MAXIMAL FAILURES OF SEQUENCE LOCALITY IN A.E.C.

SH932

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Abstract. We are interested in examples of a.e.c. with amalgamation having some (extreme) behaviour concerning types. Note we deal with $\mathfrak{t}$ being sequence-local, i.e. local for increasing chains of length a regular cardinal (for types, equality of all restrictions imply equality). For any cardinal $\theta \geq \aleph_0$ we construct an a.e.c. with amalgamation $\mathfrak{t}$ with $\text{LST}(\mathfrak{t}) = \theta, |\tau_K| = \theta$ such that \{ $\kappa : \kappa$ is a regular cardinal and $\mathcal{K}$ is not $(2^{\kappa}, \kappa)$-sequence-local \} is maximal. In fact we have a direct characterization of this class of cardinals: the regular $\kappa$ such that there is no uniform $\kappa^+$-complete ultrafilter (on any $\lambda > \kappa$). We also prove a similar result to "$(2^{\kappa}, \kappa)$-compact for types".

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0. Introduction

Recall a.e.c. (abstract elementary classes); were introduced in [Sh:88]; and their (orbital) types defined in [Sh:300], see on them [Sh:h], [Bal09]. It has seemed to me obvious that even with $\mathfrak{t}$ having amalgamation, those types in general lack the good properties of the classical types in model theory. E.g. $\text{"}(\lambda,\kappa)$-sequence-locality

**Definition 0.1.** 1) We say that an a.e.c. $\mathfrak{t}$ is a $\text{(}\lambda,\kappa)$-sequence-local (for types) when $\kappa$ is regular and for every $\leq_{\mathfrak{t}}$-increasing continuous sequence $\langle M_i : i \leq \kappa \rangle$ of models of cardinality $\lambda$ and $p, q \in \mathcal{S}(M_\kappa)$ we have $(\forall i < \kappa)(p \upharpoonright M_i = q \upharpoonright M_i) \Rightarrow p = q$. We omit $\lambda$ when we omit "$|M_i| = \lambda$".

2) We say an a.e.c. $\mathfrak{t}$ is $\text{(}\lambda,\kappa\text{)}$-local when: $\kappa \geq \text{LST}(\mathfrak{t})$ and if $M \in \mathfrak{t}_\lambda$ and $p_1, p_2 \in \mathcal{S}(M)$ and $N \leq_{\mathfrak{t}} M \wedge \|N\| \leq \kappa \Rightarrow p_1|N = p_2|N \text{ then } p_1 = p_2$.

3) We may replace $\lambda$ by $\leq \lambda, < \lambda, [\mu, \lambda]$ with the obvious meaning (and allow $\lambda$ to be infinity).

Of course, being sure is not a substitute for a proof, some examples were provided by Baldwin-Shelah [BlSh:862, §2]. There we give an example of the failure of $\text{(}\lambda,\kappa)$-sequence-locality for $\mathfrak{t}$-types in ZFC for some $\lambda, \kappa$, actually $\kappa = \aleph_0$. This was done by translating our problems to abelian group problems. While those problems seem reasonable by themselves they may hide our real problem.

Here in §1 we get $\mathfrak{t}$, an a.e.c. with amalgamation with the class $\{ \kappa : (< \infty, \kappa)$-sequence-localness fail for $\mathfrak{t}\}$ being maximal; what seems to me a major missing point up to it, see Theorem 1.3. Also we deal with "compactness of types" getting unsatisfactory results - classes without amalgamation; in [BlSh:862] this was done only in some universes of set theory but with amalgamation; see §2.

We relay on [BlSh:862] to get that $\mathfrak{t}$ has the JEP and amalgamation.

**Question 0.2.** Can $\{ \kappa : \mathfrak{t} \text{ is } (< \infty, \kappa)\text{-local}\}$ be "wild"? E.g. can it be all odd regular alephs? etc?

Note that for this the present translation theorem of [BlSh:862] is not suitable. In §2 we deal with sequence-compactness of types.

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1. An a.e.c. with maximal failure of being local

Claim 1.1. Assume

\( \oplus_1 \) (a) \( \kappa = \text{cf}(\kappa) > \theta > \aleph_0 \) or just \( \kappa = \text{cf}(\kappa) \geq \aleph_0, \theta \geq \aleph_0 \)
(b) there is no uniform \( \theta^\ast \)-complete ultra-filter \( D \) on \( \kappa \)
(c) \( \tau_\theta \) is a vocabulary of cardinality \( \theta \) consisting of \( \theta^n \)-place predicates with each \( n \) (and no more say \( \{ R_{\gamma,n} : \gamma < \theta, n < \omega \}, n = \text{arity}(R_{\gamma,n}) \)).

Then

(\( \Box \)) there are \( I_\alpha, M_{\ell,\alpha}, \pi_{\ell,\alpha} \) (for \( \ell \in \{ 1, 2 \} \) and \( \alpha \leq \kappa \)), \( g_\alpha \) (for \( \alpha < \kappa \)) satisfying:

(a) \( I_\alpha \) is a set of cardinality \( \theta^\kappa \), is \( \subseteq \)-increasing continuous with \( \alpha \)
(b) \( M_{\ell,\alpha} \), a \( \tau_\theta \)-model of cardinality \( \leq \theta^\kappa \), is increasing continuous with \( \alpha \)
(c) \( \pi_{\ell,\alpha} \) is a function from \( M_{\ell,\alpha} \) onto \( I_\alpha \), increasing continuous with \( \alpha \)
(d) \( |\pi_{\ell,\alpha}^{-1}(\{ t \})| \leq \theta^{\aleph_0} \) for \( t \in I_\alpha, \alpha \leq \kappa \) and \( \ell \in \{ 1, 2 \} \)
(e) if \( t \in I_{\alpha+1}\backslash I_\alpha \) then \( \pi_{\ell,\alpha}^{-1}(\{ t \}) \subseteq M_{\ell,\alpha+1}\backslash M_{\ell,\alpha} \)
(f) for \( \alpha < \kappa \), \( g_\alpha \) is an isomorphism from \( M_{1,\alpha} \) onto \( M_{2,\alpha} \) respecting \( \{ \pi_{1,2}, \pi_{2,\kappa} \} \) which means \( a \in M_{1,\alpha} \Rightarrow \pi_{1,\alpha}(a) = \pi_{2,\alpha}(g_\alpha(a)) \)
(g) for \( \alpha = \kappa \) there is no isomorphism from \( M_{1,\alpha} \) onto \( M_{2,\alpha} \) respecting \( \{ \pi_{1,\kappa}, \pi_{2,\kappa} \} \).

Proof. Follows from 1.2 which is just a fuller version adding to \( \tau_\theta \) unary function \( F_c \) for \( c \in G \); this is just a notational change when \( \theta^{\aleph_0} = \theta \). Otherwise see \( (\ast)_1 \) of the proof of 1.2; anyhow we shall use 1.2. \( \Box_{1,1} \)

Claim 1.2. Assuming \( \oplus_1 \) of 1.1 we have:

(\( \Box \)) there are \( I_\alpha, M_{\ell,\alpha}, \pi_{\ell,\alpha} \) (for \( \ell \in \{ 1, 2 \} \) and \( \alpha \leq \kappa \)), \( g_\alpha \) (for \( \alpha < \kappa \)) and \( G \) such that:

(a) \( G \) is an additive (so abelian) group of cardinality \( \theta^{\aleph_0} \)
(b) \( I_\alpha \) is a set, increasing continuous with \( \alpha, |I_\alpha| = \theta^\kappa \)
(c) \( M_{\ell,\alpha} \) is a \( \tau_\alpha^\kappa \)-model, increasing continuous with \( \alpha \), of cardinality \( \theta^\kappa \), where \( \tau_\alpha^\kappa = \tau_\theta \cup \{ F_c : c \in G \}, F_c \) a unary function symbol, \( \tau_\theta \) is from \( \oplus_1(c) \) of 1.1
(d) \( \pi_{\ell,\alpha} \) is a function from \( M_{\ell,\alpha} \) onto \( I_\alpha \), increasing continuous with \( \alpha \)
(e) \( F_{c_{\ell,\alpha}} \) (\( c \in G \)) is a permutation of \( M_{\ell,\alpha} \), increasing continuous with \( \alpha \)
(f) \( \pi_{\ell,\alpha}(a) = \pi_{\ell,\alpha}(F_{c_{\ell,\alpha}}(a)) \)
(g) \( F_{c_{\ell,\alpha}}(F_{c_{\ell,\alpha}}(a)) = F_{c_{\ell,\alpha}}(a) \)
(h) \( \pi_{\ell,\alpha}(a) = \pi_{\ell,\alpha}(b) \Leftrightarrow \bigvee_{c \in G} F_{c_{\ell,\alpha}}(a) = b \)
(i) for \( \alpha < \kappa \), \( g_\alpha \) is an isomorphism from \( M_{1,\alpha} \) onto \( M_{2,\alpha} \) which respects \( \{ \pi_{1,\alpha}, \pi_{2,\alpha} \} \) which means \( a \in M_{1,\alpha} \Rightarrow \pi_{1,\alpha}(a) = \pi_{2,\alpha}(g_\alpha(a)) \)
(j) there is no isomorphism from \( M_{1,\kappa} \) \( \mid \tau_\theta \) onto \( M_{2,\kappa} \) \( \mid \tau_\theta \) respecting \( \{ \pi_{1,\kappa}, \pi_{2,\kappa} \} \).

Proof. Let

(\( \ast \)) \( \sigma = \theta^{\aleph_0} \) so \( \sigma = \theta^{\aleph_0} \)}
Note that
\[ (*)_2 \quad (a) \quad |G| = \sigma \]
\[ (b) \quad (A_\beta : \beta \leq \kappa) \text{ is a } \subseteq\text{-increasing continuous, each } A_\beta \text{ a set of cardinality } \sigma^\kappa = \theta^\kappa \]
\[ (c) \quad (I_\beta : \beta \leq \kappa) \text{ is } \subseteq\text{-increasing continuous, each } I_\beta \text{ of cardinality } \sigma^\kappa = \theta^\kappa \]
\[ (d) \quad \pi_\beta \text{ is a mapping from } A_\beta \text{ onto } I_\beta \]
\[ (e) \quad \text{if } t \in I_\alpha \subseteq I_\beta \text{ then } \pi_\beta^{-1}(t) = \pi_\alpha^{-1}(t) \text{ has cardinality } |G| = \sigma \]
\[ (f) \quad \text{if } t \in I_{\alpha+1} \setminus I_\alpha \text{ then } \pi_{\alpha+1}^{-1}(t) \subseteq A_{\alpha+1} \setminus A_\alpha \]
\[ (g) \quad \text{if } \alpha \leq \beta \leq \kappa \text{ then } g_\beta \text{ maps } A_\alpha \text{ onto itself and } g_\beta \circ g_\beta \text{ is the identity.} \]

For each \( n < \omega \) and \( \beta \leq \kappa \) we define equivalence relations \( E'_n, E_n, E_{n, \beta} \) on \( ^n(A_\beta) \):
\[ (*)_3 \quad \bar{a}E'_n, b \iff \pi_\beta(\bar{a}) = \pi_\beta(\bar{b}) \text{ where of course } \pi_\beta((a_\ell : \ell < n)) = (\pi_\beta(a_\ell) : \ell < n) \]
\[ (*)_4 \quad \bar{a}E_n, b \iff \bar{a}E'_n, b \text{ and there are } k < \omega \text{ and } \bar{a}_0, \ldots, \bar{a}_k \text{ such that} \]
\[ (i) \quad \bar{a}_\ell \in ^n(A_\beta) \]
\[ (ii) \quad \bar{a} = \bar{a}_0 \]
\[ (iii) \quad \bar{b} = \bar{a}_k \]
\[ (iv) \quad \text{for each } \ell < k \text{ for some } \alpha_1, \alpha_2 < \kappa \text{ we have } g_{\alpha_1}^{-1}(g_{\alpha_2}(\bar{a}_\ell)) \text{ is well defined and equal to } \bar{a}_{\ell+1} \text{ or } g_{\alpha_2}(g_{\alpha_1}^{-1}(\bar{a}_\ell)) \text{ is well defined and equal to } \bar{a}_{\ell+1}. \]

Note:
\[ (*)_4.1 \quad (a) \quad \text{the two possibilities in } (*)_4(iv) \text{ are one as } g_{\alpha}^{-1} = g_\alpha \text{ so the first one is a special case of the second;} \]
\[ (b) \quad g_\alpha \text{ does not preserve } \bar{a}/E_{n, \beta}, \text{ in fact, } a, g_\alpha(a) \text{ are never } E_{n, \beta} \text{ equivalent;} \]
\[ (c) \quad \text{clearly they are well defined iff } (\forall \ell \leq k)[\bar{a}_\ell \in ^n(A_{\min(\alpha_1, \alpha_2)})] \text{ because if } \alpha \leq \beta \text{ then } g_\beta \text{ maps } A_\beta \text{ onto itself because } g(a_{f_1, u_1}) = a_{f_1, u_2} \Rightarrow |u_1| + 1 = |u_2| \mod 2 \]
\[ (d) \quad \text{if } \alpha \leq \beta, a \in A_\alpha, \text{ then } g_\beta \text{ maps } a/E_{n, \beta} \text{ onto itself} \]
\[ (e) \quad \text{if } \alpha, \beta \leq \kappa, \text{ then } g_\alpha, g_\beta \text{ commute (on the intersection of their domains, } A_{\min(\alpha, \beta)}. \]
Why? E.g. for clause (b) note clause (d).

Note

(*)5  (a) $E_{n,\beta}'$, $E_{n,\beta}$ are indeed equivalence relations on $n(A_\beta)$
(b) $E_{\beta,n}$ refine $E'_{\beta,n}$
(c) if $n < \omega$, $\bar{a} \in n(A_\beta)$ then $\bar{a}/E_{n,\beta}$ has at most $\sigma$ members (really exactly two but we shall use only its having $\leq 2^\sigma$ members)
(d) if $\alpha < \beta \leq \kappa$ then $E_{n,\beta}' \upharpoonright n(A_\alpha) = E_{n,\alpha}'$ and $E_{n,\beta}' \upharpoonright n(A_\alpha) = E_{n,\alpha}$
(read (*)4(iv) carefully!)
(e) if $\alpha < \beta \leq \kappa$, $\bar{a} \in n(A_\alpha)$ and $\bar{b} \in \bar{a}/E_{n,\beta}'$ then $\bar{b} \in n(A_\alpha)$
(f) if $g_\alpha(\bar{a}) = \bar{b}_\ell$ for $\ell = 1, 2$ then: $\bar{a}_1E_{n,\beta}'\bar{a}_2$ iff $\bar{b}_1E_{n,\beta}'\bar{b}_2$.

Now we choose a vocabulary $\tau^*_\alpha$ of cardinality $2^\sigma$ (but see (*)12) and for $\alpha \leq \kappa$ we choose a $\tau^*_\alpha$-model $M_{1,\alpha}$ such that:

(*)6  (a) $M_{1,\alpha}$ increasing with $\alpha$ with universe $A_\alpha$
(b) assume that $\bar{a}, \bar{b}$ are $E_{n,\alpha}$-equivalent (so $\bar{a}, \bar{b} \in n(A_\alpha)$ and $\pi_\alpha(\bar{a}) = \pi_\alpha(\bar{b})$); then $\bar{a}, \bar{b}$ realize the same quantifier free type in $M_{1,\alpha}$ iff $\bar{a}E_{n,\alpha}\bar{b}$
(c) $\tau^*_\alpha = \{E_n : n < \omega\} \cup \{F_c : c \in G\} \cup \{R_c : e \in \sigma\} \cup \{R_{n,i} : i < \sigma, n < \omega\}$

where $E_n, R_c$ are two-place predicates, $F_c$ a unary function symbol, $R_{n,i}$ is $n$-place predicate
(d) for every function $e \in \sigma$

$R^M_{e,\alpha} = \{(a_{f_1, \beta_1, u_1}, a_{f_2, \beta_2, u_2}) \in A_\alpha \times A_\alpha : f_1 = e \circ f_2 \text{ and if } i < \sigma \text{ then } i \in u_1 \text{ iff } ([i \in u_2 : e(j) = i]) \text{ is odd}\}$

recalling $f_\ell \in \sigma$

(e) $E^M_{\alpha}$ and $F^M_{\alpha} = F_c$ is defined by $f_\ell : A_\alpha \to A_\alpha$ satisfies $F_c(a_{f,\alpha,u}) := a_{f,\alpha,u+\alpha}$

(f) if $\alpha \leq \beta_\ell < \kappa$ for $\ell = 1, 2$ then $g_\beta_1^{-1}g_\beta_2|A_\alpha$ is an automorphism of $M_{1,\alpha}$.

Why is this possible? First, we shall show that for each $\alpha < \kappa$, $g_\alpha$ maps $R^M_{e,\alpha}$ onto itself.

Assume we are given a pair $(a_{f_1, \beta_1, u_1}, a_{f_2, \beta_2, u_2})$ from $A_\alpha \times A_\alpha$ so $\beta_1, \beta_2 < 1 + \alpha$ and $f_1 = e \circ f_2$ so

(*)6.1 $(a_{f_1, \beta_1, u_1}, a_{f_2, \beta_2, u_2}) \in R^M_{e,\alpha}$ iff $u_1 = \{e(j) : j \in u_2 \text{ and } (\exists \text{odd } \ell \in u_2)(e(i) = e(j))\}$.

Why? Read (*)6 carefully, in particular note that if $i \notin \{e(j) : j \in u_2\}$ then $i \notin u_1$.

(*)6.2 $(g_\alpha(a_{f_1, \beta_1, u_1}), g_\alpha(a_{f_2, \beta_2, u_2})) \in R^M_{e,\alpha}$ iff $(a_{f_1, \beta_1, u_1+\ell(f_1(\alpha))}, a_{f_2, \beta_2, u_2+G(f_2(\alpha))}) \in R^M_{e,\alpha}$ iff $u_1 + \ell(f_1(\alpha)) = \{e(j) : j \in u_2 + G(f_2(\alpha)) \text{ and } (\exists \text{odd } \ell \in (u_2 + G(f_2(\alpha)))(e(i) = e(j)))\}$. 
[Why? Inside (\(\ast\))\(_{6.2}\) the first “iff” holds by the definition of \(g_\alpha\), the second “iff” holds as in (\(\ast\))\(_{6.1}\).]

But \(f_1 = e \circ f_2\) hence

\[(\ast)_{6.3} f_1(\alpha) = e(f_2(\alpha))\]

\[(\ast)_{6.4} \text{letting } x = f_2(\alpha) < \sigma \text{ we have } u_1 = \{e(j) : j \in u_2 \text{ and } (\exists \text{odd}_{\ell}(e(i) = e(j)))\} \iff u_1 + G \{e(x) = \{e(j) : j \in u_2 + G \{x\} \text{ and } \exists \text{odd}_{\ell}(e(i) = e(j))).}\]

[Why? Check by cases according to whether \(x \in u_2\) and whether \(e(x) \in u_1\). I.e. by “\(G\) is of order two” it suffices to prove the “only if” so assume the first equality in \((\ast))_{6.2}\). If \(e(x) \notin u_1\), then just add \(e(x)\) to both sides. Similarly if \(e(x) \in u_1 \cap X \notin u_2\) and if \(e(x) \in u_1 \cap \} x \in u_2\).

So together we get equivalence, so the “first” holds.

Second, for defining the \(R^{M_{1,\alpha}}\)’s

\[(\ast)_{6.5} \text{ (a) for each } N \text{ let } E''_n \text{ be the following equivalence relation on } nG : \bar{u}_1E''_n \bar{u}_2 \text{ iff for some } v \in G, |v| \text{ is even and } \bigwedge_{\ell < n} u_{1,\ell} + G v = u_{2,\ell} \]

(b) let \(\{X_{n,1} : i < \sigma\}\) list the \(E''_n\)-equivalence classes

(c) let \(R^{M_{1,\alpha}}_{n,\iota} = \{\bar{a} \in n(A) : \bar{a} = \{a_{f,\iota,\alpha,\omega} : f < n\} \text{, then } (u_{1,\ell} : \ell < n) \in X_{n,1}\}\).

This completed the choice of \(M_{1,\alpha}\). Third, \(g_\alpha\) preserves “\(\bar{a}, \bar{b}\) are \(E_{n,\alpha}\)-equivalent”, “\(\bar{a}, \bar{b}\) are \(E'_{n,\alpha}\)-equivalent” and their negations. That is, \(\bar{a}, g_\alpha(\bar{a})\) are not \(E_{n,\alpha}\)-equivalent, but as \((\forall \beta)(g_\beta = g_\beta^{-1})\), \(\bar{a}, \bar{b}\) being \(E_{n,\alpha}\)-equivalent means that there is an even length pass from \(\bar{a}\) to \(\bar{b}\), in the graph \((\hat{c}, g_\beta(\hat{c})) : \beta \in [\gamma, \kappa) \text{ and } \hat{c} \in n(A_\gamma)\) where \(\gamma = \min\{\gamma : \bar{a}, \bar{b} \in n(A_\gamma)\}\).

Fourth, no problem in the \(M_{1,\alpha}\)’s are increasing by \((\ast))_{5}(d)\), just check that.

Fifth, \(g_\alpha\) commutes with \(F_{\kappa,\alpha}\) for \(c \in G\) because \(G\) is an Abelian group.

Sixth, we should check clause \((\ast))_{6}(f)\). Now \(g_\beta^{-1}g_\beta[A_\alpha] = (g_\beta[A_\alpha])(g_\beta^{-1}[A_\alpha])\) by \((\ast))_{2}(g)\) and it has order 2 because \(G\) is of order 2 and it maps \(E_{n,\alpha}\) to itself by the “third”, commute with \(F_{\kappa,\alpha}\) by the fifth, maps \(R^{M_{1,\alpha}}\) to itself by the “first”.

Lastly, it maps \(R^{M_{1,\alpha}}\) to itself by \((\ast))_{6}d). So we are done proving \((\ast))_{6}d).\]

\((\ast))_{7} \text{ for } \alpha < \kappa \text{ let } M_{2,\alpha} \text{ be the } \tau_\alpha^n\text{-model with universe } A_\alpha \text{ such that } g_\alpha \text{ is an isomorphism from } M_{1,\alpha} \text{ onto } M_{2,\alpha}.

Now we note

\[(\ast))_{8} \text{ for } \alpha < \beta < \kappa, M_{2,\alpha} \subseteq M_{2,\beta}.\]

[Why? By the definitions of \(M_{1,\gamma}, \gamma, E'_{n,\gamma}, E_{n,\gamma}\), in particular, the “first” and “third”, in “why \((\ast))_{6}\)”, fourth, i.e. \((\ast))_{5}(d)\) .]

\[(\ast))_{9} \text{ let } M_{2,\kappa} := \cup\{M_{2,\alpha} : \alpha < \kappa\}, \text{ well defined by } (\ast))_{8}\]

\[(\ast))_{10} \text{ let } \pi_{\ell,\beta} = \pi_{\beta} \text{ for } \ell = 1, 2 \text{ and } \beta < \kappa\]

\[(\ast))_{11} \text{ except clause (j) the demands in the conclusion of } \exists \text{ of 1.2 holds easily.}\]

[Why? Just check.]
[Why? As there is a model \( M \) of cardinality \( 2^\sigma \) with \( |\tau_M| = \theta \) omitting a quantifier free type \( p \) such that \( M \subseteq N \land M \models N \models p \). Such \( M \) exists as \( \sigma = \theta^\omega \) and clause (b) of the assumption \( \oplus_1 \) of 1.2, 1.1.]

Note

\((*)_{13} \) if \( (a_{f,\alpha,u_1},a_{f,\alpha,u_2}) \) is \( E_{2,\alpha} \)-equivalent to \((a_{f,\alpha,v_1},a_{f,\alpha,v_2}) \) then \( G \models \lnot u_1 = u_2 = v_1 - v_2 \).

[Why? By induction on the \( k \) from \((*)_{14}\).]

So to finish we assume toward contradiction

\( \exists \ h \ \text{is an isomorphism from} \ M_{1,\kappa} \ \text{onto} \ M_{2,\kappa} \ \text{which respects} \ (\pi_1,\alpha,\pi_2,\alpha) \ \text{for} \ \kappa < \kappa \).

So trivially

\((*)_{15} \) \( h(a_{f,\alpha,u}) \in \{a_{f,\alpha,v} : v \in G \} \) and \( \bar{a} \in \{}(A_n) \Rightarrow h(\bar{a}) \in \bar{a}/E_{n,\alpha} \subseteq \bar{a}/E'_{n,\alpha} \).
Why? For clause (a) use \( \circ 8 \) for the function \( e = \text{id}_\sigma \) and \( f_1 = f_2 = f \). Clause (b) follows. Clause (c) holds by \( \circ 6 \) equivalently by \( \circ 5 \). 

\( \circ 8 \) there are \( f_\sigma \in \kappa \) and \( \alpha_\sigma < \kappa \) such that:

(i) if \( f_\sigma \leq f \in \kappa \) and \( \alpha < \kappa \) then \( |u_{f_\alpha,f_\sigma}| = |u_{f_\alpha} \|

(ii) moreover if \( f_\sigma = e \circ f \) where \( e \in \kappa \) and \( f \in \kappa, \alpha < \kappa \) then \( e \upharpoonright u_{f_\alpha} \) is one-to-one from \( u_{f_\alpha} \) onto \( u_{f_\alpha} \) so \( n(f_\sigma) = n(f) \)

(iii) if \( \alpha < \kappa, f_1 = f \circ f_2, f_\sigma = e_1 \circ f_1, f_\sigma = e_2 \circ f_2 \) so \( e_1, e_2 \in \kappa \), then \( e \upharpoonright u_{f_\alpha,\sigma} \) is one-to-one onto \( u_{f_\alpha,\sigma} \).

Why? First note that clause (ii), (iii) follows from clause (i). Second, if clause (i) fails, then we can find a sequence \( \langle f_n, \alpha_n, e_n \rangle : n < \omega \rangle \) such that

\( \circ 9 \) \( n(f_\sigma) > 0 \), i.e. \( \alpha < \kappa \Rightarrow u_{f_\alpha,\sigma} \neq \emptyset \).

Why? If \( (\forall f \in \kappa)(\forall \alpha < \kappa)(u_{f_\alpha,\sigma} = \emptyset) \) then (by \( \circ 4 \)) we deduce \( h \) is the identity, contradiction. Otherwise assume \( u_{f_\alpha,\sigma} \neq 0 \) hence as in the proof of \( \circ 8 \) there is \( f_\sigma \) such that \( f_\sigma \leq f' \land f \leq f' \) so by \( \circ 5 \) and \( \circ 8 \) we have \( 0 < |u_{f_\alpha}| \leq |u_{f_\sigma,\alpha}| = |u_{f_\alpha,\sigma}| \).

\( \circ 10 \) if \( f \in \kappa, \alpha < \kappa \) and \( i \in u_{f_\alpha} \) then \( \kappa = \sup \{ \beta < \kappa : \alpha < \beta \) and \( f(\beta) = i \} \).

Why? If not, let \( \beta(\sigma) < \kappa \) be \( > \sup \{ \beta < \kappa : \alpha < \beta, f(\beta) = i \} \) and \( > \omega \) and \( > \alpha \). Let \( \Sigma = \{ (a_{f_\alpha,\sigma}a, g_{f_\beta(\sigma)}) : u \in G, i \notin u \} \). Now for every \( \beta \in \beta(\sigma, \kappa) \) the function \( g_{f_\beta} \) maps the set \( Y \) onto itself (see its definition in \( (\circ 7) \)) hence by the definition of \( E_{2, \beta(\sigma) + 1} \) (in \( \circ 4) \) it follows that \( \bar{a} \in Y \Rightarrow \bar{a}/E_{2, \beta(\sigma) + 1} \subseteq Y \) and as \( h \) respects \( (\pi_1, \beta(\sigma) + 1, \pi_2, \beta(\sigma) + 1) \) it follows that \( h(\bar{a}) \subseteq \bar{a}/E_{2, \beta(\sigma) + 1} \) and so \( \kappa > \gamma \geq \beta(\sigma) + 1 \Rightarrow g_{\gamma}^{-1}(h(\bar{a})) \in \bar{a}/E_{2, \beta(\sigma) + 1} \).

Now for \( \bar{a} \in Y \), the pairs \( \bar{a}, h(\bar{a}) \) realizes the same quantifier free type in \( M_{1, \beta(\sigma) + 1}, M_{2, \beta(\sigma) + 1} \) respectively, hence by the choice of \( M_{2, \beta(\sigma) + 1} \) the pairs \( \bar{a}, g_{\beta(\sigma) + 1}^{-1}(h(\bar{a})) \) realize the same quantifier free type in \( M_{1, \alpha} \). By \( (\circ 10)(b) \) recalling \( g_{\beta(\sigma) + 2}^{-1}(h(\bar{a})) \in \bar{a}/E_{2, \beta(\sigma) + 2} \) this implies that \( \bar{a}, g_{\beta(\sigma) + 2}^{-1}(h(\bar{a})) \) are \( E_{2, \beta(\sigma) + 1} \) equivalent. By the definition of \( E_{2, \beta(\sigma) + 1} \), \( g_{\beta(\sigma) + 2}^{-1}(h(\bar{a})) \) belongs to the closure of \( \{ \bar{a} \} \cup \{ g_{\gamma}^{-1} : \gamma \in (\beta(\sigma); \kappa) \} \) hence \( h(\bar{a}) \) belongs to it. But by an earlier sentence \( Y \) is closed under those functions so \( h(\bar{a}) \in Y \). Similarly \( h^{-1}(\bar{a}) \in Y \), hence \( h \) maps \( Y \) onto itself, recalling \( \circ 2 \) this implies \( i \notin u_{f_\alpha,\sigma} \), contradicting an assumption of \( \circ 10 \), so \( \circ 10 \) holds.]
Now fix $f_{\ast}, \alpha_{\ast}$ as in $\otimes_{8}$ for the rest of the proof, without loss of generality $f_{\ast}$ is onto $\sigma$ and let $u_{f_{\ast}, \alpha_{\ast}} = \{i_{\ast}^\ell : i < \ell(*)\}$ with $\langle i_{\ast}^\ell : \ell < \ell(*)\rangle$ increasing for simplicity. Now for every $f \in {^\kappa} \sigma$ such that $f_{\ast} \leq f$ and $\alpha < \kappa$ by $\otimes_{8}(ii),(iii)$ we know that if $e \in {^\kappa} \sigma \land f_{\ast} = e \circ f$ then $e$ is a one-to-one mapping from $u_{f_{\ast}, \alpha_{\ast}}$ onto $u_{f_{\ast}, \alpha_{\ast}}$; but so $e \restriction u_{f_{\ast}, \alpha_{\ast}}$ is uniquely determined by $f_{\ast}, \alpha_{\ast}, f, \alpha$ so let $i_{f_{\ast}, \alpha, \ell} \in u_{f_{\ast}, \alpha_{\ast}}$ be the unique $i \in u_{f_{\ast}, \alpha_{\ast}}$ such that $e(i) = i_{\ast}^\ell$ (equivalently $(\exists \alpha)(f(\alpha) = i \land f_{\ast}(\alpha) = i_{\ast}^\ell)$).

Now if $f_{\ast} \leq f \in {^\kappa} \sigma$ and $\alpha_{1}, \alpha_{2} < \kappa$ and we choose $e = id_{\sigma}$ so necessarily $f \restriction u_{f_{\ast}, \alpha_{1}} = e \circ f \restriction u_{f_{\ast}, \alpha_{2}}$, then $e \restriction \operatorname{Rang}(f \restriction u_{f_{\ast}, \alpha_{2}})$ map $u_{f_{\ast}, \alpha_{2}}$ onto $u_{f_{\ast}, \alpha_{1}}$ but $e$ is the identity so we can write $u_{f}$ instead of $u_{f_{\ast}}$, let $i_{f, \ell} = i_{f_{\ast}, \alpha, \ell}$ for $\ell < \ell(*)$, $\alpha < \kappa$.

Let

$$\mathbf{A} = \{A \subseteq \kappa : \text{for some } f, f_{\ast} \leq f \text{ and } \alpha < \kappa \text{ we have } f^{-1}\{i_{f, \ast}\} \setminus \alpha \subseteq A\}$$

$$\Box_{1} \mathbf{A} \subseteq \mathcal{P}(\kappa)\setminus[\kappa]^{<\kappa}.$$  

[Why? As $\kappa$ is regular, this means $A \in \mathbf{A} \Rightarrow A \subseteq \kappa \land \sup(A) = \kappa$ which holds by $\otimes_{10}$.]

$$\Box_{2} \kappa \in \mathbf{A}.$$  

[Why? By the definition of $\mathbf{A}$.]

$$\Box_{3} \text{if } A \in \mathbf{A} \text{ and } A \subseteq B \subseteq \kappa \text{ then } B \in \mathbf{A}.$$  

[Why? By the definition of $\mathbf{A}$.]

$$\Box_{4} \text{ if } A_{1}, A_{2} \in \mathbf{A} \text{ then } A := A_{1} \cap A_{2} \text{ belongs to } \mathbf{A}.$$  

[Why? Let $(f_{\ell}, e_{\ell}, \alpha_{\ell})$ be such that $f_{\ast} = e_{\ell} \circ f_{\ell}$ and $f_{\ell} \in {^\kappa} \sigma, \alpha_{\ell} < \kappa$ and $f_{\ell}^{-1}\{i_{f_{\ell}, 0}\} \setminus \alpha_{\ell} \subseteq A_{\ell}$ for $\ell = 1, 2$. Let $\operatorname{pr}: \sigma \times \sigma \rightarrow \sigma$ be one-to-one and onto and define $f \in {^\kappa} \sigma$ by $f(\alpha) = \operatorname{pr}(f_{1}(\alpha), f_{2}(\alpha))$. Clearly $f_{\ell} \leq f$ for $\ell = 1, 2$ hence $i_{f, 0}$ is well defined and $i_{f, 0} = \operatorname{pr}(i_{f_{1}, 0}, i_{f_{2}, 0})$. Now for every $\alpha < \kappa, f(\alpha) = i_{f, 0} \Rightarrow f(\alpha) = i_{f_{1}, 0} \land f(\alpha) = i_{f_{2}, 0} = i \Rightarrow \alpha \in A_{1} \land \alpha \in A_{2} \Rightarrow \alpha \in A_{1} \cap A_{2} \Rightarrow \alpha \in A$ so $f^{-1}\{i_{f, 0}\} \subseteq A$ hence $A \in \mathbf{A}$.]

$$\Box_{5} \text{ if } A \subseteq \kappa \text{ then } A \in \mathbf{A} \text{ or } \kappa \setminus A \in \mathbf{A}.$$  

[Why? Define $f \in {^\kappa} \sigma$: 

$$f(\alpha) = \begin{cases} 2f_{\ast}(\alpha) & \text{if } \alpha \in A \\ 2f_{\ast}(\alpha) + 1 & \text{if } \alpha \in \kappa \setminus A. \end{cases}$$

Let $i = i_{f_{\ast}, 0}$ so by the definition of $\mathbf{A}$ we have $f^{-1}\{i\} = f^{-1}\{i_{f_{\ast}, 0}\} \subseteq A$. But if $i$ is even then $f^{-1}\{i\} \subseteq A$ and $i$ is odd then $f^{-1}\{i\} \subseteq \kappa \setminus A$ so by $\Box_{3}$ we are done.]

$$\Box_{6} \mathbf{A} \text{ is a uniform ultrafilter on } \kappa.$$  

[Why? By $\Box_{1} - \Box_{5}$.]

$$\Box_{7} \mathbf{A} \text{ is } \sigma^{+} \text{-complete.}$$
[Why? Assume $B_\varepsilon \in \mathcal{A}$ for $\varepsilon < \sigma$ and let $B = \cap \{B_\varepsilon : \varepsilon < \sigma\}$. Define $A_\varepsilon \subseteq \kappa$ for $\varepsilon < \sigma$ as follows: $A_{1+\varepsilon} = \bigcap_{\varepsilon < \xi} B_\xi \setminus B_\xi$ (so is $\kappa \setminus B_0$ if $\varepsilon = 0$) for $\varepsilon < \sigma$ and $A_0 = B$. Clearly $\langle A_\varepsilon : \varepsilon < \sigma\rangle$ is a partition of $\kappa$, let $f \in {}^\kappa\sigma$ be such that $f \upharpoonright A_\varepsilon$ is constantly $\varepsilon$. Let $f' \in {}^\kappa\theta$ be such that $f \leq f' \wedge f_\varepsilon \leq f'$. Now $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$ is included in some $A_\varepsilon$. If $\varepsilon = 0$ this exemplifies $\bigcap_{\varepsilon < \sigma} B_\varepsilon \in \mathcal{A}$ as required. If $\varepsilon = 1 + \zeta < \sigma$, then $(f')^{-1}\{i_{f',0}\} \subseteq A_\varepsilon \subseteq \kappa \setminus B_\varepsilon$, contradiction to $\square_0$ because $B_\varepsilon \in \mathcal{A}$ and $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$.]

So by the assumptions of 1.2, that is, $\oplus_1(b)$ of 1.1 we get a contradiction, coming from the assumption “toward contradiction (j) of $\mathbb{B}$ of 1.2 fails”, so it holds and the other clauses were proved so we are done. \hfill $\square_{1.2}$

**Theorem 1.3.** For every $\theta$ there is an $t = t_0^\theta$ such that

- (a) $t$ is an a.e.c. with $\text{LST}(t) = \theta, |\tau_t| = \theta$
- (b) $t$ has the amalgamation property
- (c) $t$ admits intersections (see Definition 1.4 below)
- (d) if $\kappa$ is a regular cardinal and there is no uniform $\theta^+$-complete ultrafilter on $\kappa$, then: $t$ is not $(\leq 2^\kappa, \kappa)$-sequence-local for types, i.e., we can find an $\leq_t$-increasing continuous sequence $(\mathcal{M}_i : i \leq \kappa)$ of models and $p \neq q \in \mathcal{S}_t(M_0)$ such that $i < \kappa \Rightarrow p \upharpoonright M_i = q \upharpoonright M_i$ and $M_\kappa$ is of cardinality $\leq 2^\kappa$.

We shall prove 1.3 below. As in [BlSh:862, 1.2,34] the aim of the definition of “admit intersections” is to ensure types behave reasonably.

**Definition 1.4.** We say an a.e.c. $t$ admits intersections when there is a function $c_t(A, M)$ such that:

- (a) $c_t(A, M)$ is well defined iff $M \in K_t$ and $A \subseteq M$
- (b) $c_t(A, M)$ is preserved under isomorphisms and $\leq_t$-extensions
- (c) for every $M \in K_t$ and non-empty $A \subseteq M$ the set is $B = c_t(A, M)$ satisfies: $M \upharpoonright B \in K_t, M \upharpoonright B \leq_t M$ and $A \subseteq M \leq_t N \wedge M \leq_t N \Rightarrow B \subseteq M_1$; we may use $c_t(A, M)$ for $M'[c_t(A, M)].$

**Claim 1.5.** Assume $t$ is an a.e.c. admitting intersections. Then $\text{tp}_t(a_1, M, N_1) = \text{tp}_t(a_2, M, N_2)$ iff letting $M_t = N_t \cup c_t(M \cup \{a_t\})$, there is an isomorphism from $M_1$ onto $M_2$ over $M$ mapping $a_1$ to $a_2$.

*Proof.* Should be clear by the definition. \hfill $\square_{1.5}$

**Remark 1.6.** In Theorem 1.3 we can many times demand $\|M_\kappa\| = \kappa$, e.g., if $(\exists \lambda)(\kappa = 2^\lambda)$.

Note we now show that 1.3 is best possible.

**Claim 1.7.** 1) If $t$ satisfies clause (a) of 1.3, i.e. $t$ is an a.e.c. with LST-number $\leq \theta$ and $|\tau_t| \leq \theta$ and $\kappa$ fails the assumption of clause (d) of 1.3, that is there is a uniform $\theta^+$-complete ultrafilter on $\kappa$, then the conclusion of clause (d) of 1.3 fails, that is $t$ is $\kappa$-sequence local for types.

2) If $D$ is a $\theta^+$-complete ultrafilter on $\kappa$ and $t$ is an a.e.c. with $\text{LST}(t) \leq \theta$ then ultraproducts by $D$ preserve “$M \in t$”, “$M \leq_t N$”, i.e.
if $M_i, N_i (i < \kappa)$ are $\tau(\bar{\kappa})$-models and $M = \prod_{i < \kappa} M_i / D$ and $N = \prod_{i < \kappa} N_i$, then:

(a) $M \in K$ if \{ $i < \kappa : M_i \in t$ \} $\in D$
(b) $M \leq t$ $N$ if \{ $i : M_i \leq t N_i$ \} $\in D$.

Proof. Note that if $D$ is $\theta^+$-complete, then it is $\sigma^+$-complete where $\sigma = \theta^{\aleph_0}$ (and much more, it is $\theta'$-complete for the first measurable $\theta' > \theta$).

1) So assume

- (a) $\langle M_i : i \leq \kappa \rangle$ is $\leq t$-increasing
- (b) $M_\kappa = N_0 \leq t N_\ell$ for $\ell = 1, 2$
- (c) $p_\ell = t p_\ell (a, N_0, N_\ell)$ for $\ell = 1, 2$
- (d) $i < \kappa \Rightarrow p_1 | M_i = p_2 | M_i$.

We shall show $p_1 = p_2$, this is enough.

Without loss of generality

(*)$_1$ (a) $a_1 = a_2$ call it $a$
(b) $\tau_\kappa \subseteq H(\theta)$.

By (d) of (*) we have:

(d)$_+^+$ for each $i < \kappa$ there are $n_i < \omega$ and $\langle N_{i,m} : n \leq n_i \rangle$ such that

(a) $N_{i,0} = N_1$
(b) $N_{i,m} = N_2$ or just $h_i$ is an isomorphism from $N_{i,m}$ onto $N_2$ such that $h_i | (M_i \cup \{ a \})$ is the identity
(c) $a \in N_{i,\ell}$ and $M_i \leq t N_{i,\ell}$
(d) if $m < m_i$ then $N_{i,2m+1} \leq t N_{i,2m}, N_{i,2m+2}$.

As $\kappa = \text{cf}(\kappa) > \aleph_0$ without loss of generality $i < \kappa \Rightarrow n_i = n_*$. Let $\chi$ be such that $\langle M_i : i \leq \kappa \rangle, \langle N_{i,n} : n \leq n_* \rangle : i < \kappa \rangle$ and $t_{\text{LST}(t)}$ all belongs to $H(\chi)$; concerning $t_{\text{LST}(t)}$ this means $\tau_\chi$ and $\text{LST}(t)$ belongs to $H(\chi)$ hence $\{ M \in K_t : M \in \mathcal{H}(\text{LST}^+_t) \}$ and $\leq t | \mathcal{H}(\text{LST}^+_t)$ belongs to $H(\chi)$; those hold by (*$_1$)(b). Let $\mathfrak{B}$ be the ultrapower $(H(\chi), \in \mathfrak{B}) / D$ and $j_0$ the canonical embedding of $(H(\chi), \in)$ into $\mathfrak{B}$ and let $j_1$ be the Mostowski-Collapse of $\mathfrak{B}$ to a transitive set $\mathcal{H}$ and let $j = j_1 \circ j_0$. So $j$ is an elementary embedding of $(H(\chi), \in)$ into $(\mathcal{H}, \in)$ even an $\langle \psi, \theta \rangle$-elementary one.

Recall we are assuming without loss of generality $\tau_\kappa \subseteq H(\theta)$ hence $j(\tau_\kappa) = \tau_\kappa$ hence by part (2), $j$ preserves "$N \in K_t$", "$N^1 \leq t N^{2n}$". If $h$ is an isomorphism from $N'$ onto $N''$.

So $j(\langle M_i : i \leq \kappa \rangle)$ has the form $\langle M^*_i : i \leq j(\kappa) \rangle$ but $j(\kappa) > \kappa_* := \bigcup_{i < \kappa} j(i)$ by the uniformity of $D$ and let $j(\langle N_{i,n} : n \leq n_* \rangle : i < \kappa \rangle) = \langle N^*_{i,n} : n \leq n_*$ : $i < j(\kappa) \rangle$ and $j(h_i : i < \kappa) = \langle h^*_i : i < \kappa_* \rangle$.

So

(a) $j(M_\kappa)$ is a $\leq t$-embedding of $M_\kappa$ into $M^*_\kappa$ hence even into $M^*_{\kappa_*}$
(b) $M^*_\kappa \leq t N^*_{\kappa,n}$ and $j(a) \in N^*_{\kappa,n}$ for $i < \kappa, n \leq n_*$
(c) $N^*_{\kappa,0} = j(N_1)$
(d) $h_{\kappa_*}$ is an isomorphism from $N_{\kappa,n_*}$ onto $j(N_2)$
(e) $N^*_{\kappa,2m+1} \leq t N^*_{\kappa,2m}, N^*_{\kappa,2m+2}$ for $2m + 1 < n_*$
(f) \( j(a) \in N_{\kappa,m} \).

Together we are done.

2) By the representation theorem of a.e.c. [Sh:888, \S 1]. \( \square_{1.7} \)

**Proof.** Proof of 1.3

Let \( \sigma = \theta^{\aleph_0} \). Let \( G = ([\sigma]^{<\aleph_0}, \Delta) \) and let \( \langle \epsilon_i : i < \sigma \rangle \) list the members of \( G \), let \( \langle \eta_\alpha : \alpha < \sigma \rangle \) list \( \omega^\theta \).

Now \( \exists X_1 \) let \( B_{\epsilon, n} \subseteq G \) for \( \epsilon < \theta \) be such that: if \( a, b \in G \) then \( (\forall \epsilon < \theta)(\forall n < \omega)(a \in B_{\epsilon, n} \Rightarrow b = b) \); moreover, \( B_{\epsilon, n} = \{ \epsilon_\alpha : \eta_\alpha(n) = \epsilon \} \).

Let \( \tau \) have the predicates \( G, I, J, H(\text{unary}), E_1, Q(\text{binary}), R_{n, \alpha} \) \((n\text{-place; } \eta < \omega, \alpha < \theta, P_{\epsilon, n}(\text{unary; } \epsilon < \theta)\) and function symbols \( F_1(\text{unary}), F_2, \pi, + (\text{binary}) \); so \( |\tau| = \theta \). We define \( K \) as a class of \( \tau \)-models by:

\( \exists X_2 \) \( M \in K \) iff (up to isomorphism):

(\( a \)) \( (G^M, I^M, J^M, H^M) \) is a partition of \( |M| \), (recall that they are unary)

(\( b \)) \( (G^M, +^M) \) is a subgroup of the group \((|\sigma|^{<\aleph_0}, \Delta), P^M_{\epsilon, n} \subseteq G^M \) for \( \epsilon < \theta \), \( (P^M_{\epsilon, n} : \epsilon < \theta) \) be a partition of \( M \) such that \( a \neq b \in G^M \Rightarrow (\exists \epsilon < \theta)(\exists n < \omega)[a \in P^M_{\epsilon, n} \wedge b \notin P^M_{\epsilon, n}] \)

(\( c \)) \( Q^M \subseteq H^M \times J^M \) is such that \( (\forall a \in H^M)(\exists b \in Q^M)((a, b) \in Q^M) \);

(\( d \)) \( E^M_1 \) is an equivalence relation on \( H^M \) such that: if \( (a, b) \in Q^M \) and \( a_2 \in H^M \) then \( a_1 E^M_1 a_2 \Leftrightarrow (a_2, b) \in Q^M \)

(\( e \)) \( \pi^M \) is a function from \( H^M \) into \( I^M \)

(\( f \)) \( E^M_2 = \{(a, b) : a E^M_1 b \) and \( \pi^M(a) = \pi^M(b) \) so \( a, b \in H \}\)

(\( g \)) \( F^M_2(a, b) \) is a partial two-place function such that:

(\( \alpha \)) \( F^M_2(a, b) \) is well defined iff \( b \in G^M, a \in H^M \)

(\( \beta \)) \( \forall a \in H^M, (F^M_2(a, b) : b \in G^M) \text{ list } a/E^M_2 \in \{ a' \in H^M : \pi^M(a') = \pi^M(a) \} \) with no repetitions

(\( \gamma \)) \( \forall a \in H^M \) and \( b, c \in G \) then \( F^M_2(a, b + c) = F^M_2(a, b, c) \), on the + see clause (\( b \))

(\( \delta \)) \( F(a, 0_G) = a \) for \( a \in H^M \)

(\( \epsilon \)) \( \forall n < \omega \) and \( \gamma < \theta \) the relation \( R^M_{n, \gamma} \) is an \( n \)-place relation \( \subseteq \cup \{ a/E^M_1 : a \in H^M \} \).

We define \( \leq_T \) as being a submodel. Easily \( \exists X_3 \) \( \tau = (K, \leq_T) \) is an a.e.c.

For \( A \subseteq M \in K \) let

(\( a \)) \( c_1^M(A) = \) the subgroup of \((G^M, +^M)\) generated by \((A \cap G^M) \cup \{ b \in G^M : \) for some \( a_1 \neq a_2 \in A \cap H^M \) we have \( a_1 E^M_2 a_2 \) and \( F^M_2(a_1, b) = a_2 \} \)

(\( b \)) \( c_2^M(A) = (A \cap I^M) \cup \{ \pi^M(a) : a \in A \cap H^M \} \)

(\( c \)) \( c_3^M(A) = \{ a \in H^M : \) for some \( b \in c_0^M(A) \) and \( a_1' \in A \cap H^M \) we have \( a = F^M_2(a_1', b) \})

(\( d \)) \( c\ell(A, M) = c\ell_M(A) = M \cup \{ c_\ell^M(A) : \ell = 1, 2, 3 \} \).
Now this function $c\ell(A, M)$ shows that $F$ admits intersections (see Definition 1.4) so

\[ \exists_k F \text{ admits closure and } \text{LST}(F) + |\tau| = \theta. \]

Assume $\kappa$ is as in clause (d) of 1.3, we use the $M_{\ell, \alpha}(\ell = 1, 2, \alpha \leq \kappa)$ constructed in 1.2 (the relevant properties are stated in 1.2). They are not in the right vocabulary so let $M'_{\ell, \alpha}$ be the following $\tau$-model:

\[ (a) \quad \text{elements } G^{M'_{\ell, \alpha}} = G \]
\[ J^{M'_{\ell, \alpha}} = I_{\alpha} \]
\[ H^{M'_{\ell, \alpha}} = [M_{\ell, \alpha}] \]

(we assume disjointness)

\[ (b) \quad (G^{M'_{\ell, \alpha}}, +^{M'_{\ell, \alpha}}) \text{ is } G = ([\sigma]^{<\kappa_0}, \Delta) \]
\[ P'^{M'_{\ell, \alpha}} \subseteq G^{M_{\ell, \alpha}} \text{ as required in } \exists_1 \not\text{ depending on } (\ell, \alpha) \]

\[ (c) \quad F'^{M'_{\ell, \alpha}}_1 \text{ is constantly } t^*_1 \text{ on } H^{M'_{\ell, \alpha}} \]

\[ (d) \quad E'^{M'_{\ell, \alpha}}_1 = \{(a, b) : F'^{M'_{\ell, \alpha}}_1(a) = F'^{M'_{\ell, \alpha}}_1(b) \text{ so } a, b \in H^{M'_{\ell, \alpha}} \}
\]

\[ (e) \quad \pi^{M'_{\ell, \alpha}} \text{ is } \pi_{\ell, \alpha} \text{ (constructed in 1.2)} \]

\[ (f) \quad \pi^{M'_{\ell, \alpha}}_2 = \{(a, b) : aE'^{M'_{\ell, \alpha}}_2b \text{ and } \pi^{M'_{\ell, \alpha}}_2(a) = \pi^{M'_{\ell, \alpha}}_2(b) \text{ so } a, b \in H^{M'_{\ell, \alpha}} \}
\]

\[ (g) \quad F'^{M'_{\ell, \alpha}}_2(a, b) = F'^{M'_{\ell, \alpha}}_2(b) \text{ for } a \in H^{M'_{\ell, \alpha}} \]

Let $M'_{0, \alpha} = M'_{1, \alpha} | (G^{M'_{\ell, \alpha}} \cup J^{M'_{\ell, \alpha}})$ for $\ell = 1, 2$ and $\alpha \leq \kappa$ (we get the same result).

Note easily

\[ \exists_6 M_{0, \alpha} \leq_\ell M_{\ell, \alpha}, (M_{\ell, \alpha} : \alpha \leq \kappa) \text{ is } \leq_\ell \text{-increasing (check)} \]

\[ \exists_7 \text{tp}(t^*_1, M'^{M'_{1, \alpha}}_{0, \alpha}, M'^{M'_{1, \alpha}}_{1, \alpha}) = \text{tp}(t^*_2, M'^{M'_{1, \alpha}}_{0, \alpha}, M'^{M'_{1, \alpha}}_{2, \alpha}) \text{ for } \alpha < \kappa. \]

[Why? By the isomorphism from $M_{1, \alpha}$ onto $M_{2, \alpha}$ respecting $(\pi_{1, \alpha}, \pi_{2, \alpha})$ in 1.1.]

\[ \exists_8 \text{tp}(t^*_1, M'^{M'_{1, \alpha}}_{0, \alpha}, M'^{M'_{1, \alpha}}_{1, \alpha}) \neq \text{tp}(t^*_2, M'^{M'_{1, \alpha}}_{0, \alpha}, M'^{M'_{1, \alpha}}_{2, \alpha}). \]

[Why? By the non-isomorphism in 1.1; extension will not help.]

Now by the "translation theorem" of [BlSh:862, 4.7] we can find $F'$ which has all the needed properties, i.e. also the amalgamation and JEP. $\square_{1.3}$
2. Compactness of types in a.e.c.

Baldwin [Bal09] ask “can we in ZFC prove that some a.e.c. has amalgamation, JEP but fail compactness of types”. The background is that in [BlSh:862] we construct one using diamonds.

To me the question is to show this class can be very large (in ZFC).

Here we omit amalgamation and accomplish both by direct translations of problems of existence of models for theories in $L_{\kappa^+\kappa^+}$, first in the propositional logic.

So whereas in [BlSh:862] we have an original group $G^M$, here instead we have a set $P^M$ of propositional “variables” and $P^M$, set of such sentences (and relations and functions explicating this; so really we use coding but are a little sloppy in stating this obvious translation).

In [BlSh:862] we have $I^M$, set of indexes, 0 and $H$, set of Whitehead cases, $H_t$ for $t \in I^M$, here we have $I^M$, each $t \in I^N$ representing a theory $P_t^M \subseteq P^M$ and in $J^M$ we give each $t \in I^M$ some models $M^t_t : P^M \rightarrow \{\text{true, false}\}$. This is set up so that amalgamation holds.

Notation 2.1. In this section types are denoted by $p, q$ as $p, q$ are used for propositional variables.

\[ \{ \text{b2.0} \} \]

**Definition 2.2.** 1) We say that an a.e.c. $t$ has $(\leq \lambda, \kappa)$-sequence-compactness (for types) when: if $(M_i : i \leq \kappa)$ is $\leq \tau$-increasing continuous and $i < \kappa \Rightarrow |M_i| \leq \lambda$ and $p_i \in \mathcal{F}_{<\omega}(M_i)$ for $i < \kappa$ satisfying $i < j < \kappa \Rightarrow p_i = p_j|M_i$ then there is $p_\kappa \in \mathcal{F}_{<\omega}(M_\kappa)$ such that $i < \kappa \Rightarrow p_\kappa|M_i = p_i$.

2) We define “(= $\lambda, \kappa$)-sequence-compactness” similarly. Let $(\lambda, \kappa)$-sequence-compactness mean $(\leq \lambda, \kappa)$-compactness.

\[ \{ \text{b2.1} \} \]

**Question 2.3.** Can we find an a.e.c. $t$ with amalgamation and JEP such that $\{ \theta : t \text{ have } (\lambda, \theta)\text{-compactness of types for every } \lambda \}$ is complicated, say:

(a) not an end segment but with “large” members

(b) any $\{ \theta : \theta \text{ satisfies } \psi \}, \psi \in L_{\kappa^+, \kappa^+}$ (second order).

\[ \{ \text{b2.2} \} \]

**Definition 2.4.** Let $\kappa \geq \aleph_0$, we define $t = t_\kappa$ as follows:

(A) the vocabulary $t_\kappa$ consist of $F_i(i \leq \kappa), R_\kappa(\ell = 1, 2), P, \Gamma, I, J, c_i(i < \kappa), F_i(i \leq \kappa)$, (pedantically see later),

(B) the universe of $M \in K_t$ is the disjoint union of $P^M, \Gamma^M, I^M, J^M$ so $P, \Gamma, I, J$ are unary predicates

(C) (a) $P^M$ a set of propositional variables (i.e. this is how we treat them)

(b) $\Gamma^M$ is a set of sentences of one of the forms $\varphi = (p), \varphi = (q \equiv p \wedge q), \varphi = (q \equiv \neg p), \varphi = (q \equiv \bigwedge_{i<\kappa} p_i)$, so $p, q, p_i \in P^M$ but in the last case $\{p_i : i < \kappa\} \subseteq \{c_i^M : i < \kappa\}$ (or code this!)

(c) for $i < \kappa$ the function $F^M_i : \Gamma^M \rightarrow P^M$ are such that for every $i < \kappa$ and $\varphi \in \Gamma^M$ we have:

(α) if $\varphi = (p)$ and $i \leq \kappa$ then $F_{i+1}(\varphi) = p, F_0(\varphi) = c_0$

(β) if $\varphi = (r \equiv p \wedge q)$ then $F_i(\varphi)$ is $c_1$ if $i = 0$, is $p$ if $i = 1$, is $q$ if $r = 2$ is $r$ if $r \geq 3$

(γ) if $\varphi = (q \equiv \neg p)$ then $F_i(\varphi)$ is $c_2$ if $i = 0$, $p$ if $i = 1$, $q$ if $i \geq 2$
(δ) if \( \varphi = (q \equiv \bigwedge_{j<k} p_j) \) then \( F_i(\varphi) \) is \( c_3 \) if \( i = 0, \)
\( q \) if \( i = 1, p_{2^i,j} \) if \( i = j + 1 \)

(d) \( I \) a set of theories, i.e. \( R_1^M \subseteq \Gamma \times I \) and for \( t \in I \) let
\[ \Gamma_t^M = \{ \psi \in \Gamma^M : \psi R_1 t \} \subseteq \Gamma^M \]

(e) \( J \) is a set of models, i.e. \( R_2^M \subseteq (\Gamma \cup P) \times J \) and for \( s \in J \) we have
\( M_s^M \) is the model, i.e. function giving truth values to \( p \in P^M \), i.e.

\[ (a) \ M_s^M(p) \text{ is true if } p, R_2^M s; \text{ is false if } \neg p, R_2^M s \]
\[ (b) \ (\varphi, s) \in R_2^M \text{ iff computing the truth value of } \varphi \text{ in } M_s^M \]
we get truth
\( (f) \ F_n^M : J^M \rightarrow \Gamma^M \) such that \( s \in J^M \Rightarrow M_s^M \) is a model of \( \Gamma_{F_n^M(s)} \)
\( (g) \ (\forall t \in I^M) (\exists s \in J^M)(F_n^M(s) = t) \)

\[ (D) M \leq N \text{ iff } M \subseteq N \text{ are } \tau_e \text{-models from } K_{\ell}. \]

Claim 2.5. \( \mathfrak{t} \) is an a.e.c., \( LST(\mathfrak{t}) = \kappa. \)

Proof. Obvious. \( \square_{2.5} \)

Claim 2.6. \( \mathfrak{t} \) has the JEP.

Proof. Just like disjoint unions (also of the relations and functions) except for the individual constants \( c_i \) (for \( i < \kappa \)). \( \square_{2.6} \)

Claim 2.7. Assume \( M_0 \leq M_\ell \) for \( \ell = 0, 1 \) and \( |M_0| = P^{M_0} \cup \Gamma^{M_0} = P^{M_\ell} \cup \Gamma^{M_\ell} \)
for \( \ell = 1, 2 \) and \( a_\ell \in I^{M_\ell} \) for \( \ell = 1, 2. \) Then \( tp_\ell(a_1, M_0, M_1) = tp_\ell(a_2, M_0, M_2) \) iff
\[ \Gamma_{a_1}^{M_\ell} = \Gamma_{a_2}^{M_\ell}. \]

Proof. The if direction, \( \Leftarrow \)

Let \( h \) be a one to one mapping with domain \( M_1 \) such that \( h \mid M_0 = \) the identity, \( h(a_1) = a_2 \) and \( h(M_1) \cap M_2 = M_0 \cup \{a_2\}. \) Renaming without loss of generality \( h \)
is the identity. Now define \( M_2 \) as \( M_1 \cup M_2, \) as in 2.6, now \( a_1 = a_2 \) does not cause trouble because \( P^{M_0} = P^{M_\ell}, \Gamma^{M_0} = \Gamma^{M_\ell} \) for \( \ell = 1, 2. \)

The only if direction, \( \Rightarrow \)
Obvious. \( \square_{2.7} \)

Claim 2.8. Assume \( \lambda, \theta \) are such that:

\( (a) \ \theta \text{ is regular } \leq \lambda \text{ and } \lambda \geq \kappa \)
\( (b) \ \langle \Gamma_i : i \leq \theta \rangle \text{ is } \leq \text{-increasing continuous sequence of sets propositional sentences in } L_{\kappa^+ \omega} \text{ such that } \Gamma_i \text{ has a model } \iff i < \theta \}
\( (c) \ |\Gamma_\theta| \leq \lambda. \)

Then \( \mathfrak{t} \) fail \( (\lambda, \theta) \text{-sequence-compactness} \) (for types).

Remark 2.9. We may wonder but: for \( \theta = \aleph_0 \) compactness holds? Yes, but only assuming amalgamation.
Proof. Without loss of generality $|\Gamma_0| = \lambda$. Without loss of generality $\langle p_\varepsilon^* : \varepsilon < \kappa \rangle$ are pairwise distinct propositions variables appearing in $\Gamma_0$ (but not necessarily in $\Gamma_0$) and each $\psi \in \Gamma_i$ is of the form $(p)$ or $r \equiv p \land q$ or $r \equiv \neg p$ or $r \equiv \bigwedge_{i<\kappa} p_i$ where $\{p_i : i < \kappa\} \subseteq \{p_\varepsilon^* : \varepsilon < \kappa\}$.

Let $P_i$ be the set of propositional variables appearing in $\Gamma_i$ without loss of generality $|P_i| = \lambda$.

We choose a model $M_i$ for $i \leq \theta$ such that:

- $|M_i| = P_i \cup \Gamma_i$
- $P_i^M = P_i$ and $\Gamma_i^M = \Gamma_i$
- the natural relations and functions.

Let $M_\theta : P_\theta \rightarrow \{\text{true false}\}$ be a model of $\Gamma_i$.

We define a model $N_i \in K_\kappa$ for $i < \kappa$ (but not for $i = \theta$!)

- $M_i \leq_t N_i$
- $P_i^{N_i} = P_i^M$
- $\Gamma_i^{N_i} = \Gamma_i^M$
- $I_i = \{t_i\}$
- $J_i = \{s_i\}$
- $F_i^{N_i}(s_i) = t_i$
- $R_i^{N_i} = \Gamma_i \times \{t_i\}$
- $R_i^{N_i}$ is chosen such that $M_\theta^{N_i}$ is $M_i$
- $F_i^{N_i} (i < \kappa)$ are defined naturally.

Now

\[ (* )_1 \quad p_i = \text{tp}(t_i, M_i, N_i) \in S^1(M_i). \]

[Why? Trivial.]

\[ (* )_2 \quad i < j < \theta \Rightarrow p_i = p_j|M_j. \]

[Why? Let $N_{i,j} = N_i[(M_j \cup \{s_j, t_j\})].$

Easily $\text{tp}(t_j, M_i, N_{i,j}) \leq p_j$ and $\text{tp}(t_j, M_i, N_{i,j}) = p_i$ by the claim 2.7 above.]

\[ (* )_3 \quad \text{there is no } p \in S^1(M_\theta) \text{ such that } i < \theta \Rightarrow p|M_i = p_i. \]

[Why? We prove more:

\[ (* )_4 \quad \text{there is no } \langle N, t \rangle \text{ such that }\]

\[ (a) \quad M_\kappa \leq_t N \]

\[ (b) \quad t \in I^N \]

\[ (c) \quad (\forall \varphi \in \Gamma_\kappa)[\varphi R_i^N t]. \]

[Why? As then $\Gamma_\theta = \Gamma_i^M$ has a model contradiction to an assumption.]

\[ \square_{2.8} \]

\[ \text{(b2.13)} \]

Conclusion 2.10. If $\theta > \kappa$ is regular with no $\kappa^+$-complete uniform ultrafilter on $\theta$ and $\lambda = 2^\theta$, then $\text{it}$ is not $(\lambda, \theta)$-sequence-compact.
Remark 2.11. Recall if $D$ is an ultrafilter on $\theta$ then $\min\{\sigma' : D \text{ is not } \sigma'-\text{complete}\}$ is $\aleph_0$ or a measurable cardinality.

Proof. (Well known).

Let $M$ be the model with universe $2^\theta$, $P_0^M = \theta$ and $R^M \subseteq \theta \times \lambda$ be such that $\{\{\alpha < \lambda : \alpha R^M \beta \} : \beta < \lambda\} = \mathcal{P}(\theta)$, $<^M$ the well ordering of the ordinal on $\lambda$ the vocabulary has cardinality $\kappa$ and has elimination of quantifiers and Skolem functions.

Let $\Gamma_i = \text{Th}(M, \beta_{\alpha < \lambda} \cup \{\alpha < c : \alpha < \theta\} \{c \text{ a new individual constant}\}$, then $\Gamma_i : i < \theta$ is as$^1$ required in 2.12 below hence 2.8 apply. □2.10

Conclusion 2.12. In Claim 2.8 if $\lambda = \lambda^\kappa$ then we can allow $\langle \Gamma_i : i < \theta \rangle$ to be a sequence of theories in $L_{\kappa^+, \kappa^+(\tau)}\tau$ any vocabulary of cardinality $\leq \lambda$.

Proof. Without loss of generality we can add Skolem functions (each with $\leq \kappa$ places) in particular. So $\Gamma_i$ becomes universal and adding propositional variables for each quantifier free sentence and writing down the obvious sentences, we get a set of propositional sentences, we get $\Gamma_i$ as there. □2.12

I think we forgot

Observation 2.13. If $\lambda \geq \kappa > \theta = \text{cf}(\theta)$ then the condition in 2.8 holds.

Proof. Just let $\Gamma_0 = \{\forall i < \theta \neg p_i\}, \Gamma_i = \Gamma_0 \cup \{p_j : j < i\}$. □2b.21

Conclusion 2.14. 1) $C_\kappa = \{\theta : \theta = \text{cf}(\theta)\}$ and for every $\lambda$ and a.e.c. $\mathfrak{t}$ with $\text{LST}(\mathfrak{t}) \leq \kappa, |\tau_\mathfrak{t}| = \kappa$ have $(\lambda, \theta)$-compactness of type is the class $\{\theta : \theta = \text{cf}(\theta) > \kappa$ and there is a uniform $\kappa^+\text{-complete ultrafilter on } \theta\}$. 2) In $C_\kappa$ we can replace “every $\lambda^\kappa$ by $\lambda = 2^\theta + \kappa$.

Proof. Put together 2.10,2.16. □2.13

Of course, a complimentary result (showing the main claim is best possible) is:

Claim 2.15. If $\mathfrak{t}'$ is an a.e.c., $\text{LST}(\mathfrak{t}') \leq \kappa$ and on $\theta$ there is a uniform $\kappa^+\text{-complete ultrafilter on } \theta$ and $\theta$ is regular and $\lambda$ any cardinality then $\mathfrak{t}'$ has $(\lambda, \kappa)$-compactness of types.

Proof. Write down a set of sentences on $L_{\kappa^+, \kappa^+(\tau^+)}$ expressing the demands.

Let $\langle M_i : i \leq \theta \rangle$ be $<^\tau$-increasing continuous, $\|M_i\| \leq \lambda, p_i = t\mathfrak{t}_\ell(a_i, M_i, N_i)$ so $M_i \leq N_i$ such that $j < i < \theta$ implies $p_i = p_j$ $M_i$. Without loss of generality $\|N_i\| \leq \lambda$.

Let $\langle N_{i,j,\ell} : \ell \leq n_{i,j,\ell}, \pi_{i,1} \rangle$ witness $p_i = p_j$ $M_i$ for $i < j < \theta$ (i.e. $M_i \leq N_{i,j,\ell}$). Without loss of generality $\|N_{i,j,\ell}\| \leq \lambda, N_{i,j,0} = N_i, a_i \in N_{i,j,\ell}, \bigwedge_{\ell < n_{i,j,\ell}} N_{i,j,\ell+1} \subseteq N_{i,j,\ell}$ and $\pi_{i,j}$ be an isomorphism from $N_j$ onto $N_{i,j,n_{i,j}}$. Let $\tau^+ = \tau \cup \{\mathfrak{t} : \mathfrak{t} < \kappa, n < \omega\}$. Let $\langle M_i^+ : i \leq \theta \rangle$ be $\subseteq$-increasing, $M_i^+$ a $\tau^+$-expansion of $M_i$ such that $u \subseteq M_i^+ \Rightarrow M_i\{\text{cf}(M_i^+ u) \leq i \leq M_i$. Similarly $\langle N_{i,j,\ell}^+ : \ell \leq n_{i,j,\ell} \rangle$ a $\tau^+$-expansion of $N_{i,j,\ell}$ as above such that $\langle \forall \mathfrak{t} < n_{i,j,\ell} \rangle\exists \exists \in \{1, 2\}\rangle(N_{i,j,\ell}^+ \subseteq N_{i,j,\ell+1}^+ \vee N_{i,j,\ell+1}^+ \subseteq N_{i,j,\ell}^+).$

Now write down a translation of the question, “is there $p$ such that ...” □2.15

or directly as $\Gamma_i$ has Skolem functions

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1
Claim 2.16. If $D$ is a uniform $\kappa$-complete ultrafilter on $\theta$, $\langle M_i : i \leq \theta \rangle$ is $\leq_2$-increasing continuous, $p_i \in \mathcal{P}_\kappa(M_i)$ as witnessed by $(N_i, a_i)$ for $i < \kappa$, $p_i = p_j|M_i$ for $i < j < \kappa$ as witnessed by $(\pi_i, (N_{i,j,\ell} : \ell \leq m_{i,j})$ as in the proof above.

1) There is $p_\kappa \in \mathbf{S}^\kappa(M_\theta)$ such that $i < \theta \Rightarrow p_\kappa|M_i$.

2) In fact for each $i < \kappa$ let $U_i \in D$ be such that $i < j \in U_i \Rightarrow n_{i,j} = n_i^*$. Let $N_{i,\kappa,\ell} = \prod_{j \in U_i} N_{i,j,\ell}/D$. So $\langle N_{i,\kappa,\ell} : \ell \leq n_i^* \rangle$ are as above. Let $M = \prod_{i < \kappa} M_i/D, \pi_{i,\kappa} = \prod_{j \in U_i} \pi_{i,j}/D$, etc.
3. On some stability spectrums of an a.e.c.

Convention 3.1. $\mathfrak{t}$ is an a.e.c. with amalgamation.

Definition 3.2. For $\theta \geq \text{LST}(\mathfrak{t})$. We say $\mathfrak{t}$ is $(\lambda, \theta)$-stable when $M \in K^\mathfrak{t}_\lambda \Rightarrow \lvert S(M)/E^\mathfrak{t}_\lambda \rvert \leq \lambda$ where

$$p_{E^\mathfrak{t}_\lambda} \Leftrightarrow (\forall N)(N \leq \mathfrak{t} \land \|N\| \leq \theta \Rightarrow p\{N = q(\theta)\}.$$  

Theorem 3.3. Fixing $\theta$ the class $\{\lambda : \mathfrak{t} \text{ is } (\lambda, \theta)\text{-stable}\}$ behave as in [Sh:3].

Remark 3.4. See [Sh:734] = [Sh:h, V, §7] or [Sh:702] if not covered.

Definition 3.5. $\kappa_\theta(\mathfrak{t}) := \text{Min}\{\kappa \leq \theta^+ : \text{there is no sequence } (M_i : i \leq \kappa) \text{ which is } \leq_1\text{-increasing continuous, } \|M_i\| \leq \theta \text{ and } p \in \mathcal{S}(M_i) \text{ such that } p|M_{i+1} \text{ strongly } (\theta)\text{-split over } M_i\}.$

Claim 3.6. 1) If $\lambda > 2^\theta$ and $\mathfrak{t}$ is not $(\lambda, \theta)$-stable then for some $\kappa \leq \theta^+$ satisfying $\lambda^\kappa > \lambda$ we have $\kappa < \kappa_\theta(\mathfrak{t})$.

2) If $\lambda > \theta, \lambda^\kappa > \lambda$ then $\kappa < \kappa_\theta(\mathfrak{t})$ then $\mathfrak{t}$ not $(\lambda, \theta)$-stable.

Conclusion 3.7. $(\ldots, \theta)$-stability spectrum - behave as in [Sh:3].

Discussion 3.8. We can look at $\lambda \in [\theta, 2^\theta)$ using splitting rather than strongly splitting.

It seems to me the main question is

Question 3.9. Assume $(\exists \mathfrak{t} \geq \text{LS}(\mathfrak{t})(\kappa_\mathfrak{t}(\mathfrak{t}) > \aleph_0)).$

What can you say on $\text{Min}\{\theta : \kappa_\theta(h) > \aleph_0, \theta \geq \text{LST}(\mathfrak{t})\}$?

Question 3.10. Assume $\text{GCH}$ can we find an a.e.c. $\mathfrak{t}$ such that: $(\forall \theta \geq \text{LST}(\mathfrak{t})(\kappa_\mathfrak{t}(\mathfrak{t}) = \aleph_0)$ but unstable in every regular $\lambda > \text{LST}(\mathfrak{t})$?

References


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