

MAD FAMILIES AND SANE PLAYER
SH935

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ABSTRACT. We throw some light on the question: is there a MAD family (= a maximal family of infinite subsets of \mathbb{N} , the intersection of any two is finite) which is saturated (= completely separable i.e. any $X \subseteq \mathbb{N}$ is included in a finite union of members of the family or includes a member (and even continuum many members) of the family). We prove that it is hard to prove the consistency of the negation:

- (a) if $2^{\aleph_0} < \aleph_\omega$, then there is such a family
- (b) if there is no such families then some situation related to pcf holds whose consistency is large; and if $\mathfrak{a} > \aleph_1$ even unknown
- (c) if there is no inner model with measurables then there is such a family.

0. INTRODUCTION

We try to throw some light on

Problem 0.1. Is there, provably in ZFC, a completely separable MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$, see Definition 0.3(1),(3).

{0z.2}

Erdős-Shelah [ErSh:19] investigates on ZFC-existence of families $\mathcal{A} \subseteq \mathcal{P}(\omega)$ with separability properties, continuing Hechler [Hec71] which mostly uses MA; now 0.1 is Problem A of [ErSh:19], pg.209, see earlier Miller [Mil37], and see later Goldstern-Judah-Shelah [GJSh:399] on existence for larger cardinals. It seemed natural to prove the consistency of a negative answer by CS iteration making the continuum \aleph_2 but this had not worked out; the results here show this is impossible.

The celebrated matrix-tree theorem of Balcar-Pelant-Simon [BPS80], Balcar-Simon [BS89] is related to our approach. In Gruenhut-Shelah [GhSh:E64] we try to generalize it, hoping eventually to get applications, e.g. “there is a subgroup of ${}^\omega\mathbb{Z}$ which is reflexive (i.e. canonically isomorphic to the dual of its dual)” and “less”, see Problem X of [EM02], no success so far. We then had tried to use such constructions to answer 0.1 positively, but this does not work. Simon [CK96] have proved (in ZFC), that there is an infinite almost disjoint $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \subseteq \omega$ and $(\exists^\infty A \in \mathcal{A})[B \cap A \text{ infinite}] \Rightarrow (\exists A \in \mathcal{A})(A \subseteq B)$. Shelah-Steprans [ShSr:931] try to continue it with dealing with Hilbert spaces. Here \mathfrak{s} and ideals (formally $J \in \text{OB}$) are central.

Originally we have a unified proof but with some parameters for the properties. As on the one hand it was claimed this is unreadable and on the other hand we have a direct proof, which was presented (for $\mathfrak{s} < \mathfrak{a}$), in the Hebrew University and Rutgers, we use the later one. A minor price is that the proof in §2 are saying

Date: December 22, 2009 .

Research supported by the United States-Israel Binational Science Foundation (Grant No. 2006108). Publication 935. The author thanks Alice Leonhardt for the beautiful typing.

- repeat the earlier one with the following changes. The major price that some information is lost: using smaller more complicated cardinal invariants as well as some points in the proof which we hope will serve other proofs (including covering all cases) so we intend to return to it in [Sh:F951].

A relative problem of Balcar and Simon is: given an AD \mathcal{B} we look for such \mathcal{A} refining it, i.e. $(\forall A \in \mathcal{A})(\exists B \in \mathcal{B})(A \subseteq^* B)$. At present there is no difference between the two problems (i.e. in 1.1, 2.1, 2.6 covers this too)

{0z.3} Anyhow

Conclusion 0.2. 1) If $2^{\aleph_0} < \aleph_\omega$ then there is a saturated MAD family.
2) Moreover in (1) for any hereditary dense $J_* \subseteq [\omega]^{\aleph_0}$ we can find such a family $\subseteq J_*$.

{0z.4} We hope to return to this.

Definition 0.3. 1) We say \mathcal{A} is an AD (family) for B when $\mathcal{A} \subseteq [B]^{\aleph_0}$ is infinite, almost disjoint (i.e. $A_1 \neq A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2$ finite).

1A) We say \mathcal{A} is MAD for B when \mathcal{A} is AD for B and is \subseteq -maximal among such \mathcal{A} 's.

2) If $B = \omega$ we may omit it.

3) For $\mathcal{A} \subseteq [\omega]^{\aleph_0}$, $\text{id}_{\mathcal{A}}$ is the ideal generated by $\mathcal{A} \cup [\omega]^{<\aleph_0}$.

4) A MAD family \mathcal{A} is saturated when: if $B \in \text{id}_{\mathcal{A}}^+$ (see 0.5(3)) then B almost contains some member of \mathcal{A} (equivalently: if $B \in \text{id}_{\mathcal{A}}^+$ then B almost contains continuum many members of \mathcal{A} because if $B \in \text{id}_{\mathcal{A}}^+$ there is an AD family $\mathcal{B} \subseteq [B]^{\aleph_0} \cap \text{id}_{\mathcal{A}}^+$ of cardinality 2^{\aleph_0}).

{0z.5}

Remark 0.4. 1) Note that if there is a MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \in \text{id}_{\mathcal{A}}^+ \Rightarrow (\exists 2^{\aleph_0} A \in \mathcal{A})(B \cap A \text{ is infinite})$, then there is a MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \in \text{id}_{\mathcal{A}}^+ \Rightarrow (\exists 2^{\aleph_0} A \in \mathcal{A})(A \subseteq B)$ equivalently $B \in \text{id}_{\mathcal{A}}^+ \Rightarrow (\exists A \in \mathcal{A})(A \subseteq B)$.

{0z.6}

2) So the four variants of "there is $\mathcal{A} \dots$ " in 0.3(4), 0.4(1) are equivalent.

Notation 0.5. 1) For $A \subseteq \omega$ let $A^{[\ell]}$ be A if $\ell = 1$ and $\omega \setminus A$ if $\ell = 0$.

2) For $J \subseteq [\omega]^{\aleph_0}$ let $J^\perp = \{B : B \in [\omega]^{\aleph_0} \text{ and } [A \in J \Rightarrow A \cap B \text{ finite}]\}$ and also for $\bar{A} = \langle A_s : s \in S \rangle$ let $\bar{A}^\perp = \{A_s : s \in S\}^\perp$.

3) $\text{id}_{\mathcal{A}}(B)$ is the ideal of $\mathcal{P}(B)$ generated by $\mathcal{A} \cup [\omega]^{<\aleph_0}$ and $\text{id}_{\mathcal{A}}^+(B) = [B]^{\aleph_0} \setminus \text{id}_{\mathcal{A}}(B)$; if $B = \omega$ we may omit it.

4) Let $A \subseteq^* B$ means that $A \setminus B$ is finite.

5) If $\mathcal{C} \subseteq \mathcal{P}(\omega)$ and $\eta \in {}^{\mathcal{C}}2$ then $I_{\mathcal{C}, \eta}(B)$ is $\{C \subseteq B : C \subseteq^* A^{[\eta(A)]} \text{ for every } A \in \mathcal{C}\}$; if $B = \omega$ we may omit it.

6) In part 5), if ν is a function extending η then let $I_{\mathcal{C}, \nu} = I_{\mathcal{C}, \eta}$.

7) For $\mathcal{A} \subseteq \mathcal{P}(B_2)$ and $B_1 \subseteq B_2$ let $\mathcal{A} \upharpoonright B_1 = \{A \cap B_1 : A \in \mathcal{A} \text{ satisfies } A \cap B_1 \text{ is infinite}\}$.

{0z.8}

Definition 0.6. 1) Let $\text{OB} = \{I \subseteq [\omega]^{\aleph_0} : I \cup [\omega]^{<\aleph_0} \text{ is an ideal of } \mathcal{P}(\omega)\}$.

2) For $A \subseteq \omega$ let $\text{ob}(A) = \{B : B \in [\omega]^{\aleph_0} \text{ and } B \subseteq^* A\}$ so $\text{ob}(\omega) = [\omega]^{\aleph_0}$.

3) If $\eta \perp \nu$ means $(\neg \eta \leq \nu) \wedge (\neg \nu \leq \eta)$.

4) We say \mathcal{A} is AD in $J \subseteq [\omega]^{\aleph_0}$ when \mathcal{A} is AD and $\mathcal{A} \subseteq J$.

5) We say \mathcal{A} is MAD in $J \subseteq [\omega]^{\aleph_0}$ when \mathcal{A} is AD in J and is \subseteq -maximal among such \mathcal{A} 's.

6) $J \subseteq [\omega]^{\aleph_0}$ is hereditary when $A \in [\omega]^{\aleph_0} \wedge A \subseteq^* B \in J \Rightarrow A \in J$.

7) $J \subseteq [\omega]^{\aleph_0}$ is dense when $(\forall B \in [\omega]^{\aleph_0})(\exists A \in J)[A \subseteq B]$.

1. THE SIMPLE CASE: $\mathfrak{s} < \mathfrak{a}$

We here give a proof for the case $\mathfrak{s} < \mathfrak{a}$.

{4d.3}

Theorem 1.1. 1) If $\mathfrak{s} < \mathfrak{a}$ then there is a saturated MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$.
2) Moreover, given a hereditary dense $J_* \subseteq [\omega]^{\aleph_0}$ we can demand $\mathcal{A} \subseteq J_*$.

Proof. Stage A: Let $\kappa = \mathfrak{s}$, so $\text{cf}(\kappa) > \aleph_0$. For part (1) let $J_* = [\omega]^{\aleph_0}$ which is a hereditary dense subset of $[\omega]^{\aleph_0}$, i.e. as in part (2) and the reader may concentrate on this case.

Choose a sequence $\langle C_\alpha^* : \alpha < \kappa \rangle$ of subsets of ω exemplifying $\mathfrak{s} = \kappa$, i.e. $\neg(\exists B \in [\omega]^{\aleph_0}) \wedge \bigwedge_\alpha (B \subseteq^* C_\alpha^* \vee B \subseteq^* \omega \setminus C_\alpha^*)$. For $i < \kappa$ and $\eta \in {}^i 2$ let $C_\eta^* = C_i^*$, the aim of this notation is to simplify latter proofs where we say “repeat the present proof but ...”.

Stage B: For $\alpha \leq 2^{\aleph_0}$ let AP_α , the set of α -approximations, be the set of t consisting of the following objects satisfying the following conditions:

- ⊞₁ (a) $\mathcal{T} = \mathcal{T}_t$ is a subtree of ${}^{\kappa > 2} 2$, i.e. closed under initial segments
- (b) let $\text{suc}(\mathcal{T}) = \{\eta \in \mathcal{T} : \ell g(\eta) \text{ is a successor ordinal}\}$ and¹
 $\text{cl}(\mathcal{T}) = \{\eta \in {}^{\kappa \geq 2} 2 : \text{if } i < \ell g(\eta) \text{ then } \eta \upharpoonright i \in \mathcal{T}\}$
- (c) $1 \leq |\mathcal{T}| \leq \aleph_0 + |\alpha|$
- (d) $\bar{I} = \bar{I}_t = \langle I_\eta : \eta \in \text{cl}(\mathcal{T}) \rangle = \langle I_\eta^t : \eta \in \text{cl}(\mathcal{T}_t) \rangle$
- (e) $\bar{A} = \bar{A}_t = \langle A_\eta : \eta \in \text{suc}(\mathcal{T}) \rangle = \langle A_\eta^t : \eta \in \text{suc}(\mathcal{T}_t) \rangle$

such that

- (f) $A_\eta \in I_\eta \cap J_*$ or² $A_\eta = \emptyset$ and $\mathcal{S}_t = \{\eta \in \text{suc}(\mathcal{T}_t) : A_\eta \neq \emptyset\}$
- (g) $I_\eta = \{A \in [\omega]^{\aleph_0} : \text{if } i < \ell g(\eta) \text{ then } A \subseteq^* (C_{\eta \upharpoonright i}^*)^{[\eta(i)]} \text{ and if } i+1 < \ell g(\eta) \text{ then } A \cap A_{\eta \upharpoonright (i+1)} \text{ is finite}\}$, so I_η is well defined also when $\eta \in \text{cl}(\mathcal{T})$.

We let

- (h) $C_\eta^t = C_\eta^*$ (for generalizations)
- ⊞₂ $\text{AP} = \cup \{\text{AP}_\alpha : \alpha \leq 2^{\aleph_0}\}$
- ⊞₃ $s \leq_{\text{AP}} t$ iff (both are from AP and)
 - (a) $\mathcal{T}_s \subseteq \mathcal{T}_t$
 - (b) $\bar{I}_s = \bar{I}_t \upharpoonright \text{cl}(\mathcal{T}_s)$
 - (c) $\bar{A}_s = \bar{A}_t \upharpoonright \text{suc}(\mathcal{T}_s)$.

Stage C: We assert various properties of AP; of course s, t denote members of AP:

- ⊞₄ (a) \leq_{AP} partially orders AP
- (b) $\eta < \nu \in \text{cl}(\mathcal{T}_t) \Rightarrow I_\nu^t \subseteq I_\eta^t$
- (c) if $\eta \in \text{cl}(\mathcal{T}_t)$ then $I_\eta^t \in \text{OB}$, i.e. $I_\eta^t \cup [\omega]^{< \aleph_0}$ is an ideal of $\mathcal{P}(\omega)$

¹so $\text{cl}(\{<>\}) = \{<>, <0>, <1>\}$

²the case “ $A_\eta = \emptyset$ ” is not needed in this proof

- (d) $\langle A_\eta^t : \eta \in \mathcal{S}_t \rangle$ is almost disjoint (so $A_\eta^t \in \text{ob}(\omega)$ and $\eta \neq \nu \in \mathcal{S}_t \Rightarrow A_\eta^t \neq A_\nu^t$; recall that here $\mathcal{S}_t = \text{suc}(\mathcal{T}_t)$)
- (e) if $\eta \in \text{cl}(\mathcal{T}_t)$ and $lg(\eta) = \kappa$ then $I_\eta^t = \emptyset$
- (f) if $s \leq_{\text{AP}} t$ then $\text{cl}(\mathcal{T}_s) \subseteq \text{cl}(\mathcal{T}_t)$ and $\eta \in \text{cl}(\mathcal{T}_s) \Rightarrow I_\eta^s = I_\eta^t$ (and clause (b) of \boxplus_3 follow from clauses (a),(c))
- (g) if $\nu \in \text{cl}(\mathcal{T}_s) \setminus \mathcal{T}_s$ and $\eta \in \mathcal{S}_s$ and $B \in I_\nu^s$ then $B \cap A_\nu$ is finite.

[Why clause (d)? Let $\eta_0 \neq \eta_1 \in \mathcal{S}_t$, if $\eta_0 \perp \eta_1$ let $\rho = \eta_0 \cap \eta_1$ hence for some $\ell \in \{0, 1\}$ we have $\rho \hat{\langle} \ell \rangle \trianglelefteq \eta_0, \rho \hat{\langle} 1 - \ell \rangle \triangleleft \eta_1$ so $A_{\eta_\ell} \in I_{\eta_\ell}^t \subseteq I_{\rho \hat{\langle} \ell \rangle}^t \subseteq \text{ob}((C_\rho^t)^{[\ell]})$ hence $A_{\eta_0} \cap A_{\eta_1} \subseteq^* \text{ob}((C_\rho^t)^{[\ell]}) \cap \text{ob}((C_\rho^t)^{[1-\ell]}) = \emptyset$. If $\eta_0 \triangleleft \eta_1$ note that $A_{\eta_1}^t \in I_{\eta_1}^t \subseteq \text{ob}(\omega \setminus A_{\eta_0}^t)$. Also if $\eta_1 \triangleleft \eta_0$ similarly so clause (d) holds indeed.

Why Clause (e)? Recalling the choice of $\langle C_\alpha^* : \alpha < \kappa \rangle$ and $\langle C_\eta^* : \eta \in \kappa > 2 \rangle$ hence $C_{\eta \upharpoonright \alpha}^t = C_\alpha^*$. So if $B \in I_\eta^t$, then $B \in I_{\eta \upharpoonright (\alpha+1)}$ hence then $(B \subseteq^* C_\alpha^* \vee B \subseteq^* \omega \setminus C_\alpha^*)$ for every $\alpha < \kappa$, a contradiction to the choice of $\langle C_\alpha^* : \alpha < \kappa \rangle$.]

- \boxplus_5 (a) $\alpha < \beta \leq 2^{\aleph_0} \Rightarrow \text{AP}_\alpha \subseteq \text{AP}_\beta$
- (b) $\text{AP}_0 \neq \emptyset$ (e.g. use t with $\mathcal{T}_t = \{ \langle \rangle \}$)
- (c) if $\langle t_i : i < \delta \rangle$ is \leq_{AP} -increasing, $t_i \in \text{AP}_{\alpha_i}$ for $i < \delta$, $\langle \alpha_i : i < \delta \rangle$ is increasing, δ a limit ordinal and $\alpha_\delta = \cup \{ \alpha_i : i < \delta \}$ then $t_\delta = \cup \{ t_i : i < \delta \}$ naturally defined belongs to $\text{AP}_{\alpha_\delta}$ and $i < \delta \Rightarrow t_i \leq_{\text{AP}} t_\delta$

\boxplus_6 let J_t be the ideal on $\mathcal{P}(\omega)$ generated by $\{A_\eta^t : \eta \in \mathcal{S}_t\} \cup [\omega]^{< \aleph_0}$.

For $s \in \text{AP}$ and $B \in \text{ob}(\omega)$ we define:

- (*)₁ $S_B = S_B^s := \{ \eta \in \text{cl}(\mathcal{T}_s) : [B \setminus A]^{\aleph_0} \cap I_\eta^t \neq \emptyset \text{ for every } A \in J_s \}$
- (*)₂ $\text{SP}_B = \text{SP}_B^s := \{ \eta \in \mathcal{T}_s : \eta \hat{\langle} 0 \rangle \in S_B^s \text{ and } \eta \hat{\langle} 1 \rangle \in S_B^s \}$.

Note

- (*)₃ S_B is a subtree of $\text{cl}(\mathcal{T}_s)$ and $\langle \rangle \in S_B \Leftrightarrow B \in J_s^+$, and $\text{SP}_B \subseteq \mathcal{T}_s$.

[Why? The first statement holds by recalling $\boxplus_4(b)$, the second, $\langle \rangle \in S_B \Leftrightarrow B \in J_s^+$, as $I_{\langle \rangle} = \text{ob}(\omega)$, the third, $\text{SP}_B \subseteq \mathcal{T}_s$ as by the definition of $\text{cl}(\mathcal{T}_s)$ we have $\eta \hat{\langle} \ell \rangle \in \text{cl}(\mathcal{T}_s) \Rightarrow \eta \in \mathcal{T}_s$.]

- (*)₄ if $\eta \in S_B$ and $\nu_0 \triangleleft \nu_1 \triangleleft \dots \triangleleft \nu_{n-1}$ list $\{ \nu \triangleleft \eta : \nu \in \text{SP}_B \}$ so this set is finite and we let $C_s(\eta, B) := \cap \{ (C_{\nu_\ell}^s)^{[\eta(lg(\nu_\ell))]} : \ell < n \}$, then $S_{B \cap C_s(\eta, B)} = \{ \nu \in S_B : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu \}$.

[Why? Clearly $(\forall A \in I_\eta^s)(A \subseteq^* C_s(\eta, B))$ by the definition of I_η^s , see $\boxplus_1(g)$.

As $B \cap C_s(\eta, B) \subseteq B$ clearly $S_{B \cap C_s(\eta, B)} \subseteq S_B$. Also as $\eta \in S_B$ and as $(\forall A \in I_\eta^s)(A \subseteq^* C_s(\eta, B))$ clearly $\eta \in S_{B \cap C_s(\eta, B)}$ and moreover $\{ \nu \in S_{B \cap C_s(\eta, B)} : \eta \trianglelefteq \nu \} = \{ \nu \in S_B : \eta \trianglelefteq \nu \}$ by $\boxplus_4(b)$.

Also as S_B and $S_{B \cap C_s(\eta, B)}$ are subtrees clearly $\{ \nu : \nu \trianglelefteq \eta \} \subseteq S_B \cap S_{B \cap C_s(\eta, B)}$ and $\eta \trianglelefteq \nu \in \text{cl}(\mathcal{T}_s) \Rightarrow [\nu \in S_B \Leftrightarrow \nu \in S_{B \cap C_s(\eta, B)}]$.

So to prove the equality it suffices to assume $\alpha < lg(\eta), \nu \in S_B, lg(\eta \cap \nu) = \alpha, lg(\nu) > \alpha$ and $\nu \in S_{B \cap C_s(\eta, B)}$ and get a contradiction. If $\ell < n$ and $\alpha = lg(\nu_\ell)$

then $(\forall A \in I_\nu^s)[A \subseteq^* (C_{\eta \uparrow \alpha}^s)^{[1-\eta(\alpha)]}]$, so an easy contradiction. If $\alpha \notin \{\ell g(\nu_\ell) : \ell < n\}$ we can get contradiction to $\eta \uparrow \alpha \notin SP_B$. So we are done proving $(*)_4$.]

- $(*)_5$ (a) for every $\eta \in cl(\mathcal{T}_t)$ the set $\{B \in ob(\omega) : \eta \notin S_B\}$ belongs to OB
- (b) if $B = B_0 \cup \dots \cup B_n \subseteq \omega$ then $S_B = S_{B_0} \cup \dots \cup S_{B_n}$
- $(*)_6$ if $B \subseteq \omega$ and $\nu \in S_B \cap \mathcal{T}_t$ and $B \subseteq^* (C_\nu^t)^{[\ell]}$ and $(\nu \in \mathcal{S}_t \Rightarrow B \cap A_\nu^t \text{ finite})$
then $\nu \hat{\ } \langle \ell \rangle \in S_B$.

Stage D:

- \boxplus_7 if $\alpha < 2^{\aleph_0}$, $s \in AP_\alpha$ and $B \in ob(\omega) \setminus J_s$ then we can find $t \in AP_{\alpha+1}$ such that $s \leq_{AP} t$ and B contains A_η for some $\eta \in \mathcal{S}_t \setminus \mathcal{T}_s$.

This is a major point and we shall prove it in Stage F below.

Stage E: We prove the theorem.

Let $\langle B_\alpha : \alpha < 2^{\aleph_0} \rangle$ list $\mathcal{P}(\omega)$ each appear 2^{\aleph_0} times. By induction on $\alpha \leq 2^{\aleph_0}$ we choose t_α such that

- \otimes (a) $t_\alpha \in AP_\alpha$
- (b) $\beta < \alpha \Rightarrow t_\beta \leq_{AP} t_\alpha$
- (c) if $\alpha = \beta + 1$ then either $B_\alpha \in J_{t_\beta}$ or B_α contains A_η , for some $\eta \in \mathcal{S}_{t_\alpha} \setminus \mathcal{T}_{t_\beta}$.

For $\alpha = 0$ use $\boxplus_5(b)$.

For α limit use $\boxplus_5(c)$.

For $\alpha = \beta + 1$ use \boxplus_7 .

Now let $t \in AP$ be $\cup \{t_\alpha : \alpha < 2^{\aleph_0}\}$ and it is easy to check that \bar{A}_t is a saturated MAD family, enough for 1.1(1) and recalling that by $\boxplus_1(f)$ it is $\subseteq J_*$ also enough for 1.1(2).

Stage F: The rest of the proof is dedicated to

the proof of \boxplus_7 so α, s and B are given.

The proof is now splitted to cases.

Case 1: Some $\nu \in S_B$ is such that $\nu \in cl(\mathcal{T}_s) \setminus \mathcal{T}_s$.

Clearly as $\nu \in S_B$ there is $B_1 \in [B]^{\aleph_0} \cap I_\nu^s$ and recalling J_* is dense see stage A, without loss of generality $B_1 \in J_*$. Note that $\ell g(\nu) > 0$ as $\langle \rangle \in \mathcal{T}_s$ by clause (c) of \boxplus_1 .

Note that $A \in I_\nu^s \wedge \eta \in \mathcal{S}_s \Rightarrow A \cap A_\eta^s$ is finite, e.g. by the proof of $\boxplus_4(d)$ or better by $\boxplus_4(g)$.

Subcase 1A: Assume $\ell g(\nu)$ is a successor ordinal.

Let $B_2 \subseteq B_1$ be such that B_2 and $B_1 \setminus B_2$ are infinite. Now define t as follows: $\mathcal{T}_t = \mathcal{T}_s \cup \{\nu\}$, A_ρ^t is A_ρ^s if $\rho \in \text{suc}(\mathcal{T}_s)$ and is B_2 if $\rho = \nu$, lastly define I_ρ^t for $\rho \in \mathcal{T}_t$ as in clause (g) of \boxplus_1 . Easy to check that t is as required; actually $B_2 = B_1$ is O.K., too.

Subcase 1B: Assume $\ell g(\nu)$ is a limit ordinal.

Clearly $\ell g(\nu) < \kappa$ by $\boxplus_4(e)$, as $I_\nu^s \neq \emptyset$ because $B_1 \in I_\nu^s$, clearly there is $\ell \in \{0, 1\}$ such that $B'_1 := (C_\nu^s)^{[\ell]} \cap B_1$ is infinite, let $B_2 \subseteq B'_1$ be such that $B_2, B'_1 \setminus B_2$ are

infinite. We define t by $\mathcal{T}_t = \mathcal{T}_s \cup \{\nu, \nu \hat{\langle \ell \rangle}\}$, A_ρ^t is A_ρ^s if $\rho \in \text{suc}(\mathcal{T}_s)$ and is B_2 if $\rho = \nu \hat{\langle \ell \rangle}$ and I_ρ^t for $\rho \in \mathcal{T}_t$ is defined as in clause (g) of \boxplus_1 .

Easy to check that t is as required.

Case 2: $\text{SP}_B = \emptyset$ but not case 1.

Let $\nu_B^* := \cup\{\eta : \eta \in S_B\}$.

Subcase 2A: $\nu_B^* \in S_B$.

As $S_B \subseteq \text{cl}(\mathcal{T}_s)$ by the definition of S_B , as we are assuming “not case 1” necessarily $S_B \subseteq \mathcal{T}_s$ hence $\nu_B^* \in \mathcal{T}_s$ so $\ell g(\nu_B^*) < \kappa$.

We define B_2^* as $B \cap A_{\nu_B^*}$ if $A_{\nu_B^*}$ is well defined and $B_2^* = \emptyset$ otherwise; and for $\ell = 0, 1$, let $B_\ell^* := B \cap (C_{\ell g(\nu_b^*)}^*)^{[\ell]} \setminus B_2^*$.

So

(*)₇ $\langle B_0^*, B_1^*, B_2^* \rangle$ is a partition of B

hence for some $\ell = 0, 1, 2$

(*)₈ $\nu_B^* \in S_{B_\ell^*}$

easily

(*)₉ $\ell \neq 2$,

and

(*)₁₀ $\rho := \nu_{B_\ell^*}^* \hat{\langle \ell \rangle} \in S_{B_\ell^*}$.

[Why? By the definitions.]

Also as $B_\ell^* \subseteq B$ clearly

(*)₁₁ $S_{B_\ell^*} \subseteq S_B$.

But (*)₁₀ + (*)₁₁ contradicts the choice of ν_B^* .

Subcase 2B: $\nu_B^* \notin S_B$.

Necessarily $\ell g(\nu_B^*)$ is a limit ordinal $\leq \kappa$, call it δ . So $\alpha < \delta \Rightarrow \nu_B^* \upharpoonright \alpha \in S_B$ but $\rho \triangleleft \varrho \in \text{cl}(\mathcal{T}_s) \Rightarrow \rho \in \mathcal{T}_s$ hence $\alpha < \delta \Rightarrow \nu_B^* \upharpoonright \alpha \in \mathcal{T}_s$. Now for every $\alpha < \delta$ let $\nu_{B,\alpha}^* := (\nu_B^* \upharpoonright \alpha) \hat{\langle 1 - \nu_B^*(\alpha) \rangle}$, so clearly $\nu_{B,\alpha}^* \in \text{cl}(\mathcal{T}_s) \setminus S_B$ hence we can find $n = n(\alpha) < \omega$ and $A_{\alpha,0}^*, \dots, A_{\alpha,n(\alpha)-1}^* \in \{A_\rho^s : \rho \in \text{suc}(\mathcal{T}_s)\}$ such that $[B \setminus (A_{\alpha,0}^* \cup \dots \cup A_{\alpha,n(\alpha)-1}^*)]^{N_0}$ is disjoint to $I_{\nu_{B,\alpha}^*}^s$ and we can further demand under those restrictions that $n(\alpha)$ is minimal. Let $A_{\alpha,n(\alpha)}^*$ be $A_{\nu_B^* \upharpoonright \alpha}$ when defined and \emptyset otherwise. By the definitions of $I_{\nu_{B,\alpha}^*}^s, I_{\nu_B^* \upharpoonright \alpha}^s$ we have (for $\alpha < \delta$ of course):

\odot_1 $[B \cap (C_{\nu_B^* \upharpoonright \alpha}^s)^{[1 - \nu_B^*(\alpha)]} \setminus (A_{\alpha,0}^* \cup \dots \cup A_{\alpha,n(\alpha)}^*)]^{N_0}$ is disjoint to $I_{\nu_B^* \upharpoonright \alpha}^s$.

Let $\mathcal{A}^* = \{B \cap A : A = A_{\alpha,k}^* \text{ for some } \alpha < \delta, k \leq n(\alpha) \text{ and } B \cap A \text{ is infinite}\}$.

So \mathcal{A}^* is a family of pairwise almost disjoint infinite subsets of B and if \mathcal{A}^* is finite, still $B \setminus \cup\{A : A \in \mathcal{A}^*\}$ is infinite because $\mathcal{A}^* \subseteq J_s$ and we are assuming $B \notin J_s$. Now

\odot_2 there is an infinite $B_1 \subseteq B$ which is almost disjoint to any $A \in \mathcal{A}^*$.

[Why? If \mathcal{A}^* is finite by the previous paragraph. Otherwise as $|\mathcal{A}^*| \leq |\delta| + \aleph_0 \leq \kappa = \mathfrak{s}$ and by the theorem's assumption $\mathfrak{s} < \mathfrak{a}$ and by the definition of \mathfrak{a} it follows that \odot_2 holds.]

$$\odot_3 \quad B_1 \in J_s^\perp.$$

[Why? By the choice of \mathcal{A}^* and B_1 . Fully, first $B_1 \subseteq B \subseteq \omega$, second B_1 is infinite by \odot_2 , third we should prove that $[\eta \in \text{suc}(\mathcal{T}_s) \Rightarrow B_1 \cap A_\eta \text{ finite}]$. If $A_\eta \in \{A_{\alpha,m}^* : \alpha < \delta, n \leq n(\alpha)\}$ then either $A_\eta \cap B$ is finite hence $A_\eta \cap B_1 \subseteq A \cap B$ is finite or $A_\eta \cap B$ is infinite hence $A_\eta \cap B \in \mathcal{A}^*$ hence $B_1 \cap (A_\eta \cap B)$ is finite by the choice of B_1 but $B_1 \subseteq B$ hence $B_1 \cap A_\eta$ is finite. So assume $A_\eta \notin \{A_{\alpha,m}^* : \alpha < \delta, n \leq n(\alpha)\}$, so by the choice of $A_{\alpha,n(\alpha)}$ for $\alpha < \delta$ necessarily $\neg(\eta \triangleleft \nu_B^*)$. If $\nu_B^* \trianglelefteq \eta$ note that $B' \in \text{ob}(A_\eta) \Rightarrow B' \in I_\eta^s$ so if $A \cap A_\eta$ is infinite then $\eta \in \mathcal{S}_B$, contradicting the case assumption so $\neg(\nu_B^* \trianglelefteq \eta)$. Together for some $\alpha < \delta$ we have $\alpha = \ell g(\nu_B^* \cap \eta) < \delta$ and $\nu_B^* \upharpoonright \alpha \triangleleft \eta$ and we get contradiction by $\odot_1 + \odot_2$.]

We shall now prove by induction on $\alpha \leq \delta$ that $B_1 \in I_{\nu_B^* \upharpoonright \alpha}^s$. For $\alpha = 0$ recall $I_{\nu_B^* \upharpoonright \alpha}^s = [\omega]^{\aleph_0}$, for α limit $I_{\nu_B^* \upharpoonright \alpha}^s = \bigcap \{I_{\nu_B^* \upharpoonright \beta}^s : \beta < \alpha\}$ and use the induction hypothesis. For $\alpha = \beta + 1$ first note that B_1 is almost disjoint to $A_{\nu_B^* \upharpoonright \beta}$ if $\nu_B^* \upharpoonright \beta \in \text{suc}(\mathcal{T}_s)$ by \odot_3 and, second, B_1 is almost disjoint to $(C_{\nu_B^* \upharpoonright \beta}^s)^{[1-\nu_B^*(\beta)]}$ otherwise we get contradiction to \odot_2 by $(*)_6+$ the induction hypothesis; together by the definition of $I_{\nu_B^* \upharpoonright \beta}^s, I_{\nu_B^* \upharpoonright \alpha}^s$ we have $B_1 \in I_{\nu_B^* \upharpoonright \alpha}^s$. Having carried the induction, in particular $B_1 \in I_{\nu_B^* \upharpoonright \delta}^s = I_{\nu_B^*}^s$; now recalling $B_1 \notin J_s$ by \odot_3 and the choice of \mathcal{A}^*, J_s , this contradicts the subcase assumption $\nu_B^* \notin S_B$.

Case 3: None of the above.

Without loss of generality

$$\oplus_1 \quad \text{if } B_1 \subseteq B \text{ but } B_1 \notin J_s \text{ then none of the two cases above holds.}$$

We try to choose $\bar{\eta}^n = \langle \eta_\rho : \rho \in {}^n 2 \rangle$ by induction on n such that:

- (a) $\eta_\rho \in \text{SP}_B$
- (b) if $\rho = \varrho \hat{\ } \langle \ell \rangle$ then $\eta_\varrho \hat{\ } \langle \ell \rangle \trianglelefteq \eta_\rho$
- (c) $\{\nu : \nu \triangleleft \eta_\rho \text{ and } \nu \in \text{SP}_B\} = \{\eta_{\rho \upharpoonright k} : k < \ell g(\eta_\rho)\}$.

For $n = 0$, note that $\text{SP}_B \neq \emptyset$ as not case 2 (and not case 1) so we can choose $\eta_\rho \in \text{SP}_B$ with minimal length. If $n = m+1$ and $\rho \in {}^m 2$ by the induction hypothesis $\eta_\rho \in \text{SP}_B$, hence $\eta_\rho \in \mathcal{T}_s$ and by the definition of SP_B for $\ell = 0, 1$ the sequence $\eta_\rho \hat{\ } \langle \ell \rangle$ belongs to S_B .

First assume $\{\nu \in \text{SP}_B : \eta_\rho \hat{\ } \langle \ell \rangle \trianglelefteq \nu\} = \emptyset$. So $B_1 := B \cap C_s(\eta_\rho \hat{\ } \langle \ell \rangle, B) \notin J_s$ noting $C_s(\eta_\rho \hat{\ } \langle \ell \rangle, B) = \bigcap \{C_{\eta_\rho \upharpoonright k}^{[\rho \upharpoonright k]} : k \leq m\}$, recalling it is defined in $(*)_4$ from Stage C; hence $\eta_\rho \hat{\ } \langle \ell \rangle \in S_{B_1} \neq \emptyset$.

Now by $(*)_4$ we know $S_{B_1} = \{\nu \in S_B : \nu \trianglelefteq \eta_\rho \hat{\ } \langle \ell \rangle \text{ or } \eta_\rho \hat{\ } \langle \ell \rangle \trianglelefteq \nu\}$ so case 2 or case 1 holds for B_1 , contradiction to \oplus_1 .

Second, assume we have $(\exists \eta)(\eta_\rho \hat{\ } \langle \ell \rangle \trianglelefteq \eta \in \text{SP}_B)$ so choose such $\eta_\rho \hat{\ } \langle \ell \rangle$ of minimal length.

Hence we have carried the inductive choice of $\langle \bar{\eta}^n : n < \omega \rangle$.

For each $\rho \in {}^\omega 2$ let $\eta_\rho = \bigcup \{\eta_{\rho \upharpoonright n} : n < \omega\}$, clearly $\eta_\rho \in \text{cl}(\mathcal{T}_s)$. Also $\langle \eta_\rho : \rho \in {}^\omega 2 \rangle$ is without repetitions and each η_ρ belongs to $\text{cl}(\mathcal{T}_s)$, so as $|\mathcal{T}_s| < 2^{\aleph_0}$ there is $\rho \in {}^\omega 2$

such that $\eta_\rho \notin \mathcal{T}_s$. By clause (c) above we have $\{\varrho : \varrho \triangleleft \eta_\rho \text{ and } \varrho \in SP_B\} = \{\eta_\rho \upharpoonright n : n < \omega\}$.

It is enough to find B_1 such that

- \oplus_2 B_1 is infinite, $B_1 \subseteq B$ satisfies $n < \omega \Rightarrow B_1 \subseteq^* C_s(\eta_\rho \upharpoonright n, B)$ and $B_1 \cap A_{\eta_\rho \upharpoonright \alpha}$ is finite when $\alpha < \ell g(\eta_\rho)$ is a successor ordinal.

Note that $\langle C_s(\eta_\rho \upharpoonright k, B) : k < \omega \rangle$ is \subseteq -decreasing.

First, assume we can find pairwise³ distinct $\alpha_k < \ell g(\eta_\rho)$ for $k < \omega$ such that $m < \omega \wedge k < \omega \Rightarrow [B \cap C_s(\eta_\rho \upharpoonright m, B) \cap A_{\eta_\rho \upharpoonright \alpha_k}$ is infinite]. In this case without loss of generality $\alpha_n \geq \ell g(\eta_\rho \upharpoonright n)$ for $n < \omega$ and we can choose an infinite $B_0 \subseteq B$ such that $n < \omega \Rightarrow B_0 \setminus \cup \{A_{\eta_\rho \upharpoonright k} : k < n\} \subseteq^* C_s(\eta_\rho \upharpoonright \alpha_n, B) \wedge (B_0 \cap A_{\eta_\rho \upharpoonright \alpha_n} \in \text{ob}(\omega))$. So

- \oplus_3 $B_0 \subseteq B, B_0 \notin J_s$;
 \oplus_4 the set SP_{B_0} is empty.

[Why? By $(*)_4$ for each $n < \omega$ we have $S_{B_0} \subseteq \{\nu : \nu \leq \rho_n \vee \rho_n \leq \nu\}$, hence $S_{B_0} \cap \mathcal{T}_s \subseteq \{\nu : \nu \triangleleft \eta_\rho \text{ or } \eta_\rho \leq \nu\}$ but $\eta_\rho \notin \mathcal{T}_s$ so $SP_{B_0} = \emptyset$.]

- \oplus_5 S_{B_0} is not empty.

[Why? By $(*)_3$.]

By $\oplus_4 + \oplus_5$ for the set B_0 , case 2 or case 1 hold, so we get contradiction to \oplus_1 .

Second, assume that we cannot find such pairwise distinct $\alpha_k < \ell g(\eta_\rho)$. Now we choose an infinite $B_1 \subseteq B$ such that $n < \omega \Rightarrow B_1 \subseteq^* C_s(\eta_\rho \upharpoonright n, B)$ and $\alpha < \ell g(\eta_\rho) \wedge \bigwedge_{n < \omega} (C_s(\eta_{\rho_n}, B) \cap A_{\eta_\rho \upharpoonright \alpha} \cap B \text{ infinite}) \Rightarrow B_1 \cap A_{\eta_\rho \upharpoonright \alpha}$ is finite. Obviously $B_1 \in I_{\eta_\rho}$, so for it case 1 holds as exemplified by η_ρ again contradiction to \oplus_1 . $\square_{1.1}$

³in fact, necessarily $\alpha_k \in \{\ell g(\eta_\rho \upharpoonright n) : n < \omega\}$

2. THE OTHER CASES

{g.3}

Theorem 2.1. 1) If $\kappa = \mathfrak{s} = \mathfrak{a}$ and $\text{cf}([\mathfrak{s}]^{\aleph_0}, \subseteq) = \mathfrak{s}$ then there is a saturated MAD family.

2) If $\kappa = \mathfrak{s} = \mathfrak{a}$ and $\mathbf{U}(\kappa) = \kappa$, see Definition 2.2 below and $J_* \subseteq [\omega]^{\aleph_0}$ is hereditary and dense then there is a saturated MAD family $\subseteq J_*$.

{g.5}

Definition 2.2. 1) For cardinals $\partial \leq \sigma \leq \theta \leq \lambda$ let $\mathbf{U}_{\theta, \sigma, \partial}(\lambda) = \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq \sigma} \text{ such that for every } X \in [\lambda]^\theta \text{ for some } u \in \mathcal{P} \text{ we have } |X \cap u| \geq \partial\}$. If $\partial = \sigma$ we may omit ∂ ; if $\sigma = \partial = \aleph_0$ we may omit them both, and if $\sigma = \partial = \aleph_0 \wedge \theta = \lambda$ we may omit θ, σ, ∂ .

1A) In the case of our Theorem, it means: $\mathbf{U}(\kappa) = \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\kappa]^{\leq \aleph_0} \text{ and } (\forall X \in [\kappa]^\kappa)(\exists u \in \mathcal{P})(|X \cap u| \geq \aleph_0)\}$.

2) If in addition J is an ideal on θ then $\mathbf{U}_{\theta, \sigma, J}(\lambda) = \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq \sigma} \text{ such that for every function } f : \theta \rightarrow \lambda \text{ for some } u \in \mathcal{P} \text{ the set } \{i < \theta : f(i) \in u\} \text{ does not belong to } J\}$.

3) Let $\text{Pr}(\kappa, \theta, \sigma, \partial)$ mean: $\kappa \geq \theta \geq \sigma \geq \partial$ and we can find $(\bar{E}, \bar{\mathcal{P}})$ such that (if $\partial = \aleph_0$ we may omit ∂ , if $\sigma = \partial = \aleph_0$ we may omit them, if $\sigma = \partial = \aleph_0 \wedge \theta = \kappa$ we may omit θ, σ, ∂):

- (a) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha \in E \rangle$
- (b) E is a club of κ and $\gamma \in E \Rightarrow |\gamma|$ divide γ
- (c) if $u \in \mathcal{P}_\alpha$ then $u \in [\alpha]^{\leq \sigma}$ has no last member
- (d) \bullet_1 \mathcal{P}_α is \subseteq -increasing
- \bullet_2 $|\mathcal{P}_\alpha| < \kappa$
- (e) if $w \subseteq \kappa$ and $\text{otp}(w) = \theta$ and $\text{sup}(w) \in \text{acc}(E)$ then for some u, j we have:

- \bullet_1 $|u \cap w| \geq \partial$
- \bullet_2 $j \in \text{acc}(E)$
- \bullet_3 $u \in \mathcal{P}_j$
- \bullet_4 $|w \cap j| < \theta$
- (f) if $i \in \{0\} \cup E$ and $j \in \min(E \setminus (i+1))$, $w \subseteq [i, j]$, $\text{otp}(w) = \theta$ then for some set u
 - \bullet_1 $u \in \mathcal{P}_j$ and $u \subseteq (i, j)$
 - \bullet_2 $|u \cap w| \geq \partial$.

{g.7}

Explanation 2.3. The proof of 2.1 is really similar to that of 1.1. The difference is that in the proof of \odot_2 of subcase 2B of stage F, if $\ell g(\nu_B^*) = \kappa$ it does not follow that we have $|\mathcal{A}^*| < \mathfrak{a}$, so we have to do something else when $|\mathcal{A}^*| = \mathfrak{a} = \mathfrak{s}$. By the assumption $\mathbf{U}(\kappa) = \kappa$ there is a sequence $\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ of members of $[\kappa]^{\aleph_0}$ such that $u_\alpha \subseteq \alpha$ and for every $X \in [\kappa]^\kappa$ for some α , $u_\alpha \cap X$ is infinite. Now if e.g. $\ell g(\nu) = \alpha \geq \omega$ we can use u_α and apply 2.5 below to appropriate \bar{B}_ν and get \mathcal{P}_ν and add it to the family $\{C_\alpha^* : \alpha < \kappa\}$ witnessing $\mathfrak{s} = \kappa$ the family \mathcal{P}_ν as in 2.5. So now we really need to use C_ν^s rather than C_α^* .

{g.10}

Observation 2.4. If $\text{Pr}(\kappa, \theta, \sigma, \partial)$ is satisfied by $(E, \bar{\mathcal{P}})$ then we can find $(E', \bar{\mathcal{P}}')$ as in 2.2(3) but

- (d)' \bullet_2 if $j = \max(E')$ then $|\mathcal{P}'_j| \leq j$

(e) as above but $\text{sup}(w) \in E$.

Proof. Use $E' = \text{acc}(E)$ or any club $E' \subseteq \text{acc}(E)$ of κ such that $\delta \in \text{nacc}(E') \Rightarrow \text{cf}(\delta) \neq \text{cf}(\theta)$ and let \mathcal{P}'_γ be \mathcal{P}_γ if $\gamma \in \text{acc}(E')$ and be $\cup\{\mathcal{P}_\beta : \beta \in E' \cap \gamma\}$ if $\gamma \in \text{nacc}(E')$. $\square_{2.4}$

{g.19}

Observation 2.5. Assume $\bar{B}^* = \langle B_n^* : n < \omega \rangle$ satisfies $B_n^* \in [\omega]^{\aleph_0}$, $B_{n+1}^* \subseteq B_n^*$ and $|B_n^* \setminus B_{n+1}^*| = \aleph_0$ for infinitely many n 's. Then we can find \mathcal{P} such that

- (*) (a) $\mathcal{P} \subseteq [\omega]^{\aleph_0}$ is of cardinality \mathfrak{b}
- (b) if $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is an AD family, $B \subseteq \omega$ and $(\exists^\infty n)(B \cap (B_n^* \setminus B_{n+1}^*) \notin \text{id}_{\mathcal{A}})$ then for some countable (infinite) $\mathcal{P}' \subseteq \mathcal{P}$ for 2^{\aleph_0} function $\eta \in {}^{\mathcal{P}'} 2$ we have: for some $\text{id}_{\mathcal{A}}$ -positive set $A \subseteq^* B$ we have: $A \subseteq^* C^{[n(C)]}$ for every $C \in \mathcal{P}'$ and $A \subseteq^* B_n$ for every n .

Proof. Proof of 2.5 Let $\mathcal{B} = \{\bar{B} : \bar{B} = \langle B_n : n < \omega \rangle \text{ where } B_n \subseteq \omega \text{ is infinite, } B_n \supseteq B_{n+1} \text{ and } B_n \setminus B_{n+1} \text{ is infinite for infinitely many } n < \omega\}$, i.e. the set of \bar{B} satisfying the demands on \bar{B}^* .

For $\bar{B} \in \mathcal{B}$ and \mathcal{A} as in (*) (b) let $\text{pos}(\bar{B}, \mathcal{A}) = \{B \subseteq \omega : B \cap (B_n \setminus B_{n+1}) \notin \text{id}_{\mathcal{A}} \text{ for infinitely many } n\}$, i.e. the set of B such that (\bar{B}, B) satisfies the demands on (\bar{B}^*, B) for the family \mathcal{A} . So the claim says that for every $\bar{B} \in \mathcal{B}$ there is $\mathcal{P} \subseteq [\omega]^{\aleph_0}$ of cardinality \mathfrak{b} such that if $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is an AD family and $B \in \text{pos}(\bar{B}, \mathcal{A})$ then there is a countable infinite $\mathcal{P}' \subseteq \mathcal{P}$ as there.

Consider the statement:

- \boxplus if $\bar{B} \in \mathcal{B}$ then we can find \mathbf{B} such that
 - (a) $\mathbf{B} = \langle \bar{B}_\delta : \delta \in S_{\aleph_0}^{\mathfrak{b}} \rangle$ recalling $S_{\aleph_0}^{\mathfrak{b}} = \{\delta < \mathfrak{b} : \text{cf}(\delta) = \aleph_0\}$
 - (b) $\delta \in S_{\aleph_0}^{\mathfrak{b}} \Rightarrow \bar{B}_\delta \in \mathcal{B}$
 - (c) if \mathcal{A} is an AD family and $B \in \text{pos}(\bar{B}, \mathcal{A})$, then for some club E of \mathfrak{b} , for every $\delta \in E \cap S_{\aleph_0}^{\mathfrak{b}}$ we have $(\exists^\infty n)[B \cap (B_{\delta,n} \setminus B_{\delta,n+1}) \in \text{id}_{\mathcal{A}}^+]$.

Why is this statement enough? By it we can find a subset \mathcal{B}' of \mathcal{B} of cardinality \mathfrak{b} such that $\bar{B}^* \in \mathcal{B}'$ and for every $\bar{B} \in \mathcal{B}'$ for some $\mathbf{B} = \langle \bar{B}_\delta : \delta \in S_{\aleph_0}^{\mathfrak{b}} \rangle$ as in \boxplus we have $\delta \in S_{\aleph_0}^{\mathfrak{b}} \Rightarrow \bar{B}_\delta \in \mathcal{B}'$. Clearly $\mathcal{P} = \{B_n \setminus B_{n+1} : \bar{B} \in \mathcal{B}'\}$ is as required.

Why does this statement hold?

Let $\bar{f} = \langle f_\alpha : \alpha < \mathfrak{b} \rangle$ be a sequence of members of ${}^\omega \omega$ witnessing \mathfrak{b} and without loss of generality $f_\alpha \in {}^\omega \omega$ is increasing and $\alpha < \beta < \mathfrak{b} \Rightarrow f_\alpha <_{J_\omega^{\text{bd}}} f_\beta$.

For $\alpha < \mathfrak{b}$ let $C_\alpha := \cup\{(B_n \setminus B_{n+1}) \cap [0, f_\alpha(n)) : n < \omega\}$ so clearly

$$(*)_1 \quad \alpha < \beta \Rightarrow C_\alpha \subseteq^* C_\beta.$$

We choose $\alpha_\varepsilon = \alpha(\varepsilon) < \mathfrak{b}$ by induction on $\varepsilon < \mathfrak{b}$, increasing with ε as follows: for $\varepsilon = 0$ let $\alpha_\varepsilon = \min\{\alpha < \mathfrak{b} : C_\alpha \text{ is infinite}\}$, for $\varepsilon = \zeta + 1$ let $\alpha_\varepsilon = \min\{\alpha < \mathfrak{b} : \alpha > \alpha_\zeta \text{ and } C_\alpha \setminus C_{\alpha(\zeta)} \text{ is infinite}\}$ and for ε limit let $\alpha_\varepsilon = \cup\{\alpha_\zeta : \zeta < \varepsilon\}$. By the choice of \bar{f} every α_ε is well defined, see the proof of \oplus_α below.

So $\langle \alpha_\varepsilon : \varepsilon < \mathfrak{b} \rangle$ is increasing continuous with limit \mathfrak{b} . For each $\delta \in S_{\aleph_0}^{\mathfrak{b}}$ let $\langle \varepsilon(\delta, n) : n < \omega \rangle$ be increasing with limit δ and, lastly, let $\bar{B}_\delta = \langle \omega \setminus \bigcup_{m \leq n} C_{\alpha(\varepsilon(\delta, m))} : n < \omega \rangle$ so $B_{\delta, n} = \omega \setminus \bigcup_{m \leq n} C_{\alpha(\varepsilon(\delta, m))}$ hence $B_{\delta, n+1} \subseteq B_{\delta, n}$ and $B_{\delta, n} \setminus B_{\delta, n+1}$ is infinite by the choice of $\alpha_{\varepsilon(\delta, n)+1}$. Clearly $\bar{B}_\delta \in \mathcal{B}$ (really follows from the proof below).

Why is $\langle \bar{B}_\delta : \delta \in S_{\aleph_0}^{\mathfrak{b}}$ as required in \boxplus ? Clauses (a) + (b) are obvious and to check clause (c) let \mathcal{A} be an AD family and $B \subseteq \omega$ be such that

$$(*)_2 \quad u = u_B := \{n < \omega : B \cap (B_n \setminus B_{n+1}) \notin \text{id}_{\mathcal{A}}\} \text{ is infinite.}$$

It is enough to prove that for every $\alpha < \mathfrak{b}$

$$\oplus_\alpha \text{ there is } \beta \in (\alpha, \mathfrak{b}) \text{ such that } B \cap C_\beta \setminus C_\alpha \in \text{id}_{\mathcal{A}}^+.$$

[Why is it enough? As then for some club E of \mathfrak{b} , for every $\delta \in E \cap S_{\aleph_0}^{\mathfrak{b}}$ we have $(\forall \varepsilon < \delta)(\alpha_\varepsilon < \delta) \wedge (\forall \alpha < \delta)(\exists \beta)(\alpha < \beta < \delta \wedge C_\beta \setminus C_\alpha \in \text{id}_{\mathcal{A}}^+)$ hence $(\exists^\infty n)((C_{\alpha(\varepsilon(\delta, n+1))} \setminus C_{\alpha(\varepsilon(\delta, n))}) \in \text{id}_{\mathcal{A}}^+)$ which means $(\exists^\infty n)(B_{\delta, n} \setminus B_{\delta, n+1}) \in \text{id}_{\mathcal{A}}^+$ as required.]

So let us prove \oplus_α .

If \oplus_α fails, for every $\beta \in (\alpha, \mathfrak{b})$ there are $n = n(\beta)$ and $A_{\beta, 0}, \dots, A_{n(\beta)-1} \in \mathcal{A}$ such that $C_\beta \setminus C_\alpha \subseteq^* A_{\beta, 0} \cup \dots \cup A_{\beta, n(\beta)-1}$. Without loss of generality $n(\beta)$ is minimal hence by $(*)_1$ the sequence $\langle n(\beta) : \beta \in [\alpha, \mathfrak{b}) \rangle$ is non-decreasing, but $\mathfrak{b} = \text{cf}(\mathfrak{b}) > \aleph_0$, hence, for some $\alpha_* \in [\alpha, \mathfrak{b})$, the sequence $\langle n(\beta) : \beta \in [\alpha_*, \mathfrak{b}) \rangle$ is constant and let $n(\alpha_*) = n_*$.

As \mathcal{A} is AD and $C_{\alpha_*} \setminus C_\alpha \subseteq^* A_{\alpha_*, 0} \cup \dots \cup A_{\alpha_*, n_*-1}$ and $\beta \in (\alpha_*, \mathfrak{b}) \Rightarrow C_{\alpha_*} \setminus C_\alpha \subseteq C_\beta \setminus C_\alpha \subseteq A_{\beta, 0} \cup \dots \cup A_{\beta, n_*-1}$ using “ \mathcal{A} is almost disjoint” it follows that $\{A_{\alpha_*, \ell} : \ell < n_*\} \subseteq \{A_{\beta, \ell} : \ell < n_*\}$ hence they are equal.

So

$$\odot \quad \beta \in (\alpha, \mathfrak{b}) \Rightarrow C_\beta \setminus C_\alpha \subseteq^* A_{\alpha_*, 0} \cup \dots \cup A_{\alpha_*, n_*-1}.$$

For each $n \in u = u_B$ as $B \cap B_n \setminus B_{n+1} \in \text{id}_{\mathcal{A}}^+$, and $(C_\beta \setminus C_\alpha), A_{\alpha_*, 0}, \dots, A_{\alpha_*, n_*-1}$ are from $\text{id}_{\mathcal{A}}$, clearly there is $k_n \in (B \cap B_n \setminus B_{n+1}) \setminus (C_\beta \setminus C_\alpha) \setminus A_{\alpha_*, 0} \setminus \dots \setminus A_{\alpha_*, n_*-1}$ and for $n \in \omega \setminus u$ let $k_n = k_{\min(u \setminus n)}$. By the choice of f there is $\beta \in (\alpha_*, \mathfrak{b})$ such that $u_1 := \{n < \omega : k_n < f_\beta(n)\}$ is infinite. As f_β is increasing, clearly $n \in u_1 \Rightarrow k_{\min(u \setminus n)} = k_n < f_\beta(n) \leq f_\beta(\min(u \setminus n))$ so $\min(u \setminus n) \in u_1$, hence $u_2 := u_1 \cap u \subseteq \omega$ is infinite. So $\{k_n : n \in u_2\} \in [\omega]^{\aleph_0}$ is an infinite and is a subset of $C_\beta \setminus C_{\alpha_*} \setminus A_{\alpha_*, 0}, \dots, A_{\alpha_*, n_*-1}$, contradiction. $\square_{2.5}$

Proof. Proof of 2.1 We prove part (2), and part (1) follows from it. We immitate the proof of 1.1.

Stage A:

Let $\kappa = \mathfrak{s}$. Let $\mathcal{P} \subseteq [\kappa]^{\aleph_0}$ witness $\mathbf{U}(\kappa) = \kappa$, for transparency we assume $\omega \in \mathcal{P}$ and $u \in \mathcal{P} \Rightarrow \text{otp}(u) = \omega$, this holds without loss of generality as $\mathfrak{b} \leq \mathfrak{s} \leq \kappa$.

Let $\langle u_\alpha : \alpha < \kappa \rangle$ list \mathcal{P} possibly with repetitions, without loss of generality $n \leq \omega \Rightarrow u_n = \omega$ and $\alpha > \omega \Rightarrow u_\alpha \subseteq \alpha$. For $\alpha < \kappa$ let $\langle \gamma(\alpha, \ell) : \ell < \omega \rangle$ list u_α in increasing order.

Let $\langle \mathcal{U}_\alpha : \alpha < \kappa \rangle$ be a partition of κ , to sets each of cardinality κ such that $\min(\mathcal{U}_{1+\alpha}) \geq \sup(u_\alpha) + 1$ and $\omega \subseteq \mathcal{U}_0$. Let $\langle C_\alpha^* : \alpha \in \mathcal{U}_0 \rangle$ list a subset of $\mathcal{P}(\omega)$ witnessing $\mathfrak{s} = \kappa$.

If \bar{B} is as in the assumption of 2.5 and $\alpha \in (0, \kappa)$ let $\mathcal{P}_{\bar{B}}$ be as in the conclusion of 2.5 and $\langle C_{\bar{B}, \alpha, i}^* : i \in \mathcal{U}_\alpha \rangle$ list $\mathcal{P}_{\bar{B}}$.

Stage B: As in the proof of 1.1 but we use $C_\rho^s(\rho \in \mathcal{T}_s)$ which may really depend on s and where $C_\rho^t(\rho \in \mathcal{S}'_t), \bar{B}_{\nu, \beta}^t$ are defined in clauses $\boxplus_1(e)', (g), (h), (i), (j)$ below

(so the $\boxplus(e), (g), (h)$ from 1.1 are replaced) and depend just on \mathcal{T}_t, \bar{A}_t and \bar{I}_t , too⁴ where

- \boxplus_1 (a)-(d) and (f) as in 1.1 of course and
- (e) \bullet_1 as before, i.e. $\bar{A} = \langle A_\eta^t : \eta \in \text{suc}(\mathcal{T}_t) \rangle$,
 - \bullet_2 $\bar{C} = \langle C_\eta^t : \eta \in \Lambda_t \rangle$ where $\Lambda_t = \{\eta : \eta \in {}^i 2 \text{ and } i \in \mathcal{U}_0$
or $\alpha > 0, i \in \mathcal{U}_\alpha \text{ and } \eta \upharpoonright (\text{sup}(u_B)) \in I_t \text{ or (for 2.6) } \eta \in \mathcal{T}_t\}$
 - (g) as in 1.1 but replacing $C_{\eta \upharpoonright i}^*$ by $C_{\eta \upharpoonright i}^t$
 - (h) if $i \in \mathcal{U}_0$ and $\nu \in \mathcal{T}_t \cap {}^i 2$ then $C_\nu^s = C_i^*$
 - (i) if $\beta \in (0, \kappa)$ and $\nu \in \text{sup}(u_\beta) 2$ and $\langle C_{\nu \upharpoonright i}^t : i \in u_\beta \rangle$ are well defined then we let $\bar{B}_{\nu, \beta}^t = \langle B_{\nu, \beta, n}^t : n < \omega \rangle$ be defined by $B_{\nu, \beta, n}^t = \cap \{C_{\nu \upharpoonright \gamma(\beta, k)}^t : k < n\}$
 - (j) if $\beta \in (0, \kappa), i \in \mathcal{U}_\beta$ hence $i \geq \beta \geq \text{sup}(u_\beta)$ and $\rho \in {}^i 2$ and $\bar{B}_{\rho \upharpoonright \text{sup}(u_\beta), \beta}^t$ is well defined then $C_\rho^t = C_{\bar{B}_{\rho \upharpoonright \text{sup}(u_\beta), \beta}^t}^*$.

Stage C:

As in 1.1 we just add:

- \boxplus_4 (g) if $s \leq_{\text{AP}} t$ and $\bar{B}_{\nu, \beta}^s$ is well defined then $\bar{B}_{\nu, \beta}^t$ is well defined and equal to it
- (h) if $s \leq_{\text{AP}} t$ and C_ν^s is well defined then C_ν^t is well defined and equal to it, so $\Lambda_s \subseteq \Lambda_t$
- (i) C_ν^s is well defined when $\nu \in \text{cl}(\mathcal{T}_s)$.

In the proof of $\boxplus_4(e)$ use the choice of $\langle C_\nu^s : \nu \in {}^i 2, i \in \mathcal{U}_0 \rangle$, i.e. of $\langle C_\alpha^* : \alpha \in \mathcal{U}_0 \rangle$ in Stage A.

Stages D,E: As in 1.1.

Stage F: The only difference is in the proof of \odot_2 in subcase(2B). Recall

Case 2: $\text{SP}_B = \emptyset$ but not Case 1, i.e. $S_B \subseteq \mathcal{T}_s$, recall $B \subseteq \omega, B \notin \text{id}_s$.

Subcase 2B: $\nu_B^* \notin S_B$ where $\nu_B^* = \cup \{\eta : \eta \in \text{SP}_B\}$

\odot_2 there is an infinite $B_1 \subseteq B$ which is almost disjoint to any $A \in \mathcal{A}^*$.

Why \odot_2 holds? If $|\mathcal{A}^*| < \kappa$ then \mathcal{A}^* has cardinality $< \kappa = \mathfrak{s}$ hence by the theorem's assumption $|\mathcal{A}^*| < \mathfrak{s} = \mathfrak{a}$, so \odot_2 follows. So we can assume $|\mathcal{A}^*| = \kappa$ but $|\mathcal{A}^*| \leq \aleph_0 + |\ell g(\nu_B^*)|$ hence necessarily $\ell g(\nu_B^*) = \kappa$ follows as in the proof of 1.1 and let

$$W := \{\alpha < \kappa : \text{for some } \ell < n(\alpha) \text{ we have } A_{\ell, \alpha} \cap B \in \mathcal{A}^* \\ \text{(equivalently } A_{\ell, \alpha}^* \cap B \text{ is infinite) but} \\ A_{\ell, \alpha}^* \notin \{A_{\ell_1, \alpha_1} : \alpha_1 < \alpha \text{ and } \ell_1 \leq n(\alpha_1)\}\}.$$

For $\alpha \in W$ choose $\ell(\alpha) < n(\alpha)$ such that $B \cap A_{\ell(\alpha), \alpha}$ is infinite and $A_{\ell, \alpha} \notin \{A_{\ell_1, \alpha_1}^* : \alpha_1 < \alpha, \ell_1 \leq n(\alpha_1)\}$. As $n(\alpha) < \omega$ for $\alpha < \kappa$, clearly $|W| = \kappa$ because $|\mathcal{A}^*| = \kappa$, hence by the choice of \mathcal{P} there is $u_* \in \mathcal{P}$ such that $|W \cap u_*|$ is infinite;

⁴also here we require $\eta \in \text{suc}(\mathcal{T}_s) \Rightarrow A_\eta \neq \emptyset$

let $\alpha(*) \in [\omega, \kappa)$ be such that $u_{\alpha(*)} = u_*$ and let $\nu = \nu_B^* \upharpoonright \text{sup}(u_*)$; recall that $\text{otp}(u_*) = \omega$.

Recall also that $\langle \gamma_{\alpha(*)}, n : n < \omega \rangle$ list u_* in increasing order and so $v := \{n < \omega : \gamma_{\alpha(*)}, n \in W\}$ is infinite and clearly $n \in v \Rightarrow B_{\nu, \alpha(*)}, n \setminus B_{\nu, \alpha(*)}, n+1$ is infinite. So by the choice of $\mathcal{P}_{\bar{B}, \alpha(*)}^s$, i.e. 2.5 and clauses (i),(j) of \boxplus_1 for some $\beta \in \mathcal{U}_{\alpha(*)}$ so $\beta \geq \alpha(*) \geq \ell g(\nu)$ we have $B \setminus (C_{\nu_B^* \upharpoonright \beta}^s)^{[\ell]} \notin \text{id}_s$ for $\ell = 0, 1$ hence $B_1 := B \setminus (C_{\nu_B^* \upharpoonright \beta}^s)^{[\nu_B^*(\beta)]} \notin \text{id}_s$ recalling that for $\beta \in \mathcal{U}_\alpha, \alpha \neq 0$ and $\rho \in {}^\beta 2$ the set C_ρ^s depends just on $\ell g(\rho)$ and $\rho \upharpoonright \text{sup}(u_\alpha)$ (and our s).

Now consider B_1 instead of B , clearly S_{B_1} is a subset of S_B and $\nu_B^* \upharpoonright (\beta + 1)$ is not in it, but $B_1 \in \text{ob}(B)$ hence $S_{B_1} \subseteq S_B$ hence $S_{B_1} \subseteq \{\nu_B^* \upharpoonright \gamma : \gamma \leq \beta\}$ and B_1 fall under subcase (2A) as $\beta < \kappa = \ell g(\nu_B^*)$. $\square_{2.1}$

{g.21}

Theorem 2.6. *There is a saturated MAD family $\mathcal{A} \subseteq J_*$ when $\mathfrak{a} < \kappa = \mathfrak{s}, J_* \subseteq \text{ob}(\omega)$ is dense and hereditary and $\text{Pr}(\kappa, \mathfrak{a})$, see 2.2(3).*

Proof. Proof of 2.6 We immitate the proof of 1.1, 2.1. Note that $\mathfrak{b} \leq \mathfrak{a} < \mathfrak{s}$.

Stage A:

Similarly to stage A of the proof of 2.1; let $(E, \bar{\mathcal{P}}^*)$ be as in Definition 2.2(3) and Observation 2.4, as $\mathfrak{b} < \kappa$, without loss of generality $u \in \mathcal{P}_\alpha \Rightarrow \text{otp}(u) = \omega$ for $\alpha \in [\omega, \kappa)$. As we can replace E by any appropriate club E' of κ contained in $\text{acc}(E)$ see 2.4 there, without loss of generality $\text{otp}(E) = \text{cf}(\kappa)$, $\min(E) \geq \omega$ and $\gamma \in E \Rightarrow \gamma + 1 + \mathfrak{b} < \min(E)$. Let $\langle \gamma_i^* : i < \text{cf}(\kappa) \rangle$ list E in increasing order and let $\mathcal{P}_i = \mathcal{P}_{\gamma_i^*}^*$.

Let $\langle u_\gamma : \gamma < \kappa \rangle$ be such that $\langle u_\gamma : \gamma_i^* \leq \gamma < \gamma_{i+1}^* \rangle$ list \mathcal{P}_{i+1} (which includes \mathcal{P}_i) and $u_j = \omega$ for $j < \gamma_0^*$.

Let $\langle \mathcal{U}_\alpha : \alpha < \kappa \rangle$ be a partition of $\{2i : i < \kappa\}$ such that $\min(\mathcal{U}_{1+\alpha}) \geq \alpha + \omega$, $|\mathcal{U}_{1+\alpha}| = \mathfrak{b}, |\mathcal{U}_0| = \kappa, 1 \leq \alpha < \gamma_i^* \Rightarrow \mathcal{U}_\alpha \subseteq \gamma_i^*$.

Let $\langle C_i^* : i \in \mathcal{U}_0 \rangle$ list a family of subsets of ω witnessing $\mathfrak{s} = \kappa$.

Let $\mathcal{P}_{\bar{B}}, \langle C_{\bar{B}, \alpha, i}^* : i \in \mathcal{U}_\alpha \rangle$ be as in 2.1, Stage A.

Stage B:

As in 2.1, i.e. the case $\mathfrak{s} = \mathfrak{a}$, but we change $\boxplus_1(f)$

- $$\boxplus_1(f) \bullet \quad A_\eta \in I_\eta \cap J_* \text{ or } A_\eta = \emptyset \text{ and}$$
- $\mathcal{S}_t := \{\eta \in \bar{\mathcal{T}}_t : A_\eta \neq \emptyset\} \subseteq \{\eta \in \mathcal{T}_t : \ell g(\eta) = \gamma_i^* + 1 \text{ for some } i < \kappa\}$.

Stage C:

As in the proof of 2.1.

Stage D:

Here there is a minor change: we replace \boxplus_7 in 1.1, 2.1 by \boxplus_7, \boxplus_8 below

- $$\boxplus_7 \text{ if } \alpha < 2^{\aleph_0}, s \in \text{AP}_\alpha \text{ and } B \in \text{id}_s^+ \text{ then there are a limit ordinal } \xi \in \kappa \setminus E$$
- and $t \in \text{AP}_{\alpha+1}$ such that $s \leq_{\text{AP}} t$ and $|S_B^t \cap \xi 2| = 2^{\aleph_0}$.

This is proved in Stage F

\boxplus_8 if $\xi \in \kappa \setminus E$ is a limit ordinal, $\alpha < 2^{\aleph_0}$, $t \in \text{AP}_\alpha$, $B \in \text{ob}(\omega)$ and $|S_B^t \cap \xi 2| = 2^{\aleph_0}$ and $\zeta = \min(E \setminus \xi)$ then for every t_1 and $\alpha + \zeta \leq \beta < 2^{\aleph_0}$ such that $t \leq_{\text{AP}} t_1 \in \text{AP}_\beta$ there is $t_2, t_1 \leq_{\text{AP}} t_2 \in \text{AP}_{\beta+1}$ satisfying $(\exists \eta \in \text{suc}(\mathcal{T}_{t_2}))[\eta \notin \mathcal{T}_{t_1} \wedge A_\eta^{t_2} \in \text{ob}(\omega) \wedge A_\eta^{t_2} \subseteq B]$.

The proof of \boxplus_8 is like the proof of Case 1 in Stage F in the proof of 1.1 but we elaborate; we are given β, ξ, ζ and t_1 such that $t \leq_{\text{AP}} t_1 \in \text{AP}_\beta$; now we choose $\rho \in S_B^t \cap \xi 2 \setminus \mathcal{T}_{t_1}$ exists as $|S_B^t \cap \xi 2| = 2^{\aleph_0} > |\mathcal{T}_{t_1}|$. Choose B_1 such that $B_1 \subseteq B$, $B_1 \in I_\rho^t$ and without loss of generality $B_1 \in J_*$.

Note that for every $\varepsilon \in [\xi, \zeta + 1)$ either $C_\rho^{t_1}$ is well defined for every $\rho \in \varepsilon 2$ such that $\rho \trianglelefteq \rho$ and its value is the same for all such ρ (when ε is odd or $C_\rho^{t_1}$ for $\rho \trianglelefteq \rho \in \varepsilon 2$ is not well defined (when ε is even). So $\mathcal{B} = \{C_\rho^{t_1} : \rho \triangleleft \rho \in \varepsilon 2 \text{ and } C_\rho^{t_1} \text{ is well defined}\}$ is a family of $\leq |\zeta| < \kappa = \mathfrak{s}$ subsets of B_1 hence there is an infinite $B_2 \subseteq B_1$ such that $\rho \triangleleft \rho \in \varepsilon 2 \wedge (C_\rho^{t_1} \text{ well defined}) \Rightarrow B_2 \subseteq^* C_\rho^{t_1} \vee B_2 \subseteq^* \omega \setminus C_\rho^{t_1}$ and (as $B_1 \in J_*$ and J_* is hereditary) $B_2 \in J_*$.

We choose η such that $\rho \triangleleft \eta \in \varepsilon 2$ and $[\ell g(\rho) \leq \gamma < \zeta + 1 \wedge (C_{\eta|\gamma}^t \text{ is well defined}) \wedge B_2 \subseteq^* (C_{\eta|\gamma}^t)^{[\ell]} \wedge \rho \in \{0, 1\}] \Rightarrow \eta(\gamma) = \ell]$. Let us define $t_2 \in \text{AP}_{\beta+\zeta+2} := \text{AP}_{\beta+1}$ (as $\alpha + \zeta + 1 \leq \beta$) as follows:

- (a) $\mathcal{T}_{t_2} := \mathcal{T}_{t_1} \cup \{\rho : \rho \trianglelefteq \eta\}$
- (b) $A_\rho^{t_2}$ is $A_\rho^{t_1}$ if well defined, is B_2 if $\rho = \eta$ and is \emptyset if $\eta \in \text{suc}(\mathcal{T}_{t_2})$ but $A_\rho^{t_2}$ is not already defined
- (c) $C_\rho^{t_2}$ is $C_\rho^{t_1}$ if $\rho \in \mathcal{T}_{t_1}$ and we choose $C_{\eta|\varepsilon}^{t_2}$ by induction on $\varepsilon \in [\xi, \zeta + 2]$ as follows: if it is determined by \boxplus_1 we have no choice otherwise let it be $\omega^{[\eta(\varepsilon)]}$.

The other objects of t_2 are determined by those we have chosen. So \boxplus_8 holds indeed.

\boxplus_9 if $s \in \text{AP}_\alpha$ and $\rho \in \text{cl}(\mathcal{T}_s)$ then for some $t, s \leq_{\text{AP}} t \in \text{AP}_{\alpha+3}$ and $\mathcal{T}_s \subseteq \mathcal{T}_t \subseteq \mathcal{T}_s \cup \{\rho, \rho \hat{\langle} 0 \rangle, \rho \hat{\langle} 1 \rangle\}$ and $I_\rho^s \neq \emptyset \Rightarrow \rho \in \mathcal{T}_t$ and $I_\rho^s \neq \emptyset \wedge \rho < 2 \wedge I_{\rho \hat{\langle} \alpha \rangle}^t \neq \emptyset \Rightarrow \rho \hat{\langle} \ell \rangle \in \mathcal{T}_t$.

[Why? Easier than \boxplus_8 .]

Stage E:

Similar to 1.1 with the changes necessitated by the change in Stage D.

Stage F:

We prove \boxplus_7 , the proof splits to cases.

Case 1: Some $\nu \in S_B$ is such that $\nu \in \text{cl}(\mathcal{T}_s) \setminus \mathcal{T}_s$.

Let $C_{\nu, n} \in \text{ob}(\omega)$ for $n < \omega$ be such that $\cap \{C_{\nu, n}^{[\ell]} : \ell < \ell g(\rho)\} \cap B_1$ is infinite for every $\rho \in \omega^{>2}$.

We choose $\mathcal{T}_t = \mathcal{T}_s \cup \{\nu \hat{\rho} : \rho \in \omega^{>2}\}$. For $\rho \in \omega^{>2}$, we choose $C_{\nu \hat{\rho}}^t$ by induction on $\ell g(\rho)$: if $\ell g(\nu \hat{\rho}) = \ell g(\nu) + n$ is even and $n \in \{2m, 2m + 1\}$ then $C_{\nu \hat{\rho}}^t = C_{\nu, m}$, otherwise we act as in the proof of \boxplus_8 . Lastly, let $A_{\nu \hat{\rho}}^t = \emptyset$.

Easily

- (*) if $s \leq_{\text{AP}} s_1, |\mathcal{T}_{s_1}| < 2^{\aleph_0}$ then $|S_B^{s_1} \cap \ell g(\nu) + \omega 2| = 2^{\aleph_0}$.

So we are done with Case 1.

Case 2: $SP_B^s = \emptyset$ but not Case 1 and let $\nu_B^* = \cup\{\eta : \eta \in S_B\}$.

Subcase 2A: $\nu_B^* \notin S_B$

As in the proof 1.1 except \odot_2 this is as in the proof of 2.1 but in the end replace “Subcase 1B” by “Case 1”, but we elaborate.

So $\nu_{B,\alpha}^* = (\nu_B^* \upharpoonright \alpha) \wedge \langle 1 - \nu_B^*(\alpha) \rangle$ for $\alpha < \delta := \ell g(\nu_B^*)$ which is a limit ordinal and $\langle \langle A_{\alpha,n}^* : n \leq n(\alpha) \rangle : \alpha < \delta \rangle$ and $\mathcal{A} := \{B \cap A : A = A_{\alpha,k}^* \text{ for some } \alpha < \delta, k \leq n(\alpha) \text{ and } B \cap A \text{ is infinite}\}$ are as in Stage (2B) of the proof of 1.1. We have to prove:

\odot_2 there is an infinite $B_1 \subseteq B$ which is almost disjoint to any $A \in \mathcal{A}^*$.

Why \odot_2 holds? If $|\mathcal{A}^*| < \mathfrak{a}$ then by the definition of \mathfrak{a} clearly \odot_2 follows. So we can assume $|\mathcal{A}^*| \geq \mathfrak{a}$ and let

$$W := \{\alpha < \kappa : \text{for some } \ell < n(\alpha) \text{ we have } A_{\ell,\alpha} \cap B \in \mathcal{A}^* \\ \text{(equivalently } A_{\ell,\alpha} \cap B \text{ is infinite) but} \\ A_{\ell,\alpha} \notin \{A_{\ell_1,\alpha_1} : \alpha_1 < \alpha \text{ and } \ell_1 \leq n(\alpha_1)\}\}.$$

and let

$$W' = \{\alpha \in W : |\alpha \cap W| < \mathfrak{a}\}.$$

Subcase 2B(α): $\sup(W') \in \text{acc}(E')$

For $\alpha \in W$ choose $\ell(\alpha) < n(\alpha)$ such that $B \cap A_{\alpha,\ell(\alpha)}$ is infinite and $A_{\ell,\alpha} \notin \{A_{\ell_1,\alpha_1} : \alpha_1 < \alpha, \ell_1 \leq n(\alpha_1)\}$. As $n(\alpha) < \omega$ for $\alpha < \kappa$, clearly $|W| = |\mathcal{A}^*| \geq \mathfrak{a}$ hence $\text{otp}(W') = \mathfrak{a}$.

So by Definition 2.2(3), i.e. the choice of $(E, \bar{\mathcal{P}})$ there is a pair (u_*, γ_j^*) as in clause (e) there. So $u \in \mathcal{P}_j$ hence $u = u_{\alpha(*)}$ for some $\alpha(*) \in [\gamma_j, \gamma_{j+1})$ and $W \cap \sup(E \cap \gamma_j^*) = W' \cap \sup(E \cap \gamma_j^*)$ has cardinality $< \mathfrak{a}$ and let $\nu = \nu_B^* \upharpoonright \sup(u_*)$; recall $\text{otp}(u_*) = \omega$.

Recall also that $\langle \gamma_{\alpha(*)}, n : n < \omega \rangle$ list u_* in increasing order and so $v := \{n < \omega : \gamma_{\alpha(*)}, n \in W\}$ is infinite and clearly $n \in v \Rightarrow B_{\nu, \alpha(*)}, n \setminus B_{\nu, \alpha(*)}, n+1$ is infinite. So by the choice of $\mathcal{P}_{\bar{B}_v^s, \alpha(*)}$, i.e. 2.5 and clauses (i),(j) of \boxplus_1 for some $\beta \in \mathcal{U}_{\alpha(*)}$ so $\beta \geq \ell g(\nu)$ we have $B \setminus (C_{\nu_B^* \upharpoonright \beta}^s)^{[\ell]} \notin \text{id}_s$ for $\ell = 0, 1$ hence $B_1 := B \setminus (C_{\nu_B^*}^s)^{[\nu_B^*(\beta)]} \notin \text{id}_s$ recalling that for $\beta \in \mathcal{U}_{\alpha}, \alpha \neq 0$ and $\rho \in {}^\beta 2$ the set C_ρ^s depends just on $\ell g(\rho)$ and $\rho \upharpoonright \sup(u_\alpha)$ (and our s).

We finish as in the proof of 2.1.

Subcase 2B(β): $\sup(W') \in (\gamma_j^*, \gamma_j^*]$ where $\gamma_j^* \in E, \gamma^* = \sup(\gamma_j^* \cap W')$.

Apply Definition 2.2(3), clause (f) to γ, γ_j^*, W' we get $u = u_* \in \mathcal{P}_{\gamma_j^*}$ so $u = u_{\alpha(*)} \subseteq [\gamma^*, \gamma_j^*), \alpha(*) \in (\gamma^*, \gamma_j^*)$.

Let $\beta = \sup(u_*), \nu = \nu_B^* \upharpoonright \beta$ and by the choice of $\mathcal{P}_{\bar{B}_v^s}$ there is a $\mathcal{Q} \subseteq \mathcal{P}_{\bar{B}_v^s, \alpha(*)}$ of cardinality \aleph_0 and $\Lambda \subseteq {}^\omega 2$ of cardinality 2^{\aleph_0} such that for every $\rho \in \Lambda$ there is $B_\rho \in \text{ob}(B) \cap \text{id}_s^+$ such that $A \in \mathcal{Q} = B_\eta \subseteq^* A^{[\rho(A)]}$ and $B_{\eta,n} := B_n \setminus \cup \{C_{\nu_B^* \upharpoonright \gamma_{\alpha(*)}, k}^s : k < n\} \in I_{\nu_B^* \upharpoonright \gamma_{\alpha(*)}, n}^s$.

Clearly for some $v \subseteq \mathcal{U}_{\alpha(*)} \subseteq (\gamma_j^*, \gamma_{j+1}^*)$ of cardinality $\aleph_0, \nu \triangleleft \rho \in \mathcal{T}_s \wedge \ell g(\rho) \geq \sup(v) \Rightarrow \{c_{\rho \upharpoonright \varepsilon} : \varepsilon \in v\} = \mathcal{Q}$.

Let $B'_\eta \in \text{ob}(B)$ be $\subseteq^* B_{\eta,n}$ for $n < \omega$. Analyzing $S_{B'_\eta}$ and recalling $\gamma_{j+1}^* < \mathfrak{s}$ clearly $S_{B'_\eta} \cap \{\nu \in \mathcal{T}_s : \ell g(s) \leq \text{sup}(u_*)\}$ is $\{\nu_B^* \upharpoonright \gamma : \gamma < \text{sup}(u_*)\}$ and $\text{sup}(u_*) > \gamma^* + 1$, so there is no ρ such that A_ρ^2 non-empty, $\rho \in \mathcal{T}_s$ and $\text{sup}(u_*) \leq \ell g(\rho) < \gamma_j^*$, so $S_{B'_\eta} \cap \{\nu \in \mathcal{T}_s : \ell g(s) < \gamma_j^*\}$ does not depend on $\langle A_\eta^s : \eta \in T, \text{suc}(\mathcal{T}_s), \ell g(\eta) \geq \gamma_j^* \rangle$ so we can finish easily as in case 1.

Case 3: As in the proof of 2.1.

□_{2.6}

3. FURTHER DISCUSSION

The cardinal invariant \mathfrak{s} plays here a major role, so the claims depend on how \mathfrak{s} and \mathfrak{a} are compared; when $\mathfrak{s} = \mathfrak{a}$ it is not clear whether the further assumption of 2.1(1) may fail. If $\mathfrak{s} > \mathfrak{a} > \aleph_1$, it is not clear if the assumption of 2.1(2) may fail. But the first case is proved ZFC, but the others need pcf assumptions.

All this does not exclude the case $\mathfrak{s} = \aleph_{\omega+1}$, $\mathfrak{a} = \aleph_1$ hence $\mathfrak{b} = \aleph_1$, as in [Sh:668].

{4q.0}

Claim 3.1. *As Theorems 1.1, 2.1, 2.6 cover it.*

- 1) *If there is no inner model with measurable (and much more) then there is a completely separable MAD family, \mathcal{A} .*
- 2) *Moreover, if $J_* \subseteq \text{ob}(\omega)$ is dense and hereditary then we can demand $\mathcal{A} \subseteq J_*$.*
- 3) *We have to consider the conditions in 2.1, 2.6 and act as in [?].* $\square_{0.2}$

We now remark on some further possibilities.

{4q.1}

Definition 3.2. 1) We say $\mathcal{S} \subseteq \text{ob}(\omega)$ is \mathfrak{s} -free when:

- (a) for every $A \in \text{ob}(\omega)$ there is $B \in \text{ob}(A)$ such that B induces an ultrafilter on mathscrS ; i.e. $C \in \mathcal{S} \Rightarrow A \subseteq^* C \vee A \subseteq^* (\omega \setminus C)$.

1A) We say $\text{mathscrS} \subseteq \text{ob}(\omega)$ is \mathfrak{s} -free in I when $I \in \text{OB}$ and for every $A \in I$ there is $B \in \text{ob}(A)$ which induces an ultrafilter on S .

2) We say $\text{mathscrS} \subseteq \text{ob}(\omega)$ is \mathfrak{s} -richly free when clause (a) and

- (b) if $A \in \text{ob}(\omega)$ and the following set is infinite, then $\{D \cap \mathcal{S} : D \text{ an ultrafilter on } \omega \text{ containing } \text{ob}(A)\}$ it has cardinality continuum.

3) We say $\mathcal{S} \subseteq \text{ob}(\omega)$ is \mathfrak{s} -anti-free if no $B \in \text{ob}(\omega)$ induces an ultrafilter on \mathcal{S} .

4) Let \mathfrak{S} be $\{\kappa : \text{there is a } \subseteq\text{-increasing sequence } \langle \mathcal{S}_i : i < \kappa \rangle \text{ of } \mathfrak{s}\text{-richly-free families such that } \cup\{\mathcal{S}_i : i < \kappa\} \text{ is not } \mathfrak{s}\text{-free.}\}$

5) Recall $\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq \text{ob}(\omega) \text{ and no } B \in \text{ob}(\omega) \text{ induces an ultrafilter on } \mathcal{S}\}$.

6) We say $\text{ch}_{\dim}(\mathcal{B}) < \kappa$ when $(\exists \eta \in {}^{\mathcal{B}}2)(I = \text{ob}_{\mathcal{B}, \eta}) \Rightarrow \text{cf}(I, \subseteq)(I) < \kappa$.

{4q.4}

Observation 3.3. 1) If \mathcal{S} is \mathfrak{s} -free and $\mathcal{S}' \subseteq \mathcal{S}$ then \mathcal{S}' is \mathfrak{s} -free.

2) If $\mathcal{S} \subseteq \text{ob}(\omega)$ and $|\mathcal{S}| < \mathfrak{s}$ then \mathcal{S} is \mathfrak{s} -free.

3) If $\mathcal{S}_n \subseteq \text{ob}(\omega)$ is \mathfrak{s} -free for $n < \omega$ then $\cup\{\mathcal{S}_n : n < \omega\}$ is \mathfrak{s} -free.

4) $\mathfrak{s} \in \mathfrak{S}$.

5) $\kappa \in \mathfrak{S}$ iff $\text{cf}(\kappa) \in \mathfrak{S}$.

6) $\kappa \in \mathfrak{S} \Rightarrow \aleph_1 \leq \kappa \leq 2^{\aleph_0}$.

7) In Definition of \mathfrak{S} we can add “ $\cup\{\mathcal{S}_i : i < \kappa\}$ is \mathfrak{s} -anti-free”.

8) $\text{cf}(\mathfrak{s}) > \aleph_0$, in fact $\kappa \in \mathfrak{S} \Rightarrow \text{cf}(\kappa) > \aleph_0$.

{4q.7}

Definition 3.4. 1) We say $A \in \text{ob}(\omega)$ obeys $f \in {}^\omega \omega$ when: for every $n_1 < n_2$ from A we have $f(n_1) < n_2$.

2) Let $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ be a sequence of members of ${}^\omega \omega$. We say $\bar{A} = \langle A_\alpha : \alpha \in u \rangle$ obeys \bar{f} when $u \subseteq \delta$ and A obeys f_α for $\alpha \in A$.

3) $\mathfrak{a}_{\bar{f}} = \text{Min}\{|u| : \text{there are } B \in \text{ob}(\omega) \text{ and } \bar{A} = \langle A_\alpha : \alpha \in u \rangle \text{ obeying } \bar{f} \text{ such that } \{A_\alpha \cap B : \alpha \in u\} \text{ is a MAD of } B\}$.

* * *

{2q.13}

Discussion 3.5. Question: 1) Can we find such MAD for every λ ? ($\lambda = (2^{\aleph_0})^+$ is the first case and seems the problem).

2) Can we in [§1][Sh:669], let:

- (*) $\text{CON}(2^{\aleph_0} = \aleph_2 + \mu = \aleph_\omega$ (or just μ limit of cofinality \aleph_0 ?) and every MAD $\mathcal{A} \subseteq [\mu]^{\aleph_0}$, for some $u \in [\mu]^{\aleph_0}$, $|\mathcal{A} \cap \mathcal{P}(\omega)| = \aleph_1$.

E.g.

- (A) assume κ is 1.5 - huge as witness by the elementary embedding \mathbf{j} , G.C.H. for simplicity $\kappa_1 = \kappa, \kappa_2 = \mathbf{j}(\kappa), \mu_\ell = \kappa_\ell^{+\omega}$. Now we need a MAD which is Cohen absolutely so.

{2q.17}

Discussion 3.6. 1) We may like to have only $\leq \mathfrak{b}$ levels. In this case our question is:

- (*) for regular κ
- (a) if $B, \langle B_\alpha^* : \alpha < \kappa \rangle$ is \subseteq^* -decreasing, $\kappa = \text{cf}(\kappa)$ let $I_B = \{A \subseteq \omega : \alpha < \kappa \Rightarrow A \subseteq^* B_\alpha^*\}$ an ideal,
 - (b) $\mathfrak{B}_B = \{\theta : \text{in } I_B \text{ there is a } \subseteq^*\text{-increasing sequence of length } \theta \text{ with no upper bound}\}$
 - (c) $\mathfrak{b}_B = \min(B_B)$
 - (d) $\mathfrak{b}_\kappa^\circ = \sup\{\mathfrak{b}_B : B \in \text{seq}_\kappa \text{ as above}\}$
 - (e) $\mathfrak{b}_* = \text{Min}\{\theta : \text{if } \kappa \in \theta \cap \text{Reg} \text{ then } \mathfrak{b}_\kappa^\circ \leq \theta\}$.

{4q.19}

Question 3.7. If $\kappa < \mathfrak{b}$ is $\mathfrak{b}_\kappa^\circ \leq \mathfrak{b}$? I.e. $\mathfrak{b} = \mathfrak{b}_*$? Or use $\sup(\mathfrak{B})$?

2) Even if $\mathfrak{b} = \mathfrak{b}_*$, to get a slender $G \cong G^{**}$, we seem to need: (as in [Sh:F844]): restrict the f_η (generating the G_2, G_2) such that $f_\eta \upharpoonright [\eta_1^*, \eta_{i+1}^*) \in \mathcal{F}_i^{\ell g(\eta)}, \bar{\mathcal{F}}^\alpha = \langle \mathcal{F}_i^\alpha : i < \omega \rangle$ is \subseteq^* -increasing mod finite, $|\mathcal{F}_i^\alpha| \ll [n_i, n_{i+1}]$ so $\nu \triangleleft \eta \Rightarrow \wedge \rho_\nu \neq \rho_\eta \Rightarrow (\forall^\infty i)(\Sigma\{f_\eta(n), f_\nu(n) : n \in [n_i, n_{i+1}]\}) = 0$. The point is that we can continue of courses $f_\eta(n)$ is divisible by $i!$ if $n \geq n_i$ or something like that.

3) The case \mathfrak{B} has no last member are more complicated ($\sup(\mathfrak{B})$) can be singular or weakly inaccessible.

4) Alternatively: if

{4q.32}

Discussion 3.8. We can define for §1 or any variant a parallel of \mathfrak{h} . Define a game.

* * *

{4q.26}

Discussion 3.9. (080215)

We try to combine the ideas of present [S1][Sh:900] and last weak. The later one means that letting $\lambda := \mathfrak{s}$ and $\langle A_\alpha^* : \alpha < \lambda \rangle$ exemplifies it we use the tree $\mathcal{T} = \bigcup_{\alpha < \lambda} \mathcal{T}_\alpha, \mathcal{T}_\alpha = \{\eta \in {}^\alpha 2 : \text{for some infinite } B \subseteq \omega, B \in \text{id}_{\bar{A}^* \upharpoonright \alpha}\}, \text{id}_\eta = \{B \subseteq \omega :$

we have $\beta < \alpha \Rightarrow B \subseteq^* A_{\alpha, \beta}^{<\eta(B)>}\}$.

For any $C \in [\omega]^{\aleph_0}$, we can prove that there is a countable $u \subseteq \lambda$ such that for a perfect set Ξ of $\nu \in {}^u 2$, we have $(\forall w \in [u]^{<\aleph_0})(B \cap \bigcap \{A_\beta^{<\eta(B)>} : \beta \in w\})$ is infinite. But $\text{cf}(\lambda) > \aleph_0$ so $\alpha(*) = \cup\{\beta + 1 : \beta \in u\} < \lambda$, so for $\nu \in \Xi$ there is $\eta_\nu \in {}^{\alpha(*)} 2$ such that $[B]^{\aleph_0} \cap \text{id}_{<A_\beta^{<\eta_\nu(B)>} : \beta < \alpha(*)>} \neq \emptyset$.

This is not enough for each (or just may) $\eta \in \text{suc}(\mathcal{T})$ we should choose $A_\eta \in \text{id}_\eta \setminus [\omega]^{\aleph_0}$ which approximate the desired MAD (so almost disjoint along each version: incomparable η 's are automatically almost disjoint; let $\mathcal{A} = \{A_\eta^t : A_\eta^t \text{ defined}\}$, $\mathbf{t} = \langle A_\eta : \eta \in \text{suc}(\mathcal{T}) \rangle$.

A first problem how can we make it disjoint. By the above it seemed each $B \in [\omega]^{\aleph_0}$ not in the ideal $\mathcal{A}_\mathbf{t}$, well this seems reasonable, i.e. for each such B will have 2^{\aleph_0} changes, so by reasonable bookkeeping each can be treated.

But this leaves us with the problem of contradicting saturativity along some branch. This is the second idea: for each $A'_\eta, \eta \in \text{suc}(\mathcal{T})$ there is $B_\eta \subseteq \omega$ such that

- (*) (a) $\nu \in \text{suc}(\mathcal{T}) \wedge \nu \triangleleft \eta \Rightarrow A_\eta \cap B_\eta$ finite
- (b) $\eta \preceq \nu \in \text{suc}(\mathcal{T}) \Rightarrow A_\eta \subseteq B_\eta$.

This by using a maximal \subseteq^* -increasing sequence in $\text{id}_{\eta^-}^\mathbf{t}$, and if $(\text{id}_\eta, \subseteq^*)$ is not \aleph_1 -directed we terminate immediately.

{4q.31}

Definition 3.10. Definition/Choice: 1) $\lambda = \mathfrak{s}$.

2) $\bar{A}^* = \langle A_\alpha^* : \alpha < \lambda \rangle$ exemplifies this, i.e. $A_\alpha \subseteq \omega$ for $\alpha < \lambda$ (is infinite co-infinite) but for no infinite $B \subseteq \omega$ do we have $\alpha < \lambda \Rightarrow B \subseteq^* A_\alpha \vee B \subseteq^* \omega \setminus A_\alpha$.

3) [Here] for $B \subseteq \omega$ let $\text{id}_B = \{A \subseteq \omega : A \subseteq^* B\}$.

4) [Here] η^- is $\eta \upharpoonright \alpha$ when $\ell g(\eta) = \alpha + 1$.

{4q.34}

Discussion 3.11. On Forcing:

We may try to immitate [§1][Sh:668] by building or probably forcing by \mathbb{P} such that, e.g. $|\mathbb{P}| = \lambda$, when $\lambda = \aleph_{\omega+1}$, GCH holds and $S_{\aleph_1}^\lambda \notin \tilde{I}[\lambda]$.

This may apply to a family of such problems, so we may consider the general forcing of this kind. If the approximations are $\mathbf{p} = (\mathbb{P}_\mathbf{p}, \bar{\eta}, u_\mathbf{p} + \text{history})$ e.g. $u_\mathbf{p} \in [\lambda]^{\aleph_1}, \langle \eta_\alpha^\mathbf{p} : \alpha \in u_\mathbf{p} \rangle$ we need an algebra \mathfrak{B} controlling the dependence or better $u_* = \text{cl}_{\mathfrak{B}}(u_*) \subseteq u_\mathbf{p} \Rightarrow \mathbf{p} \upharpoonright u \leq_{\text{AP}} \mathbf{p}$. E.g. $\mathbf{p} <_{\text{AP}} \mathbf{q} \Rightarrow \mathbb{P}_\mathbf{q}/\mathbb{P}_\mathbf{p}$ is adding Cohens, [alternatively, try iterated creatures].

Moved from pg.18: We may use games, e.g. of length $\kappa = \aleph_1$. This simplify. Note

that in the games we can let the MAD player choose $\mathcal{C}_\alpha \subseteq \{\mathcal{A} : \mathcal{A} \subseteq [\omega]^{\aleph_0} \text{ is non-empty countable and almost disjoint}\}$ such that $(\forall A \in \text{ob}(\omega))(\exists \mathcal{A} \in \mathcal{C}_\alpha)(\mathcal{A} \subseteq \text{ob}(A))$ (or so) and the SANE player have to choose \mathcal{A}_α from \mathcal{C}_α . This leads to: for $\mathbf{x} \in \text{AP}$, all members of $\mathcal{B}_i := \cup\{\mathcal{A}_\eta^\mathbf{x} : \eta \in \text{suc}(\mathcal{T}_\mathbf{x}), \ell g(\eta) = i + 1\}$ are f_i -thin, i.e. $A \in \mathcal{A}_i \Rightarrow (\forall^\infty n)(|A \cap (0, f_i(n))| \leq n)$ where $\langle f_i : i < \omega_1 \rangle$ exemplify $\mathfrak{B} = \aleph_1$.

Note: if $\mathcal{A} \subseteq \text{ob}(\omega)$ is MAD, $|\mathcal{A}| = \aleph_1$ then there is $\mathcal{B} \subseteq \text{ob}(\omega), |\mathcal{B}| = \aleph_1$ such that $B \in \text{id}_\mathcal{A}^+ \Rightarrow (\exists^{\text{continuum}} I \in \mathcal{W}_\mathcal{B})(\text{ob}(B) \cap I \neq \emptyset)$ as if $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$, for each $\alpha \in [\omega, \omega_1)$ deal with $\{B : (\exists^{\aleph_0} i < \alpha)(B \cap A_\alpha \text{ infinite})\}$.

But define \mathfrak{S} .

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