Partial choice functions for families of finite sets

Eric J. Hall   Saharon Shelah*

Abstract

Let p be a prime. We show that ZF + "Every countable set of p-element sets has an infinite partial choice function" is not strong enough to prove that every countable set of p-element sets has a choice function, answering an open question from [1]. The independence result is obtained by way of a permutation (Fraenkel-Mostowski) model in which the set of atoms has the structure of a vector space over the field of p elements, and then the use of atoms is eliminated by citing an embedding theorem of Pincus. By way of comparison, some simpler permutation models are considered in which some countable families of p-element sets fail to have infinite partial choice functions.

1 Introduction

Let C($\aleph_0$,n) be the statement asserting that every infinite, countable set of n-element sets has a choice function. Let PC($\aleph_0$,n) be the statement asserting that every infinite, countable set C of n-element sets has an infinite partial choice function. That is, PC($\aleph_0$,n) asserts that there is a choice function whose domain is an infinite subset of C. (Recall C($\aleph_0$,n) is Form 288(n), and PC($\aleph_0$,n) is Form 373(n) in Howard and Rubin's reference [2]. Also, C($\aleph_0$,2) is Form 30, and PC($\aleph_0$,2) is Form 18.) The main result of this paper is that for any prime p, PC($\aleph_0$,p) does not imply C($\aleph_0$,p) in ZF. This answers questions left open in [1].

The independence results are obtained using the technique of permutation models (also known as Fraenkel-Mostowski models). See Jech [3] for basics about permutation models and the theory ZFA (ZF modified to allow atoms). A suitable permutation model will establish the independence of C($\aleph_0$,p) from PC($\aleph_0$,p) in the context of ZFA. This suffices by work of Pincus [4] (extending work of Jech and Sochor), which shows that once established under ZFA, the independence result transfers to the context of ZF (this is because the statement PC($\aleph_0$,p) is "injectively boundable"; see also Note 103 in [2]).

Section 2 is the proof of the main result, Theorem 2.1. Readers with some experience with permutation models may wonder whether the model used in the proof of Theorem 2.1 is unnecessarily complicated. Section 3 explains why certain simpler models which may appear promising candidates to witness the independence of PC($\aleph_0$,2) from C($\aleph_0$,2) in fact fail to do so.

*The second author's research was supported by the US-Israel Binational Science Foundation. Publication 934.

Research partially supported by NSF grant No. NSF-DMS 0600940.

2000 *Mathematics Subject Classification*: 03E25 (03E05, 15A03).

2 The main theorem

Theorem 2.1. *Let p be a prime integer. In ZF, PC($\aleph_0$,p) does not imply C($\aleph_0$,p)*

Proof. As discussed in the Introduction, it suffices describe a permutation model in which PC($\aleph_0$,p) holds and C($\aleph_0$,p) fails. Let $\mathcal{M}$ be a model of ZFAC whose set of atoms is countable and infinite; we will work in $\mathcal{M}$ unless otherwise specified. We will describe a permutation submodel of $\mathcal{M}$.

First, we set some notation for a few vector spaces over the field $\mathbb{F}_p$ with p elements. Let $W = \oplus_{i=1}^{\infty}\mathbb{F}_p$, so each element of W is a sequence $w = (w_1, w_2, w_3, \dots)$ of elements of $\mathbb{F}_p$, with at most finitely many nonzero terms. Let G be the full product $\otimes_{i=1}^{\infty}\mathbb{F}_p$ (sequences may have infinitely many nonzero elements). Finally, let $U = \mathbb{F}_p \times W$, so each element of U is a pair (a, w) with $a \in \mathbb{F}_p$, $w \in W$.

For each $w \in W$, let $U_w = \{\,(a, w) \,:\, a \in \mathbb{F}_p\,\}$, so that $\mathcal{P} = \{\,U_w \,:\, w \in W\,\}$ is a partition of U into sets of size p. We define an action such that each $g \in G$ gives an automorphism of U, and such that the G-orbits are the elements of the partition $\mathcal{P}$, as follows. For each $(a, w) \in U$ and $g \in G$, let

$$(a, w)g = (a + \sum_{i=1}^{\infty} w_i g_i, w)$$

(where w_i is the i^{th} entry in the sequence w, and likewise g_i; the product $w_i g_i$ is in the field $\mathbb{F}_p$, and the sum $a + \sum_i w_i g_i$ is a (finite) sum in $\mathbb{F}_p$). This action induces an isomorphism of G with a subgroup of $\mathrm{Aut}(U)$; we will henceforth identify G with this subgroup, think of the operation on G as composition instead of addition, and continue to let G act on the right.

Remarks. It is clear from the original definition of G that G is abelian, and all its non-identity elements have order p. As a subgroup of $\mathrm{Aut}(U)$, G may be characterized as the group of all automorphisms of U which act on each element of the partition $\mathcal{P}$ and have order p or 1. Equivalently, G is the group of all automorphisms of U which act on each element of $\mathcal{P}$ and act trivially on $U_{\mathbf{0}}$.

Now, identify the set of atoms in $\mathcal{M}$ with the vector space U. Thus, we think of each g in G as a permutation of the set of atoms. Each permutation of U extends uniquely to an automorphism of $\mathcal{M}$, and so we will also think of G as a subgroup of $\mathrm{Aut}(\mathcal{M})$.

Let $\mathcal{I}$ be a (proper) ideal on W such that

(∗1) every infinite subset of W contains an infinite member of $\mathcal{I}$, and

(∗2) $A \in \mathcal{I} \Rightarrow \mathrm{Span}(A) \in \mathcal{I}$,

where $\mathrm{Span}(A)$ is the $\mathbb{F}_p$-vector subspace of U generated by A. For proof of the existence of such an ideal, see Lemma 2.4.

Notation and definitions regarding stabilizers and supports. For $A \subset W$ and $g \in G \subset \mathrm{Aut}(\mathcal{M})$, we say "*g fixes at A*" if g fixes each atom in $\mathbb{F}_p \times A = \bigcup_{w \in A} U_w$. Let $G_{(A)}$ denote the subgroup of G consisting of elements which fix at A (i.e., $G_{(A)}$ is the pointwise stabilizer of $\bigcup_{w \in A} U_w$). When $A = \{a_1, \dots, a_n\}$ is finite, we may write $G_{(a_1,\dots,a_n)}$ for $G_{(\{a_1,\dots,a_n\})}$. If G' is a subgroup of G, then $G'_{(A)} = G' \cap G_{(A)}$. For $x \in \mathcal{M}$, we say that

A supports x if $xg = g$ for each $g \in G$ which fixes at A, and x is *symmetric* if x has a support which is a member of $\mathcal{I}$.

Let $\mathcal{N}$ be the permutation model consisting of hereditarily symmetric elements of $\mathcal{M}$. Note that the empty set supports the partition $\mathcal{P}$ of U described above, and also supports any well-ordering of $\mathcal{P}$ in $\mathcal{M}$. So in $\mathcal{N}$, $\mathcal{P}$ is a countable partition of the set U of atoms into sets of size p. However, no choice function for $\mathcal{P}$ has a support in $\mathcal{I}$, and so $\mathcal{N} \models \neg\mathrm{C}(\aleph_0,p)$.

Remarks. (1) Note, by (∗2) above, that A supports x if and only if $\mathrm{Span}(A)$ supports x, and thus A supports x if and only if any basis for $\mathrm{Span}(A)$ supports x.

(2) Let $w \in W$. Observe that for any $g \in G$, g fixes one element of U_w if and only if g fixes each element of U_w, and $G_{(w)}$ is the stabilizer subgroup of each element of U_w.

We now want to show that $\mathcal{N} \models \mathrm{CP}(\aleph_0,p)$. We first establish a couple of lemmas about supports of elements of $\mathcal{N}$.

Lemma 2.2. *Suppose $A \in \mathcal{I}$ and $x \in \mathcal{N}$. Either there is a finite set $B \subset W$ such that $B \cup A$ supports x, or the $G_{(A)}$-orbit of x is infinite.*

Proof. We give a forcing argument similar to one used in Shelah [5]. We set up a notion of forcing $\mathbf{Q}$ which adds a new automorphism of U like those found in $G_{(A)}$. Let $A^{\perp}$ be a subspace of W complementary to A (i.e., $\mathrm{Span}(A \cup A^{\perp}) = W$ and $A \cap A^{\perp} = \{\mathbf{0}\}$), and fix a basis $\{ w_i : i \in \omega \}$ for $A^{\perp}$. Conditions of $\mathbf{Q}$ shall have the following form: For any $n \in \omega$ and function $f \colon n \to \mathbb{F}_p$, let q_f be the unique automorphism of $\mathbb{F}_p \times \mathrm{Span}\{w_0, \dots w_{n-1}\} \subset U$ which fixes each U_{w_i} and maps $(0, w_i)$ to $(f(i), w_i)$. As usual, for conditions $q_1, q_2 \in \mathbf{Q}$, we let $q_1 \leq q_2$ iff $q_2 \subseteq q_1$. Thus, if $\Gamma \subset \mathbf{Q}$ is a generic filter, then $\pi = \bigcup \Gamma$ is an automorphism of $A^{\perp}$ preserving the partition $\mathcal{P}$. Easily, π extends uniquely to an automorphism of U fixing at A and preserving the partition $\mathcal{P}$, and thus we will think of such π as being an automorphism of U. Observe that $\mathbf{Q}$ is equivalent to Cohen forcing (the way we have associated each condition with a finite sequence of elements of $\mathbb{F}_p$, it is easy to think of $\mathbf{Q}$ as just adding a Cohen generic sequence in ${}^{\omega}\mathbb{F}_p$). Let $\dot{\pi}$ be a canonical name for the automorphism added by $\mathbf{Q}$. Let $(\mathbf{Q}_1, \dot{\pi}_1)$ and $(\mathbf{Q}_2, \dot{\pi}_2)$ each be copies of $(\mathbf{Q}, \dot{\pi})$.

Case 1: For some $(q_1, q_2) \in \mathbf{Q}_1 \times \mathbf{Q}_2$, $(q_1, q_2) \Vdash \check{x}\dot{\pi}_1 = \check{x}\dot{\pi}_2$.

Let $B \subset W$ be some finite support for q_1; e.g. $B = \{ w \in W : (\exists n \in \mathbb{F}_p)\ (n, w) \in \mathrm{Dom}(q_1) \cup \mathrm{Range}(q_1) \}$. Let $\Gamma \subset \mathbf{Q}_1 \times \mathbf{Q}_2$ be generic over $\mathcal{M}$ with $(q_1, q_2) \in \Gamma$, and let (π_1, π_2) be the interpretation of $(\dot{\pi}_1, \dot{\pi}_2)$ in $\mathcal{M}[\Gamma]$. For any $g \in G_{(A \cup B)}$, $(g\pi_1, \pi_2)$ is another $\mathbf{Q}_1 \times \mathbf{Q}_2$-generic pair of automorphisms. Let $\Gamma_g \subset \mathbf{Q}_1 \times \mathbf{Q}_2$ such that $(g\pi_1, \pi_2)$ is the interpretation of $(\dot{\pi}_1, \dot{\pi}_2)$ in $\mathcal{M}[\Gamma_g]$.

Note that (q_1, q_2) is in both Γ and Γ_g, so $\mathcal{M}[\Gamma] \models x\pi_1 = x\pi_2$, and $\mathcal{M}[\Gamma_g] \models xg\pi_1 = x\pi_2$. Thus, $x\pi_1 = xg\pi_1$ (if desired, one can briefly reason in an extension which contains both Γ and Γ'), and it follows that $x = xg$ (recall that automorphisms of U which preserve $\mathcal{P}$, such as g and π_1, commute).

We have shown that every $g \in G_{(A \cup B)}$ fixes x, which is to say that $A \cup B$ supports x, which completes the proof for Case 1.

Case 2: $\Vdash_{\mathbf{Q}_1 \times \mathbf{Q}_2} \check{x}\dot{\pi}_1 \neq \check{x}\dot{\pi}_2$.

Let $\mathcal{H}(\kappa)$ be the set of hereditarily of cardinality smaller than κ sets, where $\kappa > 2^{\aleph_0} + |\mathrm{TC}(x)|$, and let C be a countable elementary submodel of $\mathcal{H}(\kappa)$ with $x \in C$. It is clear that there exist infinitely many elements of $G_{(A)}$ which are mutually $\mathbf{Q}$-generic over C, and in fact there is perfect set such elements by [5] (specifically, Lemma 13, applied to the equivalence relation $\mathcal{E}$ on $G_{(A)}$ defined by $\pi_1 \mathrel{\mathcal{E}} \pi_2 \leftrightarrow x\pi_1 = x\pi_2$). More precisely, there is a perfect set $P \subset G_{(A)}$ such that for each $\pi_1, \pi_2 \in P$, (π_1, π_2) is $\mathbf{Q}_1 \times \mathbf{Q}_2$-generic over C. Thus $x\pi_1 \neq x\pi_2$ whenever $\pi_1, \pi_2 \in P$, and hence, the $G_{(A)}$-orbit of x is infinite. □

Lemma 2.3. *Let $x \in X \in \mathcal{N}$. Let $A \in \mathcal{I}$ be a support for X. If $|X| = p$, then there exists $b \in W$ such that $A \cup \{b\}$ supports x.*

Proof. Since $G_{(A)}$ fixes X, the $G_{(A)}$ orbit of x is contained in X, and hence is finite. By the previous lemma, there is a finite set $B \subset W$ such that $A \cup B$ supports x. We will show that if $|B| > 1$, then there is some B' with $|B'| < |B|$ such that $A \cup B'$ supports x; the lemma then follows by induction. Assume, without loss of generality, that

$$B \text{ is a linearly independent set disjoint from } \mathrm{Span}(A), \qquad (*)$$

and let $B = \{b_1, \ldots, b_n\}$, where this is a set of n distinct elements. Assume also that for each proper subset $C \subset B$, $A \cup C$ fails to support x (otherwise we are done easily). Then $G_{(A \cup C)}$ acts non-trivially on X for each proper $C \subset B$, and, since $G_{(A \cup C)}$ is a p-group and $|X| = p$, the $G_{(A \cup C)}$-orbit of x must be X. Let $G' = G_{(A \cup \{b_3, \ldots, b_n\})}$. Let H be the stabilizer of x in G'; that is, $H = \{\, g \in G' : xg = x \,\}$. Then $[G' : H] = |\operatorname{Orb}_{G'}(x)| = p$. Note that $G'_{(b_1, b_2)} = G_{(A \cup B)} \subset H$ since $A \cup B$ supports x. It suffices to find $b \in W$ such that $G'_{(b)} \subseteq H$, for then $A \cup \{b, b_3, \ldots, b_n\}$ supports x, as desired.

Recall (by Remark (2) above) that $G'_{(b_1, b_2)}$ is the stabilizer in G' of any ordered pair $(u_1, u_2) \in U_{b_1} \times U_{b_2}$. It follows from $(*)$ that there exist elements of G which move the p elements of U_{b_1} while fixing all elements of U_w for each $w \in A \cup \{b_2, b_3, \ldots, b_n\}$, and likewise with b_1 and b_2 switched. Thus $U_{b_1} \times U_{b_2}$ itself is the G'-orbit of the pair (u_1, u_2), and so $[G' : G'_{(b_1, b_2)}] = p^2$. Therefore, $[H : G'_{(b_1, b_2)}] = p$.

Let $h \in H \smallsetminus G'_{(b_1, b_2)}$. The natural image of h in the quotient group $H / G'_{(b_1, b_2)}$ generates that quotient group (which has order p), and therefore h generates the action of H on $U_{b_1} \times U_{b_2}$. Let $m, n \in \mathbb{F}_p$ such that

$$(0, b_1)h = (m, b_1) \quad \text{and} \quad (0, b_2)h = (n, b_2).$$

Note that if $m = 0$ or $n = 0$, then we are done easily: Say $m = 0$. Then h fixes at b_1, and consequently every element of H fixes at b_1, so $H \subseteq G'_{(b_1)}$. But then $H = G'_{(b_1)}$, since both subgroups have the same index in G', and the proof is completed by taking $b = b_1$.

On the other hand, if m and n are both nonzero, then we have inverses m^{-1} and n^{-1} in $\mathbb{F}_p$, and we let $b = m^{-1}b_1 - n^{-1}b_2$. Now we just want to show that $G'_{(b)} \subseteq H$. But since these two groups have the same index in G', it is equivalent to show $H \subseteq G'_{(b)}$. Compute:

$$\begin{aligned}(0, b)h = (0, m^{-1}b_1 - n^{-1}b_2)h = m^{-1}(0, b_1)h - n^{-1}(0, b_2)h = \\ m^{-1}(m, b_1) - n^{-1}(n, b_2) = (1, m^{-1}b_1) - (1, n^{-1}b_2) = (0, b).\end{aligned}$$

Thus h fixes at $\{b\}$, and so does every power of h. Since every element of H acts on b_1 and b_2 like a power of h, it follows that $H \subseteq G'_{(b)}$, as desired. ☐

Now, to show $\mathcal{N} \models \mathrm{CP}(\aleph_0,p)$, let $Z = \{\, X_n \,:\, n \in \omega \,\}$ be a set of p-elements sets, with Z countable in $\mathcal{N}$. Let $A \in \mathcal{I}$ be a support for a well-ordering of Z, so that A is a support for each element of Z. For each $n \in \omega$, let $x_n \in X_n$ (of course, Z might not have a choice function in $\mathcal{N}$, but we are working in $\mathcal{M}$). By Lemma 2.3, since $|X_n| = p$, there is some $s_n \in W$ such that $A \cup \{s_n\}$ supports x_n. Let $S = \{\, s_n \,:\, n \in \omega \,\}$. If S is finite, then $A \cup S \in \mathcal{I}$, and $A \cup S$ is a support for the enumeration $\langle x_n \rangle_{n\in\omega}$, so in fact Z has a choice function in $\mathcal{N}$. If S is infinite, then there is an infinite $B \in \mathcal{I}$ such that $B \subseteq S$; say $B = \{\, s_n \,:\, n \in J \,\}$. In this case, $A \cup B$ is a support for the enumeration $\langle x_n \rangle_{n\in J}$. In either case, Z has an infinite partial choice function in $\mathcal{N}$, as desired. ☐

It remains in this section to establish the existence of an ideal on $W = \oplus_{i=0}^{\infty}\mathbb{F}_p$ having the properties needed in the proof of Theorem 2.1.

Notation and definitions.

1. For $n \in \omega \smallsetminus \{0\}$, let $\log_{*p}(n)$ be the least $k \in \omega$ such that $(\log_p)^k(n) \leq 1$, where $(\log_p)^0(n) = n$ and $(\log_p)^{k+1}(n) = \log_p\big((\log_p)^k(n)\big)$.

   Note: In what follows, $\log_{*p}$ could be replaced by $\log_* = \log_{*2}$ with no effect on the arguments, except for minor changes needed in part (2) of Lemma 2.4.
2. For convenience, let $\{\, e_k \,:\, k \in \omega \,\}$ be the "standard basis" for $W = \oplus_{i=0}^{\infty}\mathbb{F}_p$; i.e.

$$e_k(i) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{else.} \end{cases}$$

3. For $w = \sum_{\ell} a_\ell e_\ell \in W$, let $\mathrm{pr}_k(w) = \sum_{\ell<k} a_\ell e_\ell$.
4. $d_k(A) = |\{\, \mathrm{pr}_k(w) \,:\, w \in A \,\}|$.
5. We say $A \subset W$ is *thin* if

$$\lim_{k\to\infty} \frac{\log_{*p}[d_k(A)]}{\log_{*p}(k)} \;=\; 0.$$

Lemma 2.4. *Let $\mathcal{I}$ be the set of thin subsets of W. Then*

(0) $\mathcal{I}$ is an ideal on W,

(1) every infinite subset of W contains an infinite member of $\mathcal{I}$, and

(2) $A \in \mathcal{I} \Rightarrow \mathrm{Span}(A) \in \mathcal{I}$.

Proof. (0) Clearly $\mathcal{I}$ is closed under subset. Suppose A_1 and A_2 are thin, and let $A = A_1 \cup A_2$. Then (for any $k \in \omega$) $d_k(A) \leq d_k(A_1) + d_k(A_2)$, so

$$\frac{\log_{*p}[d_k(A)]}{\log_{*p}(k)} \leq \frac{\log_{*p}[d_k(A_1) + d_k(A_2)]}{\log_{*p}(k)} \leq \frac{1 + \max_{i=1,2}\big(\log_{*p}[d_k(A_i)]\big)}{\log_{*p}(k)}.$$

The limit as $k \to \infty$ must be 0, so A is thin.

(1) Let $A \subseteq W$ be an infinite thin set. By König's Lemma, we can find pairwise distinct $x_n \in A$ for $n \in \omega$ such that for each $i \in \omega$, $\langle x_n(i) \rangle_{n<\omega}$ is eventually constant.

Let $n_0 = 0$. For $i \in \omega$, assuming $n_0, \dots n_i$ are chosen, we can choose n_{i+1} large enough so that

$$\begin{aligned} &\mathrm{pr}_{n_i}(x_{n_{i+1}}) = \mathrm{pr}_{n_i}(x_{n_{i+1}+t}) \quad \text{for all } t \in \omega, \\ \text{and} \quad &\log_{*p}(n_{i+1}) > i+1. \end{aligned}$$

Let $A^- = \{ x_{n_i} : i \in \omega \}$. Then $d_{n_i}(A^-) \leq i+1$, and

$$\lim_{i \to \infty} \frac{\log_{*p}(d_{n_i}(A^-))}{\log_{*p}(n_i)} \leq \lim_{i \to \infty} \frac{\log_{*p}(i+1)}{i} = 0.$$

Therefore A^- is an infinite, thin subset of A.

(2) For any $A \subset W$, observe that

$$d_k(\mathrm{Span}\, A) \leq p^{d_k(A)}.$$

Thus

$$\log_{*p}(d_k(\mathrm{Span}\, A)) \leq \log_{*p}\left(p^{d_k(A)}\right) \leq 1 + \log_{*p}(d_k(A)).$$

It follows easily that if A is thin, then $\mathrm{Span}\, A$ is also thin. □

Everything needed for Theorem 2.1 has now been proven. Note that this theorem does not say anything about the independence of C($\aleph_0$,n) from PC($\aleph_0$,n) in ZF when n is not prime. We intend to consider the case when n is not prime elsewhere.

3 Simpler models not useful for the main theorem

We consider a family of permutation models, some of which may on first consideration seem to be promising candidates to witness that PC($\aleph_0$,2) $\not\to$ C($\aleph_0$,2). However, it will turn out that PC($\aleph_0$,2) fails in every such model.

Let $\mathcal{M}$ be a model of ZFAC whose set U of atoms is countable and infinite. Let $\mathcal{P} = \{ U_n : n \in \omega \}$ be a partition of U into pairs. Let G be the group of permutations of U (equivalently, automorphisms of $\mathcal{M}$) which fix each element of $\mathcal{P}$. Let $\mathcal{I}$ be some ideal on ω. For $A \in \mathcal{I}$ and $g \in G$, we say g *fixes at* A if g fixes each element of $\bigcup_{n \in A} U_n$. Define *support* and *symmetric* by analogy with the definitions of these terms in the proof of the main theorem, and let $\mathcal{N}$ be the permutation submodel consisting of the hereditarily symmetric elements.

If $\mathcal{I}$ is the ideal of finite subsets of ω, then $\mathcal{N}$ is the "second Fraenkel model." Clearly $\mathcal{P}$ has no infinite partial choice function in the second Fraenkel model. Of course, if $\mathcal{I}$ is any larger than the finite set ideal, then $\mathcal{P}$ does have an infinite partial choice function, and it may be tempting to think that if $\mathcal{I}$ is well-chosen, then perhaps PC($\aleph_0$,2) will hold in the resulting model $\mathcal{N}$. However, we will show how to produce a set $Z = \{ X_n : n \in \omega \}$ of pairs, countable in $\mathcal{N}$, with no infinite partial choice function (no matter how $\mathcal{I}$ is chosen).

Notation: For sets A and B, let $P(A, B)$ be the set of bijections from A to B. We are interested in this when A and B are both pairs, in which case $P(A, B)$ is also a pair.

Let $X_0 = A_0$. For $i \in \omega$, let $X_{i+1} = P(X_i, A_{i+1})$. The empty set supports each pair X_i, so $Z = \{ X_n : n \in \omega \}$ is a countable set in $\mathcal{N}$. Let $S \in \mathcal{I}$; we'll show that S fails to support any infinite partial choice function for Z. Let $i = \min(\omega \setminus S)$, and let $g \in G$ be the permutation which swaps the elements of A_i and fixes all other atoms, so $g \in G_{(S)}$. This g fixes each element of X_n for $n < i$, but swaps the elements of X_i. By simple induction, g also swaps the elements of X_n for all $n > i$. It follows that for any $C \in \mathcal{M}$ which is an infinite partial choice function for Z, $Cg \neq C$, and thus S does not support C.

References

[1] O. De la Cruz, E. Hall, P. Howard, K. Keremedis, and J. Rubin. Unions and the axiom of choice. *Math. Logic Quart.* to appear.

[2] Paul Howard and Jean E. Rubin. *Consequences of the axiom of choice.* American Mathematical Society, Providence, RI, 1998. (`http://consequences.emich.edu/`).

[3] Thomas J. Jech. *The axiom of choice.* North-Holland Publishing Co., Amsterdam, 1973. Studies in Logic and the Foundations of Mathematics, Vol. 75.

[4] D. Pincus. Zermelo–Fraenkel consistency results by Fraenkel–Mostowski methods. *J. Symbolic Logic*, 37(4):721–743, Dec. 1972.

[5] Saharon Shelah. Can the fundamental (homotopy) group of a space be the rationals? *Proceedings of the American Mathematical Society*, 103:627–632, 1988.