MANY PARTITION RELATIONS BELOW DENSITY
SH918

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Abstract. We force $2^\lambda$ to be large and for many pairs in the interval $(\lambda, 2^\lambda)$ a strong version of the polarized partition relations hold. We apply this to problems in general topology. E.g., consistently, every $2^\lambda$ is successor of singular and for every Hausdorff regular space $X$, $\text{hd}(X) \leq s(X)^{+3}$, $\text{hL}(X) \leq s(X)^{+3}$ and better when $s(X)$ is regular, via a half-graph partition relations. For the case $s(X) = \aleph_0$ we get $\text{hd}(X)$, $\text{hL}(X) \leq \aleph_2$. 

Date: February 4, 2014.

2010 Mathematics Subject Classification. Primary 03E35, 03E02; Secondary: 54A25, 54A35.

Key words and phrases. set theory, independence, forcing, partition relations, topological cardinal invariants, hereditary density, hereditary Lindelöf.

The author thanks Alice Leonhardt for the beautiful typing. Research supported by the United States-Israel Binational Science Foundation (Grant No. 2002323). First Typed - 06/Dec/21.
Anotated Content

§0 Introduction, pg. 3

§1 A Criterion for Strong Polarized Partition Relations, pgs.5-11

[We give sufficient conditions for having strong versions of polarized partition relations after forcing.]

§2 The forcing, pgs.12-24

[Assume GCH for simplicity and \( p \) a parameter with \( \lambda < \mu \) regular and \( \Theta \subseteq \text{Reg} \cap [\lambda, \mu^+] \) and we define \( Q_p \) which adds \( \mu \) Cohen subsets to \( \lambda \) but have many kinds of supports, one for each \( \theta \in \Theta \), influencing the order.]

§3 Applying the criterion, pgs. 25-30

[The main result is that (cardinal arithmetic is changed just by making \( 2^\lambda = \mu \) and) using §1 we prove the strong version of polarized partition relations hold in many instances.]
0. Introduction

Out motivation is a problem in general topology and for this we get a consistency result in the partition calculus.

In Juhasz-Shelah [JuSh:899] was proved: if $\forall \mu < \lambda)(\mu^{<\lambda} < \lambda)$ then there is a c.c.c. forcing notion that adds a regular topological space, hereditarily Lindelöf of density $\lambda$.

A natural question asked there ([JuSh:899]) is:

**Problem 0.1.** Assume $\aleph_1 < \lambda \leq 2^{\aleph_0}$. Does there exist (i.e., provably in ZFC) a hereditary Lindelöf regular space of density $\lambda$?

On cardinal invariants in general topology see [Juh80].

We prove the consistency of a negative answer, in fact of stronger results by proving the consistency of strong variants of polarized partition relations (the half-graphs, see below). They are strong enough to resolve the question about hereditary density (and hereditary Lindelöf). Moreover, if $\lambda = \lambda^{<\lambda} < \mu = \mu^{<\mu}$ (and G.C.H. holds in $[\lambda, \mu)$), then there is a forcing extension making $2^\lambda \geq \mu$ neither adding new $(< \lambda)$-sequences nor collapsing cardinals such that for many pairs $\lambda_* < \mu_*$ in the interval we have the appropriate partition relations.

An earlier result is in the paper [Sh:276, Theorem 1.1, pg.357] and it states the following: if $\lambda > \kappa > \mu$ are regular cardinals, $\lambda > \kappa^{++}$, then there is a cardinal and cofinality preserving forcing that makes $2^\mu = \lambda$ and $\kappa^{++} \rightarrow (\kappa^{++}, (\kappa; \kappa)_\lambda)^2$ in addition to the main result there $2^\lambda \rightarrow [\lambda]^{<\alpha}_2$, see more in [Sh:289], [Sh:288], [Sh:481], [Sh:546]. The applied notion of forcing $[Q, \leq]$ is the following: $p \in Q$ if $p$ is a function from a subset Dom$(p) \in [\lambda]^\leq \alpha$ into Add$(\mu, 1) - \{\emptyset\}$ where Add$(\mu, 1)$ denotes the forcing adding a Cohen subset of $\mu$. $p \leq q$ if Dom$(p) \supseteq$ Dom$(q)$, $p(\alpha) \leq q(\alpha)$ for $\alpha \in$ Dom$(q)$ and $|[\alpha \in$ Dom$(q) : p(\alpha) \neq q(\alpha)]| < \mu$.

For simultaneously many $n$-place polarized partition relation Shelah-Stanley [ShSt:608] deals with it but there are problems there, so we do not rely on it.

Our main result in general topology is Theorem 3.10, by it: consistently, G.C.H. fails badly ($2^\mu$ is a successor of a limit cardinal $> \mu$ except when $\mu$ is strong limit singular and then $2^\mu = \mu^+$) and hd$(X)$, hL$(X)$ are $\leq s(X)^{+3}$ for every Hausdorff regular $X$ and $|X| \leq 2^{(hd(X))^+}, w(X) \leq 2^{(hL(X))^+}$ for any Hausdorff $X$. (Usually $s(X)^{+2}$ suffice so in particular $X$ is hereditary Lindelöf $\Rightarrow$ $X$ has density $\leq \aleph_2$).

Concerning partition relations we give a generalization of the earlier result explained above, namely, the consistency of $2^{\aleph_0} = \lambda$ and $\mu^{++} \rightarrow (\mu, (\mu; \mu)_\mu)^2$ simultaneously holding for each regular cardinal $\mu$ such that $\mu^{++} \leq \lambda$. This gives a model in which though GCH fails badly, we have strong enough partition relations implying that the hereditary density and the hereditary Lindelöf numbers of a $T_3$ space $X$ are bounded by $s(X)^{+3}$ where $s(X)$ stands for spread.

The notion of forcing $(P, \leq)$ used for the argument is defined as follows. For each regular cardinal $\mu < \lambda$ define the following equivalence relation $E_\mu$ on $\lambda$. $xE_\mu y$ iff $x + \mu = y + \mu$. Let $[x]_\mu$ denote the equivalence class of $x$, $p \in P$ if $p$ is a function from some set Dom$(p) \subseteq \lambda$ into $\{0, 1\}$ such that $|[x]_\mu \cap$ Dom$(p)| < \mu$ holds for every successor $\mu < \lambda, x < \lambda$. $p \leq q$ if $p \supseteq q$ and for every successor $\mu < \lambda$ we have $|[x]_\mu : \emptyset \neq$ Dom$(q) \cap [x]_\mu \neq$ Dom$(p) \cap [x]_\mu| < \mu$.
This notion of forcing $(P, \leq)$, in a most remarkable way, imitates concurrently several different posets $(Q, \leq)$ as defined above. Not surprisingly, in order to show that $(P, \leq)$ is cardinal and cofinality preserving, the author uses ideas similar to those in [Sh:276].

In order to prove the main claim, that is, the partition relation, we use the following trick: we find a condition $\bar{p}$ such that the dense sets we are interested in are all dense below $\bar{p}$. It suffices, therefore, to show that forcing with the part below $\bar{p}$ gives the required result, and this reduces the problem to showing that a certain notion of forcing $(R, \leq)$ forces the sought-for-partition relation where $|R|$ is small (compared to $\mu$). As $(R, <)$ is close to the poset $(Q, <)$ of [Sh:276], an elementary submodel argument similar to the one there applies.

The exposition of the method is axiomatic; the author formulates the most general situation where this method works, and then specifies it to the situation sketched above. This is not necessarily the optimal description for those who are only interested in the application given. There is, however, reason for the peculiar way of presenting this proof: we would like to include this method into the tool kit set, and simply quote it at possible later applications.

Recall (first appeared in Erdős-Hajnal [EH78], but probably raised by Galvin in letters in the mid seventies):

\textbf{Definition 0.2.} 1) $\lambda \rightarrow (\mu, \mu, \mu)_2^2$ means that:
   - for every $c : [\lambda]^2 \rightarrow \kappa$ there are $\varepsilon$ and $\alpha_i, \beta_i$ for $i < \mu$ such that:
     - (a) $\varepsilon < \kappa$
     - (b) if $i < j < \mu$ then $\alpha_i < \beta_i < \alpha_j < \lambda$
     - (c) if $i \leq j < \mu$ then $c\{\alpha_i, \beta_j\} = \varepsilon$.

2) We can replace $\mu$ by an ordinal and if $\kappa = 2$ we may omit it.

\textbf{Definition 0.3.} 1) Let $\lambda \rightarrow (\mu, (\mu, \mu)_\kappa)^2$ means that:
   - for every $c : [\lambda]^2 \rightarrow 1 + \kappa$ there are $\varepsilon$ and $\alpha_i, \beta_i$ for $i < \mu$ such that:
     - (a) $\varepsilon < \kappa$
     - (b) $\alpha_i < \beta_i < \alpha_j < \lambda$ for $i < j < \mu$
     - (c) if $\varepsilon = 0$ then $i < j \Rightarrow c\{\alpha_i, \alpha_j\} = \varepsilon$, so we can forget the $\beta_i$’s
     - (c) if $\varepsilon \geq 1$ then $i \leq j \Rightarrow c\{\alpha_i, \beta_j\} = \varepsilon$.

2) In part (1) if $\kappa = 1$ we may omit it. Above replacing $\mu$ by “$< \mu$” means “for every $\xi < \mu$ we have ....”.

We thank Shimoni Garti for many corrections and Istvan Juhasz for questions and historical remarks; we may continue this research in [Sh:F884].
1. §1 Strong polarized partition relations

We deal with sufficient conditions on a forcing notion for preserving such partition relations. For this, we use an expansion of a forcing notion. Instead of the usual pair $(Q, \leq_Q)$, namely, the underlying set and the partial order, we use a quadruple of the form $Q = (Q, \leq_Q, \leq_{pr}^Q, \text{ap}_Q)$.

The “pr” stands for pure, and the “ap” stands for apure. Both are included (as partial orders) in $Q$.

Discussion 1.1. We define (below) the notion of “$(\lambda, \theta, \xi)$-forcing” to give a sufficient condition for appropriate cases of the partition relations defined above to hold. We start with the quadruple $Q = (Q, \leq_Q, \leq_{pr}^Q, \text{ap}_Q)$ such that $q \in Q \Rightarrow \text{ap}_Q(q) \subseteq Q$ and $\leq_Q, \leq_{pr}^Q$ are quasi orders on $Q$. The idea is that if $r \in \text{ap}_Q(q)$ then $r$ and $q$ are compatible in $Q$, close to “$r$ is an a-pure extension of $q$”.

Definition 1.2. 1) We say that $Q$ is a $(\chi^{+}, \theta, \xi)$-forcing notion when $\chi^{+}, \theta$ are regular uncountable cardinals, $\xi$ an ordinal and $\oplus \leq Q$ holds; in writing $(\chi^+, \theta, < \xi)$ we mean that $\otimes$ holds for every $\xi < \zeta$; also we can replace $\chi^{+}$ by $\lambda$:

$\oplus$ (a) $Q = (Q, \leq_Q, \leq_{pr}^Q, \text{ap}_Q)$
(b) $Q = (Q, \leq_Q)$ is a forcing notion (i.e. a quasi order, so $\models_Q$ means $\models Q$ and $p \in Q$ means $p \in Q$ and $V^Q$ means $V^Q$ and $\mathcal{G}$ is the $Q$-name of the generic set)
(c) $\leq_{pr}^Q$ is a quasi order on $Q$ and $p \leq_{pr}^Q q$ implies $p \leq Q q$
(d) for $q \in Q$ we have $\uparrow q \in Q$ implies $p \leq Q q$
(e) $\text{ap}_Q$ is a function with domain $Q$
(f) $r \in \text{ap}_Q(q)$ $\Rightarrow$ $r, q$ are compatible in $Q$; moreover $\leq_{pr}^Q$ furthermore there is $r^{+} \in \text{ap}_Q(q^{+})$ such that $q^{+} \models Q \otimes^{r+} \epsilon \in \mathcal{G}_Q$ $\Rightarrow$
(g) $(Q, \leq_{pr}^Q)$ satisfies the $\chi^{+}$-c.c.
(h) if $\bar{q} = \langle q \epsilon : \epsilon < \theta \rangle$ is $\leq_{pr}^Q$-increasing then $\bar{q}$ stationary many limit ordinals $\zeta < \theta$, the sequence $\bar{q} \uparrow \zeta$ has an exact $\leq_{pr}^Q$-upper bound, see part (2) below

---

1. It is natural to demand $q \in \text{ap}_Q(q)$, but not really necessary (if we do not demand it, this just complicates a little $s(C)(d)$).

2. No harm in asking that $r \leq_{pr}^Q s$ and $s \in \text{ap}_Q(q^{+})$ and $q^{+} \leq s$ for some $s$. Why this does not follow from our assumption? By the present demand $r^{+}, q^{+}$ have a common $\leq_{pr}$-upper bound which is $s$, so $s \models q^{+} \otimes^{r^{+}} \epsilon \in \mathcal{G}_Q$ hence $r \in \mathcal{G}_Q$ so without loss of generality $r \leq s$, but this does not say $q \leq_{pr}^Q s$.

3. Note that: we can restrict ourselves to the case $q_0 \in I$, where $I$ is a dense subset of $Q$. Also we can restrict ourselves to the set of $\bar{q}$ sequences which is the set of plays of a suitable game with one player using a fixed strategy, etc.
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1.2}) There is a price for demanding a strict inequality. The price is (in 2.12 (1))

 politic. We say that

1.5

Remark

6 SAHARON SHELAH

Let

1.3

1) Here it mostly does not matter, but

ordering

<

\( \kappa \)

without loss of generality

counterexample.

We shall now prove

1.2) Assume \( Q \) satisfies clauses (a)-(e) of part (1).

Let \( q = \langle q_\varepsilon : \varepsilon < \delta \rangle \) be a \( \leq_{\mathbb{Q}} \)-increasing sequence of conditions, \( \delta < \theta \) a limit

ordinal. We say that \( q \) is an exact \( \leq_{\mathbb{Q}} \)-upper bound of \( \hat{q} \) when \( \varepsilon < \delta = \ell g(q) \Rightarrow q_\varepsilon \leq_{\mathbb{Q}} q \) and:

\((*)_{q,q} \) if \( p \in \text{ap}_Q(q) \) then for some \( \varepsilon < \delta \) and \( p' \in \text{ap}_Q(q) \), we have \( \models Q \) “if

\( q, p' \in G_Q \) then \( p \in G_Q \).”

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Remark 1.3. Can we weaken clause (i) of \( \oplus \) of 1.2(1) to “cardinality \( \leq \theta \)?

1) Here it mostly does not matter, but in one point of the proof of 1.4 it does: in

proving \( \oplus \) there, choosing \( \zeta(*) \) such that it will be possible to choose \( \varepsilon(*) \).

2) There is a price for demanding a strict inequality. The price is (in 2.12(1))

that, recalling \( \kappa = \kappa_\ell \), instead of using \( \text{ap}_y(q) = \{ r : q \leq_{\mathbb{Q}} r \in Q_y \} \) we use

\( \text{ap}_y(q) = \{ r : q \leq_{\mathbb{Q}} r \in Q_y \} \) and \( \text{supp}_y(q, r) \subseteq \text{supp}_y(p_{\text{ap}_y(q)}, q) \} \).

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Claim 1.4. If \( Q \) is a \( (\chi^+, \theta, \xi_\alpha) \)-forcing notion, \( \kappa < \theta = cf(\theta) \) and \( \chi = \chi^{<\theta} \) then

\( \chi^+ \rightarrow (\xi_\alpha, (\xi_\alpha; \xi_\alpha)_\alpha)^2 \) holds in \( V^Q \).

Remark 1.5. We can replace \( \chi^+ \) by “regular \( \chi^+ \) such that \( \alpha < \chi^+ \Rightarrow |\alpha|^{<\theta} < \chi^+ \).”

Proof. Let \( \lambda_\ast \) be large enough (so in particular \( Q, \theta, \ldots, \in \mathcal{H}(\lambda_\ast^+) \)). Choose a well

ordering \( \prec_{\lambda_\ast^+} \) of the set \( \mathcal{H}(\lambda_\ast^+) \). Recalling Definition 1.2 clearly \( \theta > \aleph_0 \), hence

without loss of generality \( \kappa \) is infinite, so \( 1 + \kappa = \kappa \).

Toward contradiction assume \( p^* \models Q \) “\( c \) is a function from \( [\chi^+]^2 \) to \( \kappa \)” is a
counterexample.

We now choose \( M \) such that

\( \oplus_1 \)

\( (a) \quad M = \langle M_\alpha : \alpha \leq \theta \rangle \)

\( (b) \quad M_\alpha = (\mathcal{H}(\lambda_\ast^\alpha), \in) \)

\( (c) \quad M_\alpha \) has cardinality \( \chi \)

\( (d) \quad [M_\alpha]^{<\theta} \subseteq M_\alpha \) if \( \alpha \) is non-limit

\( (e) \quad M_\alpha \) is \( \prec \)-increasing continuous

\( (f) \quad Q, p^*, c \) belong to \( M_\alpha \) and \( \chi + 1 \subseteq M_\alpha \)

\( (g) \quad M \upharpoonright (\alpha + 1) \in M_{\alpha+1} \).

Note that \( \chi = \chi^{<\theta} \) implies \( \theta < \chi^+ \), so let

\( \oplus_2 \) \( \delta_\ast := \min(\chi^+ \setminus M_\theta) \).

We shall now prove

\( \oplus_3 \) if \( q \in Q \) and \( \varphi(x, y) \in L_{\theta, \vartheta} \) is a formula with parameters from \( M_\theta \) such that

\( (\mathcal{H}(\lambda_\ast^\vartheta), \in, <_{\lambda_\ast^\vartheta}^\vartheta) \models \varphi[\delta_\ast, q] \) then for some pair \( (\delta, \vartheta') \in M_\theta \) we have:

\( (a) \quad \delta < \delta_\ast \)

\( (b) \quad (\mathcal{H}(\lambda_\ast^\vartheta), \in, <_{\lambda_\ast^\vartheta}^\vartheta) \models \varphi[\delta, \vartheta'] \)
(c) \( q', q \) has a common \( \leq_{\text{pr}}^{Q} \)-upper bound.

Why \( \odot_3 \) holds? Let \( \bar{r} = (r_\zeta : \zeta < \zeta^*) \) list \( Q \), each member appearing \( \chi^+ \) times, now without loss of generality \( \bar{r} \in M_\delta \) so necessarily we can find \( \zeta_1 \in \zeta^* \setminus M_\delta \) such that \( q = r_{\zeta_1} \) and let \( \zeta_2 = \min(M_\delta \cap (\zeta^* + 1) \setminus \zeta_1) \), of course \( \zeta_2 \in M_\delta \) and \( \zeta_1 < \zeta_2 \wedge \text{cf}(\zeta_2) > \chi \).

Let

\[
Y = \{ q' \in Q : (\mathcal{H}(\lambda^+_1), \varepsilon, \prec^*_\lambda) \models (\exists x)(\varphi(x, q') \land x \in \chi^+) \}.
\]

Recall that \( \chi^0 = \chi \), so

\( \odot_{3.1} \) \( Y \in M_\delta, Y \subseteq Q \) and \( q \in Y \).

Now we ask

\( \odot_{3.2} \) is there \( Z \subseteq Y \) of cardinality \( \leq \chi \) such that for every \( q'' \in Y \) for at least one \( q' \in Z \) the pair \( (q', q'') \) is \( \leq_{\text{pr}}^{Q} \)-compatible?

Assume toward contradiction that the answer is negative, then in particular \( |Y| > \chi \) and we can choose \( r_\varepsilon \in Y \) by induction on \( \varepsilon < \chi^+ \) such that \( \zeta < \varepsilon \Rightarrow \) the pair \( (r_\zeta, r_\varepsilon) \) is \( \leq_{\text{pr}}^{Q} \)-incompatible. Why? In stage \( \varepsilon \) try to use \( Z := \{ r_\zeta : \zeta < \varepsilon \} \), so \( Z \subseteq Y \) has cardinality \( \leq |\varepsilon| \leq \chi \), so some \( r_\varepsilon \in Y \) can serve as \( q'' \) in \( \odot_{3.2} \), by our assumption toward contradiction. Hence \( (r_\varepsilon : \varepsilon < \chi^+) \) contradict clause (I) of Definition 1.2(1). So the answer to \( \odot_{3.2} \) is yes, hence there is such \( Z \in M_\delta \), but \( \chi + 1 \subseteq M_\delta \) hence \( \chi \subseteq M_\delta \).

So apply the property of \( Z \), with \( q \) standing for \( q'' \), so there is \( q' \in Z \subseteq Q \cap M_\delta \) such that the pair \( (q', q) \) is \( \leq_{\text{pr}}^{Q} \)-compatible; but \( Z \subseteq Y \) hence by the definition of \( Y \) there is \( \delta \in \chi^+ \) such that \( (\mathcal{H}(\lambda^+_1), \varepsilon, \prec^*_\lambda) \models \varphi[\delta, q'] \), and as \( q' \in Z \subseteq M_\delta \) without loss of generality \( \delta \in M_\theta \), hence \( \delta \in \chi^+ \cap M_\theta \) so by the definition of \( \delta_* \) we have \( \delta < \delta_* \); so \( \odot_3 \) holds indeed.

Next (but its proof will take awhile)

\( \odot_4 \) if \( q^0 \in Q \) is above \( \text{pr} \) then for some triple \( (q^1, p, \iota) \) we have:

(a) \( q^0 \leq_{\text{pr}}^{Q} q^1 \)

(b) \( \iota < \kappa \)

(c) \( p \in \text{ap}_Q(r) \) for some \( r \) satisfying \( q^0 \leq_{\text{pr}}^{Q} r \leq_{\text{pr}}^{Q} q^1 \)

(d) if \( \iota = 0 \) then \( p \leq q^1 \)

(e) if \( q \) satisfies \( q^1 \leq_{\text{pr}}^{Q} q \) and \( \varphi(x, y) \in L_{\theta, \theta} \) is a formula with parameters from \( M_\theta \) satisfied by the pair \( (\delta_*, q) \) in the model \( (\mathcal{H}(\lambda^+_1), \varepsilon, \prec^*_\lambda) \), then we can find \( q', q'', \delta \) such that the septuple \( q = (q, p, \iota, \varphi(x, y), q', q'', \delta) \) satisfies

\( \boxtimes_q \)

1. \( \delta < \delta_* \) (hence \( \delta \in M_\theta \))

2. \( (\mathcal{H}(\lambda^+_1), \varepsilon, \prec^*_\lambda) \models \varphi[\delta, q'] \)

3. if \( \iota = 0 \) then

(a) \( q \leq_{\text{pr}}^{Q} q'' \)

(b) \( q' \leq_{\text{pr}}^{Q} q'' \)

(\( c \)) \( q'' \models \text{"c}(\delta, \delta_*) = 0 \)
null

Why? Assume toward contradiction that $\otimes 4$ fails. We let \langle S_\varepsilon : \varepsilon \leq \theta \rangle be a $\subseteq$-
increasing continuous sequence of subsets of $\theta$ with $S_\theta = \theta, |S_{\varepsilon+1} \setminus S_\varepsilon| = \theta, |S_0| = \theta$
and $\min(S_{\varepsilon+1} \setminus S_\varepsilon) \geq \varepsilon$. Now we try to choose \langle $q_\varepsilon^*, x_\varepsilon$,$\varphi_\varepsilon$ \rangle by induction on $\varepsilon < \theta$
(but $\varphi_\varepsilon$ is chosen in the $(\varepsilon + 1)$-th stage) such that:

\begin{itemize}
  \item \langle $q_\varepsilon^*, x_\varepsilon : \varepsilon \leq \theta$ \rangle is $\leq \bar{p}$-increasing
    \begin{itemize}
      \item if $\varepsilon \in (0, \kappa)$, then $q_\varepsilon \leq \bar{p}_\varepsilon$ and $q_\varepsilon'' \models "p \in G_\varepsilon \Rightarrow \gamma \subseteq \delta, \delta_\varepsilon = \varepsilon \land q' \in G_\varepsilon".$
    \end{itemize}
\end{itemize}

We show that the induction can be carried out. Assume we are stuck at $\varepsilon$. Now if $\varepsilon = 0$ we can satisfy clauses $(\alpha) + (\beta)$ and recalling $1 \leq |\text{ap}_\varepsilon(q_\varepsilon^*)| < \theta$ we can choose $x_0$ to satisfy clause $(\delta)$ and since $(\gamma)$, $(\varepsilon)$ are vacuous we are done.
Suppose $\varepsilon > 0$. For limit $\varepsilon$ we can choose $q^*_\varepsilon$ as required in clause $(\alpha)$ by clause $(\varepsilon)$ of Definition 1.2(1); also clause $(\gamma)$ is relevant but causes no problem; and lastly, we can choose $x_\varepsilon$ and since clause $(\varepsilon)$ is vacuous for limit ordinals, we are done again. So $\varepsilon$ is a successor, let $\varepsilon = \varepsilon + 1$, so $q^*_\varepsilon$ was defined. Now if we cannot choose \langle $q^*_{\varepsilon+1}$,$\varphi_\varepsilon(x,y)$ \rangle = \langle $q^*_{\varepsilon+1}$,$\varphi_\varepsilon(x,y)$ \rangle then the triple \langle $q^*_{\varepsilon+1}$,$p^*_\varepsilon$,$\iota_\varepsilon$,$\varphi_\varepsilon(x,y)$ \rangle is as required from the triple \langle $q^*_{\varepsilon+1}$,$p^*_\varepsilon$,$\iota_\varepsilon$ \rangle in $\otimes 4$. But this is impossible (by our assumption toward contradiction), so we can find \langle $q^*_{\varepsilon+1}$,$\varphi_\varepsilon(x,y)$ \rangle as required; and again we can choose $x_\varepsilon$ as for $\varepsilon = 0$.

So it is enough to get a contradiction from the assumption that we can carry out the induction. But by clause $(\gamma)$ of Definition 1.2(1) the set $S := \{ \zeta : \varepsilon < \zeta \}$ is a limit ordinal and the sequence \langle $q_\zeta^* : \varepsilon < \zeta$ \rangle has an exact $\leq \bar{p}$-upper bound is stationary.
As $S$ is stationary noting $\otimes 4(\delta)$ and recalling clause $(\beta)$ of Definition 1.2(1) which gives $|\text{ap}_\varepsilon(q_\varepsilon^*)| < \theta = \text{cf}(\theta)$ for $\varepsilon < \theta$, clearly for some limit ordinal $\zeta(\ast) \in S$ we have: if $\varepsilon < \kappa (< \theta)$ and $p \in \cup\{\text{ap}_\varepsilon(q_\varepsilon^*) : \varepsilon < \zeta(\ast)\}$ then for unboundedly many $\varepsilon < \zeta(\ast)$ we have $(p^*_\varepsilon, \iota_\varepsilon) = (p, \iota)$.

Let $\varphi(x,y) \in L_{\theta, \theta}$ express all the properties that the pair \langle $\delta_\varepsilon, q_\varepsilon^*(\ast)$ \rangle satisfies and are used below, i.e., \langle $\exists y_0, \ldots, y_\zeta(\ast)$, $x \in \chi^+ \land y = y_\zeta(\ast)$, $\bigwedge_{\varepsilon < \zeta(\ast)} y_\varepsilon \leq \bar{p}_\varepsilon y_\zeta \land \varphi_\varepsilon(x,y_{\varepsilon+1}) \land (y_{\varepsilon(\ast)}) \rangle$ is an exact $\leq \bar{p}$-upper bound of \langle $y_\varepsilon : \varepsilon < \zeta(\ast)$ \rangle.

\begin{itemize}
  \item if $\varepsilon \in (0, \kappa)$, then $q_\varepsilon \leq \bar{p}_\varepsilon$ and $q_\varepsilon'' \models "p \in G_\varepsilon \Rightarrow \gamma \subseteq \delta, \delta_\varepsilon = \varepsilon \land q' \in G_\varepsilon".$
\end{itemize}
So

\((+)(H(\lambda^+), \varepsilon, <_{\lambda^+}) \models \varphi[\delta, *_{\zeta(*)}].\)

By case 2, we can find a pair \((\delta, q')\) such that:

\(\bigcirc_{4.2} \quad \begin{array}{l}
(a) \quad \delta < \delta_4 \text{ hence } \delta \in M_0 \text{ and } q' \in M_0 \\
(b) \quad (H(\lambda^+), \varepsilon, <_{\lambda^+}) \models \varphi[\delta, q'] \\
(c) \quad q', *_{\zeta(*)} \text{ are } \leq_{\delta} Q \text{-compatible.} 
\end{array} \)

Let \(q''\) be such that:

\(\bigcirc_{4.2} \quad \begin{array}{l}
(d) \quad q' \leq_{\delta} q'' \text{ and } *_{\zeta(*)} \leq_{\delta} q''. 
\end{array} \)

Let \((q'_\zeta : \zeta \leq \zeta(*)\)) exemplify \(\varphi[\delta, q']\) and without loss of generality \({q'_\zeta : \zeta \leq \zeta(*)} \subseteq M_0\), in particular, \(\varepsilon \leq \zeta(*) \Rightarrow q'_\zeta \leq_{\delta} q'_{\zeta(*)} = q' \leq_{\delta} q''\) and, of course, \(\varepsilon < \zeta(*) \Rightarrow q'_\varepsilon \leq_{\delta} q'_{\varepsilon} \leq_{\delta} q''\).

Case 1: \(q'' \models Q \, c\{\delta, \delta_4\} = 0\).

There is \(\varepsilon < \zeta(*)\) such that \(\varepsilon_4 = 0\). We get contradiction to the choice of the \((q'_\zeta, \varphi[\delta_4, \varphi[\delta, q'_\zeta]]\).

Why? Let us check that the septuple \(q = (q'_{\varepsilon + 1}, q'_{\varepsilon + 1}, 0, \varphi[\varepsilon, q(x, y), q'_{\varepsilon + 1}, q'', \delta])\) is such that \(E_{\varepsilon} Q\) holds.

For \(\bullet 1\): Recall \(\bigcirc_{4.2}(a)\)

For \(\bullet 2\): By \(\bigcirc_{4.2}(\varepsilon)(*)\) we have \((H(\lambda^+), \varepsilon, <_{\lambda^+}) \models \varphi[\varepsilon, q'_{\varepsilon + 1}]\) by the choice of \(\varphi[\varepsilon, q(x, y)]\) and \((q'_\zeta : \zeta \leq \zeta(*)\)) we have \((H(\lambda^+), \varepsilon, <_{\lambda^+}) \models \varphi[\delta, q'_{\varepsilon + 1}]\) as required.

For \(\bullet 3(a)\): it means \(q'_{\varepsilon + 1, \zeta} \leq_{\delta} q''\) which holds as \(q'_{\varepsilon + 1} \leq_{\delta} q'_{\varepsilon} \leq_{\delta} q''\) by \(\bigcirc_{4.2}(d)\).

For \(\bullet 3(b)\): it means \(q'_{\varepsilon + 1} \leq_{\delta} q''\) which has been proved just before “Case 1”.

For \(\bullet 3(\gamma)\): it means \(q'' \models Q \, c\{\delta, \delta_4\} = 0\) which holds by the case assumption.

For \(\bullet 4\): it is vacuous. So indeed \(E_{\varepsilon} Q\) holds contradicting the choice of \((q'_\zeta, \varphi[\delta_4, \varphi[\delta, q'_\zeta]]\), see \(\bigcirc_{4.1}(\varepsilon)\).

Case 2: Not Case 1.

Choose \((q'', \varepsilon)\) such that \(q^+ \in Q, q'_{\zeta(*)} \leq_{\delta} q'' \leq_{\delta} q^+\) and \(q^+ \models Q \, c\{\delta, \delta_4\} = \varepsilon''\) where \(\varepsilon \in (0, \varepsilon_4)\), we use “not Case 1”. By clause (j) of \(\oplus\) of Definition 1.2 applied with \((q'_{\zeta(*)}, q^+\)) here standing for \((q_4, \varepsilon)\) there, we can find a pair \((s, p)\) such that

\(\bigcirc_{4.3} \quad \begin{array}{l}
(a) \quad p \in \text{ap}_{\delta} Q *_{\zeta(*)} \\
(b) \quad q'_{\zeta(*)} \leq_{\delta} p \leq_{\delta} s \\
(c) \quad s \models_{\delta} Q \, "p \in G_{\delta} \Rightarrow q^+ \in G_{\delta}\". 
\end{array} \)

As \(q'_{\zeta(*)} \) is an exact \(\leq_{\delta} Q\)-upper bound of \((q'_\zeta : \varepsilon < \zeta(*)\)) because \(\zeta(*) \in S\) and \(p \in \text{ap}_{\delta} Q *_{\zeta(*)}\), see part (2) of Definition 1.2, there is a pair \((p', \varepsilon(*)\) such that:

\(\bigcirc_{4.4} \quad \begin{array}{l}
(a) \quad \varepsilon(*) < \zeta(*) \\
(b) \quad p' \in \text{ap}_{\delta} Q *_{\zeta(*)} \\
(c) \quad s \models_{\delta} Q \, "\text{if } q'_{\zeta(*)}, p' \in G_{\delta} \text{ then } p \in G_{\delta}\". 
\end{array} \)
So by the choice of $\zeta(*)$ for some $\zeta < \zeta(*)$ which is $> \varepsilon(*)$ we have $(p_{\varepsilon}, \iota_\varepsilon) = (p', \iota)$. Let $q = (q_{\varepsilon+1}, p_{\varepsilon}, \iota_{\varepsilon}, \varphi_\varepsilon(x, y), q_\varepsilon', s, \delta)$. This septuple satisfies $\Xi_q$ because:

For $\bullet_1$: Recall $\odot_{4.2}(a)$

For $\bullet_2$: as in case 1.

For $\bullet_3$: it is vacuous.

For $\bullet_4$: it means first $q_{\varepsilon+1} \leq_{Q_p} s$ which holds as $q_{\varepsilon+1} \leq_{Q_p} q_\varepsilon(*)$ by $\odot_{4.1}(a)$ and $q_\varepsilon(*) \leq_{Q_p} s$ by $\odot_{4.3}(b)$. Second, $s \models "p_\varepsilon \in G_Q \Rightarrow \sigma_\varepsilon(\delta, \delta_\varepsilon) = \iota"$ which holds as $p_\varepsilon = p'$ and assuming $G \subseteq Q$ is generic over $V$ if $s, p' \in G$ then by $\odot_{4.3}(b)$ also $q_{\varepsilon(*)} \in G$ hence by $\odot_{4.4}(c)$ also $p \in G$ hence by $\odot_{4.3}(e)$ also $q^+ \in G$ hence by the choice of $q^+$ in the beginning of the case we have $V[G]$ satisfies $\sigma[G][\delta, \delta_\varepsilon] = \iota$.

Third, $s \models "p_\varepsilon \in G_Q \Rightarrow q_\varepsilon \in G_Q"$ which holds as $p_\varepsilon = p'$ and assuming $G \subseteq Q$ is generic over $V$, if $s, p' \in G$ then as above $q^+ \in G$ hence by the choice of $q^+$ in the beginning of the case also $q'' \in G$ hence by $\odot_{4.2}(d)$ also $q' \in G$ hence by the choice of $q'$ and of $\langle q_{\varepsilon} : \zeta \leq (\varepsilon(*)) \rangle$ we have $q_{\varepsilon} \in G$ as required.

Hence we get a contradiction to the choice of $(q_{\varepsilon+1}, \varphi_3)$. So we are done proving $\odot_4$.

Let the triple $(q_\varepsilon, p_\varepsilon, \iota_\varepsilon)$ satisfy the demands on $(q^+, p, \iota)$ in $\odot_4$ for $q^0 = p^*$ and let $r_\varepsilon$ be as guaranteed by clause (c) of $\odot_4$ so

$\odot_5$ (a) $q_\varepsilon \in Q$

(b) $(q_\varepsilon : \xi \leq \zeta)$ is $\leq_{Q_p}$-increasing

(c) $q_0 = q_\varepsilon$

(d) $\alpha_\varepsilon < \beta_\varepsilon < \delta_\varepsilon$ and $\varepsilon < \zeta \Rightarrow \beta_\varepsilon < \alpha_\varepsilon$

(e) $(q_{\varepsilon'}, q_{\varepsilon'\varepsilon}, \alpha_\varepsilon)$ is as $(q_\varepsilon', q_{\varepsilon''}, \delta)$ is guaranteed to be in clause (e) of $\odot_4$

with $q_{\varepsilon'}$ here standing for $q$ there (and of course $p_\varepsilon, \iota_\varepsilon$ here stands for $p, \iota$ there) and a suitable $\varphi$, hence

$\odot_5$ (a) $q_\varepsilon \leq_{Q_Q} q_{\varepsilon''}$

(b) $\alpha_\varepsilon, \beta_\varepsilon \leq \alpha_\varepsilon < \delta_\varepsilon$ for $\xi \leq \zeta$

(c) $\varepsilon \leq \zeta$ (in $\alpha_\varepsilon, \beta_\varepsilon$)

(d) the pair $(\alpha_\varepsilon, q_\varepsilon') \in M_0$ is similar to $(\delta_\varepsilon, q_\varepsilon)$

(e) if $\iota_\varepsilon > 0$ then $q_{\varepsilon'} \models "\sigma_{\varepsilon, \iota_\varepsilon} = \iota_\varepsilon \Rightarrow q_{\varepsilon'} \models \sigma_{\varepsilon, \iota_\varepsilon} = \iota_\varepsilon"$ for $\varepsilon < \zeta$

(f) the quadruple $(\beta_\varepsilon, \iota_\varepsilon, q_\varepsilon, q_{\varepsilon''}) \in \sigma_\varepsilon(\delta_\varepsilon, r\varepsilon) \subseteq M_0$ is similar to the quadruple $(\delta_\varepsilon, r\varepsilon, p_\varepsilon, q_{\varepsilon''})$, i.e.

$\odot_5$ (a) $\beta_\varepsilon \leq Q p_\varepsilon$ and $q_{\varepsilon'} \models "\varphi_{\alpha_\varepsilon, \beta_\varepsilon} = \iota_\varepsilon \Rightarrow q_{\varepsilon'} \models \varphi"$ for $\varepsilon < \zeta$

(c) $\varepsilon \leq \zeta$ (in $\alpha_\varepsilon, \beta_\varepsilon$)

(d) $\varphi_{\alpha_\varepsilon, \beta_\varepsilon}$

(g) $q_\varepsilon \leq_{Q_Q} q_{\varepsilon+1}$ and $q_{\varepsilon''} \leq_{Q} q_{\varepsilon+1}$.
Why can we carry out the induction? Note that \( q^\epsilon_0, \ldots, q^\epsilon_\zeta \) are chosen in the \((\zeta + 1)\)-th step.

For \( \zeta = 0 \) just let \( q_0 = q_* \) so the only relevant clauses (a),(c) are satisfied.

For \( \zeta \) limit only clause (b) is relevant and we can choose \( q_\zeta \) by clause (e) of Definition 1.2.

We are left with \( \zeta \) successor, let \( \zeta = \xi + 1 \).

We first choose \( (q^\epsilon_\zeta, q^\epsilon_{\zeta}^\prime), (\alpha_\zeta) \) as required in clause (e) of \( \oplus_5 \) using appropriate \( \varphi \) and \( \oplus_4(e) \) for our \( (q_*, p_*, \iota_*) \). Clearly in \( \oplus_5 \) clause (e) holds as well as the second statement in clause (d). In particular, (e)(\( \delta \)) comes from \( \oplus_4(e), (e)(e) \) comes from \( \varphi \), i.e. as \( \varepsilon < \zeta \rightarrow q_\xi \leq_{Q^\rho} q_\xi \).

Second, we choose \( (\beta_\zeta, r_\zeta, p_\zeta, q^\epsilon''_\zeta) \) as required in clause (f) of \( \oplus_6 \). [Why? We can find \( (\beta_\zeta, r_\zeta, p_\zeta, q^\epsilon''_\zeta) \in M_0 \) similar to \((\delta_\xi, r, p^\epsilon, q^\epsilon_\xi)\), using \( (\ast)_3 \) with \( (\delta_\xi, q^\epsilon_\xi) \) here standing for \((\delta_\xi, q, r)\) there and \( q^\epsilon''_\xi \) here standing for \( q' \) in the conclusion of \( \oplus_3 \) (and \( r_\xi, p_\zeta \) are gotten by existential quantifiers in choosing \( \varphi \) which holds as \( r_\xi, p_\zeta \) witness).

First, note that \( \alpha_\xi < \delta_\xi \) holds as \( \alpha_\zeta \in M_0 \) hence \( \alpha \) and \( \beta_\xi < \delta_\xi \) so \( \beta_\xi < \delta_\xi \) so clause (f)(\( \alpha \)) holds. Second, \( q^\epsilon''_\xi, q^\epsilon''_\zeta \) are \( \leq_{Q^\rho} \)-compatible by \( \oplus_3(c) \) hence clause (f)(\( \beta \)) holds.

Third, the parallel of (f)(\( \gamma \)) holds for \((p_*, r_\xi)\) by the choice of \( r_\xi \) and as \( q_* = q_0 \leq_{Q^\rho} q_\xi \leq_{Q^\rho} q^\epsilon''_\zeta \).

Fourth, the parallel of (f)(\( \delta \)) holds for \((q^\epsilon''_\xi, p_*)\) by (e)(\( \delta \)).

Third, as \( q^\epsilon''_\xi, q^\epsilon''_\zeta \) are \( \leq_{Q^\rho} \)-compatible there is \( q_\xi = q_{\xi + 1} \) as required in clause (g).

So we can satisfy \( \oplus_6 \).

Now we apply clause (h) of Definition 1.2(1) to the sequence \((q_\xi, p_\xi) : \varepsilon < \theta\) hence there is \( \zeta < \theta \) as there, so as \( p_\xi \in \text{aP}_{Q^\rho}(q_\xi) \) the conditions \( p_\xi, q_\xi \) are compatible in \( Q \) hence they have a common upper bound \( r \in Q \) hence by the choice of \((p_\xi, q_\xi) : \varepsilon < \theta\) above, \( r \models_{Q^\rho} "\zeta, \xi \leq \text{otp}\{\varepsilon < \zeta : q_\xi, p_\xi \in G_{Q^\rho}\}" \).

So \( r \models_{Q^\rho} "\text{the sequence } \langle (\alpha_\zeta, \beta_\zeta) : \varepsilon < \zeta \text{ and } q_\xi, p_\xi \in G_{Q^\rho} \rangle \text{ is as required}" \) noting that:

- if \( \iota_* \geq 0 \), then \( q_{\xi + 1} \models "c\{\alpha_\zeta, \beta_\zeta\} = \iota_* \) for \( \varepsilon \leq \zeta \)
- if \( \iota_* = 0 \), then \( q_{\xi + 1} \models "c\{\alpha_\zeta, \alpha_\zeta\} = \iota_* \) for \( \varepsilon \leq \zeta \).

So we are done. \[ \Box_{1.4} \]
2. Many strong polarized partition relations

We can below say more on strongly inaccessible \( \theta \in \Theta \).

\[ \text{Hypothesis 2.1.} \quad \text{Let} \ p = (\lambda, \mu, \Theta, \partial) \text{ satisfy:} \]

\[ (a) \quad \lambda = \lambda^{\lambda^<_\lambda} < \mu = \mu^{<\mu} \]

\[ (b) \quad \Theta \subseteq [\lambda, \mu] \text{ is a set of regular cardinals with } \lambda, \mu \in \Theta \]

\[ (c) \quad \partial = \langle \partial_\theta : \theta \in \Theta \rangle \text{ is an increasing sequence of cardinals such that} \]

\[ (a) \quad \partial_\theta = \text{cf}(\partial_\theta) \]

\[ (b) \quad \partial_\theta = \langle \partial_\theta \rangle^{<\partial_\theta} \]

\[ (c) \quad \partial_\theta \leq \theta \text{ and if } \theta < \kappa \text{ are from } \Theta \text{ then } \partial_\theta < \partial_\kappa \]

\[ (\delta) \quad \partial_\theta \geq \kappa \text{ if } \kappa \in (\Theta \cap \theta) \]

\[ (e) \quad \text{if } \theta = \lambda \text{ then } \partial_\theta = \lambda. \]

The reader may concentrate on (see 3.4):

\[ \text{Example 2.2. Assume} \]

\[ (a) \quad V \text{ satisfies G.C.H. from } \lambda \text{ to } \mu, \text{ i.e., } \partial \in [\lambda, \mu] \Rightarrow 2^\partial = \partial^+ \]

\[ (b) \quad \lambda = \lambda^{\lambda^<_\lambda} < \mu = \mu^{<\mu} \]

\[ (c) \quad \Theta := \{ \theta^+ : \lambda \leq \theta < \mu \} \cup \{ \lambda, \mu \} \text{ and} \]

\[ (d) \quad \partial_\theta = \theta \text{ for every } \theta \in \Theta, \text{ so in } 2.3(5) \text{ below we have } \partial^\theta = \text{ min}\{\theta^+, \mu\}. \]

For the rest of this section \( p, i.e. \lambda, \mu, \Theta, \partial \) are fixed.

\[ \text{Definition 2.3.} \]

1) For \( \kappa \in \Theta \), let \( E_\kappa \) be the equivalence relation on \( \mu \) defined by

\[ (\ast) \quad iE_{\kappa,j} \text{ if } i + \kappa = j + \kappa. \]

2) For any cardinal \( \kappa \in [\lambda, \mu] \) define \( E_{\kappa,\kappa} \) as \( E_{\kappa,\kappa} := \bigcup \{ E_\theta : \theta \in \Theta \cap \kappa \} \). For such \( \kappa \), if \( \kappa \notin \Theta \), let \( E_{\kappa,\kappa} = E_{\kappa,\kappa} \).

3) For \( i < \mu \) and \( \kappa \in \Theta \) let \( [i]_\kappa = i/E_\kappa = \text{ the } E_\kappa\text{-equivalence class of } i \), and for \( A \subseteq \mu \), let \( A/E_\kappa = \{ i/E_\kappa : i \in A \} \). For \( i < \mu, A \subseteq \mu \) we say that \( i/E_\kappa \) is represented in \( A \) if \( A\cap (i/E_\kappa) \neq \emptyset \). If \( A \subseteq B \subseteq \mu \), we say that \( i/E_\kappa \) grows from \( A \) to \( B \) if \( B \cap (i/E_\kappa) = \emptyset \). If we write functions \( p, q \) instead of \( A, B \) we mean \( \text{Dom}(p), \text{Dom}(q) \) respectively.

4) Note that for all \( i, j < \mu \) we have \( iE_{\mu,j} \). Thus, the following definition makes sense: if \( i, j < \mu \) we let \( \kappa(i,j) \) be the minimal \( \kappa \in \Theta \) such that \( iE_{\kappa,j} \).

5) Suppose \( \kappa \in \Theta \), let

\[ \partial^\kappa = \text{ min}\{ \partial_\theta : \kappa < \theta \in \Theta \} \text{ if } \kappa < \mu \text{ and } \partial^\kappa = \mu \text{ if } \kappa = \mu. \]

(Notice that \( \kappa \) is just an index in \( \partial^\kappa \), and this is not cardinal exponentiation.)

Thus, in particular,

\[ \text{Observation 2.4.} \]

1) For \( i, j < \mu \) we have: \( \kappa(i,j) \) is well defined and for \( i, j < \mu, \theta \in [\lambda, \mu] \) we have \( iE_{\theta,j} \Rightarrow \theta \geq \kappa(i,j) \) as

\[ (\ast) \quad \text{if } \theta < \kappa \text{ are both from } \Theta, \text{ then } E_\theta \text{ refines } E_\kappa \text{ and, in fact, each } E_\kappa\text{-equivalence class is the union of } \kappa \text{ many } E_\theta\text{-equivalence classes.} \]
2a) If \( \kappa < \theta \) are from \( \Theta \) then \( \partial^{\kappa} \leq \partial_\theta \); used in 2.8(1).
2b) \( \partial_\theta < \partial^\theta \) except possibly for \( \theta = \mu \) (still \( \partial_\mu \leq \partial^\mu \)); recall 2.1(c)(\( \gamma \)).
2c) \( \sup(\Theta \cap \kappa) \leq \partial_\kappa \) for \( \kappa \in \Theta \); recall 2.1(c)(\( \delta \)).
2d) \( \partial^\theta = (\partial^\theta)^{< \theta} \) for \( \theta \in \Theta \).
2e) If \( \kappa \in \Theta \) then each \( E_{\kappa, \kappa} \)-equivalence class has cardinality \( \leq \partial_\kappa \) (by (2c)); used in the proof of 2.8(3)).

3a) \( \partial_\lambda = \lambda \).
3b) If \( \theta < \kappa \) are successive elements of \( \Theta \) then \( \partial^{\theta} = \partial_\kappa \).
3c) If \( \kappa \in \Theta \) and \( \bigcup(\Theta \cap \kappa) \) is a singular cardinal, then \( \partial_\kappa \geq (\bigcup(\Theta \cap \kappa))^+ \).

**Definition 2.5.** 1) The forcing notion \( Q_\kappa = (Q_\kappa, \leq Q_\kappa) \), but we may omit \( p \) when clear from the context, is defined by:

\[
(A) \quad q \in Q \iff
\begin{align*}
& (a) \ q \text{ is a (partial) function from } \mu \text{ to } \{0, 1\} \\
& (b) \ i \leq \mu \text{ and } \kappa \in \Theta, \text{ then the cardinality of } (i/E_\kappa) \cap \text{Dom}(q) \leq \partial_\kappa \\
& \text{(note: taking } \kappa = \mu, \text{ the cardinality of } \text{Dom}(q) \leq \partial_\mu \leq \mu) \\
(B) \quad p \leq Q q \iff
\begin{align*}
& (a) \ p \leq q \text{ and } \\
& (b) \ \text{no } E_\kappa \text{-equivalence class grows from } p \text{ to } q
\end{align*}
\]

2) For \( \kappa \in \Theta \setminus \{\mu\} \) and \( p, q \in Q, \) let:

\[
(A) \quad p \leq Q q \iff p \leq q \text{ and } \\
(B) \quad p \leq Q q \iff p \leq q
\]

3) For \( \kappa = \mu \) and \( p, q \in Q, \) let:

\[
(A) \quad p \leq Q q \iff p = q \\
(B) \quad p \leq Q q \iff p \leq q
\]

4) Let \( Q_\kappa = Q_{p, \kappa} = (Q_\kappa, \leq Q_\kappa, \leq p) \), where \( \text{ap}_\kappa = \text{ap}_{p, \kappa} \) is the function with domain \( Q \) such that \( \text{ap}_\kappa(q) = \{q' : q \leq Q_\kappa q'\} \); so \( Q_\kappa \) as a forcing notion is \( Q \).

5) Let \( \leq Q_{p, \kappa} \leq Q_{p, \kappa} \leq Q_p \) be \( \leq Q_p \) for \( \kappa \in \Theta \).

**Remark 2.6.** Clearly \( Q_\kappa \) is related to \( \S \), and if \( \kappa \) is the last member of \( \Theta \cap \mu \), we can use it (enough if \( \Theta = \{\lambda, \mu\} \), but not in general, so we shall use a variant).

**Claim 2.7.** Concerning Definition 2.5

\[
(a) \quad \text{if } \kappa \in \Theta, \text{ then } \leq Q_{p, \kappa}, \leq Q_{p} \text{ are partial orderings of } Q \\
(b) \quad p \leq Q q \implies p \leq q \text{ and } p \leq Q q \implies p \leq q \\
(c) \quad \text{if } \kappa = \mu \text{ then } \leq Q_{p, \kappa} \leq Q_{p} \text{ is the equality} \\
(d) \quad \text{if } p_1, p_2 \in Q \text{ and they are compatible as functions, then } p_1 \cup p_2 \in Q;
\]
(β) moreover, letting $q = p_1 \cup p_2$, if clause (b) of 2.5(1)(B) holds between $p_k$ and $q$, for $k = 1, 2$, then $q$ is the lub, in $Q$, of $p_1$ and $p_2$

c_if $p \leq q$ and $\kappa \in \Theta$, then there are $r, s \in Q$ such that:

(α) $p \leq^p s \leq^p q$,

(β) $p \leq^p s \leq^p q$,

(γ) $q = r \cup s$,

(δ) $q$ is the $\leq$-lub of $r, s$

d_if $q \in Q$ then

(α) $0 \leq q$ (and $0$, the empty function, $\in Q_p$)

(β) $\forall r(q \leq r \equiv q \leq^p r)$;

(γ) $\kappa \in \Theta \setminus \{\mu\} \Rightarrow 0 \leq^p q$

(δ) $0 \neq q \Rightarrow 0 \leq^p q$ for any $\kappa \in \Theta \setminus \{\mu\}$

e_if $\kappa_1 \leq \kappa_2$ are both from $\Theta$, then:

$\leq^p_{\kappa_2} \leq^p_{\kappa_1}$ and $\leq^p_{\kappa_1} \subseteq \leq^p_{\kappa_2}$

(f) if $\kappa \in \Theta$ and $p \leq^p q$ and $p \leq^p r$, then:

(α) $q \cup r$ is a well defined function in $Q$

(β) $p \leq (q \cup r)$

(γ) $q \leq^p (q \cup r)$

(δ) $r \leq^p (q \cup r)$

(ε) $q \cup r$ is a $\leq$-lub of $q, r$ in $Q_p$

(g) if $\kappa \in \Theta, p \leq^p q_i (i = 1, 2)$ and $q_1, q_2$ are compatible in $Q$ (even just as functions), then $p \leq^p (q_1 \cup q_2)$

(h) if $p \leq^p q_k$ for $k = 1, 2$, and $q_1, q_2$ are compatible in $Q$ (even just as functions), then $q_k \leq^p q_1 \cup q_2$ for $k = 1, 2$

(i) (α) if $\{p_x : \varepsilon < \zeta\}$ has an $\leq$-upper bound $\text{then } \cup \{p_x : \varepsilon < \zeta\}$ is an upper bound

(β) similarly for $\leq^p_{\kappa_1}, \leq^p_{\kappa_2}$

(γ) assume $p_x \in Q$ for every $\varepsilon < \zeta$, and $p_x, p_{x'}$ has a common $\leq_{\kappa_2}$-upper bound for any $\varepsilon, \zeta < \zeta$; then the union of

$\{p_x : \varepsilon < \zeta\}$ is a $\leq_{\kappa_1}$-lub,

when $x = us, ap$ and $\zeta < \lambda$

(δ) if $\{p_x : \varepsilon < \zeta\} \subseteq Q$ has a common $\leq_{\kappa}$-upper bound and $\zeta < \partial_{\kappa}$, then

$\{p_x : \varepsilon < \zeta\}$ has a $\leq_{\kappa}$-lub - the union

(j) if $p \leq^p q$ $\text{then } \text{Dom}(q) \setminus \text{Dom}(p)$ has cardinality $< \partial_{\kappa}$

(k) if $p_1 \leq^p p_3$ and $p_1 \leq p_2 \leq p_3$ $\text{then } p_1 \leq^p p_2$ and $p_2 \leq^p p_3$

(l) if $p_1 \leq^p p_2, p_2 \leq^p q_k$ for $k = 1, 2$ and $q_1 \cup q_2$ is a function, then $q := q_1 \cup q_2$ is a $\leq$-lub of $q_1, q_2$ and $q_k \leq^p q, q_1 \leq q$

(m) assume $p_1, p_2$ are compatible in $Q$ then there is a pair $(q, t)$ such that:

$\bullet_1 p_1 \leq^p q$

$\bullet_2 p_2 \leq^p t$

$\bullet_3 q \models "t \in G \Rightarrow p_1 \in G"

$\bullet_4 q, t$ are compatible and we say $(q, t)$ is a witness for $(p_1, p_2)$
(n) if \( \langle p^\ell_\alpha : \alpha < \delta \rangle \) is \( \leq^\ell_\kappa \)-increasing for \( \ell = 1, 2, \delta \) a limit ordinal of cofinality
\[< \partial_\kappa \text{ and } \alpha < \delta \Rightarrow p^1_\alpha \leq^\kappa p^2_\alpha \] then \( \bigcup_{\alpha < \delta} p^1_\alpha \leq^\kappa \left( \bigcup_{\alpha < \delta} p^2_\alpha \right) \).

**Proof.** Straightforward. E.g.

Clause (i):

So assume \( x \in \{\text{us, pr, ap}\} \) and \( \kappa \in \Theta \) and \( \{p_\varepsilon : \varepsilon < \zeta\} \subseteq Q \) and \( q \in Q \) is an
\[\leq^\kappa_\alpha\text{-upper bound of } \{p_\varepsilon : \varepsilon < \zeta\}. \] Let \( p := \bigcup\{p_\varepsilon : \varepsilon < \zeta\} \) then we shall prove that
\( p \in Q \) and \( p \) is a \( \leq^\kappa_\alpha\) -upper bound of \( \{p_\varepsilon : \varepsilon < \zeta\}; \) this clearly suffices for proving
sub-clauses (\( \alpha \)), (\( \beta \)) of clause (i), and the \( \leq^\kappa_\alpha\)-lub part, i.e. sub-clauses (\( \gamma \)), (\( \delta \)) are
left to the reader; for (\( \gamma \)), (\( \delta \)), see 2.8(1B), (1A).

Now

\[(*)_1 \ \ p \text{ is a well defined function with domain } \subseteq \mu \text{ and } p \subseteq q. \]

Why? As \( \varepsilon < \zeta \Rightarrow p_\varepsilon \subseteq q \), i.e. as functions (by 2.5(1)(B)(a)) clearly \( p \subseteq q \), as
functions, so \( p \) is a well defined function with domain \( \subseteq \text{Dom}(q) \) but \( \text{Dom}(q) \subseteq \mu \) by 2.5(A)(a).]

\[(*)_2 \ \text{if } i < \mu \text{ and } \theta \in \Theta \text{ then the cardinality of } (i/E_\theta) \cap \text{Dom}(p) \text{ is } < \partial_\theta. \]

Why? Recall \( p \subseteq q \in Q \), see above so as \( q \in Q \) by 2.5(1)(a) we have \( (i/E_\theta) \cap \text{Dom}(p) \leq |(i/E_\theta) \cap \text{Dom}(q)| < \partial_\theta. \]

\[(*)_3 \ \ p \in Q. \]

Why? By \((*)_1 + (*)_2\) recalling 2.5(1)(A).]

\[(*)_4 \ p_\varepsilon \subseteq p \text{ for } \varepsilon < \zeta. \]

Why? By the choice of \( p. \]

\[(*)_5 \ \text{if } \varepsilon < \zeta \text{ and } \theta \in \Theta \text{ then } \{A \in \mu/E_\theta : A \text{ grows from } p_\varepsilon \text{ to } p\} \text{ has cardinality } \]< \partial_\theta. \]

Why? Because, recalling \( p \subseteq q \), this set is included in \( \{A \in \mu/E_\theta : A \text{ grows from } p_\varepsilon \text{ to } q\} \) which has cardinality \(< \partial_\theta \) because \( p_\varepsilon \subseteq q \) which holds as \( p_\varepsilon \leq^\kappa q. \]

\[(*)_6 \ p_\varepsilon \leq p \text{ for } \varepsilon < \zeta. \]

Why? By \((*)_4 + (*)_5\) recalling 2.5(1)(B).]

\[(*)_7 \ \text{if } x = \text{ us then } p \text{ is a } \leq^\alpha \text{-upper bound of } \{p_\varepsilon : \varepsilon < \zeta\}. \]

Why? By \((*)_3 + (*)_6.\]

\[(*)_8 \ \text{if } x = \text{ pr and } \varepsilon < \zeta \text{ then } p_\varepsilon \leq^\nu p. \]

Why? If \( \kappa = \mu \) then \( \leq^\nu_\kappa \) is equality and \( p_\varepsilon \leq^\nu_\kappa q \) hence \( p_\varepsilon = q \) but \( p_\varepsilon \subseteq p \subseteq q \) hence \( p_\varepsilon = p \) so this is trivial, hence assume \( \kappa < \mu \). We have to check 2.5(2)(A),
now clause (a) there holds by \((*)_6\) and clause (b) there holds as no \( E_\kappa \)-equivalence
class grows from \( p_\varepsilon \) to \( q \) (as \( p_\varepsilon \leq^\nu q \)) and \( p \subseteq q. \]

\[(*)_9 \ \text{if } x = \text{ pr then } p \text{ is a } \leq^\kappa \text{-upper bound of } \{p_\varepsilon : \varepsilon < \zeta\}. \]

Why? By \((*)_8.\]
**Claim 2.8.** Let $κ ∈ Θ$. 

1) $(Q, ≤^p_κ)$ is $(< θ^κ)$-complete and in fact if $\bar{p} = (p_α : α < θ)$ is $< θ^κ$-increasing, $θ$ a limit ordinal $< θ^κ$ then $p_θ := \cup \{p_α : α < θ\}$ is a $< θ^κ$-lub and $α ≤$-lub of $\bar{p}$; we use $κ < θ ∈ Θ ⇒ θ^κ ≤ θ_α$, see 2.4(2a).

1A) If $γ(∗) < θ^κ$ and $p_α ∈ Q$ for $α < γ(∗)$ and $p_α, p_β$ has a common $< θ^κ$-lub for any $α, β < γ(∗)$ then $p_θ := \cup \{p_α : α < γ(∗)\}$ is a $< θ^κ$-lub of $\{p_α : α < γ(∗)\}$. 

1B) If $γ(∗) < λ$ then (1A) holds for $≤^p_κ$. 

2) If $p ∈ Q$ then $Q_{p, p'} := Q_{p, p'} = \{q : p ≤^p_κ q, <^p_κ\}$ satisfies the $(\bar{q}_κ)^+ - c.c.$.

3) Moreover if $(p_α : α < θ^κ)$ is $< θ^κ$-increasing and $p_α ≤^p_κ q_α$ for $α < θ^κ$, then for some $α < β$ the conditions $q_α, q_β$ are compatible in $Q$ moreover there is $r$ such that $q_α ≤ r$ and $q_β ≤ r$ and $p_α = p_β ⇒ q_α ≤^p_κ r ∧ q_β ≤^p_κ r$.

4) Assume $p ∈ Q_{p, p}, χ = |A| < θ^κ, κ ∈ Θ$ and $p ⊨ “f$ is a function from $A ∈ V$ to $V”$. Then we can find $q$ such that:

$$p ≤^p_κ q$$

(α) if $a ∈ A$ then $I_{q, f, a} := \{r : q ≤^p_κ r \text{ and } r \text{ forces } f(a)\}$ is predense over $q$ in $Q_{q, q}$

(β) moreover some subset $I'_{q, f, a}$ of $I_{q, f, a}$ of cardinality $≤_κ$ is predense over $q$ in $Q_{q, q}$ (really follows).

**Proof.** 1) By (1A).

1A) Let $q_α, q_β$ be a common $≤^p_κ$-upper bound of $p_α, p_β$ for $α, β < γ(∗)$. Why is $p_θ ∈ Q$? Let us check Definition 2.5(1)(A).

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*compare with [ShSt:608, 1.8]*
Clearly $p_\alpha$ is a partial function from $\mu$ to $\{0,1\}$ so clause (a) there holds. For checking clause (b) there, assume $\theta \in \Theta$ and $A \in \mu/E_\theta$.

First, assume $\theta \leq \kappa$ and $A \cap \text{Dom}(p_\alpha) \neq \emptyset$ then for some $\alpha < \gamma(*)$ we have $A \cap \text{Dom}(p_\alpha) \neq \emptyset$, hence $A \cap \text{Dom}(p_\alpha) = \bigcup \{A \cap \text{Dom}(p_\beta) : \beta \in \gamma(*)\} \subseteq \bigcup \{A \cap \text{Dom}(q_\alpha,\beta) : \beta < \gamma(*)\}$, but $p_\alpha \leq q_\alpha,\beta$ and $A \cap \text{Dom}(p_\alpha) \neq \emptyset$ hence $A \cap \text{Dom}(q_\alpha,\beta) = A \cap \text{Dom}(p_\alpha)$. Together $A \cap \text{Dom}(p_\alpha) = A \cap \text{Dom}(p_\alpha)$ which, because $p_\alpha \in Q$, has cardinality $< \theta_0$ as required in clause (b) of Definition 2.5(1)(A).

Second, of course, if $A \cap \text{Dom}(p_\alpha) = \emptyset$ this holds, too.

Third, assume $\theta > \kappa$, then $\alpha < \gamma(*) \Rightarrow p_\alpha \in Q \Rightarrow |A \cap \text{Dom}(p_\alpha)| < \theta_0$, hence $|A \cap \text{Dom}(p_\alpha)| = |A \cap \bigcup \{A \cap \text{Dom}(p_\alpha)\}| \leq \sum_{\alpha < \gamma(*)} |A \cap \text{Dom}(p_\alpha)|$ which is $< \theta_0$ as $\gamma(*) < \partial^\kappa \leq \theta_0 = \text{cf}(\theta_0)$, so again the desired conclusion of clause (b) of Definition 2.5(1)(A) holds. Together indeed $p_\alpha \in Q$.

Why $\alpha < \gamma(*) \Rightarrow p_\alpha \leq p_\alpha$? We have to check 2.5(1)(B), obviously clause (a) there holds. Clause (b) there is proved as above.

Why $\alpha < \gamma(*) \Rightarrow p_\alpha \leq p_\alpha$? We have to check Definition 2.5(2)(A), now clause (a) there was just proved and clause (b) there holds as in the proof of $p_\alpha \in Q$.

Next we show that $p_\alpha$ is a $\leq p_\alpha$-lub of $p_\alpha$, so assume $q \in Q$ and $\alpha < \delta \Rightarrow p_\alpha \leq p_\alpha$.

To show $p_\alpha \leq p_\alpha$ we have to check clauses (B)(a),(b) of 2.5(1) and (A)(b) of 2.5(2).

As $p_\alpha = \bigcup\{p_\alpha : \alpha < \gamma(*)\}$, clearly $p_\alpha \subseteq q$ as a function so 2.5(1)(B)(a) above holds.

Also if $A \in \mu/E_\kappa$ and $A$ is represented in $p_\alpha$ then it is represented in $p_\alpha$ for some $\alpha < \gamma(*)$, but $p_\alpha \leq p_\alpha$ so $q \upharpoons A = p_\alpha \upharpoons A$ but $(p_\alpha|A) \subseteq (p_\alpha|A) \subseteq (q|A)$ hence $q \upharpoons A = p_\alpha \upharpoons A$ as required in 2.5(2)(A)(b).

Lastly, when $\theta \in \Theta$, 2.5(1)(B)(b) holds: if $\theta \leq \kappa$ because more was just proved and if $\theta > \kappa$ it is proved as in the proof of $p_\alpha \in Q$.

2) This is a special case of (3) when $(p_\alpha : \alpha < \partial^+ _\kappa)$ is constant (recalling 2.7(h)).

3) So in particular $p_\alpha \leq p_\alpha$ if $i < \partial^+ _\kappa$. Hence by clause (j) of Claim 2.7 the set $u_i : \text{Dom}(q_i) \setminus \text{Dom}(p_i)$ has cardinality $< \partial_\kappa$. Hence by the $\Delta$-system lemma (recalling that $(\partial_\kappa)^{< \theta_0} = \partial_\kappa$ by 2.1(c)(\beta)) for some unbounded $U \subseteq \partial^+ _\kappa$ the sequence $\langle u_i : i \in U \rangle$ is a $\Delta$-system, with heart $u_\kappa$. Moreover, since $2^{|u_\kappa|} \leq \partial^+ _\kappa = \partial_\kappa < \partial^+ _\kappa$, we can assume that $q_i|u_\kappa = q_i$ for every $i \in U$.

As each $E_{\kappa,<}\text{-class}$ has cardinality $\leq \partial_\kappa$ (see 2.4(2)(c),(e)), without loss of generality for every $i \neq j$ from $U$, if $a \in u_i \setminus u_\kappa$ then $a/ E_{\kappa,<}$ is disjoint to $u_j$. Now by 2.7(h) for every $i,j \in U$, the function $q = q_i \cup q_j$ is a $\leq p_\alpha$-lub of $q_i,q_j$ for part (2), i.e. when $p_j = p_j$. Also it is easy to check that for $i < j$, $q$ is a $\leq$-lub of $q_i,q_j$ which is $\leq p_\alpha$-above $q_j$ for part (3).

4) If $\kappa = \mu$ then $\leq p_\alpha \leq p_\alpha$ by clause 2.7(a)(\gamma), recall $Q_p = \{q \in Q : p \leq q \leq q_p\}$ so $q = p$ can serve, as $Q_p$ satisfies the $\partial^+ _\kappa$-c.c. by part (2); so we shall assume $\kappa < \mu$.

Recall that $\partial_\kappa < \partial^\kappa$ by 2.4(2)(b). As $|A| < \partial^\kappa = \text{cf}(\partial^\kappa)$, by part (1) of the claim and clause (f) of Claim 2.7 it is enough to consider the case $A = \{a\}$. Now we try to choose $p_i, r_i, b_i$ by induction on $i < \partial^+ _\kappa$, but $r_i, b_i$ are chosen in stage $i + 1$ together with $p_i+1$, such that

\begin{itemize}
  \item \textcircled{a} $p_0 = p$
  \item \textcircled{b} $(p_j : j \leq i)$ is $\leq p_\alpha$-increasing
  \item \textcircled{c} $p_{i+1} \leq p_\alpha r_i$
  \item \textcircled{d} $p_{i+1} \vdash \text{"if } r_i \in G_q \text{ then } f(a) = b_i$\end{itemize}
(e) $p_{i+1} \vDash "if \ r_i \in G_Q \ then \ for \ no \ j < i \ do \ we \ have \ r_j \in G_Q"$

(f) if $i$ is a limit, then $p_i$ is the union so a $\leq^p_{\kappa}$-lub of $(p_j : j < i)$.

For $i = 0$ just use clause (a) of $\otimes$.

For $i$ limit use clause (f) of $\oplus$ recalling part (1) of the claim and the fact that $\partial^+ \leq \partial^p$.

For $i = j + 1$, try to choose $q_i$ such that:

$$p_j \leq q_i$$

and

$$q_i \vDash "r_{i+1} \notin G_Q \ for \ i_1 < j".$$ 

If we cannot, we have succeeded, i.e. $p_i$ is as required from $q$ with $T_{p_i,f,a} = \{p_i \cup r_j : j < i\}$. If we can, let $(b_j,r_j)$ be such that $q_i \leq r_j$ and $r_j$ forces $f(a) = b_j$; clearly possible. By clause (c) of Claim 2.7 applied to the pair $(p_j,r_j)$ we choose $p_i$ such that $p_i \leq^p \kappa p_i \leq^\sup r_j$ and clearly we have carried out the induction. But if we carry the induction then we get a contradiction by part (3). So we have to be stuck for some $i < \partial^\kappa$, and as said above we then get the desired conclusion. □

**Conclusion 2.9.** Forcing with $Q_p$

(a) does not collapse cardinals except possibly cardinals from the set $\Omega_p = \{\theta : \lambda < \theta \leq \mu \ and \ for \ no \ \kappa \in \Theta \ do \ we \ have \ \partial_\kappa < \theta \leq \partial^p\}$, so $\mu \notin \Omega_p$

(b) does not change cofinalities $\notin \Omega_p$, moreover if it changes the cofinality of $\theta \in \text{Reg} \ to \ \chi < \theta$ then there is $\theta_1 \in \Omega_p$ such that $\chi \leq \theta < \theta_1$

(c) does not add new sequences of length $< \lambda$

(d) does not change $2^\theta$ for $\theta \notin [\lambda,\mu)$

(e) makes $2^\lambda = \mu$

(f) also the set $\Omega_p^\kappa := \bigcup\{\kappa_1,2^{\sup(\Theta \cap \kappa)}\}$: for some $\kappa \in \Theta, \Theta \cap \kappa$ has no last member, so $\sup(\Theta \cap \kappa)$ is strong limit and $\kappa_1 = \min(\text{Reg} \ \sup(\Theta \cap \kappa))$, is O.K. in clauses (a),(b)

(g) $Q_p$ has cardinality $\mu$ and satisfies the $\partial^+_{\mu}$-c.c., recalling $\partial_{\mu} \leq \mu$.

**Proof.** First, $Q_p$ is $(< \lambda)$-complete hence it adds no new sequences to $\lambda^V$, i.e. clause (c) holds so cardinals $\leq \lambda$ are preserved as well as cofinalities $\leq \lambda$ as well as $2^\theta$ for $\theta < \lambda$.

Second, $|Q_p| = \mu$ as $p \in Q_p \Rightarrow p$ is a function from $\text{Dom}(p) \subseteq \mu$ to $\{0,1\}$, see 2.5(1)(A)(a) and $|\text{Dom}(p)| < \partial_\mu = \mu$ by 2.5(1)(A)(b) and $\mu^{<\mu} = \mu$ by ??(a).

Third, by 2.8(2) the forcing notion $Q_p$ satisfies the $\partial^+_{\mu}$-c.c. But $Q = Q_p$ when $p = \emptyset$ so $Q$ satisfies the $\partial^+_{\mu}$-c.c. and of course $\partial_{\mu} \leq \mu$. This gives clauses (g) and (d) (recalling (c)).

Fourth, for clause (e), for any $\alpha < \mu$ let $\eta_\alpha \in \lambda^2$ be defined by $p \vDash \"\eta_\alpha(i) = \ell \ if \ i < \lambda \wedge (\alpha + i) \in \text{Dom}(p) \wedge \ell = p(\alpha + i)\"$. By density indeed $\vDash Q \ "\eta_\alpha \in \lambda^2\"$ and $\vDash Q \ "\eta_\alpha \neq \eta_\beta\"$ for $\alpha \neq \beta < \mu$, so clearly clause (e) holds.

---

\[5\] we can use $r'_j$ such that $p_j \leq^p r'_j \leq^p r_j$ such that $r_j$ is the $\leq$-lub of $r'_j,p_{i+1}$, may be helpful but not needed now.
Fifth, use 2.8(2),(4) to prove clauses (a) and (b), toward contradiction assume \( \theta \) is regular in \( V \) and \( \theta_1 \) is not in \( \Omega_p \) but \( p \forces_{\Omega} \varphi \) for \( \chi = cf(\theta) < \theta_1 \leq \theta^+ \). If \( \theta \leq \lambda \) or just \( \chi < \lambda \) use clause (c), if \( \theta > \mu \) use clause (g) so necessarily \( \lambda \leq \chi < \theta_1 \leq \theta \leq \mu \).

By the choice of \( \Omega_p \) there is \( \kappa \in \Theta \) such that \( \partial_\kappa < \theta_1 \leq \partial^\kappa \) and \( \chi + \partial_\kappa < \theta_1 \leq \theta \); now without loss of generality \( p \forces \varphi \) \( f : \chi \to \theta \) has range unbounded in \( \theta^+ \). Apply 2.8(4) with \( (p, \chi, f, \kappa) \) here standing for \( (p, A, f, \kappa) \) there and get \( q, (\mathcal{I}_{q,f,\alpha} : \alpha < \chi) \) as there. By 2.8(3) we have \( |\mathcal{I}_{q,f,\alpha}| \leq \partial_\kappa \) and \( \{ \mathcal{I}_{q,f,\alpha} : \alpha < \chi \} \) has cardinality \( \leq \chi + \partial_\kappa < \theta_1 \). In any case, in \( V \) the set \( \{ \beta : \text{for some } \alpha < \chi \text{ and } q \not\forces \varphi (\alpha) \neq \beta \} \) has cardinality \( < \theta_1 \leq \theta \), contradiction. So clauses (a),(b) holds.

We are left with clause (f), it is not really needed, still nice to have. Now if \( \theta \in \text{Reg} \cap (\lambda, \mu] \) is in \( \Omega_p \) and \( \kappa \) witness it then necessarily \( \Theta \cap \kappa \), which is not empty has no last element so if \( \theta_1 < \theta_2 \) are from \( \Theta \cap \kappa \) then \( \theta_1 \leq \partial_{\theta_2} < \partial_{\theta_2} \leq \theta_2 \) hence \( \sup(\Theta \cap \kappa) \) is strong limit.

If \( \theta = \kappa \) use clause (b). If \( \theta \geq 2^{\kappa} \) we repeat the proofs above for \( \leq_{\kappa}^{\text{pr}} \) where \( \leq_{\kappa}^{\text{pr}} = \cap \{ \leq_{\kappa}^\Theta : \kappa \in \Theta \cap \kappa \} \), \( \leq_{\kappa}^{\text{pr}} = \{ (p, q) : p \leq q \text{ and } \alpha \in \text{Dom}(p) \} \setminus \{ (p, q) : \exists \theta \in \Theta \cap \kappa \left( (\alpha/\text{End}_\kappa \cap \text{Dom}(p)) \neq \emptyset \right) \} \).

**Definition 2.10.** 1) If \( p \leq q \) and \( \kappa \in \Theta \) let \( \text{supp}(p, q) := \{ i/\text{End}_\kappa : i \in \text{Dom}(q) \} \setminus \text{Dom}(p) \} \) so of cardinality \( < \partial_\kappa \).

2) We say \( y = (\kappa, \bar{p}, \bar{u}) = (\kappa_y, \bar{p}_y, \bar{u}_y) \) is a reasonable \( p \)-parameter when:

\[ \otimes_1 \]
\[ (a) \] \( \kappa \in \Theta \) but \( \kappa < \mu \)
\[ (b) \] \( \bar{p} = (p_\alpha : \alpha < \gamma) \) is a non-empty \( \leq_{\kappa}^{\text{pr}} \) increasing continuous sequence, so we write \( \gamma = \gamma_y, \bar{p} = \bar{p}_y \) and \( p_\alpha = p_{\alpha}^y \)
\[ (c) \] \( \bar{u} = (u_\alpha : \alpha < \gamma) \) is \( \leq \) increasing continuous, so \( u_\alpha = u_{\alpha}^y, \bar{u} = \bar{u}_y \)
\[ (d) \] \( u_\alpha \subseteq \cup \{ i/\text{End}_\kappa : i \in \text{Dom}(p_\alpha) \} \) for \( \alpha < \gamma \)
\[ (e) \] \( |u_\alpha| \leq \partial^\kappa \) for \( \alpha < \gamma \)

3) For \( y \) as above we define \( Q_y \) as \( (Q_y, \leq_y, \leq_{\kappa}^{\text{pr}}, \text{ap}_y) \) (so \( Q_y = (Q_y, \leq_y) \) is \( Q_y \) as a forcing notion) where:

\[ \otimes_2 \]
\[ (a) \] \( \theta = \theta_y = \min(\Theta \setminus \kappa^+), \) notice that \( \theta \) is well defined, as \( \kappa_y < \mu \) and \( \mu \in \Theta \)
\[ (b) \] \( Q_y := \{ q : \text{for some } \alpha < \gamma \text{ we have } p_\alpha \leq_{\kappa}^{\text{pr}} q \text{ and supp}_\theta(p_\alpha, q) \subseteq u_\alpha \} \)
\[ (c) \] \( \leq_y \leq_{\theta}^{\text{pr}} \) \( Q_y \)
\[ (d) \] for \( q \in Q_y \), let \( \alpha_y(q) = \min \{ \alpha < \gamma_y : p_\alpha \leq_{\kappa}^{\text{pr}} q \text{ and supp}_\theta(p_\alpha, q) \subseteq u_\alpha \} \)
\[ (e) \] the two-place relation \( \leq_{\theta}^{\text{pr}} \) is defined by \( p \leq_{\theta}^{\text{pr}} q \) iff
\[ (a) \] \( p, q \in Q_y \)
\[ (b) \] \( p \leq_{\theta}^{\text{pr}} q \)
\[ (f) \] for \( q \in Q_y \) let \( \text{ap}_y(q) = \text{ap}_\theta(q) = \{ r \in Q_y : q \leq_{\kappa}^{\text{pr}} r \text{ and supp}_\theta(q, r) \subseteq \text{supp}_\theta(p_{\alpha_y(q)}, q) \} \).

**Observation 2.11.** Let \( y \) be a reasonable \( p \)-parameter.

0) If \( p_1 \leq p_2 \leq q_2 \leq q_1 \) and \( \kappa_1 \geq \kappa_2 \) are from \( \Theta \) then \( \text{supp}_\kappa(p_2, q_2) \subseteq \text{supp}_\kappa(p_1, q_1) \).

0A) If \( p_1 \leq p_2 \leq p_3 \) then \( \text{supp}_\kappa(p_1, p_3) = \text{supp}_\kappa(p_1, p_2) \cup \text{supp}_\kappa(p_2, p_3) \).

1) For \( q \in Q_y \) the ordinal \( \alpha_y(q) \) is well defined \( < \gamma_y \).

2) If \( q_1 \leq q_2 \) are from \( Q_y \) then \( \alpha_y(q_1) \leq \alpha_y(q_2) \).

2A) If \( q_1 \in Q_y \) and \( q_1 \leq_{\kappa}^{\text{pr}} q_2 \) then \( q_2 \in Q_y, q_1 \leq_\kappa q_2 \) and \( \alpha_y(q_1) = \alpha_y(q_2) \).
3) If \( p \leq^{pr} r \) and \( q \in ap_{\mathcal{Y}}(p) \) then \( s := q \cup r \) belongs to \( Q_{\mathcal{Y}} \), \( s \in ap_{\mathcal{Y}}(r) \) and \( q \leq^{pr} s \).

**Proof.** (0), (A) Should be easy.

1) By the definitions of \( q \in Q_{\mathcal{Y}} \) and of \( \alpha_{\mathcal{Y}}(q) \).

2) For \( \ell = 1, 2 \) letting \( \alpha = \alpha_{\mathcal{Y}}(q) \) we have \( p_{\alpha_{\mathcal{Y}}(q)} \leq^{ap} q \cap supp_{\mathcal{Y}}(p_{\alpha_{\mathcal{Y}}(q)}, q) \subseteq u_{\alpha_{\mathcal{Y}}(q)} \). If \( \alpha_{2} < \alpha_{1} \) then \( p_{\alpha_{2}} \leq p_{\alpha_{1}} \leq^{ap} q_{1} \leq q_{2} \) \( \wedge p_{\alpha_{2}} \leq^{ap} q_{2} \) hence \( p_{\alpha_{2}} \leq^{ap} q_{1} \) (by (2.7)(k)) and \( supp_{\mathcal{Y}}(p_{\alpha_{2}}, q_{1}) \subseteq supp_{\mathcal{Y}}(p_{\alpha_{2}}, q_{2}) \subseteq u_{\alpha_{2}} \) by the definition of 2.10(1) of \( supp \), contradicting the choice of \( \alpha_{1} \).

2A) We know \( p_{\alpha_{\mathcal{Y}}(q_{1})} \leq^{ap} q_{1} \) by the definition of \( \alpha_{\mathcal{Y}}(q_{1}) \) but we assume \( q_{1} \leq^{ap} q_{2} \) and \( \leq^{pr}_{\mathcal{P}, \kappa} \) is a quasi order hence \( p_{\alpha_{\mathcal{Y}}(q_{1})} \leq^{ap} q_{1} \). So by the definition \( q_{2} \in Q_{\mathcal{Y}} \wedge \alpha_{\mathcal{Y}}(q_{1}) \geq \alpha_{\mathcal{Y}}(q_{2}) \). Also clearly \( q_{1} \leq_{\mathcal{P}} q_{2} \) hence \( q_{1} \leq_{\mathcal{Y}} q_{2} \) hence by part (2), \( \alpha_{\mathcal{Y}}(q_{1}) \leq \alpha_{\mathcal{Y}}(q_{2}) \), together we are done.

3) Let \( \kappa = \kappa_{\mathcal{Y}} \) and \( \theta = \theta_{\mathcal{Y}} \), \( p_{\alpha} = p_{\alpha}^{\mathcal{Y}} \). By Definition 2.10(3)(e),(f) we know that \( p \leq^{pr}_{\mathcal{P}, \kappa} r \) and \( p \leq^{ap}_{\mathcal{P}, \kappa} q \). By Claim 2.7(f) we know that \( s \in Q_{\mathcal{P}} \) and \( p \leq^{ap}_{\mathcal{P}, \kappa} q \leq^{pr}_{\mathcal{P}, \kappa} s \) and \( p \leq^{pr}_{\mathcal{P}, \kappa} r \leq^{pr}_{\mathcal{P}, \kappa} s \) recalling \( s = q \cup r \), note

\[(*)_{1} \text{ the ordinal } \beta := \alpha_{\mathcal{Y}}(r) < \gamma_{\mathcal{Y}} \text{ is well defined.}\]

**Why?** As \( r \in Q_{\mathcal{Y}} \).

\[(*)_{2} \alpha_{\mathcal{Y}}(s) = \alpha_{\mathcal{Y}}(r) = \beta.\]

**Why?** As \( p \in Q_{\mathcal{Y}} \) the ordinal \( \alpha := \alpha_{\mathcal{Y}}(p) < \gamma_{\mathcal{Y}} \) is well defined and by part (2) we have \( \alpha \leq \beta \). So clearly \( p_{\beta} \leq^{ap}_{\mathcal{P}, \theta} r \) by the choice of \( \beta \) and \( r \leq^{ap}_{\mathcal{P}, \kappa} s \) as said above, hence by ??(e) recalling \( \kappa < \theta \), we have \( \leq^{ap}_{\mathcal{P}, \kappa} \leq^{ap}_{\mathcal{P}, \theta} \) hence \( r \leq^{ap}_{\mathcal{P}, \theta} s \), so together \( p_{\beta} \leq^{ap}_{\mathcal{P}, \theta} s \).

Also \( s = q \cup r \) hence \( supp_{\mathcal{Y}}(r, s) \subseteq supp_{\mathcal{Y}}(p, q) \) and as \( q \in ap_{\mathcal{Y}}(p) \) necessarily \( p \leq^{ap}_{\mathcal{P}, \kappa} q \) hence \( p \leq^{ap}_{\mathcal{P}, \theta} q \) hence by part (2A) \( supp_{\mathcal{Y}}(p, q) \subseteq supp_{\mathcal{Y}}(p_{\alpha_{\mathcal{Y}}(q)}, q) \subseteq u_{\alpha_{\mathcal{Y}}(q)}^{\mathcal{Y}} = u_{\alpha_{\mathcal{Y}}(\alpha)}^{\mathcal{Y}} \) but \( u_{\alpha_{\mathcal{Y}}(\alpha)}^{\mathcal{Y}} \subseteq u_{\alpha}^{\mathcal{Y}} \) as \( \alpha \leq \beta \). Together \( supp_{\mathcal{Y}}(r, s) \subseteq u_{\beta} \), and by the choice of \( \beta \) clearly \( supp_{\mathcal{Y}}(p_{\beta}, r) \subseteq u_{\beta} \) hence \( supp_{\mathcal{Y}}(p_{\beta}, s) \subseteq supp_{\mathcal{Y}}(p_{\beta}, r) \cup supp_{\mathcal{Y}}(r, s) \subseteq u_{\beta} \cup u_{\beta} = u_{\beta} \). As we have shown earlier that \( p_{\beta} \leq^{ap}_{\mathcal{P}, \theta} s \) it follows that \( s \in Q_{\mathcal{Y}} \) and \( \alpha_{\mathcal{Y}}(s) \leq \beta \). But \( r \leq^{ap}_{\mathcal{P}, \kappa} s \) hence by part (2) we know that \( \beta = \alpha_{\mathcal{Y}}(r) \leq \alpha_{\mathcal{Y}}(s) \) so necessarily \( \alpha_{\mathcal{Y}}(s) = \alpha_{\mathcal{Y}}(r) = \beta \), i.e. \((*)_{2}\) holds.

So \( p_{\alpha_{\mathcal{Y}}(s)} \leq^{ap}_{\mathcal{P}, \theta} s \) and \( supp_{\mathcal{Y}}(p_{\alpha_{\mathcal{Y}}(s)}, s) = supp_{\mathcal{Y}}(p_{\beta}, s) \subseteq u_{\beta} = u_{\alpha_{\mathcal{Y}}(s)} \) so together \( s \in Q_{\mathcal{Y}} \), the first statement in the conclusion.

Also \( q \leq^{pr}_{\mathcal{Y}} s \), for this check \((*)_{1}(\alpha) + (*)_{2}\) of Definition 2.10(3); for clause \((\alpha)\): \( q \in Q_{\mathcal{Y}} \) is assumed, \( s \in Q_{\mathcal{Y}} \) was just proved; for clause \((\beta)\) \( "q \leq^{pr}_{\mathcal{P}, \kappa} s" \) was proved in the beginning of the proof; so the third statement in the conclusion holds.

Lastly, we check that \( s \in ap_{\mathcal{Y}}(r) \), for this we have to check the two demands in 2.10(3)(f), now \( "s \in Q_{\mathcal{Y}}" \) was proved above, \( r \leq^{ap}_{\mathcal{P}, \kappa} s \) was proved in the beginning of the proof and \( "supp_{\mathcal{Y}}(r, s) \subseteq supp_{\mathcal{Y}}(p_{\alpha_{\mathcal{Y}}(s)}, s)" \) holds as \( supp_{\mathcal{Y}}(r, s) \subseteq supp_{\mathcal{Y}}(p_{\alpha_{\mathcal{Y}}(s)}, r) = supp_{\mathcal{Y}}(p_{\alpha_{\mathcal{Y}}(s)}, s) \) is as required.

\[\square\]

\[\{2B.35\}\]

**Claim 2.12.** 1) Assume \( \kappa < \theta \) are successive members of \( \Theta_{\mathcal{P}} \) and \( (\forall \alpha < \theta_{\mathcal{Y}})((|\alpha| < \partial_{\alpha} < \theta_{\mathcal{Y}})) \) and \( \mathcal{Y} \) is a reasonable \( \mathcal{P} \)-parameter, \( \kappa = \kappa_{\mathcal{Y}} \) hence \( \theta_{\mathcal{Y}} = \theta \) and \( \mathcal{P}_{\mathcal{Y}} \) is \( \leq^{pr}_{\mathcal{Y}} \)-increasing (hence also \( \leq^{ap}_{\mathcal{Y}} \)-increasing) and \( \gamma_{\mathcal{Y}} \) is a successor or a limit ordinal of cofinality \( \geq \theta_{\mathcal{Y}} \). Then \( Q_{\mathcal{Y}} \) is a \( (\partial_{\alpha}, \partial_{\alpha}, \theta_{\mathcal{Y}}, \theta_{\mathcal{Y}}) \)-forcing.

2) If in addition \( \gamma_{\mathcal{Y}} = \alpha_{\mathcal{Y}} + 1 \) then

\[ p_{\alpha_{\mathcal{Y}}} \models "G_{\mathcal{Q}} \cap Q_{\mathcal{Y}} \text{ is a subset of } Q_{\mathcal{Y}} \text{ generic over } V". \]
Proof. 1) We should check for \(Q = Q_y\) (defined in 2.10) each of the clauses of Definition 1.2. Let \(p_\alpha = p_\alpha^y\), \(u_\alpha = u_\alpha^y\).

Clause (a): Trivial, just \(Q_y\) has the right form, a quadruple.

Clause (b): \((Q_y, \leq y)\) is a forcing notion.

Why? By \(\otimes_2(b) + (c)\) from 2.10(3), i.e. \(Q_y\) is a non-empty subset of \(Q_p\) because \(\gamma_y > 0\) so \(p_0^y = p \in Q_y\) and \(\leq y\) being \(\subseteq \) is a quasi order.

Clause (c): \(\leq_{p^y}\) is a quasi order on \(Q_y\) and \(p \leq_{p^y} q \Rightarrow p \leq y q \Rightarrow p \leq p q\).

Why? The first half holds because if \(p_1 \leq_{p^y} p_2 \leq_{p^y} p_3\) then: we should check that \(p_1 \leq_{p^y} p_3\), i.e. clauses \((\alpha), (\beta)\) of \(\otimes_2(c)\) of 2.10(3) hold. Now clause \((\alpha)\) is obvious, for clause \((\beta)\) note \(p_1 \leq_{p^y} p_2 \leq_{p^y} p_3\) and \(\leq_{p^y}\) is a partial order of \(Q_p\), so \(p_1 \leq_{p^y} p_3\), and hence \((\beta)\) holds.

The second part of clause \((c)\) which says \(p \leq_{p^y} q \Rightarrow p \leq y q\) (recalling Claim 2.7(a)(\(\beta)\)) holds by the definition of \(\leq_{y}, \leq_{p^y}\) in \(\otimes_2(c), (e)\) of 2.10(3).

Clause (d)(\(\alpha\)): \(\alpha_{p^y}\) is a function with domain \(Q_y\).

Why? By \(\otimes_2(f)\) of 2.10(3).

Clause (d)(\(\beta\)): if \(q \in Q_y\) then \(q \in \alpha_{p^y}(q) \subseteq Q_y\).

Why? By \(\otimes_2(f)\) of 2.10(3) trivially \(\alpha_{p^y}(q) \subseteq Q_y\). Also we can check that \(q \in \alpha_{p^y}(q) : q \in Q_y\) by an assumption and \(q \leq_{a^p} q\) as \(\leq_{a^p}\) is a quasi order on \(Q_p\) and \(\alpha_{p^y}(q, q) \subseteq \alpha_{p^y}(p_{\alpha_y}(q), q)\) trivially because \(\alpha_{p^y}(q, q) = \emptyset\).

Clause (d)(\(\gamma\)): if \(r \in \alpha_{p^y}(q)\) and \(q \in Q_y\) then \(r, q\) are compatible in \(Q_y\).

Why? As \(r \in \alpha_{p^y}(q)\) \(\Rightarrow (q \leq_{a^p} r \wedge \{r, q\} \subseteq Q_y) \Rightarrow q \leq y r\).

Clause (d)(\(\gamma^+\)): if \(r \in \alpha_{p^y}(q)\) and \(q \leq_{p^y} q^+\) then \(q^+, r\) are compatible in \((Q_y, \leq y)\), moreover there is \(r^+ \in \alpha_{Q_y}(q^+)\) such that \(q^+ \models_{Q_y} r^+ \Rightarrow r \in G_{Q_y}\).

This follows from 2.11(3), by defining \(s = r^+ = r \cup q^+\), which gives more.

Clause (e): \((Q_y, \leq_{p^y})\) is \(\langle \delta_0 \rangle\)-complete, recalling \(\delta_0 = \delta^\infty\).

So assume \(\langle q_\epsilon : \epsilon < \delta \rangle\) is \(\leq_{p^y}\)-increasing and \(\delta\) is a limit ordinal \(\delta < \delta^\infty\); now \((Q_p, \leq_{p^y})\) is \(\langle \delta \rangle\)-complete by Claim 2.8(1) and \(\langle q_\epsilon : \epsilon < \delta \rangle\) is also \(\leq_{p^y}\)-increasing by clause \(\otimes_2(e)(\beta)\) of Definition 2.10(3) hence \(q_\delta := \cup\{q_\epsilon : \epsilon < \delta\}\) is a \(\leq_{p^y}\)-hub of the sequence by 2.8(1). Now \(\alpha_\epsilon := \alpha_{Q_y}(q_\epsilon) : \epsilon < \delta\) is an \(\leq\)-increasing sequence of ordinals \(\leq \gamma_y\) by Observation 2.11(2).

Also by an assumption of 2.12(1), the ordinal \(\gamma_y\) is a successor ordinal or limit of cofinality \(\geq \delta_0\) but then \(\delta < \text{ cf}(\gamma_y)\). So in both cases \(\alpha_\epsilon = \sup\{\alpha_\epsilon : \epsilon < \delta\}\) is an ordinal \(\leq \gamma_y\). But \(\rho^P\) is \(\leq_{p^y}\)-increasing continuous hence \(p_{\alpha_\epsilon} = \cup\{p_{\alpha_\epsilon} : \epsilon < \delta\}\) and similarly \(u_{\alpha_\epsilon} = \cup\{\alpha_\epsilon : \epsilon < \delta\}\). Now easily \(q_\delta\) is a \(\leq_{\delta_0}\)-extension of \(p_{\alpha_\epsilon}\), and \(\alpha_{p^y}(q_\delta, q_\delta) \subseteq \cup\{\alpha_{p^y}(q_{\alpha_\epsilon}, q_\delta) : \epsilon < \delta\}\) \(\subseteq \cup\{u_{\alpha_\epsilon} : \epsilon < \delta\}\) \(= u_{\alpha_\epsilon}\), which has cardinality \(\leq \delta_0\) set each hence \(q_\delta \in Q_y\). Easily \(q_\delta\) is as required.

Clause (f): \((Q_y, \leq_{p^y})\) satisfies the \(\delta_0^\infty\)-c.c.

Why? Let \(q_\epsilon \in Q_y\) for \(\epsilon < \delta_0^+\), so \(\alpha_\epsilon := \alpha_{Q_y}(q_\epsilon)\) is well defined and without loss of generality \(\alpha_\epsilon : \epsilon < \delta_0^+\) is constant or increasing; also \(p_{\alpha_\epsilon} \leq_{p^y} q_\epsilon\) so by
Definition 2.5 the set supp_0(p_α(q_ε), q_ε) has cardinality < ∂_δ, so by the ∆-system lemma, as in the proof of 2.8(3) there are ε(1) < ε(2) < ∂_δ such that:

(∗) if i_1 ∈ supp(p_α(q_ε(1)), q_ε(1)) and i_2 ∈ supp(p_α(q_ε(2)), q_ε(2)) then

(a) if i_1 = i_2 then q_ε(1)(i) = q_ε(2)(i)

(b) if i_1 ∈ E_δ i_2 then i_1, i_2 ∈ supp(p_α(q_ε(1)), q_ε(1)) ∩ supp(p_α(q_ε(2)), q_ε(2)).

So ε(1) < ε(2), α_ε(1) ≤ α_ε(2), p_α(q_ε(1)) ≤ ap θ(1), p_α(q_ε(2)) ≤ ap θ(2).

Hence q := q_ε(1) ∪ q_ε(2) belongs to Q_p is a ≤ ap_δ-hub of {q_θ(1), q_θ(2)} and q_θ(2) ≤ ap θ hence q ∈ Q_y. Also if i ∈ Dom(p_ε(1)) then i/E_δ is disjoint to Dom(p_ε(2)) by (∗)(b); this implies p_ε(2) ≤ ap y which means p_ε(2) ≤ ap Q_y by 2.10(3)(e), for ℓ = 1, 2 so q_ε(1), q_ε(2) are indeed compatible in (Q_y, ≤ ap Q_y).

Clause (g): if q = ⟨q_ε : ε < ∂_δ⟩ is ≤ ap Q_y-increasing, then for stationarily many limit ζ < ∂_δ the sequence q ∪ ζ has an exact ≤ ap Q_y-upper bound (recalling that ∂_δ here stands for θ in Definition 1.2).

Why? We prove more, that if cf(ζ) = ∂_κ and ⟨q_ε : ε < ζ⟩ is ≤ ap Q_y-increasing then the union q = ∪{q_ε : ε < ζ} is an exact ≤ ap Q_y-upper bound. This suffices as ∂_δ < ∂_κ and both are regular. Now by 2.11(2) the sequence ⟨α_ε(q_ε) : ε < ζ⟩ is ≤ ap Q_y-increasing hence ⟨u_α(q_ε) : ε < ζ⟩ is ≤ -increasing in ζ and letting α_ε = ∪{α_ε(q_ε) : ε < ζ} we have α_ε ≤ γ_ζ as γ_ζ is a successor ordinal or limit of cofinality ≥ ∂_δ; hence u_α(q_ε) = ∪{u_α(q_ε) : ε < ζ}, see 2.10(2)(c).

By the proof of clause (e) which we have proved above, clearly q ∈ Q_y and is a ≤ ap Q_y-upper bound of ⟨q_ε : ε < ζ⟩. But what about “exact”? we should check Definition 1.2(2). So assume p ∈ ap_δ(q) and we should prove that for some ε < ζ and p′ ∈ ap_δ(q_ε) we have |p| q_ε G_y. then p ∈ G_y.

Note that q_ε ≤ ap Q_y is a subset of supp(p_α(q_ε), p) ≤ u_α(q_ε) by the definition of ap_δ(q), hence u := supp(p_α(q_ε), p) is a subset of supp_δ(p_α(q_ε), p) ≤ u_α(q_ε) of cardinality < ∂_κ. As ⟨u_α(q_ε) : ε < ζ⟩ is ≤ -increasing with union u_α, for some ε < ζ we have u ≤ u_α(q_ε). Let p′ = q_ε | p_δ, Dom(p_δ) and check (as in earlier cases).

Clause (h): if ⟨q_ε : ε < ∂_δ⟩ is ≤ ap Q_y-increasing and r_ε ∈ ap_δ(q_ε) for ε < ∂_δ and ξ < ∂_δ then for some ζ < ∂_δ we have ζ r_ε q_ε G_y then ζ ≤ otp{ε < ζ : r_ε ∈ G_y}.

This follows from 2.8(3).

Clause (i): ap_δ(q) has cardinality < ∂_δ.

Should be clear as α < ∂_δ ⇒ |α| ≤ ap < ∂_δ by an assumption of the claim and α < ∂_δ ⇒ |u_α| < ∂_δ (see 2.10(3)(f)) and the definition of ap_δ(q) in ⊗_2(e) of 2.10(3).

Let α = α_γ(q) so α < γ_ζ and [ap_δ(q)] = |{s : q ≤ ap s and supp(p_α(q, s) ≤ sup(b(p_α(q, q), q))] ≤ |supp(p_α(q, q), q)] ≤ sup(b[p_α(q_ε, q), q)] ≤ α_γ(q, q) but |supp(b[p_α(q_ε, q), q]) < ∂_δ and so by an assumption of the claim |supp(b[p_α(q_ε, q), q]) = α_γ(q, q) < ∂_δ so we are done.

Clause (j): Let q_ε ≤ r, so α ≤ β where α := α_γ(q_ε), β := α_γ(r).

By 2.7(c) we can find a pair (q, p) such that q ≤ ap p ≤ p r, q_ε ≤ ap p ≤ p r, r = p ∪ q. Now check.

2) Let Q_y = {p : p_α ≤ ap p}. So clearly Q_y ≤ ap y and then (∀p ∈ Q_y)(∃q ∈ Q_y)[p ≤ q] by clause (f) of Claim 2.7, i.e. Q_y is a dense subset of Q_y (by ≤ ap Q_y ≤ ap Q_y). Really q_1 ∈ Q_y ∧ q_1 ≤ q_2 ∈ Q_y ⇒ q_2 ∈ Q_y by 2.11(2).
Suppose $I$ is a dense open subset of $Q_y$, so $I_1 := I \cap Q''$ is dense in $Q_y$.

Let $G$ be a subset of $Q$ generic over $V$ such that $p_{\check{\alpha}}$ belongs to it. If $I \cap G \neq \emptyset$ we are done, otherwise some $q_1 \in G$ is incompatible (in $Q$) with every $q \in I$. As $G$ is directed there is $q_2 \in G$ such that $p_{\check{\alpha}} \leq q_2 \land q_1 \leq q_2$. As $p_{\check{\alpha}} \leq q_2$ by clause (c) of Claim 2.7 there is a $r_2 \in Q$ such that $p_{\check{\alpha}} \leq r_2 \leq q_2$. So $r_2 \in Q''$ hence by the assumption on $I$ there is $r_3 \in I$ such that $r_2 \leq r_3$. Now as $r_3 \in I$ necessarily $p_{\check{\alpha}} \leq r_3$ and of course $p_{\check{\alpha}} \leq r_2 \leq r_3$ hence by clause (k) of Claim 2.7 we have $r_2 \leq r_3$. Recalling $r_2 \leq q_2$, it follows by clause (f) of 2.7 that there is $q_3 \in Q$ such that $q_2 \leq q_3 \land q_3 \leq q_3$ hence $q_3 \Vdash "G \cap I \neq \emptyset"$ and $q_1 \leq q_3$, contradicting the choice of $q_1$.

\[\square_{1.2}\]

**Claim 2.13.** If $\kappa \in \Theta \setminus \{\mu, \theta = \min(\Theta \setminus \kappa^+)\}$ and $\theta = \mu \Rightarrow \partial_\theta < \mu$ and $(\forall \alpha < \partial_\theta)[|\alpha|^{<\delta_\theta} < \partial_\theta]$ and $\delta < \partial_\theta$, $\sigma < \partial_\theta$ then $\Vdash_{Q_p} \theta \in \partial_\theta^{+2}$.

**Proof.** Let $\sigma < \partial_\theta$ and $\delta < \partial_\theta$ and we shall prove $\Vdash_{Q_p} \theta \in \partial_\theta^{+2}$. Toward this assume $c$ is a $Q_p$-name and $q^* \in Q_p$ forces that $c$ is a function from $[\partial_\theta^+]^2$ to $1 + \sigma$. Now we shall apply Claim 2.8(4) with $\theta$ here standing for $\kappa$ there. We choose $(p_i, u_i)$ by induction on $i < \partial_\theta^+$ such that:

1. $p_i \in Q_p$ is $\leq_{p^i}^{\mathcal{C}}$-increasing continuous with $i$ and $p_0 = q^*$
2. for every $i < j < \partial_\theta^+$ the set $I_{i,j}$ is predense above $p_{j+1}$ where $I_{i,j} = \{r : p_{j+1} \leq r, r \leq \partial_\theta^+\}$
3. moreover $I_{i,j}$ has a subset $I'_{i,j}$ of cardinality $\leq \partial_\theta$ which is predense over $p_{j+1}$
4. $u_i$ is $\subseteq$-increasing continuous and $u_i \subseteq \cup\{\alpha/E_\alpha : \alpha \in \text{Dom}(p_i)\}$ and $|u_i| \leq \partial_\theta$ for $i < \partial_\theta^+$
5. $\alpha \in u_i \Rightarrow (\alpha/E_\alpha) \subseteq u_i$
6. $q \in I'_{i,j} \Rightarrow \text{supp}_\kappa(p_{j+1}, q) \subseteq u_{j+1}$.

[Why is this possible? For $i = 0$ let $p_0 = q^*$, for $i$ limit let $u_i = \cup\{u_j : j < i\}$ and $i < \partial_\theta^+$, and we can apply 2.8(1) with $\kappa$ there standing for $\theta$ here, so if $\partial_\theta^+ < \partial_\theta^+$ this is fine, otherwise by 2.4(2)(h) necessarily $\theta = \mu \land \partial_\theta = \mu = 2^\theta$ contradicting an assumption. Lastly, if $i = i + 1$ then we have to deal with $c(\xi, \zeta)$ for $\xi < \zeta$, i.e. with $\leq \partial_\theta$ names of ordinals $< \sigma$. So we apply 2.8(4) with $(p_i, u_i, c(j, i) : j < i), \theta)$ here standing for $(p, A, f, \kappa)$ there and get $p_i, (I_{i,j}, I'_{i,j} : j < i)$ here standing for $q, (Q_{i,j}, Q'_{i,j} : a \in A)$ there. So the relevant parts of clauses (a),(b),(c) hold. Define $u_i$ as in clauses (d),(e),(f) possible as $|I'_{i,j}| \leq \partial_\theta, r \in I'_{i,j} \Rightarrow |\text{supp}_\kappa(p_{j+1}, q)| \leq \partial_\kappa < \partial_\theta$. So we are done carrying the induction.]

Let $\check{p} = \langle p_i : i < \partial_\theta^+ \rangle$ and $\check{u} = \langle u_i : i < \partial_\theta^+ \rangle$.

So this will help to translate the problem from the forcing $Q$ to the forcing $Q_y$.

We define $y = (\kappa, \langle p_\alpha : \alpha < \partial_\theta^+ \rangle, \langle u_\alpha : \alpha < \partial_\theta^+ \rangle)$, so:

$\check{y}$ is a reasonable $p$-parameter.

[Why? Check, see Definition 2.10(2).]

$Q_y$ is a $(\partial_\theta^+, \partial_\theta, < \partial_\theta)$-forcing.
Now for \( i < j < \partial^+_{\theta} \)

(\( a \)) \( I_{i,j} \) is predense in \( Q_y \)

(\( b \)) if \( q_1, q_2 \in I_{i,j} \) or just \( q \in Q_y \), then \( q_1, q_2 \) are compatible in \( Q_p \)

if they are compatible in \( Q_y \).

[Why? By Claim 2.12(1).]

Now for \( i < j < \partial^+_{\theta} \)

\[ I_{i,j} \] is predense in \( Q_y \)

if \( q_1, q_2 \in I_{i,j} \) or \( q \in Q_y \), then \( q_1, q_2 \) are compatible in \( Q_p \)

if they are compatible in \( Q_y \).

[Why? The first clause (a) holds by our definitions. For the second clause (b), assume \( q_1, q_2 \in Q_y \). If they are compatible in \( Q_y \), then clearly they are compatible in \( Q_p \). To show the other direction, let \( q \) be \( q_1 \cup q_2 \). If they are compatible in \( Q_y \), then clearly they are compatible in \( Q_p \). To show the other direction, let \( q \) be \( q_1 \cup q_2 \). If \( q \in Q_y \) we are done, since \( q_1, q_2 \leq_y q \). So we prove that \( q \in Q_y \). Denote \( \alpha_1 = \alpha_y(q_1), \alpha_2 = \alpha_y(q_2) \) and without loss of generality \( \alpha_1 \leq \alpha_2 \). So \( p_{\alpha_1} \leq_{\partial^+} q_1, p_{\alpha_2} \leq_{\partial^+} q_2 \) and also \( p_{\alpha_1} \leq_{\partial^+} p_{\alpha_2} \), and it follows from 2.7(f)(\( \delta \)) that \( p_{\alpha_2} \leq_{\partial^+} q \). Moreover, \( \text{supp}_{\partial^+}(p_{\alpha_2}, q) \subseteq \text{supp}_{\partial^+}(p_{\alpha_1}, q_1) \cup \text{supp}_{\partial^+}(p_{\alpha_2}, q_2) \subseteq u_{\alpha_1} \cup u_{\alpha_2} = u_{\alpha_2} \). Together, \( q \in Q_y \) and we are done.]

So we can define a \( Q_y \)-name \( c' \) as follows; for \( q \in Q_y \)

\[ q \Vdash_{Q_y} "c'(i,j) = t" \text{ iff } q \Vdash_{Q_p} "c(i,j) = t". \]

So by (\( * \))

\[ \Vdash_{Q_y} "c': [\partial^+]^2 \to \sigma". \]

Now by claim 1.4 for some \( Q_y \)-name and a sequence \( \langle \alpha_\varepsilon, \beta_\varepsilon : \varepsilon < \xi \rangle \) we have

\[ \Vdash_{Q_y} "\text{the sequence } \langle \alpha_\varepsilon, \beta_\varepsilon : \varepsilon < \xi \rangle \text{ is as required in Definition 0.3 (for } \partial^+_{\partial^+} \to (\xi, (\xi, \xi)) \text{)}".\]

So for each \( \varepsilon < \xi \) there is a maximal antichain \( J_\varepsilon \) of \( Q_y \) of elements forcing a value to \( \langle \alpha_\varepsilon, \beta_\varepsilon \rangle \) by \( Q_y \).

But \( Q_y \) satisfies the \( \partial^+_{\partial^+} \)-c.c. so \( |J_\varepsilon| \leq \partial_{\theta} \) hence for some \( \alpha_* < \partial^+_{\theta} \) we have:

(\( * \)) \( J_\varepsilon \subseteq \{ q : (\exists \alpha \leq \alpha_*)(p_{\alpha} \leq_{\partial^+} q) \} \) for any \( \varepsilon < \xi \)

Recall that (by 2.12)

(\( * \)) \( p_{\alpha_*} \Vdash "G_Q \cap Q_y(\alpha_*+1) \text{ is a subset of } Q_y(\alpha_*+1) \text{ generic over } V" \)

so we are done. \( \square \) 2.13

Remark 2.14. 1) We can replace the exponent 2 by \( n \geq 2 \), so getting suitable polarized partition relations; we intend to continue elsewhere.

2) For exact such results provable in ZFC see [EHMR84] and [Sh:95].
3. Simultaneous Partition Relations and General topology

Recall (to simplify results we define \( hL^+(X) > \lambda > \text{cf}(\lambda) \)) using an elaborate definition for regulars).

**Definition 3.1.** Let \( X \) be a topological space:

(a) the density of \( X \) is:
\[
d(X) = \min\{|S| : S \subseteq X \text{ and } S \text{ is dense in } X\}
\]
(b) the hereditary density of \( X \) is:
\[
hd(X) = \sup\{\lambda : X \text{ has a subspace of density } \geq \lambda\}
\]
(c) \( hd^+(X) = \sup\{\lambda^+ : X \text{ has a subspace of density } \geq \lambda\} \)
(d) \( X \) is not \( \lambda \)-Lindelöf if there is a family \( \{U_\alpha : \alpha < \lambda\} \) of open subsets of \( X \) whose union is \( X \) but \( w \subseteq \lambda \land |w| < \lambda \Rightarrow \cup\{U_\alpha : \alpha \in w\} \neq X \)
(e) the hereditarily Lindelöf number of \( X \) is:
\[
hL(X) = \sup\{\lambda : \text{ there are } x_\alpha \in X \text{ and } U_\alpha \subseteq \text{open}(X) \text{ for } \alpha < \lambda, \text{ such that } x_\alpha \in U_\alpha \text{ and } \alpha < \beta \Rightarrow x_\beta \notin U_\alpha\}
\]
(f) \( hL^+(X) = \sup\{\lambda^+ : \text{ there are } x_\alpha \in X, U_\alpha \text{ for } \alpha < \lambda \text{ as above}\} \)
(g) the spread of \( X \) is \( s(X) = \sup\{\lambda : X \text{ has a discrete subset with } \lambda \text{ points}\} \)
\[
s^+(X) = \delta(X) = \sup\{\lambda^+ : X \text{ has a discrete subspace with } \lambda \text{ points}\} \)

Our starting point was the following problem (0.1) of Juhasz-Shelah [JuSh:899].

**Problem 3.2.** Assume \( \aleph_1 < \lambda < 2^{\aleph_0} \). Does there exist a hereditarily Lindelöf Hausdorff regular space of density \( \lambda \)?

We answer negatively by a consistency result but then look again at related problems on hereditary density, Lindelöfness and spread; our main theorem is 3.10 getting consistency for all cardinals.

We also try to clarify the relationships of this and related partition relations to \( \chi \rightarrow [\theta]^{2n,2}_{\aleph_0} \), recalling that by [Sh:276], consistently, e.g. \( 2^{\aleph_0} \rightarrow [\aleph_1]^2_{\aleph_0} \) for \( n < \omega \).

Now, see 3.13 below, \( 2^{\aleph_0} \rightarrow [\aleph_1]^{2n,2}_{\aleph_0} \) implies \( 2^{\aleph_0} \rightarrow (\aleph_1, (\aleph_1; \aleph_1)_n)^2 \) and by 3.14 it implies \( \gamma < \aleph_1 \Rightarrow 2^{\aleph_0} \rightarrow (\gamma)^{2n,2}_{\aleph_0} \) see on the consistency of this Baumgartner-Hajnal in [BH73], and Galvin in [Gal75].

On cardinal invariants in general topology, in particular, \( s(X), hd(X), hL(X) \), see Juhasz [Juh80]; in particular recall the obvious.

**Observation 3.3.** For a Hausdorff topological space \( X \):

(a) \( hL(X) \geq s(X) \)
(b) \( hd(X) \geq s(X) \)
(c) for \( \lambda \) regular \( X \) is hereditarily \( \lambda \)-Lindelöf (i.e. every subspace is \( \lambda \)-Lindelöf)
\[
\text{iff there is } x_\alpha \in X, U_\alpha \text{ for } \alpha < \lambda \text{ as in (e) of Definition 3.1}
\]
(d) we choose the second statement in (c) as the definition of "\( X \) is hereditarily \( \lambda \)-Lindelöf" then 3.7, 3.9 holds also for \( \lambda \) singular.

**Conclusion 3.4.** Assume \( \lambda = \lambda^{<\lambda} < \mu = \mu^{<\mu} \) and GCH holds in \( [\lambda, \mu] \), so \( \lambda \leq \text{cf}(\theta) \leq \mu \Rightarrow \theta = \text{cf}(\theta) \) and \( \{\lambda, \mu\} \subseteq \Theta \subseteq \text{Reg} \cap [\lambda, \mu] \) and for \( \theta \in \Theta \) we let \( \partial_\theta = \partial_\theta(\Theta) \). Then

(a) \( p \) is as required in Hypothesis 2.1
(b) the forcing notion \( Q_p \) satisfies:
\(\lambda\) for any Hausdorff

\[\{t.1\}
\]

**Proof.** By 2.9 and 2.13.

\[\square_{3.4}\]

The topological consequences from 3.4 in 3.5 hold by 3.7 and 3.9 below, that is

\[\{3c.89\}\]

**Conclusion 3.5.** We can add in 3.4 that

\[\{it.9\}\]

**Remark 3.6.** 1) If \(\lambda_1 \to (\xi_1; \xi_1)^2_{\kappa_1}\) and \(\lambda_2 \geq \lambda_1, \xi_2 \leq \xi_1, \kappa_2 \leq \kappa_1\) then

\[\lambda_2 \to (\xi_2; \xi_2)^2_{\kappa_2}\].

IA) Similarly for \(\lambda \to (\xi, (\xi; \xi))\) and

\[\{it.14\}\]

**Claim 3.7.** \(X\) has a discrete subspace of size \(\mu\), i.e. \(s^+(X) \geq \mu\) (hence is not hereditarily \(\mu\)-Lindelöf) when:

\[\{top.1\}\]

**Remark 3.8.** The proofs of 3.7, 3.9 are similar to older proofs.

**Proof.** \(X\) has a subspace \(Y\) with density \(\geq \lambda\), by clause (c) of the assumption. We choose \(x_\alpha, C_\alpha\) by induction on \(\alpha < \lambda\) such that

\[\oplus\]

(a) \(x_\alpha \in Y\)

(b) \(C_\alpha = \text{the closure of } \{x_\beta : \beta < \alpha\}\)

(c) \(x_\alpha \notin C_\alpha\).

This is possible as \(Y\) has density \(\geq \lambda\).

Let \(u^1_\alpha\) be an open neighborhood of \(x_\alpha\) disjoint to \(C_\alpha\).

Let \(u^2_\alpha\) be an open neighborhood of \(x_\alpha\) whose closure, \(\text{cl}(u^2_\alpha)\) is \(\subseteq u^1_\alpha\). Why does it exist? As \(X\) is a regular (= \(T_3\)) space.

We define \(c : [\lambda]^2 \to \{0, 1\}\) as follows:

\[\{\ast\}\]

If \(\alpha < \beta\) then \(c(\alpha, \beta) = 1\) iff \(x_\beta \in u^2_\alpha\).
By the assumption $\lambda \to (\mu, (\mu; \mu))^2$ at least one of the following cases occurs.

Case 1: There is an increasing sequence $(\alpha_\varepsilon : \varepsilon < \mu)$ of ordinals $< \lambda$ such that $\varepsilon < \zeta < \mu \Rightarrow c\{\alpha_\varepsilon, \alpha_\zeta\} = 0$. This means that $\varepsilon < \zeta < \mu \Rightarrow x_{\alpha_\zeta} \notin u_{\alpha_\varepsilon}^2$. But if $\varepsilon < \zeta < \mu$ then $u_{\alpha_\varepsilon}^2$ is an open neighborhood of $x_{\alpha_\zeta}$ included in $u_{\alpha_\varepsilon}^1$ which is disjoint to $C_{\alpha_\varepsilon}$ and $x_{\alpha_\varepsilon} \in C_{\alpha_\varepsilon}$ so $x_{\alpha_\varepsilon} \notin u_{\alpha_\varepsilon}^2$.

Lastly, $x_{\alpha_\varepsilon} \in u_{\alpha_\varepsilon}^2$ by the choice of $u_{\alpha_\varepsilon}^2$. Together we are done, i.e. $(x_{\alpha_\varepsilon}, u_{\alpha_\varepsilon}^2) : \varepsilon < \mu$ is as required.

Case 2: There is a sequence $((\alpha_\varepsilon, \beta_\varepsilon) : \varepsilon < \mu)$ such that:

\( (* )_1 \varepsilon < \zeta < \mu \Rightarrow \alpha_\varepsilon < \beta_\varepsilon < \alpha_\zeta < \lambda \)

\( (* )_2 \varepsilon < \zeta \Rightarrow c\{\alpha_\varepsilon, \beta_\varepsilon\} = 1 \), really $\varepsilon < \zeta$ suffice.

So

\( (* )_3 \varepsilon < \zeta \Rightarrow x_{\beta_\varepsilon} \in u_{\alpha_\varepsilon}^2 \)

but now for every $\varepsilon < \mu$ let

\( (* )_4 y_\varepsilon := x_{\beta_\varepsilon} \) and $u^2_\varepsilon := u^2_{\beta_\varepsilon} \setminus cl(u^2_{\alpha_{\beta_\varepsilon+1}})$.

So

\( a) u^2_\varepsilon = u^2_{\beta_\varepsilon} \setminus cl(u^2_{\alpha_{\beta_\varepsilon+1}}) \) is open (as open minus closed)

\( b) y_\varepsilon \in u^2_\varepsilon \).

[Why? Recall $y_\varepsilon = x_{\beta_\varepsilon}$ belongs to $u^2_{\beta_\varepsilon}$ (by the choice of $u^2_{\beta_\varepsilon}$) and not to $u^1_{\alpha_{\beta_\varepsilon+1}}$ (as $u^1_{\alpha_{\beta_\varepsilon+1}}$ is disjoint to $C_{\alpha_{\beta_\varepsilon+1}}$ while $x_{\beta_\varepsilon} \in C_{\alpha_{\beta_\varepsilon+1}}$,) hence not to $cl(u^2_{\alpha_{\beta_\varepsilon+1}})$ being a subset of $u^1_{\alpha_{\beta_\varepsilon+1}}$. Together $y_\varepsilon$ belongs to $u^2_{\beta_\varepsilon} \setminus cl(u^2_{\alpha_{\beta_\varepsilon+1}}) = u^2_\varepsilon$.

(c) if $\varepsilon < \zeta < \mu$ then $y_\zeta \notin u^2_\zeta$.

[Why? Now $y_\zeta = x_{\beta_\zeta}$ belongs to $u^2_{\alpha_{\beta_\varepsilon+1}}$, by $(*)_3$ as $2\varepsilon + 1 < 2\zeta$ which follows from $\varepsilon < \zeta$ hence $y_\zeta$ belongs to $cl(u^2_{\alpha_{\beta_\varepsilon+1}})$ hence $y_\zeta \notin u^2_\zeta$ by the definition of $u^2_\zeta$.]

\( d) \text{ if } \zeta < \varepsilon < \mu \text{ then } y_\zeta \notin u^2_\zeta. \)

[Why? As $u^2_\varepsilon \subseteq u^2_{\beta_\varepsilon}$ and the latter is disjoint to $C_{\beta_\varepsilon}$ to which $x_{\beta_\varepsilon} = y_\zeta$ belongs.

Together $(y_\zeta, u^2_\zeta) : \varepsilon < \mu$ exemplifies that we are done.]

\[ \Box \]

Claim 3.9. $X$ has a discrete subspace of size $\mu$ when:

\( a) \lambda \to (\mu, (\mu; \mu))^2 \)

\( b) X \text{ is a Hausdorff moreover a regular } (= T_3) \text{ topological space} \)

\( c) hL^+(X) > \lambda, \text{ i.e. if } \lambda \text{ is a regular cardinal this means that } X \text{ is not hereditarily } \lambda-Lindel"of \)

Proof. Similar to 3.7. We choose $(x_\alpha, u^1_\alpha) : \alpha < \lambda$ such that $u^1_\alpha$ is an open subset of $X, x_\alpha \in u^1_\alpha$ and $u^1_\alpha \cap \{x_\beta : \beta \in (\alpha, \lambda)\} = \emptyset$. We can choose them as $hL^+(X) > \lambda$.

We then choose an open neighborhood $u^2_\alpha$ of $x_\alpha$ such that $cl(u^2_\alpha) \subseteq u^1_\alpha$. We then define $c : [\lambda]^2 \to \{0, 1\}$ as follows

\( * \text{ if } \alpha < \beta \text{ then } c[\alpha, \beta] = 1 \text{ iff } x_\alpha \in u^2_\beta. \)
We continue as in the proof of 3.7, but now, in Case 2

\((*)'_4\) \(y_\varepsilon := x_{\alpha_2}\), \(u_2 := u_{\varepsilon}^2 \setminus \text{cl}(u_{\varepsilon_{2+1}}^2)\).

\(\square_{3.9}\)

Now we come to our main result.

**Theorem 3.10. The Main Theorem**

It is consistent (using no large cardinals) that:

\((*)\) \(\alpha\) \(2^\mu = \mu^+\) if \(\mu\) is strong limit singular and always \(2^\mu\) is the successor of a singular cardinal

\((\beta)\) for every \(\mu\) we have \(\mu \leq \chi < 2^\mu \Rightarrow 2^\chi = 2^\mu\)

\((\gamma)\) \(\text{hd}(X) \geq \theta \Leftrightarrow \text{hL}(X) \geq \theta \Leftrightarrow s(X) \geq \theta\) for any limit cardinal \(\theta\)

and Hausdorff regular \((= T_3)\) topological space \(X\)

\((\delta)\) \(\text{hd}(X) \leq s(X)^{+3}\) and \(\text{hL}(X) \leq s(X)^{+3}\) for any Hausdorff regular \((= T_3)\) topological space

\((\epsilon)\) in \((\delta)\) we can replace \(s(X)^{+3}\) by \(s(X)^{+2}\) except when \(s(X)\) is regular

\((\zeta)\) in particular, if \(X\) is a (Hausdorff regular topological space which is) Lindelöf or of countable density or just \(s(X) = \aleph_0\) then

\(\text{hd}(X) + \text{hL}(X) \leq \aleph_2\)

\((\eta)\) if \(X\) is a Hausdorff space\(^6\) then \(|X| < 2^{(\text{hd}(X))^+}\)

\((\theta)\) if \(X\) is a Hausdorff space then \(w(X) \leq 2^{(\text{hL}(X))^+}\)

\((i)\) if \(2^\mu > \mu^+\) then \(\mu^{++} \rightarrow (\xi; (\xi; \mu))^2\) for \(\xi < \mu^+\).

**Remark 3.11.** In the Theorem 3.10 above:

1) If we use less sharp results in §1,§2,§3 we should above just use \((\text{hd}(X))^{n(+)}\) for large enough \(n(+).\)

2) We may like to improve clause \((\eta)\) to \(\leq 2^{\text{hd}(X)}\). If below we choose \(\mu_{\varepsilon+1}\) strongly inaccessible (so we need to assume \(V \models \text{"there are unboundedly many strong inaccessible cardinals and clause (a) is changed"}\)), nothing is lost, we have \(\lambda_{\varepsilon+1} = \mu_{\varepsilon+1}\) then we can add

\((\eta)^+\) for any Hausdorff space \(X, |X| < 2^{\text{hd}(X)}\) except (possibly) when \(\text{hd}(X)\) is strong limit singular.

3) Similarly for clause \((\theta)\) about \(w(X) \leq 2^{\text{hL}(X)}\).

4) Probably using large cardinal we can eliminate also the exceptional case in \((\eta)^+\); it seemed that a similar situation is the one in Cummings-Shelah [CuSh:541], but we have not looked into this.

5) We may wonder whether in clause \((\zeta)\) we can replace \(\aleph_2\) by \(\aleph_1\) and similarly for other cardinals, hopefully see [Sh:F884].

**Proof.** We can assume \(V\) satisfies G.C.H. We choose \(((\lambda_\varepsilon, \mu_\varepsilon) : \varepsilon \text{ an ordinal})\) such that:

\(^6\)is interesting because usually \(2^\chi = 2^{(\chi^+)}\), see clause (a)
\(a\) \(\lambda_0 = \mu_0 = \aleph_0\),
\(b\) \(\lambda_\varepsilon < \text{cf}(\mu_{\varepsilon+1}) < \mu_{\varepsilon+1}\)
\(c\) \(\lambda_{\varepsilon+1}\) is the first regular \(\geq \mu_{\varepsilon+1}\),
\(d\) for limit \(\varepsilon\) we have \(\lambda_\varepsilon\) is the first regular cardinal
\(\geq \mu_\varepsilon := \cup\{\lambda_\zeta : \zeta < \varepsilon\}\).

Now let \(p_\varepsilon = (\lambda_\varepsilon, \lambda_{\varepsilon+1}, \Theta_\varepsilon, \bar{\delta}_\varepsilon)\) where \(\Theta_\varepsilon, \bar{\delta}_\varepsilon\) are defined by \(\Theta_\varepsilon = \text{Reg} \cap [\lambda_\varepsilon, \lambda_{\varepsilon+1}], \bar{\delta}_\varepsilon = \langle \delta^\varepsilon_\theta : \theta \in \Theta_\varepsilon, \bar{\delta}_\varepsilon = \delta, \rangle\) so are chosen as in 3.4.

So \(\langle p_\varepsilon : \varepsilon \text{ an ordinal} \rangle\) is a class. We define an Easton support iteration \(\langle P_\varepsilon, Q_\varepsilon : \varepsilon \in \text{Ord} \rangle\) so \(\cup\{P_\varepsilon : \varepsilon \in \text{Ord}\}\) is a class forcing, choosing the \(P_\varepsilon\)-name \(\bar{Q}_\varepsilon\) such that \(\Vdash_{P_\varepsilon} \langle Q_\varepsilon : \varepsilon \in \text{Ord} \rangle\) is defined as in Definition 2.5 for the parameter \(p_\varepsilon\) (in the universe \(V^{P_\varepsilon}\) of course).

As in \(V^{P_\varepsilon}\) section two is applicable for \(p_\varepsilon\) so in \(V^{P_{\varepsilon+1}}\), the conclusions of 3.4, 3.5 hold and \(2^{\lambda_\varepsilon} = \lambda_{\varepsilon+1}\) so cardinal arithmetic should be clear, in particular, clause (a) holds. Of course, forcing with \(P_{\infty}/P_{\varepsilon+1}\) does not change those conclusions as it is \(\lambda_{\varepsilon+1}\)-complete.

In \(V^{P_{\infty}}\) we have enough cases of \(\theta^+ \to \langle \chi, (\xi; \chi) \rangle^2\), i.e. clause (\(\gamma\)) by 2.13. So, first, if \(\chi \geq s(X)\) belongs to \([\lambda_\varepsilon, \mu_{\varepsilon+1}]\) and is regular we have \(\chi^+ = (\chi; (\chi; \chi))^2\) and \(\text{hd}(X) \leq \chi^{+3}\). But if \(s(X) \in [\lambda_\varepsilon, \mu_{\varepsilon+1}]\) then \(s(X)^+ < \mu_{\varepsilon+1}\) recalling \(\mu_{\varepsilon}\) is singular hence \(\text{hd}(X)\), \(\text{hd}(X) \leq \chi^{+3} < \mu_{\varepsilon+1}\).

Second, if \(\chi = s(X)\) belongs to no such interval then \(\chi^+ = \lambda_\varepsilon, \chi = \mu_\varepsilon > \text{cf}(\mu_\varepsilon)\) for some \(\varepsilon\) hence recalling \(\lambda_\varepsilon = \lambda_\varepsilon^{\text{cf}^{\varepsilon}} = 2^\chi\) (in \(V^{P_{\infty}}\)) we have the conclusion. So clause (\(\delta\)) follows hence also clauses (\(\gamma\), (\(\varepsilon\)).

Let us deal with clause (\(\eta\)), let \(\chi = \text{hd}(X)\). First, if \(\chi \in [\lambda_\varepsilon, \mu_{\varepsilon+1}]\) we get \(\text{hd}(X) \leq \chi^{+3} < \mu_{\varepsilon+1}\) hence \(|X| \leq 2^\chi^{+3} = 2^\chi\) by the classical inequality of de-Groot, \(|X| \leq 2^{\text{hd}(X)}\); see [JuH80]). Second, if \(\chi\) belongs to no such interval, then \(\chi = \mu_\varepsilon \land \chi^+ = \lambda_\varepsilon, 2^\varepsilon = 2^\chi\) for some \(\varepsilon\). So \(|X| \leq 2^{\text{hd}(X)} \leq 2^\chi = 2^\varepsilon\) as required.

Clause (\(\theta\)) is proved similarly. \(\square\)

**Theorem 3.12.** If in \(V\) there is a class of (strongly) inaccessible cardinals, then in some forcing extension

\((*)\) (\(\alpha\)) \(2^\alpha = \mu^+\) when \(\mu\) is a strong limit singular cardinal and is a weakly inaccessible cardinal otherwise

\((*)\) (\(\beta\)) (i) as in Theorem 3.10.

**Proof.** As in the proof of Theorem 3.10. \(\square\)

**Claim 3.13.** Assume \(\chi \to [\theta]^2_{\alpha, 2}\) where \(\kappa \geq 2, \chi \leq 2^\lambda\) and \(\lambda = \lambda^{< \lambda} < \theta = \text{cf}(\theta)\).

Then \(\chi \to [\theta]^2_{\alpha, 2}\).

**Proof.** As in the proof of Claim 3.11.
0.2(2) holds, so assume \( \ell \in \{0, 1\} \Rightarrow \varepsilon_\ell \neq 0 \). Let \( \Lambda = \{ \eta \in \lambda^+ : \eta \triangleleft \eta_\alpha \} \). Now \( \Lambda \) has two \( \triangleleft \)-incomparable members (otherwise we get a contradiction by \( \text{cf}(\theta) > \lambda \)) say \( \nu_0, \nu_1 \in \Lambda \) are \( \triangleleft \)-incomparable and without loss of generality \( \nu_0 < \text{lex} \nu_1 \).

So

(1) if \( \nu_0 \triangleleft \eta_\alpha \) and \( \nu_1 \triangleleft \eta_\beta \) and \( \alpha < \beta \) then \( c\{\alpha, \beta\} = \varepsilon_0 \)

(2) if \( \nu_1 \triangleleft \eta_\alpha, \nu_0 \triangleleft \eta_\alpha \) and \( \alpha < \beta \) then \( c\{\alpha, \beta\} = \varepsilon_1 \).

As \( \theta \) is regular and \( \text{otp}(U) = \theta \) we can choose \( \alpha_\varepsilon, \beta_\varepsilon \) by induction on \( \varepsilon < \theta \) such that:

\[ \begin{align*}
(\alpha) & \quad \alpha_\varepsilon \in U \text{ and } \alpha_\varepsilon > \sup \{ \beta_\zeta : \zeta < \varepsilon \} \\
(b) & \quad \nu_0 \triangleleft \eta_{\alpha_\varepsilon} \\
(c) & \quad \beta_\varepsilon \in U \text{ is } \alpha_\varepsilon \\
(d) & \quad \nu_1 \triangleleft \eta_{\beta_\varepsilon}.
\end{align*} \]

So Case (c) of Definition 0.2(2) holds. So we are done.

\[ \square_{3.13} \]

Remark 3.15. If we use versions of \( \chi \to [\theta]^{\kappa,2}_{\kappa,2} \) with privilege positions for the value \( 0 \), we can get corresponding better results in 3.13, 3.14.
4. Private Appendix

Moved from pg 2, Anotated Content:

§4  Simultaneous Partitions and General Topology, pgs. 29-36

We deduce the consistency of “2^λ is a successor of singular and many partition relations hold between λ and 2^λ for every λ”. We deduce consistency results on hereditary density/Lindelöf and the spread of T_δ topological spaces (and so for Boolean Algebras).

5. Polarized partitions in Sh:918

To [Sh:918, 2.12], we can add

Claim 5.1. Assume
(a) κ ∈ Θ\{µ} 
(b) θ = min(Θ\{κ}) but θ = µ ⇒ ∂θ < µ
(c) λ = ⟨λ_ℓ : ℓ < n⟩ with λ_ℓ ≤ ∂_n^+ non-decreasing for simplicity 
(d) µ = ⟨µ_ℓ : ℓ < n⟩
(e) σ < ∂θ and λ →^+(µ)_σ, see below.

Then in V^[θ] we have λ → (µ)_σ.

Discussion 5.2. This also gives an alternative proof of a version of the main theorem, losing by replacing χ^+2 by χ^{n(σ)}, n(σ) < ω large enough but winning by having n-place partition relations for n < n(σ) or so.

Definition 5.3. Let n < ω, λ = ⟨λ_ℓ : ℓ < n⟩, µ = ⟨µ_ℓ : ℓ < n⟩ and σ be given with λ_ℓ, µ_ℓ > 0 and e an equivalence relation on n such that λ, µ respect e, i.e., ℓeκ ⇒ µ_ℓ = µ_k ∧ λ_ℓ = λ_k then λ → (µ)_σ, i.e., λ →(µ)_σ^{(1)} means: if e : \prod ℓ<µ λ_ℓ → σ then there are U_ℓ ⊆ λ_ℓ for ℓ < n such that otp(U_ℓ) = µ_ℓ and ℓeκ ⇒ sup(U_ℓ) = sup(U_k) and e| \prod \ell<n U_\ell := e\{α : α ∈ \prod \ell<n U_\ell increasing\} is constant. If e is equality we may omit it.

Observation 5.4. In 5.3 without loss of generality ⟨U_ℓ : ℓ < n⟩ are pairwise disjoint and if λ increases then without loss of generality sup(U_ℓ) < min(U_{ℓ+1}), also if λ or (e|µ_ℓ) : ℓ < n) is with no repetitions then without loss of generality ⟨conv(U_ℓ) : ℓ < n⟩ pairwise disjoint.

We may consider a more general case

Definition 5.5. Assume
(a) γ(σ) is an ordinal 
(b) e is an equivalence relation on γ(σ) 
(c) λ = ⟨λ_γ : γ < γ(σ)⟩ is a non-decreasing sequence respecting e 
(d) µ = (µ_γ : γ < γ(σ)) is a sequence respecting e 
(e) let S_γ = ∪{γ × λ_γ : γ < γ(σ)} similar to S_γ for U = ⟨u_γ : γ < γ(σ)⟩ and tp(\tilde{s}) = {{i_1, i_2} : for some γ_1, γ_2 we have (i_1, γ_1), (i_2, γ_2) ∈ tp(\tilde{s})
(f) if $\bar{s} \in n(S_{\lambda})$ let $tp(\bar{s}) = \{(i, \gamma) : \text{for some } i < n \text{ we have } s_i \in \{\gamma\} \times \lambda_i\}$

(g) we call $\bar{s}$ standard when $(i_1, \gamma_1), (i_2, \gamma_2) \in tp(\bar{s})$ $\Rightarrow \gamma_1 < \gamma_2 \Rightarrow i_1 < i_2 \Rightarrow (i_1, \gamma_1), (i_2, \gamma_2)$

(h) $t$ is a subset of $\{tp(\bar{s}) : \bar{s} \in n(S_{\lambda}) \text{ for some } n \}$ is standard

(i) $S_{\lambda, t} = \{\bar{s} \in n(t)(S_{\lambda}) : tp(\bar{s}) = t\}$ for $t \in t, S_{\lambda, n} = \{\bar{s} \in n(S_{\lambda}) : \bar{s} \text{ is standard}\}$

(j) if $\bar{n} \in \omega^\omega \text{ then } t_n(\gamma(\ast)) = \{(\ell, \gamma_\ell) : \ell < n\}$ where $\sum_{i < j} n_i \leq \ell_1 < \ell_2 < \sum_{i < j} n_i \Rightarrow \gamma_{\ell_1} = \gamma_{\ell_2}$ for $j < \ell g(\bar{n})$ and if $\bar{n} \subseteq \omega^\omega$ then $t_n(\gamma(\ast)) = \cup\{t_n(\gamma(\ast)) : \bar{n} \in n\}; \text{ so below we may write } n \text{ instead of } t \text{ and } i \text{ instead of } \{\{\}\}.$

1) We say $\bar{\lambda} \rightarrow (\bar{\mu})_3^+$ when: if $c_n : n(S_{\lambda}) \rightarrow \sigma$ for $n < \omega \text{ then }$ there is $\bar{U}$ such that

(a) $\bar{U} = \{U_\gamma : \gamma \lt \gamma(\ast)\}$

(b) $U_\gamma \subseteq \lambda_\gamma$ has order type $\mu_\gamma$

(c) $(\gamma) \sup(U_\gamma) \lt \gamma(\ast)$ respects $e$

(d) if $t \in t$ then $s_n(t(\{\bar{s} \in n(t)(S_{\lambda}) : k < \ell g(\bar{s}) \cap s_k = (i, \gamma) \Rightarrow \gamma \in U_i \text{ and }

\text{tp}(\bar{s}) = t\})$ is constant.

2) We say $\bar{\lambda} \rightarrow (\bar{\mu})_3^+$ when: if $c_n : n(S_{\lambda}) \rightarrow \sigma$ for $n < \omega \text{ then }$ there is $\bar{U}$ such that

(c) clause $(\alpha), (\beta), (\gamma)$ above hold and

(d) if $t \in t$ and $m = |t|, n < \omega \text{ and } s_i \in n(S_{\lambda})$ and $s_\ell \in S_{\lambda, t}$ and $s \cdot s_\ell \in n_{i+m}(S_{\lambda})$ for $\ell = 1, 2$ and $k < \ell g(\bar{s}) \cap s_k = (i, \gamma)$ then $\gamma \in C_i \text{ and similarly for } s_1, s_2$.

3) We say $\bar{\lambda} \rightarrow (\bar{\mu})_3^+$ when: if $\bar{f} = (f_s : s \in S_{\lambda}), f_s$ a condition in Cohen$_\upsilon(\lambda)$, $\lambda = \sup(\lambda)$, then we can find $\bar{U}$ satisfying clauses $(\alpha), (\beta), (\gamma)$ above and

(d) for each $t \in t$ the functions $\{f_s : s \in S_{\lambda} \text{ and } \text{tp}(\bar{s}) = t\}$ are pairwise compatible.

4) We say $\bar{\lambda} \rightarrow (\bar{\mu})_3^+$ when: if $\mathcal{F} = (\mathcal{F}_s : s \in S_{\lambda})$ each $\mathcal{F}_s$ a maximal antichain of Cohen$_\upsilon(\lambda), \lambda, \lambda = \sup(\lambda)$ and we check $e_n((s, f) : s \in n(S_{\lambda}), f \in \mathcal{F}_s) \rightarrow \sigma$ then we can find $\bar{U}$ satisfying clauses $(\alpha), (\beta), (\gamma)$ above and

(d) we can find $(f_s : s \in S_{\lambda})$ such that

- $f_s \in \mathcal{F}_s$

- $(f_s : s \in S_{\lambda})$ are pairwise compatible

- if $n, s_1, s_2$ are as in part one then $c_n((s_1, f_{s_1}) = c_n((s_2, f_{s_2}).$

5) Let $\bar{\lambda} \rightarrow (\bar{\mu})^+_\upsilon$ be $\lambda \rightarrow (\bar{\mu})^+_\upsilon$ when $\ell = 3$ (or is it $u$) and we use $\mu(\{\mu_n\})$.

Claim 5.6. 1) If $\theta = \theta^{\mu_0} \land \bigwedge_{\ell < n} 2^{\theta^{+\ell}} = \theta^{+\ell+1} \land \partial_\theta = \chi^+ \land \chi = \sigma$ then $\bar{\lambda} \rightarrow (\bar{\mu})^+_\upsilon$

where $\bar{\lambda} = (\theta^\ell : \ell < n), \bar{\mu} = (\chi : \ell < n)$ (in fact $\mu_\ell = (\theta^\ell(-1))$ is O.K. with $\mu_0 = \chi$).

2) Can use $\partial_\theta \leq \sigma < \theta^\upsilon$ but $\partial_\ell$ is larger.

3) The Infinite version (5.5).
Proof. See [Sh:95].

Proof. Proof of 5.1
We start as in [Sh:918, 2.12] but
\[ q^* \vdash_{\mathcal{P}_\mu} \mathfrak{c} : \prod_{\ell < n} \lambda_\ell \to \sigma^*. \]

For notational simplicity define \( \mathfrak{c}^+ : [\partial_{\mu}^+]^n \to \sigma \) such that \( \bar{\alpha} \in \prod_{\ell < n} \lambda_\ell \) with no repetitions, \( \mathfrak{c}^+(\bar{\alpha}) = \mathfrak{c}(\text{Rang}(\bar{\alpha})) \)

\[ \odot_1 q^* \vdash_{\mathcal{P}_\mu} \mathfrak{c} \]

\[ \odot_2 \text{in } \odot_1 \text{ we choose } p_i \text{ for } i < \partial_{\mu} \text{ (not } \partial_{\mu}^+) \]

\[ \odot_3 \text{ in } \odot_1(b) \text{ use:} \]

\[ (b)_i \text{ for every } u \in [i]^{n-1} \text{ the set } \mathcal{I}_{u,i} := \{ r : p+1 \leq r \text{ and } r \text{ forces a value to } \mathfrak{c}^+(u \cup \{i\}) \} \]

\[ (c)_i \mathcal{I}_{u,i} \cdots \]

Define \( y \) really \( \mathbb{Q}_\lambda, \alpha_\lambda(-) \).

After that let \( I = \{ \bar{\alpha} \in \prod_{\ell < n} \lambda_\ell : \bar{\alpha} \text{ increasing} \} \) and for \( \langle \alpha_\ell : \ell < n \rangle \) let \( \mathcal{I}_n = \mathcal{I}_{\langle \alpha_\ell, \ell < n-1 \rangle, \alpha_{n-1}} \) and choose \( q_\alpha \in \mathcal{I}_n \) and \( v(\bar{\alpha}) = \text{supp}(p_{\alpha}(q_\alpha)) \cup \{0\} \) so of cardinality \( < \partial_{\mu} \) for simplicity we are assuming \( \partial_{\mu} = \chi^+ \) so of cardinality \( \leq \chi \) let \( q_\alpha \vdash_{\mathcal{P}_\mu} \mathfrak{c}^+(\bar{\alpha}) = \Upsilon_{\bar{\alpha}} \) let \( v(\bar{\alpha}) = \{ f(\bar{\alpha}) : \zeta < \zeta_0 \leq \chi \} \) possibly with repetitions.

By \( \bar{\lambda} \to_{\mathcal{P}_\mu} (\bar{\mu})_\chi \) we can find \( \bar{U} \) such that

\[ (a) \bar{U} = \{ U_\ell : \ell < n \} \]

\[ (b) \text{otp}(\bar{U}) = \mu_\ell \]

\[ (c) \text{(sup}(\bar{U}) : \ell < n) \text{ is non-decreasing (put?)} \]

\[ (d) \langle \Upsilon_\alpha : \bar{\alpha} \in \prod_{\ell < n} U_\ell \text{ and } f_{\zeta(1)}(\bar{\alpha}) = f_{\zeta(2)}(\bar{\beta}) \rangle \text{ then } q_\alpha(f_{\zeta(1)}(\bar{\alpha})) = q_\beta(f_{\zeta(2)}(\bar{\beta})) \]

\[ (a) \text{ the function } q_* = \cup\{ q_\alpha : \bar{\alpha} \in \prod_{\ell < n} U_\ell \} \text{ is well defined.} \]

Now as \( \max(\bar{\mu}) < \partial_{\mu} \)

\[ (a) \bar{q}_* \in \mathbb{Q}_\mu \]

\[ (b) q_\alpha \leq q_* \text{ for } \bar{\alpha} \in \prod_{\ell < n} U_\ell. \]

So \( q_* \) forces that \( \langle U_\ell : \ell < n \rangle \) are as required.

Question 5.7. In [Sh:918]:

(a) assume \( \theta_0 < \ldots < \theta_{n-1} \) are from \( \Theta \) and \( \sigma < \theta_0 \).

Can we prove \( \vdash_{\mathcal{P}_\mu} \langle \theta_{\ell+1} : \ell < n \rangle \to \langle \theta_\ell : \ell < n \rangle_\sigma \)?

(b) or at least \( \vdash_{\mathcal{P}_\mu} \langle \theta_{\ell+1} : \ell < n \rangle \to \langle \theta_0 : \ell < n \rangle_{\theta_0} \)?

\{d12\}
References


[Sh:F884] Saharon Shelah, Partition relations and $T_3$-spaces complementary to Sh918.
