

**PCF ARITHMETIC WITHOUT AND WITH CHOICE**  
**SH938**

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ABSTRACT. We deal with relatives of GCH which are provable. In particular we deal with rank version of the revised GCH. Our motivation was to find such results when only weak versions of the axiom of choice are assumed but some of the results gives us additional information even in ZFC. We also start to deal with pcf for pseudo-cofinality (in ZFC with little choice).

ANOTATED CONTENT

- §0 Introduction, pg.2  
[We present introductory remarks mainly to §3,§4.]
- §1 Preliminaries, pg.3  
[We present some basic definitions and claims, mostly used later.]
- §2 Commuting ranks, pg.6  
[If we have ideals  $D_1, D_2$  on sets  $I_1, I_2$  and a  $I_1 \times I_2$ -rectangle  $\bar{\alpha}$  of ordinals, we can compute rank in two ways: one is first apply  $\text{rk}_{D_1}$  on each row and then  $\text{rk}_{D_2}(-)$  on the resulting column. In the other we first apply  $\text{rk}_{D_2}(-)$  on each column and then  $\text{rk}_{D_1}(-)$  on the resulting row. We give sufficient conditions for an inequality. We use (ZFC + DC and) weak form of choice like  $\text{AC}_{I_\ell}$  or  $\text{AC}_{\mathcal{P}(I_\ell)}$ .]
- §3 Rank systems and a Relative of GCH, pg.11  
[We give a framework to prove a relative of the main theorem of [Sh:460] dealing with ranks. We do it with weak form of choice (DC +  $\text{AC}_{<\mu}$ ),  $\mu$  a limit cardinal, this give new information also in ZFC.]
- §4 Finding systems, pg.18  
[The main result in §3 deals with an abstract setting. Here we find an example, a singular limit of measurables. Note that even under ZFC this gives information on ranks.]
- §5 Pseudo tcf, pg.21  
[We look again at the  $\text{pcf}(\bar{\alpha})$ , but only for  $\aleph_1$ -complete filters using pseudo-cofinality and the cofinalities not too small. Under such restrictions we get a parallel to the pcf theorem.]

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## 0. INTRODUCTION

In [Sh:460] and [Sh:513], [Sh:829] we prove in  $ZFC = ZF + AC$  relatives of G.C.H. Here mainly we are interested in related relatives assuming only weak forms of choice, but some results add information even working in ZFC, in particular the generalization of [Sh:460] for ranks. Always we can assume  $ZF + DC$ .

Our original motivation was

{y0.3}

**Conjecture 0.1.** Assume  $ZF + DC$  and  $\mu$  a limit cardinal such that  $AC_{<\mu}$  and  $\mu$  is strong limit. For every ordinal  $\gamma$ , for some  $\kappa < \mu$ , for any  $\alpha < \mu$  and  $\kappa$ -complete filter  $D$  on  $\alpha$  we have  $rk_D(\gamma) = \gamma$ .

Here we get an approximation to it, i.e. for  $\mu$  a limit of measurables restricting ourselves to ultrafilters; this is conclusion 4.4 deduced by applying Theorem 3.10 to Claim 4.3. Can we do it with  $\mu = \beth_\omega$ ?

Also we would like to weaken  $AC_{<\mu}$ ; this is interesting per se and as then we will be able to combine [Sh:835] + [Sh:513] - see below. We intend to try in [Sh:F955]; we start with  $\bar{J} = \langle J_n : n < \omega \rangle$  such that  $IND(\bar{J})$  or something similar.

{y0.7}

It may be illuminating to compare the present result with (see [Sh:g, V]).

**Claim 0.2.** If  $\kappa \geq \theta > \aleph_0, \lambda \geq 2^{2^\kappa}$  then the following conditions are equivalent:

- (\*)<sub>1</sub> for every  $\theta$ -complete filter  $D$  on  $\kappa$ , we have  $rk_D(\lambda^+) = \lambda^+$
- (\*)<sub>2</sub>  $\alpha < \lambda^+ \Rightarrow rk_D(\alpha) < \lambda^+$  for every  $\theta$ -complete filter  $D$  on  $\kappa$
- (\*)<sub>3</sub> there is no  $\mathcal{F} \subseteq {}^\kappa\lambda$  of cardinality  $\geq \lambda^+$  and  $\theta$ -complete filter  $D$  in  $\kappa$  such that  $f_1 \neq f_2 \in \mathcal{F} \Rightarrow f_1 \neq_D f_2$ .

Also we can in 0.2 replace  $\lambda^+$  by a cardinal of cofinality  $> 2^{2^\kappa}$ . So the result in [Sh:460] implies a weak version of the conjecture above, say on  $|rk_D(\alpha)|$ , but the present one gives more precise information. On the other hand, the present conjecture is not proved for  $\mu = \beth_\omega$ , also it seems less accomodating to the possible results with  $\aleph_\omega$  instead of  $\beth_\omega$  in [Sh:513] below  $2^{2^{\aleph_\omega}}$ .

{y0.9}

**Question 0.3.** In [Sh:908] can we prove that the rank is small?

**Discussion 0.4.** We present examples showing some limitations.

Below part (1) of the example shows that Claim 2.3 cannot be improved too much and part (2) shows that Conclusion 4.4 cannot be improved too much. In fact, in conjecture 0.1 if we demand only “ $\mu$  is a limit cardinal” then it consistently fails. This implies that we cannot improve too much other results in §3, §4.

We may wonder how to compare the result in [Sh:460] and Conjecture 0.1 even in ZFC.

{y0.4}

**Example 0.5.** 1) If  $D_\ell = \text{dual}([\kappa_\ell]^{<\kappa_\ell})$  for  $\ell = 1, 2$  (so if  $\kappa_\ell$  is regular then  $D_\ell = \text{dual}(J_{\kappa_\ell}^{\text{bd}})$ ) and  $\kappa_2 < \kappa_1$  then  $D_2$  does not 2-commute with  $D_1$ , i.e.  $\boxplus_{D_1, D_2}^2$  fail.

2) Consistently with ZFC, for every  $n$ ,  $rk_{J_{\aleph_n}^{\text{bd}}}(\aleph_\omega) > \aleph_\omega$ .

*Remark 0.6.* We cannot weaken the assumption  $\boxplus(a)$  in 2.3 too much because even assuming ZFC, if  $\boxplus(a)$  below for some  $f, g, \bar{f}, \bar{g}$  we have  $\otimes$  of 2.3 but  $rk_{D_1}(f) > rk_{D_2}(g)$ .

*Proof.* 1) Let  $A = \kappa_1$  and let  $f_2 \in {}^{\kappa_2}\text{Ord}$  be constantly 1 hence by Definition 1.10 and Claim 1.11(3) the ideal  $J_2 = J[f_2, D_2]$  is  $[\kappa_2]^{<\kappa_2}$ . Choose a function  $h : \kappa_1 \rightarrow \kappa_2$  and  $(\forall \beta < \kappa_2)(\exists^{\kappa_1} \alpha < \kappa_1)(h(\alpha) = \beta)$  and let  $\langle B_\alpha : \alpha \in A \rangle$  be such that we have  $B_\alpha := \kappa_2 \setminus h(\alpha)$ .

So if  $A_* \in D_1, B_* \in J_2^+$  then for some  $\alpha_* < \kappa_1$  we have  $A_* \supseteq \kappa_1 \setminus \alpha_*$  and  $B_* \subseteq \kappa_2, |B_*| = \kappa_2$  and choose  $t \in B_*$  and then choose  $s \in A_*$  such that  $h(s) = t+1$ , such  $s$  exists by the choice of  $h$  so  $(s, t) \in A_* \times B_*$  but  $(s, t) \notin \{s\} \times B_s$ , contradiction. 2) Assume that the sequence  $\langle 2^{\aleph_n} : n < \omega \rangle$  is increasing with supremum  $> \aleph_\omega$  and in  $\text{cf}({}^{\aleph_n}\aleph_n, <_{J_{\aleph_n}^{\text{bd}}})$  there is an increasing sequence of length  $\aleph_{\omega+1}$  for each  $n \in [1, \omega)$  hence it follows that  $\text{rk}_{J_{\aleph_n}^{\text{bd}}}(\aleph_\omega) > \text{rk}_{J_{\aleph_n}^{\text{bd}}}(\aleph_n) \geq \aleph_n$  for  $n \in [1, \omega)$ .  $\square_{0.5}$

We may hope to prove interesting things in ZF + DC by division to cases: if [Sh:835] apply fine, if not then we have a strict **p**. But we need  $\text{AC}_{<\mu}$  to prove even clause (f) in 3.1, see [Sh:F955]. We may consider that even in ZFC, probably [Sh:908] indicate that we can use weaker assumptions.

Let us say something on our program on set theory with little choice of which this work is a part. We always “know” that the axiom of choice is true. In addition we had thought that there is no interesting general combinatorial set theory without AC (though equivalence of version of choice, inner model theory and some other exist). Concerning the second point, since [Sh:497] our opinion changed and have thought that there is an interesting such set theory, with “bounded choice” related to pcf. More specifically [Sh:497] seems to prove that such theory is not empty. Then [Sh:835] suggest to look at axioms of choice “orthogonal” to “ $\mathbf{V} = \mathbf{L}[\mathbb{R}]$ ”, e.g. demand then  $\omega^{\geq \alpha}$  can be well ordered (and weaker relatives). The results say that the universe is somewhat similar to universes gotten by Easton like forcing, blowing up  $2^\lambda$  for every regular  $\lambda$  without well ordering the new  $\mathcal{P}(\lambda)$ . Continuing this Larson-Shelah [LrSh:925] generalize classical theorem on splitting a stationary subset of a regular  $\lambda$  consisting of ordinals of cofinality  $\kappa$ .

In [Sh:F955] we shall continue this work. In particular, we continue §5 to get a parallel of the pcf theorem and more. Recall that in [Sh:513] in ZFC we get connections between the existence of independent sets and a strong form of [Sh:460]. We prove related theorems on rank.

1. PRELIMINARIES

{z0.0}

**Context 1.1.** 1) We work in ZF in all this paper.

- 2) We try to say when we use DC but assuming it always makes no great harm.
- 3) We shall certainly mention the use of any additional form of choice, mainly  $\text{AC}_A$ .
- 4) In 1.2 - 1.11 we quote definitions and claims to be used, see [Sh:835]. The rest is used only in §5.

{z0.11}

**Definition 1.2.** 1) A filter  $D$  on  $Y$  is  $(\leq B)$ -complete when: if  $\langle A_t : t \in B \rangle \in {}^B D$  then  $A := \cap \{A_t : t \in B\} \in D$ . We can instead say “ $|B|^+$ -complete” even if  $|B|^+$  is not well defined.

- 1A) A filter  $D$  on  $Y$  is pseudo  $(\leq B)$ -complete when if  $\langle A_t : t \in B \rangle \in {}^B D$  then  $\cap \{A_t : t \in B\}$  is not empty (so adopt the conventions of part (1)).
- 2) For an ideal  $J$  on a set  $I$  let  $\text{dual}(J) = \{I \setminus X : X \in J\}$ , the dual ideal and  $\text{Dom}(J) = I$ , abusing notation we assume  $J$  determines  $I$ .
- 3) For a filter  $D$  on a set  $I$  let  $\text{dual}(D) = \{I \setminus X : X \in D\}$ ,  $\text{Dom}(D) = I$ . We may use properties defined for filter  $D$  for the dual ideal (and vice versa).

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4) For a filter  $D$  on  $I$  let  $D^+ = \{A \subseteq I : I \setminus A \notin D\}$  and for an ideal  $J$  on  $I$  let  $J^+ = (\text{dual}(J))^+$ .

*Remark 1.3.* It may be interesting to try to assume that relevant filters are just pseudo  $(\leq B)$ -complete instead of  $(\leq B)$ -complete. Now 1.14 clarify the connection to some extent, but presently we do not pursue this direction.

{z0.13}

**Definition 1.4.**  $\mathbf{C}$  is the class of sets  $A$  such that  $\text{AC}_A$ , the axiom of choice for  $A$  non-empty sets, holds.

{z0.15}

**Definition 1.5.** 1)  $\theta(A) = \text{Min}\{\alpha : \text{there is no function from } A \text{ onto } \alpha\}$ .

2)  $\Upsilon(A) = \text{Min}\{\alpha : \text{there is no one-to-one function from } \alpha \text{ into } A\}$  so  $\Upsilon(A) \leq \theta(A)$ .

{z0.21}

**Definition 1.6.** 1) For  $D$  a filter on  $I$  and  $f, g \in {}^I\text{Ord}$  let  $f <_D g$  or  $f < g \text{ mod } D$  means that  $\{s \in I : f(s) < g(s)\} \in D$ ; similarly for  $\leq, =, \neq$ .

2) For  $D$  a filter on  $I$  and  $f \in {}^I\text{Ord}$  and  $\alpha \in \text{Ord} \cup \{\infty\}$  we define when  $\text{rk}_D(f) = \alpha$  by induction on  $\alpha$ :

⊗ For  $\alpha < \infty$ ,  $\text{rk}_D(f) = \alpha$  iff  $\beta < \alpha \Rightarrow \text{rk}_D(f) \neq \beta$  and for every  $g \in {}^I\text{Ord}$  satisfying  $g <_D f$  there is  $\beta < \alpha$  such that  $\text{rk}_D(g) = \beta$ .

3) We can replace  $D$  by the dual ideal.

{z0.24}

**Observation 1.7.** 1) Let  $D$  be a pseudo  $\aleph_1$ -complete filter on  $I$ . If  $f, g \in {}^I\text{Ord}$  and  $f \leq_D g$  then  $\text{rk}_D(f) \leq \text{rk}_D(g)$  and so if  $f =_D g$  then  $\text{rk}_D(f) = \text{rk}_D(g)$ .

2) If  $D_\ell$  is a pseudo  $\aleph_1$ -complete filter on  $I$  for  $\ell = 1, 2$  then  $D_1 \subseteq D_2 \wedge f \in {}^I\text{Ord} \Rightarrow \text{rk}_{D_1}(f) \leq \text{rk}_{D_2}(f)$ .

*Proof.* Easy. □

{z0.23}

**Claim 1.8.** Assume  $D$  is a filter on  $I$  such that  $D$  is  $\aleph_1$ -complete or just pseudo  $\aleph_1$ -complete (see Definition 1.2(1A)).

1) [DC] For  $f \in {}^I\text{Ord}$ , in 1.6,  $\text{rk}_D(f)$  is always an ordinal, i.e.  $< \infty$ .

2) [DC] If  $\alpha \leq \text{rk}_D(f)$  then for some  $g \in \prod_{t \in I} (f(t) + 1)$  we have  $\alpha = \text{rk}_D(g)$ . If

$\alpha < \text{rk}_D(f)$  we can add  $g <_D f$  and we can demand  $(\forall y \in I)(g(y) < f(y) \vee g(y) = 0 = f(y))$ .

2A) If  $\text{rk}_D(f) < \infty$  then part (2) holds for  $f$  (without assuming DC).

3) If  $f, g \in {}^I\text{Ord}$  and  $f <_D g$  and  $\text{rk}_D(f) < \infty$  then  $\text{rk}_D(f) < \text{rk}_D(g)$ .

4) For  $f \in {}^I\text{Ord}$  we have  $\text{rk}_D(f) > 0$  iff  $\{t \in I : f(t) > 0\} \in D$ .

5) If  $f, g \in {}^I\text{Ord}$  and  $f = g + 1$  then  $\text{rk}_D(f) = \text{rk}_D(g) + 1$ .

*Proof.* Straight, e.g.

2A) We prove this by induction on  $\beta = \text{rk}_D(f)$ . If  $\beta \leq \alpha$  there is nothing to prove. If  $\beta = \alpha + 1$  by the definition, there is  $g <_D f$  such that  $\text{rk}_D(g) \geq \alpha$ , now by part (3) we have  $\text{rk}_D(g) < \text{rk}_D(f)$  which means  $\text{rk}(g) < \alpha + 1$ , so together  $\text{rk}_D(g) = \alpha$  and let  $g' \in {}^I\text{Ord}$  be defined by  $g'(s)$  is  $g(s)$  if  $g(s) < f(s)$  and is 0 if  $g(s) \geq f(s)$  so  $g' <_D f$  and  $g' \leq_D g \leq_D g'$  hence  $\text{rk}_D(g') = \text{rk}_D(g) = \alpha$  is as required.

Lastly, if  $\beta > \alpha + 1$  by the definition there is  $f' <_D f$  such that  $\text{rk}_D(f') \geq \alpha + 1$  and by 1.7(1) without loss of generality  $t \in I \Rightarrow f'(t) \leq f(t)$  and by part (3)  $\text{rk}_D(f') < \text{rk}_D(f)$  so we can apply the induction hypothesis to  $f'$ . □<sub>1.8</sub>

{z0.25}

**Claim 1.9.** 1) [AC $_{\aleph_0}$ ] If  $D$  is an  $\aleph_1$ -complete filter on  $I$  and  $f \in {}^I\text{Ord}$  and  $I = \cup\{I_n : n < \omega\}$  then  $\text{rk}_D(f) = \text{Min}\{\text{rk}_{D+I_n}(f) : n < \omega \text{ and } I_n \in D^+\}$ .

2) [AC $\mathcal{Y}$ ] If  $D$  is a  $|\mathcal{Y}|^+$ -complete filter on  $I, \mathcal{Y}$  infinite and  $f \in {}^I\text{Ord}$  and  $\cup\{I_t : t \in \mathcal{Y}\} \in D$  then  $\text{rk}_D(f) = \text{Min}\{\text{rk}_{D+I_t}(f) : t \in \mathcal{Y} \text{ and } I_t \in D^+\}$ .

*Proof.* Like [Sh:71].

1) By part (2).

2) The inequality  $\leq$  is obvious (i.e. by 1.7(2)). We prove by induction on the ordinal  $\alpha$  that  $(\forall y \in \mathcal{Y})[I_y \in D^+ \Rightarrow \text{rk}_{D+I_y}(f) \geq \alpha] \Rightarrow \text{rk}_D(f) \geq \alpha$ .

For  $\alpha = 0$  and  $\alpha$  is limit this is trivial.

For  $\alpha = \beta + 1$ , we assume  $(\forall y \in \mathcal{Y})[I_y \in D^+ \Rightarrow \text{rk}_{D+I_y}(f) \geq \alpha > \beta]$  so by Definition 1.6 it follows that  $[t \in \mathcal{Y} \wedge I_y \in D^+ \Rightarrow (\exists g)(g \in {}^I\text{Ord} \wedge g <_{D+I_y} f \wedge \text{rk}_{D+I_y}(g) \geq \beta)]$  hence, if  $y \in \mathcal{Y} \wedge I_y \in D^+$  then  $\{t \in I : f(t) = 0\} = \emptyset \text{ mod } (D + I_y)$ , i.e.  $\{y : f(y) = 0\} \cap I_y = \emptyset \text{ mod } D$ . As this holds for every  $y \in \mathcal{Y}$  and  $D$  is  $|\mathcal{Y}|^+$ -complete clearly we have  $\{t \in I : f(t) = 0\} = \emptyset \text{ mod } D$ . We can by 1.7(1) replace  $f$  by  $f' \in {}^I\text{Ord}$  when  $\{y \in I : f(y) = f'(y)\} \in D$  so without loss of generality  $t \in I \Rightarrow f(t) > 0$ .

But  $\mathcal{Y} \in \mathbf{C}$ , hence by 1.8(2A) there is a sequence  $\langle g_y : y \in \mathcal{Y}_* \rangle$  such that  $\mathcal{Y}_* := \{y \in \mathcal{Y} : I_y \in D^+\}$  and  $g_y \in {}^I\text{Ord}, g_y <_{D+I_y} f, \text{rk}_{D+I_y}(g_y) \geq \beta$  and  $t \in I \Rightarrow g_y(t) < f(t)$  so  $g_y < f$ .

As  $D$  is  $|\mathcal{Y}|^+$ -complete necessarily  $I_* := \cup\{I_y : y \in \mathcal{Y} \setminus \mathcal{Y}_*\} = \emptyset \text{ mod } D$ , but  $\cup\{I_y : y \in \mathcal{Y}\} \in D$  hence  $I_* = \cup\{I_y : y \in \mathcal{Y}_*\}$  belongs to  $D$ . Define  $g \in {}^I\text{Ord}$  by  $g(s) = \min\{g_x(s) : x \in \mathcal{Y}_* \text{ satisfies } s \in I_x\}$  if  $s \in I_*$  and 0 if  $s \in I \setminus I_*$ .

Hence  $(\cup\{I_y : y \in \mathcal{Y}_*\}) \in D$  and  $g \in {}^I\text{Ord}$  and  $g <_D f$  (and even  $g < f$ ) so by the induction hypothesis

$\odot$  it suffices to prove  $y \in \mathcal{Y}_* \Rightarrow \text{rk}_{D+I_y}(g) \geq \beta$ .

Fix  $y \in \mathcal{Y}_*$ , and for each  $x \in \mathcal{Y}_*$  let  $I_{y,x} := \{t \in I_x \cap I_y : g(t) = g_x(t)\}$  so by the choice of  $g(t)$  we have

$\boxplus_1$  if  $y \in \mathcal{Y}_*, t \in I_y$  then for some  $x \in \mathcal{Y}_*$  we have  $t \in I_{y,x} \subseteq I_y$  and  $g(t) = g_x(t)$ .

Hence

$\boxplus_2$   $\langle I_{y,x} : x \in \mathcal{Y}_* \rangle$  exists and  $\cup\{I_{y,x} : x \in \mathcal{Y}_*\} = I_y \in (D + I_y)$ .

Now

$\boxplus_3$  if  $x \in \mathcal{Y}_* \wedge I_{y,x} \in (D + I_y)^+$  then  $\text{rk}_{D+I_{y,x}}(g) \geq \beta$ .

[Why? By the choice of  $I_{y,x}$  we have  $g = g_x \text{ mod } (D + I_{y,x})$  hence  $\text{rk}_{D+I_{y,x}}(g) = \text{rk}_{D+I_{y,x}}(g_x)$ , also  $I_{y,x} \subseteq I_x$  hence  $D + I_{y,x} \supseteq D + I_x$  which by 1.7(2) implies  $\text{rk}_{D+I_{y,x}}(g_x) \geq \text{rk}_{D+I_x}(g_x)$  which is  $\geq \beta$ . Together we are done.]

By  $\boxplus_2 + \boxplus_3$  and the induction hypothesis it follows that  $y \in \mathcal{Y}_* \Rightarrow \text{rk}_{D+I_y}(g) \geq \beta$  so by  $\odot$  we are done. □<sub>1.9</sub>

{z0.27}

**Definition 1.10.** For  $I, D, f$  in 1.6 let  $J[f, D] =: \{Z \subseteq I : Y \setminus Z \in D \text{ or } Z \in D^+ \wedge \text{rk}_{D+(I \setminus Z)}(f) > \text{rk}_D(f)\}$ .

{z0.29}

**Claim 1.11.** [DC+AC $\mathcal{Y}$ ] Assume  $D$  is an  $\aleph_1$ -complete  $|\mathcal{Y}|^+$ -complete filter on  $I$ .

- 1) If  $f \in {}^I\text{Ord}$  then  $J[f, D]$  is an  $\aleph_1$ -complete and  $|\mathcal{Y}|^+$ -complete ideal on  $I$ .
- 2) If  $f_1, f_2 \in {}^Y\text{Ord}$  and  $J = J[f_1, D] = J[f_2, D]$  then  $\text{rk}_D(f_1) < \text{rk}_D(f_2) \Rightarrow f_1 < f_2 \text{ mod } J$  and  $\text{rk}_D(f_1) = \text{rk}_D(f_2) \Rightarrow f_1 = f_2 \text{ mod } J$ .
- 3) If  $f \in {}^I\text{Ord}$  is e.g. constantly 1 then  $J[f, D] = \text{dual}(D)$ .
- 4) If  $f \in {}^I\text{Ord}$  and  $A \in (J[f, D])^+$  then ( $A \in D^+$  and)  $\text{rk}_{D+A}(f) = \text{rk}_D(f)$ .

*Proof.* 1) By 1.9.

2) As  $J$  is an ideal on  $I$  (by part (1)) this should be clear by the definitions; that is, let  $A_0 := \{t \in Y : f_1(t) = f_2(t)\}$ ,  $A_1 := \{t \in Y : f_1(t) < f_2(t)\}$  and  $A_2 := \{t \in Y : f_1(t) > f_2(t)\}$ . Now  $\langle A_0, A_1, A_2 \rangle$  is a partition of  $Y$ . First, assume  $A_0 \in J^+$ , then by the definition of  $J[f_1, D]$  we have  $\neg(\text{rk}_D(f_\ell) < \text{rk}_{D+A_0}(f_\ell))$ ; i.e.  $\text{rk}_{D+A_0}(f_\ell) \leq \text{rk}_D(f_\ell)$  and so by 1.7(2) we have  $\text{rk}_D(f_\ell) = \text{rk}_{D+A_0}(f_\ell)$ .

Now as  $A_0 \in J^+$ , by the choice of  $A_0$ ,  $f_1 <_{D+A_0} f_2$  hence  $f_1 < f_2 \pmod J$  and  $\text{rk}_D(f_1) = \text{rk}_{D+A_0}(f_1) < \text{rk}_{D+A_0}(f_2) = \text{rk}_D(f_2)$ .

[Why? By the previous sentence, by 1.8(3), by the previous sentence respectively.]

Second, similarly if  $A_2 \in J^+$  then  $f_2 < f_1 \pmod{(D+A_2)}$  and  $\text{rk}_D(f_1) > \text{rk}_D(f_2)$ .

Lastly, if  $A_1 \in J^+$  then  $f_1 = f_2 \pmod{(D+A_1)}$  hence  $\text{rk}_{D+A_1}(f_1) = \text{rk}_{D+A_2}(f_2)$  and  $\text{rk}_D(f_1) = \text{rk}_D(f_1) = \text{rk}_{D+A_1}(f_2) = \text{rk}_{D+A_2}(f_2) = \text{rk}_D(f_2)$ .

By the last three sentences at most one of  $A_0, A_1, A_2$  belongs to  $J^+$  and as  $A_0 \cup A_1 \cup A_2 = Y$  at least one of  $A_0, A_1, A_2$  belongs to  $J^+$ , so easily we are done.

3) Obvious.

4) Proved inside the proof of part (2). □<sub>1.11</sub>

{z0.32}

**Definition 1.12.** 1) Let  $\text{FIL}_S^{\text{cc}}(I)$  or  $\text{FIL}_S^{\text{pcc}}(I)$  be the set of  $D$  such that:

$D$  is a filter on the set  $I$  which is  $|S|^{+}$ -complete and is  $\aleph_1$ -complete or is pseudo  $|S|^{+}$ -complete and pseudo  $\aleph_1$ -complete.

2) Let  $\text{FIL}_{\text{cc}}(I)$  or  $\text{FIL}_{\text{pcc}}(I)$  be  $\text{FIL}_{\emptyset}^{\text{cc}}$  or  $\text{FIL}_{\emptyset}^{\text{pcc}}$ .

3) Omitting  $I$  means for some  $I$  and then we let  $I = \text{Dom}(D)$ .

{z0.51}

**Definition 1.13.** 1) For a filter  $D$  on  $I$  and set  $S$  we define  $\text{comp}_{S,\gamma}(D)$  by induction on  $\gamma \in \text{Ord} \cup \{\infty\}$ .

$\gamma = 0$ :  $\text{comp}_{S,\gamma}(D) = D$

$\gamma = \text{limit}$ :  $\text{comp}_{S,\gamma}(D) = \cup\{\text{comp}_{S,\beta}(D) : \beta < \gamma\}$

$\gamma = \beta + 1$ :  $\text{comp}_{S,\gamma}(D) = \{A \subseteq Y : A \text{ belongs to } \text{comp}_{S,\beta}(D) \text{ or include the intersection of some } S\text{-sequence of members of } \text{comp}_{S,\beta}(D), \text{ i.e. } \cap\{A_s : s \in S\}, \text{ where } \langle A_s : s \in S \rangle \text{ is a sequence of members of } \text{comp}_{S,\beta}(D)\}$ .

2) Similarly for a family  $\mathcal{S}$  of sets replacing  $S$  by “some member of  $\mathcal{S}$ ”, e.g. we define  $\text{com}_{\in \mathcal{S},\gamma}(D)$  by induction on  $\gamma$  using  $(\in \mathcal{S})$ -sequences, i.e.  $S$ -sequence for some  $S \in \mathcal{S}$ .

3) If  $\gamma = \infty$  we may omit it. We say that  $D$  is a pseudo  $(\leq S, \gamma)$ -complete when  $\emptyset \notin \text{comp}_{S,\gamma}(D)$ .

{z0.55}

**Observation 1.14.** 1) If  $D$  is a filter on  $I$  and  $S$  is a set, then:

(a)  $\langle \text{comp}_{S,\gamma}(D) : \gamma \in \text{Ord} \cup \{\infty\} \rangle$  is an  $\subseteq$ -increasing sequence of filters of  $I$  (starting with  $D$ )

(b) if  $\text{comp}_{S,\gamma+1}(D) = \text{comp}_{S,\gamma}(D)$  then for every  $\beta \geq \gamma$  we have  $\text{comp}_{S,\beta}(D) = \text{comp}_{S,\gamma}(D)$

(c) there is an ordinal  $\gamma = \gamma_S(D) < \theta(\mathcal{P}(I))$  such that  $\text{comp}_{S,\gamma}(D) = \text{comp}_{S,\gamma+1}(D)$  and  $\langle \text{comp}_{S,\beta}(D) : \beta \leq \gamma \rangle$  is strictly  $\subset$ -increasing.

2) Assume  $\text{AC}_S$ . Then for any filter  $D$  on  $I$  we have  $\gamma_S(D) \leq \theta$  when  $\theta := \{\min(\lambda) : \lambda \text{ a cardinality such that } \text{cf}(\lambda) \geq \theta(S)\}$  or  $\theta = \theta(\omega^> S)$ .

3) Assume  $DC + AC_S + |S \times S| = |S|$ . Then for any filter  $D$  on  $I$  we have  $\gamma_S(D) \leq 1$  and  $\text{com}_{S,1}(D)$  is an  $(\leq S)$ -complete filter or is  $\mathcal{P}(I)$ ; the latter holds iff  $D$  is not pseudo  $(\leq S)$ -complete.

4) Similarly for “ $\in \mathcal{S}$ ” but  $AC_S$  is replaced by  $S \in \mathcal{S} \Rightarrow AC_{\mathcal{S}}$  and in part (3),  $\theta = \min\{\kappa : \kappa \text{ regular and } S \in \mathcal{S} \Rightarrow \kappa \geq \theta(\mathcal{S})\}$  or  $\theta = \sup\{\theta(\Lambda) : \Lambda \in \Xi\}$  where  $\Xi = \{\Lambda : \Lambda \text{ is a set of sequences of length } < \omega \text{ with no } \omega\text{-branch, closed under initial segments such that if } \eta \in \Lambda \text{ is not } \triangleleft\text{-maximal then } (\exists S \in \mathcal{S})(\forall \alpha)(\eta \hat{\ } \langle s \rangle \in \Lambda \Leftrightarrow s \in S)\}$ .

*Proof.* We prove the versions with  $\mathcal{S}$ , i.e. for (4). Let  $D_\gamma = \text{comp}_{\in \mathcal{S}, \gamma}(D)$  for  $\gamma \in \text{Ord}$ .

1) Clause (a) is by the definition; clause (b) is proved by induction in  $\beta \geq \gamma$ , for  $\beta = \gamma$  this is trivial, for  $\beta = \gamma + 1$  use the assumption and for  $\gamma = \beta$  use the definition and the induction hypothesis. As for clause (c) let  $\gamma_* = \min\{\gamma \in \text{Ord} \cup \{0\}; \text{ if } \gamma < \infty \text{ then } D_\gamma = D_{\gamma+1}, \text{ so } \langle D_\gamma : \gamma \leq \gamma_* \rangle \text{ is } \subset\text{-increasing continuous by clause (b), } \langle D_\gamma : \gamma \leq \gamma_* \rangle \text{ is constant and define } h : \mathcal{P}(I) \rightarrow \gamma_* \text{ by: } A \in D_{\gamma+1} \setminus D_\gamma \Rightarrow h(A) = \gamma \text{ and } h(A) = 0 \text{ otherwise there is no such } \gamma. \text{ So } h \text{ is onto } \gamma_* \text{ hence } \gamma_* < \theta(\mathcal{P}(A)) \text{ so } \gamma_* \text{ is as required on } \gamma_S(D)\}$ .

2) Let  $\gamma$  be an ordinal.

Let

$$\mathcal{T}_n^1 = \{\Lambda : \begin{array}{l} \Lambda \text{ is a set of sequences of length } \leq n, \\ \text{closed under initial segments such that for every non-maximal } \eta \in \Lambda \\ \text{for some } S \in \mathcal{S} \text{ we have} \\ \eta \hat{\ } \langle s \rangle \in \Lambda \Leftrightarrow s \in S \}. \end{array}$$

$$\mathcal{T}_{\gamma,n}^2 = \{\mathbf{x} : \begin{array}{l} \text{(a) } \mathbf{x} \text{ has the form } \langle I_\eta, \gamma_\eta : \eta \in \Lambda \rangle \\ \text{(b) } \Lambda \in \mathcal{T}_n^1 \text{ and } I_\eta \subseteq I \text{ for } \eta \in \Lambda \\ \text{(c) } I_\eta = \cap \{I_{\eta \hat{\ } \langle s \rangle} : s \text{ satisfies } \eta \hat{\ } \langle s \rangle \in \Lambda\} \text{ if } \eta \in \Lambda \\ \text{but } \eta \text{ is not } \triangleleft\text{-maximal in } \Lambda \\ \text{(d) } \eta \triangleleft \nu \in \Lambda \Rightarrow \gamma_\nu < \gamma_\eta < 1 + \gamma \\ \text{(e) } I_\eta \in D \text{ if } \eta \in \Lambda \text{ is } \triangleleft\text{-maximal in } \Lambda \\ \text{but } \ell g(\eta) < n \} \end{array}$$

$$\mathcal{T}_n^2 = \cup \{\mathcal{T}_{\gamma,n}^2 : \gamma \text{ is an ordinal}\}.$$

Let  $\mathbf{n}(\mathbf{x}) = n$  for the minimal possible  $n$  such that  $\mathbf{x} \in \mathcal{T}_n^2$  and let  $\mathbf{x} = \langle I_\eta^{\mathbf{x}}, \gamma_\eta^{\mathbf{x}} : \eta \in \Lambda_{\mathbf{x}} \rangle$ .

Let  $\mathcal{T}_\gamma^3 = \cup \{\mathcal{T}_{\gamma,n}^2 : n < \omega\}$  and let  $<_*$  be the natural order on  $\mathcal{T}_\gamma^3 : \mathbf{x} <_* \mathbf{y}$  iff  $\mathbf{n}(\mathbf{x}) < \mathbf{n}(\mathbf{y}), \Lambda_{\mathbf{x}} = \Lambda_{\mathbf{y}} \cap \mathbf{n}(\mathbf{x}) \geq (\cup \{S : S \in \mathcal{S}\})$  and  $(I_\eta^{\mathbf{x}}, \gamma_\eta^{\mathbf{x}}) = (I_\eta^{\mathbf{y}}, \gamma_\eta^{\mathbf{y}})$  for  $\eta \in \Lambda_{\mathbf{x}}$ .

Now

$$\textcircled{*} A \in D_\gamma \text{ iff there is an } \omega\text{-branch } \langle \mathbf{x}_n : n < \omega \rangle \text{ of } (\mathcal{T}_\gamma^3, <_*) \text{ such that } I_{\langle \rangle}^{\mathbf{x}_n} = A_*.$$

[Why? We prove it by induction on the ordinal  $\gamma$ . For  $\gamma = 0$  and  $\gamma$  limit this is obvious so assume we have it for  $\gamma$  and we shall prove it for  $\gamma + 1$ .

First assume  $A \in D_{\gamma+1}$  and we shall find such  $\omega$ -branch; if  $A \in D_\gamma$  this is obvious, otherwise there are  $S \in \mathcal{S}$  and a sequence  $\langle A_s : s \in S \rangle$  of members of  $D_\gamma$  such that  $A = \cap \{A_s : s \in S\}$ . So  $X_s := \{\bar{\mathbf{x}} : \bar{\mathbf{x}} \text{ witness } A_s \in D_\gamma\}$  is well defined and non-empty by the induction hypothesis, clearly the sequence  $\langle X_s : s \in S \rangle$  exists, because we can use  $AC_S$  to choose  $\langle \bar{\mathbf{x}}_s : s \in S \rangle$ .

Now define  $\bar{\mathbf{x}} = \langle \mathbf{x}_n : n < \omega \rangle$  as follows:  $\Lambda_{\mathbf{x}_n} = \{\langle \rangle\} \cup \{\langle s \rangle \hat{\ } \eta : \eta \in \Lambda_{\mathbf{x}_s, n-1} \text{ and } s \in S\}$ ,  $\gamma_{\langle \rangle}^{\mathbf{x}_n} = \cup\{\gamma_{\langle s \rangle}^{\mathbf{x}_s, n} + 1 : s \in S\}$  and  $A_{\langle \rangle}^{\mathbf{x}_n} = A$  and  $A_{\langle s \rangle}^{\mathbf{x}_n} = A_{\langle s \rangle}^{\mathbf{x}_s, n-1}$ . Now check.

Second, assume that there is such  $\omega$ -branch  $\langle \mathbf{x}_n : n < \omega \rangle$  of  $(\mathcal{T}_\gamma^3, <_*)$  such that  $I_{\langle \rangle}^{\mathbf{x}_n} = A$ . Let  $S' = \{\eta(0) : \eta \in \Lambda_{\mathbf{x}_1}\}$  so necessarily  $S \in \mathcal{S}$ . For each  $n < \omega$  and  $s \in S$  we define  $\mathbf{y}_{n,s}$  as follows  $\Lambda_s^{\mathbf{y}_{n,s}} = \{\nu : \langle s \rangle \hat{\ } \nu \in \Lambda_{\mathbf{x}_{n+1}}\}$  and for  $\nu \in \Lambda_s^n$  let  $\gamma_\nu^{\mathbf{y}_{n,s}} = \nu_{\langle s \rangle}^{\mathbf{x}_{n+1}}$  and  $A_\nu^{\mathbf{y}_{n,s}} = A_{\langle s \rangle}^{\mathbf{x}_{n+1}}$ . Now clearly  $\langle \mathbf{y}_{n,s} : n < \omega \rangle$  is an  $\omega$ -branch of  $(\mathcal{T}_\gamma^3, \leq_*)$  so by the induction hypothesis  $A_{\langle s \rangle}^{\mathbf{x}_1} \in D$ ,  $\text{comp}_{S, \gamma}(D)$  and  $A_{\langle \rangle}^{\mathbf{x}_0} = A = \cap\{A_{\langle s \rangle}^{\mathbf{x}_s} : \langle \rangle \in \Lambda_{\mathbf{x}_1}\} \in \text{comp}_{S, \gamma+1}(D)$ . So we are done.]

Now toward a contradiction assume that  $\gamma_S(D) > \theta$ , so there is  $A \in D_{\theta+1} \setminus D_\theta$  hence here is an  $\omega$ -branch  $\langle \mathbf{x}_n : n < \omega \rangle$  of  $\mathcal{T}_\gamma^3$  witnessing that  $A \in D_{\theta+1}$ , let  $\Lambda = \cup\{\Lambda_{\mathbf{x}_n} : n < \omega\}$  and  $\gamma_\eta = \gamma_\eta^{\mathbf{x}_n}$  for every  $n < \omega$  large enough. So  $\Lambda$  is well founded (recalling  $\eta \triangleleft \nu \in \Lambda \Rightarrow \gamma_\eta > \gamma_\nu$ ) and we can choose  $\langle \eta' : \eta \in \Lambda \rangle$  such that  $\gamma_{\eta'} = \sup\{\gamma_\nu + 1 : \eta \triangleleft \nu \in \Lambda \text{ and } \ell g(\nu) = \ell g(\eta) + 1\}$ . If  $\gamma_{\langle \rangle} < \theta$  we are done otherwise let  $\eta \in \Lambda$  be  $\triangleleft$ -maximal such that  $\gamma_{\eta'} \geq \theta$  hence  $\eta \triangleleft \nu \Rightarrow \gamma_\nu < \theta$ , so necessarily  $\gamma_{\eta'} = \theta = \cup\{\gamma_\nu + 1 : \eta \triangleleft \nu \in \Lambda, \ell g(\nu) = \ell g(\eta) + 1\}$ . Let  $S \in \mathcal{S}$  be such that  $\eta \hat{\ } \langle s \rangle \in \Lambda \Leftrightarrow s \in S$ , so  $\{\gamma_{\eta'} \hat{\ } \langle s \rangle : s \in S\}$  is an unbounded subset of  $\theta$  so  $\text{cf}(\theta) \leq \theta(S) < \theta$ . This takes care of the first possibility for  $\theta$  so the second case is easier.

3) It suffices to show that we can replace  $\mathbf{x} \in \mathcal{T}_2^2$  by  $\mathbf{x} \in \mathcal{T}_1^2$ . For part (3), i.e. having one  $S$  this is obvious, for part (4) we use:

(\*) if  $S \in \mathcal{S}$  and  $\bar{\Lambda} = \langle \Lambda_s : s \in S \rangle, \Lambda_s \in \Xi$  then for some  $\Lambda' \in \Xi$  we have  $|\{(s, t) : s \in S, t \in \Lambda_s\}| \leq |\Lambda'|$ .

□<sub>1.14</sub>

{z0.59}

**Definition 1.15.** 1) For a filter  $D$  on a set  $I$  and a set  $S$  let  $\gamma_S(D)$  be as in clause (c) of the Observation 1.14(1).

1A) Similarly with “ $\in \mathcal{S}$ ” instead  $S$ .

2)  $D$  is pseudo  $(S, \gamma)$ -complete if  $\emptyset \notin \text{comp}_{S, \gamma}(D)$ .

2A) Similarly with “ $\in \mathcal{S}$ ” instead  $S$ .

{z0.79}

**Observation 1.16.** 1) If  $h$  is a function from  $S_1$  onto  $S_2$  then  $\theta(S_1) \geq \theta(S_2)$  and every [pseudo]  $(\leq S_1)$ -complete filter is a [pseudo]  $(\leq S_2)$ -complete filter.

## 2. COMMUTING RANKS

The aim of this section is to sort out when two rank  $\text{rk}_{D_1}, \text{rk}_{D_2}$  do so called commute.

{c3.1}

**Definition 2.1.** Assume that  $D_\ell$  is an  $\aleph_1$ -complete filter on  $I_\ell$  for  $\ell = 1, 2$ . For  $\iota \in \{1, 2, 3, 4, 5\}$  we say  $D_2$  does  $\iota$ -commute with  $D_1$  when:  $\boxplus_\iota = \boxplus_{D_1, D_2}^\iota$  holds where:

⊞<sub>1</sub> if  $A \in D_1$  and  $\bar{B} = \langle B_s : s \in A \rangle \in {}^A(D_2)$  then we can find  $A_*, B_*$  such that:  $A_* \in D_1, B_* \in D_2$  and  $A_* \times B_* \subseteq \cup\{\{s\} \times B_s : s \in A\}$  so  $A_* \subseteq A$

⊞<sub>2</sub> if  $A \in D_1$  and  $\bar{B} = \langle B_s : s \in A \rangle \in {}^A(D_2)$  and  $J_2 = J[f_2, D_2]$  for some  $f_2 \in {}^{I_2}\text{Ord}$  then we can find  $A_*, B_*$  such that  $A_* \in D_1, B_* \in J_2^+$  and  $A_* \times B_* \subseteq \cup\{\{s\} \times B_s : s \in A\}$  so  $A_* \subseteq A$

- $\boxplus_3$  if  $A \in D_1$  and  $\bar{B} = \langle B_s : s \in A \rangle \in {}^A(D_2)$  and  $J_1 = J[f_1, D_1]$  for some  $f_1 \in {}^{I_1}\text{Ord}$  then we can find  $A_*, B_*$  such that  $A_* \in J_1^+, A_* \subseteq A, B_* \in D_2$  and  $s \in A_* \Rightarrow B_* \subseteq B_s$
- $\boxplus_4$  if  $A \in D_1$  and  $\bar{B} = \langle B_s : s \in A \rangle \in {}^A(D_2)$  and  $\bar{J}^1 = \langle J_t^1 : t \in I_2 \rangle$  satisfies  $J_t^1 \in \{J[f, D_1] : f \in {}^{I_1}\text{Ord}\}$  and  $J_2 \in \{J[f, D_2] : f \in {}^{I_2}\text{Ord}\}$  then we can find  $A_*, B_*$  such that  $B_* \in J_2^+$  and  $t \in B_* \Rightarrow A_* \in (J_t^1)^+$  and  $(s, t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$  hence  $A_* \subseteq A, A_* \in D_1^+$
- $\boxplus_5$  like  $\boxplus_4$  but we omit the sequence  $\bar{J}^1$  and the demand on  $A_*$  is  $A_* \in D_1^+$ .

{c3.2}

*Remark 2.2.* 1) This is seemingly not a commutative relation.

2) We shall first give a consequence and then give sufficient conditions.

3) We intend to generalize to systems (see 3.1 and 3.8).

4) Can we below use “ $D_\ell \in \text{FIL}_{\text{pcc}}(I_1)$ ”, see Definition 1.12? Yes, but only when we do not use  $D + A, A \in D^+$ .

{c3.3}

**Claim 2.3.**  $\text{rk}_{D_1}(f) \leq \text{rk}_{D_2}(g)$  when:

- $\oplus$  (a)  $D_\ell \in \text{FIL}_{\text{cc}}(I_\ell)$  for  $\ell = 1, 2$   
 (b)  $\bar{g} = \langle g_t : t \in I_2 \rangle$   
 (c)  $g_t \in {}^{I_1}\text{Ord}$   
 (d)  $g \in {}^{I_2}\text{Ord}$  is defined by  $g(t) = \text{rk}_{D_1}(g_t)$   
 (e)  $\bar{f} = \langle f_s : s \in I_1 \rangle$   
 (f)  $f_s \in {}^{I_2}\text{Ord}$  is defined by  $f_s(t) = g_t(s)$   
 (g)  $f \in {}^{I_1}\text{Ord}$  is defined by  $f(s) = \text{rk}_{D_2}(f_s)$
- $\boxplus$  (a)  $D_2$  does 2-commute with  $D_1$   
 (b)  $AC_{I_1}$  holds.

*Remark 2.4.* In order not to use DC in the proof we should consider  $\infty$  as a member of Ord.

*Proof.* We prove by induction on the ordinal  $\zeta$  that

$\square_\zeta$  if  $\oplus + \boxplus$  above holds for  $D_1, D_2, f, g, \bar{f}, \bar{g}$  and  $\text{rk}_{D_1}(f) \geq \zeta$  then  $\text{rk}_{D_2}(g) \geq \zeta$ .

The case  $\zeta = 0$  is trivial and the case  $\zeta$  a limit ordinal follows by the induction hypothesis. So assume that  $\zeta = \xi + 1$ .

Let

(\*)<sub>1</sub>  $A := \{s \in I_1 : f(s) > 0\}$ .

As we are assuming  $\text{rk}_{D_1}(f) > \xi \geq 0$  necessarily

(\*)<sub>2</sub>  $A \in D_1$ .

For each  $s \in A, f(s) > 0$  so applying clause (g) of  $\oplus$  we get

(\*)<sub>3</sub>  $\text{rk}_{D_2}(f_s) > 0$  when  $s \in A$

hence

(\*)<sub>4</sub>  $B_s := \{t \in I_2 : f_s(t) > 0\}$  belongs to  $D_2$  when  $s \in A$ .

So  $\langle B_s : s \in A \rangle \in {}^A(D_2)$ . Recall (see  $\boxplus(a)$  of the assumption) that  $D_2$  does 2-commute with  $D_1$ , apply it to  $A, \langle B_s : s \in A \rangle, J_2 = J[g, D_2]$ ; so we can find  $A_*, B_*$  such that

- (\*)<sub>5</sub> (a)  $A_* \in D_1$  (and  $A_* \subseteq A$ )
- (b)  $B_* \in J_2^+$  recalling  $J_2 = J[g, D_2]$  so  $B_* \in D_2^+$  and (by Definition 1.10)

$$\text{rk}_{D_2+B_*}(g) = \text{rk}_{D_2}(g)$$

- (c)  $(s, t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$ .

Now by the present assumption of  $\boxminus_\zeta$  we have

- (\*)<sub>6</sub>  $\text{rk}_{D_1}(f) \geq \zeta = \xi + 1$ .

Hence by the definition of  $\text{rk}$  we can find  $f'$  such that

- (\*)<sub>7</sub> (a)  $f' \in {}^{I_1}\text{Ord}$  and  $\text{rk}_{D_1}(f') \geq \xi$
- (b)  $f' <_{D_1} f$  so
- (c) by (\*)<sub>1</sub> without loss of generality  $s \in A \Rightarrow f'(s) < f(s)$ .

For each  $s \in A$ , clearly  $f'(s) < f(s) = \text{rk}_{D_2}(f_s) \leq \text{rk}_{D_2+B_*}(f_s)$ , recalling 1.7(2), clause (g) of  $\oplus$  and  $D_2 \subseteq D_2 + B_*$ , hence by 1.8(2) for each  $s \in I_1$  there is a function  $f'_s$  such that

- (\*)<sub>8</sub> (a)  $f'_s \in ({}^{I_2})\text{Ord}$ ,
- (b)  $f'_s < f_s \text{ mod } D_2$  if  $s \in A$
- (c)  $\text{rk}_{D_2+B_*}(f'_s) = f'(s)$ ; may require this only for  $s \in A$ .

As  $I_1 \in \mathbf{C}$  by  $\boxplus(b)$  of the assumption, clearly

- (\*)<sub>8</sub><sup>+</sup> there is such a sequence  $\bar{f}' = \langle f'_s : s \in I_1 \rangle$ .

As  $s \in A_* \wedge t \in B_* \Rightarrow f_s(t) > 0$ , see (\*)<sub>4</sub> + (\*)<sub>5</sub>, without loss of generality

- (\*)<sub>9</sub> if  $s \in A_*$  and  $t \in B_*$  then  $f'_s(t) < f_s(t)$ .

We now define  $\bar{g}' = \langle g'_t : t \in I_2 \rangle$  by

- (\*)<sub>10</sub>  $g'_t(s) = f'_s(t)$  for  $s \in I_1, t \in I_2$  so  $g'_t \in {}^{I_1}\text{Ord}$ .

So

- (\*)<sub>11</sub>  $s \in A_* \wedge t \in B_* \Rightarrow g'_t(s) < f_s(t) = g_t(s)$

hence (recalling  $A_* \in D_1$  by (\*)<sub>5</sub>(a) and 1.8(3))

- (\*)<sub>12</sub> if  $t \in B_*$  then  $g'_t <_{D_1} g_t$  hence  $\text{rk}_{D_1}(g'_t) < \text{rk}_{D_1}(g_t)$ .

Define  $g' \in ({}^{I_2})\text{Ord}$  by  $g'(t) := \text{rk}_{D_1}(g'_t)$  hence (recalling  $\text{rk}_{D_1}(g_t) = g(t)$ )

- (\*)<sub>13</sub>  $g' < g \text{ mod } D_2 + B_*$ .

Note that here  $D_1 + A_* = D_1$ , (though not so when we shall prove 2.9).

Now we apply the induction hypothesis to  $g', f', \bar{g}' := \langle g'_t : t \in I_2 \rangle, \bar{f}' := \langle f'_s : s \in I_1 \rangle, D_1 + A_*, D_2 + B_*$  and  $\xi$  and get

- (\*)<sub>14</sub>  $\xi \leq \text{rk}_{D_2+B_*}(g')$ .

[Why is this legitimate? First, we have to check that clauses (a)-(g) of  $\oplus$  hold in this instance.

Clause (a): First “ $D_1 + A_* \in \text{FIL}_{\text{cc}}(I_1)$ ” as we assume  $D_1 \in \text{FIL}_{\text{cc}}(I_1)$  and  $A_* \in D_1$ , see  $(*)_5(a)$ , actually  $A_* \in D_1^+$  suffice (used in proving ??).

Second, “ $D_2 + B_* \in \text{FIL}_{\text{cc}}(I_2)$ ” as  $D_2 \in \text{FIL}_{\text{cc}}(I_2)$  and  $B_* \in D_2^+$  by  $(*)_5(b)$ .

Clause (b): “ $\bar{g}' = \langle g'_t : t \in I_2 \rangle$ ” by our choice.

Clause (c): “ $g'_t \in {}^{I_1}\text{Ord}$ ” by  $(*)_{10}$ .

Clause (d): “ $g' \in {}^{I_2}\text{Ord}$  is defined by  $g'(t) = \text{rk}_{D_1}(g'_t)$ ” by its choice after  $(*)_{12}$ .

Clause (e): “ $\bar{f}' = \langle f'_s : s \in I \rangle$ ” by our choice in  $(*)_8^+$ .

Clause (f): “ $f'_s \in {}^{I_2}\text{Ord}$  is defined by  $f'_s(t) = g'_t(s)$  holds by  $(*)_{10}$ .

Clause (g): “ $f' \in {}^{I_1}\text{Ord}$  is defined by  $f'(s) = \text{rk}_{D_2+B_*}(f'_s)$ ” holds by  $(*)_7(a) + (*)_8(c)$ .

Now  $\square_\xi$ , the induction hypothesis, assumes “ $\text{rk}_{D_1+A_*}(f') \geq \xi$ ” which holds by  $(*)_7(a) + (*)_5(a)$ , actually  $A_* \in D_1^+$  suffice here and its conclusion is  $\xi \leq \text{rk}_{D_2+B_*}(g')$  as promised in  $(*)_{14}$ .]

Next

$$(*)_{15} \quad \xi < \text{rk}_{D_2}(g).$$

[Why?

- <sub>1</sub>  $\xi \leq \text{rk}_{D_2+B_*}(g')$  by  $(*)_{14}$
- <sub>2</sub>  $\text{rk}_{D_2+B_*}(g') < \text{rk}_{D_2+B_*}(g)$  by  $(*)_{13}$  and 1.8(3)
- <sub>3</sub>  $\text{rk}_{D_2+B_*}(g) = \text{rk}_{D_2}(g)$  by  $(*)_5(b)$ .

Together  $(*)_{15}$  holds.]

So

$$(*)_{16} \quad \zeta = \xi + 1 \leq \text{rk}_{D_2}(g)$$

as promised. Together we are done.  $\square_{2.3}$

{c3.11}

**Claim 2.5.** Assume  $D_\ell \in \text{FIL}_{\text{cc}}(I_\ell)$  for  $\ell = 1, 2$ .

If  $D_2$  does  $\iota_1$ -commute with  $D_1$  then  $D_2$  does  $\iota_2$ -commute with  $D_1$  when  $(\iota_1, \iota_2) = (1, 2), (1, 3), (4, 5)$ .

*Proof.* Obvious for (4,5) use 1.11(3).  $\square_{2.5}$

{c3.8}

**Claim 2.6.** Assume  $D_\ell \in \text{FIL}_{\text{cc}}(I_\ell)$  for  $\ell = 1, 2$ . If at least one of the following cases occurs, then  $D_2$  does 1-commute (hence 2-commute) with  $D_1$ .

Case 1:  $D_2$  is  $|I_1|^+$ -complete.

Case 2:  $D_1$  is an ultrafilter which is  $|I_2|^+$ -complete

Case 3:  $D_1, D_2$  are ultrafilters and if  $\bar{A} = \langle A_t : t \in I_2 \rangle \in {}^{I_2}(D_1)$  then for some  $A_* \in D_1$  we have  $\{t : A_t \supseteq A_*\} \in D_2$ .

*Proof.* So let  $A \in D_1$  and  $\langle B_s : s \in A \rangle \in {}^A(D_2)$  be given.

Case 1: Let  $A_* = A$  and  $B_* = \cap\{B_s : s \in A\}$ , so  $A_* \in D_1$  by an assumption and  $B_* \in D_2$  as we assume  $\{B_s : s \in A\} \subseteq D_2$  and  $D_2$  is  $|I_1|^+$ -complete (and necessarily  $|A| \leq |I_1|$ ).

Case 2: For each  $t \in I_2$  let  $A'_t := \{s \in I_1 : s \in A \text{ and } t \in B_s\}$  and let  $A''_t$  be the unique member of  $\{A'_t, I_1 \setminus A'_t\} \cap D_1$ , recalling  $D_1$  is an ultrafilter on  $I_1$ . Clearly the functions  $t \mapsto A'_t$  and  $t \mapsto A''_t$  are well defined hence the sequences  $\langle A'_t : t \in I_2 \rangle, \langle A''_t : t \in I_2 \rangle$  exist and  $\{A''_t : t \in I_2\} \subseteq D_1$ .

As  $D_1$  is  $|I_2|^+$ -complete necessarily  $A_* := \cap\{A''_t : t \in I_2\}$  belongs to  $D_1$ , and clearly  $A_* \subseteq A$ . Let  $B_* = \{t \in I_2 : A''_t = A'_t\}$ .

So now choose any  $s_* \in A_*$  (possible as  $A_* \in D_1$  implies  $A_* \neq \emptyset$ ) so  $B_{s_*} \in D_2$  and  $t \in B_{s_*} \Rightarrow s_* \in A'_t \Rightarrow s_* \in A'_t \cap A_* \Rightarrow A'_t \cap A_* \neq \emptyset \Rightarrow A''_t = A'_t \Rightarrow t \in B_*$  so  $B_{s_*} \subseteq B_*$ . So  $A_*, B_*$  are as required.

Case 3:

Like Case 2. □<sub>2.6</sub>

{c3.10}

**Claim 2.7.** 1) Assume  $D_1 \in \text{FIL}_{cc}(I_1)$  and  $D_2 \in \text{FIL}_{cc}(I_2)$ .

Then  $D_2$  does 3-commute with  $D_1$  when  $D_1$  is  $(\leq \mathcal{P}(I_2))$ -complete.

2) In part (1) if  $E \subseteq D_2$  is  $(D_2, \subseteq)$ -cofinal, it suffices to assume  $D_1$  is  $(\leq E)$ -complete.

*Remark 2.8.* For part (1) in the definition of  $(\leq \mathcal{P}(I_2))$ -complete we can use just partitions, but not in part (2).

*Proof.* 1) So let  $A \in D_1$  and  $\bar{B} = \langle B_s : s \in A \rangle \in {}^A(D_2)$  and  $J_1 = J[f_1, D_1]$  for some  $f_1 \in {}^I\text{Ord}$  be given. So  $s \mapsto B_s$  is a function from  $A \in D_1$  to  $D_2 \subseteq \mathcal{P}(I_2)$  hence recalling that by 1.11(1) the ideal  $J_1$  on  $I_1$  is  $(\leq \mathcal{P}(I_2))$ -complete, there is  $B_* \in D_2$  such that  $A_* := \{s \in A : B_s = B_*\} \in J_1^+$ . Clearly  $A_*, B_*$  are as required. 2) For  $B \in E$  let  $A_B^* = \{s \in A : B \subseteq B_s\}$ , so clearly  $\langle A_B^* : B \in E \rangle$  is a sequence of subsets of  $A \in D_1$  with union  $A$ , so for some  $B_* \in E$  the set  $A_* := \{s \in A : B_* \subseteq B_s\}$  belongs to  $J_1^+$ , so we are done. □<sub>2.7</sub>

{c3.14}

**Claim 2.9.**  $\text{rk}_{D_1}(f) \leq \text{rk}_{D_2}(g)$  when:

⊕ as in 2.3

but we replace clause (⊕) there by

- ⊕' (a)  $D_2$  does 4-commute with  $D_1$
- (b)  $\text{AC}_{I_1}$  holds.

*Proof.* We repeat the proof of 2.3 but:

First change: we replace  $(*)_5$  and the paragraph before it by the following:

So  $\bar{B} = \langle B_s : s \in A \rangle \in {}^A(D_2)$ .

Recall that  $D_2$  does 4-commute with  $D_1$ , apply this to  $A, \langle B_s : s \in A \rangle, \bar{J}^1 = \langle J_t^1 : t \in I_2 \rangle$  where  $J_t^1 := J[g_t, D_1], J_2 := J[g, D_2]$  and we get  $A_*, B_*$  such that:

- $(*)_5'$  (a)  $A_* \in D_1^+$  and  $A_* \subseteq A$
- (b)  $B_* \in J_2^+$  hence  $B_* \in D_2^+$  and  $\text{rk}_{D_2+B_*}(g) = \text{rk}_{D_2}(g)$
- (c)  $(s, t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$

(d) if  $t \in B_*$  then  $A_* \in (J_t^1)^+$  hence

$$t \in B_* \Rightarrow \text{rk}_{D_1+A_*}(g_t) = \text{rk}_{D_1}(g_t) = g(t).$$

Second change: we replace  $(*)_{12}$  and the line before, the line after it and  $(*)_{13}$  by:

Define  $g' \in {}^I_2\text{Ord}$  by  $g'(t) = \text{rk}_{D_1+A_*}(g'_t)$ .

Now

$(*)'_{12}$  if  $t \in B_*$  then

(a)  $g'_t <_{D_1+A_*} g_t$ , by  $(*)_{11}$

(b)  $\text{rk}_{D_1+A_*}(g'_t) < \text{rk}_{D_2+A_*}(g_t)$  by (a) and 1.8(3),

(c)  $\text{rk}_{D_1+A_*}(g_t) = \text{rk}_{D_1}(g_t)$  recalling  $(*)'_5(d)$  and  $J_t^1 = J[g_t, D_1]$  hence

(d)  $\text{rk}_{D_1}(g_t) = g(t)$  by clause (d) of  $\oplus$

(e)  $\text{rk}_{D_1+A_*}(g_t) = g(t)$  by (c), (d) above hence

(f)  $g'(t) < g(t)$  by the choice of  $g'$ , clause (b) and clause (e).

Hence by  $(*)'_{12}(f)$  we have

$(*)'_{13} \ g' < g \text{ mod } D_2 + B_*$ .

Concerning the rest, we quote  $(*)_5(b)$  twice but  $(*)'_5(b) = (*)_5(b)$ , and quote  $(*)_5(a)$  twice but noted there that  $(*)'_5(a)$  suffice and  $g'$  is defined before  $(*)'_{12}$  rather than apply  $(*)_{12}$ .  $\square_{2.9}$

### 3. RANK SYSTEMS AND A RELATIVE OF GCH

To phrase our theorem we need to define the framework.

**Definition 3.1.** Main Definition: We say that  $\mathbf{p} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_{\mathbf{p}}, \text{rk}_{\mathbf{p}}, \Sigma_{\mathbf{p}}, \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{p}})$   $\{m4.3\}$  is a weak (rank) 1-system when:

(a)  $\mu$  is singular

(b) each  $\mathbf{d} \in \mathbb{D}$  is (or just we can compute from it) a pair  $(I, D) = (I_{\mathbf{d}}, D_{\mathbf{d}}) = (I[\mathbf{d}], D_{\mathbf{d}}) = (I_{\mathbf{p}, \mathbf{d}}, D_{\mathbf{p}, \mathbf{d}})$  such that:

( $\alpha$ )  $\theta(I_{\mathbf{d}}) < \mu$ , on  $\theta(-)$  see Definition 1.5

( $\beta$ )  $D_{\mathbf{d}}$  is a filter on  $I_{\mathbf{d}}$

(c) for each  $\mathbf{d} \in \mathbb{D}$ , a definition of a function  $\text{rk}_{\mathbf{d}}(-)$  with domain  ${}^I[\mathbf{d}]\text{Ord}$  and range  $\subseteq \text{Ord}$ , that is  $\text{rk}_{\mathbf{p}, \mathbf{d}}(-)$  or  $\text{rk}_{\mathbf{d}}^{\mathbf{p}}(-)$

(d) ( $\alpha$ )  $\Sigma$  is a function with domain  $\mathbb{D}$  such that  $\Sigma(\mathbf{d}) \subseteq \mathbb{D}$

( $\beta$ ) if  $\mathbf{d} \in \mathbb{D}$  and  $\mathbf{e} \in \Sigma(\mathbf{d})$  then  $I_{\mathbf{e}} = I_{\mathbf{d}}$  [natural to add  $D_{\mathbf{d}} \subseteq D_{\mathbf{e}}$ ,

this is not demanded but see 3.8(2)]

(e) ( $\alpha$ )  $\mathbf{j}$  is a function from  $\mathbb{D}$  onto  $\text{cf}(\mu)$

( $\beta$ ) let  $\mathbb{D}_{\geq i} = \{\mathbf{d} \in \mathbb{D} : \mathbf{j}(\mathbf{d}) \geq i\}$  and  $\mathbb{D}_i = \mathbb{D}_{\geq i} \setminus \mathbb{D}_{i+1}$

( $\gamma$ )  $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow \mathbf{j}(\mathbf{e}) \geq \mathbf{j}(\mathbf{d})$

(f) for every  $\sigma < \mu$  for some  $i < \text{cf}(\mu)$ , if  $\mathbf{d} \in \mathbb{D}_{\geq i}$ , then  $\mathbf{d}$  is  $(\mathbf{p}, \leq \sigma)$ -complete where:

- (\*) we say that  $\mathbf{d}$  is  $(\mathbf{p}, \leq X)$ -complete (or  $(\leq X)$ -complete for  $\mathbf{p}$ ) when: if  $f \in {}^{I[\mathbf{d}]}\text{Ord}$  and  $\zeta = \text{rk}_{\mathbf{d}}(f)$  and  $\langle A_j : j \in X \rangle$  a partition<sup>1</sup> of  $I_{\mathbf{d}}$ , then for some  $\mathbf{e} \in \Sigma(\mathbf{d})$  and  $j < \sigma$  we have  $A_j \in D_{\mathbf{e}}$  and  $\zeta = \text{rk}_{\mathbf{e}}(f)$ ; so this is not the same as “ $D_{\mathbf{d}}$  is  $(\leq X)$ -complete”; we define  $(\mathbf{p}, |X|^+)$ -complete, i.e.  $(\mathbf{p}, < |X|^+)$ -complete similarly
- (g) no hole<sup>2</sup>: if  $\text{rk}_{\mathbf{d}}(f) > \zeta$  then for some pair  $(\mathbf{e}, g)$  we have:  $\mathbf{e} \in \Sigma(\mathbf{d})$  and  $g <_{D[\mathbf{e}]} f$  and  $\text{rk}_{\mathbf{e}}(g) = \zeta$
- (h) if  $f = g + 1 \bmod D_{\mathbf{d}}$  then  $\text{rk}_{\mathbf{d}}(f) = \text{rk}_{\mathbf{d}}(g) + 1$
- (i) if  $f \leq g \bmod D_{\mathbf{d}}$  then  $\text{rk}_{\mathbf{d}}(f) \leq \text{rk}_{\mathbf{d}}(g)$ .

{m4.4}

**Definition 3.2.** 1) We say  $\mathbf{p} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu)$  is a weak (rank) 2-system, (if we write system we mean 2-system) when in 3.1 we replace clauses (d),(f),(g) by:

- (d)'  $(\alpha)$   $\Sigma$  is a function with domain  $\mathbb{D}$
- $(\beta)$  for  $\mathbf{d} \in \mathbb{D}$  we have  $\Sigma(\mathbf{d}) \subseteq \{(\mathbf{e}, h) : \mathbf{e} \in \mathbb{D} \geq \mathbf{j}(\mathbf{d}) \text{ and } h : I_{\mathbf{e}} \rightarrow I_{\mathbf{d}}\}$ ; writing  $\mathbf{e} \in \Sigma(\mathbf{d})$  means then  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  for some function  $h$
- (f)' for every  $\sigma < \mu$  for some  $i < \text{cf}(\mu)$ , if  $\mathbf{d} \in \mathbb{D}_{\geq i}$ , then  $\mathbf{d}$  is  $(\mathbf{p}, \leq \sigma)$ -complete where:
- (\*) we say that  $\mathbf{d}$  is  $(\mathbf{p}, \leq X)$ -complete (for  $\mathbf{p}$ ) when: if  $f \in {}^{I[\mathbf{d}]}\text{Ord}$  and  $\zeta = \text{rk}_{\mathbf{d}}(f)$  and  $\langle A_j : j \in X \rangle$  a partition<sup>3</sup> of  $I_{\mathbf{d}}$ , then for some  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  and  $j < \sigma$  we have  $h^{-1}(A_j) \in D_{\mathbf{e}}$  and  $\zeta = \text{rk}_{\mathbf{e}}(f \circ h)$ ; we define “ $(\mathbf{p}, |X|^+)$ -complete” similarly
- (g)' no hole: if  $\text{rk}_{\mathbf{d}}(f) > \zeta$  then for some  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  and  $g \in {}^{I[\mathbf{e}]}\text{Ord}$  we have  $g < f \circ h \bmod D_{\mathbf{e}}$  and  $\text{rk}_{\mathbf{e}}(g) = \zeta$ .

{m4.4d}

**Definition/Claim 3.3.** Let  $\mathbf{p}$  be a weak rank 1-system we can define  $\mathbf{q}$  and prove it is a weak rank 2-system by  $\mathbb{D}_{\mathbf{q}} = \mathbb{D}_{\mathbf{p}}$ ,  $\text{rk}_{\mathbf{q}} = \text{rk}_{\mathbf{p}}$ ,  $\Sigma_{\mathbf{q}}(\mathbf{d}) = \{(\mathbf{e}, \text{id}_{I[\mathbf{d}]}) : \mathbf{e} \in \Sigma_{\mathbf{p}}(\mathbf{d})\}$ ,  $\mathbf{j}_{\mathbf{q}} = \mathbf{j}_{\mathbf{p}}$ ,  $\mu_{\mathbf{q}} = \mu_{\mathbf{p}}$ .

{m4.5}

**Convention 3.4.** 1) We use  $\mathbf{p}$  only for systems as in Definition 3.1 or 3.2.

2) We may not distinguish  $\mathbf{p}$  and  $\mathbf{q}$  in 3.3 so deal only with 2-systems.

{m4.5d}

*Remark 3.5.* The following is an alternative to Definition 3.2. As in 3.1 we can demand  $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow I_{\mathbf{e}} = I_{\mathbf{d}}$  but for every  $\mathbf{d}$  we have a family  $\mathcal{E}_{\mathbf{d}}$ , i.e. the function  $\mathbf{d} \mapsto \mathcal{E}_{\mathbf{d}}$  is part of  $\mathbf{p}$  and make the following additions and changes:

- $(\alpha)$   $\mathcal{E}_{\mathbf{d}}$  is a family of equivalence relations on  $I_{\mathbf{d}}$
- $(\beta)$  we replace  ${}^{I[\mathbf{d}]}\text{Ord}$  by  $\{f \in {}^{I[\mathbf{d}]}\text{Ord} : \text{eq}(f) := \{(s, t) : s, t \in I_{\mathbf{d}} \text{ and } f(s) = f(t)\} \in \mathcal{E}_{\mathbf{d}}\}$
- $(\gamma)$  if  $E_1, E_2$  are equivalence relations on  $I_{\mathbf{d}}$  such that  $E_2$  refines  $E_1$  then  $E_2 \in \mathcal{E}_{\mathbf{d}} \Rightarrow E_1 \in \mathcal{E}_{\mathbf{d}}$
- $(\delta)$  if  $\mathbf{e} \in \Sigma(\mathbf{d})$  then  $I_{\mathbf{e}} = I_{\mathbf{d}}$  and  $\mathcal{E}_{\mathbf{d}} \subseteq \mathcal{E}_{\mathbf{e}}$ .

{m4.6}

**Definition 3.6.** For  $\iota = 1, 2$  we say that  $\mathbf{p} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu)$  is a strict  $\iota$ -system when it satisfies clauses (a)-(i) from 3.1 or from 3.2 and

<sup>1</sup>as long as  $\sigma$  is a well ordered set it does not matter whether we use a partition or just a covering, i.e.  $\cup\{A_j : j \in \sigma\} = I_{\mathbf{d}}$

<sup>2</sup>we may use another function  $\Sigma$  here, as in natural examples here we use  $\Sigma(\mathbf{d}) = \{\mathbf{d}\}$  and not so in clause (f)

<sup>3</sup>as long as  $\sigma$  is a well ordered set it does not matter whether we use a partition or just a covering, i.e.  $\cup\{A_j : j \in \sigma\} = I_{\mathbf{d}}$

(j) for every  $\mathbf{d} \in \mathbb{D}$  and  $\zeta, \xi, f, j_0$  satisfying  $\boxplus$  below, there<sup>4</sup> is  $j < \text{cf}(\mu)$  such that: there are no  $\mathbf{e}, g$  satisfying  $\oplus$  below, where:

- $\oplus$ 
  - <sub>1</sub>  $\mathbf{e} \in \mathbb{D}_{\geq j}$
  - <sub>2</sub>  $g \in {}^{I[\mathbf{e}]}\zeta$
  - <sub>3</sub>  $\{g(t) : t \in I_{\mathbf{e}}\} \subseteq [\xi, \zeta_*)$  for some  $\zeta_* < \zeta$
  - <sub>4</sub>  $\text{rk}_{\mathbf{e}}(f(s)) = f(s)$  for  $s \in I_{\mathbf{d}}$  follows by
  - <sub>5</sub>  $j \geq j_0$
  - <sub>6</sub>  $\text{rk}_{\mathbf{e}}(g) > \zeta$ ,
- $\boxplus$ 
  - <sub>1</sub>  $f \in {}^{I[\mathbf{d}]}\xi$
  - <sub>2</sub>  $\text{rk}_{\mathbf{d}}(f) = \zeta$
  - <sub>3</sub>  $\xi < \zeta$
  - <sub>4</sub>  $\text{cf}(\mu) = \text{cf}(\zeta)$
  - <sub>5</sub>  $j_0 < \text{cf}(\mu)$
  - <sub>6</sub>  $s \in I_{\mathbf{d}} \wedge \mathbf{e}' \in \mathbb{D}_{\geq j_0} \Rightarrow \text{rk}_{\mathbf{e}'}(f(s)) = f(s)$ .

{m4.9d}

**Observation 3.7.** If  $\mathbf{p}$  is a strict  $\iota$ -system then  $\mathbf{p}$  is a weak  $\iota$ -system.

{m4.9}

**Definition 3.8.** 1) We say that a weak  $\iota$ -system  $\mathbf{p}$  is weakly normal when:

- <sub>1</sub> in  $(d)(\beta)$  of Definition 3.1 we add  $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow D_{\mathbf{d}} \subseteq D_{\mathbf{e}}$
- <sub>2</sub> in  $(d)'(\beta)$  of Definition 3.2 we add: if  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  then  $(\forall A \in D_{\mathbf{d}})(h^{-1}(A) \in D_{\mathbf{e}})$ .

2) We say  $\mathbf{p}$  is normal when it is weakly normal and

- <sub>3</sub> in Definition 3.1, if  $A \in D_{\mathbf{d}}^+, \mathbf{d} \in \mathbb{D}, f \in {}^{I[\mathbf{d}]}\text{Ord}$  and  $\zeta = \text{rk}_{\mathbf{d}}(f)$  then for some  $\mathbf{e} \in \Sigma(\mathbf{d})$  we have  $A \in D_{\mathbf{e}}$  and  $\text{rk}_{\mathbf{e}}(f) = \zeta$
- <sub>4</sub> in Definition 3.2, if  $\mathbf{d} \in \mathbb{D}, f \in {}^{I[\mathbf{d}]}\text{Ord}, \text{rk}_{\mathbf{d}}(f) = \zeta$  and  $A \in D_{\mathbf{d}}^+$ , then for some  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  we have  $\{s \in I_{\mathbf{e}} : h(s) \in A\} \in D_{\mathbf{e}}$  and  $\text{rk}_{\mathbf{e}}(f \circ h) = \zeta$ .

{m4.10}

**Claim 3.9.** Assume  $\mathbf{p}$  is a weak  $\iota$ -system and  $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ .

0) If  $\mathbf{p}, \mathbf{q}$  are as in 3.3, then  $\mathbf{p}$  is [weakly] normal iff  $\mathbf{q}$  is.

1) If  $f, g \in {}^{I[\mathbf{d}]}\text{Ord}$  and  $f <_{D_{\mathbf{d}}} g$  then  $\text{rk}_{\mathbf{d}}(f) < \text{rk}_{\mathbf{d}}(g)$ .

2) Assume  $\mathbf{p}$  is weakly normal; if  $f \in {}^{I[\mathbf{d}]}\text{Ord}$  and  $\text{rk}_{\mathbf{d}}(f) > 0$  then  $\{s \in I_{\mathbf{d}} : f(s) > 0\} \in D_{\mathbf{d}}^+$ .

2A) If in addition  $\mathbf{p}$  is normal then  $\{s \in I_{\mathbf{d}} : f(s) = 0\} = \emptyset \text{ mod } D_{\mathbf{d}}$ .

3)  $\text{rk}_{\mathbf{d}}(f)$  depends just on  $f/D_{\mathbf{d}}$  (and  $\mathbf{d}$  and, of course,  $\mathbf{p}$ ).

*Proof.* 0) Easy; note that by this part, below without loss of generality  $\iota_{\mathbf{p}} = 2$ .

1) Let  $f_1 \in {}^{I[\mathbf{d}]}\text{Ord}$  be defined by  $f_1(s) = f(s) + 1$ . So clearly  $f_1 \leq_{D_{\mathbf{d}}} g$  hence by clause (i) of 3.1 (equivalently 3.2) we have  $\text{rk}_{\mathbf{d}}(f_1) \leq \text{rk}_{\mathbf{d}}(g)$ . Also  $f_1 = f + 1 \text{ mod } D_{\mathbf{d}}$  hence by clause (h) of 3.1 we have  $\text{rk}_{\mathbf{d}}(f_1) = \text{rk}_{\mathbf{d}}(f) + 1$ . The last two sentences together give the desired conclusion.

1A) By clause (i) of 3.2 used twice.

2) By clause  $(g)'$  of Definition 3.2, the “no hole”, there is a triple  $(\mathbf{e}, h, g)$  as there, so  $B := \{s : s \in I_{\mathbf{e}} \text{ and } g(s) < f(h(s))\} \in D_{\mathbf{e}}$  hence by the weak normality of  $\mathbf{p}$  we have  $\{h(s) : s \in B\} \neq \emptyset \text{ mod } D_{\mathbf{d}}$  but this set includes  $\{s \in I_{\mathbf{d}} : f(s) > 0\}$ .

<sup>4</sup>can we make  $j$  depend on  $f$  (and a partition of)  $\mathbf{d}$ ? Anyhow later we use  $\mathbf{d}' \in \Sigma(\mathbf{d})$ , if  $\{\mathbf{d}\} \neq \Sigma(\mathbf{d})$ ? Also so  $\iota = 1, 2$  may make a difference.

2A) Similarly.

3) Use clause (i) of Definition 3.1 twice. □<sub>3.9</sub>

{m4.12}

**Theorem 3.10.** [ZF] Assume that  $\mathbf{p} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu)$  is a strict rank 1-system (see Main Definition 3.1) or just a strict 2-system. Then for every ordinal  $\zeta$  there is  $i < \text{cf}(\mu)$  such that: if  $\mathbf{d} \in \mathbb{D}_{\geq i}$  then  $\text{rk}_{\mathbf{d}}(\zeta) = \zeta$ , i.e.  $\text{rk}_{\mathbf{d}}(\langle \zeta : s \in I_{\mathbf{d}} \rangle) = \zeta$ .

*Proof.* We shall use the notation:

○<sub>0</sub> If there is an  $i$  as required in the theorem for the ordinal  $\zeta$  then let  $\mathbf{i}(\zeta)$  be the minimal such  $i$  (otherwise,  $\mathbf{i}(\zeta)$  is not well defined).

Without loss of generality,

○<sub>1</sub> every  $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$  is  $(\mathbf{p}, \leq (\text{cf}(\mu)))$ -complete, i.e. clause (f) of 3.1 for  $\sigma_* := \text{cf}(\mu)^+$  holds for every  $\mathbf{d} \in \mathbb{D}$ .

[Why? Let  $i_*$  be the  $i < \text{cf}(\mu)$  which exists by clause (f) for  $\sigma_*$ . Now we just replace  $\mathbb{D}$  by  $\mathbb{D}_{\geq i_*}$  (and  $\mathbf{j}$  by  $\mathbf{j}|_{\mathbb{D}_{\geq i_*}}$ , etc).]

Clearly we have

○<sub>2</sub>  $\text{rk}_{\mathbf{d}}(\zeta) \geq \zeta$  for  $\zeta$  an ordinal and  $\mathbf{d} \in \mathbb{D}$ .

[Why? We can prove this by induction on  $\zeta$  for all  $\mathbf{d} \in \mathbb{D}$ , by clauses (h) + (i) of Definition 3.1.]

As a warmup we shall note that:

○<sub>3</sub> if  $\mathbf{d} \in \mathbb{D}$  and  $\zeta < \sigma_*$  or just  $\mathbf{d} \in \mathbb{D}$  and every  $\mathbf{e}$  in the  $\Sigma$ -closure of  $\{\mathbf{d}\}$  is  $(\mathbf{p}, \leq \zeta)$ -complete then:  
 (α)  $\text{rk}_{\mathbf{d}}(\zeta) = \zeta$   
 (β)  $f \in {}^I[\mathbf{d}]\zeta \Rightarrow \text{rk}_{\mathbf{d}}(f) < \zeta$ .

[Why? Note that as  $\zeta < \sigma_*$  clearly  $\mathbf{d}$  is  $(\mathbf{p}, \leq \zeta)$ -complete by ○<sub>1</sub> and clause (f) of 3.1, so we can assume that  $\mathbf{d}$  is  $(\mathbf{p}, \leq \zeta)$ -complete. We prove the statement inside ○<sub>3</sub> by induction on the ordinal  $\zeta$  (for all  $\mathbf{d} \in \mathbb{D}$ ). Note that for  $\varepsilon < \zeta$ , “ $\mathbf{d}$  is  $(\mathbf{p}, \leq \zeta)$ -complete” implies “ $\mathbf{d}$  is  $(\mathbf{p}, \leq \varepsilon)$ -complete”, we shall use this freely.

Arriving to  $\zeta$ , to prove clause (β) let  $f \in {}^I[\mathbf{d}]\zeta$  and for  $\varepsilon < \zeta$  we define  $A_\varepsilon := \{t \in I_{\mathbf{d}} : f(t) = \varepsilon\}$ , so  $\langle A_\varepsilon : \varepsilon < \zeta \rangle$  is a well defined partition of  $I_{\mathbf{d}}$  so the sequence exists, hence as “ $\mathbf{d}$  is  $(\mathbf{p}, \leq \zeta)$ -complete” recalling (\*) from clause (f)' of 3.2 for some triple  $(\mathbf{e}, h, \varepsilon)$  we have  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  and  $\varepsilon < \zeta$  and  $h^{-1}(A_\varepsilon) \in D_{\mathbf{e}}$  and  $\text{rk}_{\mathbf{e}}(f \circ h) = \text{rk}_{\mathbf{d}}(f)$ .

Now  $f \circ h = \langle \varepsilon : t \in I_{\mathbf{d}} \rangle \text{ mod } D_{\mathbf{e}}$  hence by Claim 3.9(3) we have  $\text{rk}_{\mathbf{e}}(f \circ h) = \text{rk}_{\mathbf{e}}(\varepsilon)$ . But the assumptions on  $\mathbf{d}$  holds for  $\mathbf{e}$  hence by the induction hypothesis on  $\zeta$  we know that  $\text{rk}_{\mathbf{e}}(\varepsilon) = \varepsilon$  and  $\varepsilon < \zeta$  so together  $\text{rk}_{\mathbf{d}}(f \circ h) < \zeta$ , so clause (β) holds.

To prove clause (α) first consider  $\zeta = 0$ ; if  $\text{rk}_{\mathbf{d}}(\zeta) > 0$  by clause (g) of Definition 3.1, 3.2 there are  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  and  $g \in {}^{I[\mathbf{e}]}\text{Ord}$  such that  $g < \langle \zeta : t \in I_{\mathbf{e}} \rangle \text{ mod } D_{\mathbf{e}}$ , so for some  $t \in I_{\mathbf{e}}$  we have  $g(t) < \zeta$  but  $\zeta = 0$ , contradiction; this is close to 3.9(2).

Second, consider  $\zeta > 0$ , so by ○<sub>2</sub> we have  $\text{rk}_{\mathbf{d}}(\zeta) \geq \zeta$  and assume toward contradiction that  $\text{rk}_{\mathbf{d}}(\zeta) > \zeta$ , so by clause (g) of Definition 3.1, 3.2 there is a triple  $(\mathbf{e}, h, g)$  as there. Now apply clause (β) of ○<sub>3</sub> for  $\zeta$  (which we have already proved) recalling  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  so also  $\mathbf{e}$  is  $(\mathbf{p}, \leq \zeta)$ -complete. We get  $\text{rk}_{\mathbf{e}}(g) < \zeta$ , a contradiction. So ○<sub>3</sub> indeed holds.]

Now as in the desired equality we have already proved one inequality in  $\odot_2$ , we need to prove only the other inequality. We do it by induction on  $\zeta$ .

Case 1:  $\zeta < \mu$ .

By clause (f) for some  $i < \text{cf}(\mu)$  we have  $\mathbf{d} \in \mathbb{D} \wedge \mathbf{j}(\mathbf{d}) \geq i \Rightarrow \mathbf{d}$  is  $(\mathbf{p}, \leq \zeta)$ -complete, hence by  $\odot_3(\alpha)$  we have  $\text{rk}_{\mathbf{d}}(\zeta) = \zeta$ , as required.

Case 2:  $\zeta = \xi + 1$ .

By clause (h) of Definition 3.1 we have  $\mathbf{d} \in \mathbb{D} \Rightarrow \text{rk}_{\mathbf{d}}(\zeta) = \text{rk}_{\mathbf{d}}(\xi) + 1$ . Hence  $\mathbf{d} \in \mathbb{D}_{\geq \mathbf{i}(\xi)} \Rightarrow \text{rk}_{\mathbf{d}}(\zeta) = \text{rk}_{\mathbf{d}}(\xi) + 1 = \xi + 1 = \zeta$ , so  $\mathbf{i}(\xi)$  exemplifies that  $\mathbf{i}(\zeta)$  exists and is  $\leq \mathbf{i}(\xi)$  so we are done.

Case 3:  $\zeta$  is a limit ordinal  $\geq \mu$  of cofinality  $\neq \text{cf}(\mu)$ .

So for each  $\xi < \zeta$  by the induction hypothesis  $\mathbf{i}(\xi) < \text{cf}(\mu)$  is well defined. For  $i < \text{cf}(\mu)$  let  $u_i := \{\xi < \zeta : \mathbf{i}(\xi) \leq i\}$ , so is well defined; moreover the sequence  $\langle u_i : i < \text{cf}(\mu) \rangle$  exists and is  $\subseteq$ -increasing. If  $i < \text{cf}(\mu) \Rightarrow \sup(u_i) < \zeta$  then  $\langle \sup(u_i) : i < \text{cf}(\mu) \rangle$  is a  $\leq$ -increasing sequence of ordinals  $< \zeta$  with limit  $\zeta$ . So as  $\text{cf}(\zeta) \neq \text{cf}(\mu)$  necessarily for some  $i_* < \text{cf}(\mu)$  the set  $S := \{\xi : \xi < \zeta \text{ and } \mathbf{i}(\xi) < i_*\}$  is an unbounded subset of  $\zeta$ . We shall prove that  $\mathbf{i}(\zeta)$  is well defined and  $\leq i_*$ .

Subcase 3A:  $\text{cf}(\zeta) \geq \mu$ .

Let  $\mathbf{d} \in \mathbb{D}_{\geq i_*}$  and  $g \in {}^{I[\mathbf{d}]}\zeta$  be given. Clearly  $\text{Rang}(g)$  is a subset of  $\zeta$  of cardinality  $< \theta(I_{\mathbf{d}})$  which by clause (b)( $\alpha$ ) of 3.1 is  $< \mu \leq \text{cf}(\zeta)$  hence we can fix  $\xi \in S$  such that  $\text{Rang}(g) \subseteq \xi$ , hence by clause (i) of 3.1,  $\text{rk}_{\mathbf{d}}(g) \leq \text{rk}_{\mathbf{d}}(\xi)$  but  $\mathbf{i}(\xi) = i_*$  and  $\mathbf{d} \in \mathbb{D}_{\geq i_*}$  hence  $\text{rk}_{\mathbf{d}}(\xi) = \xi < \zeta$  so together  $\text{rk}_{\mathbf{d}}(g) < \zeta$ . As this holds for every  $\mathbf{d} \in \mathbb{D}_{\geq i_*}$  by the no-hole clause ( $g$ )' and clause (e)( $\gamma$ ) of 3.2 it follows that  $\mathbf{d} \in \mathbb{D}_{\geq i_*} \Rightarrow \text{rk}_{\mathbf{d}}(\zeta) \leq \zeta$  as required.

Subcase 3B:  $\text{cf}(\zeta) < \mu$  (but still  $\text{cf}(\zeta) \neq \text{cf}(\mu)$ ).

Let  $\langle \zeta_\varepsilon : \varepsilon < \text{cf}(\zeta) \rangle$  be an increasing sequence of ordinals from  $S$  with limit  $\zeta$ . Now let  $j_* < \text{cf}(\mu)$  be such that  $\mathbf{d} \in \mathbb{D}_{\geq j_*} \Rightarrow \mathbf{d}$  is  $(\mathbf{p}, \text{cf}(\zeta)^+)$ -complete, see clause (f) of Definition 3.1.

Now assume  $\mathbf{d} \in \mathbb{D}_{\geq \max\{i_*, j_*\}}$  and  $g \in {}^{I[\mathbf{d}]}\zeta$ . For  $\varepsilon < \text{cf}(\zeta)$  let  $A_\varepsilon = \{t \in I_{\mathbf{d}} : g(t) < \zeta_\varepsilon \text{ but } j < \varepsilon \Rightarrow g(t) \geq \zeta_j\}$  so  $\langle A_\varepsilon : \varepsilon < \text{cf}(\zeta) \rangle$  is well defined and is a partition of  $I_{\mathbf{d}}$ . Hence by clause (f) of Definition 3.2 for some  $\varepsilon < \text{cf}(\zeta)$  and  $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$  we have  $h^{-1}(A_\varepsilon) \in D_{\mathbf{e}}$  and  $\text{rk}_{\mathbf{d}}(g) = \text{rk}_{\mathbf{e}}(g \circ h)$ ; but  $\mathbf{j}(\mathbf{e}) \geq \mathbf{j}(\mathbf{d}) \geq i_*, j_*$  and by the choice of  $A_\varepsilon$  and clause (i) of 3.1 the latter is  $\leq \text{rk}_{\mathbf{e}}(\zeta_\varepsilon)$  hence as  $\mathbf{i}(\zeta_\varepsilon) \leq i_*$  the latter is  $= \zeta_\varepsilon < \zeta$ . As this holds for every  $\mathbf{d} \in \mathbb{D}_{\geq \max\{i_*, j_*\}}$  and  $g \in {}^{I[\mathbf{d}]}\zeta$ , by the no-hole clause ( $g$ )' of 3.2 necessarily  $\text{rk}_{\mathbf{d}}(\zeta) \leq \zeta$ . So  $\max\{i_*, j_*\} < \text{cf}(\mu)$  is as required, so we are done.

Case 4:  $\zeta \geq \mu$  is a limit ordinal such that  $\text{cf}(\zeta) = \text{cf}(\mu)$ .

Let  $\langle \zeta_i : i < \text{cf}(\zeta) \rangle$  be increasing with limit  $\zeta$ . Assume toward contradiction that for every  $i < \text{cf}(\mu)$  there is  $\mathbf{d}_i \in \mathbb{D}_{\geq i}$  such that  $\text{rk}_{\mathbf{d}_i}(\zeta) > \zeta$  but we do not assume that such a sequence  $\langle \mathbf{d}_i : i < \text{cf}(\mu) \rangle$  exists. Choose such  $\mathbf{d}_0$ ; as  $\text{rk}_{\mathbf{d}_0}(\zeta) > \zeta$ , clearly there are  $f_0 \in {}^{I[\mathbf{d}_0]}\zeta$  and a member  $\mathbf{d}'_0$  of  $\Sigma(\mathbf{d}_0)$  such that

$$\odot_4 \text{rk}_{\mathbf{d}'_0}(f_0) = \zeta$$

[Why? By using clause ( $g$ )' of 3.2.]

Note

$\odot_5$   $\mathbf{i}(f_0(t))$  is well defined for every  $t \in I_{\mathbf{d}'_0}$ .

[Why holds? Because  $f_0(t) < \zeta$  and the induction hypothesis.]

For  $j_1 < \text{cf}(\zeta), j_2 < \text{cf}(\mu)$  let  $A_{j_1, j_2} = \{t \in I_{\mathbf{d}'_0} : f_0(t) < \zeta_{j_1} \text{ and } (\forall j < j_1)(f_0(t) \geq \zeta_j) \text{ and } \mathbf{i}(f_0(t)) = j_2\}$ . By clause  $(f)'$  of 3.2 applied to the pair  $(\mathbf{d}'_0, f_0)$  and the partition  $\langle A_{j_1, j_2} : j_1 < \text{cf}(\zeta), j_2 < \text{cf}(\mu) \rangle$ , for some  $(\mathbf{d}_*, h_*) \in \Sigma(\mathbf{d}'_0)$  we have  $\text{rk}_{\mathbf{d}_*}(f_0 \circ h_*) = \zeta$  and for some  $j_1, j_2$  we have  $h_*^{-1}(A_{j_1, j_2}) \in D_{\mathbf{d}_*}$ . By 3.9(3) for some  $f = f_0 \circ h_* \text{ mod } D_{\mathbf{d}_*}$  and letting  $\mathbf{d} := \mathbf{d}_*$  we have

- $\odot_6$  (a)  $\mathbf{d} \in \mathbb{D}$
- (b)  $f \in {}^{I[\mathbf{d}]}\text{Ord}$
- (c)  $\text{rk}_{\mathbf{d}}(f) = \zeta$
- (d)  $t \in I_{\mathbf{d}} \Rightarrow \mathbf{i}(f(t)) = j_2 \wedge f(t) < \zeta_{j_1} \wedge (\forall j < j_1)(f(t) \geq \zeta_j)$ .

Next

$\odot_7$  clause  $\boxplus$  from 3.6 with the quintuple  $(\mathbf{d}, \zeta, \zeta_{j_1}, f, j_2)$  standing for  $(\mathbf{d}, \zeta, \xi, f, j_0)$ .

[Why? We check the six demands

- <sub>1</sub> “ $f \in {}^{I[\mathbf{d}]}\mathcal{E}$ ” which holds by  $\odot_6(b) + (d)$
- <sub>2</sub> “ $\text{rk}_{\mathbf{d}}(f) = \zeta$ ” which holds by  $\odot_6(c)$
- <sub>3</sub> “ $\zeta < \zeta$ ” which holds as  $\xi$  was chosen as  $\zeta_{j_1}$  and  $(\forall i < \text{cf}(\mu))(\zeta_i < \zeta)$
- <sub>4</sub> “ $\text{cf}(\zeta) = \text{cf}(\mu)$ ” which holds by the case assumption
- <sub>5</sub>  $j_2 < \text{cf}(\mu)$  obvious
- <sub>6</sub>  $s \in I_{\mathbf{d}} \wedge \mathbf{e}' \in \mathbb{D}_{\geq j_2} \Rightarrow \text{rk}_{\mathbf{e}'}(f(s)) = f(s)$  holds by  $\odot_6(d)$ .

So  $\odot_7$  indeed holds.]

Now by  $\odot_7$  we apply clause (j) of Definition 3.6(1) applied with  $\mathbf{d}, \zeta, \zeta_{j_1}, f, j_2$  here standing for  $\mathbf{d}, \zeta, \xi, f, j_0$  there we can find  $j$  as there. Let  $i_2 = \max\{j, j_1, j_2, \mathbf{i}(\zeta_{j_1})\}$  so  $i_2 < \text{cf}(\mu)$  and choose  $\mathbf{e}_0 \in \mathbb{D}_{\geq i_2}$  such that  $\text{rk}_{\mathbf{e}_0}(\zeta) > \zeta$  as in the beginning of the case. As  $\text{rk}_{\mathbf{e}_0}(\zeta) > \zeta$  by clause  $(g)'$  of 3.2 there are  $\mathbf{e}_1 \in \Sigma(\mathbf{e}_0)$  and  $g \in {}^{I[\mathbf{e}_1]}\zeta$  such that  $\text{rk}_{\mathbf{e}_1}(g) \geq \zeta$  so  $g < \langle \zeta : t \in I_{\mathbf{e}_1} \rangle$ . Now without loss of generality

- $\odot_8$  (a)  $\text{rk}_{\mathbf{e}_1}(g) = \zeta + 1$
- (b)  $\zeta_* = \sup\{g(t) : t \in I_{\mathbf{e}_1}\} < \zeta$
- (c)  $\mathbf{j}(\mathbf{e}_1) \geq i_2$ .

[Why? Because we can use  $g' \in {}^{I[\mathbf{e}_1]}\zeta$  defined by  $g'(t) = g(t) + 2$  for  $t \in I_{\mathbf{e}_1}$ , by clause (c) of 3.1,  $\text{rk}_{\mathbf{e}_1}(g') = \text{rk}_{\mathbf{e}_1}(g) + 2 > \zeta$ . By clause  $(e)(\gamma)$  we have  $\mathbf{j}(\mathbf{e}_1) \geq \mathbf{j}(\mathbf{e}_0) \geq i_2$ . Now we find  $(\mathbf{d}''_2, h'') \in \Sigma(\mathbf{e}_1)$  and  $g_2$  as in the proof of  $\odot_6$  and rename.]

Also without loss of generality

- $\odot_9$   $t \in I_{\mathbf{e}_1} \Rightarrow g(t) \geq \zeta_{j_1}$ .

[Why? Let  $A_0 = \{t \in I_{\mathbf{e}_1} : g(t) < \zeta_{j_1}\}, A_1 = \{t \in I_{\mathbf{e}_1} : g(t) \geq \zeta_{j_1}\}$  so by clause  $(f)'$  of 3.2 for some pair  $(\mathbf{e}_2, h) \in \Sigma(\mathbf{e}_1)$  we have  $\text{rk}_{\mathbf{e}_2}(g \circ h) = \text{rk}_{\mathbf{e}_1}(g) = \zeta + 1$  and  $(h^{-1}(A_0) \in D_{\mathbf{e}_2}) \vee (h^{-1}(A_1) \in D_{\mathbf{e}_2})$ . So if  $h^{-1}(A_0) \in D_{\mathbf{e}_2}$  then by clause (i) of 3.1,  $\text{rk}_{\mathbf{e}_1}(g \circ h) \leq \text{rk}_{\mathbf{e}_2}(\zeta_{j_1})$  but  $\mathbf{i}(\zeta_{j_1})$  is well defined  $\leq i_2 \leq \mathbf{j}(\mathbf{e}_1) \leq \mathbf{j}(\mathbf{e}_2)$  so  $\text{rk}_{\mathbf{e}_2}(\zeta_{j_1}) = \zeta_{j_1}$  together  $\text{rk}_{\mathbf{e}_2}(g \circ h) \leq \zeta_{j_1}$  contradicting the previous sentence. Hence  $h^{-1}(A_0) \notin D_{\mathbf{e}_2}$  so  $h^{-1}(A_1) \in D_{\mathbf{e}_2}$ . Let  $g' \in {}^{I[\mathbf{e}_2]}\text{Ord}$  be defined by  $g'(t)$  is  $(g \circ h)(t)$  if  $t \in h^{-1}(A_1)$  and is  $\zeta_{j_1} + 1$  if  $t \in h^{-1}(A_0)$ . By Claim 3.9(3) we have

$\text{rk}_{\mathbf{e}_2}(g') = \text{rk}_{\mathbf{e}_2}(g \circ h)$  so  $(\mathbf{e}_2, g')$  satisfies all requirements on the pair  $(\mathbf{e}_1, g)$  and  $t \in I_{\mathbf{e}_2} \Rightarrow g'(t) \geq \zeta_{j_1} > 0$ , so we have justified the non-loss of generality.]

Let  $\xi := \zeta_{j_1}$  and let  $\mathbf{e} = \mathbf{e}_1$ . By the choice of  $j$  after  $\odot_6$ , i.e. as in clause (j) of 3.6, recalling  $\mathbf{e} \in \mathbb{D}_{\geq j}$  we shall get a contradiction to the choice of  $(\mathbf{d}, \xi, \zeta, f, j_2, \mathbf{e}, g, j)$ . To justify it we have to recall by  $\odot_7$  that the quintuple  $(\mathbf{d}, \zeta, \xi, f, j_2)$  satisfies  $\boxplus$  of 3.6(j) and then we prove that the triple  $(\mathbf{e}, g, j)$  satisfies  $\oplus$  of 3.6(j).

Now  $\oplus$  of 3.6 says:

- <sub>1</sub> “ $\mathbf{e} \in \mathbb{D}_{\geq j}$ ” as  
as  $j \geq i_2, \mathbf{e}_0 \in \mathbb{D}_{\geq i_2}$  and  $\mathbf{e} = \mathbf{e}_1 \in \Sigma(\mathbf{e}_0)$
- <sub>2</sub> “ $g \in I^{[\mathbf{e}]}\zeta$ ”  
which holds as  $g \in I^{[\mathbf{e}]}\zeta$
- <sub>3</sub> “ $g(t) \in [\xi, \zeta_*)$ ”  
holds as  $g(t) < \zeta$  by •<sub>2</sub> +  $\odot_8(b)$  and  $g(t) \geq \zeta_{j_1} = \xi$  by  $\odot_9$
- <sub>4</sub> “ $\text{rk}_{\mathbf{e}}(f(s)) = f(s)$  for  $s \in I_{\mathbf{d}}$ ”  
holds by  $\odot_6(d)$
- <sub>5</sub> “ $\text{rk}_{\mathbf{e}}(g) > \zeta$ ”  
holds by  $\odot_8(a)$ .

So we really get a contradiction.  $\square_{3.10}$

{m4.14}

**Definition 3.11.** 1) We say that the pair  $(\mathbf{d}, \mathbf{e})$  commute (or 3-commute) for  $\mathbf{p}$  when  $\mathbf{d}, \mathbf{e} \in \mathbb{D}_{\mathbf{p}}$  and  $\text{rk}_{\mathbf{d}}(f) \geq \text{rk}_{\mathbf{e}}(g)$  whenever  $\mathbf{d}_* \in \Sigma(\mathbf{d})$  and  $(f, g, \bar{f}, \bar{g})$  is a  $(\mathbf{p}, \mathbf{d}, \mathbf{e})$ -rectangle, see below.

2) We say that  $(f, \mathbf{d}, g, \mathbf{e}, \bar{f}, \bar{g})$  is  $\mathbf{p}$ -rectangle or  $(f, g, \bar{f}, \bar{g})$  is a  $(\mathbf{p}, \mathbf{d}, \mathbf{e})$ -rectangle when:

- ⊗ (a)  $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$
- (b)  $\mathbf{e} \in \mathbb{D}_{\mathbf{p}}$
- (c)  $\bar{g} = \langle g_t : t \in I_{\mathbf{e}} \rangle$  and  $g_t \in I^{[\mathbf{d}]}\text{Ord}$  for  $t \in I_{\mathbf{e}}$
- (d)  $g \in I^{[\mathbf{e}]}\text{Ord}$  is defined by  $g(t) = \text{rk}_{\mathbf{d}}(g_t)$
- (e)  $f_s \in I^{[\mathbf{e}]}\text{Ord}$  is defined by  $f_s(t) = g_t(s)$
- (f)  $\bar{f} = \langle f_s : s \in I[\mathbf{d}] \rangle$
- (g)  $f \in I^{[\mathbf{d}]}\text{Ord}$  is defined by  $f(s) = \text{rk}_{\mathbf{e}}(f_s)$ .

{m4.16}

**Claim 3.12.** [Assume  $\text{ZF} + \text{AC}_{<\mu}$ ] If  $\mathbf{p} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{i}, \mu)$  be a weak rank 1-system then  $\mathbf{p}$  is a strict rank 1-system when some function  $\Sigma_1$  witness it which means:

- (\*)<sub>0</sub>  $\Sigma_1$  a function with domain  $\mathbb{D}_1$
- (\*)<sub>1</sub>  $\Sigma_1(\mathbf{d}) \subseteq \Sigma(\mathbf{d})$  for  $\mathbf{d} \in \mathbb{D}$
- (\*)<sub>2</sub> for every  $\mathbf{d}, \zeta, \xi, f, j_0$  satisfying  $\boxplus$  of 3.6, for some  $j < \text{cf}(\mu)$  for every  $\mathbf{e} \in \mathbb{D}_{\geq j}$  we have
  - (a)  $\mathbf{e}$  is  $(\mathbf{p}, \leq \Sigma_1(\mathbf{d}))$ -complete
  - (b)  $(\mathbf{d}, \mathbf{e})$  does commute (for  $\mathbf{p}$ ) see 3.11, at least for  $f \in I^{[\mathbf{d}]}\zeta$  and any  $g \in I^{[\mathbf{e}]}\xi, \zeta$
- (\*)<sub>3</sub> we strengthen clause (g) of Definition 3.1 to  
(g)<sup>+</sup> add:  $\text{rk}_{\mathbf{e}}(f) = \text{rk}_{\mathbf{d}}(f)$
- (\*)<sub>4</sub>  $\text{AC}_{I[\mathbf{d}]}$  and  $\text{AC}_{\Sigma_1(\mathbf{d})}$  whenever  $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ .

*Remark 3.13.* 1) In  $(*)_1$ , can we make  $j$  depend on  $f$  and a partition of  $I_{\mathbf{d}}$ ? Will be somewhat better.

2) We can similarly prove this for a weak rank 2-system. It is natural though not necessary to add  $(\mathbf{e}, h) \in \Sigma_1(\mathbf{d}) \Rightarrow I_{\mathbf{e}} = I_{\mathbf{d}} \wedge h = \text{id}_{I_{\mathbf{d}}}$ .

*Proof.* Let  $\mathbf{d}, \zeta, \xi, f, j_0$  satisfying  $\boxplus$  of 3.6(j) be given and we should find  $j < \text{cf}(\mu)$  such that for no pair  $(\mathbf{e}, g)$  clause  $\oplus$  there holds. Without loss of generality  $s \in I_{\mathbf{d}} \Rightarrow f(s) > 0$ .

Let  $j < \text{cf}(\mu)$  be as in  $(*)_2$  in the claim and without loss of generality  $j > j_0$  and we shall prove that  $j$  is as required in clause (j) of Definition 3.6, this is enough. So assume  $\mathbf{e} \in \mathbb{D}_{\geq j}, g \in {}^{I[\mathbf{e}]}\xi, \zeta$  and toward contradiction,  $(j, \zeta, \xi, \mathbf{e}, g)$  satisfy  $\oplus$  there. For each  $t \in I_{\mathbf{e}}$  clearly  $g(t) < \zeta = \text{rk}_{\mathbf{d}}(f)$  hence by clause  $(g)^+$  of  $(*)_2$ , see (g) of Definition 3.1, “no hole”, there are  $g_t \in {}^{I[\mathbf{d}]}\xi$  and  $\mathbf{d}_t \in \Sigma(\mathbf{d})$  such that  $g_t <_{D_{\mathbf{d}_t}} f$  and  $\text{rk}_{\mathbf{d}_t}(g_t) = g(t)$ , without loss of generality  $g_t < \max(f, 1_{I[\mathbf{d}]}) = f$  and by the  $(*)_3$ , “we add” also  $\text{rk}_{\mathbf{d}_t}(f) = \text{rk}_{\mathbf{d}}(f)$ .

As  $I_{\mathbf{e}} \in \mathbf{C}$  by  $(*)_4$ , we can choose such sequence  $\langle (g_t, \mathbf{d}_t) : t \in I_{\mathbf{e}} \rangle$ . Now  $\mathbf{e}$  is  $(\mathbf{p}, \leq \Sigma_1(\mathbf{d}))$ -complete and  $(\mathbf{d}, \mathbf{e})$  commute for  $\mathbf{p}$ , by clause (a) of  $(*)_2$  (i.e. by the choice of  $j$  and as  $\mathbf{e} \in \mathbb{D}_{\geq j}$ ), hence we can find  $\mathbf{e}_* \in \Sigma_1(\mathbf{e})$  and  $\mathbf{d}_* \in \Sigma_1(\mathbf{d})$  such that  $\text{rk}_{\mathbf{e}_*}(g) = \text{rk}_{\mathbf{e}}(g) = \zeta$  and  $\{t \in I_{\mathbf{e}} : \mathbf{d}_t = \mathbf{d}_*\}$  belongs to  $D_{\mathbf{e}_*}$ . For  $s \in I_{\mathbf{d}} = I_{\mathbf{d}_*}$  let  $f_s \in {}^{I[\mathbf{e}_*]}\text{Ord}$  be defined by  $f_s(t) = g_t(s)$  so  $f_s(t) = g_t(s) < \xi$  and let  $f' \in {}^{I[\mathbf{d}_*]}\text{Ord}$  be defined by  $f'(s) = \text{rk}_{\mathbf{e}_*}(f_s)$  and let  $\bar{f} = \langle f_s : s \in I_{\mathbf{d}_*} \rangle$ .

Fixing  $s \in I_{\mathbf{d}_*}$  we have  $t \in I_{\mathbf{e}_*} \Rightarrow f_s(t) = g_t(s) < \text{Max}\{f(s), 1\} = f(s)$ , i.e.  $f_s < \langle f(s) : t \in I_{\mathbf{e}_*} \rangle$  hence  $\text{rk}_{\mathbf{e}_*}(f_s) < \text{rk}_{\mathbf{e}_*}(f(s))$ . Now by  $\boxplus \bullet_5$  from 3.6, as  $j_0 \leq j \leq \mathbf{j}(\mathbf{e}_*)$  we have  $s \in I_{\mathbf{e}_*} \Rightarrow \text{rk}_{\mathbf{e}_*}(f(s)) = f(s)$  so  $s \in I_{\mathbf{e}_*} \Rightarrow \text{rk}_{\mathbf{e}_*}(f_s) < f(s)$ , i.e.  $f' < f$ .

Clearly  $(f', g, \bar{f}, \bar{g})$  is a  $(\mathbf{p}, \mathbf{d}_*, \mathbf{e}_*)$ -rectangle hence by clause (b) of  $(*)_2$  of the assumptions, i.e. the choice of  $s$  and Definition 3.11(2) we know that  $\text{rk}_{\mathbf{d}_*}(f') \geq \text{rk}_{\mathbf{e}_*}(g)$ .

But recall that  $\text{rk}_{\mathbf{e}_*}(g) = \text{rk}_{\mathbf{e}}(g)$  by the choice of  $\mathbf{e}_*$ . We get a contradiction by

$$(*) \quad \zeta = \text{rk}_{\mathbf{d}}(f) = \text{rk}_{\mathbf{d}_*}(f) > \text{rk}_{\mathbf{d}_*}(f') \geq \text{rk}_{\mathbf{e}_*}(g) = \text{rk}_{\mathbf{e}}(g) > \zeta.$$

[Why those inequalities? By  $\bullet_2$  of  $\boxplus$  from 3.6 we are assuming as  $\mathbf{d}_* \in \{\mathbf{d}_t : t \in I_{\mathbf{e}}\}$  and the choice of the  $\mathbf{d}_t$ 's; as  $f' <_{D_{\mathbf{d}_*}} f$  and 3.9(3); by an inequality above; by the choice of  $\mathbf{e}_*$ ; by  $\bullet_5$  of  $\oplus$  of 3.6.]  $\square$

{m4.17}

**Claim 3.14.** Like 3.12 assuming<sup>5</sup>  $\text{AC}_{\text{cf}(\mu)}$  we can weaken  $(*)_1(b)$  to

$(*)_1(b)'$  if  $(\mathbf{d}, \zeta, \xi, f, j_0)$  satisfies  $\boxplus$  of 3.6 and  $\langle \mathbf{i}(f(t) : t \in I_{\mathbf{d}}) \rangle$  is constant, and  $\kappa = \sup\{\text{cf}(f(t)) : t \in I_{\mathbf{d}}\} < \mu$ ,  $\text{cf}(\text{rk}_{\mathbf{d}}(f)) = \text{cf}(\mu)$  then:

( $\alpha$ )  $\text{cf}(\prod_{t \in I_{\mathbf{d}}} f(t)/D_{\mathbf{d}}) < \mu$  or

( $\beta$ ) for any  $j \leq \text{cf}(\mu)$ , if  $\mathbf{e} \in \mathbb{D}_{\geq j}$  and  $g \in {}^{I[\mathbf{e}]}\xi, \zeta$  then for every  $\mathbf{p}$ -rectangle  $(\mathbf{d}, \mathbf{e}, f, g, \bar{f}, \bar{g})$  as we have  $\text{rk}_{\mathbf{d}}(f) \geq \text{rk}_{\mathbf{e}}(g)$ .

*Proof.* Without loss of generality the set  $A := \{s \in I_{\mathbf{d}} : \text{cf}(f(s)) \leq \text{cf}(\mu)\}$  is  $\emptyset$  or  $I_{\mathbf{d}}$ . If the former holds, by  $\text{AC}_{\text{cf}(\mu)}$  as  $\text{cf}(\zeta) = \mu$ , we can find  $\bar{h} = \langle h_i : i < \text{cf}(\mu) \rangle$  satisfying  $h_i \in \prod_{s \in I_{\mathbf{d}}} f(s)$  such that  $\sup\{\text{rk}_{\mathbf{d}}(h_i) : i < \text{cf}(\mu)\} = \zeta$  and let

$h_* \in \prod_{s \in I_{\mathbf{d}}} f(s)$  be defined by  $h_*(s) = \cup\{f_i(s) : i < \text{cf}(\mu)\}$ , well defined as  $A = I_{\mathbf{d}}$ ,

<sup>5</sup>hard not to have it by  $(*)_3$  of 3.12

contradiction. So  $A = \emptyset$  and without loss of generality  $s \in I_{\mathbf{d}} \Rightarrow f(s)$  is a limit ordinal (as  $\text{rk}_{\mathbf{d}}(w) = w$ ).

Now let  $\langle C_s : s \in I_{\mathbf{d}} \rangle$  be such that  $C_s$  is an unbounded subset of  $f(s)$  of order type  $\text{cf}(f(s))$ . We continue as in the first paragraph of 3.12's proof but demand  $g_t \in \prod_{s \in I_{\mathbf{d}}} C_s$ .  $\square_{3.14}$

#### 4. FINDING SYSTEMS

##### §(4A) Building weak rank systems and measurable

{e5.3}

**Claim 4.1.**  $[ZF + DC]$

If  $\otimes_1$  holds and  $\mathbf{p}_{\kappa, \theta} = \mathbf{p}_{\bar{\kappa}} = \mathbf{p} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu)$  is defined in  $\otimes_2$  then  $\mathbf{p}$  is a weak rank 1-system, even weakly normal (and  $(g)^+$  of 3.12 holds) where:

- $\otimes_1$  (a)  $\bar{\kappa} = \langle \kappa_i : i < \partial \rangle$  is an increasing sequence of regular cardinals  $> \partial$  with limit  $\mu$  such that if  $i < \partial$  is a limit ordinal then  $\kappa_i = (\Sigma\{\kappa_j : j < i\})^+$
- (b)  $\theta^*$  is a cardinal or  $\infty$
- $\otimes_2$  (a)  $\mathbb{D}_i = \{J : J \text{ is a } \kappa_i\text{-complete ideal on some } \kappa = \kappa_J < \mu \text{ including } [\kappa]^{< \kappa} \text{ and satisfying } \text{cf}(J, \leq) < \theta^* \text{ (and if } \theta = \infty \text{ we stipulate this as the empty demand) such that } \beta < \kappa \Rightarrow \{\beta\} \in J\}$  and let  $\mathbb{D} = \mathbb{D}_0$
- (b) if  $\mathbf{d} = J \in \mathbb{D}_i, J$  an ideal on  $\kappa_J := \cup\{A : A \in J\}$  then we let  $I_{\mathbf{d}} = \kappa_J$  and  $D_{\mathbf{d}}$  be the filter dual to the ideal  $J$
- (c)  $\mathbf{j}(J) = \min\{i : J \text{ is not } \kappa_{i+1}^+\text{-complete}\}$
- (d)  $\Sigma(J) = \{J + B : B \supseteq A \text{ and } \kappa_J \setminus B \text{ is not in } J\}$
- (e)  $\text{rk}_J(f)$  is as in Definition 1.6.

*Proof.* So we have to check all the clauses in Definition 3.1.

Clause (a): As  $\mu = \Sigma\{\kappa_i : i < \partial\}$ , the sequence  $\langle \kappa_i : i < \partial \rangle$  is increasing and  $\kappa_0 > \partial$  (all by  $\otimes_1$ ) clearly  $\mu$  is a singular cardinal (and  $\partial = \text{cf}(\mu)$ ).

Clause (b): So let  $\mathbf{d} \in \mathbb{D}$ , so  $\mathbf{d} = J$ .

Subclause ( $\alpha$ ): So  $I_{\mathbf{d}} = \kappa_J < \mu$  hence  $\theta(I_{\mathbf{d}}) = \theta(\kappa_J) = \kappa_J^+ < \mu$  recalling  $\mu$  is a limit cardinal and the definition of  $\mathbb{D} = \mathbb{D}_0$  in clause (a) of  $\otimes_2$ .

Subclause ( $\beta$ ): Also obvious.

Clause (c): For  $f \in {}^{(\kappa_{\mathbf{d}})}\text{Ord}$ ,  $\text{rk}_{\mathbf{d}}(f)$  as defined in  $\otimes_2(e)$ , is an ordinal recalling Claim 1.8(1).

Clause (d)( $\alpha$ ):  
Trivial.

Clause (d)( $\beta$ ):

Trivially  $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow I_{\mathbf{e}} = I_{\mathbf{d}} \wedge D_{\mathbf{e}} \supseteq D_{\mathbf{d}}$ ; so also concerning “ $\mathbf{p}$  is weakly normal”, demand  $\bullet_1$  in 3.8(1) holds as well as normal.

Clause (e):

Obvious from the definitions.

Clause (f):

Let  $\sigma < \mu$  be given and choose  $i < \partial$  such that  $\sigma < \kappa_i$ . Let  $\mathbf{d} \in \mathbb{D}$  be such that  $j = \mathbf{j}(\mathbf{d}) \geq i$  so  $D = D_{\mathbf{d}}$  is a filter on some  $\kappa_J$ , so assume  $\cup\{A_\varepsilon : \varepsilon < \varepsilon^*\} = \kappa_J$  and  $\varepsilon^* < \kappa_i$ . Now  $D$  is  $\kappa_i$ -complete and (see 1.9(2)) we have  $\text{rk}_{D_{\mathbf{d}}}(f) = \min\{\text{rk}_{D+A_\varepsilon}(f) : \varepsilon < \varepsilon^* \text{ and } A_\varepsilon \in D_{\mathbf{d}}^+\}$  which is what is needed as  $A_\varepsilon \in D_{\mathbf{d}}^+ \Rightarrow \mathbf{d} + (\kappa_j \setminus A_\varepsilon) \in \Sigma(\mathbf{d})$ .

Clause (g): By 1.8(2).

Moreover, the stronger version with  $\mathbf{e} = \mathbf{d}$  holds then  $\text{rk}_{\mathbf{d}}(f'') \geq \xi$ .

Clause (h):

Easy. On the one hand, as  $g < f$ , the definition of  $\text{rk}_{\mathbf{d}}(f)$ , we have  $\text{rk}_{\mathbf{d}}(f) \geq \text{rk}_{\mathbf{d}}(g) + 1$ . On the other hand, if  $g' < f \text{ mod } D_{\mathbf{d}}$  then  $g' \leq g \text{ mod } D$  hence by clause (j) we have  $\text{rk}_{\mathbf{d}}(g') \leq \text{rk}_{\mathbf{d}}(g) < \text{rk}_{\mathbf{d}}(g) + 1$ , as this holds for every  $g' < f \text{ mod } D_{\mathbf{d}}$  we have  $\text{rk}_{\mathbf{d}}(f) \leq \text{rk}_{\mathbf{d}}(g) + 1$ . Together we are done.

Clause (i):

Obvious. □<sub>4.1</sub>

**Discussion 4.2.** Assume  $\mu$  is strong limit singular,  $\mu = \sum_{i < \kappa} \mu_i$ ,  $\kappa = \text{cf}(\mu) < \mu_1 < \mu$  and  $\mu_i$  is increasing with  $i$ . Assume that for each  $i$  there is a pair  $(D, I)$ ,  $D$  is a  $\mu_i$ -complete ultra-filter on  $I$ ,  $\theta(I) < \mu$ . This seems to be a good case, but either we have “ $D$  is a ( $\leq \theta(I)$ )-complete” so  ${}^I\text{Ord}/D$  is “dull” or  $\theta(I) > \text{completeness}(D)$  and so there are ultrafilters on  $\kappa < \mu$  so  $\mu = \sup(\text{measurables} \cap \mu)$ .

{e5.13}

**Claim 4.3.** [ZF + DC + AC<sub><μ</sub>]

Assume  $\mu$  is singular and  $\mu = \sup(\mu \cap \text{measurable})$ , (equivalently for every  $\kappa < \mu$  there is a  $\kappa$ -complete non-principal ultrafilter on some  $\kappa' < \mu$ ). Let  $\bar{\kappa} = \langle \kappa_i : i < \text{cf}(\mu) \rangle$  be increasing with limit  $\mu$ ,  $\kappa_i > \text{cf}(\mu)$  such that for  $i$  limit  $\kappa_i = (\Sigma\{\kappa_j : j < i\})^+$  and  $\kappa_i$  is measurable for  $i$  non-limit.

Then  $\mathbf{p} = \mathbf{p}_{\bar{\kappa}}^{\text{uf}}$  is a strict rank 1-system where  $\mathbf{p}$  is defined by

- ⊗ (a)  $\mathbb{D}_{\geq i} = \{J : \text{dual}(J) \text{ is a non-principal ultra-filter which is } \kappa_i\text{-complete on some } \kappa = \kappa_J < \mu\}$   
so naturally  $I_J = \kappa_J$  and  $D_J = \text{dual}(J)$
- (b)  $\mathbf{j}(J) = \min\{i : J \text{ is not } \kappa_{i+1}^+\text{-complete}\}$ , well defined
- (c)  $\Sigma(J) = \{J\}$
- (d)  $\text{rk}_J(f) = \text{rk}_{\text{dual}(J)}(f)$  as in 1.6.

*Proof.* We can check clauses (a)-(i) of 3.1 as in the proof of 4.1.

We still have to prove the “strict”, i.e. we should prove clause (j) from Definition 3.12. We prove this using Claim 3.12, we choose  $\Sigma_1(\mathbf{d}) = \{\mathbf{d}\} \subseteq \Sigma(\mathbf{d})$  for  $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ .

So in Claim 3.12, we have  $(*)_0$  by the choice of  $\Sigma_1$ , and concerning  $(*)_2$  in the proof of 4.1 we obviously can prove  $(g)^+$ , and concerning  $(*)_3$  it holds as we assume AC<sub><μ</sub>. Concerning  $(*)_1$  let  $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ ,  $\zeta, \xi, f \in {}^I[\mathbf{d}]\zeta$  be given and let  $j < \partial = \text{cf}(\mu)$  be such that  $\theta(\mathcal{P}(\kappa)) < \kappa_j$ , and let  $\mathbf{e} \in \mathbb{D}_{\geq j}$ . Now clause (a) is trivial as  $|\Sigma_1(\mathbf{d})| = 1$ , and clause (b) says that “the pair  $(\mathbf{d}, \mathbf{e})$  commute”, see Definition 3.11. So let  $(f, g, \bar{f}, \bar{g})$  be a  $(\mathbf{p}, \mathbf{d}, \mathbf{e})$ -rectangle, see Definition 3.11(2), and we should prove that  $\text{rk}_{\mathbf{e}}(g) \leq \text{rk}_{\mathbf{d}}(f)$ . let  $I_1 = I_{\mathbf{e}}, I_2 = I_{\mathbf{d}}$ .

To prove this we apply 2.3 or 2.9, but the  $f, \bar{f}$  are interchanged with  $g, \bar{g}$ ; we check  $\oplus(a) - (g)$  from 2.3. They hold by  $\oplus(a) - (f)$  of Definition 3.11.

Concerning  $\boxplus(a)$  “ $AC_{I_1}$ ” it holds as  $AC_{<\mu}$  holds and the definition of  $\mathbf{p}$ . Lastly, we should prove  $\boxplus(b)$  there which says “ $D_{\mathbf{d}}$  does 2-commute or 3-commute with  $D_e$  which holds by Case 2 of Claim 2.6.  $\square_{4.3}$

{5e.17}

*Conclusion 4.4.* [ $AC_{<\mu}$ ,  $\mu$  a singular cardinal] Assume  $\mu = \sup\{\lambda < \mu : \lambda \text{ is a measurable cardinal}\}$ . Then for every ordinal  $\zeta$  for some  $\kappa < \lambda$  we have  $\text{rk}_D(\zeta) = \zeta$  for every  $\kappa$ -complete ultrafilter on some cardinality  $< \mu$ .

*Proof.*  $\dagger$  suffices to prove this for the case  $\mu$  has cofinality  $\aleph_0$ . Now we can apply Claim 4.3 and Theorem 3.10.  $\square_I$

5. PSUEDO TRUE COFINALITY

Pseudo PCF

We try to develop pcf theory with little choice. We deal only with  $\aleph_1$ -complete filters, and replace cofinality and other basic notions by pseudo ones, see below. This is quite reasonable as with choice there is no difference.

This section main result are 5.9, existence of filters with pseudo-true-cofinality; 5.19, giving a parallel of  $J_{<\lambda}[\alpha]$ ; and ??, on generators of  $J_{<\lambda}^{[\bar{\alpha}]}$ .

In the main case we may (in addition to ZF) assume  $DC + AC_{\mathcal{P}(\mathcal{P}(Y))}$ ; this will be continued in [Sh:938].

{r1}

**Hypothesis 5.1.** ZF

{r2}

**Definition 5.2.** 1) We say that a partial order  $P$  is  $(< \kappa)$ -directed when every subset  $A$  of  $P$  of power  $< \kappa$  has a common upper bound.

1A) Similarly  $P$  is  $(\leq S)$ -directed.

2) We say that a partial order  $P$  is pseudo  $(< \kappa)$ -directed when it is  $(< \kappa)$ -directed and moreover every subset  $\cup\{P_\alpha : \alpha < \delta\}$  has a common upper bound when:

- (a) if  $\delta < \kappa$  is a limit ordinal
- (b)  $\bar{P} = \langle P_\alpha : \alpha < \delta \rangle$  is a sequence of non-empty subsets of  $P$
- (c) if  $\alpha_1 < \alpha_2, p_1 \in P_{\alpha_1}$  and  $p_2 \in P_{\alpha_2}$  then  $p_1 <_P p_2$ .

2A) For a set  $S$  we say that the partial order  $P$  is pseudo  $(\leq S)$ -directed when  $\cup\{P_s : s \in S\}$  has a common upper bound whenever

- (a)  $\langle P_s : s \in S \rangle$  is a sequence
- (b)  $P_s \subseteq P$
- (c) if  $s <_S t$  and  $f \in P_s, g \in P_t$  then  $f <_P g$
- (d) if  $s \in S$  then  $P_s$  has a common upper bound (so if  $S$  has no minimal member this is redundant).

{r3}

**Definition 5.3.** We say that a partial (or quasi) order  $P$  has pseudo true cofinality  $\delta$  when:  $\delta$  is a limit ordinal and there is a sequence  $\langle P_\alpha : \alpha < \delta \rangle$  such that

- (a)  $P_\alpha \subseteq P$  and  $\delta = \sup\{\alpha < \delta : P_\alpha \text{ non-empty}\}$
- (b) if  $\alpha_1 < \alpha_2 < \delta, p_1 \in P_{\alpha_1}, p_2 \in P_{\alpha_2}$  then  $p_1 <_P p_2$
- (c) if  $p \in P$  then for some  $\alpha < \delta$  and  $q \in P_\alpha$  we have  $p \leq_P q$ .

(comes from 2009-09-16)

[938] revision: [2009-09-16]

{r4}

*Remark 5.4.* 0) See 5.2(2) and 5.8(1).

1) We could replace  $\delta$  by a partial order  $Q$ .

2) The most interesting case is in Definition 5.6.

3) We may in Definition 5.3 demand  $\delta$  is a regular cardinal.

4) Usually in clause (a) without loss of generality  $\bigwedge_{\alpha} P_{\alpha} \neq \emptyset$ , as without loss of generality  $\delta = \text{cf}(\delta)$  using  $P'_{\alpha} = P_{f(\alpha)}$  where  $f(\alpha) =$  the  $\alpha$ -th member of  $\{\beta < \delta : P_{\beta} \neq \emptyset\}$ . Why do we allow  $P_{\alpha} = \emptyset$ ? as it is more natural in 5.17(1), but can usually ignore it.

{r5}

**Example 5.5.** Suppose we have a limit ordinal  $\delta$  and a sequence  $\langle A_{\alpha} : \alpha < \delta \rangle$  of sets with  $\prod_{\alpha < \delta} A_{\alpha} = \emptyset$ ; moreover  $u \subseteq \delta = \sup(u) \Rightarrow \prod_{\alpha \in u} A_{\alpha} = \emptyset$ . Define a partial order  $P$  by:

(a) its set of elements is  $\{(\alpha, a) : a \in A_{\alpha} \text{ and } \alpha < \delta\}$

(b) the order is  $(\alpha_1, a_1) <_P (\alpha_2, a_2)$  iff  $\alpha_1 < \alpha_2$  (and  $a_{\ell} \in A_{\alpha_{\ell}}$  for  $\ell = 1, 2$ ).

It seems very reasonable to say that  $P$  has true cofinality but there is no increasing cofinal sequence.

{r6}

**Definition 5.6.** 1) For a set  $Y$  and sequence  $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$  of ordinals and cardinal  $\kappa$  we define

$$\text{ps-tcf-fil}_{\kappa}(\bar{\alpha}) = \{D : D \text{ a } \kappa\text{-complete filter on } Y \text{ such that } (\Pi\bar{\alpha}/D) \text{ has a pseudo true cofinality}\};$$

see below.

2) We say that  $\Pi\bar{\alpha}/D$  or  $(\Pi\bar{\alpha}, D)$  or  $(\Pi\bar{\alpha}, <_D)$  has pseudo true cofinality  $\gamma$  when  $D$  is a filter on  $Y = \text{Dom}(\bar{\alpha})$  and  $\gamma$  is a limit ordinal and the partial order  $(\Pi\bar{\alpha}, <_D)$  essentially does<sup>6</sup>, i.e., there is a sequence  $\bar{\mathcal{F}} = \langle \mathcal{F}_{\beta} : \beta < \gamma \rangle$  satisfying:

$$\otimes_{\bar{\mathcal{F}}} (a) \quad \mathcal{F}_{\beta} \subseteq \{f \in {}^Y \text{Ord} : f <_D \bar{\alpha}\}$$

$$(b) \quad \mathcal{F}_{\beta} \neq \emptyset$$

$$(c) \quad \text{if } \beta_1 < \beta_2, f_1 \in \mathcal{F}_{\beta_1} \text{ and } f_2 \in \mathcal{F}_{\beta_2} \text{ then } f_1 < f_2 \text{ mod } D$$

$$(d) \quad \text{if } f \in {}^Y \text{Ord} \text{ and } f < \bar{\alpha} \text{ mod } D \text{ then for some } \beta < \gamma \text{ we have } g \in \mathcal{F}_{\beta} \Rightarrow f < g \text{ mod } D \text{ (by clause (c) this is equivalent to: for some } \beta < \gamma \text{ and some } g \in \mathcal{F}_{\beta} \text{ we have } f \leq g \text{ mod } D).$$

3)  $\text{ps-pcf}_{\kappa}(\bar{\alpha}) = \text{ps-pcf}_{\kappa\text{-comp}}(\bar{\alpha}) := \{\gamma : \text{there is a } \kappa\text{-complete filter } D \text{ on } Y \text{ such that } \Pi\bar{\alpha}/D \text{ has pseudo true cofinality } \gamma \text{ and } \gamma \text{ is minimal for } D\}$ .

4)  $\text{pcf-fil}_{\kappa, \gamma}(\bar{\alpha}) = \{D : D \text{ a } \kappa\text{-complete filter on } Y \text{ such that } \Pi\bar{\alpha}/D \text{ has true cofinality } \gamma\}$ .

5) In part (2) if  $\gamma$  is minimal we call it  $\text{ps-tcf}(\Pi\bar{\alpha}, D)$  or simply  $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ ; note that it is a well defined (regular cardinal).

<sup>6</sup>so necessarily  $\{s \in Y : \alpha_s > 0\}$  belongs to  $D$  but is not necessarily empty; if it is non-empty then  $\Pi\bar{\alpha} = \emptyset$ , so pedantically this is wrong, but we shall ignore this or assume  $\bigwedge_t \alpha_t \neq 0$  when not said otherwise.

{r7}

- Claim 5.7.** 1) If  $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ , then  $(\Pi\bar{\alpha}, <_D)$  is pseudo  $(< \lambda)$ -directed.  
 1A) If  $\theta(S) < \lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$  then  $(\Pi\bar{\alpha}, <_D)$  is pseudo  $(\leq S)$ -directed.  
 2) Similarly for any quasi order.  
 3) Assume  $\text{AC}_\alpha$  for  $\alpha < \lambda$ . If  $\text{cf}(\alpha_t) \geq \lambda = \text{cf}(\lambda)$  for  $t \in Y$  then  $(\Pi\bar{\alpha}, <_D)$  is  $\lambda$ -directed.  
 4) Assume  $\text{AC}_{Y \times \lambda}$ . If  $\text{cf}(\alpha_s) > \lambda$  for  $s \in Y$  then  $(\Pi\bar{\alpha}, <_D)$  is pseudo  $\lambda^+$ -directed.

*Proof.* As in 5.8(1) below. □<sub>5.7</sub>

{r8}

**Claim 5.8.** Let  $\bar{\alpha} = \langle \alpha_s : s \in Y \rangle$  and  $D$  is a filter on  $Y$ .

- 0) If  $\Pi\bar{\alpha}/D$  has pseudo true cofinality then  $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$  is a regular cardinal; similarly for any partial order.  
 1) If  $\Pi\bar{\alpha}/D$  has pseudo true cofinality  $\gamma_1$  and true cofinality  $\gamma_2$  then  $\text{cf}(\gamma_1) = \text{cf}(\gamma_2) = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ , similarly for any partial order.  
 2)  $\text{ps-pcf}_\kappa(\bar{\alpha})$  is a set of regular cardinals so if  $\Pi\bar{\alpha}/D$  has pseudo true cofinality then  $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$  is  $\gamma$  where  $\gamma = \text{cf}(\gamma)$  and  $\Pi\bar{\alpha}/D$  has pseudo cofinality  $\gamma$ .  
 3) Always  $\text{ps-pcf}_\kappa(\bar{\alpha})$  has cardinality  $< \theta(\{D : D \text{ a } \kappa\text{-complete filter on } Y\})$ .  
 4) If  $\bar{\beta} = \langle \beta_s : s \in Y \rangle \in {}^Y \text{Ord}$  and  $\{s : \beta_s = \alpha_s\} \in D$  then  $\text{ps-tcf}(\Pi\bar{\alpha}/D) = \text{ps-tcf}(\Pi\bar{\beta}/D)$  so one is well defined iff the other is.

*Proof.* 0) By the definitions.

- 1) Let  $\langle \mathcal{F}_\beta^\ell : \beta < \gamma_\ell \rangle$  exemplify “ $\Pi\bar{\alpha}/D$  has pseudo true cofinality  $\gamma_\ell$ ” for  $\ell = 1, 2$ .  
 Now

$$(*) \text{ if } \ell \in \{1, 2\} \text{ and } \beta_\ell < \gamma_\ell \text{ then for some } \beta_{3-\ell} < \gamma_{3-\ell} \text{ we have } g_1 \in \mathcal{F}_{\beta_\ell}^\ell \wedge g_2 \in \mathcal{F}_{\beta_{3-\ell}}^{3-\ell} \Rightarrow g_1 <_D g_2.$$

[Why? Choose  $g^\ell \in \mathcal{F}_{\beta_\ell+1}^\ell$ , choose  $\beta_{3-\ell} < \gamma_{3-\ell}$  and  $g_{3-\ell} \in \mathcal{F}_{\beta_{3-\ell}}^{3-\ell}$  such that  $g^\ell < g^{3-\ell} \text{ mod } D$ .]

Hence

$$(*) \text{ } h_1 : \gamma_1 \rightarrow \gamma_2 \text{ is well defined when } h_1(\beta_1) = \text{Min}\{\beta_2 < \gamma_2 : (\forall g_1 \in \mathcal{F}_{\beta_1}^1)(\forall g_2 \in \mathcal{F}_{\beta_2}^2)(g_1 < g_2 \text{ mod } D)\}.$$

Clearly  $h$  is non-decreasing and it is not eventually constant (as  $\cup\{\mathcal{F}_\beta^1 : \beta < \gamma_1\}$  is cofinal in  $\Pi\bar{\alpha}/D$ ) and has range unbounded in  $\gamma_2$  (similarly).

The rest should be clear.

2) Follows.

3),4) Easy. □<sub>5.8</sub>

Concerning [Sh:835]

{r9}

**Claim 5.9.** The Existence of true cofinality filter [ $\kappa > \aleph_0 + \text{DC} + \text{AC}_{<\kappa}$ ] If

- (a)  $D$  is a  $\kappa$ -complete filter on  $Y$   
 (b)  $\bar{\alpha} \in {}^Y \text{Ord}$   
 (c)  $\delta := \text{rk}_D(\bar{\alpha})$  satisfies  $\text{cf}(\delta) \geq \theta(\text{Fil}_\kappa^1(Y))$ , see below.

Then for some  $D'$  we have

- ( $\alpha$ )  $D'$  is a  $\kappa$ -complete filter on  $Y$

( $\beta$ )  $D' \supseteq D$

( $\gamma$ )  $\Pi\bar{\alpha}/D'$  has pseudo true cofinality, in fact,  $\text{ps-tcf}(\Pi\bar{\alpha}, <_D) = \text{cf}(\text{rk}_D(\bar{\alpha}))$ .

Recall from [Sh:835]

{r9a}

**Definition 5.10.** 0)  $\text{Fil}_\kappa^1(Y) = \{D : D \text{ a } \kappa\text{-complete filter on } Y\}$  and if  $D \in \text{Fil}_\kappa^1(Y)$  then  $\text{Fil}_\kappa^1(D) = \{D' \in \text{Fil}_\kappa^1(Y) : D \subseteq D'\}$ .

1)  $\text{Fil}_\kappa^4(Y) = \{(D_1, D_2) : D_1 \subseteq D_2 \text{ are } \kappa\text{-complete filters on } Y\}$ .

2)  $J[f, D]$  where  $D$  is a filter on  $Y$  and  $f \in {}^Y\text{Ord}$  is  $\{A \subseteq Y : A = \emptyset \text{ mod } D \text{ or } \text{rk}_{D+A}(f) > \text{rk}_D(f)\}$ .

{r9b}

*Remark 5.11.* 1) On the Definition of pseudo  $(< \kappa, 1 + \gamma)$ -complete  $D$  see 1.13; we may consider changing the definition of  $\text{Fil}_\kappa^1(Y)$  to  $D$  is  $\aleph_1$ -complete and pseudo  $(< \kappa, 1 + \gamma)$ -complete filter on  $Y$ .

2) Related to [Sh:835].

*Proof. Proof of the Claim 5.9*

Recall  $\{y \in Y : \alpha_y = 0\} = \emptyset \text{ mod } D$  as  $\text{rk}_D(\langle \alpha_y : y \in Y \rangle) = \delta > 0$  but  $f_1, f_2 \in {}^Y\text{Ord} \wedge (f_1 = f_2 \text{ mod } D) \Rightarrow \text{rk}_D(f_1) = \text{rk}_D(f_2)$  hence without loss of generality  $y \in Y \Rightarrow \alpha_y > 0$ .

Let  $\mathbb{D} = \{D' : D' \text{ is a filter on } Y \text{ extending } D \text{ which is } \kappa\text{-complete}\}$ . So  $\theta(\mathbb{D}) \leq \theta(\text{Fil}_{\aleph_1}^1(Y)) \leq \text{cf}(\delta)$ . For any  $\gamma < \text{rk}_D(\bar{\alpha})$  and  $D' \in \mathbb{D}$  let

(\*)<sub>2</sub> (a)  $\mathcal{F}_{\gamma, D'} = \{f \in \Pi\bar{\alpha} : \text{rk}_D(f) = \gamma \text{ and } D' \text{ is dual}(J[f, D])\}$

(b)  $\mathcal{F}_{D'} = \cup\{\mathcal{F}_{\gamma, D'} : \gamma < \text{rk}_D(\bar{\alpha})\}$

(c)  $\Xi_{\bar{\alpha}, D'} = \{\gamma < \text{rk}_D(\bar{\alpha}) : \mathcal{F}_{\gamma, D'} \neq \emptyset\}$

(d)  $\mathcal{F}_\gamma = \cup\{\mathcal{F}_{\gamma, D''} : D'' \in \mathbb{D}\}$ .

Now

(\*)<sub>3</sub> if  $\gamma < \text{rk}_D(\bar{\alpha})$  then  $\mathcal{F}_\gamma \neq \emptyset$ .

[Why? By 1.8(2) there is  $g \in {}^Y\text{Ord}$  such that  $g < f \text{ mod } D$  and  $\text{rk}_D(g) = \gamma$  and without loss of generality  $g \in \Pi\bar{\alpha}$ . Now let  $D' = \text{dual}(J[g, D])$ , so  $(D, D') \in \text{Fil}_\kappa^4(Y)$ ,  $D' \in \mathbb{D}$  and  $g \in \mathcal{F}_{\gamma, D'}$ , see 1.8(2), Claim [Sh:835, 0.10(2)], here we use  $\text{AC}_{< \kappa}$ .]

(\*)<sub>4</sub>  $\{\sup(\Xi_{\bar{\alpha}, D'}) : D' \in \mathbb{D} \text{ and } \Xi_{\bar{\alpha}, D'} \text{ is bounded in } \text{rk}_D(\bar{\alpha})\}$  is a subset of  $\text{rk}_{D'}(\bar{\gamma})$  which has cardinality  $< \theta(\mathbb{D}) \leq \theta(\text{Fil}_\kappa^1(Y)) \leq \text{cf}(\delta)$ .

[Why? The function  $D' \mapsto \sup(\Xi_{\bar{\alpha}, D'})$  witness this.]

(\*)<sub>5</sub> the set in (\*)<sub>4</sub> is bounded below  $\text{rk}_D(\bar{\alpha})$  so let  $\gamma(*) < \text{rk}_D(\bar{\alpha})$  be its supremum.

[Why? By (\*)<sub>4</sub>.]

(\*)<sub>6</sub> there is  $D' \in \mathbb{D}$  such that  $\Xi_{\bar{\alpha}, D'}$  is unbounded in  $(\Pi\bar{\alpha}, <_{D'})$ .

[Why? Choose  $\gamma < \text{rk}_D(\bar{\alpha})$  such that:  $\gamma > \gamma(*)$ . By (\*)<sub>3</sub> there for some  $f \in \mathcal{F}_{\gamma(*)}$  and  $D' \in \mathbb{D}$  we have  $f \in \mathcal{F}_{\gamma(*)}, D'$  so by the choice of  $\gamma(*)$  the set  $\Xi_{\bar{\alpha}, D'}$  cannot be bounded in  $\text{rk}_D(\bar{\alpha})$ .]

(\*)<sub>7</sub> if  $\gamma_1 < \gamma_2$  are from  $\Xi_{\bar{\alpha}, D'}$  and  $f_1 \in \mathcal{F}_{\gamma_1, D'}, f_2 \in \mathcal{F}_{\gamma_2, D'}$  then  $f_1 <_{D'} f_2$ .

[Why? By 1.8, [Sh:835, 0.10(2)].]

Together we are done: by  $(*)_6$  there is  $D' \in \mathbb{D}$  such that  $\Xi_{\bar{\alpha}, D'}$  is unbounded in  $\text{rk}_D(\bar{\alpha})$ . Let  $\bar{\mathcal{F}} = \langle \mathcal{F}_{\gamma, D'} : \gamma \in \Xi_{\bar{\alpha}, D'} \rangle$  witness that  $(\Pi\bar{\alpha}, <_{D'})$  has pseudo true cofinality, and so  $\text{ps-tcf}(\Pi\bar{\alpha}, <_D) = \text{cf}(\text{otp}(\Xi_{\bar{\alpha}, D'})) = \text{cf}(\text{rk}_D(\bar{\alpha}))$ , so we are done.  $\square_{5.9}$

So we have

{r10}

**Definition/Claim 5.12.** 1) We say that  $\delta = \text{ps-tcf}_{\bar{D}}(\bar{\alpha})$ , where  $\delta$  is a limit ordinal when, for some set  $Y$ :

- (a)  $\bar{\alpha} \in {}^Y\text{Ord}$
- (b)  $\bar{D} = (D_1, D_2)$
- (c)  $D_1 \subseteq D_2$  are  $\aleph_1$ -complete filters on  $Y$
- (d)  $\text{rk}_{D_1}(\bar{\alpha}) = \delta = \sup(\Xi_{\bar{D}, \bar{\alpha}})$  where  $\Xi_{\bar{D}, \bar{\alpha}} = \{\gamma < \text{rk}_{D_1}(\bar{\alpha}) : \text{for some } f < \bar{\alpha} \text{ mod } D_1, \text{ we have } \text{rk}_{D_1}(f) = \gamma \text{ and } D_2 = \text{dual}(J[f, D_1])\}$ .

2) If  $D_1$  is  $\aleph_1$ -complete filter on  $Y$ ,  $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$  and  $\text{cf}(\alpha_t) \geq \theta(\text{Fil}_{\aleph_1}^1(Y))$  for  $t \in Y$  then for some  $\aleph_1$ -complete filter  $D_2$  on  $Y$  extending  $D_1$  we have  $\text{ps-tcf}_{(D_1, D_2)}(\bar{\alpha})$  is well defined.

3) Moreover in part (2) there is a definition giving for any  $(Y, D_1, D_2, \bar{\alpha})$  as there, a sequence  $\langle \mathcal{F}_\gamma : \gamma < \delta \rangle$  exemplifying the value of  $\text{ps-tcf}_{\bar{D}}(\bar{\alpha})$ .

*Proof.* 2), 3) Let  $\delta := \text{rk}_{D_1}(f)$ , so by Claim 5.16 below  $\text{cf}(\delta) \geq \theta(\text{Fil}_{\aleph_1}^1(Y))$  hence has Claim 5.9 above and its proof the conclusion holds: the proof is needed for “ $\delta = \sup(\Xi_{\bar{D}, \alpha}$ ”, noting observation 5.13 below.  $\square_{5.12}$

{r10d}

**Observation 5.13.** 1) [DC] or just [AC $\aleph_0$ ].

Assume  $D$  is an  $\aleph_1$ -complete filter on  $Y$  and  $f, f_n \in {}^Y\text{Ord}$  for  $n < \omega$  and  $f(t) = \sup\{f_n(t) : n < \omega\}$ . Then  $\text{rk}_D(f) = \sup\{\text{rk}_D(f_n) : n < \omega\}$ .

*Remark 5.14.* Similarly for other amounts of completeness, see 5.18.

*Proof.* As  $\text{rk}_D(f) = \min\{\text{rk}_{D+A_n}(f) : n < \omega\}$  if  $\cup\{A_n : n < \omega\} \in D, A_n \in D^+$  by [Sh:71] or see 1.9.  $\square_{5.13}$

*Remark 5.15.* Also in 1.9(2) can use AC $_I$  only, i.e. omit the assumption DC, a marginal point here.

{r11}

**Claim 5.16.** [AC $_{<\theta}$ ] The ordinal  $\delta$  has cofinality  $\geq \theta$  when:

- ⊗ (a)  $\delta = \text{rk}_D(\bar{\alpha})$
- (b)  $\bar{\alpha} = \langle \alpha_y : y \in Y \rangle \in {}^Y\text{Ord}$
- (c)  $D$  is an  $\aleph_1$ -complete filter on  $Y$
- (d)  $y \in Y \Rightarrow \text{cf}(\alpha_y) \geq \theta$ .

*Proof.* Note that  $y \in Y \Rightarrow \alpha_y > 0$ . Toward contradiction assume  $\text{cf}(\delta) < \theta$  so  $\delta$  has a cofinal subset  $C$  of cardinality  $< \theta$ . For each  $\beta < \delta$  for some  $f \in {}^Y\text{Ord}$  we have  $\text{rk}_D(f) = \beta$  and  $f <_D \bar{\alpha}$  and without loss of generality  $f \in \prod_{y \in Y} \alpha_y$ . By AC $_{<\theta}$

there is a sequence  $\langle f_\beta : \beta \in C \rangle$  such that  $f_\beta \in \prod_{y \in Y} \alpha_y, f_\beta <_D \bar{\alpha}$  and  $\text{rk}_D(f_\beta) = \beta$ .

Define  $g \in \prod_{y \in Y} \alpha_y$  by  $g(y) = \cup\{f_\beta(y) : \beta \in C \text{ and } f_\beta(y) < \alpha_y\}$ . By clause (d) we have  $[y \in Y \Rightarrow g(y) < \alpha_y]$ , so  $g <_D \bar{\alpha}$ , hence  $\text{rk}_D(g) < \text{rk}_D(\bar{\alpha})$  but by the choice

of  $g$  we have  $\beta \in C \Rightarrow f_\beta \leq_D g$  hence  $\beta \in C \Rightarrow \beta = \text{rk}_D(f_\beta) \leq \text{rk}_D(g)$  hence  $\delta = \sup(C) \leq \text{rk}_D(g)$ , contradiction.  $\square_{5.16}$

{r12} **Observation 5.17.** 1) Assume  $(\bar{\alpha}, D)$  satisfies

- (a)  $D$  a filter on  $Y$  and  $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$  and each  $\alpha_t$  is a limit ordinal
- (b)  $\bar{\mathcal{F}} = \langle \mathcal{F}_\beta : \beta < \partial \rangle$  exemplify  $\partial = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$  so we demand just  $\partial = \sup\{\beta < \partial : \mathcal{F}_\beta \neq \emptyset\}$
- (c)  $\mathcal{F}'_\beta = \{f \in \prod_{t \in Y} \alpha_t : \text{for some } g \in \mathcal{F}_\beta \text{ we have } f = g \text{ mod } D\}$ .

Then:  $\langle \mathcal{F}'_\beta : \beta < \partial \rangle$  exemplify  $\partial = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$  that is

- ( $\alpha$ )  $\bigcup_{\beta < \gamma} \mathcal{F}'_\beta$  is cofinal in  $(\Pi\bar{\alpha}, <_D)$
- ( $\beta$ ) for every  $\beta_1 < \beta_2 < \partial$  and  $f_1 \in \mathcal{F}'_{\beta_1}$  and  $f_2 \in \mathcal{F}'_{\beta_2}$  we have  $f_1 \leq f_2$ .

2) Similarly, if  $D, \bar{\mathcal{F}}$  satisfies clauses (a),(b) above and  $D$  is  $\aleph_1$ -complete and  $\partial = \text{cf}(\partial) > \aleph_0$  then we can “correct”  $\bar{\mathcal{F}}$  to make it  $\aleph_0$ -continuous that is  $\langle \mathcal{F}''_\beta : \beta < \partial \rangle$  defined in ( $c_1$ ) + ( $c_2$ ) below satisfies ( $\alpha$ ) + ( $\beta$ ) above and ( $\gamma$ ) below and so is  $\aleph_0$ -continuous, (see below) where

- ( $c_1$ ) if  $\beta < \partial$  and  $\text{cf}(\beta) \neq \aleph_0$  then  $\mathcal{F}''_\beta = \mathcal{F}'_\beta$
- ( $c_2$ ) if  $\beta < \partial$  and  $\text{cf}(\beta) = \aleph_0$  then  $\mathcal{F}''_\beta = \{\sup\langle f_n : n < \omega \rangle : \text{for some increasing sequence } \langle \beta_n : n < \omega \rangle \text{ with limit } \beta \text{ we have } n < \omega \Rightarrow f_n \in \mathcal{F}'_{\beta_n}\}$ , see below
- ( $\gamma$ ) if  $\beta < \partial$  and  $\text{cf}(\beta) = \aleph_0$  and  $f_1, f_2 \in \mathcal{F}''_\beta$  then  $f_1 = f_2 \text{ mod } D$ .

3) This applies to an increasing sequence  $\langle \mathcal{F}_\beta : \beta < \delta \rangle, \mathcal{F}_\beta \subseteq {}^Y \text{Ord}, \delta$  a limit ordinal.

{r13} *Proof.* Straightforward.  $\square_{5.17}$

**Definition 5.18.** 0) If  $f_n \in {}^Y \text{Ord}$  for  $n < \omega$ , then  $\sup\langle f_n : n < \omega \rangle$  is defined as the function  $f$  with domain  $Y$  such that  $f(t) = \cup\{f_n(t) : n < \omega\}$ .

1) We say  $\bar{\mathcal{F}} = \langle \mathcal{F}_\beta : \beta < \lambda \rangle$  exemplifying  $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$  is weakly  $\aleph_0$ -continuous when:

if  $\beta < \partial$ ,  $\text{cf}(\beta) = \aleph_0$  and  $f \in \mathcal{F}_\beta$  then for some sequence  $\langle (\beta_n, f_n) : n < \omega \rangle$  we have  $\beta = \cup\{\beta_n : n < \omega\}, \beta_n < \beta_{n+1} < \beta, f_n \in \mathcal{F}_{\beta_n}$  and  $f = \sup\langle f_n : n < \omega \rangle$ ; so if  $D$  is  $\aleph_1$ -complete then  $\{f/D : f \in \mathcal{F}_\beta\}$  is a singleton.

{r14} 2) We say it is  $\aleph_0$ -continuous if we can replace the last “then” by “iff”.

**Theorem 5.19.** *The Canonical Filter Theorem* Assume DC and  $\text{AC}_{\mathcal{P}(Y)}$ .

Assume  $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle \in {}^Y \text{Ord}$  and  $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \theta(\mathcal{P}(Y))$  and  $\partial \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$  hence is a regular cardinal. Then there is  $D = D_\partial^\alpha$ , an  $\aleph_1$ -complete filter on  $Y$  such that  $\partial = \text{ps-tcf}(\Pi\bar{\alpha}/D)$  and  $D \subseteq D'$  for any other such  $D' \in \text{Fil}_{\aleph_1}^1(D)$ .

{r14b} *Remark 5.20.* 1) By 5.9 there are some such  $\partial$ .

2) We work to use just  $\text{AC}_{\mathcal{P}(Y)}$  and not more.

*Proof.* Let

- $\boxplus_1$  (a)  $\mathbb{D} = \{D : D \text{ is an } \aleph_1\text{-complete filters on } Y \text{ such that } (\Pi\bar{\alpha}/D) \text{ has pseudo true cofinality } \partial\}$ ,
- (b)  $D_* = \cap\{D : D \in \mathbb{D}\}$ .

Now obviously

(c)  $D_*$  is an  $\aleph_1$ -complete filter on  $Y$ .

For  $A \subseteq Y$  let  $\mathbb{D}_A = \{D \in \mathbb{D} : A \notin D\}$  and let  $\mathcal{P}_* = \{A \subseteq Y : \mathbb{D}_A \neq \emptyset\}$ . As  $\text{AC}_{\mathcal{P}(Y)}$  we can find  $\langle D_A : A \in \mathcal{P}_* \rangle$  such that  $D_A \in \mathbb{D}_A$  for  $A \in \mathcal{P}_*$ . Let  $\mathbb{D}_* = \{D_A : A \in \mathcal{P}_*\}$ , clearly

$\boxplus_2 D_* = \cap\{D : D \in \mathbb{D}_*\}$  and  $\mathbb{D}_* \subseteq \mathbb{D}$  is non-empty.

As  $\text{AC}_{\mathcal{P}_*}$  holds clearly

(\*)<sub>0</sub> we can choose  $\langle \bar{\mathcal{F}}^A : A \in \mathcal{P}_* \rangle$  such that  $\bar{\mathcal{F}}_A$  exemplifies  $D_A \in \mathbb{D}$  as in 5.17(1),(2), so in particular is  $\aleph_0$ -continuous.

For each  $\beta < \partial$  let  $\mathcal{F}_\beta^* = \cap\{\mathcal{F}_\beta^A : A \in \mathcal{P}_*\}$ , now

(\*)<sub>1</sub>  $\mathcal{F}_\beta^* \subseteq \Pi\bar{\alpha}$ .

[Why? As by 5.17(1)(c) we have  $\mathcal{F}_\beta^A \subseteq \Pi\bar{\alpha}$  for each  $A \in \mathcal{P}_*$ .]

(\*)<sub>2</sub> if  $\beta_1 < \beta_2 < \partial$ ,  $f_1 \in \mathcal{F}_{\beta_1}^*$  and  $f_2 \in \mathcal{F}_{\beta_2}^*$  then  $f_1 < f_2 \text{ mod } D_*$ .

[Why? As  $A \in \mathcal{P}_* \Rightarrow f_1 <_{D_A} f_2$  by the choice of  $\langle \mathcal{F}_\beta^* : \beta < \partial \rangle$ , hence the set  $\{t \in Y : f_1(t) < f_2(t)\}$  belongs to  $D_A$  for every  $A \in \mathcal{P}_*$  hence by  $\boxplus_2$  it belongs to  $D_*$  which means that  $f_1 <_{D_*} f_2$  as required.]

(\*)<sub>3</sub> if  $f \in \Pi\bar{\alpha}$  then for some  $\beta_f < \partial$  we have  $f' \in \cup\{\mathcal{F}_\beta^* : \beta \in [\beta_f, \partial)\} \Rightarrow f < f' \text{ mod } D_*$ .

[Why? For each  $A \in \mathcal{P}_*$  there are  $\beta, g$  such that  $\beta < \partial$ ,  $g \in \mathcal{F}_\beta^A$  and  $f < g \text{ mod } D$  hence  $\beta' \in [\beta+1, \partial) \wedge f' \in \mathcal{F}_{\beta'}^A \Rightarrow f < g < f' \text{ mod } D_A$ . Let  $\beta_A$  be the minimal such ordinal  $\beta_A < \delta$ . As  $\text{cf}(\delta) \geq \theta(\mathcal{P}(Y)) \geq \theta(\mathcal{P}_*)$ , clearly  $\beta_* = \sup\{\beta_A + 1 : A \in \mathcal{P}_*\}$  is  $< \delta$ . So  $A \in \mathcal{P}_* \wedge g \in \cup\{\mathcal{F}_\beta^* : \beta \in [\beta_*, \delta)\} \Rightarrow f <_D g$ . By  $\boxplus_2$  the ordinal  $\alpha_*$  is as required on  $\alpha_\ell$ .]

Moreover

(\*)<sub>4</sub> there is a function  $f \mapsto \beta_f$  in (\*)<sub>3</sub>.

[Why? As we can (and will) choose  $\beta_f$  as minimal  $\beta$  such that ...]

(\*)<sub>5</sub> for every  $\beta_* < \partial$  there is  $\beta \in (\beta_*, \partial)$  such that  $\mathcal{F}_\beta^* \neq \emptyset$ .

[Why? We choose by induction on  $n$ , a sequence  $\bar{\beta}_n = \langle \beta_{n,A} : A \in \mathcal{P}_* \rangle$  and a sequence  $\bar{f}_n = \langle f_{n,A} : A \in \mathcal{P}_* \rangle$  and a function  $f_n$  such that

( $\alpha$ )  $\beta_n < \partial$  and  $m < n \Rightarrow \beta_m < \beta_n$

( $\beta$ )  $\beta_0 = \beta_*$  and for  $n > 0$  we let  $\beta_n = \sup\{\beta_{m,A} : m < n, A \in \mathcal{P}_*\}$

( $\gamma$ )  $\beta_{n,A} \in (\beta_n, \partial)$  is minimal such that there is  $f_{n,A} \in \mathcal{F}_{\beta_{n,A}}^A$  satisfying  $n = m + 1 \Rightarrow f_m < f_{\beta_{n,A}} \text{ mod } D_A$

( $\delta$ )  $\langle f_{n,A} : A \in \mathcal{P}_* \rangle$  is a sequence such that each  $f_{n,A}$  are as in clause ( $\gamma$ )

( $\varepsilon$ )  $f_n \in \Pi\bar{\alpha}$  is defined by  $f_n(t) = \sup\{f_{m,A}(t) + 1 : A \in \mathcal{P}_* \text{ and } m < n\}$ .

[Why can we carry the induction? Arriving to  $n$  first,  $f_n$  is well defined  $\in \Pi\bar{\alpha}$  by clause ( $\varepsilon$ ) as  $\text{cf}(\alpha_t) \geq \theta(\mathcal{P}_*)$  for  $t \in Y$ . Second by clause ( $\gamma$ ),  $\langle \beta_{n,A} : A \in \mathcal{P}_* \rangle$  is well defined. Third by clause ( $\delta$ ) we can choose  $\langle f_{m,A} : A \in \mathcal{P}_* \rangle$  as  $\text{AC}_{\mathcal{P}_*}$ .

Lastly, the inductive construction is possibly by DC.]

Let  $\beta^* = \cup\{\beta_n : n < \omega\}$  and  $f = \sup\langle f_n : n < \omega \rangle$ . Easily  $f \in \cap\{\mathcal{F}_{\beta^*}^A : A \in \mathcal{P}_*\}$  as each  $\langle \mathcal{F}_{\beta}^A : \beta < \partial \rangle$  is  $\aleph_0$ -continuous.]

(\*)<sub>6</sub> if  $f \in \Pi\bar{\alpha}$  then for some  $\beta < \gamma$  and  $f' \in \mathcal{F}_{\beta}^*$  we have  $f < f' \bmod D^*$ .

[Why? By (\*)<sub>3</sub> + (\*)<sub>4</sub>.]

So we are done. □<sub>5.19</sub>

{r16}

**Definition 5.21.** For  $\bar{\alpha} \in {}^Y \text{Ord}$  let  $J_{<\lambda}^{\aleph_1\text{-comp}}(\bar{\alpha}) = \{X \subseteq Y : \text{ps-pcf}_{\aleph_1\text{-com}}(\bar{\alpha} \upharpoonright X) \subseteq \lambda\}$  and  $J_{\leq\lambda}^{\aleph_1\text{-comp}}$  is  $J_{<\lambda^+}^{\aleph_2\text{-comp}}$ .

{r17}

*Remark 5.22.* In 5.21, see Definition 5.6(3).

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