

**DIAMONDS**  
**SH922**

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ABSTRACT. If  $\lambda = \chi^+ = 2^\chi > \aleph_1$  then diamond on  $\lambda$  holds. Moreover, if  $\lambda = \chi^+ = 2^\chi$  and  $S \subseteq \{\delta < \lambda : \text{cf}(\delta) \neq \text{cf}(\chi)\}$  is stationary then  $\diamond_S$  holds. Earlier this was known only under additional assumptions on  $\chi$  and/or  $S$ .

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## 1. INTRODUCTION

We prove in this paper several results about diamonds. Let us recall the basic definitions and sketch the (pretty long) history of the related questions.

The diamond principle was formulated by Jensen, who proved that it holds in  $\mathbf{L}$  for every regular uncountable cardinal  $\kappa$  and stationary  $S \subseteq \kappa$ . This is a prediction principle, which asserts the following:

{0z.1}

**Definition 1.1.**  $\diamond_S$  (the set version).

Assume  $\kappa = \text{cf}(\kappa) > \aleph_0$  and  $S \subseteq \kappa$  is stationary,  $\diamond_S$  holds when there is a sequence  $\langle A_\alpha : \alpha \in S \rangle$  such that  $A_\alpha \subseteq \alpha$  for every  $\alpha \in S$  and the set  $\{\alpha \in S : A \cap \alpha = A_\alpha\}$  is a stationary subset of  $\kappa$  for every  $A \subseteq \kappa$ .

The diamond sequence  $\langle A_\alpha : \alpha \in S \rangle$  guesses enough (i.e., stationarily many) initial segments of every  $A \subseteq \kappa$ . Several variants of this principle were formulated, for example:

{0z.4}

**Definition 1.2.**  $\diamond_S^*$ .

Assume  $\kappa = \text{cf}(\kappa) > \aleph_0$  and  $S$  is a stationary subset of  $\kappa$ . Now  $\diamond_S^*$  holds when there is a sequence  $\langle \mathcal{A}_\alpha : \alpha \in S \rangle$  such that each  $\mathcal{A}_\alpha$  is a subfamily of  $\mathcal{P}(\alpha)$ ,  $|\mathcal{A}_\alpha| \leq |\alpha|$  and for every  $A \subseteq \kappa$  there exists a club  $C \subseteq \kappa$  such that  $A \cap \alpha \in \mathcal{A}_\alpha$  for every  $\alpha \in C \cap S$ .

We know that  $\diamond_S^*$  holds in  $\mathbf{L}$  for every regular uncountable  $\kappa$  and stationary  $S \subseteq \kappa$ . Kunen proved that  $\diamond_S^* \Rightarrow \diamond_S$ . Moreover, if  $S_1 \subseteq S_2$  are stationary subsets of  $\kappa$  then  $\diamond_{S_2}^* \Rightarrow \diamond_{S_1}^*$  (hence  $\diamond_{S_1}$ ). But the assumption  $\mathbf{V} = \mathbf{L}$  is heavy. Trying to avoid it, we can walk in several directions. On weaker relatives see [Sh:829] and references there. We can also use other methods, aiming to prove the diamond without assuming  $\mathbf{V} = \mathbf{L}$ .

There is another formulation of the diamond principle, phrased via functions (instead of sets). Since we use this version in our proof, we introduce the following:

{d.21}

**Definition 1.3.**  $\diamond_S$  (the functional version).

Assume  $\lambda = \text{cf}(\lambda) > \aleph_0$ ,  $S \subseteq \lambda$ ,  $S$  is stationary.  $\diamond_S$  holds if there exists a diamond sequence  $\langle g_\delta : \delta \in S \rangle$  which means that  $g_\delta \in {}^\delta \delta$  for every  $\delta \in S$ , and for every  $g \in {}^\lambda \lambda$  the set  $\{\delta \in S : g \upharpoonright \delta = g_\delta\}$  is a stationary subset of  $\lambda$ .

By Gregory [Gre76] and Shelah [Sh:108] we know that assuming  $\lambda = \chi^+ = 2^\chi$  and  $\kappa = \text{cf}(\kappa) \neq \text{cf}(\chi)$ ,  $\kappa < \lambda$ , and GCH holds (or actually just  $\chi^\kappa = \chi$  or  $(\forall \alpha < \chi)(|\alpha|^\kappa < \chi) \wedge \text{cf}(\chi) < \kappa$ ), then  $\diamond_{S_\kappa^\lambda}$  holds (recall that  $S_\kappa^\lambda = \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ ).

We have got also results which show that the failures of the diamond above a strong limit cardinal are limited. For instance, if  $\lambda = \chi^+ = 2^\chi > \mu$  and  $\mu > \aleph_0$  is strong limit, then (by [Sh:460]) the set  $\{\kappa < \mu : \diamond_{S_\kappa^\lambda}^* \text{ fails}\}$  is bounded in  $\mu$  (recall that  $\kappa$  is regular). Note that the result here does not completely subsume the earlier results when  $\lambda = 2^\chi = \chi^+$  as we get “diamond on every stationary set  $S \subseteq \lambda \setminus S_{\text{cf}(\chi)}^\lambda$ ”, but not  $\diamond_S^*$ ; this is inherent as noted in 3.4. In [Sh:829], a similar, stronger result is proved for  $\diamond_{S_\kappa^\lambda}$ : for every  $\lambda = \chi^+ = 2^\chi > \mu$ ,  $\mu$  strong limit for some finite  $\mathfrak{d} \subseteq \text{Reg} \cap \mu$ , for every regular  $\kappa < \mu$  not from  $\mathfrak{d}$  we have  $\diamond_{S_\kappa^\lambda}$ , and even  $\diamond_S$  for “most” stationary  $S \subseteq S_\kappa^\lambda$ . In fact, for the relevant good stationary sets  $S \subseteq S_\kappa^\lambda$  we get  $\diamond_S^*$ . Also weaker related results are proved there for other regular  $\lambda$  (mainly  $\lambda = \text{cf}(2^\chi)$ ).

The present work does not resolve:

**Problem 1.4.** Assume  $\chi$  is singular,  $\lambda = \chi^+ = 2^\chi$ , do we have  $\diamond_{S_{cf(\chi)}^\lambda}$ ? (You may even assume G.C.H.).

{0z.7}

However, the full analog result for 1.4 consistently fails, see [Sh:186] or [Sh:667]; that is: if G.C.H.,  $\chi > cf(\chi) = \kappa$  then we can force a non-reflecting stationary  $S \subseteq S_\kappa^{\chi^+}$  such that the diamond on  $S$  fails and cardinalities and cofinalities are preserved; also G.C.H. continue to hold. But if  $\chi$  is strong limit,  $\lambda = \chi^+ = 2^\chi$ , still we know something on guessing equalities for every stationary  $S \subseteq S_\kappa^\lambda$ ; see [Sh:667].

Note that this  $S$  (by [Sh:186], [Sh:667]) in some circumstances has to be “small”

- (\*) if ( $\chi$  is singular,  $2^\chi = \chi^+ = \lambda$ ,  $\kappa = cf(\chi)$  and) we have the square  $\square_\chi$  (i.e. there exists a sequence  $\langle C_\delta : \delta < \lambda, \delta \text{ is a limit ordinal} \rangle$ , so that  $C_\delta$  is closed and unbounded in  $\delta$ ,  $cf(\delta) < \chi \Rightarrow |C_\delta| < \chi$  and if  $\gamma$  is a limit point of  $C_\delta$  then  $C_\gamma = C_\delta \cap \gamma$ ) then  $\diamond_{S_\kappa^\lambda}$  holds, moreover, if  $S \subseteq S_\kappa^\lambda$  reflects in a stationary set of  $\delta < \lambda$  then  $\diamond_S$  holds, see [Sh:186, §3].

Also note that our results are of the form “ $\diamond_S$  for every stationary  $S \subseteq S^*$ ” for suitable  $S^* \subseteq \lambda$ . Now usually this was deduced from the stronger statement  $\diamond_S^*$ . However, the results on  $\diamond_S^*$  cannot be improved, see 3.4.

Also if  $\chi$  is regular we cannot improve the result to  $\diamond_{S_\chi^\lambda}$ , see [Sh:186] or [Sh:587], even assuming G.C.H. Furthermore the question on  $\diamond_{\aleph_2}$  when  $2^{\aleph_1} = \aleph_2 = 2^{\aleph_0}$  was raised. Concerning this we show in 3.2 that  $\diamond_{S_{\aleph_1}^{\aleph_2}}$  may fail (this works in other cases, too).

{0z.11}

**Question 1.5.** Can we deduce any ZFC result on  $\lambda$  strongly inaccessible?

By Džamonja-Shelah [DjSh:545] we know that failure of SNR helps (SNR stands for strong non-reflection), a parallel here is 2.3(2).

For  $\lambda = \lambda^{<\lambda} = 2^\mu$  weakly inaccessible we know less, still see [Sh:775], [Sh:898] getting a weaker relative of diamond, see Definition 3.5(2). Again failure of SNR helps.

On consistency results on SNR see Cummings-Džamonja-Shelah [CDSH:571], Džamonja-Shelah [DjSh:691].

We thank the audience in the Jerusalem Logic Seminar for their comments in the lecture on this work in Fall 2006. We thank the referee for many helpful remarks. In particular, as he urged for details, the proof of 2.5(1),(2) just said “like 2.3”, instead we make the old proof of 2.3(1) prove 2.5(1),(2) by minor changes and make explicit 2.5(3) which earlier was proved by “repeat of the second half of the proof of 2.3(1)”. We thank Shimoni Garti for his help in the proofreading.

{0z.13}

*Notation 1.6.* 1) If  $\kappa = cf(\kappa) < \lambda = cf(\lambda)$  then we let  $S_\kappa^\lambda := \{\delta < \lambda : cf(\delta) = \kappa\}$ .  
2)  $\mathcal{D}_\lambda$  is the club filter on  $\lambda$  for  $\lambda$  a regular uncountable cardinal.

## 2. DIAMOND ON SUCCESSOR CARDINALS

Recall (needed only for part (2) of the Theorem 2.3):

{ya.1}

**Definition 2.1.** 1) We say  $\lambda$  on  $S$  has  $\kappa$ -SNR or  $\text{SNR}(\lambda, S, \kappa)$  or  $\lambda$  has strong non-reflection for  $S$  in  $\kappa$  when  $S \subseteq S_\kappa^\lambda := \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$  so  $\lambda = \text{cf}(\lambda) > \kappa = \text{cf}(\kappa)$  and there are  $h : \lambda \rightarrow \kappa$  and a club  $E$  of  $\lambda$  such that for every  $\delta \in S \cap E$  for some club  $C$  of  $\delta$ , the function  $h \upharpoonright C$  is one-to-one and even increasing; (note that without loss of generality  $\alpha \in \text{nacc}(E) \Rightarrow \alpha$  successor and without loss of generality  $E = \lambda$ , so  $\mu \in \text{Reg} \cap \lambda \setminus \kappa^+ \Rightarrow \text{SNR}(\mu, S \cap \mu, \kappa)$ ). If  $S = S_\kappa^\lambda$  we may omit it.

{ya.2}

*Remark 2.2.* Note that by Fodor's lemma if  $\text{cf}(\delta) = \kappa > \aleph_0$  and  $h$  is a function from some set  $\supseteq \delta$  and the range of  $h$  is  $\subseteq \kappa$  then the following conditions are equivalent:

- (a)  $h$  is one-to-one on some club of  $\delta$
- (b)  $h$  is increasing on some club of  $\delta$
- (c)  $\text{Rang}(h \upharpoonright S)$  is unbounded in  $\kappa$  for every stationary subset  $S$  of  $\delta$ .

Our main theorem is:

{ya.4}

**Claim 2.3.** Assume  $\lambda = 2^\chi = \chi^+$ .

- 1) If  $S \subseteq \lambda$  is stationary and  $\delta \in S \Rightarrow \text{cf}(\delta) \neq \text{cf}(\chi)$  then  $\diamond_S$  holds.
- 2) If  $\aleph_0 < \kappa = \text{cf}(\chi) < \chi$  and  $\diamond_S$  fails where  $S = S_\kappa^\lambda$  (or just  $S \subseteq S_\kappa^\lambda$  is a stationary subset of  $\lambda$ ) then we have  $\text{SNR}(\lambda, \kappa)$  or just  $\lambda$  has strong non-reflection for  $S \subseteq S_\kappa^\lambda$  in  $\kappa$ .

{ya.6}

**Definition 2.4.** 1) For a filter  $D$  on a set  $I$  let  $\text{Dom}(D) := I$  and  $S$  is called  $D$ -positive when  $S \subseteq I \wedge (I \setminus S) \notin D$  and  $D^+ = \{S \subseteq \text{Dom}(D) : S \text{ is } D\text{-positive}\}$  and we let  $D + A = \{B \subseteq I : B \cup (I \setminus A) \in D\}$  (so if  $D = \mathcal{D}_\lambda$ , the club filter on the regular uncountable  $\lambda$  then  $D^+$  is the family of stationary subsets of  $X$ ).

2) For  $D$  a filter on a regular uncountable cardinal  $\lambda$  which extends the club filter, let  $\diamond_D$  means: there is  $\bar{f} = \langle f_\alpha : \alpha \in S \rangle$  which is a diamond sequence for  $D$  (or a  $D$ -diamond sequence) which means that  $S \in D^+$  and for every  $g \in {}^\lambda \lambda$  the set  $\{\alpha < \lambda : g \upharpoonright \alpha = f_\alpha\}$  belongs to  $D^+$ ; so  $\bar{f}$  is also a diamond sequence for the filter  $D + S$ , (clearly  $\diamond_S$  means  $\diamond_{\mathcal{D}_\lambda + S}$  for  $S$  a stationary subset of the regular uncountable  $\lambda$ ).

A somewhat more general version of the theorem is

{ya.5}

**Claim 2.5.** 1) Assume  $\lambda = \chi^+ = 2^\chi$  and  $D$  is a  $\lambda$ -complete filter on  $\lambda$  which extends the club filter. If  $S \in D^+$  and  $\delta \in S \Rightarrow \text{cf}(\delta) \neq \text{cf}(\chi)$  then we have  $\diamond_{D+S}$ .

2) We have  $\diamond_D$  when:

- (a)  $\lambda = \lambda^{< \lambda}$
- (b)  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  lists  $\cup \{\alpha \lambda : \alpha < \lambda\}$
- (c)  $S \in D^+$
- (d)  $\bar{u} = \langle u_\alpha : \alpha \in S \rangle$  and  $u_\alpha \subseteq \alpha$  for every  $\alpha \in S$
- (e)  $\chi = \sup\{|u_\alpha|^+ : \alpha < \lambda\} < \lambda$
- (f)  $D$  is a  $\chi^+$ -complete filter on  $\lambda$  extending the club filter
- (g)  $(\forall g \in {}^\lambda \lambda)(\exists \delta \in S)[\delta = \sup\{\alpha \in u_\delta : g \upharpoonright \alpha \in \{f_\beta : \beta \in u_\delta\}\}]$ .

3) Assume  $\lambda = \chi^+ = 2^\chi$  and  $\aleph_0 < \kappa = \text{cf}(\chi) < \chi$ ,  $S \subseteq S_\kappa^\lambda$  is stationary, and  $D$  is a  $\lambda$ -complete filter extending the club filter on  $\lambda$  to which  $S$  belongs. If  $\diamond_D$  fails then  $\text{SNR}(\lambda, S, \kappa)$ .

*Proof.* Proof of 2.3 Part (1) follows from 2.5(1) for  $D$  the filter  $\mathcal{D}_\lambda + S$ . Part (2) follows from 2.5(3) for  $D$  the filter  $\mathcal{D}_\lambda + S$ .  $\square_{2.3}$

*Proof.* Proof of 2.5

Proof of part (1)

Clearly we can assume

$\otimes_0$   $\chi > \aleph_0$  as for  $\chi = \aleph_0$  the statement is empty.

Let

$\otimes_1$   $\langle f_\alpha : \alpha < \lambda \rangle$  list the set  $\{f : f \text{ is a function from } \beta \text{ to } \lambda \text{ for some } \beta < \lambda\}$ .

For each  $\alpha < \lambda$  clearly  $|\alpha| \leq \chi$  so let

$\otimes_{2,\alpha}$   $\langle u_{\alpha,\varepsilon} : \varepsilon < \chi \rangle$  be  $\subseteq$ -increasing continuous with union  $\alpha$  such that  $\varepsilon < \chi \Rightarrow |u_{\alpha,\varepsilon}| \leq \aleph_0 + |\varepsilon| < \chi$ .

For  $g \in {}^\lambda\lambda$  let  $h_g \in {}^\lambda\lambda$  be defined by

$\otimes_{3,g}$   $h_g(\alpha) = \text{Min}\{\beta < \lambda : g \upharpoonright \alpha = f_\beta\}$ .

Let  $\text{cd}$ ,  $\langle \text{cd}_\varepsilon : \varepsilon < \chi \rangle$  be such that

$\otimes_4$  (a)  $\text{cd}$  is a one-to-one function from  ${}^\chi\lambda$  onto  $\lambda$  such that  $\text{cd}(\bar{\alpha}) \geq \sup\{\alpha_\varepsilon : \varepsilon < \chi\}$  (when  $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \chi \rangle$ )  
 (b) for  $\varepsilon < \chi$ ,  $\text{cd}_\varepsilon$  is a function from  $\lambda$  to  $\lambda$  such that  $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \chi \rangle \in {}^\chi\lambda \Rightarrow \text{cd}_\varepsilon(\text{cd}(\bar{\alpha})) = \alpha_\varepsilon$

(they exist as  $\lambda = \lambda^\chi$ , in the present case this holds as  $2^\chi = \chi^+ = \lambda$ ).

Now we let (for  $\beta < \lambda, \varepsilon < \chi$ ):

$\otimes_5$   $f_{\beta,\varepsilon}^1$  be the function from  $\text{Dom}(f_\beta)$  into  $\lambda$  defined by  $f_{\beta,\varepsilon}^1(\alpha) = \text{cd}_\varepsilon(f_\beta(\alpha))$  so  $\text{Dom}(f_{\beta,\varepsilon}^1) = \text{Dom}(f_\beta)$ .

Without loss of generality

$\otimes_6$   $\alpha \in S \Rightarrow \alpha$  is a limit ordinal.

For  $g \in {}^\lambda\lambda$  and  $\varepsilon < \chi$  we let

$$S_g^\varepsilon = \left\{ \delta \in S : \begin{array}{l} \delta = \sup\{\alpha \in u_{\delta,\varepsilon} : \text{for some } \beta \in u_{\delta,\varepsilon} \\ \text{we have } g \upharpoonright \alpha = f_{\beta,\varepsilon}^1\} \end{array} \right\}.$$

Next we shall show

$\otimes_7$  for some  $\varepsilon(*) < \chi$  for every  $g \in {}^\lambda\lambda$  the set  $S_g^{\varepsilon(*)}$  is a  $D$ -positive subset of  $\lambda$ .

Proof of  $\textcircled{7}$  Assume this fails, so for every  $\varepsilon < \chi$  there is  $g_\varepsilon \in {}^\lambda\lambda$  such that  $S_{g_\varepsilon}^\varepsilon$  is not  $D$ -positive and let  $E_\varepsilon$  be a member of  $D$  disjoint to  $S_{g_\varepsilon}^\varepsilon$ . Define  $g \in {}^\lambda\lambda$  by  $g(\alpha) := \text{cd}(\langle g_\varepsilon(\alpha) : \varepsilon < \chi \rangle)$  and let  $h_g \in {}^\lambda\lambda$  be as in  $\textcircled{3}, g$ , i.e.  $h_g(\alpha) = \text{Min}\{\beta : g \upharpoonright \alpha = f_\beta\}$ .

Let  $E_* = \{\delta < \lambda : \delta \text{ is a limit ordinal such that } \alpha < \delta \Rightarrow h_g(\alpha) < \delta\}$ , clearly it is a club of  $\lambda$  hence it belongs to  $D$  and so  $E = \bigcap \{E_\varepsilon : \varepsilon < \chi\} \cap E_*$  belongs to  $D$  as  $D$  is  $\lambda$ -complete and  $\chi + 1 < \lambda$ .

As  $S$  is a  $D$ -positive subset of  $\lambda$  there is  $\delta_* \in E \cap S$ . For each  $\alpha < \delta_*$  as  $\delta_* \in E \subseteq E_*$  clearly  $h_g(\alpha) < \delta_*$  and  $\alpha$  as well as  $h_g(\alpha)$  belong to  $\bigcup \{u_{\delta_*, \varepsilon} : \varepsilon < \chi\} = \delta_*$ , but  $\langle u_{\delta_*, \varepsilon} : \varepsilon < \chi \rangle$  is  $\subseteq$ -increasing hence  $\varepsilon_{\delta_*, \alpha} = \min\{\varepsilon : \alpha \in u_{\delta_*, \varepsilon} \text{ and } h_g(\alpha) \in u_{\delta_*, \varepsilon}\}$  is not just well defined but also  $\varepsilon \in [\varepsilon_{\delta_*, \alpha}, \chi) \Rightarrow \{\alpha, h_g(\alpha)\} \subseteq u_{\delta_*, \varepsilon}$ . As  $\text{cf}(\delta_*) \neq \text{cf}(\chi)$ , by an assumption on  $S$ , it follows that for some  $\varepsilon(*) < \chi$  the set  $B := \{\alpha < \delta_* : \varepsilon_{\delta_*, \alpha} < \varepsilon(*)\}$  is unbounded below  $\delta_*$ .

So

- (a)  $\alpha \in B \Rightarrow \{\alpha, h_g(\alpha)\} \subseteq u_{\delta_*, \varepsilon(*)}$  and
- (b)  $\alpha \in B \Rightarrow g \upharpoonright \alpha = f_{h_g(\alpha)} \Rightarrow \bigwedge_{\varepsilon < \chi} [g_\varepsilon \upharpoonright \alpha = f_{h_g(\alpha), \varepsilon}^1] \Rightarrow g_{\varepsilon(*)} \upharpoonright \alpha = f_{h_g(\alpha), \varepsilon(*)}^1$ .

But  $\delta_* \in E \subseteq E_{\varepsilon(*)}$  hence  $\delta_* \notin S_{g_{\varepsilon(*)}}^{\varepsilon(*)}$  by the choice of  $E_{\varepsilon(*)}$ , but by (a) + (b) and the definition of  $S_{g_{\varepsilon(*)}}^{\varepsilon(*)}$  recalling  $\delta_* \in S$  we have  $\sup(B) = \delta_* \Rightarrow \delta_* \in S_{g_{\varepsilon(*)}}^{\varepsilon(*)}$ , (where  $h_g(\alpha)$  plays the role of  $\beta$  in the definition of  $S_g^\varepsilon$  above), contradiction. So the proof of  $\textcircled{7}$  is finished.

Let  $\chi_* = (|\varepsilon(*)| + \aleph_0)$  hence  $\delta \in S \Rightarrow |u_{\delta, \varepsilon(*)}| \leq \chi_*$  and  $\chi_*^+ < \lambda$  as  $\chi_* < \chi < \lambda$  because  $\aleph_0, \varepsilon(*) < \chi < \lambda$ . Now we apply 2.5(2) which is proved below with  $\lambda, S, D, \chi_*^+, \langle f_{\beta, \varepsilon(*)}^1 : \beta < \lambda \rangle, \langle u_{\delta, \varepsilon(*)} : \delta \in S \rangle$  here standing for  $\lambda, S, D, \chi, \bar{f}, \bar{u}$  there. The conditions there are satisfied hence also the conclusion which says that  $\diamond_D$  holds.  $\square_{2.5(1)}$

*Proof.* Proof of 2.5(2)

Let

- $\boxtimes_1$   $\langle \text{cd}_\varepsilon : \varepsilon < \chi \rangle$  and  $\text{cd}$  be as in  $\textcircled{4}$  in the proof of part (1), possible as we are assuming  $\chi < \lambda = \lambda^{< \lambda}$
- $\boxtimes_2$  for  $\beta < \lambda$  and  $\zeta < \chi$  let  $f_{\beta, \zeta}^2$  be the function with domain  $\text{Dom}(f_\beta)$  such that  $f_{\beta, \zeta}^2(\alpha) = \text{cd}_\zeta(f_\beta(\alpha))$
- $\boxtimes_3$  for  $g \in {}^\lambda\lambda$  define  $h_g \in {}^\lambda\lambda$  as in  $\textcircled{3}$  in the proof of part (1), i.e.  $h_g(\alpha) = \text{Min}\{\beta : g \upharpoonright \alpha = f_\beta\}$ .

If  $2^{< \chi} < \lambda$  our life is easier but we do not assume this. For  $\delta \in S$  let  $\xi_\delta^*$  be a cardinal, and let  $\langle (\alpha_{\delta, \xi}^1, \alpha_{\delta, \xi}^2) : \xi < \xi_\delta^* \rangle$  list the set  $\{(\alpha_1, \alpha_2) \in u_\delta \times u_\delta : \text{Dom}(f_{\alpha_2}) = \alpha_1\}$ , note that  $\xi_\delta^* < \chi$ , recalling  $|u_\delta| < \chi$  by clause (e) of the assumption. We now try to choose  $(\bar{v}_\varepsilon, g_\varepsilon, E_\varepsilon)$  by induction on  $\varepsilon < \chi$ , (note that  $\bar{v}_\varepsilon$  is defined from  $\langle g_\zeta : \zeta < \varepsilon \rangle$  (see clause (e) of  $\boxtimes_4$  below) so we choose just  $(g_\varepsilon, E_\varepsilon)$ ), such that:

- $\boxtimes_4$  (a)  $E_\varepsilon$  is a member of  $D$  and  $\langle E_\zeta : \zeta \leq \varepsilon \rangle$  is  $\subseteq$ -decreasing with  $\zeta$
- (b)  $\bar{v}_\varepsilon = \langle v_\delta^\varepsilon : \delta \in S \cap E'_\varepsilon \rangle$  when  $E'_\varepsilon = \bigcap \{E_\zeta : \zeta < \varepsilon\} \cap \lambda$  so is  $\lambda$  if  $\varepsilon = 0$
- (c)  $\langle v_\delta^\zeta : \zeta \leq \varepsilon \rangle$  is  $\subseteq$ -decreasing with  $\zeta$  for each  $\delta \in S \cap E'_\varepsilon$
- (d)  $g_\varepsilon \in {}^\lambda\lambda$

- (e)  $v_\delta^\varepsilon = \{\xi < \xi_\delta^* : \text{if } \zeta < \varepsilon \text{ then } g_\zeta \upharpoonright \alpha_{\delta,\xi}^1 = f_{\alpha_{\delta,\xi}^2, \zeta}^2\}$   
 (so if  $\varepsilon$  is a limit ordinal then  $v_\delta^\varepsilon = \bigcap_{\zeta < \varepsilon} v_\delta^\zeta$  and  $\varepsilon = 0 \Rightarrow v_\delta^\varepsilon = \xi_\delta^*$ )
- (f) if  $\delta \in E'_\varepsilon \cap S$  then  $v_\delta^{\varepsilon+1} \subsetneq v_\delta^\varepsilon$  or  $\delta > \sup\{\alpha_{\delta,\xi}^1 : \xi \in v_\delta^{\varepsilon+1}\}$ .

Next

$\oplus_1$  we cannot carry the induction, that is for all  $\varepsilon < \chi$ .

Why? Assume toward contradiction that  $\langle (\bar{v}_\varepsilon, g_\varepsilon, E_\varepsilon) : \varepsilon < \chi \rangle$  is well defined. Let  $E := \bigcap \{E_\varepsilon : \varepsilon < \chi\}$ , it is a member of  $D$  as  $D$  is  $\chi^+$ -complete. Define  $g \in {}^\lambda \lambda$  by  $g(\alpha) := \text{cd}(\langle g_\varepsilon(\alpha) : \varepsilon < \chi \rangle)$ . Let  $E_* = \{\delta < \lambda : \delta \text{ a limit ordinal such that } h_g(\alpha) < \delta \text{ and } \delta > \sup(\text{Dom}(f_\alpha) \cup \text{Rang}(f_\alpha)) \text{ for every } \alpha < \delta\}$ , so  $E_*$  is a club of  $\lambda$  hence it belongs to  $D$ . By assumption (g) of the claim the set

$$S_g := \{\delta \in S : \delta = \sup\{\alpha \in u_\delta : (\exists \beta \in u_\delta)(f_\beta = g \upharpoonright \alpha)\}\}$$

is  $D$ -positive, so we can choose  $\delta \in E \cap E_* \cap S_g$ . Hence  $B := \{\alpha \in u_\delta : (\exists \beta \in u_\delta)(f_\beta = g \upharpoonright \alpha)\}$  is an unbounded subset of  $u_\delta$  and let  $h : B \rightarrow u_\delta$  be  $h(\alpha) = \min\{\beta \in u_\delta : f_\beta = g \upharpoonright \alpha\}$ , clearly  $h$  is a function from  $B$  into  $u_\delta$ . Now  $\alpha \in B \wedge \zeta < \chi \Rightarrow f_{h(\alpha)} = g \upharpoonright \alpha \wedge \zeta < \chi \Rightarrow f_{h(\alpha), \zeta}^2 = g_\zeta \upharpoonright \alpha$ , so for  $\alpha \in B$  the pair  $(\alpha, h(\alpha))$  belongs to  $\{(\alpha_{\delta,\xi}^1, \alpha_{\delta,\xi}^2) : \xi \in v_\delta^\varepsilon\}$  for every  $\varepsilon < \chi$ . Hence for any  $\varepsilon < \chi$  we have  $B \subseteq \{\alpha_{\delta,\xi}^1 : \xi \in v_\delta^\varepsilon\}$  so  $\delta = \sup\{\alpha_{\delta,\xi}^1 : \xi \in v_\delta^\varepsilon\}$ .

So for the present  $\delta$ , in clause (f) of  $\boxtimes_4$  the second possibility never occurs.

So clearly  $\langle v_{\delta_*}^\varepsilon : \varepsilon < \chi \rangle$  is strictly  $\subseteq$ -decreasing, i.e. is  $\subseteq$ -decreasing which is impossible as  $|v_{\delta_*}^0| = \xi_{\delta_*}^* < \chi$ . So we have proved  $\oplus_1$  hence we can assume

$\oplus_2$  there is  $\varepsilon < \chi$  such that we have defined our triple for every  $\zeta < \varepsilon$  but we cannot define for  $\varepsilon$ . So we have  $\langle (\bar{v}_\zeta, g_\zeta, E_\zeta) : \zeta < \varepsilon \rangle$ .

As in  $\boxplus_4(e)$ , let

$\odot_1$   $E'_\varepsilon$  be  $\lambda$  if  $\varepsilon = 0$  and  $\bigcap \{E_\zeta : \zeta < \varepsilon\}$  if  $\varepsilon > 0$  and let  $S_* := S \cap E'_\varepsilon$ .

Clearly  $\bar{v}_\varepsilon$  is well defined, see clauses (b),(e) of  $\boxtimes_4$ , and for  $\delta \in S_*$  let  $\mathcal{F}_\delta = \{f_{\alpha_{\delta,\xi}^2, \varepsilon}^2 : \xi \in v_\delta^\varepsilon\}$ , so each member is a function from some  $\alpha \in u_\delta \subseteq \delta$  into some ordinal  $< \delta$ .

Let

$\odot_2$   $S_1^* := \{\delta \in S_* : \text{there are } f', f'' \in \mathcal{F}_\delta \text{ which are incompatible as functions}\}$

$\odot_3$   $S_2^* := \{\delta \in S_* : \delta \notin S_1^* \text{ but the function } \cup \{f : f \in \mathcal{F}_\delta\} \text{ has domain } \neq \delta\}$

$\odot_4$   $S_3^* = S_* \setminus (S_1^* \cup S_2^*)$ .

For  $\delta \in S_3^*$  let  $g_\delta^* = \cup \{f : f \in \mathcal{F}_\delta\}$ , so by the definition of  $\langle S_\ell^* : \ell = 1, 2, 3 \rangle$  clearly  $g_\delta^* \in {}^\delta \delta$ . Now if  $\langle g_\delta^* : \delta \in S_3^* \rangle$  is a diamond sequence for  $D$  then we are done.

So assume that this fails, so for some  $g \in {}^\lambda \lambda$  and member  $E$  of  $D$  we have  $\delta \in S_3^* \cap E \Rightarrow g_\delta^* \neq g \upharpoonright \delta$ . Without loss of generality  $E$  is included in  $E'_\varepsilon$ . But then we could have chosen  $(g, E)$  as  $(g_\varepsilon, E_\varepsilon)$ , recalling  $\bar{v}_\varepsilon$  was already chosen. Easily the triple  $(g_\varepsilon, E_\varepsilon, \bar{v}_\varepsilon)$  is as required in  $\oplus_1$ , contradicting the choice of  $\varepsilon$  in  $\oplus_2$  so we are done proving part (2) of Theorem 2.3 hence also part (1).  $\square_{2.5(2)}$

*Proof.* Proof of part (3)

We use  $\text{cd}, \text{cd}_\varepsilon$  (for  $\varepsilon < \chi$ ),  $\langle \langle u_{\alpha, \varepsilon} : \varepsilon < \chi \rangle : \alpha < \lambda \rangle, \langle f_\alpha : \alpha < \lambda \rangle, \langle f_{\alpha, \varepsilon}^1 : \alpha < \lambda, \varepsilon < \chi \rangle$  and  $S_g^\varepsilon$  for  $\varepsilon < \kappa$  as in the proof of part (1).

Recall  $\kappa$ , a regular uncountable cardinal, is the cofinality of the singular cardinal  $\chi$  and let  $\langle \chi_\gamma : \gamma < \kappa \rangle$  be increasing with limit  $\chi$ . For every  $\gamma < \kappa$  we ask: The  $\gamma$ -Question: Do we have: for every  $g \in {}^\lambda \lambda$ , the following is a  $D$ -positive subset of  $\lambda$ :

$\{\delta \in S : S_\gamma[g] \cap \delta$  is a stationary subset of  $\delta\}$  where  $S_\gamma[g] := \{\zeta < \lambda : \text{cf}(\zeta) \in [\aleph_0, \kappa), \text{sup}(u_{\zeta, \chi_\gamma}) = \zeta$  and for arbitrarily large  $\alpha \in u_{\zeta, \chi_\gamma}$  for some  $\beta \in u_{\zeta, \chi_\gamma}$  and  $\varepsilon < \chi_\gamma$ , we have  $\text{Dom}(f_\beta) = \alpha$  and  $g \upharpoonright \alpha = f_{\beta, \varepsilon}^1\}$ .

Case 1: For some  $\gamma < \kappa$ , the answer is yes.

Choose  $\langle C_\delta : \delta \in S \rangle$  such that  $C_\delta$  is a club of  $\delta$  of order type  $\text{cf}(\delta) = \kappa$ .

For  $\delta \in S \subseteq S_\kappa^\lambda$  let  $u_\delta := \cup\{u_{\alpha, \chi_\gamma} : \alpha \in C_\delta\}$ .

Clearly

$$\boxplus_2 \quad |u_\delta| \leq \kappa + \chi_\gamma < \chi$$

$$\boxplus_3 \quad \text{for every } g \in {}^\lambda \lambda \text{ for } D\text{-positively many } \delta \in S, \text{ we have } \delta = \text{sup}\{\alpha \in u_\delta : g \upharpoonright \alpha \in \{f_{\beta, \varepsilon}^1 : \varepsilon < \chi_\gamma \text{ and } \beta \in u_\delta\}\}.$$

Why  $\boxplus_3$  holds? Given  $g \in {}^\lambda \lambda$ , let  $h_g \in {}^\lambda \lambda$  be defined by  $h_g(\alpha) = \min\{\beta < \lambda : g \upharpoonright \alpha = f_\beta\}$ , so  $h_g(\alpha) \geq \alpha$  (but is less than  $\lambda$ ). Let  $E_g = \{\delta < \lambda : \delta \text{ is a limit ordinal such that } (\forall \alpha < \delta) h_g(\alpha) < \delta\}$ , so  $E_g$  is a club of  $\lambda$  and let  $E'_g$  be the set of accumulation points of  $E_g$ , so  $E'_g$ , too, is a club of  $\lambda$ . By the assumption of this case, the set  $S' := \{\delta \in S : \delta \cap S_\gamma[g] \text{ is a stationary subset of } \lambda\}$  is  $D$ -positive, hence  $S'' := S' \cap E'_g$  is a  $D$ -positive subset of  $\lambda$ . Let  $\delta \in S''$ , by  $E'_g$ 's definition, we can find  $B_\delta^0 \subseteq E_g \cap \delta$  unbounded in  $\delta$ , so without loss of generality  $B_\delta^0$  is closed. But  $S_\gamma[g] \cap \delta$  is a stationary subset of  $\delta$ , recalling  $\delta \in S''$ , so  $B_\delta^1 = B_\delta^0 \cap S_\gamma[g] \cap C_\delta$  is a stationary subset of  $\delta$  as  $B_\delta^0, C_\delta$  are closed unbounded subsets of  $\delta$ .

Clearly  $\zeta \in B_\delta^1 \Rightarrow \zeta \in C_\delta \Rightarrow u_{\zeta, \chi_\gamma} \subseteq u_\delta$  by the definitions of  $B_\delta^1$  and  $u_\delta$ . Also  $\zeta \in B_\delta^1 \Rightarrow \zeta \in S_\gamma[g] \Rightarrow (\zeta \text{ is a limit ordinal}) \wedge \zeta = \text{sup}(u_{\zeta, \chi_\gamma}) = \text{sup}\{\alpha \in u_{\zeta, \chi_\gamma} : (\exists \beta \in u_{\zeta, \chi_\gamma})(\exists \varepsilon < \chi_\gamma)(g \upharpoonright \alpha = f_{\beta, \varepsilon}^1)\} \Rightarrow ((\zeta \text{ is a limit ordinal}) \wedge \zeta = \text{sup}\{\alpha \in u_\delta \cap \zeta : (\exists \beta \in u_\delta \setminus \alpha)(\exists \varepsilon < \chi_\gamma)(g \upharpoonright \alpha = f_{\beta, \varepsilon}^1)\})$ .

As  $B_\delta^1$  is unbounded in  $\delta$  being stationary we are done proving  $\boxplus_3$ .

Now without loss of generality every  $\delta \in S$  is divisible by  $\chi$  hence  $\delta = \chi_\gamma \delta$  and let  $u'_\delta = u_\delta \cup \{\chi_\gamma \alpha + \varepsilon : \alpha \in u_\delta, \varepsilon < \chi_\gamma\}$ , so  $u'_\delta$  is an unbounded subset of  $\delta$ , and let  $f'_\beta = f_{\alpha, \varepsilon}^1$  when  $\beta = \chi_\gamma \alpha + \varepsilon, \varepsilon < \chi_\gamma$ . So translating what we have is:

$$\boxplus_4 \quad (a) \quad \langle f'_\alpha : \alpha < \lambda \rangle \text{ is a sequence of members of } \cup\{\beta^\lambda : \beta < \lambda\}$$

$$(b) \quad \text{for } \delta \in S, u'_\delta \text{ is an unbounded subset of } \delta \text{ of cardinality } \leq \chi_\gamma \times \chi_\gamma = \chi_\gamma (< \chi)$$

$$(c) \quad \text{for every } g \in {}^\lambda \lambda \text{ for } D\text{-positively many } \delta \in S \text{ we have } \delta = \text{sup}\{\alpha \in u'_\delta : (\exists \beta \in u'_\delta)(g \upharpoonright \alpha = f'_\beta)\}.$$

Now we can apply part (2) with  $\langle f'_\alpha : \alpha < \lambda \rangle, \langle u'_\delta : \delta \in S \rangle$  replacing  $\bar{f}, \langle u_\delta : \delta \in S \rangle$ .

So as there we can prove  $\diamond_S$ , hence we are done.

Case 2: For every  $\gamma < \kappa$  the answer is no.



Let  $(g_\gamma, E_\gamma)$  exemplify that the answer for  $\gamma$  is no; so  $g_\gamma \in {}^\lambda\lambda$  and  $E_\gamma \in D$ . Let  $E = \bigcap_{\gamma < \kappa} E_\gamma$ , so  $E$  is a member of  $D$ . Let  $g \in {}^\lambda\lambda$  be defined by  $g(\alpha) = \text{cd}(\langle g_\gamma(\alpha) : \gamma < \kappa \rangle^{\wedge}(0)_\chi)$ , i.e.  $\text{cd}_\varepsilon(g(\alpha))$  is  $g_\gamma(\alpha)$  if  $\gamma < \kappa$  and is 0 if  $\varepsilon \in [\kappa, \chi)$ .

Let

$$E_g := \{\delta < \lambda : \delta \text{ a limit ordinal such that if } \alpha < \lambda \text{ then } h_g(\alpha) < \delta \text{ and } \delta > \sup(\text{Dom}(f_\alpha) \cup \text{Rang}(f_\alpha))\}.$$

We now define  $h : \lambda \rightarrow \kappa$  as follows

$\boxplus_5$  for  $\beta < \lambda$

(a) if  $\text{cf}(\beta) \notin [\aleph_0, \kappa)$  or  $\beta \notin E_g$  then  $h(\beta) = 0$

(b) otherwise

$$h(\beta) = \min\{\gamma < \kappa : \beta = \sup\{\alpha_1 \in u_{\beta, \chi_\gamma} : \text{for some } \alpha_2 \in u_{\beta, \chi_\gamma} \text{ and } \varepsilon < \chi_\gamma \text{ we have } g \upharpoonright \alpha_1 = f_{\alpha_2, \varepsilon}^1\}\}.$$

Now

$\boxplus_6$   $h : \lambda \rightarrow \kappa$  is well defined.

Why  $\boxplus_6$  holds? Let  $\beta < \lambda$ . If  $\text{cf}(\beta) \notin [\aleph_0, \kappa)$  or  $\beta \notin E_g$  then  $h(\beta) = 0 < \kappa$  by clause (a) of  $\boxplus_5$ . So assume  $\text{cf}(\beta) \in [\aleph_0, \kappa)$  and  $\beta \in E_g$ . Let  $\langle \gamma_{\beta, \varepsilon}^1 : \varepsilon < \text{cf}(\beta) \rangle$  be increasing with limit  $\beta$  and let  $\gamma_{\beta, \varepsilon}^2 = \min\{\gamma : g \upharpoonright \gamma_{\beta, \varepsilon}^1 = f_\gamma\}$ , so  $\varepsilon < \text{cf}(\beta) \Rightarrow \gamma_{\beta, \varepsilon}^2 < \beta$  as  $\beta \in E_g$ . But  $\langle u_{\beta, \chi_\zeta} : \zeta < \text{cf}(\chi) \rangle$  is  $\subseteq$ -increasing with union  $\beta$  so for each  $\varepsilon < \text{cf}(\beta)$  there is  $\zeta = \zeta_{\beta, \varepsilon} < \text{cf}(\chi)$  such that  $\{\gamma_{\beta, \varepsilon}^1, \gamma_{\beta, \varepsilon}^2\} \subseteq u_{\beta, \chi_\zeta}$ . As  $\text{cf}(\beta) < \kappa = \text{cf}(\chi)$  for some  $\zeta < \kappa$  the set  $\{\varepsilon < \text{cf}(\beta) : \zeta_{\beta, \varepsilon} < \zeta\}$  is unbounded in  $\text{cf}(\beta)$ . So  $\zeta$  can serve as  $\gamma$  in clause (b) of  $\boxplus_5$  so  $h(\beta)$  is well defined, in particular is less than  $\kappa$  so we have proved  $\boxplus_6$ .

$\boxplus_7$  if  $\delta \in S \cap E_\gamma$  then for some club  $C$  of  $\delta$  the function  $h \upharpoonright C$  is increasing.

Why  $\boxplus_7$  holds? If not, then by Fodor's lemma for some  $\gamma < \kappa$  the set  $\{\delta' \in \delta \cap S : h(\delta') \leq \gamma\}$  is a stationary subset of  $\delta$ , and we get contradiction to the choice of  $E_\gamma$  so  $\boxplus_7$  holds indeed.

So  $h$  is as promised in the claim.  $\square$

Note

{ya.12}

**Observation 2.6.** If  $\kappa_* < \lambda$  are regular,  $S_{\kappa_*}^\lambda$  strongly does not reflect in  $\lambda$  for every  $\kappa \in \text{Reg} \cap \kappa_*$  and  $\Pi(\text{Reg} \cap \kappa_*) < \lambda$ , then :

(a)  $S_{< \kappa_*}^\lambda$  can be divided to  $\leq \Pi(\text{Reg} \cap \kappa_*)$  sets, each not reflecting in any  $\delta \in S_{< \kappa_*}^\lambda$  in particular

(b)  $S_{\aleph_0}^\lambda$  can be divided to  $\leq \Pi(\text{Reg} \cap \kappa_*)$  sets each not reflecting in any  $\delta \in S_{< \kappa_*}^\lambda$ .

{ya.14}

*Remark 2.7.* 1) Of course if  $\lambda$  has  $\kappa$ -SNR then this holds for every regular  $\lambda' \in (\kappa, \lambda)$ .

2) We may state the results, using  $\lambda_\kappa^*$  (see below).

**Definition 2.8.** For each regular  $\kappa$  let  $\lambda_\kappa^* = \text{Min}\{\lambda : \lambda \text{ regular fails to have } \kappa\text{-SNR}\}$ , and let  $\lambda_\kappa^*$  be  $\infty$  (or not defined) if there is no such  $\lambda$ . {ya.16}

3. CONSISTENT FAILURE ON  $S_1^2$ 

A known question was:

**Question 3.1.** For  $\theta \in \{\aleph_0, \aleph_1\}$  do we have  $(2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 \Rightarrow \diamond_{S_\theta^{\aleph_2}})$ ? {2b.1}

So for  $\theta = \aleph_0$  the answer is yes (by 2.3(1)), but what about  $\theta = \aleph_1$ ? We noted some years ago that easily:

**Claim 3.2.** Assume  $\mathbf{V} \models \text{GCH}$  or even just  $2^{\aleph_\ell} = \aleph_{\ell+1}$  for  $\ell = 0, 1, 2$ . Then some forcing notion  $\mathbb{P}$  satisfies {2b.4}

- (a)  $\mathbb{P}$  is of cardinality  $\aleph_3$
- (b) forcing with  $\mathbb{P}$  preserves cardinals and cofinalities
- (c) in  $\mathbf{V}^{\mathbb{P}}$ ,  $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ ,  $2^{\aleph_2} = \aleph_3$
- (d) in  $\mathbf{V}^{\mathbb{P}}$ ,  $\diamond_S$  fails where  $S = \{\delta < \aleph_2 : \text{cf}(\delta) = \aleph_1\}$ , moreover
  - (\*) there is a sequence  $\bar{A} = \langle A_\delta : \delta \in S \rangle$  where  $A_\delta$  an unbounded subset of  $\delta$  of order type  $\omega_1$  satisfying
  - (\*\*) if  $\bar{f} = \langle f_\delta : \delta \in S \rangle$ ,  $f_\delta \in {}^{(A_\delta)}(\omega_1)$ , then there is  $f \in {}^{(\omega_2)}(\omega_1)$  such that  $\delta \in S \Rightarrow \delta > \sup(\{\alpha \in A_\delta : f(\alpha) \leq f_\delta(\alpha)\})$ .

*Remark 3.3.* Similarly for other cardinals.

*Proof.* There is an  $\aleph_1$ -complete  $\aleph_3$ -c.c. forcing notion  $\mathbb{P}$  not collapsing cardinals, not changing cofinalities, preserving  $2^{\aleph_\ell} = \aleph_\ell$  for  $\ell = 0, 1, 2$  and  $|\mathbb{P}| = \aleph_3$  such that in  $\mathbf{V}^{\mathbb{P}}$ , we have (\*), in fact more<sup>1</sup> than (\*) holds - see [Sh:587]. Let  $\mathbb{Q}$  be the forcing of adding  $\aleph_2$  Cohen or just any c.c.c. forcing notion of cardinality  $\aleph_2$  adding  $\aleph_2$  reals (can be  $\mathbb{Q}$ , a  $\mathbb{P}$ -name). Now we shall show that  $\mathbb{P} * \mathbb{Q}$ , equivalently  $\mathbb{P} \times \mathbb{Q}$  is as required:

Clause (a):

$|\mathbb{P} * \mathbb{Q}| = \aleph_3$ ; trivial.

Clause (b):

Preserving cardinals and cofinalities; obvious as both  $\mathbb{P}$  and  $\mathbb{Q}$  do this.

Clause (c): Easy.

Clause (d): In  $\mathbf{V}^{\mathbb{P}}$  we have (\*) as exemplified by say  $\bar{A} = \langle A_\delta : \delta \in S \rangle$ . We shall show that  $\mathbf{V}^{\mathbb{P} * \mathbb{Q}} \models \text{“}\bar{A} \text{ satisfies } (**)\text{”}$ . Otherwise in  $\mathbf{V}^{\mathbb{P} * \mathbb{Q}}$  we have  $\bar{f} = \langle f_\delta : \delta \in S \rangle$  say in  $\mathbf{V}[G_{\mathbb{P}}, G_{\mathbb{Q}}]$  a counterexample then in  $\mathbf{V}[G_{\mathbb{P}}]$  for some  $q \in \mathbb{Q}$  and  $\bar{f}$  we have

$$\mathbf{V}[G_{\mathbb{P}}] \models (q \Vdash_{\mathbb{Q}} \text{“}\bar{f} = \langle f_\delta : \delta \in S \rangle \text{ where } f_\delta : A_\delta \rightarrow \omega_1 \text{ for each } \delta \in S \text{ form a counterexample to } (**)\text{”}).$$

Now in  $\mathbf{V}[G_{\mathbb{P}}]$  we can define  $\bar{g} = \langle g_\delta^1 : \delta \in S \rangle \in \mathbf{V}[G_{\mathbb{P}}]$  where  $g_\delta^1$  a function with domain  $A_\delta$ , by

$$g_\delta^1(\alpha) = \{i : q \Vdash_{\mathbb{Q}} f_\delta(\alpha) \neq i\}.$$

<sup>1</sup>I.e. there is  $\bar{A} = \langle A_\delta : \delta \in S \rangle$  where  $A_\delta$  is an unbounded subset of  $\delta$  of order type  $\omega_1$  satisfying:

⊕ if  $\bar{f} = \langle f_\delta : \delta \in S \rangle$ ,  $f_\delta \in {}^{(A_\delta)}\omega_1$  then there is  $f \in {}^{(\omega_2)}\omega_1$  such that for every  $\delta \in S_{\aleph_1}^{\aleph_2}$  for every  $\alpha \in A_\delta$  large enough we have  $f(\alpha) = f_\delta(\alpha)$ .

So in  $\mathbf{V}[G_{\mathbb{P}}]$  we have  $q \Vdash_{\mathbb{Q}} \text{“} \bigwedge_{\delta \in S} (\forall \alpha \in A_{\delta}) f_{\delta}(\alpha) \in g_{\delta}^1(\alpha) \text{”}$ . Also  $g_{\delta}^1(\alpha)$  is a countable

subset of  $\omega_1$  as  $\mathbb{Q}$  satisfies the c.c.c.

For  $\delta \in S$  we define a function  $g_{\delta} : A_{\delta} \rightarrow \omega_1$  by letting  $g_{\delta}(\alpha) = (\sup(g_{\delta}^1(\alpha)) + 1)$  hence  $g_{\delta}(\alpha) < \omega_1$  so  $\langle g_{\delta} : \delta \in S \rangle$  is as required on  $\bar{f}$  in  $(**)$  in  $\mathbf{V}[G_{\mathbb{P}}]$ , of course. Apply clause  $(**)$  in  $\mathbf{V}[G_{\mathbb{P}}]$  to  $\langle g_{\delta} : \delta \in S \rangle$  so we can find  $g : \omega_2 \rightarrow \omega_1$  such that  $\bigwedge_{\delta \in S} \delta > \sup\{\alpha \in A_{\delta}, g_{\delta}(\alpha) > g(\alpha)\}$ . Now  $g$  is as required also in  $\mathbf{V}[G_{\mathbb{P}}][G_{\mathbb{Q}}]$ .  $\square_{3.2}$

{2b.7} We may wonder can we strengthen the conclusion of 2.3 to  $\diamond_S^*$  (of course the demand in clause (e) and (f) in claim 3.4 below are necessary, i.e. otherwise  $\diamond_S^*$  holds). The answer is not as: (the restriction in (e) and in (f) are best possible).

**Observation 3.4.** Assume  $\lambda = \lambda^{<\lambda}$ ,  $S \subseteq S_{\kappa}^{\lambda}$ .

Then for some  $\mathbb{P}$

- (a)  $\mathbb{P}$  is a forcing notion
- (b)  $\mathbb{P}$  is of cardinality  $\lambda^+$  satisfying the  $\lambda^+$ -c.c.
- (c) forcing with  $\mathbb{P}$  does not collapse cardinals and does not change cofinality
- (d) forcing with  $\mathbb{P}$  adds no new  $\eta \in \lambda^>\text{Ord}$
- (e)  $\diamond_S^*$  fails for every stationary subset  $S$  of  $\lambda$  such that
  - ( $\alpha$ )  $S \subseteq S_{\kappa}^{\lambda}$  when  $(\exists \mu < \lambda)[\mu^{<\kappa}>_{\text{tr}} = \lambda]$   
or just
  - ( $\beta$ )  $\alpha \in S \Rightarrow |\alpha|^{<\text{cf}(\alpha)>_{\text{tr}}} > |\alpha|$
- (f)  $(D\ell)_S$ , see below, fails for every  $S \subseteq S_{\kappa}^{\lambda}$  when  $\alpha \in S \Rightarrow |\alpha|^{<\text{cf}(\alpha)>_{\text{tr}}} = \lambda$ .

{2b.9} Recalling

**Definition 3.5.** 1) For  $\mu \geq \kappa = \text{cf}(\kappa)$  let  $\mu^{<\kappa}>_{\text{tr}} = \{|\mathcal{T}| : \mathcal{T} \subseteq \kappa^{\geq \mu} \text{ is closed under initial segments (i.e. a subtree) such that } |\mathcal{T} \cap \kappa^{> \mu}| \leq \mu\}$ .

2) For  $\lambda$  regular uncountable and stationary  $S \subseteq \lambda$  let  $(D\ell)_S$  mean that there is a sequence  $\bar{\mathcal{P}} = \langle \mathcal{P}_{\delta} : \delta \in S \rangle$  witnessing it which means:

- (\*) $_{\bar{\mathcal{P}}}$  (a)  $\mathcal{P}_{\delta} \subseteq {}^{\delta}\delta$  has cardinality  $< \lambda$
- (b) for every  $f \in {}^{\lambda}\lambda$  the set  $\{\delta \in S : f \upharpoonright \delta \in \mathcal{P}_{\delta}\}$  is stationary

(for  $\lambda$  successor it is equivalent to  $\diamond_S$ ; for  $\lambda$  strong inaccessible it is trivial).

*Proof.* Proof of 3.4 Use  $\mathbb{P} =$  adding  $\lambda^+$ ,  $\lambda$ -Cohen subsets.

The proof is straight.  $\square_{3.4}$

{2b.11}

*Remark 3.6.* The consistency results in 3.4 are best possible, see [Sh:829].

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