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Results in Mathematics

$\aleph_n$-Free Modules With Trivial Duals

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Abstract. In the first part of this paper we introduce a simplified version of a new Black Box from Shelah [11] which can be used to construct complicated $\aleph_n$-free abelian groups for any natural number $n \in \mathbb{N}$. In the second part we apply this prediction principle to derive for many commutative rings R the existence of $\aleph_n$-free R-modules M with trivial dual $M^* = 0$, where $M^* = \mathrm{Hom}(M, R)$. The minimal size of the $\aleph_n$-free abelian groups constructed below is $\beth_n$, and this lower bound is also necessary as can be seen immediately if we apply GCH.

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1. Introduction

The existence of almost free R-modules M over countable principal ideal domains (but not fields) with trivial dual $M^* := \mathrm{Hom}(M, R) = 0$ is a well-known fact by results using strong prediction principles like diamonds in $V = L$. Only note that any (non-trivial) R-module with endomorphism ring $\mathrm{End}\, M = R$ is such an example. For every regular cardinal $\kappa > |R|$ (which is not weakly compact) we can find (strongly) κ-free R-modules M of size κ with $\mathrm{End}\, M = R$. But from the singular compactness theorem follows, that such modules M do not exist for singular cardinals, see e.g. [4] or [8]. Thus we want to get rid of additional set theoretic restrictions and work exclusively with ZFC:

If we restrict to $\aleph_1$-free R-modules (meaning that all countable submodules are free) and do not care about the size of M, then we have an abundance of such modules and those of minimal cardinal $2^{\aleph_0}$ can be constructed by applications of

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the (ordinary) Black Box, see for example [8]. However, it is much harder to find examples like this of size $\aleph_1$ (recall that $\aleph_1$ may be much smaller than $2^{\aleph_0}$ and hence we do not have that many possible extension of a module to eliminate unwanted homomorphisms. A first example of an $\aleph_1$-free R-module M of size $\aleph_1$ with trivial dual was given in Eda [3]. Some years later we improved this result showing the existence of such modules with endomorphism ring R, see [7, 2] or [8]. If we want to replace $\aleph_1$ by $\aleph_2$ or any higher cardinal, then we necessarily encounter additional set theoretic restriction, see [6]. If we require only the existence of indecomposable abelian groups, then their κ-freeness is restricted to small cardinals; see [10]. Thus, in order to construct $\aleph_n$-free R-modules M with trivial dual we must relax the restriction on the size of M. Clearly (note that GCH is not excluded, in which case $\aleph_n = \beth_n$), the size of M must be at least $\beth_n$. These cardinals are defined inductively as $\beth_0 = 2^{\aleph_0}$ and $\beth_{n+1} = 2^{\beth_n}$; see Jech [9]. Using a recent refined Black Box from Shelah [11] which takes care of additional freeness of the module, we can give a reasonably short proof of the existence of $\aleph_n$-free R-modules M with trivial dual $M^* = 0$ of cardinality $|M| = \beth_n$ for any natural number n; see Theorem 4.3 and Corollary 4.4. It remains an open question if we can go any further and pass $\aleph_\omega$ or possibly replace $M^* = 0$ by $\operatorname{End} M = R$. In this context it is also worthwhile to recall (from [10]) that there are models of ZFC in which $\aleph_{\omega^2+1}$-free implies $\aleph_{\omega^2+2}$-free.

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2. The Combinatorial Black Box for $\overline{\lambda}$

The new Black Box depends on a finite sequence of cardinals satisfying some cardinal conditions. Thus let $k_* < \omega$ and $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{k_*} \rangle$ be a sequence of cardinals such that for $\chi_l := \lambda_l^{\aleph_0}$ $(l \leq k_*)$ the following $\blacksquare$-*conditions* holds.

$$\chi_{l+1}^{\chi_l} = \chi_{l+1} \ (l < k_*). \tag{2.1}$$

We will also say that $\overline{\lambda}$ is a $\blacksquare$-sequence and note that $\chi_l = \chi_l^{\aleph_0} < \chi_{l+1}$.

Condition (2.1) is used to enumerate all maps which we want to predict before constructing the modules. If λ is any cardinal, then we can define inductively a $\overline{\lambda}$-sequence: Let $\chi_1 = \lambda^{\aleph_0}$ and if λ_l is defined for $l < k_*$, then choose a suitable $\lambda_{l+1} > \lambda_l$ with (2.1), e.g. put $\lambda_{l+1} = \chi_l^{\chi_l}$. The sequence $\langle \beth_1, \ldots, \beth_{k_*} \rangle$ is an example of such a $\blacksquare$-sequence.

If λ is a cardinal, then ${}^{\omega\uparrow}\lambda$ will denote all *order preserving* maps $\eta : \omega \to \lambda$ (which we also call *infinite branches*) on λ, while ${}^{\omega\uparrow>}\lambda$ denotes the family of all order preserving *finite branches* $\eta : n \to \lambda$ on λ, where the natural number n, λ and ω (the first infinite ordinal) are considered as sets, e.g. $n = \{0, \ldots, n-1\}$, thus the finite branch η has length n.

For the sake of generality we first consider any sequence $\overline{\lambda} = \langle \lambda_1, \dots, \lambda_{k_*} \rangle$ of cardinals such that $\lambda_l^{\aleph_0} = \chi_l$ $(l \le k_*)$ (which later will be strengthened to a $\blacksquare$-sequence). Moreover, we associate with $\overline{\lambda}$ two sets Λ and Λ_*. Thus let

$$\Lambda = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow}\lambda_{k_*}.$$

For the second set we replace the m-th (and only the m-th) coordinate ${}^{\omega\uparrow}\lambda_m$ by the finite branches ${}^{\omega\uparrow>}\lambda_m$, thus

$$\Lambda_m = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow>}\lambda_m \times \cdots \times {}^{\omega\uparrow}\lambda_{k_*} \text{ for } m \le k_* \text{ and let } \Lambda_* = \bigcup_{m \le k_*} \Lambda_m. \quad (2.2)$$

The elements of Λ, Λ_* will be written as sequences $\overline{\eta} = (\eta_1, \dots, \eta_{k_*})$ with $\eta_l \in {}^{\omega\uparrow}\lambda$ or $\eta_l \in {}^{\omega\uparrow>}\lambda$, respectively. Using these $\overline{\eta}$s as support of elements of the module will make enough room for linear independence which will then give $\aleph_n$-freeness.

With each member of Λ we can associate a subset of Λ_*:

Definition 2.1. *If $\overline{\eta} = (\eta_1, \dots, \eta_{k_*}) \in \Lambda$ and $m \le k_*, n < \omega$, then let $\overline{\eta}\upharpoonright\langle m, n\rangle$ be the following element in Λ_m (thus in Λ_*)*

$$(\overline{\eta}\upharpoonright\langle m, n\rangle)_l = \begin{cases} \eta_l & \text{if } m \ne l \le k_* \\ \eta_m \upharpoonright n & \text{if } l = m. \end{cases}$$

We associate with $\overline{\eta}$ its support $[\overline{\eta}] = \{\overline{\eta}\upharpoonright\langle m, n\rangle \mid m \le k_*, n < \omega\}$ *which is a countable subset of Λ_*. Similarly, for $m \le k_*$ also let $[\overline{\eta}\upharpoonright m] = \{\overline{\eta}\upharpoonright\langle m, n\rangle \mid n < \omega\} \subseteq [\overline{\eta}]$.*

Definition 2.2. *Let $\overline{C} = \langle C_1, \dots, C_{k_*} \rangle$ be a sequence of sets C_m satisfying $|\, C_m \,| \le \chi_m$ for all $m \le k_*$. We let $C = \bigcup_{m \le k_*} C_m$ and define a* set-trap (for $\Lambda, \overline{C}$) *as a map $\varphi_{\overline{\eta}} : [\overline{\eta}] \to C$ with a label $\overline{\eta} \in \Lambda$.*

The following lemma will be used for the inductive proof of our next theorem.

Lemma 2.3. *Let λ be an infinite cardinal, $\chi = \lambda^{\aleph_0}$ and $\mathfrak{P}$ a set of size $|\, \mathfrak{P} \,| = \chi$. Then there is a sequence $\langle \Phi_\eta \mid \eta \in {}^{\omega\uparrow}\lambda \rangle$ such that*

(a) *$\Phi_\eta = \langle \Phi_{\eta n} \mid n < \omega \rangle$, with $\Phi_{\eta n} \in \mathfrak{P}$,*
(b) *If $\overline{f} = \{f_\nu \mid f_\nu \in \mathfrak{P}, \nu \in {}^{\omega\uparrow>}\lambda\}$, $\alpha \in \lambda$ and $\rho \in {}^{\omega\uparrow>}\lambda$, then there is $\eta \in {}^{\omega\uparrow}\lambda$ such that $0\eta = \alpha$, $\rho \subset \eta$ and $\Phi_{\eta n} = f_{\eta \upharpoonright n}$ for all $n < \omega$.*

Proof. Since $|\, \mathfrak{P} \,| = \chi = \lambda^{\aleph_0} = |\, {}^\omega\lambda \,|$, we can fix an embedding

$$\pi : \mathfrak{P} \hookrightarrow {}^\omega\lambda.$$

And since $|\, {}^{\omega>}\lambda \,| = \lambda$ there is also a list ${}^{\omega>}\lambda = \langle \mu_\alpha \mid \alpha < \lambda \rangle$ with enough repetitions for each $\eta \in {}^{\omega>}\lambda$:

$$\{\alpha < \lambda \mid \mu_\alpha = \eta\} \subseteq \lambda \text{ is unbounded.}$$

Moreover we define for each $n < \omega$ a coding map

$$\pi_n : {}^n\mathfrak{P} \longrightarrow {}^{n^2}\lambda \subseteq {}^{\omega>}\lambda$$

$$\overline{\varphi} = \langle \varphi_0, \dots, \varphi_{n-1} \rangle \mapsto \overline{\varphi}\pi_n = (\varphi_0 \pi \restriction n)^\wedge \dots {}^\wedge(\varphi_{n-1}\pi \restriction n).$$

Finally let $X \subseteq {}^{\omega\uparrow}\lambda$ be the collection of all order preserving maps $\eta : \omega \longrightarrow \lambda$ such that the following holds.

$$\exists\ \overline{\varphi} = \langle \varphi_i \mid i < \omega \rangle \in {}^{\omega}\mathfrak{P} \text{ and } \exists\ \kappa < \omega \text{ with } (\overline{\varphi} \restriction n)\pi_n = \mu_{n\eta} \text{ for all } n > k. \quad (2.3)$$

By definition of π_n it follows that $\overline{\varphi}$ is uniquely determined by (2.3). (Just note that, $\mu_{n\eta}$ determines $\varphi_m \pi \restriction n$ for all $n > k, m$.)

We now prove the two statements of the lemma. For (a) we consider any $\eta \in {}^{\omega\uparrow}\lambda$. If $\eta \notin X$, then we can choose arbitrary members $\Phi_{\eta n} \in \mathfrak{P}$, and if $\eta \in X$, then choose the uniquely determined sequence $\overline{\varphi}$ from (2.3) and let $\Phi_{\eta n} = \varphi_n$, so $\Phi_\eta = \overline{\varphi}$.

For (b) we consider some $\overline{f} = \{f_\nu \mid f_\nu \in \mathfrak{P}, \nu \in {}^{\omega\uparrow>}\lambda\}$ and $\rho \in {}^{\omega\uparrow>}\lambda$. In this case we must define an extension $\eta = \langle \alpha_n \mid n < \omega \rangle \in {}^{\omega\uparrow}\lambda$ of ρ. Thus put $0\eta = \alpha$, $\alpha_n = n\rho$ for $n < \lg(\rho)$. And if $n \geq \lg(\rho)$, then using the above and that the list of μ_αs is unbounded, we can choose inductively $\alpha_n > \alpha_{n-1}$ with $\langle f_{\eta \restriction m} \mid m < n \rangle \pi_n = \mu_{\alpha_n}$.

Finally we check statement (b). Using (2.3) it will follow that the sequence η belongs to X:

If $\overline{\varphi} = \langle f_{\eta \restriction i} \mid i < \omega \rangle \in {}^{\omega}\mathfrak{P}$ and $k = \lg(\rho)$, then we have

$$(\overline{\varphi} \restriction n)\pi_n = \langle f_{\eta \restriction m} \mid m < n \rangle \pi_n = \mu_{\alpha_n} = \mu_{n\eta} \text{ for all } n > k$$

and $\Phi_{n\eta} = \varphi_n = f_{\eta \restriction n}$ for all $n < \omega$ is immediate. □

The $\overline{\lambda}$-Black Box 2.4. *Let $\langle \lambda_1, \dots, \lambda_{k_*} \rangle$ be a ■-sequence satisfying* (2.1)*, Λ, Λ_* as above and C as in Definition* 2.2*. Then there is a family of set-traps $\langle \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle$ satisfying the following*

PREDICTION PRINCIPLE: *If $\varphi : \Lambda_* \to C$ is any map with the trap-condition $\Lambda_m \varphi \subseteq C_m$ $(m \leq k_*)$ and $\alpha \in \lambda_{k_*}$, then for some $\overline{\eta} \in \Lambda$ there is a set-trap $\varphi_{\overline{\eta}}$ with $\varphi_{\overline{\eta}} \subseteq \varphi$ and $0\eta_{k_*} = \alpha$.*

Proof. The proof of Theorem 2.4 will follow by induction on k_*. Temporarily we will attach parameters k_* to the above symbols like $\Lambda^{k_*}, \overline{C}^{k_*}, \varphi^{k_*}_{\overline{\eta}}, \overline{\eta}^{k_*}, \dots$.

The first step is $k_* = 1$. In this case the claim is a special case of Lemma 2.3. Indeed, we have $\Lambda^{k_*} = {}^{\omega\uparrow}\lambda_{k_*}$ and $\Lambda^{k_*}_* = {}^{\omega\uparrow>}\lambda_{k_*}$ and $\overline{\eta} \upharpoonleft \langle m, n \rangle = \eta_{k_*} \restriction n$ holds. We put $\mathfrak{P} = C^{k_*} = C^{k_*}_{k_*}$ and note that $|\,\mathfrak{P}\,| \leq \chi_{k_*} = \chi^{\aleph_0}_{k_*}$. The trap functions $\varphi_{\overline{\eta}}$ are defined by

$$(\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi_{\overline{\eta}} = \Phi_{\eta_{k_*} n}$$

and with $f_\nu = \nu\varphi$ condition (b) of Lemma 2.3 reads as

$$(\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi_{\overline{\eta}} = \Phi_{\eta_{k_*} n} = f_{\eta_{k_*} \restriction n} = (\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi$$

and the prediction principle in Theorem 2.4 is clear.

The induction step $k_* = k + 1$:
Suppose that Theorem 2.4 is shown for k. We must find a family of traps $\{\varphi^{k_*}_{\overline{\eta}} \mid \overline{\eta} \in$

$\Lambda^{k_*}\}$ for $\Lambda^{k_*}, \overline{C}^{k_*}$ and verify the prediction principle in Theorem 2.4. By induction hypothesis there is such a family $\{\varphi^k_{\overline{\eta}} \mid \overline{\eta} \in \Lambda^k\}$ for $\Lambda^k, \overline{C}^k$.

Let $\chi_{k_*} = \chi$, $\lambda_{k_*} = \lambda$ and recall that $\chi = \lambda^{\aleph_0}$. Moreover, by assumption $\left| C^{k_*} \right| \leq \chi$. We now consider $\mathfrak{P} = \operatorname{map}(\Lambda^k, C^{k_*}_{k_*})$ which has size $|\mathfrak{P}| = \left| C^{k_*}_{k_*} \right|^{|\Lambda^k|} \leq \chi^{\chi_k} = \chi$ by condition (2.1) of the $\blacksquare$-sequence.

If $\overline{\eta} \in \Lambda^{k_*}$, then let $\varphi^{k_*}_{\overline{\eta}} : [\overline{\eta}] \to C^{k_*}$ be the following map in (2.4). Recall that $\overline{\eta} = \langle \eta_1, \dots, \eta_{k_*} \rangle$ and thus $\eta_{k_*} \in {}^{\omega\uparrow}\lambda$ and $\Phi_{\eta_{k_*} n} \in \operatorname{map}(\Lambda^k, C^{k_*}_{k_*})$ by Lemma 2.3. If $\overline{\eta}' = \langle \eta_1, \dots, \eta_k \rangle \in \Lambda^k$, then $\overline{\eta}' \Phi_{\eta_{k_*} n}$ is a well-defined element of $C^{k_*}_{k_*}$ and $\varphi^k_{\overline{\eta}'}$ is given by induction hypothesis. We can now define

$$(\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi^{k_*}_{\overline{\eta}} = \begin{cases} \overline{\eta}' \Phi_{\eta_{k_*} n} & \text{if } m = k_* \\ (\overline{\eta}' \upharpoonleft \langle m, n \rangle)\varphi^k_{\overline{\eta}'} & \text{if } m < k_*. \end{cases} \tag{2.4}$$

In order to show the prediction principle we consider an arbitrary map $\varphi : \Lambda^{k_*}_* \to C^{k_*}$ satisfying the trap-condition $\Lambda_m \varphi \subseteq C_m$ for all $m \leq k_*$. We want to find $\overline{\eta} \in \Lambda^k$ and $\mu \in {}^{\omega\uparrow}\lambda$ such that $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle \in \Lambda^{k_*}$ satisfies $\varphi \restriction [\overline{\eta}^*] = \varphi^{k_*}_{\overline{\eta}^*}$ and $0\eta^*_{k_*} = \alpha$ (which is the claim of Theorem 2.4).

First we search for μ and define for each $\nu \in {}^{\omega\uparrow>}\lambda$ a map $f_\nu : \Lambda^k \to C^{k_*}_{k_*}$ from $\mathfrak{P}$ depending on φ. If $\overline{\eta} \in \Lambda^k$, then $\overline{\eta}^\wedge \langle \nu \rangle \in \Lambda^{k_*}_*$, thus

$$\overline{\eta} f_\nu := (\overline{\eta}^\wedge \langle \nu \rangle)\varphi \tag{2.5}$$

is well-defined. By the Lemma 2.3 we find $\mu \in {}^{\omega\uparrow}\lambda$ such that

$$0\mu = \alpha \text{ and } f_{\mu \restriction n} = \Phi_{\mu n} : \Lambda^k \to C^{k_*}_{k_*} \text{ for all } n \in \omega.$$

By Lemma 2.3(b), (2.4) and (2.5) we have for any $\overline{\eta}$ and $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle$ that

$$(\overline{\eta}^* \upharpoonleft \langle k_*, n \rangle)\varphi^{k_*}_{\overline{\eta}^*} = \overline{\eta}\Phi_{\mu n} = \overline{\eta} f_{\mu \restriction n} = (\overline{\eta}^\wedge \langle \mu \restriction n \rangle)\varphi = (\overline{\eta}^* \upharpoonleft \langle k_*, n \rangle)\varphi$$

and $0\mu = \alpha$ which is the prediction as required for $m = k_*$.

Now we consider the case when $m < k_*$ and define a map $\varphi' : \Lambda^k_* \to C^k$ depending on φ and μ. If $\overline{\eta}' \in \Lambda^k_*$, then $\overline{\eta}'^\wedge \langle \mu \rangle \in \Lambda^{k_*}_*$ (because $\mu \in {}^{\omega\uparrow}\lambda$), thus

$$\overline{\eta}' \varphi' := (\overline{\eta}'^\wedge \langle \mu \rangle)\varphi$$

is well-defined and by induction hypothesis on the traps $\varphi^k_{\overline{\eta}'}$ there is some $\overline{\eta} \in \Lambda^k$ such that

$$(\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi^k_{\overline{\eta}} = (\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi' \text{ for } m \leq k \text{ and } n < \omega.$$

Now let $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle \in \Lambda^{k_*}$. By the last displayed equation and (2.4) we have for $m < k_*$ that

$$(\overline{\eta}^* \upharpoonleft \langle m, n \rangle)\varphi^{k_*}_{\overline{\eta}^*} = (\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi^k_{\overline{\eta}} = (\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi' = (\overline{\eta}^* \upharpoonleft \langle m, n \rangle)\varphi.$$

Thus $\varphi^{k_*}_{\overline{\eta}^*}$ predicts φ with $0\eta^*_{k_*} = 0\mu = \alpha$ as suggested above. □

Definition 2.5. *Let $F : \Lambda \to \Lambda_*$ be a given map. A subset $\Omega \subseteq \Lambda$ is* free (*with respect to F*) *if there is an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \alpha_* \rangle$ of Ω (we write $\Omega_\alpha = \{\overline{\eta}^\beta \mid \beta < \alpha\}$) and there are $\ell_\alpha \leq k_*, n_\alpha < \omega$ $(\alpha < \alpha_*)$ such that for $\alpha < \alpha_*$ and $n_\alpha \leq n$*

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \Omega_\alpha F.$$

Moreover, Ω is κ-free (with respect to F) for some cardinal κ if the above holds for all subsets of Ω of cardinality $< \kappa$.

This is to say, that every newly chosen element $\overline{\eta}^\alpha$ picks up some unused element from Λ_* in its support. Note that the enumeration of Ω in Definition 2.5 does not permit repetitions. We want to show the following

Freeness-Proposition 2.6. *With the notions from Theorem 2.4 and Definition 2.5 the set Λ is $\aleph_{k_*}$-free with respect to any function $F : \Lambda \to \Lambda_*$. For any $k < k_*$, $\Omega \subseteq \Lambda$ of cardinality $|\,\Omega\,| \leq \aleph_k$ and $\langle u_{\overline{\eta}} \subseteq \{1, \ldots, k_*\} \mid |\,u_{\overline{\eta}}\,| > k, \overline{\eta} \in \Omega \rangle$ we can find an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_k \rangle$ of Ω, $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $n_\alpha < \omega$ $(\alpha < \aleph_k)$ such that*

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \Omega_\alpha F \text{ for all } n \geq n_\alpha.$$

Proof. The proof follows by induction on k. We begin with $k = 0$, hence we may assume that $|\,\Omega\,| = \aleph_0$. Let $\Omega = \{\overline{\eta}^\alpha \mid \alpha < \omega\}$ be an enumeration without repetitions. From $0 = k < |\,\overline{u}_{\overline{\eta}}\,|$ follows $\overline{u}_{\overline{\eta}} \neq \emptyset$ and we can choose any $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ for all $\alpha < \omega$. To be definite we may choose $\ell_\alpha = \min u_{\overline{\eta}^\alpha}$. If $\alpha \neq \beta < \omega$, then $\overline{\eta}^\alpha \neq \overline{\eta}^\beta$ and there is $n_{\alpha,\beta} \in \omega$ such that $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \neq \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle$ for all $n \geq n_{\alpha\beta}$. Since $\Omega_\alpha F$ is finite, we may enlarge $n_{\alpha,\beta}$, if necessary, such that $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \Omega_\alpha F$ for all $n \geq n_{\alpha,\beta}$. If $n_\alpha = \max_{\beta < \alpha} n_{\alpha,\beta}$, then $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \Omega_\alpha F$ for all $n \geq n_\alpha$. Hence case $k = 0$ is settled and we let $k' = k + 1$ and assume that the proposition holds for k.

Let $|\,\Omega\,| = \aleph_{k'}$ and choose an $\aleph_{k'}$-filtration $\Omega = \bigcup_{\delta < \aleph_{k'}} \Omega_\delta$ with $\Omega_0 = \emptyset$ and $|\,\Omega_1\,| = \aleph_k$. The crucial idea comes from [11]: We can also assume that this chain is closed, meaning that for any $\delta < \aleph_{k'}$, $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$ and $\overline{\eta} \in \Omega$ with

$$\{\eta_m \mid m \leq k_*\} \subseteq \{\nu_m, \nu'_m, (\overline{\nu}F)_m, (\overline{\nu}'F)_m \mid m \leq k_*\}$$

follows $\overline{\eta} \in \Omega_\delta$. Thus, if $\overline{\eta} \in \Omega_{\delta+1} \setminus \Omega_\delta$, then the set

$$u^*_{\overline{\eta}} = \{\ell \leq k_* \mid \exists n < \omega, \overline{\nu} \in \Omega_\delta \text{ such that } \overline{\eta} \upharpoonright \langle \ell, n \rangle = \overline{\nu} \upharpoonright \langle \ell, n \rangle \text{ or } \overline{\eta} \upharpoonright \langle \ell, n \rangle = \overline{\nu}F\}$$

is empty or a singleton. Otherwise there are $n, n' < \omega$ and distinct $\ell, \ell' \leq k_*$ with $\overline{\eta} \upharpoonright \langle \ell, n \rangle \in \{\overline{\nu} \upharpoonright \langle \ell, n \rangle, \overline{\nu}F\}$ and $\overline{\eta} \upharpoonright \langle \ell', n' \rangle \in \{\overline{\nu}' \upharpoonright \langle \ell', n' \rangle, \overline{\nu}'F\}$ for certain $\overline{\nu}, \overline{\nu}' \in \Omega_\delta$. Hence $\{\eta_m \mid m \leq k_*\} \subseteq \{\nu_m, \nu'_m, (\overline{\nu}F)_m, (\overline{\nu}'F)_m \mid m \leq k_*\}$, and the closure property implies the contradiction $\overline{\eta} \in \Omega_\delta$.

If $\delta < \aleph_{k'}$, then let $D_\delta = \Omega_{\delta+1} \setminus \Omega_\delta$ and $u'_{\overline{\eta}} := u_{\overline{\eta}} \setminus u^*_{\overline{\eta}}$ must have size $> k' - 1 = k$. Thus the induction hypothesis applies and we find an enumeration $\overline{\eta}^{\delta\alpha}$ $(\alpha < \aleph_k)$ of D_δ as in the proposition. Finally we put these chains for each $\delta < \aleph_{k'}$ together with the induced ordering to get an enumeration $\langle \overline{\eta}^\alpha \mid \alpha < \aleph_{k'} \rangle$ of Ω satisfying the proposition. □

3. The Black Box for $\aleph_n$-free modules

Let R be a commutative ring with $\mathbb{S}$ a countable multiplicatively closed subset such that the following holds.

(i) The elements of $\mathbb{S}$ are not zero-divisors, i.e. if $s \in \mathbb{S}, r \in R$ and $sr = 0$, then $r = 0$.

(ii) $\bigcap_{s\in\mathbb{S}} sR = 0$.

We also say that R is an $\mathbb{S}$-ring. If (i) holds, then R is $\mathbb{S}$-*torsion-free* and if (ii) holds, then R is $\mathbb{S}$-*reduced*, see [8]. To ease notations we use the letter $\mathbb{S}$ only if we want to emphasize that the argument depends on it. If M is an R-module, then these definitions naturally carry over to M. Finally we enumerate $\mathbb{S} = \{s_n \mid n < \omega\}$, let $s_0 = 1$ and put $q_n = \prod_{i\le n} s_i$, thus $q_{n+1} = q_n s_{n+1}$.

Similar to the Black Box in [1], we first define the basic R-module B, which is

$$B = \bigoplus_{\overline{\eta}\in\Lambda_*} Re_{\overline{\eta}}.$$

Definition 3.1. *If $U \subset \Lambda_*$, then we get a* canonical summand $B_U = \bigoplus_{\overline{\eta}\in U} Re_{\overline{\eta}}$ *of B, and in particular, let $B_{\overline{\eta}} = B_{[\overline{\eta}]}$ and $B_{\overline{\eta}\restriction m} = B_{[\overline{\eta}\restriction m]}$ be the* canonical summand of $\overline{\eta}$ *and* $\overline{\eta}\restriction m$ $(\overline{\eta} \in \Lambda)$, *respectively.*

We have several R–free summands

$$B_{\overline{\eta}\restriction m} = \bigoplus_{n<\omega} Re_{\overline{\eta}\restriction\langle m,n\rangle} \text{ and } B_{\overline{\eta}} = \bigoplus_{m\le k_*} B_{\overline{\eta}\restriction m}.$$

The $\mathbb{S}$-topology (generated by the basis sB $(s \in \mathbb{S})$) of neighbourhoods of 0 is Hausdorff on B and (as usual) we can consider the $\mathbb{S}$-completion $\widehat{B}$ of B; see [8] for elementary facts on the elements of $\widehat{B}$. Let $\widetilde{B} = \bigoplus_{\overline{\eta}\in\Lambda_*} \widehat{R}e_{\overline{\eta}}$. Every element $b \in \widehat{B}$ has a natural Λ_*-support $[b]_{\Lambda_*} \subseteq \Lambda_*$ which are those $\overline{\eta} \in \Lambda_*$ which contribute to the sum-representation $b = \sum_{\overline{\eta}\in\Lambda_*} b_{\overline{\eta}} e_{\overline{\eta}}$ with coefficients $0 \ne b_{\overline{\eta}} \in \widehat{R}$. Thus let $[b]_{\Lambda_*} = \{\overline{\eta} \in \Lambda_* \mid b_{\overline{\eta}} \ne 0\}$. Note that $[b]_{\Lambda_*}$ is at most countable and $b \in \widetilde{B}$ iff $[b]_{\Lambda_*}$ is finite. As in the earlier Black Boxes (see [8]) we use conditions on the support (given by the prediction) to select (carefully) elements from $\widehat{B}$ added to B to get the final structure M, such that

$$B \subseteq M \subseteq_* \widehat{B}$$

which is an $\mathbb{S}$-pure submodule of $\widehat{B}$, thus satisfying

$$M \cap s\widehat{B} \subseteq sM$$

for all $s \in \mathbb{S}$. Thus $\mathbb{S}$-topological arguments can be used carelessly switching between these three modules.

We will now use B, Λ_*, Λ to define the Black Box for $\aleph_n$-free R-modules. As in [1] we will also use the notion of a trap.

Definition 3.2. *Let G be any R-module. A* trap (for B, G) *is a partial R-homomorphism of B into G with a label $\overline{\eta} \in \Lambda$, say $\varphi_{\overline{\eta}} : \mathrm{Dom}(\varphi_{\overline{\eta}}) \to G$, such that $B_{\overline{\eta}} \subseteq \mathrm{Dom}(\varphi_{\overline{\eta}}) \subseteq B$.*

The $\overline{\lambda}$-Black Box 3.3. *Given a $\blacksquare$-sequence $\overline{\lambda} = \langle \lambda_1, \dots, \lambda_{k_*} \rangle$ with* (2.1) *and an R-module G of size $|G| \leq \chi_1$, let Λ, Λ_* be as above. Then there is a family of traps $\varphi_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda)$ with the following property:*
The prediction: *If $\varphi : B \to G$ is an R-homomorphism and $\alpha \in \lambda_{k_*}$, then there is $\overline{\eta} \in \Lambda$ with $0\eta_{k_*} = \alpha$ and $\varphi_{\overline{\eta}} \subseteq \varphi$.*

Proof. The theorem is an immediate consequence of Theorem 2.4. We view the set maps in Theorem 2.4 as the restrictions of the R-homomorphisms in Theorem 3.3 to the canonical R-basis $\{e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*\}$ of B. There is a one-to-one correspondence between these maps and thus Theorem 3.3 follows. $\square$

4. The R-modules

Let R be an $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced commutative ring of size $|R| < 2^{\aleph_0}$, $\chi_{k_*} = \lambda_{k_*}^{\aleph_0} = \lambda_{k_*}$ be as before, $B = \bigoplus_{\overline{\nu} \in \Lambda_*} Re_{\overline{\nu}}$ the R-module freely generated by $\{e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_*\}$ and

$$\Lambda_* = \bigcup_{\overline{\eta} \in \Lambda} [\overline{\eta}] \text{ with } [\overline{\eta}] = \{\overline{\eta} \upharpoonright \langle m, n \rangle \mid m \leq k_*, n < \omega\}.$$

We also choose any bijection

$$\delta : \lambda_{k_*} \longrightarrow \Lambda_*.$$

Thus we can write the basis elements of B in the form $e_{\delta(\alpha)}$ for any $\alpha \in \lambda_{k_*}$.

From [5] follows that the $\mathbb{S}$-adic completion $\widehat{R}$ of R has $2^{\aleph_0}$ algebraically independent elements over R, and in particular $\left|\widehat{R}\right| = 2^{\aleph_0}$.

Next we define particular elements in $\widehat{B}$. If $\overline{\eta} \in \Lambda$, then let

$$y_{\overline{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} (\sum_{m=1}^{k_*} e_{\overline{\eta} \upharpoonright \langle m,n \rangle} + b_{\overline{\eta}n} e_{\delta(0\eta_{k_*})})$$

where $b_{\overline{\eta}n} \in R$. Moreover let $y_{\overline{\eta}} = y_{\overline{\eta}0}$. We will choose $\pi_{\overline{\eta}} \in \widehat{R}$ and write $\pi_{\overline{\eta}} = \sum_{n<\omega} q_n b_{\overline{\eta}n}$ and let $\pi_{\overline{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} b_{\overline{\eta}n}$. Thus

$$y_{\overline{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} (\sum_{m=1}^{k_*} e_{\overline{\eta} \upharpoonright \langle m,n \rangle}) + \pi_{\overline{\eta}k} e_{\delta(0\eta_{k_*})}$$

and from

$$s_{k+1} y_{\overline{\eta}k+1} = \sum_{n \geq k+1} \frac{q_n}{q_k} (\sum_{m=1}^{k_*} e_{\overline{\eta} \upharpoonright \langle m,n \rangle} + b_{\overline{\eta}n} e_{\delta(0\eta_{k_*})})$$

and $y_{\overline{\eta}k} - s_{k+1}y_{\overline{\eta}k+1} = \sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,k\rangle} + b_{\overline{\eta}k}e_{\delta(0\eta_{k_*})}$, follows

$$s_{k+1}y_{\overline{\eta}k+1} = y_{\overline{\eta}k} - \sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,k\rangle} - b_{\overline{\eta}k}e_{\delta(0\eta_{k_*})}. \tag{4.1}$$

We want to define an R-module M with $B \subseteq M \subseteq_* \widehat{B}$ which is $\mathbb{S}$-pure in $\widehat{B}$. Thus M/B is $\mathbb{S}$-torsion-free and $\mathbb{S}$-divisible. It follows that for any non-trivial homomorphism $\sigma : M \to R$ there is $\overline{\nu} \in \Lambda_*$ with $e_{\overline{\nu}}\sigma \neq 0$. If $\overline{\eta} \in \Lambda$, then we will adjoin to B for some suitable $\pi_{\overline{\eta}} \in \widehat{R}$ the element $y_{\overline{\eta}} = \sum_{n<\omega} q_n(\sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,n\rangle}) + \pi_{\overline{\eta}}e_{\delta(0\eta_{k_*})}$. This will follow with the help of the next

Proposition 4.1. *Let R be an $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced commutative ring of size $< 2^{\aleph_0}$. Then for any $\overline{\eta} \in \Lambda$ there are $\pi_{\overline{\eta}} \in \widehat{R}$ and*

$$y_{\overline{\eta}} = \sum_{n<\omega} q_n(\sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,n\rangle}) + \pi_{\overline{\eta}}e_{\delta(0\eta_{k_*})} \tag{4.2}$$

with no homomorphism $\varphi : \langle B, y_{\overline{\eta}}\rangle_* \longrightarrow R$ such that $\varphi\restriction B_{[\overline{\eta}]} = \varphi_{\overline{\eta}}$ and $e_{\delta(0\eta_{k_*})}\varphi \neq 0$.

Proof. Let $e = e_{\delta(0\eta_{k_*})}$ and choose pairwise distinct elements $\pi_\alpha \in \widehat{R}$ $(\alpha < 2^{\aleph_0})$. Moreover let $y = \sum_{n<\omega} q_n(\sum_{m=1}^{k_*} e_{\overline{\eta}\restriction\langle m,n\rangle})$ and put $y_\alpha = y + \pi_\alpha e$. Suppose that for each $\alpha < 2^{\aleph_0}$ there is a homomorphism $\varphi_\alpha : \langle B, y_\alpha\rangle_* \longrightarrow R$ with $\varphi_\alpha\restriction B_{[\overline{\eta}]} = \varphi_{\overline{\eta}}$ and $e\varphi_\alpha \neq 0$. By a pigeon hole argument there are distinct $\alpha, \beta < 2^{\aleph_0}$ with the same images $y_\alpha\varphi_\alpha = y_\beta\varphi_\beta$ and also $e\varphi_\alpha = e\varphi_\beta =: c \neq 0$. But this implies

$$0 = y_\alpha\varphi_\alpha - y_\beta\varphi_\beta = (y + \pi_\alpha e)\varphi_\alpha - (y + \pi_\beta e)\varphi_\beta$$
$$= y\varphi_{\overline{\eta}} + \pi_\alpha e\varphi_\alpha - y\varphi_{\overline{\eta}} - \pi_\beta e\varphi_\beta = (\pi_\alpha - \pi_\beta)c.$$

And from $\pi_\alpha - \pi_\beta \neq 0$ follows $c = 0$, a contradiction. □

Finally we define the R-module

$$M = \langle B, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda\rangle_* \subseteq \widehat{B}. \tag{4.3}$$

Here we let $y_{\overline{\eta}}$ be as in (4.2) and apply Proposition 4.1.

First we will take care of the freeness of M by applying the set-theoretic version of freeness, i.e. Proposition 2.6. In order to apply our results to rings which are not necessarily PIDs, we more generally say that an R-module M is κ-free if any subset of size $< \kappa$ is contained in a free R-submodule of M.

Freeness-Proposition 4.2. *The module M as defined in* (4.3) *is $\aleph_{k_*}$-free.*

Proof. Besides the Λ_*-support $[g]_{\Lambda_*}$ (discussed at the beginning of the last section) any element g of the module $M = \langle B, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle_*$ has a refined natural finite support $[g]$ arriving from the definition (4.3). It consists of all those elements of Λ and Λ_* contributing to g. We observe that g is generated by elements $y_{\overline{\eta}}$ and $e_{\overline{\eta}\upharpoonright\langle m,n\rangle}$ and simply collect the $\overline{\eta}$s and $\overline{\eta}\upharpoonright\langle m,n\rangle$ needed. Clearly $[g]$ is a finite subset of $\Lambda \cup \Lambda_*$. Hence any submodule H of M has a natural support $[H]$ taking the union of supports of its elements and if $|H| < \kappa$ for any cardinal $\kappa > |R|$, then there is a subset $\Omega \subseteq \Lambda$ of size $|\Omega| < \kappa$ such that H is a submodule of the pure R-submodule

$$M_\Omega = \langle e_{\overline{\eta}\upharpoonright\langle m,n\rangle}, e_{\delta(0\eta_{k_*})}, y_{\overline{\eta}} \mid \overline{\eta} \in \Omega, m \le k_*, n < \omega \rangle_* \subseteq \widehat{B},$$

which also has size $< \kappa$. Thus, in order to show $\aleph_{k_*}$-freeness of M, we only must consider any $\Omega \subseteq \Lambda$ of size $|\Omega| < \aleph_{k_*}$ and show the freeness of the module M_Ω. We may assume that $|\Omega| = \aleph_{k_*-1}$. Let $F : \Lambda \to \Lambda_*$ be the map which assignes to $\overline{\eta} \in \Lambda$ the element $\overline{\eta}F = \delta(0\eta_{k_*}) \in \Lambda_*$.

By Proposition 2.6 we can express the generators of M_Ω of the form

$$M_\Omega = \langle e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}, e_{\overline{\eta}^\alpha F}, y_{\overline{\eta}^\alpha n} \mid \alpha < \aleph_{k_*-1}, m \le k_*, n < \omega \rangle$$

and find a sequence of pairs $(\ell_\alpha, n_\alpha) \in (k_* + 1) \times \omega$ such that for $n \ge n_\alpha$

$$\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha, n\rangle \notin \{\overline{\eta}^\beta\upharpoonright\langle \ell_\alpha, n\rangle \mid \beta < \alpha\} \cup \{\overline{\eta}^\beta F \mid \beta < \alpha\}. \tag{4.4}$$

Let $M_\alpha = \langle e_{\overline{\eta}^\gamma\upharpoonright\langle m,n\rangle}, e_{\overline{\eta}^\gamma F}, y_{\overline{\eta}^\gamma n} \mid \gamma < \alpha, m \le k_*, n < \omega \rangle$ for any $\alpha < \aleph_{k_*-1}$; thus

$$\begin{aligned} M_{\alpha+1} &= M_\alpha + \langle e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}, e_{\overline{\eta}^\alpha F}, y_{\overline{\eta}^\alpha n} \mid m \le k_*, n < \omega \rangle \\ &= M_\alpha + \langle e_{\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha,n\rangle} \mid n < n_\alpha \rangle + \langle y_{\overline{\eta}^\alpha n} \mid n \ge n_\alpha \rangle \\ &\quad + \langle e_{\overline{\eta}^\alpha F}, e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle} \mid \ell_\alpha \ne m \le k_*, n < \omega \rangle. \end{aligned}$$

Hence any element in $M_{\alpha+1}/M_\alpha$ can be represented in $M_{\alpha+1}$ modulo M_α of the form

$$\sum_{n \ge n_\alpha} r_n y_{\overline{\eta}^\alpha n} + \sum_{n < n_\alpha} r'_n e_{\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha,n\rangle} + r e_{\overline{\eta}^\alpha F} + \sum_{n<\omega} \sum_{\ell_\alpha \ne m \le k_*} r''_{mn} e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}.$$

Moreover, the summands involving the $e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}$s have disjoint supports. Now condition (4.4) applies recursively. And by the disjointness (identifying $e_{\overline{\eta}^\alpha F}$ with one of the $e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}$s if possible) it also follows that all coefficients r, r'_n, r''_{mn} must be zero, showing that the set

$$\{e_{\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha,k\rangle}, e_{\overline{\eta}^\alpha F}, e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle} \mid k < n_\alpha, \ell_\alpha \ne m \le k_*, n < \omega\} \setminus M_\alpha$$

freely generates $M_{\alpha+1}/M_\alpha$. Thus M_Ω has an ascending chain with only free factors; it follows that M_Ω is free. □

We can finally show the

Theorem 4.3. *Let R be an $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced commutative ring of size $< 2^{\aleph_0}$. Then for any $\blacksquare$-sequence $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{k_*} \rangle$ with (2.1) there exists an $\aleph_{k_*}$-free R-module M of size χ_{k_*} with trivial dual $\mathrm{Hom}(M, R) = 0$. In particular, if R is a principal ideal domain but not a field of size $< 2^{\aleph_0}$, then there is an $\aleph_{k_*}$-free R-module M of size χ_{k_*} with trivial dual.*

Proof. If M is the R-module above, then M is $\aleph_{k_*}$-free by Proposition 4.2. Obviously M has size χ_{k_*}. If $\varphi : M \longrightarrow R$ is a non-trivial homomorphism, then there is $\overline{\nu} \in \Lambda_*$ such that for some basis element $e_{\overline{\nu}}\varphi \neq 0$. By the Black Box 3.3 there is $\overline{\eta} \in \Lambda$ with $\delta(0\eta_{k_*}) = \overline{\nu}$ and $\varphi \restriction B_{[\overline{\eta}]} = \varphi_{\overline{\eta}}$. We apply Proposition 4.1 to see that this is a contradiction. Hence $\mathrm{Hom}(M, R) = 0$ follows. $\square$

Corollary 4.4. *If n is a natural number, then we find $\aleph_n$-free abelian groups of size $\beth_n$ with trivial dual.*

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