof of Theorem 2.1.* This completes the construction of the ccc poset  $P$  ( $P_\theta$  as above). Let  $G \subset (P \times \prod_{\lambda \in \Lambda} T_\lambda)$  be generic. It follows that  $V[G \cap P]$  is a model of Martin's Axiom and  $\mathfrak{c} = \theta$ . Furthermore by applying Lemma 2.4 with  $\mu = \omega$  and Lemma 2.3, we have that  $P_2 = \prod_{\lambda \in \Lambda} T_\lambda$  is  $\omega_1$ -distributive in the model  $V[G \cap P]$ . Therefore all subsets of  $\mathbb{N}$  in the model  $V[G]$  are also in the model  $V[G \cap P]$ .

Fix any  $\lambda \in \Lambda$  and let  $\rho_\lambda$  denote the generic branch in  $T_\lambda$  given by  $G$ . Let  $G^\lambda$  denote the generic filter on  $P \times \prod_{\lambda \neq \mu \in \Lambda} T_\mu$  and work in the model  $V[G^\lambda]$ . It follows easily by Lemma 2.4 and Lemma 2.3, that  $T_\lambda$  is a  $\lambda^+$ -Souslin tree in this model. Therefore by Proposition 2.2,  $K_\lambda = \bigcap_{\alpha < \lambda^+} x_{\rho_\lambda(\alpha)}^*$  is a tie-set of  $\mathfrak{bd}$ -type  $(\lambda^+, \lambda^+)$  in  $V[G]$ . By the definition of the iteration in  $P$ , it follows that condition (4) of Lemma 2.2 is also satisfied, hence the tie-set  $K = \bigcap_{\xi \in C} x_\xi^*$  meets  $K_\lambda$  in a single point  $z_\lambda$ . A simple genericity argument confirms that conditions (5) and (6) of Proposition 2.2 also holds, hence  $z_\lambda$  is a tie-point of  $K_\lambda$ .

It follows from Corollary 2.6 that there are no *unwanted* tie-sets in  $\beta\mathbb{N} \setminus \mathbb{N}$  in  $V[G]$ , at least if there are none in  $V[G \cap P]$ . Since  $\mathfrak{p} = \mathfrak{c}$  in  $V[G \cap P]$ , it follows from Proposition 1.3 that indeed there are no such tie-sets in  $V[G \cap P]$ .  $\square$

Unfortunately the next result shows that the construction does not provide us with our desired variety of tie-points (even with variations in the definition of the iteration). We do not know if  $\mathfrak{bd}$ -type can be improved to  $\delta$ -type (or simply exclude tie-points altogether).

**Proposition 2.12.** *In the model constructed in Theorem 2.1, there are no tie-points with  $\mathfrak{bd}$ -type  $(\kappa_1, \kappa_2)$  for any  $\kappa_1 \leq \kappa_2 < \mathfrak{c}$ ,*

*Proof.* Assume that  $\beta\mathbb{N} \setminus \mathbb{N} = A \overset{x}{\times} B$  and that  $\delta(\mathcal{I}_A) = \kappa_1$  and  $\delta(\mathcal{I}_B) = \kappa_2$ . It follows from Corollary 2.6 that we can assume that  $\kappa_1 = \kappa_2 = \lambda^+$  for some  $\lambda \in \Lambda$ . Also, following the proof of Corollary 2.6, there are  $P \times T_\lambda$ -names  $\mathcal{J}_A = \{\tilde{a}_\alpha : \alpha \in \lambda^+\}$  and  $P \times T_{\lambda^+}$ -names  $\mathcal{J}_B = \{\tilde{b}_\beta : \beta \in \lambda^+\}$  such that the valuation of these names by  $G$  result in increasing (mod finite) chains in  $\mathcal{I}_A$  and  $\mathcal{I}_B$  respectively whose downward closures are dense. Passing to  $V[G \cap P]$ , since  $T_\lambda$  has the  $\theta$ -cc, there is a Boolean subalgebra  $\mathcal{B} \in [\mathcal{P}(\mathbb{N})]^{<\theta}$  such that each  $\tilde{a}_\alpha$  and  $\tilde{b}_\beta$  is a name of a member of  $\mathcal{B}$ . Furthermore, there is an infinite  $C \subset \mathbb{N}$  such that  $C \notin x$  and each of  $b \cap C$  and  $b \setminus C$  are infinite for all  $b \in \mathcal{B}$ . Since  $C \notin x$ , there is a  $Y \subset \mathbb{N}$  (in  $V[G]$ ) such that  $C \cap Y \in \mathcal{I}_A$  and  $C \setminus Y \in \mathcal{I}_B$ . Now choose  $t_0 \in T_\lambda$  which forces this about  $C$  and  $Y$ . Back in  $V[G \cap P]$ , set

$$\mathcal{A} = \{b \in \mathcal{B} : (\exists t_1 \leq t_0) t_1 \Vdash_{T_\lambda} "b \in \mathcal{J}_A \cup \mathcal{J}_B"\}.$$

Since  $V[G \cap P]$  satisfies  $\mathfrak{p} = \theta$  and  $\mathcal{A}^\downarrow$  is forced by  $t_0$  to be dense in  $[\mathbb{N}]^\omega$ , there must be a finite subset  $\mathcal{A}'$  of  $\mathcal{A}$  which covers  $C$ . It also follows easily then that there must be some  $a, b \in \mathcal{A}'$  and  $t_1, t_2$  each below  $t_0$  such that  $t_1 \Vdash_{T_{\lambda^+}} "a \in \mathcal{J}_A"$ ,  $t_2 \Vdash_{T_{\lambda^+}} "b \in \mathcal{J}_B"$ , and  $a \cap b$  is infinite. The final contradiction is that we will now have that  $t_0$  fails to force that  $C \cap a \subset^* Y$  and  $C \cap b \subset^* (\mathbb{N} \setminus Y)$ .  $\square$

### 3. $T$ -INVOLUTIONS

In this section we strengthen the result in Theorem 2.1 by making each  $K \cap K_\lambda$  a symmetric tie-point in  $K_\lambda$  (at the expense of weakening Martin's Axiom in  $V[G \cap P]$ ). This is progress in producing involutions with some control over the fixed point set but we are still not able to make  $K$  the fixed point set of an involution. A poset is said to be  $\sigma$ -linked if there is a countable collection of linked (elements are pairwise compatible) which union to the poset. The statement  $\text{MA}(\sigma\text{-linked})$

is, of course, the assertion that Martin's Axiom holds when restricted to  $\sigma$ -linked posets.

Our approach is to replace  $T$ -splitting towers by the following notion. If  $f$  is a (partial) involution on  $\mathbb{N}$ , let  $\min(f) = \{n \in \mathbb{N} : n < f(n)\}$  and  $\max(f) = \{n \in \mathbb{N} : f(n) < n\}$  (hence  $\text{dom}(f)$  is partitioned into  $\min(f) \cup \text{fix}(f) \cup \max(f)$ ).

**Definition 3.1.** A sequence  $\mathfrak{T} = \{(A_\xi, f_\xi) : \xi \in W\}$  is a tower of  $T$ -involutions if  $W$  is a set of ordinals and for  $\xi < \nu \in W$  and  $t \in T$

- (1)  $A_\nu \subset^* A_\xi$ ;
- (2)  $f_\xi^2 = f_\xi$  and  $f_\xi \upharpoonright (\mathbb{N} \setminus \text{fix}(f_\xi)) \subset^* f_\nu$ ;
- (3)  $f_\xi[x_t] =^* x_t$  and  $\text{fix}(f_\xi) \cap x_t$  is infinite;
- (4)  $f_\xi([n, m]) = [n, m]$  for  $n < m$  both in  $A_\xi$ .

Say that  $\mathfrak{T}$ , a tower of  $T$ -involutions, is **full** if  $K = K_{\mathfrak{T}} = \bigcap \{\text{fix}(f_\xi)^* : \xi \in W\}$  is a tie-set with  $\beta\mathbb{N} \setminus \mathbb{N} = A \mathop{\times}_K B$  where  $A = K \cup \bigcup \{\min(f_\xi)^* : \xi \in W\}$  and  $B = K \cup \bigcup \{\max(f_\xi)^* : \xi \in W\}$ .

If  $\mathfrak{T}$  is a tower of  $T$ -involutions, then there is a natural involution  $F_{\mathfrak{T}}$  on  $\bigcup_{\xi \in W} (\mathbb{N} \setminus \text{fix}(f_\xi))^*$ , but this  $F_{\mathfrak{T}}$  need not extend to an involution on the closure of the union - even if the tower is full.

In this section we prove the following theorem.

**Theorem 3.1.** *Assume GCH and that  $\Lambda$  is a set of regular uncountable cardinals such that for each  $\lambda \in \Lambda$ ,  $T_\lambda$  is a  $<\lambda$ -closed  $\lambda^+$ -Souslin tree. Let  $T$  denote the tree sum of  $\{T_\lambda : \lambda \in \Lambda\}$ . There is forcing extension in which there is  $\mathfrak{T}$ , a full tower of  $T$ -involutions, such that the associated tie-set  $K$  has  $\mathfrak{bd}$ -type  $(\mathfrak{c}, \mathfrak{c})$  and such that for each  $\lambda \in \Lambda$ , there is a tie-set  $K_\lambda$  of  $\mathfrak{bd}$ -type  $(\lambda^+, \lambda^+)$  such that  $F_{\mathfrak{T}}$  does induce an involution on  $K_\lambda$  with a singleton fixed point set  $\{z_\lambda\} = K \cap K_\lambda$ . Furthermore, for  $\mu \leq \lambda < \mathfrak{c}$ , if  $\mu \neq \lambda$  or  $\lambda \notin \Lambda$ , then there is no tie-set of  $\mathfrak{bd}$ -type  $(\mu, \lambda)$ .* main3

**Question 3.1.** Can the tower  $\mathfrak{T}$  in Theorem 3.1 be constructed so that  $F_{\mathfrak{T}}$  extends to an involution of  $\beta\mathbb{N} \setminus \mathbb{N}$  with  $\text{fix}(F) = K_{\mathfrak{T}}$ ?

We introduce  $T$ -tower extending forcing.

**Definition 3.2.** If  $\mathfrak{T} = \{(A_\xi, f_\xi) : \xi \in W\}$  is a tower of  $T$ -involutions and  $Y$  is a subset of  $\mathbb{N}$ , we define the poset  $Q = Q(\mathfrak{T}, Y)$  as follows. Let  $E_Y$  be the (possibly empty) set of minimal elements of  $T$  such that there is some finite  $H \subset W$  such that  $x_t \cap Y \cap \bigcap_{\xi \in H} \text{fix}(f_\xi)$  is finite. Let  $D_Y = E_Y^\perp = \{t \in T : (\forall s \in E_Y) t \perp s\}$ . A tuple  $q \in Q$  if  $q = (a^q, f^q, T^q, H^q)$  where: tower

- (1)  $H^q \in [W]^{<\omega}$ ,  $T^q \in [T]^{<\omega}$ , and  $n^q = \max(a^q) \in A_{\alpha^q}$  where  $\alpha^q = \max(H^q)$ ,
- (2)  $f^q$  is an involution on  $n^q$ ,
- (3)  $(A_{\alpha^q} \setminus n^q) \subset A_\xi$  for each  $\xi \in H^q$ ,
- (4)  $\text{fin}(T^q) \subset n^q$ ,
- (5)  $f_\xi \upharpoonright (\mathbb{N} \setminus (\text{fix}(f_\xi) \cup n^q)) \subset f_{\alpha^q}$  for  $\xi \in H^q$ ,
- (6)  $f_{\alpha^q}[x_t \setminus n^q] = x_t \setminus n^q$  for  $t \in T^q$ ,

We define  $p < q$  if  $n^p \leq n^q$ , and for  $t \in T^p$  and  $i \in [n^p, n^q]$ :

- (7)  $a^p = a^q \cap n^p$ ,  $T^p \subset T^q$ , and  $H^p \subset H^q$ ,
- (8)  $a^q \setminus a^p \subset A_{\alpha^p}$ ,
- (9)  $f_{\alpha^p}(i) \neq i$  implies  $f^q(i) = f_{\alpha^p}(i)$ ,
- (10)  $f^q([n, m]) = [n, m]$  for  $n < m$  both in  $a^q \setminus a^p$ ,
- (11)  $f^q(x_t \cap [n^p, n^q]) = x_t \cap [n^p, n^q]$ ,
- (12) if  $t \in D^p$  and  $i \in x_t \cap \text{fix}(f^q)$ , then  $i \in Y$

It should be clear that the involution  $f$  introduced by  $Q(\mathfrak{T}, Y)$  satisfies that for each  $t \in D_Y$ ,  $\text{fix}(f) \cap x_t \subset^* Y$ , and, with the help of the following density argument, that  $\mathfrak{T} \cup \{(\gamma, A, f)\}$  is again a tower of  $T$ -involutions where  $A$  is the infinite set introduced by the first coordinates of the conditions in the generic filter.

density

**Lemma 3.2.** *If  $W \subset \gamma$ ,  $Y \subset \mathbb{N}$ , and  $\mathfrak{T} = \{(A_\xi, f_\xi) : \xi \in W\}$  is a tower of  $T$ -involutions and  $p \in Q(\mathfrak{T}, Y)$ , then for any  $\tilde{T} \in [T]^{<\omega}$ ,  $\zeta \in W$ , and any  $m \in \mathbb{N}$ , there is a  $q < p$  such that  $n^q \geq m$ ,  $\zeta \in H^q$ ,  $T^q \supset \tilde{T}$ , and  $\text{fix}(f^q) \cap (x_t \setminus n^p)$  is not empty for each  $t \in T^p$ .*

*Proof.* Let  $\beta$  denote the maximum  $\alpha^p$  and  $\zeta$  and let  $\eta$  denote the minimum. Choose any  $n^q \in A_{\alpha^q} \setminus m$  large enough so that

- (1)  $f_{\alpha^p}[x_t \setminus n^q] = x_t \setminus n^q$  for  $t \in \tilde{T}$ ,
- (2)  $f_\eta \upharpoonright (\mathbb{N} \setminus (n^q \cup \text{fix}(f_\eta))) \subset f_\beta$ ,
- (3)  $A_\beta \setminus A_\eta$  is contained in  $n^q$ ,
- (4)  $n^q \cap [i]_{T^p} \cap \text{fix}(f_{\alpha^p})$  is non-empty for each  $i \in \mathbb{N}$  such that  $[i]_{T^p}$  is in the finite set  $\{[i]_{T^p} : i \in \mathbb{N}\} \setminus \text{fin}(T^p)$ ,
- (5) if  $i \in x_t \cap n^q \setminus n^p$  for some  $t \in D_Y \cap T^p$ , then  $Y$  meets  $[i]_{T^p} \cap n^q \setminus n^p$  in at least two points.

Naturally we also set  $H^q = H^p \cup \{\zeta\}$  and  $T^q = T^p \cup \tilde{T}$ . The choice of  $n^q$  is large enough to satisfy (3), (4), (5) and (6) of Definition 3.2. We will set  $a^q = a^p \cup \{n^q\}$  ensuring (1) of Definition 3.2. Therefore for any  $f^q \supset f^p$  which is an involution on  $n^q$ , we will have that  $q = (a^q, f^q, T^q, H^q)$  is in the poset. We have to choose  $f^q$  more carefully to ensure that  $q \leq p$ . Let  $S = [n^p, n^q] \cap \text{fix}(f_{\alpha^p})$ , and  $S' = [n^p, n^q] \setminus S$ . We choose  $\bar{f}$  an involution on  $S$  and set  $f^q = f^p \cup (f_{\alpha^p} \upharpoonright S') \cup \bar{f}$ . We leave

it to the reader to check that it suffices to ensure that  $\bar{f}$  sends  $[i]_{T^p} \cap S$  to itself for each  $t \in T^p$  and that  $\text{fix}(\bar{f}) \cap x_t \subset Y$  for each  $t \in T^p \cap D_Y$ . Since the members of  $\{[i]_{T^p} \cap S : i \in \mathbb{N}\}$  are pairwise disjoint we can define  $\bar{f}$  on each separately.

For each  $[i]_{T^p} \cap S$  which has even cardinality, choose two points  $y_i, z_i$  from it so that if there is a  $p \in D_Y \cap T^p$  such that  $[i]_{T^p} \subset x_t$ , then  $\{y_i, z_i\} \subset Y$ . Let  $\bar{f}$  be any involution on  $[i]_{T^p} \cap S$  so that  $y_i, z_i$  are the only fixed points. If  $[i]_{T^p} \cap S$  has odd cardinality then choose a point  $y_i$  from it so that if  $[i]_{T^p}$  is contained in  $x_t$  for some  $t \in D_Y \cap T^p$ , then  $y_i \in Y \cap [i]_{T^p} \cap S$ . Set  $\bar{f}(y_i) = y_i$  and choose  $\bar{f}$  to be any fixed-point free involution on  $[i]_{T^p} \cap S \setminus \{y_i\}$ .  $\square$

Let  $P_\theta$  now be the finite support iteration defined as in Definition 2.6 except for two important changes. For  $\gamma \in C$ , we replace  $T$ -splitting towers by the obvious inductive definition of towers of  $T$ -involutions when we replace the posets  $\dot{Q}(\mathcal{S}_{C \cap \gamma}, \dot{Y}_\gamma)$  by  $\dot{Q}(\mathfrak{T}_{C \cap \gamma}, \dot{Y}_\gamma)$ . For  $\gamma \notin C$  we require that  $\Vdash_{P_\gamma}$  “ $\dot{Q}_\gamma$  is  $\sigma$ -linked.”

Special (parity) properties of the family  $\{x_t : t \in T\}$  are needed to ensure that  $\Vdash_{P_\gamma}$  “ $\dot{Q}(\mathcal{S}_{C \cap \gamma}, \dot{Y}_\gamma)$  is ccc ” even for cases when  $\text{cf}(\gamma)$  is not  $\omega_1$ .

The proof of Theorem 3.1 is virtually the same as the proof of Theorem 2.1 (so we skip) once we have established that the iteration is ccc.

**Lemma 3.3.** *For each  $\gamma \in C$ ,  $P_{\gamma+1}$  is ccc.*

main3ccc

*Proof.* We again define  $P_\alpha^*$  to be those  $p \in P_\alpha$  for which there is an  $n \in \mathbb{N}$  such that for each  $\beta \in \text{dom}(p) \cap C$ , there are  $n \in a^\beta \subset n+1$ ,  $f^\beta \in n^n$ ,  $T^\beta \in [T]^{<\omega}$ , and  $H^\beta = \text{dom}(p) \cap C \cap \beta$  such that  $p \upharpoonright \beta \Vdash_{P_\beta}$  “ $p(\beta) = (a^\beta, f^\beta, T^\beta, H^\beta)$ ”. However, in this proof we must also make some special assumptions in coordinates other than those in  $C$ . For each  $\xi \in \gamma \setminus C$ , we fix a collection  $\{\dot{Q}(\xi, n) : n \in \omega\}$  of  $P_\xi$ -names so that

$$1 \Vdash_{P_\xi} \text{“}\dot{Q}_\xi = \bigcup_n \dot{Q}(\xi, n) \text{ and } (\forall n) \dot{Q}(\xi, n) \text{ is linked.} \text{”}$$

The final restriction on  $p \in P_\alpha^*$  is that for each  $\xi \in \alpha \setminus C$ , there is a  $k_\xi \in \omega$  such that  $p \upharpoonright \xi \Vdash_{P_\xi}$  “ $p(\xi) \in \dot{Q}(\xi, k_\xi)$ ”.

Just as in Lemma 2.11, Lemma 3.2 can be used to show by induction that  $P_\alpha^*$  is a dense subset of  $P_\alpha$ . This time though, we also demand that  $\text{dom}(f^{p(0)}) = n \times T^{p(0)}$  is such that  $T^\beta \subset T^{p(0)}$  for all  $\beta \in \text{dom}(p) \cap C$  and some extra argument is needed because of needing to decide values in the name  $\dot{Y}_\gamma$  as in the proof of Lemma 3.2. Let  $p \in P_{\beta+1}$  and assume that  $P_\beta^*$  is dense in  $P_\beta$ . By density, we may assume that  $p \upharpoonright$

$\beta \in P_\beta^*$ ,  $H^{p(\beta)} \subset \text{dom}(p)$ ,  $T^{p(\beta)} \subset T^{p(0)}$ , and that  $p \upharpoonright \beta$  has decided the members of the set  $D_{\dot{Y}_\beta} \cap T^{p(\beta)}$ . We can assume further that for each  $t \in D_{\dot{Y}_\beta} \cap T^{p(\beta)}$ ,  $p \upharpoonright \beta$  has forced a value  $y_t \in \dot{Y}_\beta \cap x_t \setminus \bigcup\{x_s : s \in T^p \text{ and } s \not\leq t\}$  such that  $y_t > n^{p(\beta)}$ . We are using that  $T$  is not finitely branching to deduce that if  $t \in D_{\dot{Y}_\beta}$ , then  $p \upharpoonright \beta \Vdash_{P_\beta}$  “ $\dot{Y}_\beta \cap x_t \setminus \bigcup\{x_s : s \in T^p \text{ and } s \not\leq t\}$  is non-empty” (which follows since  $\dot{Y}_\beta$  must meet  $x_s$  for each immediate successor  $s$  of  $t$ ). Choose any  $m$  larger than  $y_t$  for each  $t \in T^{p(\beta)}$ . Without loss of generality, we may assume that the integer  $n^*$  witnessing that  $p \upharpoonright \beta \in P_\beta^*$  is at least as large as  $m$  and that  $n^* \in \bigcap_{\xi \in H^{p(\beta)}} A_\xi$ . Construct  $\bar{f}$  just as in Lemma 3.2, except that this time there is no requirement to actually have fixed points so one member of  $\dot{Y}_\beta$  in each appropriate  $[i]_{T^{p(\beta)}}$  is all that is required. Let  $\zeta = \max(\text{dom}(p) \cap \beta)$ . No new forcing decisions are required of  $p \upharpoonright \beta$  in order to construct a suitable  $\bar{f}$ , hence this shows that  $p \upharpoonright \beta \cup \{(\beta, q)\}$  (where  $q$  is constructed below  $p(\beta)$  as in Corollary 3.2 in which  $H^{p(\zeta)} \cup \{\zeta\}$  is added to  $H^q$ ) is the desired extension of  $p$  which is a member of  $P_{\beta+1}^*$ .

Now to show that  $P_{\gamma+1}$  is ccc, let  $\{p_\alpha : \alpha \in \omega_1\} \subset P_{\gamma+1}^*$ . Clearly we may assume that the family  $\{p_\alpha(0) : \alpha \in \omega_1\}$  are pairwise compatible and that there is a single integer  $n$  such that, for each  $\alpha \in \omega_1$ ,  $\text{dom}(p_\alpha(0)) = n \times T^\alpha$  for some  $T^\alpha \in [T]^{<\omega}$ . Also, we may assume that there is some  $(a, h)$  such that, for each  $\alpha$ ,

$$p_\alpha \upharpoonright \gamma \Vdash_{P_\gamma} “p(\gamma) = (a, h, T^\alpha, H^\alpha)”$$

where  $H^\alpha = \text{dom}(p_\alpha) \cap C \cap \gamma$ .

The family  $\{\text{dom}(p_\alpha) \cap \gamma : \alpha \in \omega_1\}$  may be assumed to form a  $\Delta$ -system with root  $R$ . For each  $\xi \in R$ , we may assume that, if  $\xi \notin C$ , there is a single  $k_\xi \in \omega$  such that, for all  $\alpha$ ,  $p_\alpha \upharpoonright \xi \Vdash_{P_\xi}$  “ $p_\alpha(\xi) \in \dot{Q}(\xi, k_\xi)$ ”, and if  $\xi \in C$ , then there is a single  $(a_\xi, h_\xi)$  such that  $p_\alpha \upharpoonright \xi \Vdash_{P_\xi}$  “ $p_\alpha(\xi) = (a_\xi, h_\xi, T^\alpha, H^\alpha \cap \xi)$ ”. For convenience, for each  $\xi \notin C$  let  $\dot{r}_\xi$  be a  $P_\xi$ -name of a function from  $\omega \times \dot{Q}_\xi^2$  such that, for each  $k \in \omega$ ,

$$1 \Vdash_{P_\xi} “\dot{r}_\xi(k, q, q') \leq q, q' \ (\forall q, q' \in \dot{Q}(\xi, k))”.$$

Fix any  $\alpha < \beta < \omega_1$  and let  $H = H^\alpha \cup H^\beta$ . Recall that  $p_\alpha(0)$  and  $p_\beta(0)$  are compatible. Recursively define a  $P_\xi$ -name  $q(\xi)$  for  $\xi \in$



$\text{dom}(p_\alpha) \cup \text{dom}(p_\beta)$  so that  $q \upharpoonright \xi \Vdash_{P_\xi}$

$$"q(\xi) = \begin{cases} (n, T^\alpha \cup T^\beta, f^{p_\alpha(0)} \cup f^{p_\beta(0)}) & \xi = 0 \\ \dot{r}_\xi(k_\xi, p_\alpha(\xi), p_\beta(\xi)) & \xi \in R \setminus C \\ p_\alpha(\xi) & \xi \in \text{dom}(p_\alpha) \setminus (R \cup C) \\ p_\beta(\xi) & \xi \in \text{dom}(p_\beta) \setminus (R \cup C) \\ (a_\xi, h_\xi, T^\alpha \cup T^\beta, H \cap \xi) & \xi \in C. \end{cases}."$$

Now we check that  $q \in P_\xi$  by induction on  $\xi \in \gamma + 1$ .

The first thing to note is that not only is this true for  $\xi = 1$ , but also that  $q(0) \Vdash_{Q_0}$  " $\text{fin}(T^\alpha \cup T^\beta) \subset n$ ". Since  $p_\alpha$  and  $p_\beta$  are each in  $P_{\gamma+1}^*$ , this shows that condition (4) of Definition 3.2 will hold in all coordinates in  $C$ .

We also prove, by induction on  $\xi$ , that  $q \upharpoonright \xi$  forces that for  $\eta < \delta$  both in  $H \cap \xi$  and  $t \in T^\alpha \cup T^\beta$ ,  $f_\delta[x_t \setminus n] = x_t \setminus n$ ,  $f_\eta \upharpoonright (\mathbb{N} \setminus (\text{fix}(f_\eta) \cup n)) \subset f_\delta$  and  $A_\delta \setminus n \subset A_\eta$ .

Given  $\xi \in H$  and the assumption that  $q \upharpoonright \xi \in P_\xi$ , and  $\alpha = \alpha^{q(\xi)} = \max(H \cap \xi)$ , condition (3), (5), and (6) of Definition 3.2 hold by the inductive hypothesis above. It follows then that  $q \upharpoonright \xi \Vdash_{P_\xi}$  " $q(\xi) \in \dot{Q}_\xi$ ". By the definition of the ordering on  $\dot{Q}_\xi$ , given that  $H \cap \xi = H^{q(\xi)}$  and  $T^\alpha \cup T^\beta = T^{q(\xi)}$ , it follows that the inductive hypothesis then holds for  $\xi + 1$ .

It is trivial for  $\xi \in \text{dom}(q) \setminus C$ , that  $q \upharpoonright \xi \in P_\xi$  implies that  $q \upharpoonright \xi \Vdash_{P_\xi}$  " $q(\xi) \in \dot{Q}_\xi$ ". This completes the proof that  $q \in P_{\gamma+1}$ , and it is trivial that  $q$  is below each of  $p_\alpha$  and  $p_\beta$ .  $\square$

*Remark 1.* If we add a trivial tree  $T_1$  to the collection  $\{T_\lambda : \lambda \in \Lambda\}$  (i.e.  $T_1$  has only a root), then the root of  $T$  has a single extension which is a maximal node  $t$ , and with no change to the proof of Theorem 3.1, one obtains that  $F$  induces an automorphism on  $x_t^*$  with a single fixed point. Therefore, it is consistent (and likely as constructed) that  $\beta\mathbb{N} \setminus \mathbb{N}$  will have symmetric tie-points of type  $(\mathfrak{c}, \mathfrak{c})$  in the model  $V[G \cap P]$  and  $V[G]$ .

*Remark 2.* In the proof of Theorem 2.1, it is easy to arrange that each  $K_\lambda$  ( $\lambda \in \Lambda$ ) is also  $K_{\mathfrak{T}_\lambda}$  for a  $(T_\lambda$ -generic) full tower,  $\mathfrak{T}_\lambda$ , of  $\mathbb{N}$ -involutions. However the generic sets added by the forcing  $P$  will prevent this tower of involutions from extending to a full involution.

#### 4. QUESTIONS

In this section we list all the questions with their original numbering.

**Question 1.1.** Can there be a tie-point in  $\beta\mathbb{N} \setminus \mathbb{N}$  with  $\delta$ -type  $(\kappa, \lambda)$  with  $\kappa \leq \lambda$  less than the character of the point?

**Question 1.2.** Can  $\beta\mathbb{N} \setminus \mathbb{N}$  have tie-points of  $\delta$ -type  $(\omega_1, \omega_1)$  and  $(\omega_2, \omega_2)$ ?

**Question 1.3.** Does  $\mathfrak{p} > \omega_1$  imply there are no tie-points of  $\mathfrak{b}$ -type  $(\omega_1, \omega_1)$ ?

**Question 1.4.** If  $F$  is an involution on  $\beta\mathbb{N} \setminus \mathbb{N}$  such that  $K = \text{fix}(F)$  has empty interior, is  $K$  a (symmetric) tie-set?

**Question 1.5.** Is there some natural restriction on which compact spaces can (or can not) be homeomorphic to the fixed point set of some involution of  $\beta\mathbb{N} \setminus \mathbb{N}$ ?

**Question 1.6.** If  $F$  is an involution of  $\mathbb{N}^*$ , is the quotient space  $\mathbb{N}^*/F$  (in which each  $\{x, F(x)\}$  is collapsed to a single point) a homeomorphic copy of  $\beta\mathbb{N} \setminus \mathbb{N}$ ?

**Question 3.1.** Can the tower  $\mathfrak{T}$  in Theorem 3.1 be constructed so that  $F_{\mathfrak{T}}$  extends to an involution of  $\beta\mathbb{N} \setminus \mathbb{N}$  with  $\text{fix}(F) = K_{\mathfrak{T}}$ ?

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