

MODELS OF PA: STANDARD SYTEMS WITHOUT MINIMAL
ULTRAFILTERS
SH944

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ABSTRACT. We prove that \mathbb{N} , the standard model of arithmetic, has an uncountable elementary extension N such that there is no ultrafilter on the Boolean Algebra of subsets of \mathbb{N} represented in N which is minimal (i.e. as in Rudin-Keisler order for partitions represented in N).

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0. INTRODUCTION

Enayat [Ena08], Question III, asked (see Definition 0.4(1)):

{q0.7}

Question 0.1. Can we prove in ZFC that there is an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that \mathcal{A} carries no minimal ultrafilter?

He proved the existence of examples, for the stronger notion “2-Ramsey ultrafilter”. In [Sh:937] we prove that there is an arithmetically closed Borel set $\mathbf{B} \subseteq \mathcal{P}(\mathbb{N})$ such that any expansion \mathbb{N}^+ of \mathbb{N} by any uncountably many members of \mathbf{B} has this property, i.e. the family of definable subsets of \mathbb{N}^+ carries no 2-Ramsey ultrafilter.

We deal here with Question 0.1, proving that there is such a family of cardinality \aleph_1 , this implies the version in the abstract; (since it is well-known that every arithmetically closed family of cardinality at most \aleph_1 can be realized as the standard system of some elementary extension of \mathbb{N} , as shown by Knight and Nadel [KN82]). We use forcing but the result is proved in ZFC. On other problems from [Ena08] see Enayat-Shelah [EnSh:936] and [Sh:924], [Sh:937].

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{0z.1}

Notation 0.2. 1) Let $\text{pr}:\omega \times \omega \rightarrow \omega$ be the standard pairing function (i.e. $\text{pr}(n, m) = \binom{n+m}{2} + n$, so one to one onto two-place function).

2) Let \mathcal{A} denote a subset of $\mathcal{P}(\omega)$.

3) Let $\text{BA}(\mathcal{A})$ be the Boolean algebra which $\mathcal{A} \cup \{\omega\}^{<\aleph_0}$ generates.

4) Let D denote a non-principal ultrafilter on \mathcal{A} , meaning that $D \subseteq \mathcal{A}$ and there is a unique non-principal ultrafilter D' on the Boolean algebra $\text{BA}(\mathcal{A})$ satisfying $D = D' \cap \mathcal{A}$, notice that in Definition 0.4 below the distinction between an ultrafilter on \mathcal{A} and on $\text{BA}(\mathcal{A})$ makes a difference.

5) τ denotes a vocabulary extending $\tau_{\text{PA}} = \tau_{\mathbb{N}} = \{0, 1, +, \times, <\}$, usually countable.

6) $\text{PA}(\tau)$ is Peano arithmetic for the vocabulary τ . A model N of $\text{PA}(\tau)$ is called ordinary if $N \upharpoonright \tau_{\text{PA}}$ extends \mathbb{N} ; usually the models will be ordinary.

7) $\varphi(N, \bar{a})$ is $\{b : N \models \varphi[b, \bar{a}]\}$ where $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_N)$ and $\bar{a} \in {}^{\ell g(\bar{y})}N$.

8) $\text{Sym}(A)$ is the set (or group) of permutations of N .

9) For sets u, v of ordinals let $\text{OP}_{v,u}$, “the order preserving function from u to v ” be defined by: $\text{OP}_{v,u}(\alpha) = \beta$ iff $\beta \in v, \alpha \in u$ and $\text{otp}(v \cap \beta) = \text{otp}(u \cap \alpha)$.

10) We say $u, v \subseteq \text{Ord}$ form a Δ -system pair when $\text{otp}(u) = \text{otp}(v)$ and $\text{OP}_{v,u}$ is the identity on $u \cap v$.

{0z.2}

Definition 0.3. 1) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\text{ar-cl}(\mathcal{A}) = \{B \subseteq \omega : B \text{ is first order defined in } (\mathbb{N}, A_1, \dots, A_n) \text{ for some } n < \omega \text{ and } A_1, \dots, A_n \in \mathcal{A}\}$. This is called the arithmetic closure of \mathcal{A} .

2) For a model N of $\text{PA}(\tau)$ let the standard system of N , $\text{SSy}(N)$ be $\{\varphi(M, \bar{a}) \cap \mathbb{N} : \varphi(x, \bar{y}) \in \mathbb{L}(\tau) \text{ and } \bar{a} \in {}^{\ell g(\bar{y})}M\}$ so $\subseteq \mathcal{P}(\omega)$ for any ordinary model M isomorphic to N , see 0.2(6).

{0z.7}

Definition 0.4. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

0) Let $\text{cd}_0 : \mathcal{H}(\aleph_0) \rightarrow \omega$ be one to one, and interpreting $\mathcal{H}(\aleph_0)$ inside \mathbb{N} it is (first order) definable by a bounded formula in \mathbb{N} , i.e. $\{\text{cd}_0(x, y) : x \in y \in \mathcal{H}(\aleph_0)\}$ is, and it maps $\mathbb{N} \times \mathbb{N}$ into \mathbb{N} . For $h \in {}^\omega\omega$ let $\text{cd}(h) = \{\text{pr}(n, h(n)) : n < \omega\}$, where pr is the standard pairing function of ω , see 0.2(1) and generally for $H \subseteq \mathcal{H}(\aleph_0)$ we let $\text{cd}(H) := \{\text{cd}_0(x) : x \in H\}$; this applies, e.g. to $h \in {}^{[\omega]^k}\omega$.

1) D , an ultrafilter on \mathcal{A} , is called minimal when: if $h \in {}^\omega\omega$ and $\text{cd}(h) \in \mathcal{A}$ then for some $X \in D$ we have $h \upharpoonright X$ is constant or one-to-one.

- 2) D , an ultrafilter on \mathcal{A} , is called Ramsey when: if $k < \omega$ and $h : [\omega]^k \rightarrow \{0, 1\}$ and $\text{cd}(h) \in \mathcal{A}$ then for some $X \in D$ we have $h \upharpoonright [X]^k$ is constant. Similarly k -Ramsey.
 3) D a non-principal ultrafilter on \mathcal{A} is called a Q -point when if $h \in {}^\omega\omega$ is increasing and $\text{cd}(h) \in \mathcal{A}$ then for some increasing sequence $\langle n_i : i < \omega \rangle$ we have $i < \omega \Rightarrow h(2i) \leq n_i < h(2i + 1)$ and $\{n_i : i < \omega\} \in D$.

Remark 0.5. In [Sh:937] we also use the following notions:

- 1) D is called 2.5-Ramsey or self-definably closed when: if $\bar{h} = \langle h_i : i < \omega \rangle$ and $h_i \in {}^\omega(i + 1)$ and $\text{cd}(\bar{h}) = \{\text{cd}(i, \text{cd}(n, h_i(n)) : i < \omega, n < \omega\}$ belongs to \mathcal{A} then for some $g \in {}^\omega\omega$ we have: $\text{cd}(g) \in \mathcal{A}$ and $(\forall i)[g(i) \leq i \wedge \{n < \omega : h_i(n) = g(i)\} \in D]$; this follows from 3-Ramsey and implies 2-Ramsey.
 2) D is weakly definably closed when: if $\langle A_i : i < \omega \rangle$ is a sequence of subsets of ω and $\{\text{pr}(n, i) : n \in A_i \text{ and } i < \omega\} \in \mathcal{A}$ then $\{i : A_i \in D\} \in D$, (follows from 2-Ramsey).

{0z.18}

Definition 0.6. 1) $\mathbb{L}(\mathbf{Q})$ is first order logic when we add the quantifier \mathbf{Q} where $(\mathbf{Q}x)\varphi$ means that there are uncountable many x 's satisfying φ .

2) $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ is defined parallelly.

See on those logics Keisler [Kei71]. We shall use Laver forcing in the proof of Theorem 1.1, so let us define this forcing notion.

{0z.21}

Definition 0.7. Let $T \subseteq {}^\omega > \omega$ be a subtree. For $a \in T$ let $\text{suc}_T(a) = \{a \hat{\ } \langle i \rangle \in T : i \in \omega\}$. The trunk $\text{tr}(T)$ of T is a maximal element $a \in T$ such that $a \leq_T b$ or $b \leq_T a$ for every $b \in T$.

Such a tree T will be called a Laver tree iff $s = \text{tr}(T)$ and for every $t \in T$ such that $s \leq t$, the set $\text{suc}_T(t)$ is infinite.

We define the forcing notion \mathbb{Q} (= Laver forcing) as follows. A condition $T \in \mathbb{Q}$ is a Laver tree. If $S, T \in \mathbb{Q}$ then $S \leq_{\mathbb{Q}} T$ iff $S \supseteq T$. If $\mathbf{G} \subseteq \mathbb{Q}$ is generic, then $\eta[\mathbf{G}] := \{a \in {}^\omega > \omega : \exists T \in \mathbf{G}, a \text{ is the trunk of } T\}$ will be called a Laver real.

{a1.6}

Claim 0.8. If \boxtimes then \boxplus where:

- \boxtimes (a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$ is a CS iteration
- (b) $k(*) < \omega$ and $\beta(k) < \alpha(*) < \omega_1$ for $k < k(*)$
- (c) each \mathbb{Q}_α is a Laver forcing (in $\mathbf{V}^{\mathbb{P}_\alpha}$) and η_α its generic
- (d) $h \in ({}^\omega\omega)^\mathbf{V}$
- (e) $p \in \mathbb{P}_{\alpha(*)}$
- (f) $p \Vdash_{\mathbb{P}_{\alpha(*)}} \text{“} \underline{B}_k \subseteq \omega \text{ and } |\underline{B}_k \cap [\eta_{\beta(k)}(n+1), \eta_{\beta(k)}(n+2)]| \leq h(\eta_{\beta(k)}(n)) \text{ for every } n \text{ large enough” for } k < k(*)$

\boxplus for some p_1, p_2 and B_k^* for $k < k(*)$ we have

- (a) $\mathbb{P}_{\alpha(*)} \Vdash \text{“} p \leq p_\ell \text{” for } \ell = 1, 2$
- (b) $B_k^* \subseteq \omega$ (from \mathbf{V})
- (c) $p_1 \Vdash \text{“} \underline{B}_k \subseteq^* B_k^* \text{”}$
- (d) $p_2 \Vdash \text{“} \underline{B}_k \subseteq^* (\omega \setminus B_k^*) \text{”}$.

Proof. Clearly letting $B_* = \cup\{\underline{B}_k : k < k(*)\}$ we have

- (*) $p \Vdash_{\mathbb{P}_{\alpha(*)}} \text{“for every large enough } n \text{ the set } \underline{B}_* \cap [\eta_0(n+1), \eta_0(n+2)] \text{ has } \leq \eta_0(n) \text{ members”}$.

Now by the properties of iterating Laver forcing ([Lav76] or see [Sh:f, Ch.VI]), we have:

(*) if $\mathbf{G}_1 \subseteq \mathbb{P}_1$ is generic over \mathbf{V} and $\eta = \eta_0[\mathbf{G}_1]$ then

$\Vdash_{\mathbb{P}_{\alpha(*)}/\mathbf{G}_1}$ “ if $\underline{B} \subseteq \omega$ and in $\underline{B} \cap [\eta(n), \eta(n+1))$
 there are $\leq \eta(n)$ elements for every n large enough
then for some $B' \in \mathbf{V}[\mathbf{G}_1]$, $B' \subseteq \omega$, $\underline{B} \subseteq B'$ and
 $B' \cap [\eta(n), \eta(n+1))$ has $\leq (\eta(n))^n$ members for every n large enough”.

Now this applies in particular to $\underline{B} = \underline{B}_*$ getting \underline{B}' . Hence without loss of generality $\alpha(*) = 1$ so we can replace \mathbb{P}_1 by \mathbb{Q}_0 , Laver forcing; also for a dense set of $p \in \mathbb{Q}_0$ we have: if $\eta \in p$ is of length $n+1$ so an increasing sequence of natural numbers, then $p^{[\eta]} := \{\nu \in p : \nu \triangleleft \eta \text{ or } \eta \triangleleft \nu\}$ forces a value b_η to $\underline{B}' \cap [0, \eta(n))$ so necessarily $|b_\eta| \leq \eta(n-1)$ when $n > 1$.

By thinning p , without loss of generality if $\eta \in p$ and $u_\eta = \{n : \eta \hat{\ } \langle n \rangle \in p\}$ is infinite (equivalently is not a singleton) then $\langle b_{\eta \hat{\ } \langle n \rangle} : n \in u_\eta \rangle$ is a Δ -system.

The rest of the proof should be easy, too. $\square_{0.8}$

1. NO MINIMAL ULTRAFILTER ON THE STANDARD SYSTEM

{a1.3}

Theorem 1.1. *Assume that \mathbb{N}_* is an expansion of \mathbb{N} with countable vocabulary or \mathbb{N}_* is an ordinary model of PA_τ , for some countable $\tau \supseteq \tau_{PA}$ such that \mathbb{N}_* is countable. Then there is M such that*

- (a) $\mathbb{N}_* \prec M$
- (b) $\|M\| = \aleph_1$
- (c) $\text{SSy}(M)$, the standard system of M , see Definition 0.3, has no minimal ultrafilter on it, see Definition 0.4; moreover
- (d) there is no Q -point on $\text{SSy}(M)$
- (e) $\text{SSy}(M)$ is arithmetically closed.

Proof. Stage A:

Without loss of generality \mathbb{N}_* is the Skolem Hull of \emptyset as we can expand it by \aleph_0 individual constants.

We shall choose a sentence $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau^*)$ with $\tau^* \supseteq \tau(\mathbb{N}_*)$ and prove that it has a model, and for every model M^+ of ψ , the model $M^+ \upharpoonright \tau(\mathbb{N}_*)$ is as required. By the completeness theorem for $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ it is enough to prove that ψ has a model in some forcing extension; of course it is crucial that ψ can be explicitly defined hence $\in \mathbf{V}$.

Stage B:

Recall $\text{cd} = \text{cd}_0 : \mathcal{H}(\aleph_0) \rightarrow \omega$ be one-to-one onto and definable in \mathbb{N} by a bounded formula in the natural sense; see 0.4.

Let $\mathbf{V}_0 = \mathbf{V}$ and $\lambda = (2^{\aleph_0})^+$.

Let $\mathbb{R}_0 = \text{Levy}(\aleph_1, 2^{\aleph_0})$, let $\mathbf{G}_0 \subseteq \mathbb{R}_0$ be generic over \mathbf{V}_0 and let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_0]$, i.e. in $\mathbf{V}_0^{\mathbb{R}_0}$ we have CH.

In \mathbf{V}_1 we have $\lambda = \aleph_2$ and let \mathbb{R}_1 be \mathbb{P}_{ω_2} where $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ is a CS iteration, each \mathbb{Q}_α is a Laver forcing; there are many other possibilities, let $\eta_\alpha \in {}^\omega \omega$ (increasing) be the $\mathbb{P}_{\alpha+1}$ -name of the \mathbb{Q}_α -generic real and $\nu_\alpha = \langle \text{cd}(\eta_\alpha \upharpoonright n) : n < \omega \rangle$. Let $\mathbf{G}_1 \subseteq \mathbb{R}_1$ be generic over \mathbf{V}_1 and $\mathbf{V}_2 = \mathbf{V}_1[\mathbf{G}_1]$ and let $\eta_\alpha = \eta_\alpha[\mathbf{G}_1], \nu_\alpha = \langle \text{cd}(\eta_\alpha \upharpoonright n) : n < \omega \rangle = \nu_\alpha[\mathbf{G}_1]$.

Let D^2 be a non-principal ultrafilter on ω in the universe \mathbf{V}_2 .

\boxplus_1 In the universe \mathbf{V}_2 let $M_1 = \mathbb{N}_*^\omega / D^2$, let $a_\alpha = \eta_\alpha / D^2 \in M_1$

and note

\boxplus_2 $\text{SSy}(M_1) = \mathcal{P}(\mathbb{N})^{\mathbf{V}_2}$ hence is arithmetically closed

\boxplus_3 let $f_1 \in \mathbf{V}_2$ be the function from $\lambda = \omega_2^{\mathbf{V}_1} = \omega_2^{\mathbf{V}_2}$ into M_1 defined by $f_1(\alpha) = a_\alpha$.

Stage C:

In \mathbf{V}_1 (yes, not in \mathbf{V}_2) let the forcing notion $\mathbb{R}_2 := \mathbb{P}_{\omega_2}^+$ and the set K be defined as follows (so $\mathbf{B} \in \mathbf{V}_1$ below, which is equivalent to $\mathbf{B} \in \mathbf{V}_0$, similarly for u ; so in $\boxplus_4(\alpha)$, \mathcal{A} is a \mathbb{P}_{ω_2} -name):

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- \boxplus_4 (α) $K := \{(\alpha, u, \underline{A}) : u \subseteq \lambda \text{ is countable, } \alpha \in u, \underline{A} = \mathbf{B}(\dots, \eta_\beta, \dots)_{\beta \in u},$
 \mathbf{B} a Borel function from ${}^{\text{otp}(u)}(\omega^\omega)$ to $\mathcal{P}(\omega)$ such that
 $\Vdash_{\mathbb{P}_{\omega_2}}$ “ $\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2))$ has $\leq \eta_\alpha(n)$ members; moreover
 $0 = \lim_n (|\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2))| / \eta_\alpha(n)$ ” $\}$
- (β) $\mathbf{p} \in \mathbb{P}_{\omega_2}^+$ iff
 (a) $\mathbf{p} = (p, h) = (p_{\mathbf{p}}, h_{\mathbf{p}})$
 (b) $p \in \mathbb{P}_{\omega_2}$
 (c) h a function from some finite subset $K_{\mathbf{p}}$ of K to ω_1
 (d) if $(\alpha_\ell, u_\ell, \underline{A}_\ell) \in K_{\mathbf{p}}$ for $\ell = 1, 2$ and $h(\alpha_1, u_1, \underline{A}_1) = h(\alpha_2, u_2, \underline{A}_2)$
 and $u_1 \subseteq \alpha_2$ then $p \Vdash_{\mathbb{P}_{\omega_2}}$ “ $\underline{A}_1 \cap \underline{A}_2$ is finite”
- (γ) $\mathbb{P}_{\omega_2}^+ \Vdash \mathbf{p} \leq \mathbf{q}$ iff:
 (a) $\mathbb{P}_{\omega_2} \Vdash p_{\mathbf{p}} \leq p_{\mathbf{q}}$
 (b) $h_{\mathbf{p}} \subseteq h_{\mathbf{q}}$.

Now

- (\ast)₀ if $p \in \mathbb{P}_{\omega_2}, \alpha < \omega_2$ and $p \Vdash$ “ $\underline{A} \subseteq \omega$ satisfies $\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2))$
 has $\leq \eta_\alpha(n)$ members for every n large enough and $0 = \lim (|\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2))| / \eta_\alpha(n) : n < \omega$ ” then we can find a triple (q, u, \underline{A}') such that
 (α) $\mathbb{P}_{\omega_2} \Vdash “p \leq q”$
 (β) $\text{Dom}(q) = u$
 (γ) u a countable set of ordinals $< \lambda$ (in \mathbf{V}_1 equivalently in \mathbf{V}_0)
 (δ) $q \Vdash “\underline{A} = \underline{A}'”$
 (ε) $\underline{A}' = \mathbf{B}(\dots, \eta_{\alpha_i}, \dots)_{i < \text{otp}(u)}$ where α_i is the i -th member of u , for
 some Borel function ${}^{\text{otp}(u)}(\omega^\omega)$ to $\mathcal{P}(\omega)$ so $\mathbf{B} \in \mathbf{V}_1$ equivalently \mathbf{V}_0
 (ζ) $q(\alpha_i) = \mathbf{B}_i(\dots, \eta_{\alpha_j}, \dots)_{j < i}$ for every $i < \text{otp}(u)$ for some Borel function \mathbf{B}_i
 from ${}^i(\omega^\omega)$ to Laver forcing, of course, \mathbf{B}_i is from \mathbf{V}_0 .

[Why? Standard proof.]

- (\ast)₁ $\mathbb{P}_{\omega_2}^+$ satisfies the \aleph_2 -c.c.

[Why? We need a property of the iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ stated in Claim 0.8. In more detail, given a sequence $\langle \mathbf{p}_\alpha : \alpha < \omega_2 \rangle$ of members of $\mathbb{P}_{\omega_2}^+$, for each $\alpha < \omega_2$, let $\mathbf{p}_\alpha = (p_\alpha, h_\alpha)$; and without loss of generality for each $(\alpha_1^*, u_1^*, \underline{A}_1^*) \in K_{\mathbf{p}_{\alpha_1^*}}$ for some u^1, \underline{A}^1 , the tuple $(p_\alpha, u, \underline{A}^1)$ is like (q, u, \underline{A}') in (\ast)₀, (β) – (ζ) and $(\alpha, u, \underline{A}) \in \text{Dom}(h_\alpha) \Rightarrow u \subseteq \text{Dom}(p_\alpha)$. Letting $u_\alpha = \text{Dom}(p_\alpha)$, we can find a stationary $S \subseteq \{\delta < \omega_2 : \text{cf}(\delta) = \aleph_1\}$ and $p_*, \gamma(*)$ such that:

- $u_\delta \cap \delta = u_*$ for $\delta \in S$ and $u_\alpha \subseteq \delta$ for $\alpha < \delta \in S$
- $p_\delta \upharpoonright \delta \leq p_* \in \mathbb{P}_\delta$ for $\delta \in S$
- without loss of generality $p_\delta \upharpoonright \delta = p_*$ for $\delta \in S$
- $\text{otp}(u_\delta) = \gamma(*)$ for $\delta \in S$
- if $\delta_1, \delta_2 \in S$ then the order preserving function $\text{OP}_{u_{\delta_2}, u_{\delta_1}}$ from u_{δ_1} onto u_{δ_2} maps \mathbf{p}_{δ_1} to \mathbf{p}_{δ_2} .

Let $\delta(*) = \text{Min}(S)$ and $\mathbf{G}_{\delta(*)}^1 \subseteq \mathbb{P}_{\delta(*)}$ be generic over \mathbf{V}_1 such that $p_* \in \mathbf{G}_{\delta(*)}^1$. Now we apply the conclusion of Claim 0.8 to $\mathbb{P}_{\omega_2}/\mathbf{G}_{\delta(*)}$, the rest should be clear.

For $\delta \in S$, let $\alpha_0(*) = \text{otp}(u_\delta \setminus \delta_*)$, \mathbf{h}_δ be the order preserving function from α_δ onto $u_\delta \setminus \delta$ and $(p'_\delta, h'_\delta) \in \mathbb{P}_{\alpha_\delta}$ be such that \mathbf{b}_δ maps (p'_δ, h'_δ) to (p_δ, h_δ) . Clearly $\alpha_\delta, p'_\delta, h'_\delta$ are the same for all $\delta \in S$ so call them $\alpha(*), p', h'$ and applying 0.8 with $p', (\{\alpha, \underline{A}\}: \text{for some } u \text{ the tuple } (\alpha, u, \underline{A}) \text{ belongs to } \text{Dom}(h)\}$ here stands for $p, \{(\alpha_k, \beta_k) : k < k(*)\}$ there and get p'_1, p'_2 as there.

Let $\delta_1 < \delta_2$ be from S , let q_{δ_1} be $\mathbf{h}_{\delta_1}(p'_1)$, q_{δ_2} be $\mathbf{h}_{\delta_2}(p'_2)$. Easily $p_{\delta_\ell} \leq q_{\delta_\ell}$ and $q_{\delta_1} \cup q_{\delta_2}$ is a common upper bound of $p_{\delta_1}, p_{\delta_2}$ in $\mathbb{P}_{\omega_2}^+/\mathbf{G}_{\delta(*)}^1$.

(*)₂ $\mathbb{P}_{\omega_2}^+$ collapses ω_1 to \aleph_0 .

[Why? Easy but also we can use $\mathbb{P}_{\omega_2}^+ \times \text{Levy}(\aleph_0, \aleph_1)$ instead.]

(*)₃ the function $p \mapsto (p, \emptyset)$ is a complete embedding of \mathbb{P}_{ω_2} into $\mathbb{P}_{\omega_2}^+$.

[Why? Should be clear.]

Stage D: Let $\mathbf{G}_2 = \mathbf{G}_1^+ \subseteq \mathbb{P}_{\omega_2}^+$ be generic over $\mathbf{V}_1, \mathbf{V}_3 = \mathbf{V}_1[\mathbf{G}_2]$ and by (*)₃ without loss of generality $\mathbf{G}_1 = \{p : (p, h) \in \mathbf{G}_2\}$. So $\mathbf{V}_3 = \mathbf{V}_1[\mathbf{G}_2]$ is a generic extension of \mathbf{V}_2 and let $f_2 = \cup\{h : (p, h) \in \mathbf{G}_2\}$.

So

(*)₄ in \mathbf{V}_3 if $f_2(\alpha_1, u_1, \underline{A}_1) = f_2(\alpha_2, u_2, \underline{A}_2)$ and $u_1 \subseteq \alpha_2$, then $\underline{A}_1[\mathbf{G}_1] \cap \underline{A}_2[\mathbf{G}_1]$ is finite.

In \mathbf{V}_3 let M_2 be an elementary submodel of $(\mathcal{H}(\sqsupset_\omega), \in, \dots, \mathbf{V}_\ell \cap \mathcal{H}(\sqsupset_\omega), \dots)_{\ell=0,1,2}$ of cardinality $\lambda = \aleph_1^{\mathbf{V}_3}$ which includes $\{\alpha : \alpha \leq \lambda\} = \{\alpha : \alpha \leq \omega_1^{\mathbf{V}_3}\}, \{M_1, f_1, f_2, \mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2\}$ and (the universe of) M_1 , see end of stage B, note that $\|M_2\| \subseteq |M_2|$.

Let f_0 be a one-to-one function from M_1 onto M_2 , let M_3 be a model such that f_0 is an isomorphism from M_1 onto M_3 . Lastly, let M_4 be M_3 expanded by $c_0 = \lambda = \omega_2^{\mathbf{V}_1} = \omega_1^{\mathbf{V}_3}, c_1^{M_4} = \omega_1^{\mathbf{V}}, c_2^{M_4} = M_1, d_{0,\ell}^{M_4} = \mathbf{G}_\ell, d_{1,\ell} = \mathbb{R}_\ell, d^{M_4} = \mathbb{N}_*, \langle d_{2,n}^{M_4} : n < \omega \rangle$ list the members of $\mathbb{N}_*, Q_0^{M_4} = |\mathbb{N}_*|, \in^{M_2} = \in^{\mathbf{V}_3} \upharpoonright |M_2|, F_0^M = f_0, F_1^{M_4} = f_0 \circ f_1$, see end of Stage B, $F_2^{M_4} = f_2, P_\ell^M = \mathbf{V}_\ell \cap M_2$ for $\ell = 0, 1, 2$ (so F_ℓ is a unary function symbol, P_ℓ is a unary predicate) and lastly $<_*^M$, a linear order of $|M_2| = |M_4|$ of order type $\omega_1^{\mathbf{V}_3}$.

We define the sentence ψ : it is the conjunction of the following countable sets and singletons of sentences of $\mathbb{L}_{\aleph_1, \aleph_0}(\mathbf{Q})$ in the vocabulary $\tau(M_4)$ such that $M^+ \models \psi$ iff:

- (A) $M^+ \upharpoonright \tau(\mathbb{N}_*)$ is isomorphic to \mathbb{N}_* , of course, $M^+ \upharpoonright \tau(\mathbb{N}_*)$ has universe $Q_0^{M^+}$
- (B) M^+ is uncountable, moreover $M^+ \models (\mathbf{Q}x) (x \text{ an ordinal } < c_0)$
- (C) $<_*^{M^+}$ is a linear order
- (D) every proper initial segment by $<_*^{M^+}$ is countable
- (E) $(|M^+|, \in^{M^+})$ is a model ZFC⁻ (even a model of $\text{Th}(\mathcal{H}(\sqsupset_\omega)^{\mathbf{V}_3}, \in)$)
- (F) the function $F_1^{M^+} : \{a : M^+ \models \text{“}a \text{ an ordinal } < c_0\text{”}\} \rightarrow M^+$ is one-to-one
- (G) $M^+ \models \text{“}K \text{ is as above”}$
- (H) $F_2^{M^+} : K^{M^+} \rightarrow \{a : M \models \text{“}a \text{ an ordinal } < c_1\text{”}\}$ is as above

- (I) $M^+ \models$ “for every B we have $B \in \mathcal{P}(\mathbb{N}) \wedge P_2(B)$ iff $B = A \cap \mathbb{N}$ for some definable subset of A in the model c_2 ”.

It is easy to check that

- (*)₅ $\psi \in \mathbf{V}_0$
 (*)₆ $M_4 \models \psi$ in \mathbf{V}_3 .

Hence as the completeness theorem for $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ gives absoluteness

- (*)₇ ψ has a model in $\mathbf{V} = \mathbf{V}_0$ call it M_5 .

By renaming without loss of generality

- (*)₈ (a) if $M_5 \models$ “ a is the n -th natural number” then $a = n$
 (b) if $M_5 \models$ “ $A \subseteq \omega$ ” then $A = \{n : M^+ \models “n \in A”\}$
 (c) if $M_5 \models$ “ $b \in {}^\omega\omega$ ” then $b = \{(n_1, n_2) : M^+ \models f(n_1) = n_2\}$
 (*)₉ let $N'_* = M_5 \upharpoonright \tau(\mathbb{N}_*)$, so isomorphic to N_* , let $N = M \upharpoonright \{\in\}$
 (*)₁₀ (a) let M'_1 be $c_2^{M_5}$ naturally defined
 (b) so M'_1 is a model of $\text{Th}(N'_*) = \text{Th}(N_*)$, $N'_* \prec M'_1$ and $\|M'_1\| = \aleph_1$
 (c) let \mathcal{A} be $\text{SSy}(M)$, the standard system of M

Clearly

- (*)₁₁ (a) $N \models$ “ZC”
 (b) M is a model of $\text{Th}(\mathbb{N}_*)$ and $N_* \prec M$
 (*)₁₂ let $\mathbb{R}'_\ell = d_{1, \ell}^{M^+}$ and $\mathbf{G}'_\ell = d_{2, \ell}^{M^+}$ and let $\mathbf{V}'_\ell = (P_\ell^{M^+}, \in^{M^+})$ for $\ell = 0, 1, 2$.

Stage E:

Clearly M is an uncountable elementary extension of \mathbb{N}_* , by clauses (A),(B) of Stage D and without loss of generality $\|M\| = \aleph_1$, so M satisfies clauses (a),(b) of Theorem 1.1. To prove clause (e) recall \boxplus_2 and clause (I) above hence $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is arithmetically closed; this implies \mathcal{A} is a Boolean subalgebra. Also clause (d) implies clause (c), anyhow to prove them, assume toward contradiction that D is an ultrafilter on \mathcal{A} which is minimal or just a Q -point. Let $X = \{a : N \models “a$ is an ordinal $< \omega_1”\}$, so X is really an uncountable set. For each $a \in X$ define a sequence $\rho_a \in {}^\omega\omega$ by $\rho(n) = k$ iff $M^+ \models “F_1(a)(n) = k”$.

Clearly ρ_a is an increasing sequence in ${}^\omega\omega$, hence by the assumption toward contradiction, there is $A_a \in D \subseteq \mathcal{A}$ such that $A_a \cap [\rho_a(n+1), \rho_a(n+2))$ has at most one element (or just $\leq \rho_a(n)$ elements) for each $n < \omega$.

So for some element \underline{A}_a of N , $N \models “\underline{A}_a$, in \mathbf{V}'_1 , is a \mathbb{R}_1 -name of a subset of ω and $\underline{A}_a[\mathbf{G}'_1] = A_a”$.

Clearly $M^+ \models$ “for some countable subset u of $\omega_2^{\mathbf{V}'_1} = \omega_1^{\mathbf{V}'_3}$ from \mathbf{V}'_1 and Borel function \mathbf{B} from \mathbf{V}'_1 we have $A_a = \mathbf{B}_a(\dots, \rho_b, \dots)_{b \in u_a}$ (so some $p \in \mathbf{G}'_2$ forces \underline{A}_a satisfies this)”. So using $F_2^{M^+}$ there are $a_1 \neq a_2$ from X such that the parallel of clause $(\beta)(d)$ of stage C holds, see clause (G) of stage D, so two members of D are almost disjoint, contradiction. $\square_{1.1}$

{a1.9}

Remark 1.2. 1) Note that in 1.1 we can replace \mathbb{Q}_0 by any forcing notion similar enough, see [RoSh:470].

2) We can strengthen 1.1 by replacing “ Q -point” by a weaker statement.

Similarly we can weaken the demands on how “thin” is \mathcal{B} in 0.8 and in the proof of 1.1.

2. PRIVATE APPENDIX

Moved 2017.7.17 from Definition 0.3, pg.2:

3) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\text{Sc-cl}(\mathcal{A})$ be the Scott closure of \mathcal{A} , see [KS06] which means adding the finite sets and closing under the Boolean operations (union of 2, complement) and $A \mapsto \{x : (\forall y)(\text{pr}(x, y) \in A)\}$ and $A \mapsto \{x : \text{the set } \{y : \text{pr}(x, y) \in A\} \text{ codes a well founded tree}\}$.

Moved from pg. 2:

Recently, solving a long standing problem on models of Peano arithmetic, (appearing as Problem 7 in the book [KS06]), Ali Enayat proved (and other results as well):

{q0.1}

Claim 2.1. ([Ena08]) *For some arithmetically closed family \mathcal{A} of subsets of ω , the model $\Omega_{\mathcal{A}} = (\mathbb{N}, A)_{A \in \mathcal{A}}$ satisfies*

- (a) *it has no conservative extension (i.e. one in which the intersection of any definable subset with \mathbb{N} belongs to \mathcal{A}).*

Motivated by this he asked:

{q0.4}

Question 2.2. Question I:

Is there $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that some model of $\text{Th}(\Omega_{\mathcal{A}})$ has no elementary end extension?

Moved 09.11.23 from 0.8:

- ⊠ (a) is as in [BsSh:242]
 - (b) $\Vdash_{\mathbb{Q}_0}$ “ $\eta_\alpha \in {}^\omega\omega$ is increasing enumerating the generic for \mathbb{Q}_α ”
 - (c) $h \in ({}^\omega\omega)^{\mathbf{V}}$
 - (d) $f \in {}^\omega\omega$ is defined $f(n) = \eta(n+1)$
 - (e) $g \in {}^\omega\omega$ is defined by $g(n) = h(\eta(n))$
 - (f) $\Vdash_{\mathbb{Q}_0}$ “ \mathbb{Q}_1 is an (f, g) -bounding forcing notion”
 - (g) $\mathbb{Q} = \mathbb{Q}_0 * \mathbb{Q}_1$
 - (h) $p^c \Vdash_{\mathbb{Q}}$ “ $\underline{B} \subseteq \omega$ and $|B \cap [\eta(n), \eta(n+1)]| \leq h(\eta(n))$ ”
- ⊞ for some p_1, p_2, B_1, B_2 we have
- (a) $p \leq_{\mathbb{Q}} p_\ell$ for $\ell = 1, 2$
 - (b) $B_1, B_2 \subseteq \omega$ are almost disjoint
 - (c) $p_\ell \Vdash$ “ $\underline{B} \subseteq^* B_\ell$ ” for $\ell = 1, 2$.

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