STABLE THEORIES AND REPRESENTATION OVER SETS [COSH:919]

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Abstract

In this paper we give characterizations of the stable, superstable, and ω-stable theories, in terms of an external property called representation. In the sense of the representation property, the mentioned classes of first-order theories can be regarded as "not very complicated".

Introduction

Our motivation to investigate the properties under consideration in this paper comes from the following

> **Thesis:** It is very interesting to find dividing lines and it is a fruitful approach in investigating quite general classes of models. A "natural" dividing property "should" have equivalent internal, syntactical, and external properties. (see [Sheb] for more)

The main results presented in this paper are:

Characterization of stable theories (Theorem 13):

For a complete first-order theory T, the following conditions are equivalent

1. T is stable
2. T is representable in $\mathrm{Ex}^1_{0,|T|}(\mathfrak{k}^{\mathrm{eq}})$ (cf. definitions 2, 11, and 4).
3. For some cardinals $\mu_1, \kappa_1, \mu_2, \kappa_2$, it holds that T is representable in $\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathfrak{k}^{\mathrm{eq}}))$ (cf. definition 11).
4. T is representable in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ for some cardinals μ, κ (cf. definition 10).
5. T is representable in $\mathrm{Ex}^0_{0,|T|}(\mathfrak{k}^{\mathrm{eq}})$.

Characterization of ω-stable theories (Theorem 40):

For a complete first-order theory T, the following conditions are equivalent

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We thank Assaf Hasson for his constructive remarks.

(1) T is ω-stable.
(2) T is representable in $\mathrm{Ex}^2_{\omega\omega}(\mathfrak{k}^{\mathrm{eq}})$.
(3) T is representable in $\mathrm{Ex}^1_{\omega,2}(\mathfrak{k}^{\mathrm{eq}})$.
(4) T is representable in $\mathrm{Ex}^{0,\mathrm{lf}}_{\omega,2}(\mathfrak{k}^{\mathrm{eq}})$ (see definition 11)

Characterization of superstable theories (Theorem 42):

For a complete first-order theory T the following conditions are equivalent

(1) T is superstable.
(2) T is representable in $\mathrm{Ex}^2_{2^{|T|},\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$.
(3) T is representable in $\mathrm{Ex}^1_{2^{|T|},2}(\mathfrak{k}^{\mathrm{eq}})$.
(4) T is representable in $\mathrm{Ex}^{0,\mathrm{lf}}_{2^{|T|},2}(\mathfrak{k}^{\mathrm{eq}})$.
(5) T is representable in $\mathrm{Ex}^2_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ for some cardinal μ.
(6) T is representable in $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ for some cardinals μ, κ.

Discussion 0.1. *It would seem natural to conjecture that if* $\mathfrak{k}^{\mathrm{eq}}$ *is replaced by* $\mathfrak{k}^{\mathrm{or}}$ *in the above properties, we can find an analogous characterization for dependent theories. However, such characterization would imply strong theorems on existence of indiscernible sequences. Lately (see* [KS]*), some dependent theories were discovered for which it is "quite hard to find indiscernible subsequences", rendering the naïve conjecture false.*

1. Preliminaries

The main model-theoretic definitions that we will use are

Definition 1. A theory T is called κ-stable iff for every model M of T and $A \subseteq M$, $|A| \leq \kappa$ it holds for every $m < \omega$ that $|\mathbf{S}^m(A, M)| \leq \kappa$.

T is called stable if T is κ-stable for some cardinal κ. T is called superstable if there exists λ, such that T is κ-stable for every $\kappa \geq \lambda$.

1.1. **Structure classes and representation.**

Convention 1. *$\mathfrak{k}$ denotes a class of structures with a given dictionary (signature) $\sigma_{\mathfrak{k}}$. A structure $I = \langle \sigma, |I|, \models \rangle$ is a triple of dictionary, universe(domain) and the interpretation relation for formulas in the language (usually logical closure of atomic formulas). The dictionary may contain constants, partial finitary and infinitary functions and finitary and infinitary relations.*

Discussion 1.1. *We will always have a first-order complete theory T in the background. We separate the notions of "structure" and "model" to structures that belong to a given class of structures and structures from* $\mathrm{EC}(T)$*, respectively.*

Now we reach the central definition

Definition 2. For structures M and I, and sets of formulas Δ. The function $f : M \to I$ is called a Δ-representation of M in I if

$$\mathrm{tp}_{\mathrm{qf}}(f(\bar a), \emptyset, I) = \mathrm{tp}_{\mathrm{qf}}(f(\bar b), \emptyset, I) \ \Rightarrow\ \mathrm{tp}_\Delta(\bar a, \emptyset, M) = \mathrm{tp}_\Delta(\bar b, \emptyset, M)$$

for any two sequences $\bar a, \bar b \in {}^{<\omega}M$. (qf $= \mathcal{L}_I^{\mathrm{qf}}$ denotes the quantifier free formulas in I.)

- *We say that a structure M is Δ-represented in $\mathfrak{k}$ if there exists an $I \in \mathfrak{k}$ such that M is Δ-represented in I.*
- *For two classes of structures $\mathfrak{k}_0, \mathfrak{k}$ we say that $\mathfrak{k}_0$ is Δ-represented in $\mathfrak{k}$ if every $M \in \mathfrak{k}_0$ is Δ-represented in $\mathfrak{k}$.*
- *We say that a first-order theory T is Δ-represented in $\mathfrak{k}$ if* $\mathrm{EC}(T)$ *is Δ-represented in $\mathfrak{k}$.*

Definition 3. For a structure I we say that[1] $\langle \bar a_t : t \in I\rangle$ is a Δ-indiscernible structure over A if for all $\bar s, \bar t \subseteq I$ with the same quantifier-free type it holds that[2]

$$\mathrm{tp}_\Delta(\bar a_{\bar t}, A, M) = \mathrm{tp}_\Delta(\bar a_{\bar s}, A, M)$$

For a class of structures $\mathfrak{k}$, model $M \models T$ and subset $A \subseteq M$ let

$$\mathrm{Ind}_\Delta(\mathfrak{k}, A, M) = \{\mathbf{a} : \mathbf{a} = \langle \bar a_t : t \in I\rangle \text{ is a } \Delta\text{-indiscernible structure over } A,\ I \in \mathfrak{k}\}$$

Convention 2. *We omit the respective symbol from the above notation in the specific cases $\Delta = \mathcal{L}(\tau_M)$, $M = \mathfrak{C}$ and $A = \emptyset$.*

Definition 4. $\mathfrak{k}^{\mathrm{eq}}$ denotes the class of structures structures of the language $\{=\}$.

1.2. The free algebras $M_{\mu,\kappa}$.

Definition 5. The free algebra generated by a structure I and the functions (where for each $F_{\alpha,\beta}$ is a β-ary function symbol) $\langle F_{\alpha,\beta} : \alpha < \kappa, \beta < \mu\rangle$ is a structure denoted $M_{\mu,\kappa}(I)$, whose dictionary, $\tau_{\mu,\kappa}$ contains the equality relation, a unary relation I for the (given) set of atoms I, the dictionary of I and the β-ary functions $\langle F_{\alpha,\beta} : \alpha < \kappa\rangle$ for every $\beta < \mu$.universe is[3]:

$$M_{\mu,\kappa}(I) = \bigcup_{\gamma \in \mathrm{Ord}} M_{\mu,\kappa,\gamma}(I)$$

Where $M_\zeta = M_{\mu,\kappa,\zeta}(I)$ is defined as follows:

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[1] This definition also appears in [Shea],[She87, II]

[2] for a sequence $\bar s$ we denote $\bar a_{\bar s} := a_{s_0}{}^\frown \ldots {}^\frown a_{s_{\lg(\bar s)-1}}$

[3] This defines a set and not a proper class by remark 1.

- $M_0 := I$
- *For limit* ζ*:* $M_\zeta(I) = \bigcup_{\xi<\zeta} M_\xi(I)$
- *For* $\zeta = \gamma + 1$

$$M_\zeta = M_\gamma \cup \left\{F_{\alpha,\beta}(\overline{b}) : \overline{b} \in {}^{\beta} M_\gamma,\ \alpha < \kappa,\ \beta < \mu\right\}$$

Where $F_{\alpha,\beta}(\overline{b})$ *is treated as a formal object. The interpretation of* I *and its dictionary is by reduction. The* β*-ary function* $F_{\alpha,\beta}(\overline{x})$ *is interpreted as the mapping* $\overline{a} \mapsto F_{\alpha,\beta}(\overline{a})$ *for all* $\overline{a} \in^{\beta} |M_{\mu,\kappa}(I)|$*, where* $F_{\alpha,\beta}(\overline{a})$ *on the right side of the mapping is the formal object.*

Definition 6. We denote $\lambda_\mu = \begin{cases} \mu & \mu = \mathrm{cf}\mu \\ \mu^+ & \text{otherwise} \end{cases}$ for every cardinal μ.

Remark 1. *Since* $\lambda_\mu \geq \mu$ *is regular, for all* $\beta < \mu$ *and sequence of terms* $\sigma_i(\overline{c}_i) \in M_{\lambda_\mu}$*,* $(i < \beta)$ *there exists* $\gamma < \lambda_\mu$ *such that* $\sigma_i(\overline{c}_i) \in M_\gamma$ *for all* $i < \beta$*. Therefore* $F_{\alpha,\beta}\left(\langle \sigma_i(\overline{c}_i) : i < \beta\rangle\right) \in M_{\gamma+1} \subseteq M_{\lambda_\mu}$*, hence* $M_{\mu,\kappa}(S) = M_{\mu,\kappa,\lambda_\mu}(S)$ *and particularly* $M_{\mu,\kappa}(S)$ *is a set (though defined as a class).*

Observation 1.2. $\|M_{\mu,\kappa}(S)\| \leq (\kappa + |S|)^{<\lambda_\mu}$*.* $|M_\gamma| \leq (\kappa + |S|)^{<\lambda_\mu}$ *can be proved by induction on* $\gamma \leq \lambda_\mu$*.*

Definition 7. For a sequence $\overline{a} \subseteq M_{\mu,\kappa}(S)$ we define its closure under subterms as the set $\mathrm{cl}(\overline{a})$ defined by induction on the construction of the term and the sequence length as

$\mathrm{cl}(\overline{a}) := \overline{a}$ for $\overline{a} \subseteq S$. If $\lg(\overline{a}) = 1$ and $a_0 = F_{\alpha,\beta}(\overline{b})$ then $\mathrm{cl}(\overline{a}) := \{a_0\} \cup \bigcup\{\mathrm{cl}(b_i) : i < \beta\}$. Otherwise, $\mathrm{cl}(\overline{a}) := \bigcup\{\mathrm{cl}(a_i) : i < \lg(\overline{a})\}$.

Definition 8. We say that $\lambda \geq \aleph_0$ is closed under μ-terms if for every set S, cardinal κ and sequence $\overline{a} \in {}^{<\lambda}\left(M_{\mu,\kappa}(S)\right)$ the closure of $\overline{a}$ under subterms has power less than λ.

Remark 2. *If* λ *is closed under* μ*-terms then for every set* S *and sequence* $\overline{a} \in {}^{<\lambda}\left(M_{\mu,\kappa}(S)\right)$ *there exist* $\chi < \lambda$*, and a term* $\overline{\sigma}(\overline{x}_\chi) \in \tau_{\mu,\kappa}(\overline{x}_\lambda)$ *(where* $\overline{x}_\chi = \overline{x}_\lambda \restriction \chi$*) and a sequence* $\overline{b} \in^{\chi} S$ *such that* $\overline{a} = \overline{\sigma}(\overline{b})$*.*

Observation 1.3. *All regular* $\lambda \geq \mu$ *are closed under* μ*-terms.*

Definition 9. Denote $\theta_{\mu,\kappa} := \left|M_{\mu,\kappa}(\overline{x}_{\lambda_\mu})\right| = (\lambda_\mu + \kappa)^{<\lambda_\mu}$.

Let $\mathcal{M}(S)$ a free algebra. we shall say that $\mathcal{A} \subseteq \mathcal{L}_{\mathcal{M}}$ is a minimal system of terms for $\mathcal{M}$, if and only if for every $\tau(\overline{v}) \in \mathcal{M}(S)$ there exists a single $\tau'(\overline{x}) \in \mathcal{A}$ such that for some $\overline{u} \in S$ without repetitions it holds that $\tau(\overline{v}) = \tau'(\overline{u})$.

Observation 1.4. *Every free algebra has a minimal system of terms. This follows from the axiom of choice.*

1.3. Extensions of classes of structures.

Discussion 1.5. *For a class of structures $\mathfrak{k}$, we define several classes of structures that are based on $\mathfrak{k}$.*

Definition 10. $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ is the class of structures defined by extending each $I \in \mathfrak{k}$ to a structure whose dictionary is

$$\mathbb{I}^+ := \langle I, \langle P_\alpha : \alpha < \mu \rangle, \langle F_\beta : \beta < \kappa \rangle \rangle$$

By additional relations and functions $P_\alpha, F_\beta \notin \tau_I$ for all $\alpha < \mu,\ \beta < \kappa$, such that:

- $\left\langle P_\alpha^{\mathbb{I}^+} : \alpha < \mu \right\rangle$ *is a partition on* $|I|$.
- $\left\langle F_\beta^{\mathbb{I}^+} : \beta < \kappa \right\rangle$ *are partial unary functions.*

Definition 11. $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k})$ is the class of structures in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ for which the closure of every element under the new functions if finite. (lf stands for "locally finite").

$\mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k})$ is the class of structures in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ for which $F_\beta(P_\alpha) \subseteq P_{<\alpha} := \bigcup_{\gamma<\alpha} P_\gamma$ holds for every $\alpha < \mu, \beta < \kappa$.

$\mathrm{Ex}^2_{\mu,\kappa}(\mathfrak{k})$ is the class of structures $\mathcal{M}_{\mu,\kappa}(I)$, where $I \in \mathfrak{k}$.

1.4. Some properties of representation and extension classes.

Let us note several properties of representation

Observation 1.6. *Let M, I, J be structures. If $f : J \to I$ is a Δ-representation of M in I and $g : I \to J$ is an $\mathcal{L}^{\mathrm{qf}}_I$-representation of I in J, then $g \circ f$ is a Δ-representation of M in J.*

Observation 1.7. $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k}) \supseteq \mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}) \supseteq \mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k})$.

Observation 1.8. $\mathfrak{k}$ *is qf-representable in all the extension classes of* $\mathfrak{k}$ *defined above, for any two cardinals* μ, κ.

In the following observations we use Ex^* *to mean that each claim holds for one of* Ex^0, $\mathrm{Ex}^{0,\mathrm{lf}}$, Ex^1, Ex^2 *at a time.*

Observation 1.9. *The classes* $\mathrm{Ex}^*_{\mu_1+\mu_2,\kappa_1+\kappa_2}(\mathfrak{k})$ *and* $\mathrm{Ex}^*_{\mu_2,\kappa_2}(\mathrm{Ex}^*_{\mu_1,\kappa_1}(\mathfrak{k}))$ *are qf-representable in each-other.*

Observation 1.10. *If* $\mu_2 \leq \mu_1,\ \kappa_2 \leq \kappa_1$ *then* $\mathrm{Ex}^*_{\mu_2,\kappa_2}(\mathfrak{k})$ *is qf-representable in* $\mathrm{Ex}^*_{\mu_1,\kappa_1}(\mathfrak{k})$.

Observation 1.11. $\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}))$ *is qf-representable in* $\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathfrak{k}))$.

Observation 1.12. $\mathrm{Ex}^1_{2^\kappa,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *is qf-representable in* $\mathrm{Ex}^1_{0,\kappa}(\mathfrak{k}^{\mathrm{eq}})$.

Proof. Let $\langle I, \langle P_\alpha : \alpha < 2^\kappa\rangle, \langle F_\beta : \beta < \kappa\rangle\rangle$ be the dictionary of $\mathbb{I}^+$. Without loss of generality $\left|P_0^{\mathbb{I}^+}\right| \geq 2$ (every model such that $\left|P_0^{\mathbb{I}^+}\right| = 1$ can be represented in such a model). We select two distinct $t_0, t_1 \in P_0^{\mathbb{I}^+}$ and let $h : 2^\kappa \to \mathcal{P}(\kappa)$ a bijection. Consider the structure $I' = \langle I, \langle F_\beta : \beta < \kappa\rangle, \langle G_\beta : \beta < \kappa\rangle\rangle$ whose universe is $|I^+|$, $F_\beta^{I'} = F_\beta^{I^+}$ and also define for all $\gamma < \kappa$, $x \in P_\gamma^{I^+}$,

$$G_\beta^{I'}(x) = \begin{cases} t_0 & \gamma \in h(\beta) \\ t_1 & \gamma \notin h(\beta) \end{cases}$$

it is easy to verify that the identity is a $\mathcal{L}^{\mathrm{qf}}_{\mathbb{I}^+}$-representation of $\mathbb{I}^+$ in I'. □

Observation 1.13. $\mathrm{Ex}^2_{\kappa\omega}(\mathfrak{k}^{\mathrm{eq}})$ *is qf-representable in* $\mathrm{Ex}^1_{\kappa,2}(\mathfrak{k}^{\mathrm{eq}})$ *for* $\kappa \geq \aleph_0$.

Proof. Let $\langle \tau_\alpha(\overline{x}_\alpha) : \alpha < \kappa\rangle$ be a minimal system of terms of $\mathcal{M}_{\kappa\omega}(S)$ (see definition 9, ω is the upper bound on function symbol arities, κ is the number of functions). W.l.o.g $\lg(\overline{x}_0) = 1$, $\tau_0(x_0) = x_0$. Consider the structure $\mathbb{I}^+$ whose dictionary is $\langle I,\, f_{\mathrm{last}}, f_{\mathrm{head}}, \langle P_\beta : \beta < \kappa\rangle\rangle$ and has universe

$$\{\langle \alpha, i, s_0 \dots s_i\rangle : \alpha < \kappa,\ i < \lg(\overline{x}_\alpha),\ s_0 \dots s_{i-1} \in S\}$$

Let $\langle\langle \alpha_\beta, i_\beta\rangle : i < \kappa\rangle$ enumerate the pairs $\{\langle \alpha, i\rangle : \alpha < \kappa,\ i < \lg(\overline{x}_\alpha)\}$ in increasing lexical order. Let $P_\beta^{\mathbb{I}^+}$ the set of sequences in $|\mathbb{I}^+|$ whose head is $\langle \alpha_\beta, i_\beta\rangle$, and let

$$\begin{aligned} f_{\mathrm{last}}^{\mathbb{I}^+}(\langle \alpha, i, s_0 \dots s_i\rangle) &:= \langle 0, 0, s_i\rangle \\ f_{\mathrm{head}}^{\mathbb{I}^+}(\langle \alpha, i, s_0 \dots s_i\rangle) &:= \langle \alpha, i-1, s_0 \dots s_{i-1}\rangle \quad (i > 0) \end{aligned}$$

we define a map $h : \mathcal{M}_{\kappa\omega}(S) \to |\mathbb{I}^+|$ as follows:

$$h(\tau_\alpha(\overline{v})) = \left\langle \alpha, \lg(\overline{x}_\alpha) - 1, v_0, \dots, v_{\lg(\overline{x}_\alpha)-1}\right\rangle$$

That h is a qf-representation of $\mathcal{M}_{\kappa\omega}(S)$ in $\mathbb{I}^+$ is easy to verify.

□

Definition 12. We say that a function f with domain and range contained in a structure I is a partial automorphism when for every sequence $\overline{a} \in |I|$ it holds that $\mathrm{tp}_{\mathrm{qf}}(\overline{a}, \emptyset, I) = \mathrm{tp}_{\mathrm{qf}}(f(\overline{a}), \emptyset, I)$.

2. Stable theories

The central result for this section is

Theorem 13. *for a complete first-order theory T, the following conditions are equivalent*

1. *T is stable*
2. *T is representable in $\mathrm{Ex}^1_{0,|T|}(\mathfrak{k}^{\mathrm{eq}})$ (cf. definitions 2, 11, and 4).*
3. *T is representable in $\mathrm{Ex}^1_{|T|^+,|T|}(\mathfrak{k}^{\mathrm{eq}})$.*
4. *T is representable in $\mathrm{Ex}^1_{2^{|T|},|T|}(\mathfrak{k}^{\mathrm{eq}})$.*
5. *For some cardinals $\mu_1, \kappa_1, \mu_2, \kappa_2$, it holds that T is representable in $\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathfrak{k}^{\mathrm{eq}}))$ (cf. definition 11).*
6. *T is representable in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ for some cardinals μ, κ (cf. definition 10).*
7. *T is representable in $\mathrm{Ex}^0_{0,|T|}(\mathfrak{k}^{\mathrm{eq}})$.*

Proof. $1 \Rightarrow 3$ is by theorem 30. $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ are immediate by 1.10. $4 \Rightarrow 5$, $7 \Rightarrow 6$ are immediate. $5 \Rightarrow 1$ by theorem 15. $3 \Rightarrow 2$ by observations 1.6,1.12 giving us equivalence of conditions $1-5$. $2 \Rightarrow 7$ by 1.7. We leave $6 \Rightarrow 1$ without a complete proof, since it is very similar to $5 \Rightarrow 1$. □

Stability of representable theories.

Discussion 2.1. *We shall first prove the first direction of the main theorem. Namely, that a theory which is representable in $\mathrm{Ex}^2_{\mu_2,\kappa_2}\left(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}^{\mathrm{eq}})\right)$ is stable. The method relies on the combinatorial properties of models of stable theories, particularly that all order indiscernibles are indiscernible sets.*

For this subsection we assume that T is representable in $\mathrm{Ex}^2_{\mu_2,\kappa_2}\left(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}^{\mathrm{eq}})\right)$, for some cardinals $\$\mu_1, \mu_2, \kappa_1, \kappa_2\$$.

Theorem 14. *For cardinals μ, λ, if the following holds:*
a. $\mu \geq \lambda_{\mu_2} + \mu_1^+$
b. $\lambda > \mu + \theta_{\mu_2,\kappa_2} + \kappa_1$ *a regular cardinal (see definition 9)*
g. $\lambda > \chi^{<\mu}$ *for all cardinals* $\chi < \lambda$

then, for every sequence $\bar{b} = \left\langle \bar{b}_\alpha : \alpha < \lambda \right\rangle \subseteq \mathfrak{C}$ of length $< \mu$ and $\alpha < \lambda$ there exists $S \in [\lambda]^\lambda$ such that $\left\langle \bar{b}_\alpha : \alpha \in S \right\rangle$ is an indiscernible set.

Proof. Let $M \models T$ such that $\bar{\mathbf{b}} \in |M|$ and assume that $f : M \to \mathbb{I}^+ := (\mathcal{M}_{\mu_2,\kappa_2}(I), P_\alpha, F_\beta)_{\alpha<\kappa_1,\beta<\mu_1}$ is a representation, $I = \bigcup_{\alpha<\kappa_1} P_\alpha$ and let $\bar{a}_\alpha = f(\bar{b}_\alpha)$ for all $\alpha < \lambda$.

w.l.o.g we can add the following assumptions

- Each $\bar{a}_\alpha$ is closed under subterms in $\mathcal{M}_{\mu_2,\kappa_2}(I)$: $\mu \geq \mu_2$ is regular, so it is closed under μ_2-terms.
- The set $\{F_\beta : \beta < \mu_1\}$ is closed under composition. (including the empty composition = the identity)

- Each $\overline{a}_\alpha$ is closed under the partial functions F_β: To find the closure of $\overline{a}_\alpha$ under the functions we need to add at most μ_1 elements, so the closure of $\overline{a}_\alpha$ is $< \mu$.
- $\lg \overline{a}_\alpha = \xi = |\xi|$ for all $\alpha < \lambda$: since $\lambda = \bigcup_{\xi<\mu} \{\alpha < \lambda : \xi = \lg \overline{a}_\alpha\}$ and $\lambda > \mu$ is regular, and by reordering.

The rest of the proof is by taking subsequences of the original sequence, while preserving the length λ, as follows (in brackets we note the common property of the sought subsequence):

First subsequence (sequences constructed by the same terms) : By 2, for each $i < \xi, \alpha < \lambda$ there exist terms $\sigma_{\alpha,i}(\overline{x}_{\alpha,i}) \in M_{\mu_2,\kappa_2}(\overline{x}_{\lambda_{\mu_2}})$ and sequences $\overline{t}_{\alpha,i} \in {}^{<\lambda_{\mu_2}} I$ such that $a_{\alpha,i} = \sigma_{\alpha,i}(\overline{t}_{\alpha,i})$. since $\lambda > [\theta_{\mu_2,\kappa_2}]^\xi$ is regular, there exist $\langle \sigma_i(\overline{x}_i) : i < \xi \rangle \in$, $S_0 \in [\lambda]^\lambda$ such that $\langle \sigma_{\alpha,i}(\overline{x}_{\alpha,i}) : i < \xi \rangle = \langle \sigma_i(\overline{x}_i) : i < \xi \rangle$ for all $\alpha \in S_0$.

Second subsequence (the quantifier free type of $\overline{a}_\alpha$ relative to the P_α): since $(\kappa_1)^\xi < \lambda$, there exists a $S_1 \in [S_0]^\lambda$ such that the function

$$\alpha \mapsto \left\{(i,\beta) \in \xi \times \kappa_1 : a^i_\alpha \in P_\beta\right\}$$

is constant on S_1 (denote this constant as the relation R_1).

Third subsequence - (the quantifier free type of $\overline{a}_\alpha$ relative to the F_α): since $\xi^{\mu_1+\xi} \leq \xi^{<\mu} < \lambda$, there exists a $S_2 \in [S_1]^\lambda$ such that the function

$$\alpha \mapsto \left\{(\beta,\zeta_0,\zeta_1) : \zeta_0,\zeta_1 < \xi,\ \beta < \mu_1,\ F_\beta(a^{\zeta_0}_\alpha) = a^{\zeta_1}_\alpha\right\}$$

is constant on S_2 (denote this constant as the relation R_2).

Final subsequence: By the Δ-system lemma theorem (48) there exist $S_3 \in [S_2]^\lambda$, $U \subseteq \xi$, $E \subseteq \xi \times \xi$ such that:

- $\overline{a}_\alpha \restriction U = \overline{a}_\beta \restriction U$ for all $\alpha, \beta \in S_3$.
- E is an equivalence relation such that for all $\alpha \in S_3$: $a^i_\alpha = a^j_\alpha \leftrightarrow (i,j) \in E$.
- $a^i_\alpha = a^j_\beta \rightarrow i,j \in U$ for all $\alpha \neq \beta \in S_3$.

We now show that for any finite $\overline{u}, \overline{v} \subseteq S_3$ of length ℓ without repetition, it holds that $\overline{a}_{\overline{v}}$ and $\overline{a}_{\overline{u}}$ have the same quantifier-free type in $\mathbb{I}^+$.

Let $\varphi(\overline{x}_{\ell\times\xi})$ an atomic formula. By symmetry, it suffices to show that $\varphi(\overline{a}_{\overline{u}}) \rightarrow \varphi(\overline{a}_{\overline{v}})$.

- Case $\varphi(\overline{x}_{\ell\times\xi}) =$ "$\sigma_1(\overline{x}_{\ell\times\xi}) = \sigma_2(\overline{x}_{\ell\times\xi})$":
  proof is carried by induction on the complexity of the term σ_1.

- ○ For $\sigma_1(\overline{x}_{\ell\times\xi}) = F_{\alpha,\beta}(\overline{\sigma}^*_1(\overline{x}_{\ell\times\xi}))$ it follows from properties of the free algebra that for some sequence of terms $\overline{\sigma}^*_2(\overline{x}_{\ell\times\xi})$ it holds that $\sigma_2(\overline{x}_{\ell\times\xi}) = F_{\alpha,\beta}(\overline{\sigma}^*_2(\overline{x}_{\ell\times\xi}))$ and also $\sigma^*_{1,i}(\overline{a}_{\overline{u}}) = \sigma^*_{2,i}(\overline{a}_{\overline{u}})$ for all $i < \alpha$. The induction hypothesis implies that $\sigma^*_{1,i}(\overline{a}_{\overline{v}}) = \sigma^*_{2,i}(\overline{a}_{\overline{v}})$ and thus $\sigma_1(\overline{a}_{\overline{v}}) = \sigma_2(\overline{a}_{\overline{v}})$ as required.
- ○ For $\sigma_1(\overline{x}_{\ell\times\xi}) = F_{\alpha^*_1}(\sigma^*_1(\overline{x}_{\ell\times\xi}))$, the validity of $\varphi(\overline{a}_{\overline{u}})$ implies that $\sigma_2(\overline{a}_{\overline{u}}) = \sigma_1(\overline{a}_{\overline{u}}) \in I$. It is easy to verify (by induction on the complexity of the term) that the terms $\sigma_s (s = 1, 2)$ contains only symbols from $\overline{x}_{\ell\times\xi}, F_\alpha$ (since $\mathrm{Dom}(F_\alpha) \subseteq I$). Now, for a finite sequence of ordinals $\overline{\alpha}$, denote $F_{\overline{\alpha}} := F_{\alpha_0} \circ \ldots \circ F_{\alpha_{\lg(\overline{\alpha})}}$, ($F_{\langle\rangle}$ - the identity). It is easy to verify that the term $\sigma_s(\overline{x}_{\ell\times\xi})$ takes the form $F_{\overline{\alpha}_s}(x_{i_s,\zeta_s})$. for some sequence $\overline{\alpha}$.

  And the formula φ can be rewritten as:

  $$F_{\overline{\alpha}_1}(x_{i_1,\zeta_1}) = F_{\overline{\alpha}_2}(x_{i_2,\zeta_2})$$

  Since the family $\langle F_\alpha : \alpha < \mu_1\rangle$ is closed under composition (see above), there exists an $\beta_s < \mu_1$ such that $F_{\overline{\alpha}_s} = F_{\beta_s}$. The sequences $\overline{a}_{u_{i_s}}$ are closed under $\langle F_\alpha : \alpha < \mu_1\rangle$, hence for some $\zeta^*_s < \xi$ it holds that $F_{\beta_s}(a_{u_{i_s},\zeta_s}) = a_{u_{i_s},\zeta^*_s}$ and $a_{u_{i_1},\zeta^*_1} = a_{u_{i_2},\zeta^*_2}$. The former implies $\langle\beta_s, \zeta_s, \zeta^*_s\rangle \in R_2$ and the latter implies that $\zeta^*_1, \zeta^*_2 \in U$ and $\langle\zeta^*_1, \zeta^*_2\rangle \in E$. Now, since $\overline{a}_{v_{i_1}} \restriction U = \overline{a}_{v_{i_2}} \restriction U$ it follows that $F_{\beta_s}(a_{v_{i_s},\zeta_s}) = a_{v_{i_s},\zeta^*_s}$ and $a_{v_{i_1},\zeta^*_1} = a_{v_{i_2},\zeta^*_2}$ so easily $\models \varphi(\overline{a}_{\overline{v}})$.
- $\varphi(\overline{x}_{\ell\times\xi}) = P_\alpha(\sigma(\overline{x}_{\ell\times\xi}))$: $\models \varphi(\overline{a}_{\overline{v}})$ implies that $\sigma(\overline{x}_{\ell\times\xi}) = F_{\overline{\alpha}}(x_{i,\zeta})$ for some $i < \ell, \zeta < \xi$. Now by the closure of the functions under composition, formula is equivalent to $P_\alpha(F_\beta(x_{i,\zeta}))$. And for some ζ^* we get that $F_\beta(a_{u_i,\zeta}) = a_{u_i,\zeta^*}$ and $P_\alpha(a_{u_i,\zeta^*})$ implying $\langle\beta, \zeta, \zeta^*\rangle \in R_2$ and $\langle\alpha, \zeta^*\rangle \in R_1$, respectively. Similar arguments give $\models \varphi(\overline{a}_{\overline{u}})$.

□

Theorem 15. *If T is representable in $\mathrm{Ex}^2_{\mu_2,\kappa_2}(\mathrm{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}^{\mathrm{eq}}))$, then T is stable.*

Proof. Assume towards contradiction that T is unstable. Recall from classification theory that

Theorem 16. *(the order property) T is unstable if and only if there exist a formula $\varphi(\overline{x}, \overline{y})$ and a sequence $\langle\overline{a}_n : n < \omega\rangle$ such that $\models \varphi(\overline{a}_i, \overline{a}_j)^{\mathrm{if}(i<j)}$ holds for all $i, j < \omega$. (see* [She90, II.2.13]*)*

By this fact, and compactness, we can construct a sequence $\langle\overline{a}_i : i < \lambda\rangle$, where

$$\lambda = \beth_2\left(\mu + \theta_{\mu_2,\kappa_2} + \kappa_1\right)^+,\ \mu = \lambda_{\mu_2} + \mu_1^+$$

such that $\models \varphi(\overline{a}_i, \overline{a}_j)^{\text{if}(i<j)}$ holds for all $i, j < \lambda$.

Now by the assumptions let $f : M \to \mathbb{I}^+$ be a represent M in $\text{Ex}^2_{\mu_2,\kappa_2}(\text{Ex}^1_{\mu_1,\kappa_1}(\mathfrak{k}^{\text{eq}}))$. It is easily verified that the conditions in 14 hold. Hence, there exists $S \in [\lambda]^\lambda$ such that $\{\overline{a}_i : i \in S\}$ is an indiscernible set and particularly $\models \varphi(\overline{a}_i, \overline{a}_j) \leftrightarrow \varphi(\overline{a}_j, \overline{a}_i)$ holds for all $i, j \in S$, contradicting the assumption. $\square$

Stability implies representability.

Discussion 2.2. *We turn to proving the other direction. We recall several facts about stable theories (see* [She90, II, III]*).*

Definition 17. The formula $\varphi(\overline{x}, \overline{a})$ divides over a set A iff there exists a sequence $\langle \overline{a}_n : n < \omega \rangle$ such that $\text{tp}(\overline{a}_n, A) = \text{tp}(\overline{a}, A)$ for all $n < \omega$, but there exists an $m < \omega$ such that $\models \neg \exists \overline{x} \bigwedge_{n \in w} \varphi(\overline{x}, \overline{a}_n)$ holds for all $w \in [\omega]^m$.

The type $p(\overline{x})$ forks over A if there exist formulas $\varphi_i(\overline{x}, \overline{a}_i)$ $(i < n)$, such that for all $i < n$, φ_i divides over A and $p(\overline{x}) \vdash \bigvee_{i<n} \varphi_i(\overline{x}, \overline{a}_i)$.

Fact 18. *(Monotonicity of forking) If* $\text{p}(\overline{x})$ *forks over* A *and* $B \subseteq A$, $q \vdash p$ *then* q *forks over* B.

For a stable T,

(1) *(Symmetry)* $\text{tp}(\overline{a}, A \cup \overline{b})$ *does not fork over* A *iff* $\text{tp}(\overline{b}, A \cup \overline{a})$ *does not fork over* A.

(2) *(Transitivity) For sets* $A \subseteq B \subseteq C$ *such that* $\text{tp}(\overline{a}, C)$ *does not fork over* B, *and* $\text{tp}(\overline{a}, B)$ *does not fork over* A *it holds that* $\text{tp}(\overline{a}, C)$ *does not fork over* A.

Fact 19. *(Forking preserved under elementary maps) If* $\text{p}(\overline{x})$ *forks over* A, *and* f *is an elementary map in* M, $\text{Dom}(f) \supseteq \text{Dom}(p) \cup A$, *then* $f(p)$ *forks over* $f(A)$. *(see* [She90, III.1.5]*)*

Definition 20. We say that $\mathbb{I} \subseteq \mathfrak{C}$ is strongly independent over A if

$\circledast$ *For any* $a \in \mathbb{I}$, *the type* $\text{tp}(a, A \cup \mathbb{I} \setminus \{a\}, M)$ *is the unique extension in* $\mathbf{S}(A \cup \mathbb{I} \setminus \{a\})$ *of* $\text{tp}(a, A, M)$ *which does not fork over* A.

Definition 21. We say that a sequence $\langle \mathbb{I}_\alpha : \alpha < \gamma \rangle$ is a strongly independent decomposition of M of length γ iff for all $\alpha < \gamma$, it holds that $\mathbb{I}_\alpha$ is strongly independent over $\mathbb{I}_{<\alpha}$ (in M), and that $|M| = \mathbb{I}_{<\gamma}$.

Claim 22. Let $a_1, a_2 \in \mathfrak{C}$, $A \supseteq B_1, B_2$ such that $\text{tp}(a_i, A \cup \{a_{3-i}\})$ does not fork over B_i and $\text{tp}(a_i, A)$ is the unique non-forking extension of $\text{tp}(a_i, B_i)$ in $\mathbf{S}(A)$. Then $(*)_1 \Leftrightarrow (*)_2$ where:

$(*)_i$ *The type* $\mathrm{tp}(a_i, B_i)$ *has a unique extension to* $A \cup \{a_{3-i}\}$ *which is non-forking.*

Proof. By symmetry it suffices to show that $\neg(*)_2 \Rightarrow \neg(*)_1$.

Assume that $\mathrm{tp}(a_2, B_2)$ has two different non-forking extensions $p_1, p_2 \in \mathbf{S}(A \cup \{a_1\})$.

Since both types are complete, there exists a formula $\varphi = \varphi(x, a_1, \overline{c})$ over $A \cup \{a_1\}$ such that $\varphi \in p_1,\ \neg\varphi \in p_2$. Let b_1, b_2 realize p_1, p_2, respectively.

$\mathrm{tp}(b_i, A) = p_i \restriction A$ is a non-forking extension of p, by uniqueness it follows that $p_1 \restriction A = p_2 \restriction A$. Hence, for $i < 2$ there exist elementary maps F_i in $\mathfrak{C}$ such that $F_i \restriction A = \mathrm{id}_A,\ F_i(b_i) = a_2$.

Let $q_i \in \mathbf{S}(A \cup \{b_i\})$ be a non-forking extension of $\mathrm{tp}(a_1, B_1)$.

Then $F_i(q_i) \in \mathbf{S}(A \cup \{a_2\})$ is a non-forking extension of $\mathrm{tp}(a_1, B_1)$ ($F_i \restriction A = \mathrm{id}_A$, and non-forking is preserved under elementary maps).

Now, note that $\models \varphi(b_1, a_1, \overline{c}) \wedge \neg\varphi(b_2, a_1, \overline{c})$ which implies $\varphi(a_2, x, \overline{c}) \in F_1(q_1)$ and also $\neg\varphi(a_2, x, \overline{c}) \in F_2(q_2)$. This implies that $F_1(q_1)$, $F_2(q_2)$ are distinct extensions of $\mathrm{tp}(a_1, B_1)$, as needed. □

Claim 23. If $\langle \mathbb{I}_\alpha : \alpha < \gamma \rangle$ is a strongly independent decomposition of M, then every order-preserving refinement of this partition is also a strongly independent decomposition of M.

Remark 3. *An order-preserving refinement is a partition* $\langle \mathbb{J}_\alpha : \alpha < \gamma' \rangle$ *which refines* $\langle \mathbb{I}_\alpha : \alpha < \gamma \rangle$ *such that for all* $\alpha < \beta < \gamma,\ \alpha', \beta' < \gamma',\ \mathbb{I}_\alpha \supseteq \mathbb{J}_{\alpha'}\ \mathbb{I}_\beta \supseteq \mathbb{J}_{\beta'}$ *imply* $\alpha' < \beta'$.

Proof. Using the basic properties of non-forking □

Fact 24. *For stable* T*, distinct types* $p, q \in \mathbf{S}(B)$ *non-forking over* $A \subseteq B$*, there exists* $E \in \mathrm{FE}(A)$ *such that*

$$p(x) \cup q(y) \vdash \neg E(x, y)$$

(see [She90, III.2.9(2)] *)*

Definition 25. We say that a formula $\varphi(x, \overline{c})$ (with parameters from $\mathfrak{C}$) is almost over $A \subseteq \mathfrak{C}$ iff for some $E(x, y) \in \mathrm{FE}(A)$ and some $d \in \mathfrak{C}$ it holds that $T \models E(x, d) \leftrightarrow \varphi(x, \overline{c})$.

A formula is over $A \subseteq \mathfrak{C}$ iff it is equivalent in T to a formula with parameters taken only from A.

Remark 4. *It should be clear that in the following, the notions "formula over* A*" and "formula with parameters from* A*" are interchangeable.*

Theorem 26. *For stable T, Let $A \subset B$ such that for every formula φ over B which is almost over A, φ is equivalent (in T) to a formula over A. If $p, q \in \mathbf{S}(B)$ are distinct and non-forking over A, there exists a $\varphi_*(x, \overline{c})$ over A such that $p \vdash \varphi_*$, $q \vdash \neg\varphi_*$.*

Proof. By 24, there exists an equivalence relation $E \in FE(A)$ such that $p(x) \cup q(y) \vdash \neg E(x, y)$.

Let $\{b_i : i < n(E)\} \subseteq \mathfrak{C}$ enumerate representatives for all the distinct equivalence classes of E and let

$$w := \{i < n(E) : p(x) \cup \{E(x, b_i)\} \text{ is consistent}\}$$

W.l.o.g assume that b_i realizes p for all $i \in w$. Let $\varphi(x) := \bigvee_{i \in w} E(x, b_i)$. It can be easily verified that $p(x) \vdash \varphi(x)$ and similarly, $q(x) \vdash \neg\varphi(x)$. We will show that $\varphi(x)$ is preserved by every $f \in \mathrm{Aut}(\mathfrak{C}/B)$:

Since p is over B and E is a formula over B, they are preserved by f and so:

- $p(x) \cup \{E(x, b_i)\} \Leftrightarrow p(x) \cup \{E(x, f(b_i))\}$ holds for all $i < n(E)$.
- $\neg E(b_i, b_j)$ holds for every $i, j < n(E)$, $i \neq j$ and hence also $\neg E(f(b_i), f(b_j))$.

Hence, f can be regarded as a permutation on $\{b_i/E : i \in w\}$, the equivalence classes of E in $\mathfrak{C}$:

$$f(\varphi(\mathfrak{C})) = f(\bigcup_{i \in w} b_i/E) = \bigcup_{i \in w} f(b_i)/E = \varphi(\mathfrak{C})$$

Consequently, $\models \varphi(x) \equiv f(\varphi(x))$. Now we use a well known fact:

Fact 27. *A formula $\varphi(\overline{x}, \overline{c})$ is equivalent to a formula over B if and only if $\varphi(\overline{x}, f(\overline{c})) \equiv \varphi(\overline{x}, \overline{c})$ holds for every $f \in \mathrm{Aut}(\mathfrak{C}/B)$. (see [She90, III.2.3(2)])*

Giving the required equivalent formula. □

Lemma 28. *Let $\mu = |T|^+$. For a stable T, for each model $M \models T$ there exists a strongly independent decomposition of length $\gamma \leq \mu$*

Proof. We choose by induction a sequence $\langle \mathbb{I}_\alpha : \alpha < \mu \rangle$ such that $\mathbb{I}_\alpha$ is strongly independent over $\mathbb{I}_{<\alpha}$, and is also maximal in $|M| \setminus \mathbb{I}_{<\alpha}$ with respect to this property for every $\alpha < \mu$.

Assume towards contradiction that the elements of M were not exhausted after μ iterations, then there exists an $a \in M \setminus \mathbb{I}_{<\mu}$. Recall that for a stable theory $\kappa(T) \leq \mu$ (see [She90, III;3.2,3.3]) and so, by the definition of $\kappa(T)$ there exists a set

$$B \subseteq \mathbb{I}_{<\mu},\ |B| < \kappa(T) \leq \mu$$

Such that $p(x) := \mathrm{tp}(a, \mathbb{I}_{<\mu})$ is non-forking over B, and by regularity of μ there exists $\alpha_0(*) < ||$ such that $\mathbb{I}_{<\alpha_0(*)} \supseteq B$.

Now, let

$$\Gamma := \left\{\varphi(x;\overline{c}) : B \text{ almost over } \varphi(x,\overline{c}),\ \varphi(x;\overline{y}) \in \mathcal{L},\ \overline{c} \in^{\lg \overline{y}} \mathbb{I}_{<\mu}\right\}$$

Recall that

Fact 29. *(see [She90] III;2.2(2)) There are (up to logical equivalence mod T) at most $|T| + |A|$ formulas almost over A.*

Hence, let $\Gamma_* \subseteq \Gamma,\ |\Gamma_*| \leq |B| + |T| < \mathrm{cf}(\mu)$ represent all formulas almost over B up to logical equivalence. By regularity of μ there exists $\alpha_1(*) < \mu$ such that $\overline{b} \subseteq \mathbb{I}_{<\alpha(*)}$ for all $\varphi(x, \overline{b}) \in \Gamma_*$. Let $\alpha(*) = \max\{\alpha_0(*),\ \alpha_1(*)\}$.

- We will prove that $p \restriction \mathbb{I}_{\leq\alpha(*)}$ is the unique extension in $\mathbf{S}(\mathbb{I}_{<\alpha(*)})$ of $p \restriction \mathbb{I}_{<\alpha(*)}$.

Since p does not fork over B, there exists a $q \in \mathbf{S}(\mathbb{I}_{\leq\alpha(*)})$ which extends $p \restriction \mathbb{I}_{<\alpha(*)}$, and does not fork over $\mathbb{I}_{<\alpha(*)}$.

By the transitivity of non-forking, q does not fork over B. Assume towards contradiction that $p \neq q$. By fact 24 there exists $E \in FE(B)$ such that $q(x) \cup p(y) \vdash \neg E(x, y)$, and particularly $q(x) \vdash \neg E(x, a)$.

The formula $E(x, a)$ is almost over B. By choice of $\alpha_1(*)$, there exist a $\overline{b} \subseteq \mathbb{I}_{<\alpha(*)}$ and $\varphi(x, \overline{b})$ logically equivalent to $E(x, a)$.

Now since $E(a, a)$ holds, we also get $\models \varphi(a, \overline{b})$, and since $\overline{b} \subseteq \mathbb{I}_{<\alpha(*)}$, we get $\varphi(x, \overline{b}) \in \mathrm{tp}(a, \mathbb{I}_{<\alpha(*)}) = q \restriction \mathbb{I}_{<\alpha(*)}$, a contradiction.

We proved that $\mathrm{tp}(a, \mathbb{I}_{\leq\alpha(*)})$ is the unique non-forking extension of $\mathrm{tp}(a, \mathbb{I}_{<\alpha(*)})$ in $\mathbf{S}\left(\mathbb{I}_{\leq\alpha(*)} \setminus \{b\}\right)$. Recall from the choice of $\mathbb{I}_{\alpha(*)}$ that for all $b \in \mathbb{I}_{\alpha(*)}$, $\mathrm{tp}(b, \mathbb{I}_{\leq\alpha(*)} \setminus \{b\})$ is the unique extension in $\mathbf{S}\left(\mathbb{I}_{\leq\alpha(*)} \setminus \{b\}\right)$ which does not fork over $\mathbb{I}_{<\alpha(*)}$.

Now, claim 22 implies that $\mathrm{tp}(b, \mathbb{I}_{\leq\alpha(*)} \setminus \{b\} \cup \{a\})$ is the unique non-forking extension of $\mathrm{tp}(b, \mathbb{I}_{<\alpha(*)})$ in $\mathbf{S}(\mathbb{I}_{\leq\alpha(*)} \setminus \{b\} \cup \{a\})$.

This implies that the condition $\circledast$ above holds for $\mathbb{I}_{\alpha(*)} \cup \{a\}$ over $\mathbb{I}_{<\alpha(*)}$, contradicting the maximality of $\mathbb{I}_{\alpha(*)}$. $\square$

Theorem 30. *A stable first order theory T is representable in* $\mathrm{Ex}^1_{|T|^+,|T|}(\mathfrak{k}^{\mathrm{eq}})$.

Proof. Let $M \models T$. By lemma 28 we get a strongly independent decomposition of M: $\left\langle \mathbb{I}_\alpha : \alpha < \gamma \leq |T|^+ \right\rangle$. By claim 23 we can assume w.l.o.g that $|\mathbb{I}_1| = |\mathbb{I}_0| = 1$.

Define $\mathbb{I}^+ = \left\langle F_i^*,\ P_\alpha,\ F_{\varphi(x,\overline{y}),j} : \alpha < \gamma,\ i < |T|,\ \varphi(x,\overline{y}) \in \mathcal{L},\ j < k_{\varphi(x,\overline{y})} \right\rangle \in \mathfrak{k}^{\mathrm{eq}}$ as follows:

$|\mathbb{I}^+| = |M|$, $P_\alpha^{\mathbb{I}^+} = \mathbb{I}_\alpha$ for all $\alpha < \gamma$.

Now define the functions $\left\langle (F_i^*)^{\mathbb{I}^+} : i < |T| \right\rangle$: $\mathrm{Dom}(F_i^*) = |M| \setminus (\mathbb{I}_0 \cup \mathbb{I}_1)$ for all $i < |T|$. Fix $\alpha > 1,\ a \in \mathbb{I}_\alpha$. Let $F_i(a) = b_i,\ (i < |T|)$ where $\langle b_i : i < |T| \rangle$ enumerates (possibly with repetitions) B from lemma 34 (substitute $A = \mathbb{I}_{<\alpha}$ there).

Now, recall

Fact 31. *(Type definability in stable theories) Let $A \subseteq M$, $|A| \geq 2$. For every formula $\varphi(\overline{x}, \overline{y})$ there exists another formula $\psi_\varphi(\overline{y}, \overline{z})$ such that: For all $\overline{b} \in \mathfrak{C}$ there exists a $\overline{c}$ such that*

$$\models \varphi(\overline{b}, \overline{a}) \Leftrightarrow \models \psi_\varphi(\overline{a}, \overline{c})$$

for all $\overline{a} \in A$ (see [She90, II.2.2]*).*

And for each $\varphi(x, \overline{y}) \in \mathcal{L}$ let $\psi_\varphi(\overline{y}, \overline{z}_\varphi)$ as above, and define the partial unary functions $\left\{ F^{\mathbb{I}^+}_{\varphi(x,\overline{y}),j}(x) : j < k_{\varphi(x,\overline{y})} \right\}$ as follows: $\mathrm{Dom}\left(F^{\mathbb{I}^+}_{\varphi(x,\overline{y}),j}\right) = |M| \setminus (\mathbb{I}_0 \cup \mathbb{I}_1)$. Let $\overline{c}_a \in^{\lg(\overline{z}_\varphi)} \mathbb{I}_{<\alpha}$ for all $2 \leq \alpha < \mu$ and $a \in \mathbb{I}_\alpha$ such that $\models \varphi[a, \overline{b}] \Leftrightarrow \models \psi_\varphi[\overline{b}, \overline{c}_a]$ and define for all $j < \lg(\overline{z}_\varphi)$ the function

$$F^{\mathbb{I}^+}_{\varphi(x,\overline{y}),j}(a) := (\overline{c}_a)_j$$

Thus we have defined $\mathbb{I}^+$ and we define $f : M \to \mathbb{I}^+$ as $f(a) = a$ for all $a \in |M|$. We turn to prove that f is indeed a representation.

By 33 it is suffices to prove:

For every partial automorphism h of $\mathbb{I}^+$ with domain and range closed under functions

$$\mathrm{tp}(h(\overline{a}), \emptyset, M) = \mathrm{tp}(\overline{a}, \emptyset, M)$$

holds for all $\overline{a} \subseteq \mathrm{Dom}(h)$.

Let $D_\alpha = \mathbb{I}_\alpha \cap \mathrm{Dom}(h)$, $R_\alpha = \mathbb{I}_\alpha \cap \mathrm{Rang}(h)$. It is easily verified that for $\alpha < \gamma$, $h \restriction D_\alpha$ is a partial automorphism of $\mathbb{I}^+$ from D_α onto R_α. We will prove by induction on $\alpha < \gamma$ that

$$h\left(\mathrm{tp}\left(\overline{a}, D_{<\alpha}, M\right)\right) = \mathrm{tp}\left(h(\overline{a}), R_{<\alpha}, M\right) \qquad \boxtimes_{\alpha,n}$$

holds for all $n < \omega$, and $\overline{a} \in D_\alpha$ of length n without repetitions.

$\boxtimes_{\alpha,n}$ holds for $\alpha < 2$ since by the definition, $|\mathbb{I}_\alpha| = 1$.

Now let $\alpha \geq 2$ and assume that $\boxtimes_{\beta,n}$ holds for all $n < \omega$, $\beta < \alpha$. We prove by induction on $n < \omega$ that $\boxtimes_{\alpha,n}$ holds.

First, $\boxtimes_{\alpha,1}$: Let $a \in \mathbb{I}_\alpha$, $\varphi = \varphi(x, \overline{c})$ a formula over $D_{<\alpha}$. W.l.o.g assume $\models \varphi[a, \overline{c}]$. by the definition of the functions it follows $\models \psi_\varphi[\overline{c}, F_{\varphi,0}(a) \dots F_{\varphi, \lg \overline{z}-1}(a)]$. This formula contains only constants from $D_{<\alpha}$, so by the induction hypothesis, $\models \psi_\varphi[h(\overline{c}), h(F_{\varphi,0}(a)) \dots h(F_{\varphi,\lg \overline{z}-1}(a))]$ holds. Since h is a partial automorphism (with closed range and domain) of $\mathbb{I}^+$, h commutes with the functions on $\mathbb{I}^+$ so $\models \psi_\varphi[h(\overline{c}), F_{\varphi,0}(h(a)) \dots F_{\varphi,\lg \overline{z}-1}(h(a))]$ holds. By the definitions of $F_{\varphi,j}$ ($j < \lg \overline{z}$), ψ_φ we get $M \models \varphi[h(a), h(\overline{c})]$, as needed.

For $n > 1$ we continue by induction, but first we state a lemma about $\mathbb{I}^+$ (to be proven later)

Lemma 32. *If $A \subseteq \mathbb{I}^+$ is closed under the partial functions $\mathbb{I}^+$ then $A \cap \mathbb{I}_\alpha$ Is strongly independent over $A \cap \mathbb{I}_{<\alpha}$.*

Now, let $\overline{a} \in D_\alpha$ of length n and $b \in D_\alpha \backslash \overline{a}$. By the induction hypothesis (on n), it follows that $h \restriction (D_{<\alpha} \cup \overline{a})$ is elementary. By 32, D_α is strongly independent over $D_{<\alpha}$. Hence, $\mathrm{tp}(b, D_{\leq\alpha} \backslash \{b\})$ does not fork over $D_{<\alpha}$ and particularly $\mathrm{tp}(b, D_{<\alpha} \cup \overline{a})$ does not fork over $D_{<\alpha}$.

By the induction hypothesis, $h \restriction (D_{<\alpha} \cup \overline{a})$ is elementary, and so $q := h(\mathrm{tp}(b, D_{<\alpha} \cup \overline{a}))$ does not fork over $h(D_{<\alpha}) = R_{<\alpha}$. Note that $q \supseteq h(\mathrm{tp}(b, D_{<\alpha}))$ and by $\boxtimes_{\alpha,1}$ (see above) $h(\mathrm{tp}(b, D_{<\alpha})) = \mathrm{tp}(h(b), R_{<\alpha})$ holds. Hence, q extends $\mathrm{tp}(h(b), R_{<\alpha})$ to a type over $R_{<\alpha} \cup \overline{a}$ and does not fork over $R_{<\alpha}$. Therefore there exists an extension $q \subseteq q' \in \mathbf{S}(R_{\leq\alpha} \backslash h(b))$ which does not fork over $R_{<\alpha}$.

Since R_α is closed under the partial functions, it follows from lemma 32 that R_α is strongly independent over $R_{<\alpha}$, meaning that $q' = \mathrm{tp}(h(b), R_{\leq\alpha} \backslash \{h(b)\})$. Now we reduce both types to the domain $R_{<\alpha} \cup h(\overline{a})$ to get

$$\mathrm{tp}(h(a), R_{<\alpha} \cup h(\overline{a})) = h(\mathrm{tp}(b, D_{<\alpha} \cup \overline{a})$$

and the induction step on n:

$$\mathrm{tp}(h(b^\frown \overline{a}), R_{<\alpha}) = h(\mathrm{tp}(b^\frown \overline{a}, D_{<\alpha}))$$

Hence, f is a representation. $\square$

Proof of lemma 32:

Proof. Let $A_\alpha = \mathbb{I}_\alpha \cap A$, $a \in A_\alpha$, $B := \{F_i^*(a) : i < |T|\}$. We prove that $\mathrm{tp}(a, A \cap \mathbb{I}_{\leq\alpha} \backslash \{a\})$ is the unique non-forking extension in $\mathbf{S}(A \cap \mathbb{I}_{\leq\alpha} \backslash \{a\})$ of $\mathrm{tp}(a, A \cap \mathbb{I}_{<\alpha})$.

Since $A_{<\alpha}$ is closed under the F_i^*, it follows that $B \subseteq A_{<\alpha}$, and $\mathrm{tp}(a, \mathbb{I}_{<\alpha})$ does not fork over B (see lemma 34). Now, by transitivity of non forking $\mathrm{tp}(a, \mathbb{I}_{\leq\alpha} \backslash \{a\})$ which is a non-forking extension of $\mathrm{tp}(a, \mathbb{I}_{<\alpha})$, does not fork over B either. By the definition of B, we also get that every formula over $\mathbb{I}_{<\alpha}$ which is almost over B is equivalent to a formula over B (again, see lemma 34).

Now, by monotonicity of non-forking we get that $\operatorname{tp}(a, A_{\leq\alpha}\setminus\{a\}) \subseteq \operatorname{tp}(a, \mathbb{I}_{\leq\alpha}\setminus\{a\})$ does not fork over $A_{<\alpha}$.

To prove uniqueness, let $q_0 \in \mathbf{S}(A_{\leq\alpha}\setminus\{a\})$ a non-forking extension of $\operatorname{tp}(a, A_{<\alpha})$. q_0 has a non-forking extension $q \in \mathbf{S}(\mathbb{I}_{\leq\alpha}\setminus\{a\})$. By transitivity, q does not fork over B . Recall that the functions F_i^* are defined so that every formula over $\mathbb{I}_{<\alpha}$ and almost over B is equivalent to a formula over B. The types $q \restriction \mathbb{I}_{<\alpha}$, $\operatorname{tp}(a, \mathbb{I}_{<\alpha})$ are both non-forking over B. Since q extends $\operatorname{tp}(a, A_{<\alpha}) \supseteq \operatorname{tp}(a, B)$ we get that $q \restriction \mathbb{I}_{<\alpha}$, $\operatorname{tp}(a, \mathbb{I}_{<\alpha})$ (both non-forking over B) agree on all formulas over B, and by theorem 26 this implies $q \restriction \mathbb{I}_{<\alpha} = \operatorname{tp}(a, \mathbb{I}_{<\alpha})$. Now, since q is a non-forking extension of $\operatorname{tp}(a, \mathbb{I}_{<\alpha})$ and $\mathbb{I}_\alpha$ is strongly independent over $\mathbb{I}_{<\alpha}$ we get that $q = \operatorname{tp}(a, \mathbb{I}_{\leq\alpha}\setminus\alpha)$ and so

$$q_0 = q \restriction (A_{\leq\alpha}\setminus\{a\}) = \operatorname{tp}(a, A_{\leq\alpha}\setminus\{a\})$$

as required. □

Claim 33. Let $\mathbb{I}^+ \in \operatorname{Ex}^1_{\mu,\kappa}(\mathfrak{t}^{\mathrm{or}})$ and $f : M \to \mathbb{I}^+$ a function such that

$$h(f(\overline{a})) = f(\overline{b}) \to \operatorname{tp}(\overline{a}, \emptyset, M) = \operatorname{tp}(\overline{b}, \emptyset, M)$$

holds for every partial automorphism h of $\mathbb{I}^+$ which has closed domain and range under the partial functions and sequences $\overline{a}, \overline{b} \in M$. Then f is a representation.

Proof. Let f be as described above. Now assume towards contradiction that f is not a representation. Therefore there exist $\overline{a}, \overline{b} \in M$ which have different types in M such that the map $f(\overline{a}) \mapsto f(\overline{b})$ is a partial automorphism of $\mathbb{I}^+$. It is possible to extend this partial automorphism to one with domain and range closed under the partial functions, contrary to the definition of f. □

Lemma 34. *For stable T, Let $A \subset M \models T$. there exists $B \subseteq A$ such that:*

- $|B| \leq |T|$
- *For every $\varphi(\overline{x}, \overline{c})$ over A which is almost over B there exists $\theta(\overline{x}, \overline{d})$ over B such that $\models \forall\overline{x}\left(\theta(\overline{x}, \overline{d}) \leftrightarrow \varphi(\overline{x}, \overline{c})\right)$.*
- $\operatorname{tp}(a, A)$ *does not fork over B.*

Proof. First, define an increasing sequence B_n by induction on n.

Let $|B_0| < \kappa(T) \leq |T|^+$, $B_0 \subseteq A$ such that $\operatorname{tp}(a, A)$ does not fork over B_0.

Now assume B_n was defined and let

$$S_n := \{\varphi(\overline{x}, \overline{c}) \in \mathcal{L}_T : \overline{c} \subseteq A, \varphi(\overline{x}, \overline{c}) \text{ is almost over } B_n\}$$

By fact 29 there exist at most $|T| + |B_n| = |T|$ non equivalent formulas almost over B_n. Therefore w.l.o.g $|S_n| \leq |T|$ and define B_{n+1} as follows:

$$B_{n+1} := B_n \cup \{\overline{c} : \varphi(\overline{x}, \overline{c}) \in S_n\}$$

That the required properties of $B := \bigcup_{n<\omega} B_n$ hold is easily verified. □

3. ω-STABLE THEORIES

In this section we will prove the following result:

Characterization of ω-stable theories(40): *for a complete first-order theory T, the following conditions are equivalent*

1. *T is ω-stable.*
2. *T is representable in* $\mathrm{Ex}^{2}_{\omega\omega}(\mathfrak{k}^{\mathrm{eq}})$.
3. *T is representable in* $\mathrm{Ex}^{1}_{\omega,2}(\mathfrak{k}^{\mathrm{eq}})$.
4. *T is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{\omega,2}(\mathfrak{k}^{\mathrm{eq}})$ *(see definition 11)*

Proof. Theorem 40 gives $1 \Rightarrow 2$. Claim 1.13 gives $2 \Rightarrow 3$. $3 \Rightarrow 4$ is immediate and $4 \Rightarrow 1$ follows from claim 35. □

Claim 35. If T is representable in $\mathrm{Ex}^{0,\mathrm{lf}}_{\omega,2}(\mathfrak{k}^{\mathrm{eq}})$ then T is $\omega-$stable.

Proof. By the definition of representation and ω-stability it suffices to show that for every structure $I \in \mathrm{Ex}^{0,\mathrm{lf}}_{\omega,2}(\mathfrak{k}^{\mathrm{eq}})$, $\left|\mathbf{S}^{n}_{\mathrm{qf}}(A)\right| \leq \aleph_0$ for every countable $A \subseteq I$, $n < \omega$. We give a proof sketch. Fix a countable $A \subseteq I$.

1. W.l.o.g A is closed under the functions of I.
2. $\mathrm{tp}_{\mathrm{qf}}(\overline{a}, A) \in \mathbf{S}^{n}_{\mathrm{qf}}(A)$ is determined by formulas of the following types
$$\begin{array}{rcl} P_\alpha(\tau(b)) & & (b \in \overline{x}) \\ \tau_1(b_0) & = & \tau_2(b_1)\ (b_0, b_1 \in \overline{x}) \\ \tau_1(b_0) & = & b_1\ (b_0 \in \overline{x},\ b_1 \in A) \end{array}$$
For τ, τ_1, τ_2 terms in the dictionary of I, and so necessarily unary.
3. Moreover, since I is locally finite, $\mathrm{tp}_{\mathrm{qf}}(\overline{a}, A)$ is determined by a finite subset of these formulas.

So, the number of n-ary types over A is at most $|A|^{<\omega} \leq \aleph_0$ □

Convention 3. *We assume for the rest of this chapter that T is stable in $\aleph_0$.*

Claim 36. Let $p \in \mathbf{S}(A)$, there exists a finite $B \subseteq A$ such that p does not fork over B (see [Pil83, 5.7])

Claim 37. Let $p \in \mathbf{S}(A)$. For a given $B \supseteq A$ there are only finitely many non-forking extensions of p in $\mathbf{S}(B)$. (see [Pil83, 5.27])

Corollary 38. *For $p \in \mathbf{S}(A)$ there exists a finite $B \subseteq A$ such that p is the unique non-forking extension of $p \restriction B$ to $\mathbf{S}(A)$.*

Proof. By 37 there are finitely many extensions of $p \restriction B$ in $\mathbf{S}(A)$, therefore there exists a finite $B_0 \subseteq A$ such that $q_0 \restriction B_0 \neq q_1 \restriction B_0$ holds for every distinct $q_0, q_1 \in \mathbf{S}(A)$ non-forking extensions of p. Also p does not fork over some finite $B_1 \subseteq A$. Now, the conclusion easily follows for $B = B_0 \cup B_1$. $\square$

Claim 39. Let $M \models T$, and $\mathbb{I}_0 \subseteq |M|$ an indiscernible set (possibly finite). There exists a sequence of sets $\langle \mathbb{I}_n : 0 < n < \omega \rangle$ such that

(1) *For all $a \in \mathbb{I}_n$, $n < \omega$ there exists a finite $B_a \subseteq \mathbb{I}_{<n}$ such that* $\mathrm{tp}(a, \mathbb{I}_{\leq n} \setminus \{a\})$ *is the unique non-forking extension of* $\mathrm{tp}(a, B_a)$ *in* $\mathbf{S}(\mathbb{I}_{\leq n} \setminus \{a\})$.
(2) $\mathbb{I}_n \cap \mathbb{I}_{<n} = \emptyset$ *and also* $\mathbb{I}_{<\omega} = |M|$.

Proof. Consider the family $\mathcal{F}$ of sequences $\langle \mathbb{I}_n : n < \omega \rangle$ from M which fulfill the first condition. Consider the following partial order on $\mathcal{F}$

$$\langle \mathbb{I}_n : n < \omega \rangle \preceq \langle \mathbb{J}_n : n < \omega \rangle := \bigwedge_{n<\omega} (\mathbb{I}_n \subseteq \mathbb{J}_n)$$

First, every increasing chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$ by taking unions element-by-element. By Zorn's lemma $\mathcal{F}$ contains a maximal element $\langle \mathbb{I}_n : n < \omega \rangle$. Assume towards contradiction that there exists $a \in M \setminus \mathbb{I}_{<\omega}$. By corollary 38 there exists a finite $B_a \subseteq \mathbb{I}_{<\omega}$ such that $\mathrm{tp}(a, \mathbb{I}_{<\omega})$ is the unique non-forking extension of $\mathrm{tp}(a, B_a)$ in $\mathbf{S}(\mathbb{I}_{<\omega})$. Since $B_a \subseteq A \subseteq \mathbb{I}_{<\omega}$ and $q \in \mathbf{S}(A)$ is a non-forking extension of $\mathrm{tp}(a, A)$, by transitivity of non-forking there exists $q \subseteq q' \in \mathbf{S}(\mathbb{I}_{<\omega})$ which does not fork over B_a. From the definition of B_a it follows that $q' = \mathrm{tp}(a, \mathbb{I}_{<\omega})$, and so $\mathrm{tp}(a, A)$ is the unique non-forking extension of $\mathrm{tp}(a, B_a)$ in $\mathbf{S}(A)$. Now, since B_a is finite there exists $0 < n_* < \omega$ such that $B_a \subseteq \mathbb{I}_{<n_*}$. On the other hand, from the choice of $\langle \mathbb{I}_n : n < \omega \rangle$ there exists a finite $B_b \subseteq \mathbb{I}_{<n_*}$ such that $\mathrm{tp}(b, \mathbb{I}_{\leq n_*} \setminus \{b\})$ is the unique non-forking extension of $\mathrm{tp}(b, B_b)$ in $\mathbf{S}(\mathbb{I}_{\leq n_*} \setminus \{b\})$. From claim 22 we get for all $b \in \mathbb{I}_{n_*}$ that $\mathrm{tp}(b, \mathbb{I}_{\leq n_*} \setminus \{b\} \cup \{a\})$ is the unique non-forking extension of $\mathrm{tp}(b, B_b)$ in $\mathbf{S}(\mathbb{I}_{\leq n_*} \setminus \{b\} \cup \{a\})$. Hence, $\mathbb{I}'_n = \mathbb{I}_n$ $(n \neq n_*)$, $\mathbb{I}'_{n_*} = \mathbb{I}_{n_*} \cup \{a\}$ belongs to $\mathcal{F}$ contradicting the choice of $\langle \mathbb{I}_n : n < \omega \rangle$. $\square$

Theorem 40. *Let $M \models T$, $\lambda = \|M\|$, $\mathbb{I}_0$ a set of indiscernibles in M. Then we M can be represented in $\mathcal{M}_{\omega\omega}(\mathbb{I}_0 \cup \lambda) \in \mathrm{Ex}^2_{\omega,\omega}(\mathfrak{k}^{\mathrm{eq}})$ by an extension of the identity function on $\mathbb{I}_0$.*

Proof. Let $\langle \mathbb{I}_n : n < \omega \rangle$ as in claim 39. Let $g : \|M\| \to \lambda$ a one-to-one function. T is ω-stable and so $\mathbf{S}^m(\emptyset)$ is countable for all $m < \omega$. for convenience we use the symbols $\{F_{p,n} : n < \omega,\ p \in \mathbf{S}^{<\omega}(\emptyset)\}$ as the function symbols of $\mathcal{M}_{\omega\omega}(\mathbb{I}_0 \cup \lambda)$, such that for each m-type p, $F_{p,n}$ is an m-ary function symbol.

We define an increasing sequence of one-to-one functions $f_i : \mathbb{I}_{\leq i} \to \mathcal{M}(\mathbb{I}_0 \cup \lambda)$ by induction on $n < \omega$:

Define f_0 as the identity on $\mathbb{I}_0$.

Assume that f_n was defined and now define $f_{n+1} \supseteq f_n$ as follows. For each $a \in \mathbb{I}_{n+1}$ recall B_a from claim 39. Let $\overline{c}_a \in^{\ell} (\mathbb{I}_{\leq n})$ enumerate B_a. Now define $p \in \mathbf{S}^{\ell+1}(\emptyset)$ and $f_{n+1}(a)$ as follows:

$$\begin{aligned} p &:= \operatorname{tp}(a^\frown \overline{c}_a, \emptyset, M) \\ f_{n+1}(a) &:= F_{p,n}(f_n(\overline{c}_a), g(a)) \end{aligned}$$

Let $f = \bigcup_{n<\omega} f_n$. We will use (proof is omitted) an analogue of claim 33 to show that f is a representation:

Claim 41. If $f : M \to \mathcal{M}(S)$ is a function such that

$$h(f(\overline{a})) = f(\overline{b}) \to \operatorname{tp}(\overline{a}, \emptyset, M) = \operatorname{tp}(\overline{b}, \emptyset, M)$$

for every partial automorphism h of $\mathcal{M}(S)$ with domain and range closed under subterms and $\overline{a}, \overline{b} \in M$. Then f is a representation.

First note that $a \in \mathbb{I}_n$ and also $f(a) = F_{p,n}(f(\overline{c}_a), g(a))$, so $p = \operatorname{tp}(a \frown \overline{c}_a, \emptyset, M)$ and $\operatorname{tp}(a, \mathbb{I}_{\leq n} \setminus \{a\})$ is the unique non-forking extension of $\operatorname{tp}(a, \overline{c}_a)$.

We now show that f fulfills the conditions of the claim. Let h a partial automorphism of $\mathcal{M}(\mathbb{I}_0 \cup \lambda)$ with domain and range closed under the functions. Fix $n < \omega$ and sequences $\overline{a}, \overline{b} \in \mathbb{I}_{\leq n}$ such that $h(f(\overline{a})) = f(\overline{b})$. Since f is one-to-one, w.l.o.g $\overline{a}, \overline{b}$ are without repetition. We prove that $\operatorname{tp}(\overline{a}, \emptyset, M) = \operatorname{tp}(\overline{b}, \emptyset, M)$ by induction on n:

For $n = 0$: the claim holds since $\mathbb{I}_0$ is an indiscernible set.

For $n = m + 1$:: Proof by induction on $\ell = |\overline{a} \cap \mathbb{I}_n| = |\overline{b} \cap \mathbb{I}_n|$ (the latter equality is easy to verify).

For $\ell = 0$:: This is the claim of the induction hypothesis (on n).

For $\ell = \ell_0 + 1$:: Let $a_0, b_0 \in \mathbb{I}_n$, $\overline{a}_1, \overline{b}_1 \in^{\ell} \mathbb{I}_n$, $\overline{b}_1, \overline{b}_2 \in^{<\omega} \mathbb{I}_{<n}$ such that $h(f(a_0\overline{a}_1\overline{a}_2)) = f(b_0\overline{b}_1\overline{b}_2)$. By the definition there exist $\overline{c}_{a_0}, \overline{c}_{b_0}$ such that $f(a_0) = F_{p,n}(f(\overline{c}_{a_0}), g(a_0))$, $f(b_0) = F_{p',n}(f(\overline{c}_{b_0}), g(b_0))$ for some

sequences and types. Since $\mathrm{Dom}(h)$ is closed under subterms we get:

$$F_{p',n}(f(\overline{c}_{b_0}), g(b_0)) = f(b_0) = h(f(a_0)) = \\ h(F_{p,n}(f(\overline{c}_{a_0}), g(a_0))) = F_{p,n}(h(f(\overline{c}_{a_0})), h(g(a_0)))$$

and by the definition of the free algebra $p' = p$ and $h(f(\overline{c}_{a_0})) = f(\overline{c}_{b_0})$. The induction hypothesis implies that the map G defined as

$$G(\overline{a}_1) = \overline{b}_1,\ G(\overline{a}_2) = \overline{b}_2,\ G(\overline{c}_{a_0}) = \overline{c}_{b_0}$$

is elementary. Now, let $q = \mathrm{tp}\,(a_0, \overline{a}_1 \cup \overline{a}_2 \cup \overline{c}_{a_0})$. Since $\mathrm{tp}(a_0 \frown \overline{c}_{a_0}) = p = p' = \mathrm{tp}(b_0 \frown \overline{c}_{b_0})$ holds, it follows that $G(q) \restriction \overline{c}_{b_0} = \mathrm{tp}(b_0, \overline{c}_{b_0})$. The definition of $\mathbb{I}_n$ implies that $\mathrm{tp}(a_0, \mathbb{I}_{\leq n} \setminus \{a_0\})$ is non-forking over $\overline{c}_{a_0}$, and so is $\mathrm{tp}(a_0, \overline{a}_1 \cup \overline{a}_2 \cup \overline{c}_{a_0})$. On the other hand, since G is elementary, $G(q)$ does not fork over $\overline{c}_{b_0}$. Let $\mathbf{S}(\mathbb{I}_{\leq n} \setminus \{b_0\}) \ni q' \supseteq G(q)$ a non-forking extension. Since $\mathrm{tp}(b_0, \mathbb{I}_{\leq n} \setminus \{b_0\})$ is the unique non-forking extension of $\mathrm{tp}(b_0, \overline{c}_{b_0})$, and by transitivity q' is also a non-forking extension, it follows that $q' = \mathrm{tp}(b_0, \mathbb{I}_{\leq n} \setminus \{b_0\})$ and after reduction ($\overline{b}$ is without repetitions, so $b_0 \notin \overline{b}_1$ and$\overline{b}_1 \cup \overline{b}_2 \cup \overline{b}_{a_0} \subseteq \mathbb{I}_{\leq n} \setminus \{b_0\}$):

$$G(q) = q' \restriction \overline{b}_1 \cup \overline{b}_2 \cup \overline{c}_{b_0} = \mathrm{tp}\left(b_0, \overline{b}_1 \cup \overline{b}_2 \cup \overline{b}_{a_0}\right)$$

Hence, $G \cup \{(a_0, b_0)\}$ is elementary and the proof is complete.

□

4. Superstable theories

The main theorem in this section is

Theorem 42. *For a first-order, complete theory T the following are equivalent*

- **1.:** *T is superstable.*
- **2.:** *T is representable in* $\mathrm{Ex}^{2}_{2^{|T|},\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$.
- **3.:** *T is representable in* $\mathrm{Ex}^{1}_{2^{|T|},2}(\mathfrak{k}^{\mathrm{eq}})$.
- **4.:** *T is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{2^{|T|},2}(\mathfrak{k}^{\mathrm{eq}})$.
- **5.:** *T is representable in* $\mathrm{Ex}^{2}_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinal μ.*
- **6.:** *T is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinals μ, κ.*

Proof. $2 \Rightarrow 5$, $4 \Rightarrow 6$ are immediate. $2 \Rightarrow 3$ is direct from 1.13. $3 \Rightarrow 4$ direct from 1.7. $5 \Rightarrow 6$ since $\mathrm{Ex}^{2}_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ is qf-representable in $\mathrm{Ex}^{1}_{\mu,2}(\mathfrak{k}^{\mathrm{eq}}) \subseteq \mathrm{Ex}^{0,\mathrm{lf}}_{\mu,2}(\mathrm{k}^{\mathrm{eq}})$. The rest follows from theorem 44 giving $1 \Rightarrow 2$ and theorem 43 giving $6 \Rightarrow 1$. □

Theorem 43. *If T is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinals μ, κ then T is superstable.*

Proof. Let T be non superstable and we will choose $M \models T$ which cannot be represented in $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$. Let $\lambda > \kappa$. Let T_1 a skolemization of T. We choose $M_1 \models T_1$ and sequence $\mathbf{a} = \left\langle \overline{a}_\eta : \eta \in {}^{\leq\omega}\lambda \right\rangle$ such that

1. $\overline{a}_\eta \in M_1,\ \lg(\overline{a}_\eta) = n_{\lg(\eta)}$
2. $\varphi_\eta(\overline{x}, \overline{y}) \in \mathcal{L}_T$
3. $\left\langle \overline{a}_{\eta^\frown\langle\alpha\rangle} : \alpha < \lambda \right\rangle$ is an indiscernible sequence over $\left\{ \overline{a}_\rho : \rho \in {}^{\leq\omega}\lambda,\ \rho \ntrianglelefteq \eta \right\}$ for all $\eta \in {}^{<\omega}\lambda$.
4. If $\nu \in {}^{\omega}\lambda,\ \eta = \nu \restriction n$ then

$$M_1 \models \varphi(\overline{a}_\nu, \overline{a}_{\eta^\frown\langle\alpha\rangle}) \Leftrightarrow \nu(n) = \alpha$$

Let M_2 be the closure of $\left\langle \overline{a}_\eta : \eta \in {}^{\leq\omega}\lambda \right\rangle$ in M_1, this is a Skolem closure and so $M_2 \models T_1$.

Now assume towards contradiction that $\mathtt{I} \in \mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ is such that $f : M_2 \restriction \tau_T \to \mathtt{I}$ is a representation. For a sufficiently thin club $E \subseteq \lambda$ we can choose $\alpha_n \in E,\ \alpha_{n+1} > \alpha_n,\ \cup\{\alpha_n : n < \omega\} = \delta$ and $\eta = \langle \alpha_n : n < \omega \rangle$ and it is easy to get a contradiction to 4 in both cases. $\square$

Theorem 44. *Every superstable T is representable in $\mathrm{Ex}^{2}_{2^{|T|},\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$.*

Proof. Let T superstable. Let $M \models T$. We choose $B_n, \langle a_s, u_s : s \in S_n \rangle$ by induction on $n < \omega$ such that:

1. $S_n \cap S_k = \emptyset\ (k < n)$
2. $\langle a_s : s \in S_n \rangle \subseteq M$
3. $B_n = \mathrm{acl}(\{a_s : s \in \cup\{S_k : k < n\}\}) \subseteq M$
4. $\langle a_s : s \in S_n \rangle$ is without repetitions and independent over B_n.
5. for all $s \in S$, $u_s \subseteq S_{<n}$ is finite such that $t \in u_s \Rightarrow u_t \subseteq u_s$ and $\mathrm{tp}(a_s, B_n)$ does not fork over $\{a_t : t \in u_s\}$.
6. $\langle a_s : s \in S_n \rangle$ is maximal under conditions 1-5.

$\circledast_1$: It is possible to carry the induction.
$\circledast_2$: $|M| = \{a_s : s \in S_{<\omega}\}$

Let

$$I = \{u : u \subseteq u_s \cup \{s\}\,,\ s \in S_{<\omega},\ \forall t\, (t \in u \to u_t \subseteq u)\}$$

Let $\langle v_\alpha : \alpha < \alpha(*) \rangle$ enumerate I such that

$\circledast$: (1) $v_\alpha \subseteq v_\beta \Rightarrow \alpha \leq \beta$ (2) $\alpha < \beta \wedge v_\beta \subseteq S_{<n} \Rightarrow v_\alpha \subseteq S_{<n}$

We choose a model M_{v_α} by induction on α such that

1. $M_{v_\alpha} \prec \mathfrak{C}$ has power $\leq \aleph_0 + |T|$

(2) $v_\alpha \subseteq v_\beta \Rightarrow M_{v_\alpha} \prec M_{v_\beta}$

(3) $\operatorname{tp}\left(M_{v_\alpha}, M \cup \{M_\beta : \beta < \alpha\}\right)$ does not fork over $\cup\left\{M_{v_\beta} : v_\beta \subseteq v_\alpha,\ \beta < \alpha\right\} \cup \{\overline{a}_s : s \in v_\alpha\}$

The induction is //clearly// possible.

Now $\langle M_v : v \in I\rangle$ is a stable system of models (except for I being closed to finite subsets). (see [She90, XII, p.598]). For all $v \in I$ let $\overline{b}_v$ enumerate M_v such that if $v_s = u_s \cup \{s\}$ then $b_v^0 = a_s$ (we use superscript indexes in $\overline{b}_v$). Therefore, $\left\langle \overline{b}_v : v \in I \right\rangle$ is without repetitions.

For all $\alpha < \omega \times \omega$ we define I_α as follows:
$I_0 = \{\emptyset\}$,
$I_k = \{v \in I : v \subseteq S_{<1},\ |v| = k\}\ \ (k < \omega)$,
$I_{\omega n + k} = \{v \in I : v \nsubseteq S_{<n},\ v \subseteq S_{<n+1},\ |v| = k+1\}\ \ (k < \omega,\ 0 < n < \omega)$.

Now clearly, $w \subseteq v \in I_\alpha \Rightarrow w \in I_{<\alpha}$ for all w, v.

For all $\alpha < \omega \times \omega$ let $B_\alpha = \cup\{M_v : v \in I_{<\alpha}\}$.

So,

- B_α is increasing and continuous.
- $B_{\alpha+1} = \cup\left\{\overline{b}_v : v \in I_\alpha\right\} \cup B_\alpha$.
- $v \in I_\alpha \wedge w \subseteq v \Rightarrow w \in v$ and most important,

⊠**:** for all $\alpha < \omega \times \omega$ and $v \in I_\alpha$ the type

$$p_v := \operatorname{tp}\left(\overline{b}_v, \cup\left\{\overline{b}_u : u \in I_{<\alpha+1} \wedge u \neq v\right\}\right)$$

does not fork over $\cup\left\{\overline{b}_w : w \subseteq v\right\}$, and p_v is the unique non-forking extension of $p_v \restriction \cup\left\{\overline{b}_w : w \subseteq v\right\}$ in $\mathbf{S}^{\lg(\overline{b}_v)}\left(\cup\left\{\overline{b}_u : u \in I_{<\alpha+1} \wedge u \neq v\right\}\right)$.

The proof is carried by basic properties of stable systems.

Now define an equivalence relation E_α on I_α such that $v_1 E_\alpha v_2$ iff $v_1, v_2 \in I_\alpha$and there exists an isomorphism $f_{v_1,v_2} : M_{v_1} \to M_{v_2}$ such that f_{v_1,v_2} maps $\overline{b}_{v_1}$ to $\overline{b}_{v_2}$ (element-by-element, and this implies f_{v_1,v_2} is unique). And for some bijection $g_{v_1,v_2} : v_1 \to v_2$ which preserves being in I_β for all $\beta < \alpha$, such that f_{v_1,v_2} maps $\overline{b}_{w_1}$to $\overline{b}_{g_{v_1,v_2}(w_1)}$ for all $w_1 \subset v_1$. Let $\left\langle I_{\alpha,i} : i < i(\alpha) \leq 2^{|T|}\right\rangle$ enumerate the equivalence classes of E_α.

We get that if

(α)**:** The sets $\{v_0 \dots v_{n-1}\}$, $\{u_0 \dots u_{n-1}\} \subset I$ are closed under subsets.
(β)**:** $\bigwedge_{\alpha,i}\left[v_l \in I_{\alpha,i} \Leftrightarrow u_l \in I_{\alpha,i}\right]$.
(γ)**:** $u_{l(1)} \subset u_{l(2)} \Leftrightarrow v_{l(1)} \subset u_{l(2)}$ for all $l(1), l(2) < n$
(δ)**:** For all $v_{l(1)} \subseteq v_{l(2)}$ it holds that $g_{v_{l(2)},u_{l(2)}}$ maps $v_{l(1)}$ onto $u_{l(1)}$.

Then the sequences $\overline{b}_{v_0}{}^\frown \dots \overline{b}_{v_{n-1}}$ and $\overline{b}_{u_0}{}^\frown \dots \overline{b}_{u_{n-1}}$ realize the same complete type over $\emptyset$. (This follows from the definitions of the equivalence relations E_α and $\boxtimes$ above)

W.l.o.g $I \cap S = \emptyset$. We define a structure $\mathcal{A}$ with universe $|\mathcal{A}| = I \cup S$ as follows:

$P_\alpha^{\mathcal{A}} := I_\alpha$, $P_{\alpha,i}^{\mathcal{A}} := I_{\alpha,i}, S_n^{\mathcal{A}} := S_n$.

Define partial functions $F_\epsilon^{\mathcal{A}}$ for all $\epsilon < |T|$, $n < \omega$ as $F_\epsilon^{\mathcal{A}}(s) = b^\epsilon_{u_s \cup \{s\}}$. Define partial functions G_n such that $\{G_n(u) : n\} = \{v : v \subseteq u\}$. Now, the function f well defined as $f(a) = s \Leftrightarrow a = a_s$ is a representation of M in $\mathcal{A}$. (proof should be clear from the definitions) $\square$

5. Characterization of $\kappa(T)$

Theorem 45. *For a complete theory T, and regular κ the following are equivalent:*

(1) $\kappa(T) \leq \kappa$.
(2) *T is representable in* $\mathrm{Ex}^2_{2^{|T|},\kappa}(\mathfrak{k}^{\mathrm{eq}})$.
(3) *T is representable in* $\mathrm{Ex}^2_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$, *for some μ.*
(4) *T is representable in* $\mathrm{Ex}^1_{2^{|T|},\kappa}(\mathfrak{k}^{\mathrm{eq}})$.
(5) *T is representable in* $\mathrm{Ex}^1_{2^{|T|},\kappa}(\mathfrak{k}^{\mathrm{eq}})$, *for some μ.*

6. Miscellaneous

This final section completes the proofs of several claims that were used in the main theorems above.

Theorem 46. *(Fodor) Let λ a regular cardinal, $f : \lambda \to \lambda$ such that $f(\alpha) < \alpha$ for all $0 < \alpha < \lambda$, Then $\{\alpha < \lambda : f(\alpha) = \beta\}$ is a stationary set of λ for some $\beta < \lambda$. (see [Jec78])*

Corollary 47. *If $f : \lambda \to \mu$, $\lambda > \mu$ regular, then $f^{-1}(\{\alpha\})$ is a stationary set of λ for some $\alpha < \mu$.*

Theorem 48. *Let λ a regular cardinal, $|W| = \lambda, |S_t| < \mu$ $(t \in W)$ such that $\chi < \lambda \to \chi^{<\mu} < \lambda$. Then:*

(1) *(The Δ-system lemma) There exist $W' \subseteq W$, $|W'| = \lambda$ and S such that $s \neq t \to S_t \cap S_s = S$ holds for all $s, t \in W'$.*
(2) *If $\langle z_t^\alpha : \alpha < \alpha(t)\rangle$ enumerates S_t:*
    (a) *$t \in W' \to \alpha(t) = \alpha_0$ holds for some α_0.*
    (b) *For some $U \subseteq \alpha_0$ it holds that $s, t \in W' \Rightarrow S_t \restriction U = S_s \restriction U$, $U = \{\alpha < \alpha_0 : z_t^\alpha = z_s^\alpha\}$.*

(c) *For some equivalence relation E on α_0 it holds that $t \in W' \Rightarrow z_t^\alpha = z_t^\beta \leftrightarrow (\alpha, \beta) \in E$.*

Proof.

(1) See [Jec78, She90].

(2) The map $t \to \alpha(t)$ fulfills the assumptions of Fodor's theorem ($\alpha(t) < \mu < \lambda$), therefore (a) holds for some $W_0 \subseteq W$. By part 1 there exist $S \subseteq \{z_t^\alpha : \alpha < \alpha_0, t \in W_0\}$, $W_1 \subseteq W_0$ such that $S = \overline{z}_t \cap \overline{z}_s$ for all $t \neq s$. Define a map $W_1 \ni t \mapsto U_t$ where: $U_t = \{\alpha < \alpha_0 : z_t^\alpha \in S\}$, Since the range has cardinality $2^{|\alpha_0|} \leq 2^{<\mu} < \lambda$ this map also fulfills the assumptions of Fodor's theorem, and we get that for some $W_2 \subseteq W_1$, U it holds that $t \in W_2 \to U_t = U$. The range of the map $t \to S_t \restriction U$ is ${}^U S$ whose power is $\leq |\alpha_0|^{|\alpha_0|} < \lambda$, and by another use of Fodor's theorem we get $W_3 \subseteq W_2$ such that (b) holds. The map $t \to E_t$ where $E_t = \left\{(\alpha, \beta) : z_t^\alpha = z_t^\beta,\ \alpha, \beta < \alpha_0\right\}$ has power at most $|\alpha_0|^{|\alpha_0|}$ and again by Fodor's theorem the result holds for some E and $W' \subseteq W_3$.

□

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