

ON  $\text{CON}(\mathfrak{d}_\lambda > \text{COV}_\lambda(\text{MEAGRE}))$   
SH945

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ABSTRACT. We prove the consistency of: for suitable strongly inaccessible cardinal  $\lambda$  the dominating number, i.e., the cofinality of  ${}^\lambda\lambda$ , is strictly bigger than  $\text{cov}_\lambda(\text{meagre})$ , i.e. the minimal number of nowhere dense subsets of  ${}^\lambda 2$  needed to cover it. This answers a question of Matet.

0. INTRODUCTION

Cardinal invariants on the continuum have a long tradition of research. For a topologist, it can be viewed as investigating the space  $\beta(\omega)$ , the Stone Čzech compactification of  $\omega$ . This point of view is taken, for example, in the celebrated paper of Van Douwen [vD84].

For set theorists, it is interesting to check the relationship between the relevant cardinal invariants. In this context, it is natural to generalize the problems to higher cardinals, above  $\aleph_0$ . One finds out, very soon, that for the class of (strongly) inaccessible cardinals, the generalizations are more reasonable and have more affinity to the  $\aleph_0$  case.

We shall define three cardinal invariants (but the paper deals, actually, just with two of them):

**Definition 0.1.** The bounding and dominating numbers. {z1}

Let  $\lambda$  be an inaccessible cardinal.

Let  $f, g \in {}^\lambda\lambda$

- (a)  $f \leq^* g$  if  $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$
- (b)  $A \subseteq {}^\lambda\lambda$  is unbounded if there is no  $h \in {}^\lambda\lambda$  so that  $f \in A \Rightarrow f \leq^* h$
- (c)  $A \subseteq {}^\lambda\lambda$  is dominating when for every  $f \in {}^\lambda\lambda$  there exists  $g \in A$  so that  $f \leq^* g$
- (d) the bounding number for  $\lambda$ , denoted by  $\mathfrak{b}_\lambda$ , is  $\min\{|A| : A \text{ is unbounded in } {}^\lambda\lambda\}$
- (e) the dominating number for  $\lambda$ , denoted by  $\mathfrak{d}_\lambda$ , is  $\min\{|A| : A \text{ is dominating in } {}^\lambda\lambda\}$ .

Notice that the usual definitions of  $\mathfrak{b}$  and  $\mathfrak{d}$  are  $\mathfrak{b}_{\aleph_0}$  and  $\mathfrak{d}_{\aleph_0}$  according to Definition 0.1. The definition of  $\text{cov}_\lambda(\text{meagre})$  involves some topology.

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**Definition 0.2.** The meagre covering number.

Let  $\lambda$  be a regular cardinal

- (a)  ${}^\lambda 2$  is the space of functions from  $\lambda$  into 2
- (b)  $({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$ , for  $\nu \in \bigcup_{\alpha < \lambda} {}^\alpha 2$
- (c)  $\mathcal{U} \subseteq {}^\lambda 2$  is open in the topology  $({}^\lambda 2)_{< \lambda}$ , iff for every  $\eta \in \mathcal{U}$  there exists  $i < \lambda$  so that  $({}^\lambda 2)^{[\eta \upharpoonright i]} \subseteq \mathcal{U}$
- (d)  $\text{cov}_\lambda(\text{meagre})$  is the minimal cardinality of a family of meagre subsets of  $({}^\lambda 2)_{< \lambda}$ , which covers this space.

The paper deals with the relationship between  $\mathfrak{d}_\lambda$  and  $\text{cov}_\lambda(\text{meagre})$ . Matet asked (a personal communication) whether  $\mathfrak{d}_\lambda \leq \text{cov}_\lambda(\text{meagre})$  is provable in ZFC. We give here a negative answer.

For  $\lambda$  a supercompact cardinal and  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , we force large  $\mathfrak{d}_\lambda$  i.e.,  $\mathfrak{d}_\lambda = \mu$  and small covering number (i.e.,  $\text{cov}_\lambda(\text{meagre}) = \kappa$ ). A similar result should hold also for a wider class of cardinals and we intend to return to this subject.

We try to use standard notation. We use  $\theta, \kappa, \lambda, \mu, \chi$  for cardinals  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$  for ordinals. We use also  $i$  and  $j$  as ordinals. We adopt the Cohen convention that  $p \leq q$  means that  $q$  gives more information, in forcing notions. The symbol  $\triangleleft$  is preserved for “being an initial segment”. Also recall  ${}^B A = \{f : f \text{ a function from } B \text{ to } A\}$  and let  ${}^{> \alpha} A = \bigcup \{{}^\beta A : \beta < \alpha\}$ , some prefer  ${}^{< \alpha} A$ , but  ${}^{> \alpha} A$  is used systematically in the author’s papers. At last,  $J_\lambda^{\text{bd}}$  denotes the ideal of the bounded subsets of  $\lambda$ .

The picture of cardinal invariants related to uncountable  $\lambda$  is related but usually quite different than the one for  $\aleph_0$ , they are more similar if  $\kappa$  is “large” enough, mainly strongly inaccessible.

Let us sketch some known results. These results are related to the inequality number and the covering number for category. Recall:

{z17}

**Definition 0.3.** The inequality number.

Let  $\kappa$  be an infinite cardinal. The inequality number of  $\kappa$ ,  $\mathfrak{e}_\kappa$ , is the minimal cardinal  $\lambda$  such that there is a set  $\mathcal{F} \subseteq {}^\lambda \lambda$  of cardinality  $\lambda$  such that there is no  $g \in {}^\lambda \lambda$  satisfying  $(\forall f \in \mathcal{F})(\exists^\kappa \alpha < \lambda)(f(\alpha) = g(\alpha))$ .

For  $\kappa = \aleph_0$ ,  $\mathfrak{e}_\kappa = \text{cov}_{\aleph_0}(\text{meagre})$ ; see Bartosynski (in [Bar87]) and Miller (in [Mil82]).

Now

- (a) the statement  $\mathfrak{e}_\kappa = \text{cov}_\kappa(\text{meagre})$  is valid for  $\kappa > \aleph_0$ , in the case that  $\kappa$  is strongly inaccessible, by [Lan92]. But if  $\kappa$  is a successor cardinal, it may fail
- (b) if  $\mathfrak{d}_\kappa$  is only finitely many cardinals away from  $\kappa$ , then  $\mathfrak{e}_\kappa = \mathfrak{d}_\kappa$ . This can be found in Matet-Shelah [MtSh:804]
- (c) if  $\kappa < \kappa^{< \kappa}$ , then  $\text{cov}_\kappa(\mathcal{M}) = \kappa^+$ . This is due to Landver (in [Lan92])
- (d) it is consistent to get (a) and (b) together, so that  $\text{cov}_\kappa(\text{meagre}) < \mathfrak{e}_\kappa$ . This follows from Cummings-Shelah (in [CuSh:541]).

## 1. THE FORCING

{a2}

**Theorem 1.1.** *Assume*

- (a)  $\lambda$  is supercompact
- (b)  $\lambda < \kappa = \text{cf}(\kappa) = \kappa^{<\kappa} < \mu = \text{cf}(\mu) = \mu^\lambda$
- (c)  $\kappa > \lambda^+$  and<sup>1</sup>  $\delta(*) = (\lambda^+)^{\lambda^+}$  ordinal exponentiation and  $\mathcal{U}_* = \{\delta(*)^{\alpha+1} : \alpha < \kappa\}$  not used till  $(*)_8$  in the proof of 1.3.

Then for some forcing notion  $\mathbb{P}$  not collapsing cardinals  $\geq \lambda$ ,  $\lambda$  is still supercompact in  $\mathbf{V}^{\mathbb{P}}$  and  $\text{cov}_\lambda(\text{meagre}) = \kappa, \mathfrak{d}_\lambda = \mu$ .

*Proof.* By 1.3 below. □<sub>1.1</sub>

Recall {a5}

**Definition 1.2.** 1) We say that a forcing notion  $\mathbb{P}$  is  $\alpha$ -strategically complete when for each  $p \in \mathbb{P}$  in the following game  $\mathfrak{D}_\alpha(p, \mathbb{P})$  between the players COM and INC, the player COM has a winning strategy.

A play lasts  $\alpha$  moves; in the  $\beta$ -th move, first the player COM chooses  $p_\beta \in \mathbb{P}$  such that  $p \leq_{\mathbb{P}} p_\beta$  and  $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$  and second the player INC chooses  $q_\beta \in \mathbb{P}$  such that  $p_\beta \leq_{\mathbb{P}} q_\beta$ .

The player COM wins a play if he has a legal move for every  $\beta < \alpha$ .

2) We say that a forcing notion  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete when it is  $\alpha$ -strategically complete for every  $\alpha < \lambda$ .

**Lemma 1.3.** 1) If  $\lambda$  is supercompact then after some preliminary forcing of cardinality  $\lambda$ ,  $\lambda$  is still supercompact and  $\square_\lambda$  below holds. {a7}

2) If  $\lambda$  is strongly inaccessible and  $\square_\lambda$  below holds and  $\lambda^+ < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , then for some  $\lambda^+$ -c.c.,  $(< \lambda)$ -strategically complete forcing notion  $\mathbb{P}$  we have  $\Vdash_{\mathbb{P}} \text{“}\mathfrak{d}_\lambda = \mu, \text{cov}_\lambda(\text{meagre}) = \kappa\text{”}$

where

$\square_\lambda$  for any regular cardinal  $\chi > \lambda$  and forcing notion  $\mathbb{P} \in \mathcal{H}(\chi)$  which is  $(< \lambda)$ -strategically complete (see Definition 1.2(2)) the following set  $\mathcal{S} = \mathcal{S}_{\mathbb{P}}$  is a stationary subset of  $[\mathcal{H}(\chi)]^{<\lambda}$ :

$\mathcal{S}$  is the set of  $N$ 's such that for some  $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N, N' = N'_N, M = M_N, \mathbf{G} = \mathbf{G}_N$  we have:

- (a)  $N \prec (\mathcal{H}(\chi)^{\mathbf{V}}, \in)$  and  $\mathbb{P} \in N$
- (b) the Mostowski collapse  $N'$  of  $N$  is  $\subseteq \mathcal{H}(\chi_N)$ , and let  $\mathbf{j}_N : N \rightarrow N'$  be the unique isomorphism
- (c)  $N \cap \lambda = \lambda_N$  and  $(\lambda_N)^{>N} \subseteq N$  and  $\lambda_N$  is strongly inaccessible
- (d)  $N' \subseteq M := (\mathcal{H}(\chi_N), \in)$  so both  $N'$  and  $M$  are transitive
- (e)  $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$  is generic over  $N'$  for the forcing notion  $\mathbf{j}(\mathbb{P})$
- (f)  $M = N'[\mathbf{G}]$ . {a7.3}

*Remark 1.4.* 1) Recall that:

<sup>1</sup>The assumption  $\kappa > \lambda^+$  is technical, to allow  $\kappa = \lambda^+$  we should just use  $\delta(*)\kappa$  instead of  $\kappa$ .

- (a)  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \delta, \beta < \delta \rangle$  be a  $(< \lambda)$ -support iteration of  $(< \lambda)$ -strategically complete forcing notions, then  $\mathbb{P}_\delta$  is also  $\lambda$ -strategically complete.
- (b) If  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete forcing notion then  $(\lambda > \text{Ord})^{\mathbf{V}} = (\lambda > \text{Ord})^{\mathbf{V}^{\mathbb{P}}}$ , and consequently  $\lambda$  is strongly inaccessible in  $\mathbf{V}^{\mathbb{P}}$ .

- 2) In part (1) the “ $\lambda^+ < \kappa$ ” rather than “ $\lambda < \kappa$ ” is not essential, see in the proof.  
 3) Is the use of  $\bar{g} \upharpoonright \mathcal{U}_*$  rather than  $\bar{g}$  in the proof necessary? See on this [Sh:F979].

{a8}

**Definition 1.5.** We may say  $(N, \lambda_N, \chi_N, \mathbf{j}_N, N'_N, M_N, \mathbf{G}_N)$  is a witness for  $(N, \mathbb{P})$  when clauses (a)-(f) from 1.3 hold.

*Proof. Proof of Claim 1.3* 1) This is essentially by Laver [Lav78] using Laver’s diamond.

2) We use a  $(< \lambda)$ -support iteration  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu + \kappa, \beta < \mu + \kappa \rangle$  such that

(A) if  $\alpha < \mu$  then  $\mathbb{Q}_\alpha$  is the  $(\mathbb{P}_\alpha$ -name of the) dominating forcing,  $\mathbb{Q}_\lambda^{\text{dom}}$ , i.e.  $(\mathbb{Q}_\lambda^{\text{dom}})^{\mathbf{V}^{\mathbb{P}_\alpha}}$  where in the universe  $\mathbf{V}^{\mathbb{P}_\alpha}$  the forcing  $\mathbb{Q} = \mathbb{Q}_\lambda^{\text{dom}}$  is

( $\alpha$ )  $p \in \mathbb{Q}$  iff

- (a)  $p = (\eta, f) = (\eta^p, f^p)$   
 (b)  $\eta \in {}^\varepsilon \lambda$  for some  $\varepsilon < \lambda$ , ( $\eta$  is called the trunk of  $p$ )  
 (c)  $f \in {}^\lambda \lambda$   
 (d)  $\eta \triangleleft f$

( $\beta$ )  $p \leq_{\mathbb{Q}} q$  iff

- (a)  $\eta^p \trianglelefteq \eta^q$   
 (b)  $f^p \leq f^q$ , i.e.  $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$   
 (c) if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^p(\varepsilon) \in [f^q(\varepsilon), \lambda)$ ; this follows

(B) fix  $\bar{\theta} = \langle \theta_\alpha : \alpha < \lambda \rangle$  with  $\theta_\alpha = (2^{|\alpha| + \aleph_0})^+$ , or any sequence of cardinals  $\in \text{Reg} \cap \lambda$ , increasing fast enough

(C) if  $\alpha \in [\mu, \mu + \kappa]$  then  $\mathbb{Q}_\alpha$  is the  $\bar{\theta}$ -dominating forcing, i.e.,  $(\mathbb{Q}_{\bar{\theta}})^{\mathbf{V}^{\mathbb{P}_\alpha}}$  where in the universe  $\mathbf{V}^{\mathbb{P}_\alpha}$  the forcing notion  $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}$  is defined as follows:

( $\alpha$ )  $p \in \mathbb{Q}$  iff

- (a)  $p = (\eta, f) = (\eta^p, f^p)$   
 (b)  $\eta \in \prod_{\zeta < \ell g(\eta)} \theta_\zeta$  and  $\ell g(\eta)$  is an ordinal  $< \lambda$   
 (c)  $f \in \prod_{\zeta < \lambda} \theta_\zeta$   
 (d)  $\eta \triangleleft f$

( $\beta$ ) order: as in (A)( $\beta$ ).

Let  $f_\alpha$  be the generic object for  $\mathbb{Q}_\alpha$  for  $\alpha < \mu$  and  $g_i$  be the generic object for  $\mathbb{Q}_{\mu+i}$  for  $i < \kappa$ .

Now:

- (\*)<sub>1</sub> for  $\alpha \leq \mu + \kappa$  the forcing notion  $\mathbb{P}_\alpha$  is  $(< \lambda)$ -strategically complete and, when  $\alpha < \mu + \kappa$ ,  $\mathbb{Q}_\alpha$  is  $(< \lambda)$ -strategically complete<sup>2</sup>, in fact

<sup>2</sup>for this,  $\theta_\alpha > \alpha$  is enough

- ( $\alpha$ ) for  $\alpha \in [\mu, \mu + \kappa)$  it is not  $(< \lambda)$ -complete but it is  $(< \lambda)$ -strategically complete, and even  $\lambda$ -strategically complete; simply, in a play, COM can keep having the trunk being of length  $\geq$  length of the play so far
- ( $\alpha$ )<sup>+</sup> moreover, COM can guarantee that in limit stage  $\beta$  of the game,  $\langle p_\alpha : \alpha < \beta \rangle$  has a lub
- ( $\beta$ ) for  $\alpha \in [0, \mu), \mathbb{Q}_\alpha$  is  $(< \lambda)$ -complete even for directed systems (hence  $\mathbb{P}_\beta$  for  $\beta \leq \mu$  is)
- ( $\beta$ )<sup>+</sup> moreover, for such systems there is a lub.

[Why? We prove this by induction on  $\alpha$  for  $\mathbb{P}_\alpha$ , using 1.4.]

- (\*)<sub>2</sub> for each  $\alpha \leq \mu + \kappa, \mathbb{P}_\alpha$  and for  $\alpha < \mu + \kappa$ , the forcing notions  $\mathbb{Q}_\alpha$  satisfy a strong form of the  $\lambda^+$ -c.c., (see [Sh:80] for definition, preservation and history; or pedantically [Sh:546, §1])

hence

- (\*)<sub>3</sub> (a) forcing with  $\mathbb{P}_{\mu+\kappa}$  collapses no cardinal, changes no cofinality, and adds no sequence to  ${}^{\lambda > \mathbf{V}}$ ;
- (b)  $(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_{\mu+\kappa}]} = \cup\{(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_{\mu+i}]} : i < \kappa\}$
- (c)  $(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\mu]} = \cup\{(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]} : \alpha < \mu\}$ .

[Why? By (\*)<sub>2</sub> + (\*)<sub>1</sub> clause (a) holds, for clauses (b),(c) use also the support in the iteration being  $< \lambda$  recalling that  $\mu, \kappa$  are regular  $> \lambda$ .]

- (\*)<sub>4</sub> in  $\mathbf{V}^{\mathbb{P}_\mu}, \mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$  as witnessed by  $\bar{f} = \langle \bar{f}_\alpha : \alpha < \mu \rangle$ , in fact  $\Vdash_{\mathbb{P}_{\alpha+1}} \bar{f}_\alpha \in {}^\lambda \lambda$  dominates  $(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]}$  modulo  $J_\lambda^{\text{bd}}$ .

[Why? Easy using (\*)<sub>3</sub>(c).]

- (\*)<sub>5</sub>  $\Vdash_{\mathbb{P}_{\mu+i+1}} \bar{g}_i \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_{\mu+i}]}$ , the order being modulo  $J_\lambda^{\text{bd}}$ .

[Why? As in  $\mathbf{V}^{\mathbb{P}_{\mu+i}}$  for each  $g \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  the set  $\{(\eta, f) \in \mathbb{Q}_{\bar{g}} : \text{for every } \varepsilon \in [\varepsilon, \lambda) \text{ we have } g(\varepsilon) \leq f(\varepsilon)\}$  is a dense open subset of  $\mathbb{Q}_{\mu+i}$ .]

- (\*)<sub>6</sub>  $\Vdash_{\mathbb{P}_{\mu+\kappa}} \bar{g} = \langle g_i : i < \kappa \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing and cofinal in  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_\lambda^{\text{bd}}})$ .

[Why? By (\*)<sub>5</sub> noting that  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_{\mu+\kappa}]} = \cup\{(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_{\mu+i}]} : i < \kappa\}$  which holds by (\*)<sub>3</sub>(b).]

Now

- (\*)<sub>7</sub>  $\Vdash_{\mathbb{P}_{\mu+\kappa}} \text{cov}_\lambda(\text{meagre}) \leq \kappa$ .

[Why? As we can look at  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  instead<sup>3</sup> of  ${}^\lambda 2$  and for each  $\varepsilon < \lambda, i < \kappa$  the set  $B_{\varepsilon, i} = \{\eta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon : \text{for every } \zeta \in [\varepsilon, \lambda) \text{ we have } \eta(\zeta) \leq g_i(\zeta) < \theta_\zeta\}$  is closed nowhere dense, and by (\*)<sub>6</sub>  $\mathbf{V}^{\mathbb{P}_{\mu+\kappa}} \models \prod_{\zeta < \lambda} \theta_\zeta = \cup\{B_{\varepsilon, i} : \varepsilon < \lambda, i < \kappa\}$ .]

<sup>3</sup>E.g. let  $F : {}^\lambda 2 \rightarrow \prod_{\varepsilon < \lambda} \theta_\varepsilon$  be  $F(\eta) = \rho$  iff  $\eta \in {}^\lambda 2$  and for every  $\varepsilon < \lambda, \rho(\varepsilon) = 0$  iff  $(\forall i < \theta_\varepsilon)(\eta \sum_{\zeta < \varepsilon} \theta_\zeta + i) = 0$  and  $\rho(\varepsilon) = 1 + i$  iff  $\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + i) = 1 \wedge (\forall j < i)(\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + j) = 0)$ . Now if  $\prod_{\varepsilon} \theta_\varepsilon = \cup\{\mathcal{U}_i : i < \kappa\}$ , each  $\mathcal{U}_i$  closed nowhere dense then  $\langle F^{-1}(\mathcal{U}_i) : i < \kappa \rangle$  witnesses  $\text{cov}_\lambda(\text{meagre}) \leq \kappa$ .

Now we come to the main and last point

- (\*)<sub>8</sub> letting  $\mathcal{U}_* = \{\lambda^+(\gamma + 1) : \gamma < \kappa\}$ , it is forced, i.e.  $\Vdash_{\mathbb{P}_{\mu+\kappa}}$ , that  $\mathbf{V}' := \mathbf{V}[\bar{f}, \bar{g}|\mathcal{U}_*]$  satisfies:
- (a)  $\mathbf{V}'$  has the same cardinals as  $\mathbf{V}$
  - (b) the cofinality of a cardinal is the same in  $\mathbf{V}'$  and  $\mathbf{V}$
  - (c)  $({}^{\lambda>} \text{Ord})^{\mathbf{V}'} = ({}^{\lambda>} \text{Ord})^{\mathbf{V}}$
  - (d) if  $\theta \geq \mu$  then  $(2^\theta)^{\mathbf{V}'} = (2^\theta)^{\mathbf{V}}$
  - (e)  $(2^\lambda)^{\mathbf{V}'} = \mu$
  - (f)  $\bar{g}|\mathcal{U}_*$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing cofinal in  $(\prod_{i<\lambda} \theta_i)^{\mathbf{V}'}$ .

[Why? Straight forward.]

- (\*)<sub>9</sub> it is forced, i.e.  $\Vdash_{\mathbb{P}_{\mu+\kappa}}$  that no  $f \in ({}^\lambda \lambda)^{\mathbf{V}'}$  dominate  $\{f_\alpha : \alpha < \mu\}$ .

We shall note that it suffices to prove (\*)<sub>9</sub> for proving 1.3, and that (\*)<sub>9</sub> holds, thus finishing.

Why it suffices? As  $\langle f_\alpha : \alpha < \mu \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing and  $\text{cf}(\mu) = \mu > \lambda$ , this implies  $\mathfrak{d}_\lambda \geq \mu$ , and this is the last piece missing. The rest of the proof is dedicated to proving that (\*)<sub>9</sub> holds.

Let  $\mathbf{G}_\mu \subseteq \mathbb{P}_\mu$  be generic over  $\mathbf{V}$  and so  $\langle f_\alpha : \alpha < \mu \rangle = \langle f_\alpha[\mathbf{G}_\mu] : \alpha < \mu \rangle$  is well defined. Now  $\mathbb{P}_{\mu+\kappa}/\mathbf{G}_\mu$  is just the limit of the  $(< \lambda)$ -support iteration of  $\langle \mathbb{P}_{\mu+i}/\mathbf{G}_\mu, \mathbb{Q}_{\mu+j} : i \leq \kappa, j < \kappa \rangle$ . Let  $p \in \mathbb{P}_{\mu+\kappa}/\mathbf{G}_\mu$ . For  $i \leq \kappa$  let  $\mathbb{P}_{0,i} = \mathbb{P}_{\mu+i}/\mathbf{G}, \mathbb{Q}_{0,i}$  be the  $\mathbb{P}_{0,i}$ -name of  $\mathbb{Q}_{\mu+i}$ , i.e., of  $\mathbb{Q}_{\bar{g}}$  in the universe  $\mathbf{V}[\mathbf{G}_\mu]^{\mathbb{P}_{0,i}}$ .

We shall apply  $\square_\lambda$ . Let  $\gamma(*) = \kappa$  (but we shall use  $\gamma(*)$  since the proof applies to any  $\gamma(*)$  of cofinality  $> \lambda$ ). The condition  $\square_\lambda$  is preserved by forcing by  $\mathbb{P}_\mu$  recalling (\*)<sub>1</sub>( $\beta$ ) so  $\mathbf{V}[\mathbf{G}_\mu] = \mathbf{V}^{\mathbb{P}_\mu}$  satisfies  $\square_\lambda$ . So it suffices to prove:

- (\*)'<sub>9</sub> if  $\mathbf{V}$  satisfies  $\square_\lambda$  and  $\mathbf{q} = \langle \mathbb{P}_{0,i}, \mathbb{Q}_{0,j} : i \leq \gamma(*), j < \gamma(*) \rangle$  is a  $(< \lambda)$ -support iteration, such that for every  $j < \gamma(*)$  the forcing notion  $\mathbb{Q}_{0,j}$  is  $(\mathbb{Q}_{\bar{g}})^{\mathbf{V}[\mathbb{P}_{0,j}]}$  and  $\gamma(*)$  is a regular cardinal  $> \delta(*), \lambda^+$  or just  $\lambda^+ \cdot \gamma(*) = \gamma(*) \wedge \text{cf}(\gamma(*)) \geq \lambda^+$  then it is forced, i.e.  $\Vdash_{\mathbb{P}_{0,\gamma(*)}}$ , that no  $f \in ({}^\lambda \lambda)^{\mathbf{V}[\bar{g}_0|\mathcal{U}_*]}$  dominate  $({}^\lambda \lambda)^{\mathbf{V}}$  letting  $\bar{g}_0 = \langle g_{0,i} : i < \gamma(*) \rangle$  where  $g_{0,i} \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  is the name of the generic for  $\mathbb{Q}_{0,i}$  so  $g_{0,i} = g_{\mu+i}$ .

[Why does (\*)'<sub>9</sub> suffice? We apply it with  $\mathbf{V}, \gamma(*)$  in (\*)'<sub>9</sub> standing for  $\mathbf{V}^{\mathbb{P}_\mu} = \mathbf{V}[\mathbf{G}_\mu], \kappa$  here. So in  $\mathbf{V}[\mathbf{G}_\mu]^{\mathbb{P}_{0,\gamma(*)}} = \mathbf{V}^{\mathbb{P}_{\mu+\kappa}}$ , letting  $f_\alpha = f_\alpha[\mathbf{G}_\mu]$  for  $\alpha < \mu$  we have:

- (a)  $f_\alpha \in {}^\lambda \lambda$ , for every  $\alpha < \mu$
- (b)  $\bar{f} = \langle f_\alpha : \alpha < \mu \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing cofinal in  $\mathbf{V}$
- (c)  $\{f_\alpha : \alpha < \mu\}$  has no common  $\leq_{J_\lambda^{\text{bd}}}$ -upper bound in  $\mathbf{V}[\bar{f}, \bar{g}_0|\mathcal{U}_*]$ .

This implies that  $\Vdash_{\mathbb{P}_{\mu+\kappa}}$  “ $\mathbf{V}[\bar{f}, \bar{g}|\mathcal{U}_*]$  satisfies  $\mathfrak{d}_\lambda \geq \mu$ ” as required.]

For  $i \leq \gamma(*)$  let  $\mathbb{P}_{1,i}$  be the completion of  $\mathbb{P}_{0,i}$  and let  $\mathbb{P}'_i = \mathbb{P}_{2,i}$  be the complete subforcing of  $\mathbb{P}_{1,\delta(*)^{(i+1)}}$  generated by  $g'_j = \langle g'_j : j < i \rangle = \langle g_{0,\delta(*)^{(j+1)}} : j < i \rangle$ .

We shall use the nice properties of  $\mathbb{P}'_i, \bar{g}'_i$ .

Note that

- $\boxplus_1$  (a)  $\langle g'_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$ , i.e., if  $\mathbf{G}$  is a subset of  $\mathbb{P}'_{\gamma(*)}$  generic over  $\mathbf{V}$  and  $g'_\gamma = g'_\gamma[\mathbf{G}]$  then  $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_\gamma : \gamma < \gamma(*) \rangle]$
- (b) if  $g''_\gamma \in \prod_{\zeta < \lambda} \theta_\zeta$  for  $\gamma < \gamma(*)$  and the set  $\{(\gamma, \zeta) : \gamma < \gamma(*) \text{ and } \zeta < \lambda \text{ and } g''_\gamma(\zeta) \neq g'_\gamma(\zeta)\}$  has cardinality  $< \lambda$  then  $\langle g''_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$
- (c)  $\mathbb{P}'_{\gamma+1}/\mathbb{P}'_\gamma$  is equivalent to  $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\mathbb{P}'_\gamma]}$
- (d) if  $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$  is an increasing sequence of ordinals  $< \gamma(*)$ , then  $\langle g'_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$
- (e) if  $\bar{\zeta} = \langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$  is an increasing sequence of ordinals  $< \gamma(*)$ , then the sequence  $\langle g_{\mathbf{h}(\gamma, \bar{\zeta})} : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}_{0, \gamma(*)}$  where we define  $\mathbf{h}(\gamma, \bar{\zeta}) < \gamma(*)$  for  $\gamma < \gamma(*)$  by induction on  $\gamma$  as:  
 $\cup\{\mathbf{h}(\beta, \bar{\zeta}) + 1 : \beta < \gamma\}$  if  $\beta \notin \mathcal{U}_*$  and  $\delta(*) (\zeta(\gamma) + 1)$  if  $\beta \in \mathcal{U}_*$ .

[Why? The serious point is clause (d) and (e) which is done similarly. For this it suffices to show that: if  $\langle g_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}_{0, \gamma(*)}$  and  $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$  is as there then not only  $\langle g_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$  but also  $\langle g_{\delta(*) (\zeta(\gamma) + 1)} : \gamma < \gamma(*) \rangle$  is. This holds and it straightforward translates to saying that the sequence  $\langle \delta(*) (\gamma + 1) : \gamma < \gamma(*) \rangle$  and  $\langle \delta(*) (\zeta(\gamma) + 1) : \gamma < \gamma(*) \rangle$  realizes the same  $\mathbb{L}_{\lambda^+, \lambda}$ -type in the structure  $(\gamma(*), <)$ , which holds by Kino [Kin66]. See more in [Sh:F976].

We shall use  $\boxplus_1$  freely.]

To prove  $(*)'_9$  assume toward contradiction that this fails, so  $\mathbb{P}'_{\gamma(*)}$  satisfies the  $\lambda^+$ -c.c. and for some  $\mathbb{P}'_{\gamma(*)}$ -name  $\underline{f}$  and  $\lambda$ -Borel function  $\mathbf{B}$  and  $\rho \in {}^\lambda \gamma(*)$ , moreover  $\rho \in {}^\lambda \mathcal{U}_*$  we have (noting: the “moreover” holds as  $f \in ({}^\lambda \lambda)^{\mathbf{V}[\bar{g}_0 | \mathcal{U}_*]}$ )

$$\otimes_0 p^* \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\underline{f} \in {}^\lambda \lambda \text{ and dominates } ({}^\lambda \lambda)^{\mathbf{V}}\text{” and } \underline{f} = \mathbf{B}(\langle g_{\rho(i)} : i < \lambda \rangle).$$

Now we choose  $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda \rangle$  such that

- $\otimes_1$  (a)  $N_\varepsilon$  is as in  $\square_\lambda$  for the forcing notion  $\mathbb{P}'_{\gamma(*)}$
- (b)  $\bar{N} \upharpoonright \varepsilon \in N_\varepsilon$  hence  $\bigcup_{\zeta < \varepsilon} N_\zeta \subseteq N_\varepsilon$  and  $\lambda_\varepsilon := N_\varepsilon \cap \lambda > \lambda_\varepsilon^- := \Sigma\{\lambda_\zeta : \zeta < \varepsilon\}$
- (c)  $\bar{\theta}, \mathbf{q}, p^*, \underline{f}, \mathbf{B}, \rho$  belong to  $N_\varepsilon$
- (d) let  $\delta(\varepsilon) = \text{otp}(\delta(*) \cap N_\varepsilon)$
- (e)  $\kappa_\varepsilon = \kappa_\varepsilon^{< \kappa_\varepsilon}$  where  $\kappa_\varepsilon = \text{otp}(\kappa_\varepsilon \cap N_\varepsilon)$ .

We can find  $f^* \in {}^\lambda \lambda$ , i.e.  $\in ({}^\lambda \lambda)^{\mathbf{V}}$ , such that

$$\otimes_2 \text{ for arbitrarily large } \varepsilon < \lambda \text{ for some } \zeta \in [\lambda_\varepsilon^-, \lambda_\varepsilon) \text{ we have } f^*(\zeta) > \lambda_\varepsilon.$$

For  $\varepsilon < \lambda$  let  $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, N'_\varepsilon, \mathbf{G}_\varepsilon)$  be a witness for  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$  recalling Definition 1.5 so  $\lambda_\varepsilon \in (\varepsilon, \lambda)$  is strongly inaccessible and  $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda_\zeta$ , recalling  $\otimes_1$  and  $\delta(\varepsilon) = \mathbf{j}_\varepsilon(\delta(*)$ ), etc.

Let

- ⊗<sub>3</sub>  $u_\varepsilon = N_\varepsilon \cap \gamma(*)$ ,  $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < i(\varepsilon) \rangle$  list  $u_\varepsilon$  in increasing order and for  $i < \text{otp}(u_\varepsilon)$ , equivalently  $i < \mathbf{j}_\varepsilon(\gamma(*))$  let  $\eta_i^\varepsilon = (\mathbf{j}_\varepsilon(\underline{g}'_i))^{N'_\varepsilon}[\mathbf{G}_\varepsilon] \in \prod_{\zeta < \lambda_\varepsilon} \theta_\zeta$  and let  $\bar{\eta}^\varepsilon = \langle \eta_i^\varepsilon : i < \text{otp}(u_\varepsilon) \rangle$ .

Note

- ⊗<sub>4</sub> (a)  $\bar{\eta}^\varepsilon$  is generic for  $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$ , moreover  
 (b) for each  $\varepsilon < \lambda$ , if we change  $\eta_i^\varepsilon(\zeta)$  (legally, i.e.  $< \theta_\zeta$ ) for  $< \lambda_\varepsilon$  pairs  $(i, \zeta) \in \text{otp}(u_\varepsilon) \times \lambda_\varepsilon$  and get  $\bar{\eta}'$ , then also  $\bar{\eta}'$  is generic for  $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$  and  $N'_\varepsilon[\bar{\eta}'] = M_\varepsilon$   
 (c) like  $\boxplus_1$  with  $\mathbf{V}, \mathbb{P}'_{\gamma(*)}, \lambda$  there standing for  $N_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$  here.

Hence

- ⊗<sub>4</sub> for  $\varepsilon < \lambda$ , if  $\bar{\eta}' = \langle \nu_i : i < i(\varepsilon) \rangle$  where  $i(\varepsilon) = \text{otp}(u_\varepsilon)$  is as in ⊗<sub>4</sub>(b), and  $q \in \mathbb{P}'_{\gamma(*)}$  satisfies  $i < i(\varepsilon) \Rightarrow q \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\underline{g}'_{\gamma_i(\varepsilon)} \upharpoonright \lambda_\varepsilon = \nu_i\text{”}$  then  $q$  is  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$ -generic naturally and  $q \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\mathbf{j}_\varepsilon \text{ can be extended naturally to an isomorphism from } N_\varepsilon[\underline{G}_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle \underline{g}_\gamma : \gamma \in u_\varepsilon \rangle] \text{ onto } N'_\varepsilon[\bar{\eta}']\text{”}$ .

[Why? Should be clear, see  $\boxplus_1 + \otimes_4(c)$ .]

By the assumption toward contradiction, ⊗<sub>0</sub>, and  $\mathbb{P}'_{\gamma(*)}$  being  $(< \lambda)$ -strategically closed recalling  $(*)_1(\beta)^+$ , there are  $\zeta(*), p^{**}$  and  $p^+$  such that (recall  $p^* \in \mathbb{P}'_{\gamma(*)} = \mathbb{P}_{0, \gamma(*)}$ ):

- ⊗<sub>5</sub> (a)  $p^* \leq p^{**} \in \mathbb{P}'_{\gamma(*)}$  and  $p^{**} \leq p^+ \in \mathbb{P}_{0, \gamma(*)}$   
 (b)  $\zeta(*) < \lambda$   
 (c)  $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}f^*(\zeta) < \underline{f}(\zeta) \text{ whenever } \zeta(*) \leq \zeta < \lambda\text{”}$   
 (d) if  $\gamma \in \text{Dom}(p^+)$  then  $\eta^{p^+(\gamma)}$  is an object (not just a  $\mathbb{P}'_{\gamma}$ -name) of length  $\geq \zeta(*)$  (recall that  $\eta^{p^+(\gamma)}$  is the trunk of the condition, see clause (a)(b) above).

Note that possibly  $\text{Dom}(p^+) \not\subseteq \cup \{u_\varepsilon : \varepsilon < \lambda\}$ . Choose  $\varepsilon(*) < \lambda$  such that  $\lambda_{\varepsilon(*)} > \zeta(*) + |\text{Dom}(p^+)|$  and  $\gamma \in \text{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$  recalling clause (d) of ⊗<sub>5</sub> and  $|\text{Dom}(p^+)| < \lambda$  as  $p^+ \in \mathbb{P}_{0, \gamma(*)}$  and  $\mathbb{P}_{0, \gamma(*)}$  is the limit of a  $(< \lambda)$ -support iteration.

By ⊗<sub>2</sub> we can add  $(\exists \zeta)[\lambda_{\varepsilon(*)}^- \leq \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$ . Our intention is to find  $q \in \mathbb{P}_{0, \gamma(*)}$  above  $p^+$  which is above some  $q' \in \mathbb{P}'_{\gamma(*)}$  which is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ -generic and forces it to include a generic subset of  $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$  which is induced by some  $\bar{\eta}'$  as in ⊗<sub>4</sub>(b). Toward this in ⊗<sub>6</sub> below the intention is that  $p_{i(*)}^+$  will serve as  $q$ .

Let  $i(*) = i(\varepsilon(*))$  and  $\gamma_i = \gamma_{2,i} = \gamma_{\delta(*)+(i+1)}(\varepsilon(*))$  for  $i < i(*)$  so  $\langle \gamma_i : i < i(*) \rangle$  list  $u_{\varepsilon(*)} \cap \mathcal{U}_*$  in increasing order and let  $\gamma_{i(*)} = \gamma(*)$  so  $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in u_{\varepsilon(*)}\} = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$  and  $N_{\varepsilon(*)} \models \text{“}i(*) \text{ is a regular cardinal } > \lambda_\varepsilon\text{”}$  hence  $i(*)$  is really a regular cardinal so call it  $\sigma$ . Now we define a game  $\supset$  as follows<sup>4</sup>:

- ⊗<sub>2</sub> (A) each play lasts  $i(*) + 1$  moves and in the  $i$ -th move,

<sup>4</sup>The idea is to scatter the  $\eta_{\gamma_i}^{\varepsilon(*)}$ 's. Why not use the original places? as then we have a problem in ⊗<sub>10</sub>.

- (a) if  $i = j + 1$  the antagonist player chooses  $\xi(j) < \sigma$  such that  $j_1 < j \Rightarrow \zeta(j_1) < \xi(j)$
- (b) then, if  $i = j + 1$  the protagonist chooses  $\zeta(j) \in (\xi(j), \sigma) \cap \mathcal{U}_*$ , but there are more restrictions implicit in  $\boxplus_3$
- (c) in any case the protagonoist chooses  $p_i^+, \bar{\nu}^i$  such that  $\boxplus_3$  below holds;
- (B) in the end of the play the protagonist wins the play iff he always has a legal move and in the end  $\{\zeta(i) : i < i(*)\} \in N'_{\varepsilon(*)}$ ; where
- $\boxplus_3$  (a)  $p_i^+ \in \mathbb{P}_{0, \gamma_i}$
- (b) if  $j < i$  then  $\mathbb{P}_{0, \gamma_i} \models "p_j^+ \leq p_i^+"$
- (c) if  $\gamma \in \cup\{\text{Dom}(p_j^+) : j < i\}$  then  
 $p_i^+ \upharpoonright \gamma \Vdash_{\mathbb{P}_{0, \gamma_i}} " \eta^{p_i^+(\gamma)}$  has length  $\geq i(*)$  and  $\geq \lambda_{\varepsilon(*)}$ ”  
 moreover  $\eta^{p_i^+(\gamma_j)}$  is an object,  $\eta^{p_i^+(\gamma_j)}$  for  $j < i$
- (d)  $\mathbb{P}_{0, \gamma_i} \models "p^+ \upharpoonright \gamma_i \leq p_i^+"$
- (e)  $\bar{\nu}^i = \langle \nu_{\gamma_j} : j < i \rangle$  and  $\nu_{\gamma_j} \in \prod_{\iota < \lambda_{\varepsilon(*)}} \theta_\iota$
- (f) for  $j < i$  we have  $\nu_{\gamma_j} \leq \eta^{p_i^+(\gamma_j)}$  so  $p_i^+ \upharpoonright \gamma_j \Vdash " \nu_{\gamma_j} \triangleleft g'_{\gamma_j} "$  recalling  $\boxplus_1$
- (g) for  $j < i$  we have (recall  $\bar{\eta}^\varepsilon$  from  $\otimes_3$ )
- ( $\alpha$ )  $\nu_{\gamma_j} = \eta_{\gamma_{\zeta(j)}}^{\varepsilon(*)}$  recalling  $\eta_{\gamma_j}^{\varepsilon(*)}$  is from  $\otimes_3$  or
- ( $\beta$ )  $\gamma_j \in \text{Dom}(p^+)$  and  $\{\iota < \lambda_{\varepsilon(*)} : \eta_{\zeta(j)}^{\varepsilon(*)}(\iota) \neq \nu_{\gamma_j}(\iota)\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ .

We shall prove

- $\otimes_6$  in the game  $\mathfrak{D}$
- (a) the antagonist has no winning strategy
- (b) in any move the protagonist has a legal move, moreover for any  $\zeta(i) \in (\xi(i), \sigma)$  large enough the protagonist can choose it.

Why  $\otimes_6$  suffice:

By clause (a) of  $\otimes_6$  we can choose a play  $\langle (\xi(i), \zeta(i), p_i^+, \bar{\nu}^i) : i \leq i(*) \rangle$  in which the protagonist wins. Recalling  $\mathbb{P}'_{\gamma(*)} \triangleleft \mathbb{P}_{1, \gamma(*)}$  and  $\mathbb{P}_{0, \gamma(*)}$  is a dense subforcing of  $\mathbb{P}_{1, \gamma(*)}$ , clearly

- $\otimes_7$  there is  $p$  such that
- (a)  $p \in \mathbb{P}'_{\gamma(*)}$
- (b) if  $\mathbb{P}'_{\gamma(*)} \models "p \leq p'"$  then  $p', p^+$  are compatible in  $\mathbb{P}_{0, \gamma(*)}$
- (c)  $p$  is above  $p^{**}$  and it forces  $g'_i \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < \gamma(*)$ .

Then on the one hand

- $\otimes'_7$   $p \in \mathbb{P}'_{\gamma(*)}$  being above  $p^{**}$  forces  $f^\gamma \upharpoonright [\zeta(*), \lambda) < \underline{f} \upharpoonright [\zeta(*), \lambda)$  hence  $f^* \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)}) < \underline{f} \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)})$  recalling that  $\zeta(*) < \lambda_{\varepsilon(*)}$ .

On the other hand,

- $\otimes''_7$   $p$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ -generic.

[Why? As it forces  $\eta_{\gamma_{1,i}} \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < i(*)$  and  $\langle \nu_{\gamma_i} : i < i(*) \rangle$  is (see  $\textcircled{4}$ ) “almost equal” to  $\langle \eta_{\zeta(i)}^{\varepsilon(*)} : i < i(*) \rangle$  which is a subsequence of the sequence from  $\textcircled{3}$  and recalling clause (g) of  $\textcircled{3}$ . That is  $\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} \subseteq \cup\{\{(i, \iota) : \iota < \lambda_{\varepsilon(*)} \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} : \gamma \in u_{\varepsilon(*)} \cap \text{Dom}(p^+)\}$  so is the union of  $\leq |\text{Dom}(p^+)| < \lambda_{\varepsilon(*)}$  sets each of cardinality  $< \lambda_{\varepsilon(*)}$  hence is of cardinality  $< \lambda_{\varepsilon(*)}$ . Hence by  $\textcircled{4}(c) + \textcircled{1}(d)$  the sequence  $\bar{\nu}^{i(*)}$  is generic for  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ .]

As  $f \in N_{\varepsilon(*)}$  it follows from  $\textcircled{7}''$  that

$$\textcircled{7}''' \ p \Vdash \text{“} f \upharpoonright \lambda_{\varepsilon(*)} \text{ is a function from } \lambda_{\varepsilon(*)} \text{ to } \lambda_{\varepsilon(*)}\text{”}.$$

Together  $\textcircled{7}' + \textcircled{7}'''$  gives a contradiction by the choice of  $f^*$  in  $\textcircled{2}$  and of  $\varepsilon(*)$  above, hence it is enough to use  $\textcircled{6}$ .

Why  $\textcircled{6}$  is true:

Let us prove  $\textcircled{6}$ ; first for clause (a) choose any strategy  $\mathbf{st}$  for the antagonist and fix a partial strategy  $\mathbf{st}'$  for the protagonist choosing  $(p_i^+, \bar{\nu}^i)$  from the previous choices and  $\zeta(i)$  if relevant and possible. So the only freedom left is for the protagonist to choose the  $\zeta(i)$ . So we have in  $\mathbf{V}$  a function  $F : \sigma^{>}(i(*)) \rightarrow \sigma$  such that:

(\*) $_F$  playing the game such that the antagonist uses  $\mathbf{st}$  and the protagonist uses  $\mathbf{st}'$ , arriving to the  $i$ -th move,  $\bar{\zeta} = \langle \zeta(j) : j < i \rangle$  is well defined and for the protagonist any choice  $\zeta_i \in (F(\bar{\zeta}), \sigma)$  is legal.

Now we have to find an increasing sequence  $\bar{\zeta} = \langle \zeta(i) : i < i(*) \rangle$  such that  $F(\bar{\zeta} \upharpoonright i) < \zeta(i) \in \mathcal{U}_*$  and  $\bar{\zeta} \in N'_{\varepsilon(*)}$ . As  $F \in \mathcal{H}(\chi_\varepsilon)$  and  $\mathcal{H}(\chi_\varepsilon) = N'_\varepsilon[\mathbf{G}_\varepsilon]$  where  $\mathbf{G}_\varepsilon$  is a subset of  $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}) \in N'_\varepsilon$  and  $\mathbf{j}_\varepsilon(\mathbb{P}_{0,\gamma(*)})$  satisfies the  $\lambda_\varepsilon^+$ -c.c. and  $\sigma = \text{cf}(\sigma) > \lambda_\varepsilon$  this is possible. We are left with proving  $\textcircled{6}(b)$ .

Case 1:  $i = 0$ .

$$\text{Let } p_0^+ = p^+ \upharpoonright \gamma_0.$$

Case 2:  $i$  limit.

By clauses (a) and (b), there is  $p_i^+ \in \mathbb{P}_{0,\gamma_i}$  which is an upper bound (even l.u.b.) of  $\{p_j^+ : j < i\}$  and it is easily as required. Also  $\bar{\nu}^i$  is well defined and as required.

Case 3:  $i = j + 1$  and  $\gamma_j \notin \text{Dom}(p^+)$ .

Clearly  $\gamma_i = \gamma_j + \delta(*)$  and  $\gamma_j \in u_{\varepsilon(*)}$ . As in case 4 below but easier by the properties of the iteration.

Case 4:  $i = j + 1$  and  $\gamma_j \in \text{Dom}(p^+)$

Again  $\gamma_i = \gamma_j + \delta(*)$  and  $\gamma_j \in u_{\varepsilon(*)}$ . First we find  $p'_j$  such that:

- $\textcircled{8}$  (a)  $p_j^+ \leq p'_j \in \mathbb{P}_{0,\gamma_j}$
- (b) if  $\gamma \in \text{Dom}(p_j^+)$  then  $p'_j \upharpoonright \gamma \Vdash \text{“} \ell g(\eta^{p'_j(\gamma)}) > i \text{”}$
- (c)  $p'_j$  forces  $^5$  a value to the pair  $(\eta^{p^+(\gamma_i)}, f^{p^+(\gamma_j)} \upharpoonright \lambda_{\varepsilon(*)})$ ; we call this pair  $q_j$ .

<sup>5</sup>recall that  $\eta^{p^+(\gamma)}$  is an object, not a name and  $p_j^+$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic

This should be clear.

Second

$\otimes_9$   $p_j^+$  hence  $p'_j$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic and  $\langle \nu_{\gamma_{j(1)}} : j(1) < j \rangle$  induces the generic.

[Why? As in the proof of  $\otimes_7''$  above when we assume that we have carried the induction, by  $\boxplus_2$ , clause (g) and  $\otimes_4$ .]

Now

- $\otimes_{10}$  (a)  $f^{q_j} \in (\prod_{\zeta < \lambda_{\varepsilon(*)}} \theta_\zeta)^{N'_{\varepsilon(*)}[p^j]}$   
 (b) for some  $\zeta \in (\xi(i), \sigma)$  we have
- $f^{q_j} \leq \eta_\zeta^{\varepsilon(*)}$
  - $f^{q_j} \in N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$
  - $\langle \zeta(j_1) : j_1 \langle j \rangle \in N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$ .
- (c)  $\eta^{q_j} \triangleleft f^{q_j}$ .

[Why? Clause (a) follows from clause (b) and clause (b) should be clear by  $\otimes_9$  as we can choose  $\zeta(i)$  large enough recalling  $\otimes_6$ . Also clause (c) follows from (b).]

Now we choose  $\zeta(j)$  as in clause (b) of  $\otimes_{10}$  and  $\nu_j \in \prod_{\varepsilon < \lambda_{\varepsilon(*)}} \theta_\varepsilon$  such that  $\eta^{p^+(j)} \triangleleft$

$\nu_j$ ,  $f^{q_j} \leq \nu_j$  and  $\{\iota < \lambda_{\varepsilon(*)} : \nu_j(\iota) \neq \eta_{\zeta(j)}^{\varepsilon(*)}\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ . Next choose  $p_i^+ \in \mathbb{P}'_{\gamma(*)}$  such that  $p_i^+ \upharpoonright \gamma_j = p'_j$ ,  $\eta^{p_i^+(\gamma_i)} = \nu_j$  and  $f^{p_i^+(\gamma_i)} \upharpoonright [\lambda_\varepsilon, \lambda] = f^{p^+(\gamma)} \upharpoonright [\lambda_\varepsilon, \lambda]$ .

So we have carried the induction hence proved  $\otimes_6$  so we are done.  $\square_{1.3}$

## 2. PRIVATE APPENDIX

We have used proving  $\otimes_4(c)$  inside the proof of 1.3

{a9}

**Claim/Definition 2.1.** Assume  $\lambda$  is strongly inaccessible,  $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$  is an increasing sequence of regular cardinals such that  $\theta_\varepsilon > 2^{\sup\{\theta_\zeta : \zeta < \varepsilon\}}$  for  $\varepsilon < \lambda$ . Assume further  $\mathbb{Q} = \langle \mathbb{P}_\beta, \mathbb{Q}_\gamma : \gamma < \gamma(*), \beta \leq \gamma(*) \rangle$  is a  $(< \lambda)$ -support iteration and  $\mathbb{Q}_\gamma$  is  $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}^{\mathbb{P}_\gamma}}$  the  $\bar{\theta}$ -dominating forcing (see in the proof above).

1) For  $u \subseteq \gamma \leq \gamma(*)$  we define  $\mathbb{P}_u$  and prove  $\mathbb{P}_u \subseteq \mathbb{P}_\gamma$  and moreover  $\mathbb{P}_u \triangleleft \mathbb{P}_\gamma$  and  $\mathbb{P}_u$  does not depend on  $\gamma$  by induction on  $\gamma : \mathbb{P}_u = \{p : p \in \mathbb{P}_\gamma \text{ with support } \subseteq u \text{ and for each } \beta \in \text{Dom}(p), p(\beta) \text{ is a } \mathbb{P}_{u \cap \beta}\text{-name of a condition in } \mathbb{Q}_\beta\}$ .

2)  $\mathbb{P}_u$  is isomorphic to  $\mathbb{P}_{\text{otp}(u)}$  naturally.

*Remark 2.2.* Saharon recheck. This is a special case of [Sh:F979]. Alternate.

*Proof.* The main point is:

(\*) assume  $\langle p_i : i < i(*) \rangle$  is a maximal antichain of  $\mathbb{P}_u$  then it is a maximal antichain of  $\mathbb{P}_\gamma$ .

Why? Without loss of generality  $\eta^{p_i(\alpha)}$ ,  $\alpha \in \text{Dom}(p_i)$ , is a real object (not a name).

Case 1:  $\gamma = 0$ .

Trivial.

Case 2:  $\beta := \sup(u) < \gamma$ .

Use  $\mathbb{P}_\beta \triangleleft \mathbb{P}_\gamma$ .

Case 3:  $\gamma$  is a limit ordinal, neither Case 1 nor Case 2.

So necessary  $\text{cf}(\gamma) < \lambda$ ; let  $\partial = \text{cf}(\gamma)$  and let  $\langle \gamma_\varepsilon : \varepsilon < \partial \rangle$  be increasing continuous with limit  $\gamma$  and let  $\gamma_\partial = \gamma$  and let  $\beta_\varepsilon = \min(u \cup \{\gamma\} \setminus \gamma_\varepsilon)$  for  $\varepsilon \leq \partial$ , so  $\beta = \gamma = \gamma_\partial$  and  $\varepsilon < \partial \Rightarrow \beta_\varepsilon < \gamma$ .

For any  $q \in \mathbb{P}_\gamma$  we choose  $p_\varepsilon$  by induction on  $\varepsilon \leq \partial$  such that

- $\otimes$  (a)  $p_\varepsilon \in N \cap \mathbb{P}_{u \cap \beta_\varepsilon}$
- (b)  $p_\varepsilon$  witness  $\mathbb{P}_v \triangleleft \mathbb{P}_\beta$  for  $q \upharpoonright \gamma_\varepsilon$  which means
  - (\*) if  $\alpha \in \text{Dom}(p_\varepsilon) \upharpoonright \alpha \in \mathbb{P}_{\alpha \cap u}, p_\varepsilon \upharpoonright \alpha \leq r \in \mathbb{P}_\gamma, q \upharpoonright \alpha \leq_{\mathbb{P}_\alpha} r$  and  $p_\varepsilon \upharpoonright (\alpha + 1) \leq_{\mathbb{P}_{(\alpha+1) \cap u}} p \in \mathbb{P}_{(\alpha+1) \cap u}$  and  $p \upharpoonright \alpha \leq_{\mathbb{P}_\alpha} r$  then  $r, p$  are compatible in  $\mathbb{P}_{\alpha+1}$
- (c) if  $\zeta < \varepsilon$  then  $p_\zeta \leq_{\mathbb{P}_{u \cap \beta_\varepsilon}} p_\varepsilon$ .

Case 4:  $\gamma = \beta + 1, \beta \in u$ .

Let  $v = u \cap \beta$  and let  $\mathbf{G}_\beta$  be a subset of  $\mathbb{P}_\beta$  generic over  $\mathbf{V}$  so  $\mathbf{G}_v = \mathbf{G}_\beta \cap \mathbb{P}_v$  is a subset of  $\mathbb{P}_v$  generic over  $\mathbf{V}$ . So we are interested in the forcing notion  $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}^{\mathbf{G}_v}}$ . But in  $\mathbf{V}[G]$  the cardinal  $\lambda$  is weakly compact (and even supercompact). But easily this implies that " $\langle p_i : i < i_* \leq \lambda \rangle$  is predense in  $\mathbb{Q}_{\bar{\theta}}$ " is  $\lambda$ -Borel, so is absolute from  $\mathbf{V}[G_v]$  to  $\mathbf{V}[G_\beta]$ , i.e.,  $\mathbf{V}^{\mathbb{P}_v}$  to  $\mathbf{V}^{\mathbb{P}_\beta}$ .  $\square$

{a10}

*Remark 2.3. Concluding Remarks:* 0) So see  $(*)_7$  in the proof of 1.3, if  $\aleph_0 \leq \theta_\varepsilon \leq \lambda$  for  $\varepsilon < \lambda$ , then  $\text{cov}_\lambda(\text{meagre}) \leq \text{cf}(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_\lambda^{\text{bd}}})$ .

1) In  $\otimes_0$  we can require  $\gamma(*) < \lambda^+$ . Why? If  $\kappa = \lambda^+$  so  $\gamma(*) = \kappa^+$  then by " $\mathbb{P}'_{\gamma(*)}$  satisfies the  $\lambda^+$ -c.c." so if  $p^* \Vdash_{\mathbb{P}'_\kappa} \text{"} f \text{ dominates } (\lambda \lambda)^{\mathbf{V}} \text{"}$  then for some  $\gamma(*) < \kappa, f$

is a  $\mathbb{P}'_{\gamma(*)}$ -name. Generally we can use a parallel of nep see [Sh:630]. We may treat this more generally.

2) We may control the various  $\text{cf}(\prod_{\varepsilon < \lambda} \theta_\varepsilon, < J_\lambda^{\text{bd}})$ , see [Sh:F923].

3) We may consider

**Definition 2.4.** Assume  $\lambda$  is strongly inaccessible ( $> \aleph_0$ ),  $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$  where  $\theta_\varepsilon = \text{cf}(\theta_\varepsilon) < \lambda$  and  $\langle \kappa_\varepsilon : \varepsilon < \lambda \rangle$  are increasing continuous sequences of cardinals of length  $\lambda$  with limit  $\lambda$ . We say that the forcing notion  $\mathbb{Q}$  is  $(\bar{\theta}, \bar{\kappa})$ -centered when: there is a function  $f$  witnessing it which means

{a11}

- (\*)  $f$  is a function from  $\mathbb{Q}$  to  $\lambda$  such that: if  $\kappa_\varepsilon \leq \alpha < \kappa_{\varepsilon+1}$  then  $\{p \in \mathbb{Q} : f(p) = \alpha\}$  is  $\theta_\varepsilon$ -directed.

## 3. PRIVATE APPENDIX

Moved 2009/3/26 from end of §1 (right paper ? of 935?) (ref 4d.3)

{4d.6}

**Claim 3.1.** *If  $\kappa = \mathfrak{s}$  and  $\text{cf}(\kappa) < \mathfrak{a}$  then the conclusions of ?? holds.*

*Proof.* Similar to the proof of ??. Let  $\langle \gamma_i^x : i < \text{cf}(\kappa) \rangle$  be increasing with limit  $\kappa$ . In the proof we add

$$\boxplus_1 (h) \text{ if } \ell g(\eta) \notin \{\gamma_i^* : i < \text{cf}(\kappa)\} \text{ then } A_\eta = \emptyset.$$

□

{b7} §2 Weakly compact

**Theorem 3.2.** *If  $\lambda$  is weakly compact,  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , then for some  $\lambda^+$ -c.c. ( $< \lambda$ )-strategically complete forcing notion  $\mathbb{P}$ , we have  $\Vdash_{\mathbb{P}} \mathfrak{d}_\lambda = \mu$ ,  $\text{cov}_\lambda(\text{meagre}) = \kappa$ .*

*Proof. Stage A:*

We define  $\mathbb{Q}_{\mu+\kappa} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu + \kappa, \beta < \kappa \rangle$  as in the proof of 3.2 and let  $f_\alpha, g_i$  be as there and the proof up to  $(*)_7$  including  $(*)_7$  works.

**Stage B:** For every set  $u \subseteq \mu + \kappa$  we define  $\bar{\mathbb{Q}}_u = \langle P_\alpha^u, \mathbb{Q}_\beta^u : \alpha \in u \cup \{\mu + \kappa\} \text{ and } \beta \in u \rangle$  by induction on  $\text{otp}(u)$  such that  $p \in \mathbb{P}_\alpha$  iff  $\text{Dom}(p) \subseteq u \cap \alpha$  and for every  $\beta \in \text{Dom}(\beta)$  the object  $p(\beta)$  is a  $\mathbb{P}_\alpha^u$ -name of a member of  $\mathbb{Q}_\beta$ , noting  $\mathbb{P}_\alpha^u = \mathbb{P}_{\mu+\kappa}^{u \cap \alpha}$ .

Also the order is natural

$$\boxplus_1 \text{ if } u \subseteq \mu + \kappa \text{ then } \mathbb{P}_{\mu+\kappa}^u \leq \mathbb{P}_{\mu+\kappa}.$$

[Why? As each  $\mathbb{Q}_\beta$  is absolutely enough defined and is  $2 - \kappa$ -linked.]

Add details? Use conditions  $p$  in  $\mathbb{P}_\alpha^\circledast$  ( $\alpha \leq \mu + \kappa$ )

- $\boxplus_2$  (a)  $\mathbb{P}_\alpha^* := \{f \in \mathbb{P}_\alpha : \text{the trunk } p(\beta_1) \text{ is an object for every } \beta \in \text{Dom}(p)\}$
  - (b)  $\mathbb{P}_\alpha^*$  is a dense subset of  $\mathbb{P}_\alpha$
  - (c)  $\mathbb{P}_\alpha^{u,*}$  is defined similarly
  - (d)  $\mathbb{P}_\alpha^{u,*}$  is a dense subset of
- $$\boxplus_3 \mathbb{P}_{\mu+\kappa}^* = \cup \{\mathbb{P}_{\mu+\kappa}^{u,*} : u \subseteq \mu + \kappa, |u| \leq \lambda\}.$$

**Stage C:**

$(*)_8 \Vdash_{\mathbb{P}_{\mu+\kappa}}$  “no  $f \in {}^\lambda \lambda$  dominate  $\{f_\alpha : \alpha < \mu\}$ ”.

Toward contradiction assume

$$\boxminus_1 p^* \Vdash_{\mathbb{P}_{\mu+\kappa}} \text{“} f \in {}^\lambda \lambda \text{ is dominate } \{f_\alpha : \alpha < \lambda\} \text{ equivalently } ({}^\lambda \lambda)^{\mathbf{V}[\mathbb{P}_\lambda]} \text{”}.$$

Let  $u \in [\lambda + \kappa]^{< \lambda}$  be such that  $p^* \in \mathbb{P}_{\mu+\kappa}^u$  and  $f$  is a  $\mathbb{P}_{\lambda+\kappa}^u$ -name, i.e. for every  $\alpha < \lambda$  there is a maximal antichain  $\mathcal{I}$  of  $\mathbb{P}_{\mu+\kappa}^u$  of conditions forcing a value to  $f(\alpha)$  and is  $\subseteq \mathbb{P}_{\mu+\kappa}^{u,*}$ . As  $|u| = \lambda < \text{cf}(\mu)$  there is  $\beta_* \in (\sup(u \cap \mu), \mu)$  and let  $v = u \cup \{\beta_*\}$  and let  $f_* = f_{\beta_*}[\mathbf{G}_{\mathbb{P}_\mu}]$ .

Let  $\mathbf{G} \subseteq \mathbb{P}_{\mu+\kappa}^{u \cap \mu}$  be generic over  $\mathbf{V}$  such that  $p_* \upharpoonright \mu = p^* \upharpoonright (u \cap \mu) \in \mathbf{G}$ . We continue as there. □<sub>3.2</sub> □

4. ON  $\lambda^+$ -C.C. 2-LINKED  $\lambda$ -NEP ITERATION

{c11}

**Definition 4.1.** 1) Let  $K_{\lambda,2}$  be the class of  $\mathbf{p}$  consisting of

- (a)  $\bar{Q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha(*), \beta < \beta(*) \rangle$  be  $(< \lambda)$ -support iteration
- (b)  $\vdash_{\mathbb{P}_\beta}$  “ $\mathbb{Q}_\beta \rightarrow \lambda$  and  $\mathbb{Q}_\alpha$  is strategically  $(< \lambda)$ -complete
- (c)  $\xi$  is a limit ordinal  $< \lambda$
- (d)  $\vdash_{\mathbb{P}_\beta}$  “if  $\langle p_\zeta^\ell : \zeta < \xi \rangle$  is  $\leq_{\mathbb{Q}_\beta}$ -increasing for  $\ell = 1, 2$  and  $\mathfrak{c}_\beta(p_\zeta^1) = \mathfrak{c}_\beta(p_\zeta^2)$  for  $\zeta < \xi$  then  $\{p_\zeta^\ell : \ell = 1, 2 \text{ and } \zeta < \xi\}$  has a common upper bound
- (e) the set element of  $\mathbb{Q}_\beta$  are  $\subseteq \mathcal{H}_{< \lambda}(\chi_\beta)$  and it has a generic  $f_\beta \in {}^\lambda \lambda$  for simplicity
- (f)  $\mathfrak{c}_\beta$  is definable using  $\langle \eta_\gamma; \gamma \in u_\beta \rangle$  as parameters  $u_\beta \in [\beta]^{\leq \lambda}$ .

**Definition 4.2.** 1)  $K'_{\lambda,2}$  is defined similarly but now  $\mathfrak{c}_\beta(\langle p_\varepsilon^1 : \varepsilon < \zeta \rangle) = \mathfrak{c}_\beta(\langle p_\varepsilon^2 : \varepsilon < \zeta \rangle)$  for every  $\zeta < \xi$ . The amount of absoluteness, see below.

2) In ?? and ??(1) let  $\mathbb{P}_\alpha = \mathbb{P}'_\alpha = \mathbb{P}_{\mathbf{p},\alpha}, \alpha(*) = \alpha_{\mathbf{p}}(*)$ , etc.

{c15}

**Definition 4.3.** For  $\mathbf{p} \in K_{\lambda,2}$ .

- 1) We say  $u$  is  $\mathbf{p}$ -closed when  $u \subseteq \alpha_{\mathbf{p}}(*)$  and  $\beta \in u \Rightarrow u_{\mathbf{p},\beta} \subseteq u$ .
- 2) For  $\mathbf{p}$ -closed we define  $bbQ_u = \mathbb{Q}_{\mathbf{p},u} = \langle \mathbb{P}_{\mathbf{p},u,\alpha}^*, \mathbb{Q}_{\mathbf{p},u,\beta} : \alpha \in u \cup \{\alpha_{\mathbf{p}}(*)\} \text{ and } \beta \in u \rangle$ . The definition is by induction on  $\text{otp}(u)$ :

$$p \in \mathbb{P}_{\mathbf{p},u,\alpha}^* = \{f : f \text{ is a function with domain } \in [\alpha \cap u]^{< \lambda} \\ \text{and for every } \beta \in \text{Dom}(f), f \upharpoonright \beta \in \mathbb{P}_{\mathbf{p},u,\beta}^* \text{ and there is} \\ \langle p_{\beta,\zeta}, \Upsilon_\zeta : \zeta < \varepsilon \rangle \text{ such that } f \upharpoonright \beta \vdash_{\mathbb{P}_{\mathbf{p},u,\beta}^*} \\ \text{“}\langle p_{\beta,\zeta} : \zeta < \xi \rangle \text{ is } \leq_{\mathbb{Q}_\beta[v[p_{\mathbf{p},u,\beta}^*]} \text{-increasing} \\ \text{and } \mathfrak{c}_\beta(p_{\beta,\zeta}) = \Upsilon_\zeta \text{”}\}.$$

**Claim 4.4.** Assume  $\mathbf{p} \in K_{\lambda,2}$ . If  $u \subseteq v$  are  $\mathbf{p}$ -closed then  $\alpha \in u \Rightarrow \mathbb{P}_{\mathbf{p},u,\alpha} < \mathbb{P}_{\mathbf{p},v,\alpha}$ .

*Proof.* FILL. □

## 5. PRIVATE APPENDIX

{a5x}

**Question 5.1.** : (here? complete) Assume  $\langle \mathbb{P}_1^*, \mathbb{Q}_i^* : i < i(*) \rangle$  is a  $(< \lambda)$ -support iteration with limit  $\mathbb{P}_{i(*)}$ .

- 1) If  $\mathbb{Q}_i^*$  is the  $\theta$ -dominating forcing in  $\mathbf{V}^{\mathbb{P}_i^*}$ , as in the proof of ?? and  $u \leq i(*)$  and  $\langle \mathbb{P}_i^u, \mathbb{Q}_i^u : i \in u \rangle$  is defined similarly with limit  $\mathbb{P}'_u$  then  $\mathbb{P}_i^u \triangleleft \mathbb{P}_{i(*)}$ .
- 2) Like Suslin forcing/nep.

*Proof.* Let  $P_{u \cap j} = P_{\min(u \setminus j)}^u$ . We prove this by induction on  $i(*)$ . For  $i(*) = 0$  this is obvious. Generally clearly we interpret the iteration such that every  $p$  satisfies:  $i \in \text{Dom}(p) = \text{trunk}(p(i))$  is an object (not just a  $\mathbb{P}_i$ -name) and the names in the  $f^p(\alpha)$ ,  $(\alpha < \lambda)$  of  $p$  use maximal antichains of cardinality  $\leq \lambda$ .

Now

- $\oplus_1$  if  $p \in \mathbb{P}_u$  then  $p \in \mathbb{P}_{i(*)}$ ,  $j < i(*) \Rightarrow \mathbb{P}_{u \cap j}^u \triangleleft \mathbb{P}_j$
- $\oplus_2$   $\mathbb{P}_u^u \subseteq \mathbb{P}_{i(*)}$  as partial order (similarly)
- $\oplus_3$   $\mathbb{P}_u^u \triangleleft \mathbb{P}_{i(*)}$ .

[Why? Let  $\langle p_\varepsilon : \varepsilon < \varepsilon(*) \leq \lambda \rangle$  be a maximal antichain of  $\mathbb{P}_u^u$  and  $q \in \mathbb{P}_{i(*)}$ .] □

## 6

Of course we need “no dominated  $f \in {}^\lambda\lambda$  is added”. On  $\mathbb{Q}_\theta$  O.K., but iterations?

Our true problem is as follows: for characteristic  $\lambda_\varepsilon$  having chosen the  $\mathbb{Q}$  we iterate (e.g.  $\mathbb{Q}_{\bar{\theta}}$  or see below, or can use a family  $\{\mathbb{Q}_*^i : i < i^*\}$  of definitions of  $\lambda^+$ -c.c. ( $< \lambda$ )-strategically).

Now repeating here the argument in the end of §1, the problem is that when we like to say that the value  $q_i \in Q_{\gamma_i}$  forced for  $(\eta^{p^+(\gamma_i)}, f^{p^+(\gamma_i)} \upharpoonright \lambda_\varepsilon)$  to be satisfied by a “variant of  $\eta_{\gamma_i}^\varepsilon = \text{changing } < \lambda_\varepsilon \text{ places}$ ”. But  $f^{p^+(\gamma_i)} \upharpoonright \lambda_\varepsilon$  depends on  $\eta_{\gamma_j}^\varepsilon, j \in (i, i(\varepsilon))$ .

So let us use in each  $\lambda_\varepsilon$  an iteration of length say  $\chi_\varepsilon$  of  $\mathbb{Q}^i$  (or of  $\mathbb{Q}^i$ 's (those are definitions) each occurring  $\chi_\varepsilon$  times and  $(\forall \alpha < \chi_\varepsilon)(|\alpha|^{\lambda_\varepsilon} < \chi_\varepsilon)$ ).

Now we will not use  $\langle \eta_{\gamma_i}^\varepsilon : i < \text{otp}(u_\varepsilon) \rangle$ , rather we choose  $\xi(i) < \chi_\varepsilon$  such that  $Q_{\xi(i)}^\varepsilon$  (is of the right kind and)  $\langle \xi(i) : \text{otp}(u_\varepsilon) \rangle$  has to increase fast enough such that  $\eta_{\xi(i)}^\varepsilon$  is generic also for the condition  $q_i$ . We can use a game and it suffices that the nice guy does not lose. Hence it suffices to consider

- (\*)  $\langle \bar{\xi}^\alpha : \alpha < \chi_\varepsilon \rangle, \bar{\xi}^\alpha$  increases of length  $< \lambda_\varepsilon^+$  such that for every  $F : \lambda_\varepsilon^+ \rightarrow \chi_\varepsilon$  and  $\beta < \lambda_\varepsilon^+$  there is  $\alpha$  such that  $\ell g(\bar{\xi}^\alpha) \geq \beta$  and  $\bar{\zeta} \triangleleft \bar{\xi}^\alpha \Rightarrow \xi_{\ell g(\bar{\zeta})}^\alpha$ .

So the question is

- assume  $\langle P_\alpha, \mathbb{Q}_\alpha : \alpha < \chi_\varepsilon \rangle$  is  $(< \lambda)$ -support iteration  $u \subseteq \chi_\varepsilon$  (with much space (a) is  $\mathbb{P}_\mu \triangleleft \mathbb{P}_{\chi_\varepsilon}$ ?)
- a weaker version: in the iteration, on  $\mathbb{Q}_\alpha$  has a strange memory
  - part (1):  $v_\alpha \subseteq \alpha, |v_\alpha| \leq \lambda_\varepsilon$  such that  $\beta \in v_\alpha \Rightarrow v_\beta \triangleleft v_\alpha$ .
  - part (2): one condition  $p_\alpha$  from  $\mathbb{P}_\alpha$  (e.g. chosen).

We may use

**Definition 6.1.** For  $\bar{\theta}$  as before (or constantly  $\lambda$ ) let  $\mathbb{Q}_{\bar{\theta}}^{\text{ed}} = \{(\eta, \mathcal{F}) : \mathcal{F} \subseteq \prod_{\varepsilon < \lambda} \theta_\varepsilon, |\mathcal{F}| < \lambda, \eta \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ for some } \zeta < \lambda \text{ and } \varepsilon < \ell g(\eta) \wedge f \in \mathcal{F} \Rightarrow \eta(\varepsilon) \neq f(\varepsilon)\}$ .

order:  $(\eta_1, \mathcal{F}_1) \subseteq (\eta_2, \mathcal{F}_2)$  iff  $\eta_1 \triangleleft \eta_2 \wedge \mathcal{F}_1 \subseteq \mathcal{F}_2$ .

Problem: For FS iteration of definable nep c.c.c. forcing (e.g. Suslin) i.e. absolute maximal anti-chain we have  $\mathbb{P}_u \triangleleft \mathbb{P}_\alpha$ . How does it generalize to  $(< \lambda)$ -support iteration?

Question: Other uses of the machinery of §1?

## 7. PRIVATE APPENDIX

Moved from pgs.4,5:

First Version:

Let  $\langle q_{\varepsilon, \zeta} : \zeta < \lambda \rangle$  be a maximal antichain of  $\mathbb{P}'_\kappa$  such that  $q_{\varepsilon, \zeta}$  forces a value  $f(\varepsilon)$ , say  $q_{\varepsilon, \zeta} \Vdash_{\mathbb{P}'_\kappa} \text{“}\xi_{\varepsilon, \zeta}\text{”}$ . Let  $u = \cup\{\text{Dom}(q_{\varepsilon, \zeta}) : \varepsilon, \zeta < \lambda\}$  so without loss of generality  $u \in [\kappa]^\lambda$ . Let  $\langle u_\varepsilon : \varepsilon < \lambda \rangle$  be an increasing continuous sequence of members of  $[u]^{<\lambda}$  with union  $u$ ,  $|u_\varepsilon| \leq |\varepsilon|$ . For each  $\varepsilon$  let  $\langle \bar{\eta}_\gamma^\varepsilon : \gamma < \gamma_\varepsilon \rangle$  list the  $\{\bar{\eta} : \eta = \langle \eta_j : j \in u \rangle, \eta_j \in \prod_{\zeta < \varepsilon} \theta_\zeta\}$ . So  $\gamma_\varepsilon < \theta_\varepsilon$  and  $\bar{\eta}_\gamma^\varepsilon = \langle \eta_{\gamma, j}^\varepsilon : j \in u_\varepsilon \rangle$ .

For each  $\varepsilon < \lambda$  and  $\gamma < \gamma_\varepsilon$  let

$$\mathcal{I}_{\varepsilon, \gamma} = \{q : q \in \mathbb{P}'_\kappa \text{ and } u_\varepsilon \subseteq \text{Dom}(q) \text{ and } j \in u_\varepsilon \Rightarrow q \upharpoonright j \Vdash_{\mathbb{P}'_\kappa} \text{“trunk}(q(j)) = \eta_{\gamma, j}^\varepsilon\text{”}\}.$$

Note

□  $\mathcal{I}_{\varepsilon, \gamma}$  is  $(< \theta_\varepsilon)$ -directed.

Let

□  $\mathcal{J}_{\varepsilon, \gamma}$  for  $\gamma < \lambda$  is  $\{q : q \in \mathbb{P}'_\kappa, u_\varepsilon \subseteq \text{Dom}(q), j \in u_\varepsilon = q \upharpoonright j \Vdash \text{trunk}(q) = \eta_{\gamma, j}^\varepsilon$  and  $q$  forces a value of  $f(\varepsilon)\}$ .

Now we use “in  $\mathbf{V}^{\mathbb{P}^\mu}$ ,  $\lambda$  is still weakly compact” to find  $\gamma_\varepsilon < \lambda$  such that

⊙ for every  $q \in \mathcal{I}_{\varepsilon, \gamma}$  there are  $\gamma < \gamma_\varepsilon$  and  $q' \in \mathcal{J}_{\varepsilon, \gamma}$  such that  $q \leq q'$ .

Now define  $g$

$$g(\varepsilon) = \text{Min}\{\alpha < \lambda : \text{for every } \gamma < \gamma_\varepsilon \text{ and } q \in \mathcal{I}_{\varepsilon, \gamma} \text{ for some } \alpha' < \alpha \text{ we have } q \Vdash f(\varepsilon) \neq \alpha'\}.$$

This is O.K. as  $\text{UI}_{\varepsilon, \gamma_\varepsilon}$  is the union of  $\leq \Sigma\{2^{|u_\varepsilon| + |\gamma'|} : \gamma' < \gamma\}$  directed sets.

Now  $g \in \mathbf{V}[\mathbf{G}_{\mathbb{P}^\mu}]$  so for some  $\alpha < \mu$ ,  $g <_{J_\lambda^{\text{bd}}} f_\alpha$ . But this means  $\Vdash_{\mathbb{P}_\kappa}$  “ $f$  is not dominated by  $f_\alpha$  (and more: every stationary set modulo the clubs).”

So we are done.

So it is enough to prove  $(*)_9$  below. But before this...

## 8. PRIVATE APPENDIX

**Discussion 8.1.** : 1)  $D$ -completeness for  $D$  a filter on  $\lambda$ . In preservation under iteration  $D$  is reasonable with regular ultrafilter, in every move only finitely many cardinals are active (832)?  
 2) We can define  $\mathbb{P}_u''$ ,  $u \subseteq \chi$ . When is  $\mathbb{P}_u'' \subseteq \mathbb{P}_\kappa''$ ? That is, is being maximal antichains preserved.

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