

NON FORKING GOOD FRAMES WITHOUT LOCAL CHARACTER

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ABSTRACT. We continue [Sh:h].II, studying stability theory for abstract elementary classes. In [Sh E46], Shelah obtained a non-forking relation for an AEC, (K, \preceq) , with LST -number at most λ , which is categorical in λ and λ^+ and has less than 2^{λ^+} models of cardinality λ^{++} , but at least one. This non-forking relation satisfies the main properties of the non-forking relation on stable first order theories.

Here we improve this non-forking relation such that it satisfies the local character, too. Therefore it satisfies the main properties of the non-forking relation on superstable first order theories.

Using results of [Sh:h].II, we conclude that the function $\lambda \rightarrow I(\lambda, K)$, which assigns to each cardinal λ , the number of models in K of cardinality λ , is not arbitrary.

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1. PRELIMINARIES

Familiarity with AEC's is assumed.

Hypothesis 1.1.

- (1) (K, \preceq) is an AEC.
- (2) λ is a cardinal.
- (3) The Lowenheim Skolem Tarski number of (K, \preceq) , $LST(K, \preceq)$, is at most λ .

Definition 1.2. Suppose $M_0 \prec N$ in K_λ . We say that N is *universal* over M_0 if for every $M_1 \succ M_0$, there is an embedding of M_1 into N over M_0 , namely, that fixes M_0 .

The following proposition is a version of Fodor's Lemma (there is no mathematical reason to choose this version, but we think that it is convenient).

Proposition 1.3. *There exist no $\langle M_\alpha : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha : \alpha \in \lambda^+ \rangle$, $\langle f_\alpha : \alpha \in \lambda^+ \rangle$, S such that the following conditions are satisfied:*

- (1) *The sequences $\langle M_\alpha : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha : \alpha \in \lambda^+ \rangle$ are \preceq -increasing continuous sequences of models in K_λ .*
- (2) *For every $\alpha < \lambda^+$, $f_\alpha : M_\alpha \rightarrow N_\alpha$ is a \preceq -embedding.*
- (3) *$\langle f_\alpha : \alpha \in \lambda^+ \rangle$ is an increasing continuous sequence.*
- (4) *S is a stationary subset of λ^+ .*
- (5) *For every $\alpha \in S$, there is $a \in M_{\alpha+1} - M_\alpha$ such that $f_{\alpha+1}(a) \in N_\alpha$.*

Proof. Suppose there are such sequences. Denote $M = \bigcup \{f_\alpha[M_\alpha] : \alpha \in \lambda^+\}$. By clauses (4),(5), $\|M\| = K_{\lambda^+}$. $\langle f_\alpha[M_\alpha] : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha \cap M : \alpha \in \lambda^+ \rangle$ are filtrations of M . So they are equal on a club of λ^+ . Hence there is $\alpha \in S$ such that $f_\alpha[M_\alpha] = N_\alpha \cap M$. Hence $f_\alpha[M_\alpha] \subseteq N_\alpha \cap f_{\alpha+1}[M_{\alpha+1}] \subseteq N_\alpha \cap M = f_\alpha[M_\alpha]$ and so this is a chain of equivalences. Especially $f_{\alpha+1}[M_{\alpha+1}] \cap N_\alpha = f_\alpha[M_\alpha]$, in contradiction to condition (5). \dashv

2. NON-FORKING FRAMES

The following definition, Definition 2.1 is an axiomatization of the non-forking relation in a superstable first order theory. If we subtract axiom 2.1(3)(c), we get the basic properties of the non-forking relation in $(K_\lambda, \preceq \upharpoonright K_\lambda)$ where (K, \preceq) is stable in λ .

Sometimes we do not find a natural independence relation with respect to all the types. So first we extend the notion of an AEC in λ by adding a new function S^{bs} which assigns a collection of basic (because they are basic for our construction) types to each model in K_λ , and then add an independence relation \amalg on basic types.

We do not assume *the amalgamation property* in general, but we assume the amalgamation property in $(K_\lambda, \preceq \upharpoonright K_\lambda)$. This is a reasonable assumption, because it is proved in [Sh:h].I, that if an AEC is categorical in λ and the amalgamation property fails in λ , then under a plausible set theoretic assumption, there are 2^{λ^+} models in K_{λ^+} .

Definition 2.1. $\mathfrak{s} = (K, \preceq, S^{bs}, \amalg)$ is a *good λ -frame* if:

- (1) (K, \preceq) is an AEC in λ .
- (2) (a) (K, \preceq) satisfies the joint embedding property.
(b) (K, \preceq) satisfies the amalgamation property.
(c) There is no \preceq -maximal model in K .
- (3) S^{bs} is a function with domain K , which satisfies the following axioms:
 - (a) $S^{bs}(M) \subseteq S^{na}(M) = \{ga - tp(a, M, N) : M \prec N \in K, a \in N - M\}$.
 - (b) S^{bs} respects isomorphisms: if $ga - tp(a, M, N) \in S^{bs}(M)$ and $f : N \rightarrow N'$ is an isomorphism, then $ga - tp(f(a), f[M], N') \in S^{bs}(f[M])$.

- (c) Density of the basic types: if $M, N \in K_\lambda$ and $M \prec N$, then there is $a \in N - M$ such that $ga - tp(a, M, N) \in S^{bs}(M)$.
 - (d) Basic stability: for every $M \in K$, the cardinality of $S^{bs}(M)$ is $\leq \lambda$.
- (4) the relation \amalg satisfies the following axioms:
- (a) \amalg is set of quadruples (M_0, M_1, a, M_3) where $M_0, M_1, M_3 \in K$, $a \in M_3 - M_1$ and for $n = 0, 1$ $ga - tp(a, M_n, M_3) \in S^{bs}(M_n)$ and it respects isomorphisms: if $\amalg(M_0, M_1, a, M_3)$ and $f : M_3 \rightarrow M'_3$ is an isomorphism, then $\amalg(f[M_0], f[M_1], f(a), M'_3)$.
 - (b) Monotonicity: if $M_0 \preceq M_0^* \preceq M_1^* \preceq M_1 \preceq M_3 \preceq M_3^*$, $M_1^* \cup \{a\} \subseteq M_3^{**} \preceq M_3^*$, then $\amalg(M_0, M_1, a, M_3) \Rightarrow \amalg(M_0^*, M_1^*, a, M_3^{**})$. From now on, ' $p \in S^{bs}(N)$ does not fork over M ' will be interpreted as 'for some a, N^+ we have $p = ga - tp(a, N, N^+)$ and $\amalg(M, N, a, N^+)$ '. See Proposition 2.2.
 - (c) Local character: for every limit ordinal $\delta < \lambda^+$ if $\langle M_\alpha : \alpha \leq \delta \rangle$ is an increasing continuous sequence of models in K_λ , and $ga - tp(a, M_\delta, M_{\delta+1}) \in S^{bs}(M_\delta)$, then there is $\alpha < \delta$ such that $ga - tp(a, M_\delta, M_{\delta+1})$ does not fork over M_α .
 - (d) Uniqueness of the non-forking extension: if $M, N \in K$, $M \preceq N$, $p, q \in S^{bs}(N)$ do not fork over M , and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
 - (e) Symmetry: if $M_0, M_1, M_3 \in K_\lambda$, $M_0 \preceq M_1 \preceq M_3$, $a_1 \in M_1$, $ga - tp(a_1, M_0, M_3) \in S^{bs}(M_0)$, and $ga - tp(a_2, M_1, M_3)$ does not fork over M_0 , then there are $M_2, M_3^* \in K_\lambda$ such that $a_2 \in M_2$, $M_0 \preceq M_2 \preceq M_3^*$, $M_3 \preceq M_3^*$, and $ga - tp(a_1, M_2, M_3^*)$ does not fork over M_0 .
 - (f) Existence of non-forking extension: if $M, N \in K$, $p \in S^{bs}(M)$ and $M \prec N$, then there is a type $q \in S^{bs}(N)$ such that q does not fork over M and $q \upharpoonright M = p$.
 - (g) Continuity: let $\delta < \lambda^+$ and $\langle M_\alpha : \alpha \leq \delta \rangle$ be an increasing continuous sequence of models in K and let $p \in S(M_\delta)$. If for every $\alpha \in \delta$, $p \upharpoonright M_\alpha$ does not fork over M_0 , then $p \in S^{bs}(M_\delta)$ and does not fork over M_0 .

Proposition 2.2. *If $\amalg(M_0, M_1, a, M_3)$ and the types $ga - tp(b, M_1, M_3^*)$, $ga - tp(a, M_1, M_3)$ are equal, then we have $\amalg(M_0, M_1, a, M_3)$.*

Proof. Since $ga - tp(b, M_1, M_3^*) = ga - tp(a, M_1, M_3)$, there is an amalgamation (id_{M_3}, f, M_3^{**}) of M_3 and M_3^* over M_1 with $f(b) = a$. By Definition 2.1(3)(b) (monotonicity) $\amalg(M_0, M_1, a, M_3^{**})$. Using again Definition 2.1(3)(b), we get $\amalg(M_0, M_1, a, f[M_3^*])$. Therefore by Definition 2.1(3)(a), $\amalg(M_0, M_1, a, M_3^*)$. \dashv

Definition 2.3.

- (1) $\mathfrak{s} = (K, \preceq, S^{bs}, nf)$ is an *almost good* λ -frame if \mathfrak{s} satisfies the axioms of a good λ -frame except maybe local character, but \mathfrak{s} satisfies weak local character.
- (2) \mathfrak{s} satisfies *weak local character* when there is a 2-ary relation, \prec^* on K_λ which is included in $\prec \upharpoonright K_\lambda$ such that:
 - (a) for each $M_0 \in K_\lambda$ there is $M_1 \in K_\lambda$ with $M_0 \prec^* M_1$,
 - (b) if $M_0 \prec^* M_1 \preceq M_2 \in K_\lambda$ then $M_0 \prec^* M_2$,
 - (c) if $\langle N_\alpha : \alpha < \delta + 1 \rangle$ is a \prec^* -increasing continuous sequence of models in K_λ , then for some $a \in N_{\delta+1}$ and some ordinal $\alpha < \delta$, $p =: ga - tp(a, N_\delta, N_{\delta+1})$ is a basic type, which does not fork over N_α .

In the following definition ‘na’ means non-algebraic.

Definition 2.4. We define a function S^{na} with domain K_λ by $S^{na}(M) := \{ga - tp(a, M, N) : M \preceq N, a \in N - M\}$.

Definition 2.5. Let \mathfrak{s} be an almost good λ -frame. \mathfrak{s} is *full* if $S^{bs} = S^{na}$.

The following theorem says that the stability property in λ is satisfied and presents sufficient conditions for a universal model. The stability in λ can actually derived from [JrSh 875, Theorem 2.20].

Theorem 2.6.

- (1) *Suppose:*
 - (a) \mathfrak{s} is an almost good λ -frame (so indirectly, we assume basic stability).
 - (b) $\langle M_\alpha : \alpha \leq \lambda \rangle$ is an increasing continuous sequence of models in K_λ .
 - (c) $M_{\alpha+1}$ realizes $S^{bs}(M_\alpha)$.
 - (d) $M_\alpha \prec^* M_{\alpha+1}$.

Then M_λ is universal over M_0 .
- (2) *There is a model in K_λ which is universal over λ .*
- (3) *For every $M \in K_\lambda$, $|S(M)| \leq \lambda$.*

Proof. Obviously (1) \Rightarrow (2) \Rightarrow (3). Why does (1) hold? We have to prove that letting $M_0 \prec N$, N can be embedded in M_λ over M_0 . Toward a contradiction assume that:

(*) There is no an embedding from N into M_λ over M_0 .

Let cd be a bijection from $\lambda \times \lambda$ onto λ . Now we choose $N_\alpha, A_\alpha, \langle a_{\alpha,\beta} : \beta < \lambda \rangle, f_\alpha$ by induction on α such that:

- (1) $N_0 = N, f_0 = id_{M_0}$
- (2) $\langle N_\alpha : \alpha < \lambda \rangle$ is an increasing continuous sequence of models in K_λ .
- (3) $\langle f_\alpha : \alpha < \lambda \rangle$ is an increasing continuous sequence of functions.
- (4) $f_\alpha : M_\alpha \hookrightarrow N_\alpha$ is an embedding.
- (5) $N_\alpha = \{a_{\alpha,\beta} : \beta < \lambda\}$.
- (6) $A_\alpha = \{cd(\gamma, \beta) : \gamma \preceq \alpha, ga - tp(a_{\gamma,\beta}, f_\alpha[M_\alpha], N_\alpha) \in S^{bs}(f_\alpha[M_\alpha])\}$.

(7) $a_{\gamma,\beta} \in f_{\alpha+1}[M_{\alpha+1}]$ where $(\gamma, \beta) = cd^{-1}(\text{Min}(A_\alpha))$.

Why can we carry out the induction? For $\alpha = 0$ or limit, there is no problem. Suppose we have chosen $N_\alpha, A_\alpha, \langle a_{\alpha,\beta} : \beta < \lambda \rangle, f_\alpha$. If $f_\alpha[M_\alpha] = N_\alpha$, then $f_\alpha^{-1} \upharpoonright N_0$ is an embedding over M_0 , in contradiction to (*). Thus $f_\alpha[M_\alpha] \neq N_\alpha$. Therefore there is a type in $S^{bs}(f_\alpha[M_\alpha])$ which N_α realizes. Hence $A_\alpha \neq \emptyset$. So by the definition of a type, there is no problem to find $N_{\alpha+1}, A_{\alpha+1}, \langle a_{\alpha+1,\beta} : \beta < \lambda \rangle, f_{\alpha+1}$.

Why is this enough? Define $N_\lambda := \bigcup \{N_\alpha : \alpha < \lambda\}$, $f_\lambda := \bigcup \{f_\alpha : \alpha < \lambda\}$. By smoothness, $f_\lambda[M_\lambda] \preceq N_\lambda$. But $f_\lambda[M_\lambda] \neq N_\lambda$ (otherwise $f_\lambda^{-1} \upharpoonright N_0$ is an embedding over M_0 , in contradiction to (*)). So by *weak local character*, there is $c \in N_\lambda - f_\lambda[M_\lambda]$ and there is a $\gamma \in \lambda$ such that $ga - tp(c, f_\lambda[M_\lambda], N_\lambda)$ does not fork over $f_\gamma[M_\gamma]$. Without loss of generality, $c \in N_\gamma$, because we can increase γ . Therefore there is $\beta \in \lambda$ such that $c = a_{\gamma,\beta}$. Hence $ga - tp(a_{\gamma,\beta}, f_\gamma[M_\gamma], N_\gamma) \in S^{bs}(f_\gamma[M_\gamma])$. Define an injection $g : [\gamma, \lambda) \rightarrow \lambda$ by $g(\alpha) := \text{Min}(A_\alpha)$. For each $\alpha \in [\gamma, \lambda)$, $cd(\gamma, \beta) \in A_\alpha$. So $g(\alpha) < cd(\gamma, \beta)$, (otherwise by (7) $a_{\gamma,\beta} \in f_{\alpha+1}[M_{\alpha+1}] \subset f_\lambda[M_\lambda]$, but $a_{\gamma,\beta} = c \notin f_\lambda[M_\lambda]$), and g is an injection from $[\gamma, \lambda)$ to $cd(\gamma, \beta)$ which is impossible. Thus (*) implies a contradiction. \dashv

3. NON-FORKING AMALGAMATION

Hypothesis 3.1. \mathfrak{s} is an almost good λ -frame.

In this section we present a theorem from [JrSh 875], which says that we can derive a non-forking relation on models, from the non-forking relation on elements. First we have to define the conjugation property.

Definition 3.2.

(1) Let $p = ga - tp(a, M, N)$. Let f be an isomorphism of M (i.e. f is an injection with domain M , and the relations and functions on $f[M]$ are defined such that $f : M \hookrightarrow f[M]$ is an isomorphism). Define $f(p) = ga - tp(f(a), f[M], f^+[N])$, where f^+ is an extension of f (and the relations and functions on $f^+[N]$ are defined such that $f^+ : N \hookrightarrow f^+[N]$ is an isomorphism).

(2) Let p_0, p_1 be types, $n < 2 \rightarrow p_n \in S(M_n)$. We say that p_0, p_1 are *conjugate* if there is an isomorphism $f : M_0 \hookrightarrow M_1$ such that $f(p_0) = p_1$.

Claim 3.3.

- (1) In Definition 3.2, $f(p)$ does not depend on the choice of f^+ .
- (2) The conjugation relation is an equivalence relation.

Proof. Easy. \dashv

Definition 3.4. Let \mathfrak{s} be an almost good λ -frame. \mathfrak{s} is said to satisfy the *conjugation property*, when: if $p \in S^{bs}(M_1)$ does not fork over M_0 , then there is an isomorphism $f : M_1 \rightarrow M_0$ such that $f(p) = p \upharpoonright M_0$.

Remark 3.5. If \mathfrak{s} satisfies the conjugation property, then K_λ is categorical.

Now we present the properties that a non-forking relation should satisfy.

Definition 3.6. Let $NF \subseteq {}^4K_\lambda$ be a relation. We say \otimes_{NF} when the following axioms are satisfied:

- (a) If $NF(M_0, M_1, M_2, M_3)$, then $n \in \{1, 2\} \rightarrow M_0 \preceq M_n \preceq M_3$ and $M_1 \cap M_2 = M_0$.
- (b) The monotonicity axiom: if $NF(M_0, M_1, M_2, M_3)$ and $N_0 = M_0, n < 3 \rightarrow N_n \preceq M_n \wedge N_0 \preceq N_n \preceq N_3, (\exists N^*)[M_3 \preceq N^* \wedge N_3 \preceq N^*]$, then $NF(N_0, N_1, N_2, N_3)$.
- (c) The existence axiom: for every $N_0, N_1, N_2 \in K_\lambda$, if $l \in \{1, 2\} \rightarrow N_0 \preceq N_l$ and $N_1 \cap N_2 = N_0$, then there is N_3 such that $NF(N_0, N_1, N_2, N_3)$.
- (d) The uniqueness axiom: suppose for $x = a, b$ we have $NF(N_0, N_1, N_2, N_3^x)$. Then there is a joint embedding of N^a, N^b over $N_1 \cup N_2$.
- (e) The symmetry axiom: $NF(N_0, N_1, N_2, N_3) \leftrightarrow NF(N_0, N_2, N_1, N_3)$.
- (f) The long transitivity axiom: for $x = a, b$, let $\langle M_{x,i} : i \preceq \alpha^* \rangle$ be an increasing continuous sequence of models in K_λ . Suppose $i < \alpha^* \rightarrow NF(M_{a,i}, M_{a,i+1}, M_{b,i}, M_{b,i+1})$. Then $NF(M_{a,0}, M_{a,\alpha^*}, M_{b,0}, M_{b,\alpha^*})$.

Definition 3.7. Let NF be a relation such that \otimes_{NF} . We say that NF respects the frame \mathfrak{s} when: if $NF(M_0, M_1, M_2, M_3)$ and $ga - tp(a, M_0, M_1) \in S^{bs}(M_0)$, then $ga - tp(a, M_2, M_3)$ does not fork over M_0 .

Theorem 3.8. *Suppose:*

- (1) K is categorical in λ .
- (2) \mathfrak{s} is an almost good λ -frame which satisfies the conjugation property.
- (3) $I(\lambda^{++}, K) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$.
- (4) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.
- (5) The ideal $WdId(\lambda^+)$ is not saturated in λ^{++} .

Then there is a relation NF such that \otimes_{NF} and NF respects the frame \mathfrak{s} .

Proof. By [JrSh 875]: by Corollary [JrSh 875, 4.18], $K^{3,ug}$ is dense with respect to \preceq_{bs} . Hence by Theorem [JrSh 875, 5.15], there is a unique relation, NF , with \otimes_{NF} . Now see Definition [JrSh 875, 5.3]. \dashv

4. A FULL GOOD λ -FRAME

Hypothesis 4.1. \mathfrak{s} is an almost good λ -frame which satisfies the conjugation property.

Definition 4.2. $nf^{NF} := \{(M_0, M_1, a, M_3) : M_0, M_1, M_3 \in K_\lambda, M_0 \preceq M_1 \preceq M_3, a \in M_3 - M_1 \text{ and for some } M_2 \in K_\lambda, M_0 \preceq M_2, a \in M_2 - M_0 \text{ and } NF(M_0, M_1, M_2, M_3)\}$.

The following theorem is similar to Claim [Sh:h, 9.5.2].III.

Theorem 4.3. *Let \mathfrak{s} be an almost good λ -frame which satisfies the conjugation property. Then $\mathfrak{s}^{NF} = (K, \preceq, S^{na}, nf^{NF})$ is a full good λ -frame.*

Proof. We will prove the conditions in Definition 2.1:

1. Trivial.
2. (a),(b),(c) are trivial. (d) (basic stability) is satisfied by Theorem 2.6(3).
3. (a) is trivial.
(b) is OK by the monotonicity of NF , i.e. Definition 3.6(b).

Axiom (c) (local character) is the heart of the matter. Let j be a limit ordinal, let $\langle N_i : i \preceq j+1 \rangle$ be an increasing continuous sequence of models in K_λ and let $p =: ga - tp(c, N_j, N_{j+1}) \in S^{na}(N_j)$. We have to find $i < j$ such that p does not fork over N_i in the sense of nf^{NF} , i.e. $nf^{NF}(N_i, c, N_j, N_{j+1})$. It is enough to find an increasing continuous sequence $\langle M_i : i \preceq j \rangle$ such that for each $i \preceq j$, $N_i \preceq M_i$ and $NF(N_i, N_{i+1}, M_i, M_{i+1})$ (so $NF(N_i, N_j, M_i, M_j)$) and $N_{j+1} \preceq M_j$ (for some $i < j$ $c \in M_i$, so $nf^{NF}(N_i, c, N_j, N_{j+1})$). Without loss of generality, $cf(j) = j$. We try to construct $\langle N_{\alpha,i} : i \preceq j+1 \rangle$ by induction on $\alpha \in \lambda^+$, such that:

- (1) For each $\alpha \in \lambda^+$, $\langle N_{\alpha,i} : i \preceq j+1 \rangle$ is an increasing continuous sequence of models in K_λ .
- (2) For each $i \preceq j$, $\langle N_{\alpha,i} : \alpha < \lambda^+ \rangle$ is an \prec^* -increasing continuous sequence of models in K_λ and $N_{\alpha,j+1} \preceq N_{\alpha+1,j+1}$.
- (3) $N_{0,i} = N_i$.
- (4) For each $i < j$ and $\alpha < \lambda^+$, we have $NF(N_{\alpha,i}, N_{\alpha,i+1}, N_{\alpha+1,i}, N_{\alpha+1,i+1})$.
- (5) For each $\alpha \in S =: \{\delta \in \lambda^+ : cf(\delta) = j\}$, we have $N_{\alpha,j+1} \cap N_{\alpha+1,j} \neq N_{\alpha,j}$.

If we succeed, then by clauses (2) and (5), the quadruple

$$\langle N_{\alpha,j} : \alpha < \lambda^+ \rangle, \langle N_{\alpha,j+1} : \alpha < \lambda^+ \rangle, \langle id_{N_{\alpha_j}} : \alpha < \lambda^+ \rangle, S$$

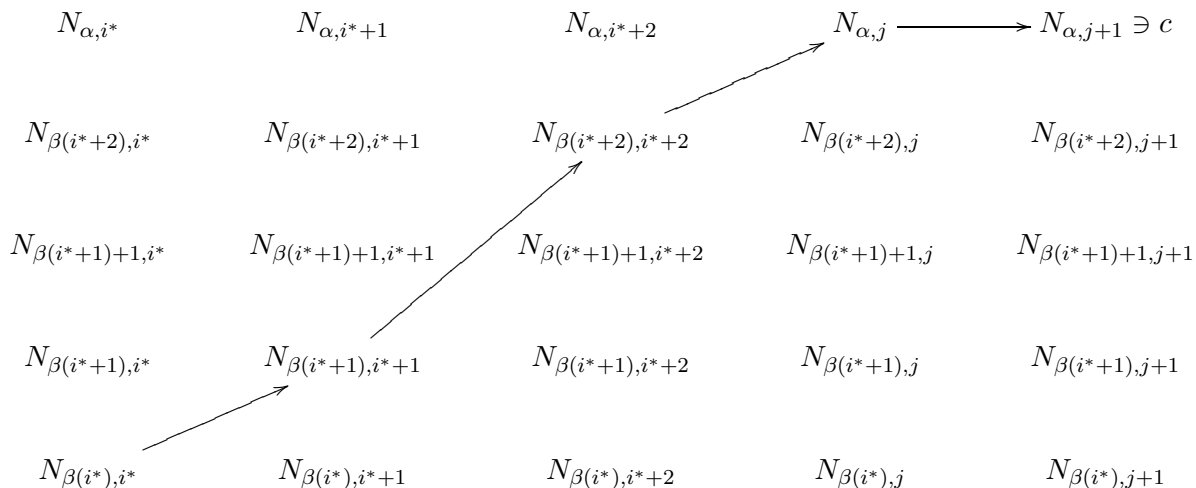
forms a counterexample to Claim 1.3, so it is impossible to carry out this construction.

Where will we get stuck? For $\alpha = 0$, we will not get stuck, see item (3).

For α limit, just (1),(2) are relevant, and we just have to take unions and use smoothness.

So we will get stuck at some successor ordinal. Suppose we have defined $\langle N_{\alpha,i} : i \preceq j+1 \rangle$. Can we find $\langle N_{\alpha+1,i} : i \preceq j+1 \rangle$? If $\alpha \notin S$, then it is easier, so assume $\alpha \in S$. Let $\langle \beta(i) : i \preceq j+1 \rangle$ be an increasing continuous sequence of ordinals such that $\beta(j) = \alpha$. If $N_{\alpha,j} = N_{\alpha,j+1}$, then we can define $M_i := N_{\alpha,i}$ and the local character is proved ($N_j \preceq N_{\alpha,j} = M_j$, so see the beginning of the proof). So without loss of generality, $N_{\alpha,j+1} \neq N_{\alpha,j}$.

In the following diagram, the arrows describe the \prec^* -increasing continuous sequence $\langle N_{\beta(i),i} : i \leq j \rangle \frown \langle N_{\alpha,j+1} \rangle$. A model that appears at the right and above another model is bigger than it.



By weak local character, there is an element c and an ordinal i^* such that $ga - tp(c, N_{\alpha,j}, N_{\alpha,j+1})$ does not fork over $N_{\beta(i^*),i^*}$.

By Definition 2.1(b) (the monotonicity axiom), $ga - tp(c, N_{\alpha,j}, N_{\alpha,j+1})$ does not fork over N_{α,i^*+1} and so $ga - tp(c, N_{\alpha,i^*+1}, N_{\alpha,j+1}) \in S(N_{\alpha,i^*+1})$. So there is an increasing continuous sequence $\langle N_{\alpha+1,i}^{temp} : i \leq j \rangle$ such that for $i < j$ we have $NF(N_{\alpha,i}, N_{\alpha,i+1}, N_{\alpha+1,i}^{temp}, N_{\alpha+1,i+1}^{temp})$, and there is $a \in N_{\alpha+1,i^*+1}^{temp}$ such that $ga - tp(a, N_{\alpha,i^*+1}, N_{\alpha+1,i^*+1}^{temp}) = ga - tp(c, N_{\alpha,i^*+1}, N_{\alpha,j+1})$. [Why? For $i \leq i^*$ define $N_{\alpha+1,i}^{temp} = N_{\alpha,i}$. Choose $N_{\alpha+1,i^*+1}^{temp}$ which is isomorphic to $N_{\alpha,j+1}$ over N_{α,i^*+1} and $N_{\alpha+1,i^*+1}^{temp} \cap N_{\alpha,j+1} = N_{\alpha,i^*+1}$. For $i \in (i^*+1, j]$ choose $N_{\alpha+1,i+1}^{temp}$ such that $NF(N_{\alpha,i}, N_{\alpha,i+1}, N_{\alpha+1,i}^{temp}, N_{\alpha+1,i+1}^{temp})$. If i is limit, then define $N_{\alpha+1,i}^{temp} := \bigcup \{N_{\alpha+1,\varepsilon} : \varepsilon < i\}$. Now by the long transitivity of NF we have $NF(N_{\alpha,i^*+1}, N_{\alpha,j}, N_{\alpha+1,i^*+1}^{temp}, N_{\alpha+1,j}^{temp})$ and so since NF respects s , the type $ga - tp(a, N_{\alpha,j}, N_{\alpha+1,j}^{temp})$ does not fork over N_{α,i^*+1} . So by Definition 2.1(e), (the uniqueness of the non-forking extension), $ga - tp(a, N_{\alpha,j}, N_{\alpha+1,j}^{temp}) = ga - tp(c, N_{\alpha,j}, N_{\alpha,j+1})$. Hence by the definition of the equality between types, without loss of generality, there is a model $N_{\alpha+1,j+1}$ such that $N_{\alpha,j+1} \preceq N_{\alpha+1,j+1}$, there is an embedding $f : N_{\alpha+1,j}^{temp} \hookrightarrow N_{\alpha+1,j+1}$ over $N_{\alpha,j}$ and $f(a) = c$. Now for $i \leq j$ define $N_{\alpha+1,i} := f[N_{\alpha+1,i}^{temp}]$. Why is (5) satisfied? $c \in N_{\alpha,j+1} \cap N_{\alpha+1,i+1} - N_{\alpha,i+1}$. By (4) and the long transitivity of NF , we have $NF(N_{\alpha,i+1}, N_{\alpha,j}, N_{\alpha+1,i+1}, N_{\alpha+1,j})$, so $c \notin N_{\alpha,j}$, but since $N_{\alpha+1,i+1} \subset N_{\alpha+1,j}$ we have $c \in N_{\alpha+1,j}$. Hence $c \in N_{\alpha,j+1} \cap N_{\alpha+1,j} - N_{\alpha,j}$. Hence we can carry out the construction.

(d) Uniqueness: suppose for $n < 2$, $ga - tp(a^n, M_0, M_1^n)$ does not depend on n , and $NF(M_0, M_2, M_1^n, M_3^n)$, see the diagram below. We have to prove

that $ga - tp(a^n, M_2, M_3^n)$ does not depend on n . By the definition of the equality between types, there is an amalgamation f^0, f^1, M_1 of M_1^0, M_1^1 over M_0 . So there are models $M_3^{n,+}$ and embeddings $f_n^+ : M_3^n \hookrightarrow M_3^{n,+}$, such that for $n < 2$ we have $NF(f_n[M_1^n], f_n^+[M_3^n], M_1, M_3^{n,+})$ and $f_n \subset f_n^+$. Since $M_2 \cap M_1^n = M_0$, without loss of generality, $f_n^+ \upharpoonright M_2 = id_{M_2}$ (we can change the names of the elements in $M_2 - M_0$, i.e. $M_2 - M_1^n$). By the long transitivity axiom of NF , we have $NF(M_0, M_2, M_1, M_3^{n,+})$. So by the uniqueness of NF , there is a joint embedding g^0, g^1, M_3 of M_3^0, M_3^1 over $M_1 \cup M_2$. So $g^0 \circ f_0^+, g^1 \circ f_1^+, M_3$ is an amalgamation of M_3^0, M_3^1 over M_2 . Since $a_n \in M_1^n$, $(g^n \circ f_n^+)(a_n) = f_n(a_n)$ and so it does not depend on n (since f_0, f_1 are witnesses for $ga - tp(a_1, M_0, M_1^n)$ does not depend on n). So $ga - tp(a^n, M_2, M_3^n)$ does not depend on n .

$$\begin{array}{ccccc}
 & & M_1 & \longrightarrow & M_3^{n,+} & \xrightarrow{g^n} & M_3 \\
 & & \uparrow & & \uparrow & & \\
 & & f_n & & f_n^+ & & \\
 & & M_1^n & \longrightarrow & M_3^n & & \\
 & & \uparrow & & \uparrow & & \\
 & & M_0 & \longrightarrow & M_2 & &
 \end{array}$$

- (e) Symmetry: by the symmetry of NF , i.e. Definition 3.6(e).
- (f) By the corresponding axiom of NF , i.e. Definition 3.6(c).
- (g) Continuity: it is easy to see that continuity follows by local character, because by definition, s^{NF} is full. ⊣

Now we can present the main theorem: we get a good λ -frame.

Theorem 4.4. *Let (K, \preceq) be an AEC such that:*

- (1) K is categorical in λ, λ^+ and $1 \leq I(\lambda^{+2}, K) < \mu_{unif}(\lambda^{+2}, 2^{\lambda^+})$.
- (2) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$, and $WDmId(\lambda^+)$ is not saturated in λ^{+2} .

Then:
there is an almost good λ -frame, \mathfrak{s} with complete... $(K_{\mathfrak{s}}, \preceq_{\mathfrak{s}}) = ((K_\lambda, \preceq)$ and a type is basic if it is minimal. Moreover, if \mathfrak{s} satisfies the conjugation property, then there is a good λ -frame with $(K_s, \preceq_s) = ((K, \preceq)$.

Remark 4.5. Background on Weak Diamond appears in [DS] and in Chapter 13 of [Gr:book]. Concerning $\mu_{unif}(\mu^+, 2^\mu)$, see the last chapter of [Sh:h], [JrSh 875] or [JrSh 966]. It is "almost 2^{μ^+} ": $1 < \mu_{unif}(\mu^+, 2^\mu)$, If $\beth_\omega \preceq \mu$, then $\mu_{unif}(\mu^+, 2^\mu) = 2^{\mu^+}$ and in any case it is not clear if $\mu_{unif}(\mu^+, 2^\mu) < 2^{\mu^+}$ is consistent. There are more claims which say that it is a "big cardinal".

Proof. By Theorem [Sh E46, 0.2] there is such an almost good frame. So by Theorem 4.3 we have the "moreover". ⊣

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While in [Sh:h].II we obtained a good λ^+ -frame, here we obtained a λ -good frame. Why is this important? In Section 1 of [Sh:h].III, Shelah defined weakly dimensionality of a good frame, and proved that it is equal to the categoricity in the successor cardinal. Since here we assume categoricity in λ^+ , the good λ -frame we obtained here is weakly dimensional.

5. THE FUNCTION $\lambda \rightarrow I(\lambda, K)$ IS NOT ARBITRARY

In this section, we prove, under set theoretical assumptions, that there is no AEC, (K, \preceq) , which is categorical in $\lambda, \lambda^+ \dots \lambda^{+(n-1)}$, but has no model of cardinality λ^{+n} . The main results of Section 4 enables to prove only a weaker version of this theorem. But we can prove this theorem, using results of [Sh E46] and [Sh:h].II.

By the last section in [Sh:h].II (alternatively, see Corollary [JrSh 875, 12.6]):

Fact 5.1. *Suppose:*

- (1) $n < \omega$,
- (2) $\mathfrak{s} = (K, \preceq, S^{bs}, nf)$ is a good λ -frame ,
- (3) For each $m < n$, $I(\lambda^{+(2+m)}, K) < \mu_{unif}(\lambda^{+(2+m)}, 2^{\lambda^{+(1+m)}})$,
- (4) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}} < \dots < 2^{\lambda^{+(1+n)}}$,
- (5) For each $m < n$, the ideal $WDmId(\lambda^{+1+m})$ is not saturated in $\lambda^{+(2+m)}$.

Then there is a model in K of cardinality $\lambda^{+(2+n)}$.

By the following theorem, if $f : card \rightarrow card$ is a class function (from the cardinals to the cardinals) with $f(\lambda) = f(\lambda^+) = \dots f(\lambda^{+(n-1)}) = 1$ and $f(\lambda^{+n}) = 0$, then under specific set theoretical assumptions (clauses (4),(5), below), f cannot be the spectrum of categoricity of any AEC.

Theorem 5.2. *There are no K, \preceq, n, λ such that*

- (1) $n \geq 3$ is a natural number,
- (2) (K, \preceq) is an AEC,
- (3) K is categorical in λ^{+m} for each $m < n$, but $K_{\lambda^{+n}} = \emptyset$,
- (4) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}} < \dots < 2^{\lambda^{+(n-1)}}$,
- (5) For each $m < n - 2$, $WDmId(\lambda^{+1+m})$ is not saturated in $\lambda^{+(2+m)}$.

Before we prove Theorem 5.2, we prove a weaker version of it:

Proposition 5.3. *The same as Theorem 5.2, but here we assume, in addition, that if $M_0 \preceq M_1 \preceq M_2$, $a \in M_2 - M_1$ and $ga - tp(a, M_0, M_2)$ is minimal, then the types $ga - tp(a, M_1, M_2), ga - tp(a, M_0, M_2)$ are conjugate.*

Proof. By Theorem 4.4, there is a good λ -frame with $(K_s, \preceq_s) = ((K, \preceq)$. Hence by Fact 5.1, there is a model in K of cardinality $\lambda^{+(n)}$. \dashv

Remark 5.4. To our opinion, by Claim [Sh E46, 7.4](p. 76), it is reasonable to assume that \mathfrak{s} satisfies the conjugation property.

Now we prove Theorem 5.2:

Proof. By Theorem [Sh:h, II.3.7](p. 297), there is a good λ^+ -frame, \mathfrak{s} such that its AEC is (K, \preceq) . Now use Fact 5.1, where λ^+ stands for λ . \dashv

REFERENCES

- [DS] Keith J. Devlin and Saharon Shelah. A weak version of \diamond which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel Journal of Mathematics*, **29**:239-247,1978
- [Gr:book] Rami Grossberg. *Classification of Abstract Elementary Classes*. A book in preparation.
- [JrSh 875] Adi Jarden and Saharon Shelah. Good frames with a weak stability. Submitted to A.P.A.L. Preprint available at <http://front.math.ucdavis.edu/0901.0852>.
- [JrSh 966] Adi Jarden and Saharon Shelah. Existence of uniqueness triples without stability. Work in progress.
- [Sh E46] Saharon Shelah. Categoricity of an abstract elementary class in two successive cardinals. Preprint available at <http://front.math.ucdavis.edu>.
- [Sh:h] Saharon Shelah. *Classification Theory for Abstract Elementary Classes*. *Studies in Logic*. College Publications. www.collegepublications.co.uk 2009. Binds together papers 88r,300,600,705,838 with introduction E53.

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