

**MODELS OF PA: STANDARD SYTEMS WITHOUT MINIMAL
ULTRAFILTERS
SH944**

SAHARON SHELAH

ABSTRACT. We prove that \mathbb{N} has an uncountable elementary extension N such that there is no ultrafilter on the Boolean Algebra of subsets of \mathbb{N} represented in N which is minimal (i.e. Ramsey for partitions represented in N).

0. INTRODUCTION

Enayat [Ena06] asked (see Definition 0.4(1)):

Question 0.1. Question III: Can we prove in ZFC that there is an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that \mathcal{A} carries no minimal ultrafilter?

{q0.7}

He proved it for the stronger notion “2-Ramsey ultrafilter”. In [Sh:937] we prove that there is an arithmetically closed Borel set $\mathbf{B} \subseteq \mathcal{P}(\mathbb{N})$ such that any expansion \mathbb{N}^+ of \mathbb{N} by any uncountably many members of \mathbf{B} has this property, i.e. the family of definable subsets of \mathbb{N}^+ carries no 2-Ramsey ultrafilter.

We deal here with the question 0.1, proving that there is such a family of cardinality \aleph_1 , this implies the version in the abstract; we use forcing but the result is proved in ZFC. On other problems from [Ena06] see Enayat-Shelah [EnSh:936] and [Sh:924], [Sh:937].

{0z.1}

Notation 0.2. 1) Let $\text{pr}:\omega \times \omega \rightarrow \omega$ be the standard pairing function (i.e. $\text{pr}(n, m) = \binom{n+m}{2} + n$, so one to one onto two-place function).

2) Let \mathcal{A} denote a subset of $\mathcal{P}(\omega)$.

3) Let $\text{BA}(\mathcal{A})$ be the Boolean algebra which $\mathcal{A} \cup \{\omega\}^{<\aleph_0}$ generates.

4) Let D denote a non-principal ultrafilter on \mathcal{A} , meaning that $D \subseteq \mathcal{A}$ and there is a unique non-principal ultrafilter D' on the Boolean algebra $\text{BA}(\mathcal{A})$ satisfying $D = D' \cap \mathcal{A}$, but in Definition 0.4 below the distinction between an ultrafilter on \mathcal{A} and on $\text{BA}(\mathcal{A})$ makes a difference.

5) τ denotes a vocabulary extending $\tau_{\text{PA}} = \tau_{\mathbb{N}} = \{0, 1, +, \times, <\}$, usually countable.

6) $\text{PA}(\tau)$ is Peano arithmetic for the vocabulary τ . A model N of $\text{PA}(\tau)$ is called ordinary if $N \upharpoonright \tau_{\text{PA}}$ extends \mathbb{N} ; usually the models will be ordinary.

7) $\varphi(N, \bar{a})$ is $\{b : N \models \varphi[b, \bar{a}]\}$ where $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_N)$ and $\bar{a} \in {}^{\ell g(\bar{y})}N$.

8) $\text{Per}(A)$ is the set (or group) of permutations of N .

9) For sets u, v of ordinals let $\text{OP}_{v,u}$, “the order preserving function from u to v ” be defined by: $\text{OP}_{v,u}(\alpha) = \beta$ iff $\beta \in v, \alpha \in u$ and $\text{otp}(v \cap \beta) = \text{otp}(u \cap \alpha)$.

Date: February 15, 2010.

The author thanks Alice Leonhardt for the beautiful typing. First version (F922) typed July 2008. The author would like to thank the Israel Science Foundation (Grant No. 710/07) and the US-Israel Binational Science Foundation (Grant No. 2006108) for partial support of this research.

10) We say $u, v \subseteq \text{Ord}$ form a Δ -system pair when $\text{otp}(u) = \text{otp}(v)$ and $\text{OP}_{v,u}$ is the identity on $u \cap v$.

{0z.2}

Definition 0.3. 1) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\text{ar-cl}(\mathcal{A}) = \{B \subseteq \omega : B \text{ is first order defined in } (\mathbb{N}, A_1, \dots, A_n) \text{ for some } n < \omega \text{ and } A_1, \dots, A_n \in \mathcal{A}\}$. This is called the arithmetic closure of \mathcal{A} .

2) For a model N of $\text{PA}(\tau)$ let the standard system of N , $\text{StSy}(N)$ be $\{\varphi(M, \bar{a}) \cap \mathbb{N} : \varphi(x, \bar{y}) \in \mathbb{L}(\tau) \text{ and } \bar{a} \in {}^{\ell g(\bar{y})}M\}$ so $\subseteq \mathcal{P}(\omega)$ for any ordinary model M isomorphic to N , see 0.2(6).

3) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\text{Sc-cl}(\mathcal{A})$ be the Scott closure of \mathcal{A} , see [KS06] which means adding the finite sets and closing under the Boolean operations (union of 2, complement) and $A \mapsto \{x : (\forall y)(\text{pr}(x, y) \in A)\}$ and $A \mapsto \{x : \text{the set } \{y : \text{pr}(x, y) \in A\} \text{ code a well founded tree}\}$.

{0z.7}

Definition 0.4. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

0) For $h \in {}^\omega\omega$ let $\text{cd}(h) = \{\text{pr}(n, h(n)) : n < \omega\}$, where pr is the standard pairing function of ω , see 0.2(1).

1) D , an ultrafilter on \mathcal{A} , is called minimal when: if $h \in {}^\omega\omega$ and $\text{cd}(h) \in \mathcal{A}$ then for some $X \in D$ we have $h \upharpoonright X$ constant or one-to-one.

2) D , an ultrafilter on \mathcal{A} , is called Ramsey when: if $k < \omega$ and $h : [\omega]^k \rightarrow \{0, 1\}$ and $\text{cd}(h) \in \mathcal{A}$ then for some $X \in D$ we have $h \upharpoonright [X]^k$ is constant. Similarly k -Ramsey.

3) D a non-principal ultrafilter on \mathcal{A} is called a Q -point when if $h \in {}^\omega\omega$ is increasing and $\text{cd}(h) \in \mathcal{A}$ then for some increasing sequence $\langle n_i : i < \omega \rangle$ we have $i < \omega \Rightarrow h(2i) \leq n_i < h(2i + 1)$ and $\{n_i : i < \omega\} \in D$.

Remark 0.5. In [Sh:937] we use also

1) D is called 2.5-Ramsey or self-definably closed when: if $\bar{h} = \langle h_i : i < \omega \rangle$ and $h_i \in {}^\omega(i + 1)$ and $\text{cd}(\bar{h}) = \{\text{cd}(i, \text{cd}(n, h_i(n)) : i < \omega, n < \omega\}$ belongs to \mathcal{A} then for some $g \in {}^\omega\omega$ we have: $\text{cd}(g) \in \mathcal{A}$ and $(\forall i)[g(i) \leq i \wedge \{n < \omega : h_i(n) = g(i)\} \in D]$; this follows from 3-Ramsey and implies 2-Ramsey.

2) D is weakly definably closed when: if $\langle A_i : i < \omega \rangle$ is a sequence of subsets of ω and $\{\text{pr}(n, i) : n \in A_i \text{ and } i < \omega\} \in \mathcal{A}$ then $\{i : A_i \in D\} \in D$, (follows from 2-Ramsey).

{0z.18}

Definition 0.6. 1) $\mathbb{L}(\mathbf{Q})$ is first order logic when we add the quantifier \mathbf{Q} where $(\mathbf{Q}x)\varphi$ means that there are uncountable many x 's satisfying φ .

2) $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ is defined paralelly.

See on those logics Keisler [Kei71].

1. NO MINIMAL ULTRAFILTER ON THE STANDARD SYSTEM

{a1.3}

Theorem 1.1. *Assume that \mathbb{N}_* is an expansion of \mathbb{N} with countable vocabulary or \mathbb{N}_* is an ordinary model of PA_τ , for some countable $\tau \supseteq \tau_{PA}$ such that \mathbb{N}_* is countable. Then there is M such that*

- (a) $\mathbb{N}_* \prec M$
- (b) $\|M\| = \aleph_1$
- (c) $\text{StSy}(M)$, the standard system of M , see 0.3, has no minimal ultrafilter on it, see Definition 0.4; moreover
- (d) there is no Q -point on $\text{StSy}(M)$
- (e) $\text{StSy}(M)$ is arithmetically closed.

Proof. Stage A:

Without loss of generality \mathbb{N}_* is the Skolem Hull of \emptyset .

We shall choose a sentence $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau^*)$ with $\tau^* \supseteq \tau(\mathbb{N}_*)$ and prove that it has a model, and for every model M^+ of ψ , the model $M^+ \upharpoonright \tau(\mathbb{N}_*)$ is as required. By the completeness theorem for $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ it is enough to prove that ψ has a model in some forcing extension; of course it is crucial that ψ can be explicitly defined hence $\in \mathbf{V}$.

Stage B:

Let $\text{cd}: \mathcal{H}(\aleph_0) \rightarrow \omega$ be one-to-one onto and definable in \mathbb{N} in the natural sense.

Let $\mathbf{V}_0 = \mathbf{V}$ and $\lambda = (2^{\aleph_0})^+$.

Let $\mathbb{R}_0 = \text{Levy}(\aleph_1, 2^{\aleph_0})$, let $\mathbf{G}_0 \subseteq \mathbb{R}_0$ be generic over \mathbf{V}_0 and let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_0]$, i.e. in $\mathbf{V}_0^{\mathbb{R}_0}$ we have CH.

In \mathbf{V}_1 let \mathbb{R}_1 be \mathbb{P}_{ω_2} where $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ is CS iteration, each \mathbb{Q}_α is a Laver forcing; there are many other possibilities, let $\eta_\alpha \in {}^\omega \omega$ (increasing) be the $\mathbb{P}_{\alpha+1}$ -name of the \mathbb{Q}_α -generic real and $\nu_\alpha = \langle \text{cd}(\eta_\alpha \upharpoonright n) : n < \omega \rangle$. Let $\mathbf{G}_1 \subseteq \mathbb{R}_1$ be generic over \mathbf{V}_1 and $\mathbf{V}_2 = \mathbf{V}_1[\mathbf{G}_1]$ and let $\eta_\alpha = \eta_\alpha[\mathbf{G}_1], \nu_\alpha = \langle \text{cd}(\eta_\alpha \upharpoonright n) : n < \omega \rangle = \nu_\alpha[\mathbf{G}_1]$.

Let D^2 be a non-principal ultrafilter on ω in the universe \mathbf{V}_2 .

\boxplus_1 In the universe \mathbf{V}_2 let $M_1 = \mathbb{N}_*/D^2$, let $a_\alpha = \eta_\alpha/D^2 \in M_1$

and note

\boxplus_2 $\text{StSy}(M_1) = \mathcal{P}(\mathbb{N})^{\mathbf{V}_2}$ hence is arithmetically closed

\boxplus_3 let $f_1 \in \mathbf{V}_2$ be the function from $\lambda = \omega_2^{\mathbf{V}_1} = \omega_2^{\mathbf{V}_2}$ into M_1 defined by $f_1(\alpha) = a_\alpha$.

Stage C:

In \mathbf{V}_1 (yes, not in \mathbf{V}_2) let the forcing notion $\mathbb{R}_2 := \mathbb{P}_{\omega_2}^+$ and the set K be defined as follows (so $\mathbf{B} \in \mathbf{V}_1$ below, which is equivalent to $\mathbf{B} \in \mathbf{V}_0$, similarly for u):

- \boxplus_4 (α) $K := \{(\alpha, u, \underline{A}) : u \subseteq \lambda \text{ is countable, } \alpha \in u, \underline{A} = \mathbf{B}(\dots, \eta_\beta, \dots)_{\beta \in u}, \mathbf{B} \text{ a Borel function from } {}^{\text{otp}(u)}(\omega^\omega) \text{ to } \mathcal{P}(\omega) \text{ such that } \Vdash_{\mathbb{P}_{\omega_2}^+} \text{“} \underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2)) \text{ has } \leq \eta_\alpha(n) \text{ members; moreover } 0 = \lim_n (|\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2)) / \eta_\alpha(n)|”}\}$
- (β) $\mathbf{p} \in \mathbb{P}_{\omega_2}^+$ iff

modified:2010-02-16

(944) revision:2010-02-15

- (a) $\mathbf{p} = (p, h) = (p_{\mathbf{p}}, h_{\mathbf{p}})$
- (b) $p \in \mathbb{P}_{\omega_2}^1$
- (c) h a function from some finite subset $K_{\mathbf{p}}$ of K to ω_1
- (d) if $(\alpha_\ell, u_\ell, \mathcal{A}_\ell) \in K_{\mathbf{p}}$ for $\ell = 1, 2$ and $h(\alpha_1, u_1, \mathcal{A}_1) = h(\alpha_2, u_2, \mathcal{A}_2)$ and $u_1 \subseteq \alpha_2$ then $p \Vdash_{\mathbb{P}_{\omega_2}} \text{“}\mathcal{A}_1 \cap \mathcal{A}_2 \text{ is finite”}$
- (γ) $\mathbb{P}_{\omega_2}^+ \models \mathbf{p} \leq \mathbf{q}$ iff:
 - (a) $\mathbb{P}_{\omega_2}^1 \models p_{\mathbf{p}} \leq p_{\mathbf{q}}$
 - (b) $h_{\mathbf{p}} \subseteq h_{\mathbf{q}}$.

Now

- (*)₀ if $p \in \mathbb{P}_{\omega_2}, \alpha < \omega_2$ and $p \Vdash \text{“}\mathcal{A} \subseteq \omega \text{ satisfies } \mathcal{A} \cap [\eta_\alpha(n), \eta_\alpha(n+1)) \text{ has } \leq \eta_\alpha(n) \text{ members for every } n \text{ large”}$ then we can find a triple (q, u, \mathcal{A}') such that
 - (α) $\mathbb{P}_{\omega_2} \models \text{“}p \leq q\text{”}$
 - (β) $\text{Dom}(q) = u$
 - (γ) u a countable set of ordinals $< \omega_2$ (in \mathbf{V}_1 equivalently in \mathbf{V}_0)
 - (δ) $q \Vdash \text{“}\mathcal{A} = \mathcal{A}'\text{”}$
 - (ε) $\mathcal{A}' = \mathbf{B}(\dots, \eta_{\alpha_i}, \dots)_{i < \text{otp}(u)}$ where α_i is the i -th member of u , for some Borel function ${}^{\text{otp}(u)}(\omega) \text{ to } \mathcal{P}(\omega)$ so $\mathbf{B} \in \mathbf{V}_1$ equivalently \mathbf{V}_0
 - (ζ) $q(\alpha_i) = \mathbf{B}_i(\dots, \eta_{\alpha_j}, \dots)_{j < i}$ for every $i < \text{otp}(u)$ for some Borel function \mathbf{B}_i from ${}^i(\omega)$ to Laver forcing.

[Why? Standard proof.]

- (*)₁ $\mathbb{P}_{\omega_2}^+$ satisfies the \aleph_2 -c.c.

[Why? We need a property of the iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ stated in 1.2 below. In more detail, given a sequence $\langle \mathbf{p}_\alpha : \alpha < \omega_2 \rangle$ of members of $\mathbb{P}_{\omega_2}^+$, for each $\alpha < \omega_2$, let $\mathbf{p}_\alpha = (p_\alpha, h_\alpha)$; and without loss of generality each p_α is like q in (*₀), (β), (γ), (ζ) and $(\alpha, u, \mathcal{A}) \in \text{Dom}(h) \Rightarrow u \subseteq \text{Dom}(p_\alpha)$. Letting $u_\alpha = \text{Dom}(p_\alpha)$, we can find a stationary $S \subseteq \{\delta < \omega_2 : \text{cf}(\delta) = \aleph_1\}$ and p_* such that

- $u_\delta \cap \delta = u_*$ for $\delta \in S$
- $p_\delta \upharpoonright \delta \leq p_* \in \mathbb{P}_\delta$ for $\delta \in S$
- without loss of generality $p_\delta \upharpoonright \delta = p_*$ for $\delta \in S$
- $\text{otp}(u_\delta) = \gamma(*)$ for $\delta \in S$
- if $\delta_1, \delta_2 \in S$ then the order preserving function $\text{OP}_{u_{\delta_2}, u_{\delta_1}}$ from u_{δ_1} onto u_{δ_2} maps \mathbf{p}_{δ_1} to \mathbf{p}_{δ_2} .

Let $\delta(*) = \text{Min}(S)$ and $\mathbf{G}_{\delta(*)} \subseteq \mathbb{P}_{\delta(*)}$ be generic over \mathbf{V}_1 such that $p_* \in \mathbf{G}_{\delta(*)}$. Now we apply the 1.2 to $\mathbb{P}_{\omega_2}/\mathbf{G}_{\delta(*)}$, the rest is clear.]

- (*)₂ $\mathbb{P}_{\omega_2}^+$ collapse ω_1 to \aleph_0 .

[Why? Easy but also we can use $\mathbb{P}_{\omega_2}^+ \times \text{Levy}(\aleph_0, \aleph_1)$ instead.]

- (*)₃ the function $p \mapsto (p, \emptyset)$ is a complete embedding of \mathbb{P}_{ω_2} into $\mathbb{P}_{\omega_2}^+$.

Stage D: Let $\mathbf{G}_2 = \mathbf{G}_1^+ \subseteq \mathbb{P}_{\omega_1}^+$ be generic over $\mathbf{V}_1, \mathbf{V}_3 = \mathbf{V}_1[\mathbf{G}_2]$ and without loss of generality $\mathbf{G}_1 = \{p : (p, h) \in \mathbf{G}_2\}$. So \mathbf{V}_3 is a generic extension of \mathbf{V}_2 and let $f_2 = \cup\{h : (p, h) \in \mathbf{G}_2\}$.

So

(*)₄ in \mathbf{V}_3 if $f_2(\alpha_1, u_1, A_1) = f_2(\alpha_1, u_2, A_2)$ and $u_1 \subseteq \alpha_2$, then $A_1[\mathbf{G}_1] \cap A_2[\mathbf{G}_1]$ is finite.

In \mathbf{V}_3 let M_2 be an elementary submodel of $(\mathcal{H}(\mathfrak{Q}_\omega), \in)$ of cardinality $\lambda = \aleph_1^{\mathbf{V}_3}$ which includes $\{\alpha : \alpha \leq \lambda\} = \{\alpha : \alpha \leq \omega_1^{\mathbf{V}_3}\}, \{M_1, H, f_1, f_2, \mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2\}$ and (the universe of) M_1 , see end of stage B.

Let f_0 be a one-to-one function from M_1 onto M_2 , let M_3 be a model such that f_0 is an isomorphism from M_1 onto M_3 . Lastly, let M_4 be M_3 expanded by $c_0 = \lambda = \omega_2^{\mathbf{V}_1} = \omega_1^{\mathbf{V}_3}, c_1^{M_4} = \omega_1^{\mathbf{V}_1}, c_2^{M_4} = M_1, d_{0,\ell}^{M_4} = \mathbf{G}_\ell, d_{1,\ell} = \mathbb{R}_\ell, d^{M_4} = \mathbb{N}_*, \langle d_{2,n}^{M_4} : n < \omega \rangle$ list the members of $\mathbb{N}_*, \in^{M_2} \in \mathbf{V}_3 \upharpoonright |M_2|, F_0^{M_4} = f_0, F_1^{M_4} = f_0 \circ f_1$, see end of Stage B, $F_2^{M_4} = f_2, P_\ell^M = \mathbf{V}_\ell \cap M_2$ for $\ell = 0, 1, 2$ (so F_ℓ is a unary function symbol, P_ℓ is a unary predicate) and $<_*^M$, a linear order of $|M_2| = |M_4|$ of order type $\omega_1^{\mathbf{V}_3}$.

We define the sentence ψ : it is the conjunction of the following countable sets and singletons of first order sentences in the vocabulary $\tau(M_4)$ such that $M^+ \models \psi$ iff:

- (A) $M^+ \upharpoonright \tau(\mathbb{N}_*) \models \text{Th}(\mathbb{N}_*)$
- (B) M^+ is uncountable, moreover $M^+ \models (\mathbf{Q}x) (x \text{ an ordinal } < c_0)$
- (C) $<_*^{M^+}$ is a linear order
- (D) every proper initial segment by $<_*^{M^+}$ is countable
- (E) $(|M^+|, \in^{M^+})$ is a model ZFC⁻ (even a model of $\text{Th}(\mathcal{H}(\mathfrak{Q}_\omega)^{\mathbf{V}_3}, \in)$)
- (F) $F_1^{M^+} : \{a : M^+ \models \text{"}a \text{ an ordinal } < c_0\text{"}\} \rightarrow M^+$ is one-to-one
- (G) $M^+ \models \text{"}K \text{ is as above"}$
- (H) $F_2^{M^+} : K^{M^+} \rightarrow \{a : M \models \text{"}a \text{ an ordinal } < c_1\text{"}\}$ is as above
- (I) $M_5 \models \text{"for every } B \text{ we have } B \in \mathcal{P}(\mathbb{N}) \wedge P_2(B) \text{ iff } B = A \cap \mathbb{N} \text{ for some definable subset of } A \text{ in the model } c_2\text{"}$.

Easy to check that

- (*)₅ $\psi \in \mathbf{V}_0$
- (*)₆ $M_4 \models \psi$ in \mathbf{V}_3 .

Hence as the completeness theory for $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ give absoluteness

- (*)₇ ψ has a model in $\mathbf{V} = \mathbf{V}_0$ call it M_5
- (*)₈ let $M = M_5 \upharpoonright \tau(\mathbb{N}_*)$, let $\mathbf{V}'_3 = N = M \upharpoonright \{\in\}$
- (*)₉ let \mathcal{A} be $\text{StSy}(M)$, the standard system of M
- (*)₁₀ let $\mathbf{V}'_\ell = (P_\ell^{M^+}, \in^{M^+})$ for $\ell = 0, 1, 2$.

Clearly

- (*)₁₁ (a) $N \models \text{"}ZC\text{"}$
- (b) M is a model of $\text{Th}(\mathbb{N}_*)$
- (*)₁₂ let $\mathbb{R}'_\ell = d_{1,\ell}^{M^+}$ and $\mathbf{G}'_\ell = d_{2,\ell}^{M^+}$.

By renaming without loss of generality

- (*)₁₃ (a) if $N^+ \models$ “ a is the n -th natural number” then $a = n$
- (b) if $N_5 \models$ “ $A \subseteq \omega$ ” then $A = \{n : M^+ \models “n \in A”\}$
- (c) if $N_5 \models$ “ $b \in {}^\omega\omega$ ” then $b = \{(n_2, n_2) : M^+ \models f(n_1) = n_2\}$
- (d) $\mathbb{N}_* \subseteq M$.

Stage E:

Clearly M is an uncountable elementary extension of \mathbb{N}_* , by clauses (A),(B) of Stage D, so M satisfies clauses (a),(b) of Theorem 1.1. To prove clause (e) recall \mathbb{E}_2 and clause (I) above hence $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is arithmetically closed; this implies \mathcal{A} is a Boolean subalgebra. Also clause (d) implies clause (c), anyhow to prove them, assume toward contradiction that D is an ultrafilter on \mathcal{A} which is minimal or just a Q -point. Let $X = \{a : N \models “a \text{ is an ordinal } < \omega_1”\}$, so X is really an uncountable set. For each $a \in X$ define a sequence $\rho_a \in {}^\omega\omega$ by $\rho(n) = k$ iff $M^+ \models “F_1(a)(n) = k”$.

Clearly ρ_a is an increasing sequence in ${}^\omega\omega$, hence by the assumption toward contradiction, there is $A_a \in D \subseteq \mathcal{A}$ such that $A_a \cap [\rho_a(n), \rho_a(n))$ has at most one element (or just $\leq \rho_a(n)$ elements) for each $n < \omega$.

So for some element \underline{A}_a of N , $N \models “\underline{A}_a$, in \mathbf{V}'_1 , is a \mathbb{R}_1 -name of a subset of ω and $\underline{A}_a[\mathbf{G}'_1] = A_a”$.

Clearly $M^+ \models$ “for some countable subset u of $\omega_2^{\mathbf{V}'_1} = \omega_1^{\mathbf{V}'_3}$ from \mathbf{V}'_1 and Borel function \mathbf{B} from \mathbf{V}'_1 we have $A_a = \mathbf{B}_a(\dots, \rho_b, \dots)_{b \in u_a}$ (so some $p \in \mathbf{G}_2^+$ forces \underline{A}_a satisfies this)”. So using $F_2^{M^+}$ there are $a_1 \neq a_2$ from X such that the parallel of clause $(\beta)(d)$ of stage C holds, see clause (G) of stage D, so two members of D are almost disjoint, contradiction. $\square_{1.1}$

{a1.6}

Claim 1.2. *If \boxtimes then \boxplus where:*

- \boxtimes (a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$ is a CS iteration
- (b) $k(*) < \omega$ and $\beta(k) < \alpha(*) < \omega_1$ for $k < k(*)$
- (c) each \mathbb{Q}_α is a Laver forcing (in $\mathbf{V}^{\mathbb{P}_\alpha}$) and η_α is generic
- (d) $h \in ({}^\omega\omega)^\mathbf{V}$
- (e) $p \in \mathbb{P}_{\alpha(*)}$
- (f) $\Vdash_{\mathbb{P}_{\alpha(*)}} “\underline{B}_k \subseteq \omega$ and $|\underline{B}_k \cap [\eta_{\beta(k)}(n+1), \eta_{\beta(k)}(n+2))| \leq h(\eta_{\beta(k)}(n))$ for every n large enough” for $k < k(*)$
- \boxplus for some p_1, p_2 and B_k^* for $k < k(*)$ we have
 - (a) $\mathbb{P}_{\alpha(*)} \models “p \leq p_\ell”$ for $\ell = 1, 2$
 - (b) $B_k^* \subseteq \omega$ (from \mathbf{V})
 - (c) $p_1 \Vdash “\underline{B}_k \subseteq^* B_k^*”$
 - (d) $p_2 \Vdash “\underline{B}_k \subseteq^* (\omega \setminus B_k^*)”$.

Proof. Clearly letting $B_* = \cup\{B_k : k < k(*)\}$ we have

- (*) $p \Vdash_{\mathbb{P}_{\alpha(*)}}$ “for every large enough n the set $B \cap [\eta_0(n+1), \eta_0(n+2))$ has $\leq \eta_0(n)$ members”.

Now by the properties of iterating Laver forcing ([Lav76] or see [Sh:f, Ch.VI]), we have:

(*) if $\mathbf{G}_1 \subseteq \mathbb{P}_1$ is generic over \mathbf{V} and $\eta = \eta_0[\mathbf{G}_1]$ then

$\Vdash_{\mathbb{P}_{\alpha(*)}/\mathbf{G}_1}$ “ if $\underline{B} \subseteq \omega$ and in $\underline{B} \cap [\eta(n), \eta(n+1))$
 there are $\leq \eta(n)$ elements for every n large enough
then for some $B' \in \mathbf{V}[\mathbf{G}_1], B' \subseteq \omega, \underline{B} \subseteq B'$ and
 $B' \cap [\eta(n), \eta(n+1))$ has $\leq (\eta(n))^n$ members for every n large enough.

Now this applies in particular to $\underline{B} = \underline{B}_*$ getting B' . Hence without loss of generality $\alpha(*) = 1$ so we can replace \mathbb{P}_1 by \mathbb{Q}_0 , Laver forcing; also for a dense set of $p \in \mathbb{Q}_0$ we have: if $\eta \in p$ is of length $n+1$ so an increasing sequence of natural number, then $p^{[n]} := \{\nu \in p : \nu \triangleleft \eta \text{ or } \eta \triangleleft \nu\}$ forces a value b_η to $\underline{B}' \cap [0, \eta(n))$ so necessarily $|b_\eta| \leq \eta(n-1)$ when $n > 1$.

By thinning p , without loss of generality if $\eta \in p$ and $u_\eta = \{n : \eta \hat{\ } \langle n \rangle \in p\}$ is infinite (equivalently is not a singleton) then $\langle b_{\eta \hat{\ } \langle n \rangle} : n \in u_\eta \rangle$ is a Δ -system.

The rest is easy, too. □

{a1.9}

Remark 1.3. 1) Note that in 1.1 we can replace \mathbb{Q}_0 by any forcing notion similar enough, see [RoSh:470].

2) We can strengthen 1.1 by replacing “ Q -point” by a weaker statement.

Similarly we can weaken the demands on how “thin” is \underline{B} in 1.2 and in the proof of 1.1.

2. PRIVATE APPENDIX

Moved from pg. 2:

Recently, solving a long standing problem on models of Peano arithmetic, (appearing as Problem 7 in the book [KS06]), Ali Enayat proved (and other results as well):

{q0.1}

Claim 2.1. ([Ena06]) *For some arithmetically closed family \mathcal{A} of subsets of ω , the model $\Omega_{\mathcal{A}} = (\mathbb{N}, A)_{A \in \mathcal{A}}$ satisfies*

- (a) *it has no conservative extension (i.e. one in which the intersection of any definable subset with \mathbb{N} belongs to \mathcal{A}).*

Motivated by this he asked:

{q0.4}

Question 2.2. Question I:

Is there $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that some model of $\text{Th}(\Omega_{\mathcal{A}})$ has no elementary end extension?

Moved 09.11.23 from 1.2:

- ⊠ (a) is as in [BsSh:242]
- (b) $\Vdash_{\mathbb{Q}_0}$ “ $\eta_\alpha \in {}^\omega\omega$ is increasing enumerating the generic for \mathbb{Q}_α ”
- (c) $h \in ({}^\omega\omega)^{\mathbf{V}}$
- (d) $f \in {}^\omega\omega$ is defined $f(n) = \eta(n+1)$
- (e) $g \in {}^\omega\omega$ is defined by $g(n) = h(\eta(n))$
- (f) $\Vdash_{\mathbb{Q}_0}$ “ \mathbb{Q}_1 is an (f, g) -bounding forcing notion”
- (g) $\mathbb{Q} = \mathbb{Q}_0 * \mathbb{Q}_1$
- (h) $p^c \Vdash_{\mathbb{Q}}$ “ $B \subseteq \omega$ and $|B \cap [\eta(n), \eta(n+1)]| \leq h(\eta(n))$ ”
- ⊞ for some p_1, p_2, B_1, B_2 we have
 - (a) $p \leq_{\mathbb{Q}} p_\ell$ for $\ell = 1, 2$
 - (b) $B_1, B_2 \subseteq \omega$ are almost disjoint
 - (c) $p_\ell \Vdash$ “ $B \subseteq^* B_\ell$ ” for $\ell = 1, 2$.

REFERENCES

- [Ena06] Ali Enayat, *A standard model of Peano arithmetic with no conservative elementary extension*, preprint (2006).
- [Kei71] H. Jerome Keisler, *Model theory for infinitary logic. Logic with countable conjunctions and finite quantifiers*, Studies in Logic and the Foundations of Mathematics, vol. 62, North-Holland Publishing Co., Amsterdam-London, 1971.
- [KS06] R. Kossak and J. Schmerl, *The structure of models of Peano arithmetic*, Oxford University Press, 2006.
- [Lav76] Richard Laver, *On the consistency of Borel’s conjecture*, Acta Math. **137** (1976), 151–169.
- [Sh:f] Saharon Shelah, *Proper and improper forcing*, Perspectives in Mathematical Logic, Springer, 1998.
- [BsSh:242] Andreas Blass and Saharon Shelah, *There may be simple P_{\aleph_1} - and P_{\aleph_2} -points and the Rudin-Keisler ordering may be downward directed*, Annals of Pure and Applied Logic **33** (1987), 213–243.

- [RoSh:470] Andrzej Roslanowski and Saharon Shelah, *Norms on possibilities I: forcing with trees and creatures*, *Memoirs of the American Mathematical Society* **141** (1999), no. 671, xii + 167, math.LO/9807172.
- [Sh:924] Saharon Shelah, *Can the order determine the addition for models of PA*, *Mathematical Logic Quarterly* **submitted**.
- [EnSh:936] Ali Enayat and Saharon Shelah, *An improper arithmetically closed Borel subalgebra of $P(\omega)$ mod FIN*, *Topology and its Applications* **preprint**.
- [Sh:937] Saharon Shelah, *Models of expansions of \mathbb{N} with no end extensions*, *Mathematical Logic Quarterly* **57** (2011), 341–365, 0808.2960.

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL, AND, DEPARTMENT OF MATHEMATICS, HILL CENTER - BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA

E-mail address: shelah@math.huji.ac.il

URL: <http://shelah.logic.at>