MAXIMAL FAILURES OF SEQUENCE LOCALITY IN A.E.C. SH932

SAHARON SHELAH

ABSTRACT. We are interested in examples of a.e.c. with amalgamation having some (extreme) behaviour concerning types. Note we deal with $\mathfrak{k}$ being sequence-local, i.e. local for increasing chains of length a regular cardinal (for types, equality of all restrictions imply equality). For any cardinal $\theta \geq \aleph_0$ we construct an a.e.c. with amalgamation $\mathfrak{k}$ with $\mathrm{LST}(\mathfrak{k}) = \theta, |\tau_{\mathfrak{K}}| = \theta$ such that $\{\kappa : \kappa$ is a regular cardinal and $\mathfrak{K}$ is not $(2^\kappa, \kappa)$-sequence-local$\}$ is maximal. In fact we have a direct characterization of this class of cardinals: the regular κ such that there is no uniform κ^+-complete ultrafilter (on any $\lambda > \kappa$). We also prove a similar result to "$(2^\kappa, \kappa)$-compact for types".

Date: December 1, 2017.
1991 *Mathematics Subject Classification.* Primary: 03C48,03E17; Secondary: 03E05, 03E50.
Key words and phrases. model theory, abstract elementary classes, types, locality.
The author thanks Alice Leonhardt for the beautiful typing. First typed October 20, 2007.

0. Introduction

Recall a.e.c. (abstract elementary classes); were introduced in [Sh:88]; and their (orbital) types defined in [Sh:300], see on them [Sh:h], [Bal09]. It has seemed to me obvious that even with $\mathfrak{k}$ having amalgamation, those types in general lack the good properties of the classical types in model theory. E.g. "(λ,κ)-sequence-locality where

{z1}

Definition 0.1. 1) We say that an a.e.c. $\mathfrak{k}$ is a (λ,κ)-sequence-local (for types) when κ is regular and for every $\le_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i \le \kappa\rangle$ of models of cardinality λ and $p,q \in \mathscr{S}(M_\kappa)$ we have $(\forall i<\kappa)(p \restriction M_i = q \restriction M_i) \Rightarrow p=q$. We omit λ when we omit "$\|M_i\|=\lambda$".
2) We say an a.e.c. $\mathfrak{k}$ is (λ,κ)-local when: $\kappa \ge$ LST$(\mathfrak{k})$ and if $M \in \mathfrak{k}_\lambda$ and $p_1,p_2 \in \mathscr{S}(M)$ and $N \le_{\mathfrak{k}} M \wedge \|N\| \le \kappa \Rightarrow p_1{\restriction}N = p_2{\restriction}N$ then $p_1=p_2$.
3) We may replace λ by $\le\lambda, <\lambda, [\mu,\lambda]$ with the obvious meaning (and allow λ to be infinity).

Of course, being sure is not a substitute for a proof, some examples were provided by Baldwin-Shelah [BlSh:862, §2]. There we give an example of the failure of (λ,κ)-sequence-locality for $\mathfrak{k}$-types in ZFC for some λ,κ, actually $\kappa=\aleph_0$. This was done by translating our problems to abelian group problems. While those problems seem reasonable by themselves they may hide our real problem.

Here in §1 we get $\mathfrak{k}$, an a.e.c. with amalgamation with the class $\{\kappa : (<\infty,\kappa)$-sequence-localness fail for $\mathfrak{k}\}$ being maximal; what seems to me a major missing point up to it, see Theorem 1.3. Also we deal with "compactness of types" getting unsatisfactory results - classes without amalgamation; in [BlSh:862] this was done only in some universes of set theory but with amalgamation; see §2.

We relay on [BlSh:862] to get that $\mathfrak{k}$ has the JEP and amalgamation.

{z2}

Question 0.2. Can $\{\kappa : \mathfrak{k}$ is $(<\infty,\kappa)$-local$\}$ be "wild"? E.g. can it be all odd regular alephs? etc?

Note that for this the present translation theorem of [BlSh:862] is not suitable.

In §2 we deal with sequence-compactness of types.

We thank Will Boney for a correction.

1. An a.e.c. with maximal failure of being local

{e1}

Claim 1.1. *Assume*

$\circledast_1$ (a) $\kappa = \text{cf}(\kappa) > \theta \geq \aleph_0$ *or just* $\kappa = \text{cf}(\kappa) \geq \aleph_0, \theta \geq \aleph_0$

(b) *there is no uniform θ^+-complete ultra-filter D on κ*

(c) *τ_θ is a vocabulary of cardinality θ consisting of θ n-place predicates with each n (and no more say $\{R_{\gamma,n} : \gamma < \theta, n < \omega\}, n = \text{arity}(R_{\gamma,n})$).*

Then

$\boxplus$ *there are $I_\alpha, M_{\ell,\alpha}, \pi_{\ell,\alpha}$ (for $\ell = 1,2$ and $\alpha \leq \kappa$), g_α (for $\alpha < \kappa$) satisfying:*

(a) *I_α, a set of cardinality θ^κ, is $\subseteq$-increasing continuous with α*

(b) *$M_{\ell,\alpha}$, a τ_θ-model of cardinality $\leq \theta^\kappa$, is increasing continuous with α*

(c) *$\pi_{\ell,\alpha}$ is a function from $M_{\ell,\alpha}$ onto I_α, increasing continuous with α*

(d) $|\pi^{-1}_{\ell,\alpha}\{t\}| \leq \theta^{\aleph_0}$ *for $t \in I_\alpha, \alpha \leq \kappa$ and $\ell = 1,2$*

(e) *if $t \in I_{\alpha+1} \backslash I_\alpha$ then $\pi^{-1}_{\ell,\alpha}\{t\} \subseteq M_{\ell,\alpha+1} \backslash M_{\ell,\alpha}$*

(f) *for $\alpha < \kappa, g_\alpha$ is an isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$ respecting $(\pi_{1,2}, \pi_{2,\kappa})$ which means $a \in M_{1,\alpha} \Rightarrow \pi_{1,\alpha}(a) = \pi_{2,\alpha}(g_\alpha(a))$*

(g) *for $\alpha = \kappa$ there is no isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$ respecting $(\pi_{1,\alpha}, \pi_{2,\alpha})$.*

Proof. Follows from 1.2 which is just a fuller version adding to τ_θ unary function F_c for $c \in G$; this is just a notational change when $\theta^{\aleph_0} = \theta$. Otherwise see $(*)_{12}$ of the proof of 1.2; anyhow we shall use 1.2. $\square_{1.1}$

{e2}

Claim 1.2. *Assuming $\circledast_1$ of 1.1 we have:*

$\boxplus$ *there are $I_\alpha, M_{\ell,\alpha}, \pi_{\ell,\alpha}$ (for $\ell = 1,2, \alpha \leq \kappa$) and g_α (for $\alpha < \kappa$) and G such that:*

(a) *G is an additive (so abelian) group of cardinality $\theta^{\aleph_0}$*

(b) *I_α is a set, increasing continuous with $\alpha, |I_\alpha| = \theta^\kappa$*

(c) *$M_{\ell,\alpha}$ is a τ^+_θ-model, increasing continuous with α, of cardinality θ^κ where $\tau^+_\theta = \tau_\theta \cup \{F_c : c \in G\}, F_c$ a unary function symbol, τ_θ is from $\circledast_1(c)$ of 1.1*

(d) *$\pi_{\ell,\alpha}$ is a function from $M_{\ell,\alpha}$ onto I_α increasing continuous with α*

(e) *$F^{M_{\ell,\alpha}}_c (c \in G)$ is a permutation of $M_{\ell,\alpha}$, increasing continuous with α*

(f) $\pi_{\ell,\alpha}(a) = \pi_{\ell,\alpha}(F^{M_{\ell,\alpha}}_c(a))$

(g) $F^{M_{\ell,\alpha}}_{c_1}(F^{M_{\ell,\alpha}}_{c_2}(a)) = F^{M_{\ell,\alpha}}_{c_1+c_1}(a)$

(h) $\pi_{\ell,\alpha}(a) = \pi_{\ell,\alpha}(b) \Leftrightarrow \bigvee\limits_{c \in G} F^{M_{\ell,\alpha}}_c(a) = b$

(i) *for $\alpha < \kappa, g_\alpha$ is an isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$ which respects $(\pi_{1,\alpha}, \pi_{2,\alpha})$ which means $a \in M_{1,\alpha} \Rightarrow \pi_{1,\alpha}(a) = \pi_{2,\alpha}(f_\alpha(a))$*

(j) *there is no isomorphism from $M_{1,\kappa} \restriction \tau_\theta$ onto $M_{2,\kappa} \restriction \tau_\theta$ respecting $(\pi_{1,\kappa}, \pi_{2,\kappa})$.*

Proof. Let

$(*)_0$ $\sigma = \theta^{\aleph_0}$ so $\sigma = \sigma^{\aleph_0}$

$(*)_1$ (a) let $G = ([\sigma]^{<\aleph_0}, \Delta)$, i.e., the family of finite subsets of σ with the operation of symmetric difference. This is an abelian group satisfying $\forall x(x + x = 0)$

(b) let $\langle a_{f,\alpha,u} : f \in {}^\kappa\sigma, \alpha < \kappa, u \in G\rangle$ be a sequence without repetitions

(c) for $\beta \le \kappa$ let $A_\beta = \{a_{f,\alpha,u} : f \in {}^\kappa\sigma, \alpha < 1 + \beta$ and $u \in G\}$

(d) for $\beta \le \kappa$ let $I_\beta = ({}^\kappa\sigma) \times (1 + \beta)$

(e) let $\pi_\beta(a_{f,\alpha,u}) = (f, \alpha)$ when $\alpha < 1 + \beta \le \kappa$

(f) for each $\beta < \kappa$ we define a permutation g_β (of order 2) of A_β by $g_\beta(a_{f,\alpha,u}) = a_{f,\alpha,u+_G\{f(\beta)\}}$ hence $a \in A_\beta \Rightarrow \pi_\beta(g_\beta(a)) = \pi_\beta(a)$.

Note that

$(*)_2$ (a) $|G| = \sigma$

(b) $\langle A_\beta : \beta \le \kappa\rangle$ is a $\subseteq$-increasing continuous, each A_β a set of cardinality $\sigma^\kappa = \theta^\kappa$

(c) $\langle I_\beta : \beta \le \kappa\rangle$ is $\subseteq$-increasing continuous, each I_β of cardinality $\sigma^\kappa = \theta^\kappa$

(d) π_β is a mapping from A_β onto I_β

(e) if $t \in I_\alpha \subseteq I_\beta$ then $\pi_\beta^{-1}\{t\} = \pi_\alpha^{-1}\{t\}$ has cardinality $|G| = \sigma$

(f) if $t \in I_{\alpha+1} \backslash I_\alpha$ then $\pi_{\alpha+1}^{-1}\{t\} \subseteq A_{\alpha+1} \backslash A_\alpha$

(g) if $\alpha \le \beta \le \kappa$ then g_β maps A_α onto itself and $g_\beta \circ g_\beta$ is the identity.

For each $n < \omega$ and $\beta \le \kappa$ we define equivalence relations $E'_{n,\beta}, E_{n,\beta}$ on ${}^n(A_\beta)$:

$(*)_3$ $\bar a E'_{n,\beta} \bar b$ iff $\pi_\beta(\bar a) = \pi_\beta(\bar b)$ where of course $\pi_\beta(\langle a_\ell : \ell < n\rangle) = \langle \pi_\beta(a_\ell) : \ell < n\rangle$

$(*)_4$ $\bar a E_{n,\beta} \bar b$ <u>iff</u> $\bar a E'_{n,\beta} \bar b$ and there are $k < \omega$ and $\bar a_0, \dots, \bar a_k$ such that

(i) $\bar a_\ell \in {}^n(A_\beta)$

(ii) $\bar a = \bar a_0$

(iii) $\bar b = \bar a_k$

(iv) for each $\ell < k$ for some $\alpha_1, \alpha_2 < \kappa$ we have $g_{\alpha_2}^{-1}(g_{\alpha_1}(\bar a_\ell))$ is well defined and equal to $\bar a_{\ell+1}$ <u>or</u> $g_{\alpha_2}(g_{\alpha_1}^{-1}(\bar a_\ell))$ is well defined and equal to $\bar a_{\ell+1}$.

Note:

$(*)_{4.1}$ (a) the two possibilities in $(*)_4(iv)$ are one as $g_\alpha^{-1} = g_\alpha$ so the first one is a special case of the second;

(b) g_α <u>does not preserve</u> $\bar a/E_{n,\beta}$!, in fact, $a, g_\alpha(a)$ are never $E_{n,\beta}$ equivalent;

(c) clearly they are well defined <u>iff</u> $(\forall \ell \le k)[\bar a_\ell \in {}^n(A_{\min\{\alpha_1,\alpha_2\}})]$ because if $\alpha \le \beta$ then g_β maps A_β onto itself because $g(a_{f_1,\alpha,u_2}) = a_{f_1,\alpha,u_2} \Rightarrow |u_1| + 1 = |u_2| \mod 2$

(d) if $\alpha \le \beta, a \in A_\alpha$, then g_β maps $a/E_{n,\beta}$ onto itself

(e) if $\alpha, \beta \le \kappa$, <u>then</u> g_α, g_β commute (on the intersection of their domains, $A_{\min\{\alpha,\beta\}}$.

[Why? E.g. for clause (b) note clause (d).]

Note

$(*)_5$ (a) $E'_{n,\beta}, E_{n,\beta}$ are indeed equivalence relations on ${}^n(A_\beta)$

(b) $E_{\beta,n}$ refine $E'_{\beta,n}$

(c) if $n < \omega, \bar{a} \in {}^n(A_\beta)$ then $\bar{a}/E'_{n,\beta}$ has at most σ members (really exactly two but we shall use only its having $\le 2^\sigma$ members)

(d) if $\alpha < \beta \le \kappa$ then $E'_{n,\beta} \restriction {}^n(A_\alpha) = E'_{n,\alpha}$ and $E_{n,\beta} \restriction {}^n(A_\alpha) = E_{n,\alpha}$ (read $(*)_4(iv)$ carefully!)

(e) if $\alpha < \beta \le \kappa, \bar{a} \in {}^n(A_\alpha)$ and $\bar{b} \in \bar{a}/E'_{n,\beta}$ then $\bar{b} \in {}^n(A_\alpha)$

(f) if $g_\alpha(\bar{a}_\ell) = \bar{b}_\ell$ for $\ell = 1, 2$ then: $\bar{a}_1 E'_{n,\beta} \bar{a}_2$ iff $\bar{b}_1 E'_{n,\beta} \bar{b}_2$.

Now we choose a vocabulary τ^*_θ of cardinality 2^σ (but see $(*)_{12}$) and for $\alpha \le \kappa$ we choose a τ^*_θ-model $M_{1,\alpha}$ such that:

$(*)_6$ (a) $M_{1,\alpha}$ increasing with α with universe A_α

(b) assume that $\bar{a}, \bar{b}$ are $E'_{n,\alpha}$-equivalent (so $\bar{a}, \bar{b} \in {}^n(A_\alpha)$ and $\pi_\alpha(\bar{a}) = \pi_\alpha(\bar{b})$); then $\bar{a}, \bar{b}$ realize the same quantifier free type in $M_{1,\alpha}$ iff $\bar{a} E_{n,\alpha} \bar{b}$

(c) $\tau^*_\theta = \{E_n : n < \omega\} \cup \{F_c : c \in G\} \cup \{R_e : e \in {}^\sigma\sigma\} \cup \{R_{n,i} : i < \sigma, n < \omega\}$ where E_n, R_e are two-place predicates, F_c a unary function symbol, $R_{n,i}$ is n-place predicate

(d) for every function $e \in {}^\sigma\sigma$

$$\begin{aligned} R_e^{M_{1,\alpha}} = \{(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \in A_\alpha \times A_\alpha : {} & f_1 = e \circ f_2 \text{ and} \\ & \text{if } i < \sigma \text{ then } i \in u_1 \\ & \text{iff } (|\{j \in u_2 : e(j) = i\}| \text{ is odd})\} \end{aligned}$$

recalling $f_\ell \in {}^\kappa\sigma$

(e) $E_n^{M_{1,\alpha}} = E_{n,\alpha}$ and $F_c^{M_{1,\alpha}} = F_c$ is defined by $F_\ell : A_\alpha \to A_\alpha$ satisfies $F_c(a_{f,\alpha,u}) := a_{f,\alpha,u+_G c}$

(f) if $\alpha \le \beta_\ell < \kappa$ for $\ell = 1, 2$ then $g^{-1}_{\beta_2} g_{\beta_1} \restriction A_\alpha$ is an automorphism of $M_{1,\alpha}$.

[Why is this possible? First, we shall show that for each $\alpha < \kappa, g_\alpha$ maps $R_e^{M_{1,\alpha}}$ onto itself.

Assume we are given a pair $(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2})$ from $A_\alpha \times A_\alpha$ so $\beta_1, \beta_2 < 1 + \alpha$ and $f_1 = e \circ f_2$ so

$(*)_{6.1}$ $(a_{f_1,\beta_1,u_1}, a_{f_2,\beta_2,u_2}) \in R_e^{M_{1,\alpha}}$ iff $u_1 = \{e(j) : j \in u_2$ and $(\exists^{\rm odd}\iota \in u_2)(e(\iota) = e(j))$.

[Why? Read $(*)_6$ carefully, in particular note that if $i \notin \{e(j) : j \in u_2\}$ then $i \notin u_1$.]

$(*)_{6.2}$ $(g_\alpha(a_{f_1,\beta_1,u_1}), g_\alpha(a_{f_2,\beta_2,u_2})) \in R_e^{M_{1,\alpha}}$ iff
$(a_{f_1,\beta_1,u_1+\{f_1(\alpha)\}}, a_{f_2,\beta_2,u_2+_G\{f_2(\alpha)\}}) \in R_e^{M_{1,\alpha}}$ iff
$u_1 +_G \{f_1(\alpha)\} = \{e(j) : j \in u_2 +_G \{f_2(\alpha)\}$ and $(\exists^{\rm odd}\iota \in (u_2 +_G f_2(\alpha))(e(\iota) = e(j))\}$.

[Why? Inside $(*)_{6.2}$ the first "iff" holds by the definition of g_α, the second "iff" holds as in $(*)_{6.1}$.]

But $f_1 = e \circ f_2$ hence

- $(*)_{6.3}$ $f_1(\alpha) = e(f_2(\alpha))$
- $(*)_{6.4}$ letting $x = f_2(\alpha) < \sigma$ we have $u_1 = \{e(j) : j \in u_2$ and $(\exists^{\rm odd}\iota \in u_2)(e(\iota) = e(j))\}$ <u>iff</u> $u_1 +_G \{e(x)\} = \{e(j) : j \in u_2 +_G \{x\}$ and $\exists^{\rm odd}\iota \in (u_2 +_G \{x\})(e(\iota) = e(j))\}$.

[Why? Check by cases according to whether $x \in u_2$ and whether $e(x) \in u_1$. I.e. by "G is of order two" it suffices to prove the "only if" so assume the first equality in $(*)_{6.2}$. If $e(x) \notin u_1$, then just add $e(x)$ to both sides. Similarly if $e(x) \in u_1 \cap X \notin u_2$ and if $e(x) \in u_1 \wedge x \in u_2$.]

So together we get equivalence, so the "first" holds.

Second, for defining the $R^{M_{1,\alpha}}_{n,i}$'s

$(*)_{6.5}$
- (a) for each N let E''_n be the following equivalence relation on ${}^nG : \bar{u}_1 E''_n \bar{u}_2$ <u>iff</u> for some $v \in G, |v|$ is even and $\bigwedge\limits_{\ell<n} u_{1,\ell} +_G v = u_{2,\ell}$
- (b) let $\langle \mathcal{X}_{n,i} : i < \sigma\rangle$ list the E''_n-equivalence classes
- (c) let $R^{M_{1,\alpha}}_{n,i} = \{\bar{a} \in {}^n(A_\alpha)$: if $\bar{a} = \langle a_{f_\ell,\alpha_\ell,u_\ell} : \ell < n\rangle$, <u>then</u> $\langle u_\ell : \ell < n\rangle \in \mathcal{X}_{n,i}\}$.

This completed the choice of $M_{1,\alpha}$. Third, g_α preserves "$\bar{a}, \bar{b}$ are $E_{n,\alpha}$-equivalent", "$\bar{a}, \bar{b}$ are $E'_{n,\alpha}$-equivalent" and their negations. That is, $\bar{a}, g_\alpha(\bar{a})$ are not $E_{n,\alpha}$-equivalent, but as $(\forall\beta)(g_\beta = g_\beta^{-1}), \bar{a}, \bar{b}$ being $E_{n,\alpha}$-equivalent means that there is an even length pass from $\bar{a}$ to $\bar{b}$, in the graph $\{(\bar{c}, g_\beta(\bar{c})) : \beta \in [\gamma,\kappa)$ and $\bar{c} \in {}^n(A_\gamma)\}$ where $\gamma = \min\{\gamma : \bar{a}, \bar{b} \in {}^n(A_\gamma)\}$.

Fourth, no problem in the $M_{1,\alpha}$'s are increasing by $(*)_5(d)$, just check that.

Fifth, g_α commutes with $F^{M_{1,\alpha}}_c$ for $c \in G$ because G is an Abelian group.

Sixth, we should check clause $(*)_6(f)$. Now $g^{-1}_{\beta_2} g_{\beta_1} \restriction A_\alpha = (g_{\beta_2} \restriction A_\alpha)(g_{\beta_1} \restriction A_\alpha)$ by $(*)_2(g)$ and it has order 2 because G is of order 2 and it maps $E^{M_{1,\alpha}}_n$ to itself by the "third", commute with $F^{M_{1,\alpha}}_c$ by the fifth, maps $R^{M_{1,\alpha}}_e$ to itself by the "first".

Lastly, it maps $R^{M_{1,\alpha}}$ to itself by $(*)_{6.4}$. So we are done proving $(*)_6$.

- $(*)_7$ for $\alpha < \kappa$ let $M_{2,\alpha}$ be the τ^*_θ-model with universe A_α such that g_α is an isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$.

Now we note

- $(*)_8$ for $\alpha < \beta < \kappa, M_{2,\alpha} \subseteq M_{2,\beta}$.

[Why? By the definitions of $M_{1,\gamma}, g_\gamma, E'_{n,\gamma}, E_{n,\gamma}$, in particular, the "first" and "third", in "why $(*)_6$", fourth, i.e. $(*)_5(d)$.]

- $(*)_9$ let $M_{2,\kappa} := \cup\{M_{2,\alpha} : \alpha < \kappa\}$, well defined by $(*)_8$
- $(*)_{10}$ let $\pi_{\ell,\beta} = \pi_\beta$ for $\ell = 1, 2$ and $\beta < \kappa$
- $(*)_{11}$ except clause (j) the demands in the conclusion of $\boxplus$ of 1.2 holds easily.

[Why? Just check.]

- $(*)_{12}$ it is O.K. to use a vocabulary of cardinality $2^\sigma = 2^{\theta^{\aleph_0}}$.

[Why? As there is a model M of cardinality 2^σ with $|\tau_M| = \theta$ omitting a quantifier free type p such that $M \subset N \wedge M \equiv N \Rightarrow N$ realizes p. Such M exists as $\sigma = \theta^{\aleph_0}$ and clause (b) of the assumption $\circledast_1$ of 1.2, 1.1.]

Note

$(*)_{13}$ if $(a_{f,\alpha,u_1}, a_{f,\alpha,u_2})$ is $E_{2,\alpha}$-equivalent to $(a_{f,\alpha,v_1}, a_{f,\alpha,v_2})$ then $G \models$ "$u_1 - u_2 = v_1 - v_2$".

[Why? By induction on the k from $(*)_4$.]

So to finish we assume toward contradiction

$\boxtimes$ h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ which respects $(\pi_{1,\alpha}, \pi_{2,\alpha})$ for $\alpha < \kappa$.

So trivially

$\circledast_1$ $h(a_{f,\alpha,u}) \in \{a_{f,\alpha,v} : v \in G\}$ and $\bar{a} \in {}^n(A_\alpha) \Rightarrow h(\bar{a}) \in \bar{a}/E_{n,\alpha} \subseteq \bar{a}/E'_{n,\alpha}$.

[Why? As h respect $(\pi_{1,\kappa}, \pi_{2,\kappa})$ see $(*)_1(e)$ and $(*)_{10}$ clearly $h(\bar{a}) \in \bar{a}/E'_{n,\alpha}$. But h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ hence by $(*)_6(b)$ we have $h(\bar{a}) \in (\bar{a}/E_{n,\alpha})$.]

$\circledast_2$ for $f \in {}^\kappa\sigma$ and $\alpha < \kappa$ let $u_{f,\alpha} \in G$ be the $u \in G$ such that $h(a_{f,\alpha,\emptyset}) = a_{f,\alpha,u}$

$\circledast_3$ for $f \in {}^\kappa\sigma, \alpha < \kappa$ and $v \in G$ we have $h(a_{f,\alpha,v}) = a_{f,\alpha,v+_G u_{f,\alpha}}$.

[Why? By $\circledast_1$ clearly h maps any finite sequence $\bar{b} \in {}^n(A_{1,\kappa})$ to an $E_{n,\alpha}$-equivalent sequence for each $\alpha < \kappa$. Now apply this to the pair $(a_{f,\alpha,\emptyset}, a_{f,\alpha,u})$ recalling $(*)_{13}$.]

$\circledast_4$ we define a partial order $\le$ on ${}^\kappa\sigma$ as follows:
$f_1 \le f_2$ <u>iff</u> there is a function $e \in {}^\sigma\sigma$ witnessing it; which means $f_1 = e \circ f_2$

$\circledast_5$ if $\alpha_1, \alpha_2 < \kappa$ and $f_1 \le f_2$ (are from ${}^\kappa\sigma$) then $|u_{f_1,\alpha_1}| \le |u_{f_2,\alpha_2}|$.

[Why? This follows from $\circledast_6$ below.]

$\circledast_6$ if $e \in {}^\sigma\sigma, f_2 \in {}^\kappa\theta$ and $f_1 = e \circ f_2 \in {}^\kappa\sigma$ and $\alpha_1, \alpha_2 < \kappa$ then
$u_{f_1,\alpha_1} \subseteq \{e(i) : i \in u_{f_2,\alpha_2}\}$.

[Why? Choose $\alpha < \kappa$ such that $\alpha > \alpha_1, \alpha > \alpha_2$ so $a_{f_1,\alpha_1,\emptyset}, a_{f_2,\alpha_1,\emptyset} \in M_{\ell,\alpha}$ for $\ell = 1,2$. Recall that h maps $R_e^{M_{1,\alpha}}$ onto $R_e^{M_{2,\alpha}}$ by $\boxtimes$ and $R_e^{M_{2,\alpha}} = R_e^{M_{1,\alpha}}$ because g_α maps $R_e^{M_{1,\alpha}}$ onto itself (see the proof of $(*)_6$ above, the "first" in that proof). Now see $(*)_6(d)$, i.e. the definition of $R_e^{M_{1,\alpha}}$, i.e. obviously $(a_{f_1,\alpha_1,\emptyset}, a_{f_2,\alpha_2,\emptyset}) \in R_e^{M_{1,\alpha}}$ so as h is an isomorphism we have $(h(a_{f_1,\alpha_1,\emptyset}), h(a_{f_2,\alpha_2,\emptyset})) \in R_e^{M_{2,\alpha}}$ so by the previous sentence and the definitions of $u_{f_\ell,\alpha_\ell} (\ell = 1,2)$ in $\circledast_2$ we have $(a_{f_1,\alpha_1,u_{f_1,\alpha_1}}, a_{f_2,\alpha_2,u_{f_2,\alpha_2}}) \in R_e^{M_{1,\alpha}}$ which by the definitions of $R_e^{M_{1,\alpha}}$ in $(*)_6(d)$ implies $u_{f_1,\alpha_1} \subseteq \{e(i) : i \in u_{f_2,\alpha_2}\}$ as promised.]

$\circledast_7$ (a) $|u_{f,\alpha_1}| = |u_{f,\alpha_2}|$ for $\alpha_1, \alpha_2 < \kappa, f \in {}^\kappa\sigma$
(b) $\mathbf{n}(f) = |u_{f,\alpha}|$ is well defined
(c) if $f_1 \le f_2$ then $\mathbf{n}(f_1) \le \mathbf{n}(f_2)$.

[Why? For clause (a) use $\circledast_6$ for the function $e=\ \mathrm{id}_\sigma$ and $f_1=f_2=f$. Clause (b) follows. Clause (c) holds by $\circledast_6$ equivalently by $\circledast_5$.]

$\circledast_8$ there are $f_*\in{}^\kappa\sigma$ and $\alpha_*<\kappa$ such that:

(i) if $f_*\le f\in{}^\kappa\sigma$ and $\alpha<\kappa$ then $|u_{f_*,\alpha_*}|=|u_{f,\alpha}|$

(ii) moreover if $f_*=e\circ f$ where $e\in{}^\sigma\sigma$ and $f\in{}^\kappa\sigma,\alpha<\kappa$ then $e\restriction u_{f,\alpha}$ is one-to-one from $u_{f,\alpha}$ onto $u_{f_*,\alpha}$ so $\mathbf{n}(f_*)=\mathbf{n}(f)$

(iii) if $\alpha<\kappa, f_1=e\circ f_2, f_*=e_1\circ f_1, f_*=e_2\circ f_2$ so $e,e_1,e_2\in{}^\sigma\sigma$, then $e\restriction u_{f_2,\alpha}$ is one-to-one onto $u_{f_1,\alpha}$.

[Why? First note that clause (ii), (iii) follows from clause (i). Second, if clause (i) fails, then we can find a sequence $\langle(f_n,\alpha_n,e_n):n<\omega\rangle$ such that

(α) $\alpha_n<\kappa, f_n\in{}^\kappa\sigma$ for $n<\omega$

(β) $f_n\le f_{n+1}$ say $f_n=e_n\circ f_{n+1}$ and $e_n\in{}^\sigma\sigma$

(γ) $(e_n,f_{n+1},\alpha_{n+1})$ witness that (f_n,α_n) does not satisfy the demand (i) on (f_*,α_*) hence $\mathbf{n}(f_n)<\mathbf{n}(f_{n+1})$.

Let $u_n=u_{f_n,\alpha_n}$ for $n<\omega$. For $n<\omega$ and $i<\sigma$ let $A_{n,i}=\langle\alpha<\kappa:f_n(\alpha)=i\rangle$, so $\langle A_{n,i}:i<\sigma\rangle$ is a partition of κ and $\alpha\in A_{n+1,i}\Rightarrow\alpha\in A_{n,e_n(i)}$. So letting $A_\eta=\cap\{A_{n,\eta(n)}:n<\omega\}$ for $\eta\in{}^\omega\sigma$ clearly $\langle A_\eta:\eta\in{}^\sigma\sigma\rangle$ is a partition of κ.

As we have $\sigma=\sigma^{\aleph_0}$ by $(*)_0$, there is a sequence $\langle e^n:n<\omega\rangle$ satisfying $e^n\in{}^\sigma\sigma$ and $f\in{}^\kappa\sigma$ such that $f_n=e^n\circ f$ for each $n<\omega$. So $n<\omega\Rightarrow f_n\le f$ which by $\circledast_7(c)$ implies $\mathbf{n}(f_n)\le\mathbf{n}(f)$. As $\langle\mathbf{n}(f_n):n<\omega\rangle$ is increasing, easily we get a contradiction.]

$\circledast_9$ $\mathbf{n}(f_*)>0$, i.e. $\alpha<\kappa\Rightarrow u_{f_*,\alpha}\ne\emptyset$.

[Why? If $(\forall f\in{}^\kappa\sigma)(\forall\alpha<\kappa)(u_{f,\alpha}=\emptyset)$ then (by $\circledast_3$) we deduce h is the identity, contradiction. Otherwise assume $u_{f,\alpha}\ne\emptyset$ hence as in the proof of $\circledast_8$ there is f' such that $f_*\le f'\wedge f\le f'$ so by $\circledast_5$ and $\circledast_8$ we have $0<|u_{f,\alpha}|\le|u_{f',\alpha}|=|u_{f_*,\alpha_*}|$.]

$\circledast_{10}$ if $f\in{}^\kappa\sigma,\alpha<\kappa$ and $i\in u_{f,\alpha}$ then $\kappa=\sup\{\beta<\kappa:\alpha<\beta$ and $f(\beta)=i\}$.

[Why? If not, let $\beta(*)<\kappa$ be $>\sup\{\beta<\kappa:\alpha<\beta,f(\beta)=i\}$ and $>\omega$ and $>\alpha$. Let $Y=\{(a_{f,\alpha,u},a_{f,\beta(*),u}):u\in G,i\notin u\}$. Now for every $\beta\in(\beta(*),\kappa)$ the function g_β maps the set Y onto itself (see its definition in $(*)_2$(f)) hence by the definition of $E_{2,\beta(*)+1}$ (in $(*)_4$) it follows that $\bar a\in Y\Rightarrow\bar a/E_{2,\beta(*)+1}\subseteq Y$ and as h respects $(\pi_{1,\beta(*)+1},\pi_{2,\beta(*)+1})$ it follows that $h(\bar a)\subseteq\bar a/E'_{2,\beta(*)+1}$ and so $\kappa>\gamma\ge\beta(*)+1\Rightarrow g^{-1}_\gamma(h(\bar a))\in\bar a/E'_{2,\beta(*)+1}$.

Now for $\bar a\in Y$, the pairs $\bar a,h(\bar a)$ realizes the same quantifier free type in $M_{1,\beta(*)+1},M_{2,\beta(*)+1}$ respectively, hence by the choice of $M_{2,\beta(*)+1}$ the pairs $\bar a,g^{-1}_{\beta(*)+2}h(\bar a))$ realize the same quantifier free type in $M_{1,\alpha}$. By $(*)_6(b)$ recalling $g^{-1}_{\beta(*)+2}(h(\bar a))\in\bar a/E'_{2,\beta(*)+1}$ this implies that $\bar a,g^{-1}_{\beta(*)+2}(h(\bar a))$ are $E_{2,\beta(*)+1}$-equivalent. By the definition of $E_{2,\beta(*)+1},g^{-1}_{\beta(*)+2}(h(\bar a))$ belongs to the closure of $\{\bar a\}$ under $\{g^{\pm1}_\gamma:\gamma\in(\beta(*);\kappa)\}$ hence $h(\bar a)$ belongs to it. But by an earlier sentence Y is closed under those functions so $h(\bar a)\in Y$. Similarly $h^{-1}(\bar a)\in Y$, hence h maps Y onto itself, recalling $\circledast_2$ this implies $i\notin u_{f,\alpha}$, contradicting an assumption of $\circledast_{10}$, so $\circledast_{10}$ holds.]

Now fix f_*, α_* as in $\circledast_8$ for the rest of the proof, without loss of generality f_* is onto σ and let $u_{f_*,\alpha_*} = \{i^*_\ell : i < \ell(*)\}$ with $\langle i^*_\ell : \ell < \ell(*)\rangle$ increasing for simplicity. Now for every $f \in {}^\kappa\sigma$ such that $f_* \le f$ and $\alpha < \kappa$ by $\circledast_8(ii),(iii)$ we know that if $e \in {}^\sigma\sigma \wedge f_* = e \circ f$ then e is a one-to-one mapping from $u_{f,\alpha}$ onto u_{f_*,α_*}; but so $e \restriction u_{f,\alpha}$ is uniquely determined by (f_*,α_*,f,α) so let $i_{f,\alpha,\ell} \in u_{f,\alpha}$ be the unique $i \in u_{f,\alpha}$ such that $e(i) = i^*_\ell$ (equivalently $(\exists\alpha)(f(\alpha) = i \wedge f_*(\alpha) = i^*_\ell)$).

Now if $f_* \le f \in {}^\kappa\sigma$ and $\alpha_1,\alpha_2 < \kappa$ and we choose $e = \mathrm{id}_\sigma$ so necessarily $f \restriction u_{f,\alpha_1} = e \circ f \restriction u_{f,\alpha_2}$, then $e \restriction \mathrm{Rang}(f \restriction u_{f,\alpha_2})$ map u_{f,α_2} onto u_{f,α_1} but e is the identity so we can write u_f instead of $u_{f,\alpha}$ let $i_{f,\ell} = i_{f,\alpha,\ell}$ for $\ell < \ell(*), \alpha < \kappa$.

Let

$$\mathcal{A} = \{A \subseteq \kappa : \text{ for some } f, f_* \le f \text{ and } \alpha < \kappa \text{ we have } f^{-1}\{i_{f,0}\}\backslash\alpha \subseteq A\}$$

$\boxdot_1$ $\mathcal{A} \subseteq \mathcal{P}(\kappa)\backslash[\kappa]^{<\kappa}$.

[Why? As κ is regular, this means $A \in \mathcal{A} \Rightarrow A \subseteq \kappa \wedge \sup(A) = \kappa$ which holds by $\circledast_{10}$.]

$\boxdot_2$ $\kappa \in \mathcal{A}$.

[Why? By the definition of $\mathcal{A}$.]

$\boxdot_3$ if $A \in \mathcal{A}$ and $A \subseteq B \subseteq \kappa$ then $B \in \kappa$.

[Why? By the definition of $\mathcal{A}$.]

$\boxdot_4$ if $A_1, A_2 \in \mathcal{A}$ then $A =: A_1 \cap A_2$ belongs to $\mathcal{A}$.

[Why? Let $(f_\ell, e_\ell, \alpha_\ell)$ be such that $f_* = e_\ell \circ f_\ell$ and $f_\ell \in {}^\kappa\sigma, \alpha_\ell < \kappa$ and $f^{-1}_\ell\{i_{f_\ell,0}\}\backslash\alpha_\ell \subseteq A_\ell$ for $\ell = 1,2$. Let pr:$\sigma \times \sigma \to \sigma$ be one-to-one and onto and define $f \in {}^\kappa\sigma$ by $f(\alpha) = \mathrm{pr}(f_1(\alpha), f_2(\alpha))$. Clearly $f_\ell \le f$ for $\ell = 1,2$ hence $i_{f,0}$ is well defined and $i_{f,0} = \mathrm{pr}(i_{f_1,0}, i_{f_2,0})$. Now for every $\alpha < \kappa, f(\alpha) = i_{f,0} \Rightarrow f_1(\alpha) = i_{f_1,0} \wedge f_2(\alpha) = i \Rightarrow \alpha \in A_1 \wedge \alpha \in A_2 \Rightarrow \alpha \in A_1 \cap A_2 \Rightarrow \alpha \in A$ so $f^{-1}\{i_{f,0}\} \subseteq A$ hence $A \in \mathcal{A}$.]

$\boxdot_5$ if $A \subseteq \kappa$ then $A \in \mathcal{A}$ or $\kappa\backslash A \in \mathcal{A}$.

[Why? Define $f \in {}^\kappa\sigma$:

$$f(\alpha) = \begin{cases} 2f_*(\alpha) & \text{if } \alpha \in A \\ 2f_*(\alpha)+1 & \text{if } \alpha \in \kappa\backslash A. \end{cases}$$

Let $i = i_{f,0}$ so by the definition of $\mathcal{A}$ we have $f^{-1}\{i\} = f^{-1}\{i_{f,0}\} \in \mathcal{A}$. But if i is even then $f^{-1}\{i\} \subseteq A$ and i is odd then $f^{-1}\{i\} \subseteq \kappa\backslash A$ so by $\boxdot_3$ we are done.]

$\boxdot_6$ $\mathcal{A}$ is a uniform ultrafilter on κ.

[Why? By $\boxdot_1 - \boxdot_5$.]

$\boxdot_7$ $\mathcal{A}$ is σ^+-complete.

[Why? Assume $B_\varepsilon \in \mathcal{A}$ for $\varepsilon < \sigma$ and let $B = \cap\{B_\varepsilon : \varepsilon < \sigma\}$. Define $A_\varepsilon \subseteq \kappa$ for $\varepsilon < \sigma$ as follows: $A_{1+\varepsilon} = \bigcap\limits_{\zeta<\varepsilon} B_\zeta \backslash B_\zeta$ (so is $\kappa \backslash B_0$ if $\varepsilon = 0$) for $\varepsilon < \sigma$ and $A_0 = B$. Clearly $\langle A_\varepsilon : \varepsilon < \sigma\rangle$ is a partition of κ, let $f \in {}^\kappa\sigma$ be such that $f \restriction A_\varepsilon$ is constantly ε. Let $f' \in {}^\kappa\theta$ be such that $f \le f' \wedge f_* \le f'$. Now $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$ is included in some A_ε. If $\varepsilon = 0$ this exemplifies $\bigcap\limits_{\varepsilon<\sigma} B_\varepsilon \in \mathcal{A}$ as required. If $\varepsilon = 1 + \zeta < \sigma$, then $(f')^{-1}\{i_{f',0}\} \subseteq A_\varepsilon \subseteq \kappa \backslash B_\varepsilon$, contradiction to $\boxdot_6$ because $B_\varepsilon \in \mathcal{A}$ and $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$.]

So by the assumptions of 1.2, that is, $\circledast_1(b)$ of 1.1 we get a contradiction, coming from the assumption "toward contradiction (j) of $\boxplus$ of 1.2 fails", so it holds and the other clauses were proved so we are done. $\square_{1.2}$

{e3}
Theorem 1.3. *For every θ there is an $\mathfrak{k} = \mathfrak{k}^*_\theta$ such that*

$\otimes$ (a) *$\mathfrak{k}$ is an a.e.c. with $\mathrm{LST}(\mathfrak{k}) = \theta, |\tau_{\mathfrak{k}}| = \theta$*

(b) *$\mathfrak{k}$ has the amalgamation property*

(c) *$\mathfrak{k}$ admits intersections (see Definition 1.4 below)*

(d) *if κ is a regular cardinal and there is no uniform θ^+-complete ultrafilter on κ, <u>then</u>: $\mathfrak{k}$ is not $(\le 2^\kappa, \kappa)$-sequence-local for types, i.e., we can find an $\le_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i \le \kappa\rangle$ of models and $p \ne q \in \mathscr{S}_{\mathfrak{k}}(M_\kappa)$ such that $i < \kappa \Rightarrow p \restriction M_i = q \restriction M_i$ and M_κ is of cardinality $\le 2^\kappa$.*

We shall prove 1.3 below. As in [BlSh:862, 1.2,§4] the aim of the definition of "admit intersections" is to ensure types behave reasonably.

{e4}
Definition 1.4. We say an a.e.c. $\mathfrak{k}$ admits intersections <u>when</u> there is a function $c\ell_{\mathfrak{k}}$ such that:

(a) $c\ell_{\mathfrak{k}}(A, M)$ is well defined iff $M \in K_{\mathfrak{k}}$ and $A \subseteq M$

(b) $c\ell_{\mathfrak{k}}(A, M)$ is preserved under isomorphisms and $\le_{\mathfrak{k}}$-extensions

(c) for every $M \in K_{\mathfrak{k}}$ and non-empty $A \subseteq M$ the set is $B = c\ell_{\mathfrak{k}}(A, M)$ satisfies: $M{\restriction}B \in K_{\mathfrak{k}}, M{\restriction}B \le_{\mathfrak{k}} M$ and $A \subseteq M_1 \le_{\mathfrak{k}} N \wedge M \le_{\mathfrak{k}} N \Rightarrow B \subseteq M_1$; we may use $cl_{\mathfrak{k}}(A, M)$ for $M{\restriction}c\ell_{\mathfrak{k}}(A, M)$.

{e5}
Claim 1.5. *Assume $\mathfrak{k}$ is an* a.e.c. *admitting intersections. <u>Then</u>* $\mathrm{tp}_{\mathfrak{k}}(a_1, M, N_1) = \mathrm{tp}_{\mathfrak{k}}(a_2, M, N_2)$ *iff letting $M_\ell = N_\ell{\restriction}c\ell_{\mathfrak{k}}(M \cup \{a_\ell\})$, there is an isomorphism from M_1 onto M_2 over M mapping a_1 to a_2.*

Proof. Should be clear by the definition. $\square_{1.5}$

{e5d}
Remark 1.6. In Theorem 1.3 we can many times demand $\|M_\kappa\| = \kappa$, e.g., if $(\exists\lambda)(\kappa = 2^\lambda)$.

Note we now show that 1.3 is best possible.

{e6}
Claim 1.7. *1) If $\mathfrak{k}$ satisfies clause (a) of 1.3, (i.e. $\mathfrak{k}$ is an a.e.c. with* LST*-number $\le \theta$ and $|\tau_{\mathfrak{k}}| \le \theta$) and κ fails the assumption of clause (d) of 1.3, that is there is a uniform θ^+-complete ultrafilter on κ, <u>then</u> the conclusion of clause (d) of 1.3 fails, that is $\mathfrak{k}$ is κ-sequence local for types.*
2) If D is a θ^+-complete ultrafilter on κ and $\mathfrak{k}$ is an a.e.c. with $\mathrm{LST}(\mathfrak{k}) \le \theta$ *<u>then</u> ultraproducts by D preserve "$M \in \mathfrak{k}$", "$M \le_{\mathfrak{k}} N$", i.e.*

$\boxtimes$ *if* $M_i, N_i (i < \kappa)$ *are* $\tau(\mathfrak{K})$*-models and* $M = \prod\limits_{i<\kappa} M_i/D$ *and* $N = \prod\limits_{i<\kappa} N_i$ *then:*

(a) $M \in K$ *if* $\{i < \kappa : M_i \in \mathfrak{k}\} \in D$

(b) $M \leq_{\mathfrak{k}} N$ *if* $\{i : M_i \leq_{\mathfrak{k}} N_i\} \in D$.

Proof. Note that if D is θ^+-complete, <u>then</u> it is σ^+-complete where $\sigma = \theta^{\aleph_0}$ (and much more, it is θ'-complete for the first measurable $\theta' > \theta$).

1) So assume

$\boxplus$ (a) $\langle M_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{k}}$-increasing

(b) $M_\kappa = N_0 \leq_{\mathfrak{k}} N_\ell$ for $\ell = 1, 2$

(c) $p_\ell = \mathbf{tp}_{\mathfrak{k}}(a_\ell, N_0, N_\ell)$ for $\ell = 1, 2$

(d) $i < \kappa \Rightarrow p_1 \restriction M_i = p_2 \restriction M_i$.

We shall show $p_1 = p_2$, this is enough.

Without loss of generality

$(*)_1$ (a) $a_1 = a_2$ call it a

(b) $\tau_{\mathfrak{k}} \subseteq \mathcal{H}(\theta)$.

By (d) of $\boxplus$ we have:

$(d)^+$ for each $i < \kappa$ there are $n_i < \omega$ and $\langle N_{i,m} : n \leq n_i \rangle$ such that

(α) $N_{i,0} = N_1$

(β) $N_{i,m_i} = N_2$ or just h_i is an isomorphism from N_{i,m_i} onto N_2 such that $h_i \restriction (M_i \cup \{a\})$ is the identity

(γ) $a \in N_{i,\ell}$ and $M_i \leq_{\mathfrak{k}} N_{i,\ell}$

(δ) if $m < m_i$ then $N_{i,2m+1} \leq_{\mathfrak{k}} N_{i,2m}, N_{i,2m+2}$.

As $\kappa = \text{cf}(\kappa) > \aleph_0$ without loss of generality $i < \kappa \Rightarrow n_i = n_*$. Let χ be such that $\langle M_i : i \leq \kappa \rangle, \langle \langle N_{i,n} : n \leq n_* \rangle : i < \kappa \rangle$ and $\mathfrak{k}_{\text{LST}(\mathfrak{k})}$ all belongs to $\mathcal{H}(\chi)$; concerning $\mathfrak{k}_{\text{LST}(\mathfrak{k})}$ this means τ_χ and $\text{LST}(\mathfrak{k})$ belongs to $\mathcal{H}(\chi)$ hence $\{M \in K_{\mathfrak{k}} : M \in \mathcal{H}(\text{LST}^+_{\mathfrak{k}})\}$ and $\leq_{\mathfrak{k}} \restriction \mathcal{H}(\text{LST}^+_{\mathfrak{k}})$ belongs to $\mathcal{H}(\chi)$; those hold by $(*)_1(b)$. Let $\mathfrak{B}$ be the ultrapower $(\mathcal{H}(\chi), \in)^\kappa/D$ and $\mathbf{j}_0$ the canonical embedding of $(\mathcal{H}(\chi), \in)$ into $\mathfrak{B}$ and let $\mathbf{j}_1$ be the Mostowski-Collapse of $\mathfrak{B}$ to a transitive set $\mathcal{H}$ and let $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$. So $\mathbf{j}$ is an elementary embedding of $(\mathcal{H}(\chi), \in)$ into $(\mathcal{H}, \in)$ and even an $\mathbb{L}_{\theta^+,\theta^+}$-elementary one. Recall we are assuming without loss of generality $\tau_{\mathfrak{k}} \subseteq \mathcal{H}(\theta)$ hence $\mathbf{j}(\tau_{\mathfrak{k}}) = \tau_{\mathfrak{k}}$ hence by part (2), $\mathbf{j}$ preserves "$N \in K_{\mathfrak{k}}$", "$N^1 \leq_{\mathfrak{k}} N^2$". "$h$ is an isomorphism from N' onto N''".

So $\mathbf{j}(\langle M_i : i \leq \kappa \rangle)$ has the form $\langle M^*_i : i \leq \mathbf{j}(\kappa) \rangle$ but $\mathbf{j}(\kappa) > \kappa_* := \bigcup\limits_{i<\kappa} \mathbf{j}(i)$ by the uniformity of D and let $\mathbf{j}(\langle \langle N_{i,n} : n \leq n_* \rangle : i < \kappa \rangle = \big\langle \langle N^*_{i,n} : n \leq n^* \rangle : i < \mathbf{j}(\kappa) \big\rangle$ and $\mathbf{j}(\langle h_i : i < \kappa \rangle) = \langle h^*_i : i < \kappa_* \rangle$.

So

(a) $\mathbf{j} \restriction M_\kappa$ is a $\leq_{\mathfrak{k}}$-embedding of M_κ into M^*_κ hence even into $M^*_{\kappa_*}$

(b) $M^*_{\kappa_*} \leq_{\mathfrak{k}} N^*_{i,n}$ and $\mathbf{j}(a) \in N^*_{i,n}$ for $i < \kappa, n \leq n_*$

(c) $N^*_{i,0} = \mathbf{j}(N_1)$

(d) h_{κ_*} is an isomorphism from N_{κ_*,n_*} onto $\mathbf{j}(N_2)$

(e) $N^*_{\kappa_*,2m+1} \leq_{\mathfrak{k}} N^*_{\kappa_*,2m}, N^*_{\kappa_*,2m+2}$ for $2m + 1 < n_*$

(f) $\mathbf{j}(a) \in N_{\kappa_*,m}$.

Together we are done.
2) By the representation theorem of a.e.c. [Sh:88r, §1]. $\square_{1.7}$

Proof. Proof of 1.3
Let $\sigma = \theta^{\aleph_0}$. Let $G = ([\sigma]^{<\aleph_0}, \Delta)$ and let $\langle c_i : i < \sigma\rangle$ list the members of G, let $\langle \eta_\alpha : \alpha < \sigma\rangle$ list ${}^\omega\theta$.
Now

$\boxtimes_1$ let $B_{\varepsilon,n} \subseteq G$ for $\varepsilon < \theta$ be such that: if $a, b \in G$ then $(\forall\varepsilon < \theta)(\forall n < \omega)(a \in B_{\varepsilon,n} \equiv b \in B_{\varepsilon,n}) \Rightarrow a = b$; moreover, $B_{\varepsilon,n} = \{c_\alpha : \eta_\alpha(n) = \varepsilon\}$.

Let τ have the predicates G, I, J, H(unary), E_1, Q(binary), $R_{n,\alpha}$ (n-place; $n < \omega, \alpha < \theta$), $P_{\varepsilon,n}$(unary; $\varepsilon < \theta$) and function symbols F_1(unary), $F_2, \pi, +$(binary); so $|\tau| = \theta$. We define K as a class of τ-models by:

$\boxtimes_2$ $M \in K$ iff (up to isomorphism):
- (a) $\langle G^M, I^M, J^M, H^M\rangle$ is a partition of $|M|$, (recall that they are unary)
- (b) $(G^M, +^M)$ is a subgroup of the group $([\sigma]^{<\aleph_0}, \Delta)$, $P^M_{\varepsilon,n} \subseteq G^M$ for $\varepsilon < \theta, \langle P^M_{\varepsilon,n} : \varepsilon < \theta\rangle$ be a partition of M such that $a \neq b \in G^M \Rightarrow (\exists\varepsilon < \theta)(\exists n < \omega)[a \in P^M_{\varepsilon,n} \wedge b \notin P^M_{\varepsilon,n}]$
- (c) $Q^M \subseteq H^M \times J^M$ is such that $(\forall a \in H^M)(\exists^{\leq 1} b)((a, b) \in Q^M)$;
- (d) E^M_1 is an equivalence relation on H^M such that: if $(a_1, b) \in Q^M$ and $a_2 \in H^M$ then $a_1 E^M_1 a_2 \Leftrightarrow (a_2, b) \in Q^M$
- (e) π^M is a function from H^M into I^M
- (f) $E^M_2 = \{(a, b) : a E^M_1 b$ and $\pi^M(a) = \pi^M(b)$ so $a, b \in H\}$
- (g) F^M_2 is a partial two-place function such that:
  - (α) $F^M_2(a, b)$ is well defined iff $b \in G^M, a \in H^M$
  - (β) for $a \in H^M, \langle F^M_2(a, b) : b \in G^M\rangle$ list $a/E^M_2 = \{a' \in H^M\colon \pi^M(a') = \pi^M(a)\}$ with no repetitions
  - (γ) if $a \in H^M$ and $b, c \in G$ then $F^M_2(a, b +^M c) = F^M_2(F^M_2(a, b), c)$, on the $+$ see clause (b)
  - (δ) $F(a, 0_{G^M}) = a$ for $a \in H^M$
  - (ε) for $n < \omega$ and $\gamma < \theta$ the relation $R^M_{n,\gamma}$ is an n-place relation $\subseteq \cup\{{}^n(a/E^M_1) : a \in H^M\}$.

We define $\leq_{\mathfrak{k}}$ as being a submodel. Easily

$\boxtimes_3$ $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ is an a.e.c.

For $A \subseteq M \in K$ let

- (a) $c\ell^1_M(A) =$ the subgroup of $(G^M, +^M)$ generated by $(A \cap G^M) \cup \{b \in G^M :$ for some $a_1 \neq a_2 \in A \cap H^M$ we have $a_1 E^M_2 a_2$ and $F^M_2(a_1, b) = a_2\}$
- (b) $c\ell^2_M(A) = (A \cap I^M) \cup \{\pi^M(a) : a \in A \cap H^M\}$
- (c) $c\ell^3_M(A) = \{a \in H^M :$ for some $b \in c\ell^0_M(A)$ and $a'_1 \in A \cap H^M$ we have $a = F^M_2(a'_1, b)\}$
- (d) $c\ell(A, M) = c\ell_M(A) = M \restriction (\cup\{c\ell^\ell_M(A) : \ell = 1, 2, 3\})$.

Now this function $c\ell(A, M)$ shows that $\mathfrak{k}$ admits intersections (see Definition 1.4) so

$\boxtimes_4$ $\mathfrak{k}$ admits closure and $\mathrm{LST}(\mathfrak{k}) + |\tau_{\mathfrak{k}}| = \theta$.

Assume κ is as in clause (d) of 1.3, we use the $M_{\ell,\alpha}(\ell = 1, 2, \alpha \le \kappa)$ constructed in 1.2 (the relevant properties are stated in 1.2). They are not in the right vocabulary so let $M'_{\ell,\alpha}$ be the following τ-model:

$\boxtimes_5$ (a) <u>elements</u> $G^{M'_{\ell,\alpha}} = G$

$I^{M'_{\ell,\alpha}} = I_\alpha$

$J^{M'_{\ell,\alpha}} = \{t^*_\ell\}, t^*_\ell$ just a new element

$H^{M'_{\ell,\alpha}} = |M_{\ell,\alpha}|$

(we assume disjointness)

(b) $(G^{M'_{\ell,\alpha}}, +^{M'_{\ell,\alpha}})$ is $G = ([\sigma]^{<\aleph_0}, \Delta)$

$P^{M'_{\ell,\alpha}}_\varepsilon \subseteq G^M$ as required in $\boxtimes_1$ <u>not</u> depending on (ℓ, α)

(c) $F^{M'_{\ell,\alpha}}_1$ is constantly t^*_ℓ on $H^{M'_{\ell,\alpha}}$

(d) $E^{M'_{\ell,\alpha}}_1 = \{(a, b) : F^{M'_{\ell,\alpha}}_1(a) = F^{M'_{\ell,\alpha}}_1(b)$ so $a, b \in H^{M'_{\ell,\alpha}}$

(e) $\pi^{M'_{\ell,\alpha}}$ is $\pi_{\ell,\alpha}$ (constructed in 1.2)

(f) $E^{M'_{\ell,\alpha}}_2 = \{(a, b) : aE^{M'_{\ell,\alpha}}_1 b$ and $\pi^{M'_{\ell,\alpha}}(a) = \pi^{M'_{\ell,\alpha}}(b)$ so $a, b \in H^{M'_{\ell,\alpha}}\}$

(g) $F^{M'_{\ell,\alpha}}_2(a, b) = F^{M_{\ell,\alpha}}_b(b)$ for $a \in H^{M'_{\ell,\alpha}}$

(h) $R^{M'_{\ell,\alpha}}_{\gamma,n}$ for $n < \omega, \gamma < \sigma$ list the relations of $M_{\ell,\alpha}$.

Let $M'_{0,\alpha} = M'_{\ell,\alpha} \restriction (G^{M'_{\ell,\alpha}} \cup I^{M'_{\ell,\alpha}})$ for $\ell = 1, 2$ and $\alpha \le \kappa$ (we get the same result). Note easily

$\boxtimes_6$ $M_{0,\alpha} \le_{\mathfrak{k}} M_{\ell,\alpha}, \langle M_{\ell,\alpha} : \alpha \le \kappa\rangle$ is $\le_{\mathfrak{k}}$-increasing (check)

$\boxtimes_7$ $\mathrm{tp}_{\mathfrak{k}}(t^*_1, M'_{0,\alpha}, M'_{1,\alpha}) = \mathrm{tp}_{\mathfrak{k}}(t^*_2, M'_{0,\alpha}, M'_{2,\alpha})$ for $\alpha < \kappa$.

[Why? By the isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$ respecting $(\pi_{1,\alpha}, \pi_{2,\alpha})$ in 1.1.]

$\boxtimes_8$ $\mathrm{tp}_{\mathfrak{k}}(t^*_1, M'_{0,\kappa}, M'_{1,\kappa}) \ne \mathrm{tp}_{\mathfrak{k}}(t^*_2, M'_{0,\kappa}, M'_{2,\kappa})$.

[Why? By the non-isomorphism in 1.1; extension will not help.]

Now by the"translation theorem" of [BlSh:862, 4.7] we can find $\mathfrak{k}'$ which has all the needed properties, i.e. also the amalgamation and JEP. $\square_{1.3}$

2. Compactness of types in a.e.c.

Baldwin [Bal09] ask "can we in ZFC prove that some a.e.c. has amalgamation, JEP but fail compactness of types". The background is that in [BlSh:862] we construct one using diamonds.

To me the question is to show this class can be very large (in ZFC).

Here we omit amalgamation and accomplish both by direct translations of problems of existence of models for theories in $\mathbb{L}_{\kappa^+,\kappa^+}$, first in the propositional logic. So whereas in [BlSh:862] we have an original group G^M, here instead we have a set P^M of propositional "variables" and P^M, set of such sentences (and relations and functions explicating this; so really we use coding but are a little sloppy in stating this obvious translation).

In [BlSh:862] we have I^M, set of indexes, 0 and H, set of Whitehead cases, H_t for $t \in I^M$, here we have I^M, each $t \in I^N$ representing a theory $P^M_t \subseteq P^M$ and in J^M we give each $t \in I^M$ some models $\mathcal{M}^M_s : P^M \to \{\text{true,false}\}$. This is set up so that amalgamation holds.

Notation 2.1. In this section types are denoted by $\mathbf{p}, \mathbf{q}$ as p, q are used for propositional variables.

{b2.0}

Definition 2.2. 1) We say that an a.e.c. $\mathfrak{k}$ has $(\leq \lambda, \kappa)$-sequence-compactness (for types) <u>when</u>: if $\langle M_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous and $i < \kappa \Rightarrow \|M_i\| \leq \lambda$ and $\mathbf{p}_i \in \mathscr{S}^{<\omega}(M_i)$ for $i < \kappa$ satisfying $i < j < \kappa \Rightarrow \mathbf{p}_i = \mathbf{p}_j \restriction M_i$ <u>then</u> there is $\mathbf{p}_\kappa \in \mathscr{S}^{<\omega}(M_\kappa)$ such that $i < \kappa \Rightarrow p_\kappa \restriction M_i = p_i$.
2) We define "$(= \lambda, \kappa)$-sequence-compactness" similarly. Let (λ, κ)-sequence-compactness mean $(\leq \lambda, \kappa)$-compactness.

{2b.1}

Question 2.3. Can we find an a.e.c. $\mathfrak{k}$ with amalgamation and JEP such that $\{\theta : \mathfrak{k}$ have (λ, θ)-compactness of types for every $\lambda\}$ is complicated, say:

- (a) not an end segment but with "large" members
- (b) any $\{\theta : \theta$ satisfies $\psi\}, \psi \in \mathbb{L}_{\kappa^+,\kappa^+}$ (second order).

{b2.2}

Definition 2.4. Let $\kappa \geq \aleph_0$, we define $\mathfrak{k} = \mathfrak{k}_\kappa$ as follows:

- (A) the vocabulary $\tau_{\mathfrak{k}}$ consist of $F_i (i \leq \kappa), R_\ell (\ell = 1, 2), P, \Gamma, I, J, c_i \ (i < \kappa), F_i (i \leq \kappa)$, (pedantically see later),
- (B) the universe of $M \in K_{\mathfrak{k}}$ is the disjoint union of P^M, Γ^M, I^M, J^M so P, Γ, I, J are unary predicates
- (C)
  - (a) P^M a set of propositional variables (i.e. this is how we treat them)
  - (b) Γ^M is a set of sentences of one of the forms $\varphi = (p), \varphi = (r \equiv p \wedge q), \varphi = (q \equiv \neg p), \varphi = (q \equiv \bigwedge\limits_{i<\kappa} p_i)$, so $p, q, p_i \in P^M$
    but in the last case $\{p_i : i < \kappa\} \subseteq \{c^M_i : i < \kappa\}$ (or code this!)
  - (c) for $i < \kappa$ the function $F^M_i : \Gamma^M \to P^M$ are such that for every $i < \kappa$ and $\varphi \in \Gamma^M$ we have:
    - (α) if $\varphi = (p)$ and $i \leq \kappa$ then $F_{1+i}(\varphi) = p, F_0(\varphi) = c_0$
    - (β) if $\varphi = (r \equiv p \wedge q)$ <u>then</u> $F_i(\varphi)$ is c_1 if $i = 0$, is p if $i = 1$, is q if $r = 2$ is r if $r \geq 3$
    - (γ) if $\varphi = (q \equiv \neg p)$ then $F_i(\varphi)$ is c_2 if $i = 0$, p if $i = 1$, q if $i \geq 2$

(δ) if $\varphi = (q \equiv \bigwedge_{j<\kappa} p_j)$ then $F_i(\varphi)$ is c_3 if $i = 0$,
q if $i = 1, p_{2+j}$ if $i = j + 1$

(d) I a set of theories, i.e. $R_1^M \subseteq \Gamma \times I$ and for $t \in I$ let $\Gamma_t^M = \{\psi \in \Gamma^M : \psi R_1^M t\} \subseteq \Gamma^M$

(e) J is a set of models, i.e. $R_2^M \subseteq (\Gamma \cup P) \times J$ and for $s \in J$ we have $\mathcal{M}_s^M$ is the model, i.e. function giving truth values to $p \in P^M$, i.e.

(α) $\mathcal{M}_s^M(p)$ is true if $p_i R_2^M s$; is false if $\neg p R_2^M s$

(β) $(\varphi, s) \in R_2^M$ iff computing the truth value of φ in $\mathcal{M}_s^M$ we get truth

(f) $F_\kappa^M : J^M \to I^M$ such that $s \in J^M \Rightarrow \mathcal{M}_s^M$ is a model of $\Gamma_{F_\kappa^M(s)}$

(g) $(\forall t \in I^M)(\exists s \in J^M)(F_\kappa^M(s) = t)$

(D) $M \leq_{\mathfrak{k}} N$ <u>iff</u> $M \subseteq N$ are $\tau_{\mathfrak{k}}$-models from $K_{\mathfrak{k}}$.

{b2.3}

Claim 2.5. *$\mathfrak{k}$ is an a.e.c., $LST(\mathfrak{k}) = \kappa$.*

Proof. Obvious. $\square_{2.5}$

{b2.5}

Claim 2.6. *$\mathfrak{k}$ has the JEP.*

Proof. Just like disjoint unions (also of the relations and functions) except for the individual constants c_i (for $i < \kappa$). $\square_{2.6}$

{b2.6}

Claim 2.7. *Assume $M_0 \leq_{\mathfrak{k}} M_\ell$ for $\ell = 0, 1$ and $|M_0| = P^{M_0} \cup \Gamma^{M_0} = P^{M_\ell} \cup \Gamma^{M_\ell}$ for $\ell = 1, 2$ and $a_\ell \in I^{M_\ell}$ for $\ell = 1, 2$. <u>Then</u> $\mathbf{tp}_{\mathfrak{k}}(a_1, M_0, M_1) = \mathbf{tp}_{\mathfrak{k}}(a_2, M_0, M_2)$ <u>iff</u> $\Gamma_{a_1}^{M_1} = \Gamma_{a_2}^{M_2}$.*

Proof. <u>The if direction, $\Leftarrow$</u>

Let h be a one to one mapping with domain M_1 such that $h \restriction M_0 =$ the identity, $h(a_1) = a_2$ and $h(M_1) \cap M_2 = M_0 \cup \{a_2\}$. Renaming without loss of generality h is the identity. Now define M_3 as $M_1 \cup M_2$, as in 2.6, now $a_1 = a_2$ does not cause trouble because $P^{M_0} = P^{M_\ell}, \Gamma^{M_0} = \Gamma^{M_\ell}$ for $\ell = 1, 2$.

<u>The only if direction, $\Rightarrow$</u>

Obvious. $\square_{2.7}$

{b2.11}

Claim 2.8. *Assume λ, θ are such that:*

(a) *θ is regular $\leq \lambda$ and $\lambda \geq \kappa$*

(b) *$\langle \Gamma_i : i \leq \theta \rangle$ is $\subseteq$-increasing continuous sequence of sets propositional sentences in $\mathbb{L}_{\kappa^+,\omega}$ such that [Γ_i has a model $\Leftrightarrow i < \theta$]*

(c) *$|\Gamma_\theta| \leq \lambda$.*

<u>Then</u> $\mathfrak{k}$ fail (λ, θ)-sequence-compactness (for types).

Remark 2.9. We may wonder but: for $\theta = \aleph_0$ compactness holds? Yes, but only assuming amalgamation.

Proof. Without loss of generality $|\Gamma_0| = \lambda$. Without loss of generality $\langle p^*_\varepsilon : \varepsilon < \kappa\rangle$ are pairwise distinct propositions variables appearing in Γ_0 (but not necessarily $\in \Gamma_0$) and each $\psi \in \Gamma_i$ is of the form (p) or $r \equiv p \wedge q$ or $r \equiv \neg p$ or $r \equiv \bigwedge\limits_{i<\kappa} p_i$ where $\{p_i : i < \kappa\} \subseteq \{p^*_\varepsilon : \varepsilon < \kappa\}$.

Let P_i be the set of propositional variables appearing in Γ_i without loss of generality $|P_i| = \lambda$.

We choose a model M_i for $i \le \theta$ such that:

$\boxplus$ (a) $|M_i| = P_i \cup \Gamma_i$
 (b) $P^M = P_i$ and $\Gamma^{M_i} = \Gamma_i$
 (c) the natural relations and functions.

Let $\mathcal{M}_i : P_i \to \{\text{true false}\}$ be a model of Γ_i.

We define a model $N_i \in K_{\mathfrak{k}}$ for $i < \kappa$ (but not for $i = \theta$!)

$\boxtimes$ (a) $M_i \le_{\mathfrak{k}} N_i$
 (b) $P^{N_i} = P^{M_i}$
 (c) $\Gamma^{N_i} = \Gamma^{M_i}$
 (d) $I^M = \{t_i\}$
 (e) $J^M = \{s_i\}$
 (f) $F^{N_i}_\kappa(s_i) = t_i$
 (g) $R^{N_i}_1 = \Gamma_i \times \{t_i\}$
 (h) $R^{N_i}_2$ is chosen such that $\mathcal{M}^{N_i}_{s_i}$ is $\mathcal{M}_i$
 (i) $F^{N_i}_i$ $(i < \kappa)$ are defined naturally.

Now

$(*)_1$ $\mathbf{p}_i = \mathbf{tp}_{\mathfrak{k}}(t_i, M_i, N_i) \in \mathbf{S}^1(M_i)$.

[Why? Trivial.]

$(*)_2$ $i < j < \theta \to \mathbf{p}_i = \mathbf{p}_j{\restriction}M_j$.

[Why? Let $N_{i,j} = N_j{\restriction}(M_j \cup \{s_j, t_j\})$.]

Easily $\mathbf{tp}(t_j, M_i, N_{i,j}) \le p_j$ and $\mathbf{tp}(t_j, M_i, N_{i,j}) = p_i$ by the claim 2.7 above.]

$(*)_3$ there is no $p \in \mathbf{S}^1(M_\theta)$ such that $i < \theta \Rightarrow p{\restriction}M_i = p_i$.

Why? We prove more:

$(*)_4$ there is no (N, t) such that
 (a) $M_\kappa \le_{\mathfrak{k}} N$
 (b) $t \in I^N$
 (c) $(\forall \varphi \in \Gamma^{M_\kappa})[\varphi R^N_1 t]$.

[Why? As then $\Gamma_\theta = \Gamma^M$ has a model contradiction to an assumption.] $\square_{2.8}$

So e.g.

{b2.13} *Conclusion* 2.10. If $\theta > \kappa$ is regular with no κ^+-complete uniform ultrafilter on θ and $\lambda = 2^\theta$, <u>then</u> $\mathfrak{k}$ is not (λ, θ)-sequence-compact.

Remark 2.11. Recall if D is an ultrafilter on θ then $\min\{\sigma' : D \text{ is not } \sigma'\text{-complete}\}$ is $\aleph_0$ or a measurable cardinality.

Proof. (Well known).

Let M be the model with universe $2^\theta, P_0^M = \theta$ and $R^M \subseteq \theta \times \lambda$ be such that $\{\{\alpha < \lambda : \alpha R^M \beta\} : \beta < \lambda\} = \mathcal{P}(\theta), <^M$ the well ordering of the ordinal on λ the vocabulary has cardinality κ and has elimination of quantifiers and Skolem functions.

Let $\Gamma_i = \text{Th}(M,\beta)_{\beta<\lambda} \cup \{\alpha < c : \alpha < \theta\}$ (c a new individual constant), then $\langle \Gamma_i : i \le \theta\rangle$ is as[1] required in 2.12 below hence 2.8 apply. $\square_{2.10}$

{b2.17}

Conclusion 2.12. In Claim 2.8 if $\lambda = \lambda^\kappa$ then we can allow $\langle \Gamma_i : i \le \theta\rangle$ to be a sequence of theories in $\mathbb{L}_{\kappa^+,\kappa^+}(\tau), \tau$ any vocabulary of cardinality $\le \lambda$.

Proof. Without loss of generality we can add Skolem functions (each with $\le \kappa$ places) in particular. So Γ_i becomes universal and adding propositional variables for each quantifier free sentence and writing down the obvious sentences, we get a set of propositional sentences, we get Γ_i as there. $\square_{2.12}$

I think we forgot

{2b.21}

Observation 2.13. *If $\lambda \ge \kappa \ge \theta = \text{cf}(\theta)$ <u>then</u> the condition in 2.8 holds.*

Proof. Just let $\Gamma_0 = \{\bigvee_{i<\theta} \neg p_i\}, \Gamma_i = \Gamma_0 \cup \{p_j : j < i\}$. $\square$

{2b.22}

Conclusion 2.14. 1) $\mathbf{C}_\kappa = \{\theta : \theta = \text{cf}(\theta)$ and for every λ and a.e.c. $\mathfrak{k}$ with $\text{LST}(\mathfrak{k}) \le \kappa, |\tau_{\mathfrak{k}}| = \kappa$ have (λ,θ)-compactness of type$\}$ is the class $\{\theta : \theta = \text{cf}(\theta) > \kappa$ and there is a uniform κ^+-complete ultrafilter on $\theta\}$.
2) In $\mathbf{C}_\kappa$ we can replace "every λ" by $\lambda = 2^\theta + \kappa$.

Proof. Put together 2.10,2.16. $\square_{2.13}$

Of course, a complimentary result (showing the main claim is best possible) is:

{b2.23}

Claim 2.15. *If $\mathfrak{k}'$ is an a.c.c., $\text{LST}(\mathfrak{k}') \le \kappa$ and on θ there is a uniform κ^+-complete ultrafilter on θ and θ is regular and λ any cardinality <u>then</u> $\mathfrak{k}'$ has (λ,κ)-compactness of types.*

Proof. Write down a set of sentences on $\mathbb{L}_{\kappa^+,\kappa^+}(\tau^+_{\mathfrak{k}})$ expressing the demands.

Let $\langle M_i : i \le \theta\rangle$ be $<_{\mathfrak{k}}$-increasing continuous, $\|M_i\| \le \lambda, p_i = \mathbf{tp}_{\mathfrak{k}}(a_i, M_i, N_i)$ so $M_i \le_{\mathfrak{k}} N_i$ such that $i < j < \theta \Rightarrow p_i = p_j \restriction M_i$. Without loss of generality $\|N_i\| \le \lambda$.

Let $\langle N_{i,j,\ell} : \ell \le n_{i,j,\ell}\rangle, \pi_{i,1}$ witness $p_i = p_j \restriction M_i$ for $i < j < 0$ (i.e. $M_i \le_{\mathfrak{k}} N_{i,j,\ell}$ (without loss of generality $\|N_{i,j,\ell}\| \le \lambda$), $N_{i,j,0} = N_i, a_i \in N_{i,j,\ell}$, $\bigwedge\limits_{\ell < n_{i,j,\ell}} (N_{i,j,\ell} \le_{\mathfrak{k}} N_{i,j,\ell+1} \vee N_{i,j,\ell+1} \le_{\mathfrak{k}} N_{i,j,\ell}$ and $\pi_{i,j}$ be an isomorphism from N_j onto $N_{i,j,n_{i,j}}$ over M_i mapping a_j to a_i.

Let $\tau^+ = \tau \cup \{F_{\varepsilon,n} : \varepsilon < \kappa, n < \omega\}$, $\text{arity}(F_{\varepsilon,n}) = n$. Let $\langle M_i^+ : i \le \theta\rangle$ be $\subseteq$-increasing, M_i^+ a τ^+-expansion of M_i such that $u \subseteq M_i^+ \Rightarrow M_i \restriction c\ell_{M_i^+}(u) \le_{\mathfrak{k}} M_i$. Similarly $(N^{+,\varepsilon}_{i,j,\ell} : \ell \le n_{i,j,\ell}); \varepsilon = 1, \ell$ such that $N^{+,\varepsilon}_{i,j,\ell}$ is a τ^+-expansion of $N_{i,j,\varepsilon}$ as above such that $(\forall \ell < n_{i,j,\ell})(\exists \varepsilon \in \{1,2\})(N^{+,\varepsilon}_{i,j,\ell} \subset N^{+,\varepsilon}_{i,j,\ell+1} \vee N^{+,\varepsilon}_{i,j,\ell+1} \subseteq N^{+,\varepsilon}_{i,j,\ell})$.

Now write down a translation of the question, "is there p such that ..." $\square_{2.15}$

[1]or directly as Γ_i has Skolem functions

{b2.26}

Claim 2.16. *If D is a uniform κ-complete ultrafilter on $\theta, \langle M_i : i \leq \theta\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, $p_i \in \mathscr{S}^\alpha_{\mathfrak{k}}(M_i)$ as witnessed by (N_i, a_i) for $i < \kappa, p_i = p_j \restriction M_i$ for $i < j < \kappa$ as witnessed by $(\pi_i, \langle N_{i,j,\ell} : \ell \leq m_{i,j}\rangle$ as in the proof above.*
1) There is $p_\kappa \in \mathbf{S}^\alpha(M_\theta)$ such that $i < \theta \Rightarrow p_\kappa \restriction M_i$.
2) In fact for each $i < \kappa$ let $\mathcal{U}_i \in D$ be such that $i < j \in \mathcal{U}_i \Rightarrow n_{i,j} = n^*_i$. Let $N_{i,\kappa,\ell} = \prod\limits_{j \in \mathcal{U}_i} N_{i,j,\ell}/D$. So $\langle N_{i,\kappa,\ell} : \ell \leq n^*_\ell\rangle$ are as above. Let $M = \prod\limits_{i<\kappa} M_i/D, \pi_{i,\kappa} = \prod\limits_{j \in \mathcal{U}_i} \pi_{i,j}/D$, etc.

3. On some stability spectrums of an a.e.c. {a1.5}

Convention 3.1. $\mathfrak{k}$ is an a.e.c. with amalgamation. {a1.7}

Definition 3.2. For $\theta \geq$ LST($\mathfrak{k}$). We say $\mathfrak{k}$ is (λ, θ)-stable <u>when</u> $M \in K^{\mathfrak{k}}_\lambda \Rightarrow |\mathbf{S}(M)/E^\theta_M| \leq \lambda$ where

$$pE^\theta_M q \Leftrightarrow (\forall N)(N \leq_{\mathfrak{k}} M \wedge \|N\| \leq \theta \Rightarrow p{\restriction}N = q{\restriction}\theta).$$

Theorem 3.3. *Fixing θ the class $\{\lambda : \mathfrak{k}$ is (λ, θ)-stable$\}$; behave as in* [Sh:3]. {a1.11}

Remark 3.4. See [Sh:734] = [Sh:h, V,§7] or [Sh:702] if not covered. {a1.13}

Definition 3.5. $\kappa_\theta(\mathfrak{k}) :=$ Min$\{\kappa \leq \theta^+$: there is no sequence $\langle M_i : i \leq \kappa \rangle$ which is $\leq_{\mathfrak{k}}$-increasing continuous, $\|M_i\| \leq \theta$ and $p \in \mathscr{S}(M_\kappa)$ such that $p{\restriction}M_{i+1}$ strongly (θ)-split over $M_i\}$. {a1.17}

Claim 3.6. *1) If $\lambda > 2^\theta$ and $\mathfrak{k}$ is not (λ, θ)-stable <u>then</u> for some $\kappa \leq \theta^+$ satisfying $\lambda^\kappa > \lambda$ we have $\kappa < \kappa_\theta(\mathfrak{k})$.*
2) If $\lambda > \theta, \lambda^\kappa > \lambda$ <u>then</u> $\kappa < \kappa_\theta(\mathfrak{k})$ <u>then</u> $\mathfrak{k}$ not (λ, θ)-stable. {a1.19}

Conclusion 3.7. $(-, \theta)$-stability spectrum - behave as in [Sh:3]. {a1.21}

Discussion 3.8. We can look at $\lambda \in [\theta, 2^\theta)$ using splitting rather than strongly splitting. {a1.23}

It seems to me the main question is

Question 3.9. Assume $(\exists \theta \geq$ LS($\mathfrak{k}$)$)(\kappa_\theta(\mathfrak{k}) > \aleph_0)$. {a1.26}
What can you say on Min$\{\theta : \kappa_\theta(\mathfrak{h}) > \aleph_0, \theta \geq$ LST($\mathfrak{k}$)$\}$?

Question 3.10. Assume GCH can we find an a.e.c. $\mathfrak{k}$ such that: $(\forall \theta \geq$ LST($\mathfrak{k}$)$)(\kappa_\theta(\mathfrak{k}) = \aleph_0)$ <u>but</u> unstable in every regular $\lambda >$ LST($\mathfrak{k}$)? {a1.31}

References

[Bal09] John Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, Providence, RI, 2009.

[Sh:h] Saharon Shelah, *Classification Theory for Abstract Elementary Classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[Sh:3] ______, *Finite diagrams stable in power*, Annals of Mathematical Logic **2** (1970), 69–118.

[Sh:88] ______, *Classification of nonelementary classes. II. Abstract elementary classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Mathematics, vol. 1292, Springer, Berlin, 1987, Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T., pp. 419–497.

[Sh:88r] ______, *Abstract elementary classes near* $\aleph_1$, Chapter I. 0705.4137. arxiv:0705.4137.

[Sh:300] ______, *Universal classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Mathematics, vol. 1292, Springer, Berlin, 1987, Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T., pp. 264–418.

[Sh:702] ______, *On what I do not understand (and have something to say), model theory*, Mathematica Japonica **51** (2000), 329–377, arxiv:math.LO/9910158.

[Sh:734] ______, *Categoricity and solvability of A.E.C., quite highly*, arxiv:0808.3023.

[BlSh:862] John Baldwin and Saharon Shelah, *Examples of non-locality*, Journal of Symbolic Logic **73** (2008), 765–782.

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

E-mail address: shelah@math.huji.ac.il
URL: http://shelah.logic.at