(945) revision:2017-11-02 modified:2017-11-05

ON CON($\mathfrak{d}_\lambda >$ COV$_\lambda$(MEAGRE)) SH945

SAHARON SHELAH

ABSTRACT. We prove the consistency of: for suitable strongly inaccessible cardinal λ the dominating number, i.e., the cofinality of ${}^\lambda\lambda$, is strictly bigger than $\text{cov}_\lambda(\text{meagre})$, i.e. the minimal number of nowhere dense subsets of ${}^\lambda 2$ needed to cover it. This answers a question of Matet.

Date: November 2, 2017.

2010 *Mathematics Subject Classification.* Primary 03E35, 03E55; Secondary: 03E17.

Key words and phrases. set theory, independence, forcing, cardinal invariants, inaccessible.

The author thanks Alice Leonhardt for the beautiful typing. This research was supported by the United States-Israel Binational Science Foundation. First version (in TeX Sept/16/2005 as F741). First version in LaTeX January 21, 2009.

(945) revision:2017-11-02 modified:2017-11-05

§ 0. Introduction

Cardinal invariants of the continuum have a long tradition of research. For a topologist, it can be viewed as investigating the space $\beta(\omega)$, the Stone-Čech compactification of ω. This point of view is taken, for example, in the celebrated paper of van Douwen [vD84]. For the set theoretic perspective see the recent excellent surveys, Blass [Bla], Bartoszyński [Bar10].

For set theorists, it is interesting to check the relationship between the relevant cardinal invariants. In this context, it is natural to generalize the problems to higher cardinals, above $\aleph_0$. One finds out, very soon, that for the class of (strongly) inaccessible cardinals, the generalizations are more reasonable and have more affinity to the $\aleph_0$ case. See Landver [Lan92], Cummings-Shelah [CuSh:541], Matet-Shelah [MtSh:804].

We shall define three cardinal invariants (but the paper deals, actually, just with two of them):

{z1}

Definition 0.1. The bounding and dominating numbers.

Let λ be an inaccessible cardinal.

Let $f, g \in {}^\lambda\lambda$

- (a) $f \leq^* g$ if $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$
- (b) $A \subseteq {}^\lambda\lambda$ is unbounded if there is no $h \in {}^\lambda\lambda$ so that $f \in A \Rightarrow f \leq^* h$
- (c) $A \subseteq {}^\lambda\lambda$ is dominating when for every $f \in {}^\lambda\lambda$ there exists $g \in A$ so that $f \leq^* g$
- (d) the bounding number for λ, denoted by $\mathfrak{b}_\lambda$, is $\min\{|A| : A$ is unbounded in ${}^\lambda\lambda\}$
- (e) the dominating number for λ, denoted by $\mathfrak{d}_\lambda$, is $\min\{|A| : A$ is dominating in ${}^\lambda\lambda\}$.

Notice that the usual definitions of $\mathfrak{b}$ and $\mathfrak{d}$ are $\mathfrak{b}_{\aleph_0}$ and $\mathfrak{d}_{\aleph_0}$ according to Definition 0.1. The definition of cov_λ(meagre) involves some topology.

{z1} {cov.1}

Definition 0.2. The meagre covering number.

Let λ be a regular cardinal

- (a) ${}^\lambda 2$ is the space of functions from λ into 2
- (b) $({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$, for $\nu \in {}^{\lambda>}2 := \bigcup\limits_{\alpha<\lambda} {}^\alpha 2$
- (c) $\mathscr{U} \subseteq {}^\lambda 2$ is open in the topology $({}^\lambda 2)_{<\lambda}$, iff for every $\eta \in \mathscr{U}$ there exists $i < \lambda$ so that $({}^\lambda 2)^{[\eta \restriction i]} \subseteq \mathscr{U}$
- (d) cov_λ(meagre) is the minimal cardinality of a family of meagre subsets of $({}^\lambda 2)_{<\lambda}$, which covers this space.

This paper deals with the relationship between $\mathfrak{d}_\lambda$ and cov_λ(meagre). Matet asked (a personal communication) whether $\mathfrak{d}_\lambda \leq \text{cov}_\lambda$(meagre) is provable in ZFC. We give here a negative answer.

For λ a supercompact cardinal and $\lambda < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$, we force large $\mathfrak{d}_\lambda$ i.e., $\mathfrak{d}_\lambda = \mu$ and small covering number (i.e., $\text{cov}_\lambda(\text{meagre}) = \kappa$). A similar result should hold also for a wider class of cardinals and we intend to return elsewhere to this subject.

A point which in a previous version was just a step along the way, the referee asked to justify fully. This was done but eventually is separated to [**?**]. A posteriori the point is that in the parallel case for $\lambda = \aleph_0$, for full memory FS iteration such a claim is true. In fact, by Judah-Shelah [JdSh:292], if $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \alpha(*), \beta < \alpha(*)\rangle$ is FS iteration of Suslin-c.c.c. forcing notion, $\underset{\sim}{\mathbb{Q}}_\beta$ with the generic $\underset{\sim}{\eta}_\beta \in {}^\omega\omega$ and for notational transparency, its definition is with no parameter and $\zeta : \beta(*) \to \alpha(*)$ is increasing and $\mathbb{P} = \langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\beta : \alpha \le \beta(*), \beta < \beta(*)\rangle$ is FS iteration, but $\underset{\sim}{\mathbb{Q}}'_\beta$ is defined exactly as $\underset{\sim}{\mathbb{Q}}_{\zeta(\beta)}$ but now in $\mathbf{V}^{\mathbb{P}'_\beta}$ rather then in $\mathbf{V}^{\mathbb{P}_{\zeta(\beta)}}$ then $\Vdash_{\mathbb{P}_{\alpha(*)}}$ "$\langle \underset{\sim}{\eta}_{\zeta(\beta)} : \beta < \beta(*)\rangle$ is generic for $\mathbb{P}'_{\beta(*)}$ over $\mathbf{V}$".

Now this is not clear to us for $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions. The solution is essentially to change the iteration: to use a "quite generic" $(< \lambda)$-support iteration which "includes" the one we like and use the complete subforcing it generates; see [Sh:1126].

We try to use standard notation. We use $\theta, \kappa, \lambda, \mu, \chi$ for cardinals and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ for ordinals. We use also i and j as ordinals. We adopt the Cohen convention that $p \le q$ means that q gives more information, in forcing notions. The symbol $\triangleleft$ is preserved for "being an initial segment". Also recall ${}^B A = \{f : f$ a function from B to $A\}$ and let ${}^{\alpha>}A = \cup\{{}^\beta A : \beta < \alpha\}$, some prefer ${}^{<\alpha}A$, but ${}^{\alpha>}A$ is used systematically in the author's papers. Lastly, $J^{\rm bd}_\lambda$ denotes the ideal of the bounded subsets of λ.

The picture of cardinal invariants related to uncountable λ is related but usually quite different than the one for $\aleph_0$, they are more similar if κ is "large" enough, mainly strongly inaccessible.

Let us sketch some known results. These results are related to the unequality number and the covering number for category. Recall:

Definition 0.3. The unequality number. {z14}

Let κ be an infinite cardinal. The unequality number of $\kappa, \mathfrak{e}_\kappa$, is the minimal cardinal λ such that there is a set $\mathscr{F} \subseteq {}^\kappa\kappa$ of cardinality λ such that there is no $g \in {}^\kappa\kappa$ satisfying $(\forall f \in \mathscr{F})(\exists^\kappa \alpha < \kappa)(f(\alpha) = g(\alpha))$.

For $\kappa = \aleph_0, \mathfrak{e}_\kappa =$ cov$_{\aleph_0}$(meagre); see Bartoszyński (in [Bar87]) and Miller (in [Mil82]).

Now

- (a) the statement $\mathfrak{e}_\kappa =$ cov$_\kappa$(meagre) is valid for $\kappa > \aleph_0$, in the case that κ is strongly inaccessible, by [Lan92]. But if κ is a successor cardinal, it may fail
- (b) if $\kappa < \kappa^{<\kappa}$, then cov$_\kappa$(meagre) $= \kappa^+$. This is due to Landver (in [Lan92]).

We intend also to address:

Problem 0.4. Can we replace "supercompact" by "strongly inaccessible"? {z15}

Problem 0.5. 1) Can we prove the consistency of cov$_\lambda$(meagre) $< \mathfrak{b}_\lambda$? {z16}
2) For λ strongly inaccessible (or just Laver indestructible supercompact) is there a non-trivial λ^+-c.c. $(< \lambda)$-strategically complete forcing notion $\mathbb{Q}$ which is ${}^\lambda\lambda$-bounding?

We thank the referee, Shimoni Garti and Haim Horowitz for helpful comments. We say more in subsequent works [Sh:1004] and in preparation [Sh:F1199].

Recall

modified:2017-11-05

(945) revision:2017-11-02

{z26}

Definition 0.6. Let κ be supercompact. We say $f : \kappa \to \mathscr{H}(\kappa)$ is a Laver diamond (for κ) when for every $x \in \mathbf{V}$ there are a normal fine ultrafilter on $I = [\lambda]^{<\kappa}$ for some λ such that the Mostowski collapse $\mathbf{j}$ on $\mathbf{V}^I/D$ maps $\langle f(\sup(u \cap \kappa)) : u \in I\rangle/D$ to x; (we can use elementary embeddings instead of an ultrafilter).

{z29}

Notation 0.7. If $\mathbb{P}$ is a forcing notion in $\mathbf{V}$ then $\mathbf{V}^{\mathbb{P}}$ denotes $\mathbf{V}[\mathbf{G}]$ for $\mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$; but in superscript we may write $\mathbf{V}[\mathbb{P}]$ instead.

The parallel to dominating = Hechler forcing is:

{z20}

Definition 0.8. 1) Fix $\lambda = \lambda^{<\lambda}$, the forcing $\mathbb{Q} = \mathbb{Q}^{\rm dom}_\lambda$ is defined by:

- (α) $p \in \mathbb{Q}$ iff
  - (a) $p = (\eta, f) = (\eta^p, f^p)$
  - (b) $\eta \in {}^\varepsilon\lambda$ for some $\varepsilon < \lambda$, (η is called the trunk of p)
  - (c) $f \in {}^\lambda\lambda$
  - (d) $\eta \triangleleft f$
- (β) $p \le_{\mathbb{Q}} q$ iff
  - (a) $\eta^p \trianglelefteq \eta^q$
  - (b) $f^p \le f^q$, i.e. $(\forall \varepsilon < \lambda) f^p(\varepsilon) \le f^q(\varepsilon)$
  - (c) if[1] $\ell g(\eta^p) \le \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$.

2) The generic is $\underset{\sim}{\eta} = \cup\{\eta^p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}\}$.

For transparency

{z23x}

{z23}

Convention 0.9. Below $\lambda, \bar\theta$ are as in 0.10 below.

A parallel to the forcing in [Sh:326], really a c.c.c. version of it is.

{z23}

Definition 0.10. Let λ be inaccessible, $\bar\theta = \langle \theta_\varepsilon : \varepsilon < \lambda\rangle$ be a sequence of regular cardinals $< \lambda$ satisfying $\theta_\varepsilon > \varepsilon$.

1) We define the forcing notion $\mathbb{Q} = \mathbb{Q}_{\bar\theta}$ by

- (α) $p \in \mathbb{Q}$ iff
  - (a) $p = (\eta, f) = (\eta^p, f^p)$
  - (b) $\eta \in \prod\limits_{\zeta < \varepsilon} \theta_\zeta$ for some $\varepsilon < \lambda$, (η is called the trunk of p)
  - (c) $f \in \prod\limits_{\zeta < \lambda} \theta_\zeta$
  - (d) $\eta \triangleleft f$
- (β) $p \le_{\mathbb{Q}} q$ iff
  - (a) $\eta^p \trianglelefteq \eta^q$
  - (b) $f^p \le f^q$, i.e. $(\forall \varepsilon < \lambda) f^p(\varepsilon) \le f^q(\varepsilon)$
  - (c) if $\ell g(\eta^p) \le \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$.

2) The generic is $\underset{\sim}{\eta} = \cup\{\eta^p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\bar\theta}}\}$.

{z25}

Remark 0.11. The forcing is parallel to the creature forcing from [Sh:326, §1,§2] which are ${}^\omega\omega$-bounding.

[1] Actually this follows from $(\alpha) + (\beta)(a), (b)$; that is, if $\ell g(\eta^p) \le \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) = f^q(\varepsilon)$ by clause $(\alpha)(d)$ and $f^p(\varepsilon) \le f^q(\varepsilon)$ by clause $(\beta)(b)$.

(945) revision:2017-11-02 modified:2017-11-05

{z32}

Definition 0.12. For an ordinal $\alpha_* = \alpha(*)$ let $\mathbf{Q}_{\lambda,\bar\theta,\alpha(*)}$ be the class of objects $\mathbf{q}$ consisting of (omitting α_* means for some α_* and $\ell g(\mathbf{q}) = \alpha_{\mathbf{q}} = \alpha_*$):

(a) $\bar u = \langle u_\alpha : \alpha < \alpha_* \rangle$ and $\mathscr{P} = \langle \mathscr{P}_\alpha : \alpha < \alpha_* \rangle$ where $\mathscr{P}_\alpha \subseteq [u_\alpha]^{\le \lambda}, u_\alpha \subseteq \alpha$, without loss of generality $\mathscr{P}_\alpha$ is closed under subsets (but is not necessarily an ideal)

(b) $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \alpha_*, \beta < \alpha_* \rangle$ is a $(<\lambda)$-support iteration let $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\mathbf{q},\alpha(\mathbf{q})}$ and $\mathbb{P}_{0,\alpha} = \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_{0,\alpha} = \underset{\sim}{\mathbb{Q}}_\alpha$

(c) each of $\mathbb{P}_\alpha$ is strategically $(<\lambda)$-complete and λ^+-c.c.

(d) $\underset{\sim}{\eta}_\beta \in \Pi\bar\theta$ is the generic of $\underset{\sim}{\mathbb{Q}}_\beta$ where $\underset{\sim}{\eta}_\beta$, the generic of $\underset{\sim}{\mathbb{Q}}_p$ (defined in clause (e) below) is $\cup\{\eta_p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_\beta}\}$

(e) if $\mathbf{G} \subseteq \mathbb{P}_\beta$ is generic over $\mathbf{V}$ then $\underset{\sim}{\eta}_\alpha[\mathbf{G}]$ in $(\Pi\bar\theta, <_{J^{\rm bd}_\lambda})$ dominate every $\nu \in \mathbf{V}[\langle \underset{\sim}{\eta}_\gamma : \gamma \in u \rangle]$ when $u \in \mathscr{P}_\alpha$; moreover, in $\mathbf{V}[\mathbf{G}], \underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}]$ is the subforcing of $\mathbb{Q}_{\bar\theta}$ consisting of the $p \in \mathbb{Q}_\beta$ such that: for some $\bar s, \underset{\sim}{f}, \eta_p$ (so $\eta_p = \eta$, etc.) we have

 (α) $p = (\eta, f) = (\eta_p, f_p)$ so $\eta \in \prod\limits_{\varepsilon<\zeta} \theta_\varepsilon$ for some $\zeta < \lambda$

 (β) $\bar s = \langle (u_i, f_i) : i < i_* \rangle$

 (γ) $i_* < \lambda, u_i \in \mathscr{P}_\beta, \eta \triangleleft f_i \in \Pi\bar\theta$ and $f_i \in \mathbf{V}[\langle \underset{\sim}{\eta}_\gamma[\mathbf{G}] : \gamma \in u_i \rangle]$

 (δ) $f = \sup\{f_i : i < i_*\}$, i.e. $\varepsilon < \lambda \Rightarrow f(\varepsilon) = \cup\{f_i(\varepsilon) : i < i_*\}$

(f) for $\alpha \le \alpha_*, \mathbb{P}_{2,\alpha}$ is the completion of $\mathbb{P}_\alpha$; we can express it via transforming $\mathbb{P}_\alpha$ to a complete Boolean Algebra, or say:

 $(*)_1$ the elements of $\mathbb{P}_{2,\alpha}$ are of the form $\mathbf{B}(\ldots, \underset{\sim}{\eta}_\gamma, \ldots)_{i<\ell(*)}$ where:

  (α) $i(*) \le \lambda$

  (β) $\gamma_i \in \mathscr{U}$ for $i < i_*$

  (γ) $\mathbf{B}$ is a λ-Borel function from ${}^{i(*)}(\Pi\bar\theta)$ into $\{0,1\} = \{\text{false}, \text{true}\}$; $\mathbf{B}$ is from $\mathbf{V}$, of course, such that $\nVdash_{\mathbb{P}_{\mathbf{q}}}$ "$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma_i}, \ldots)_{i<i(*)} = 0$"

 $(*)_2$ the order is natural: $\mathbb{P}_{2,\alpha} \models$ "$\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(*)} \le \mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}, \ldots)_{i<i(2)}$" iff $\Vdash_{\mathbb{P}_\alpha}$ "if $\mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}[\underset{\sim}{\mathbf{G}}, \ldots)_{i<i(1)}$ is equal to 1 then so is $\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(1)}$"

(g) for $\mathscr{U} \subseteq \alpha_*$ let $\mathbb{P}_{\mathscr{U}}$ be the subforcing of $\mathbb{P}_{2,\alpha(\mathbf{q})}$ consist of $\{\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma(i)}, \ldots)_{i<i(*)} \in \mathbb{P}_{\alpha(\mathbf{q})} : i(*) \le \lambda$ and $\gamma_i \in \mathscr{U}$ for every $i < i(*)\}$.

Claim 0.13. *1) For any sequence $\langle u_\alpha, \mathscr{P}_\alpha : \alpha < \alpha_* \rangle$ as above, i.e. as in clause (a) of Definition 0.12, there is one and only one $\mathbf{q}$ as above and the $\mathbb{P}, \mathbb{P}^\bullet_{\mathbf{q},\mathscr{U}}$'s are as demanded.* {z35} {z32}
2) For every $\alpha \le \alpha_*$ the set $\mathbb{P}^\bullet_\alpha$ of $p \in \mathbb{P}_\alpha$ satisfying the following is dense:

(a) *$\eta_p, i_p, \langle u_{p,i} : i < i_p \rangle$ are objects (not just $\mathbb{P}_\alpha$-names)*

(b) *each $\underset{\sim}{f}_i$ has the form $\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma)i,1)}, \ldots)_{j<j(*)\le\lambda}$ where $\{\gamma(i,j) : j < j(i)\} \subseteq u_{p,i}$.*

3) Above for every $v \subseteq \alpha$ and $j_* < \lambda$ the set of $p \in \mathbb{P}^\bullet_\alpha$ such that $v \subseteq \mathrm{dom}(p) \wedge (\forall\beta \in \mathrm{dom}(p))(\ell g(\eta_{p(\beta)}) > j_*)$ is dense.
4) $\mathbb{P}_{\mathbf{q},1,\alpha} \lessdot \mathbb{P}_{\mathbf{q},2,\alpha}$ moreover $\mathbb{P}_{\mathbf{q},1,\alpha}$ is dense in $\mathbb{P}^\bullet_{\mathbf{q},2,\alpha}$ and $\mathscr{U}_1 \subseteq \mathscr{U}_2 \subseteq \alpha_{\mathbf{q}} \Rightarrow \mathbb{P}_{\mathbf{q},\mathscr{U}_1} \lessdot \mathbb{P}_{\mathbf{q},\mathscr{U}_2} \lessdot \mathbb{P}_{\mathbf{q},\alpha}$ so $\mathbb{P}_{\mathbf{q},\{\beta:\beta<\alpha\}} = \mathbb{P}_{\mathbf{q},2,\alpha}$ and $|\mathbb{P}_{\mathbf{q},\mathscr{U}}| \le |\mathscr{U}|^\lambda$.
5) If $\alpha < \alpha_*$ and $u \in \mathscr{P}_\alpha$ then $\underset{\sim}{\eta}_\alpha \in \Pi\bar\theta$ dominate every $\nu \in (\Pi\bar\theta)^{\mathbf{V}[\underset{\sim}{\bar\eta} \restriction u]}$.

(945) revision:2017-11-02 modified:2017-11-05

6) Assume $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q}}$ *is generic over* $\mathbf{V}, \eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ *and* $\eta'_\alpha \in (\Pi\bar{\theta})^{\mathbf{V}[\mathbf{G}]}$ *for* $\alpha < \alpha_*$ *and* $\{(\alpha, \varepsilon) : \alpha < \alpha_*, \varepsilon < \alpha$ *and* $\eta_\alpha(\varepsilon) \neq \eta'_\alpha(\varepsilon)\}$ *has cardinality* $< \lambda$. *Then for some (really unique)* $\mathbf{G}'$ *we have* $\mathbf{G}' \subseteq \mathbb{P}_{\mathbf{q}}$ *is generic over* $\mathbf{V}$ *and* α.

Proof. See [**?**]. $\square_{0.13}$

{z38} **Theorem 0.14.** *For any ordinal* α_* *there is a quadruple* $(\mathbf{q}, \delta_*, \mathscr{U}, h)$ *such that:*

(A) (a) $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar{\theta}}$ *and let* $\delta_* = \ell g(\mathbf{q})$

(b) $\mathscr{U} \subseteq \delta_*$ *has order type* α_*

(c) h *is the order preserving function from* α_* *onto* $\mathscr{U}$

(d) *if* $\alpha \in \mathscr{U}$ *then* $\mathscr{U} \cap \alpha \in \mathscr{P}_{\mathbf{Q},\alpha}$

(B) *if* $\mathscr{U}_1 \subseteq \mathscr{U}, \mathscr{U}_2 \subseteq \mathscr{U}$ *is an initial segment of* $\mathscr{U}, \mathrm{otp}(\mathscr{U}_1) = \mathrm{otp}(\mathscr{U}_2)$ *and* g *is the order preserving function from* $\mathscr{U}_1$ *onto* $\mathscr{U}_2$, *then* g *induces an isomorphism* $\hat{g}$ *from* $\mathbb{P}_{\mathbf{q},\mathscr{U}_2}$ *onto* $\mathbb{P}_{\mathbf{q},\mathscr{U}_2}$ *mapping* $\underset{\sim}{\eta}_\beta$ *to* $\underset{\sim}{\eta}_{g(\beta)}$ *for* $\beta \in \mathscr{U}_2$.

Proof. By [Sh:1126]. $\square_{0.14}$

(945) revision:2017-11-02 modified:2017-11-05

§ 1. The forcing

We intend to prove here

{a2}

Theorem 1.1. *Assume*

(a) λ *is supercompact*

(b) $\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mathrm{cf}(\mu) = \mu^\lambda$.

Then for some forcing notion $\mathbb{P}$ not collapsing cardinals $\geq \lambda, \lambda$ is still supercompact in $\mathbf{V}^{\mathbb{P}}$ and $\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa, \mathfrak{d}_\lambda = \mu$.

Proof. By 1.3 below. $\square_{1.1}$ {a7}

Recall

{a5}

Definition 1.2. 1) We say that a forcing notion $\mathbb{P}$ is α-strategically complete when for each $p \in \mathbb{P}$ in the following game $\Game_\alpha(p, \mathbb{P})$ between the players COM and INC, the player COM has a winning strategy.

A play lasts α moves; in the β-th move, first the player COM chooses $p_\beta \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\beta$ and $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$ and second the player INC chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq_{\mathbb{P}} q_\beta$.

The player COM wins a play if he has a legal move for every $\beta < \alpha$.

2) We say that a forcing notion $\mathbb{P}$ is $(< \lambda)$-strategically complete when it is α-strategically complete for every $\alpha < \lambda$.

{a7}

Lemma 1.3. *1) If λ is supercompact then after some preliminary forcing of cardinality λ, λ is still supercompact and $\boxdot_\lambda$ below holds.*

1A) The statement $\boxdot_\lambda$ holds in $\mathbf{V}^{\mathbb{P}}$ when $\mathbf{V}$ satisfies $\boxdot_\lambda$ and $\mathbb{P}$ is a $(< \lambda)$-strategically complete forcing notion and in $\mathbb{P}$ any directed system of cardinality $< \lambda$ of conditions in $\mathbb{P}$ has an upper bound.

2) If λ is strongly inaccessible and $\boxdot_\lambda$ below holds and $\lambda^+ < \kappa = \mathrm{cf}(\kappa) < \mu = \mu^\lambda$ and μ is regular (for transparency), then for some λ^+-c.c., $(< \lambda)$-strategically complete forcing notion $\mathbb{P}$ we have $\Vdash_{\mathbb{P}}$ "$\mathfrak{d}_\lambda = \mu$, $\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$"

{a8}

Definition 1.4. For λ supercompact we define $\boxdot_\lambda$ by:

$\boxdot_\lambda$ for any regular cardinal $\chi > \lambda$ and forcing notion $\mathbb{P} \in \mathscr{H}(\chi)$ which is $(< \lambda)$-strategically complete (see Definition 1.2(2)) the following set $\mathscr{S} = \mathscr{S}_{\mathbb{P}} = \mathscr{S}_{\chi,\mathbb{P}}$ is a stationary subset of $[\mathscr{H}(\chi)]^{<\lambda}$: {a5}

$\mathscr{S} = \mathscr{S}_{\mathbb{P}} = \mathscr{S}_{\chi,\mathbb{P}}$ is the set of N's such that for some $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N, \mathbb{A} = \mathbb{A}_N, M = M_N, \mathbf{G} = \mathbf{G}_N$ we have (and we may say $(\lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ is a witness for $N \in \mathscr{S}_{\chi,\mathbb{P}}$ or for $(N, \mathbb{P}, \chi)$):

(a) $N \prec (\mathscr{H}(\chi)^{\mathbf{V}}, \in)$ and $\mathbb{P} \in N$

(b) the Mostowski collapse of N is $\mathbb{A}$ and let $\mathbf{j}_N : N \to \mathbb{A}$ be the unique isomorphism

(c) $N \cap \lambda = \lambda_N$ and ${}^{(\lambda_N)>}N \subseteq N$ and λ_N is strongly inaccessible

(d) $\mathbb{A} \subseteq M := (\mathscr{H}(\chi_N), \in)$ so both $\mathbb{A}$ and M are transitive

(e) $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$ is generic over $\mathbb{A}$ for the forcing notion $\mathbf{j}_N(\mathbb{P})$

(f) $M = \mathbb{A}[\mathbf{G}]$.

modified:2017-11-05

(945) revision:2017-11-02

Recall that:

{a7.3}

Fact 1.5.

(a) if $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta \rangle$ is a $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions, <u>then</u> $\mathbb{P}_\delta$ is also $< \lambda$-strategically complete; (see e.g. [Sh:546]).

(b) If $\mathbb{P}$ is $(< \lambda)$-strategically complete forcing notion <u>then</u> $({}^{\lambda >}\mathrm{Ord})^{\mathbf{V}} = ({}^{\lambda >}\mathrm{Ord})^{\mathbf{V}^{\mathbb{P}}}$, and consequently λ is strongly inaccessible in $\mathbf{V}^{\mathbb{P}}$

(c) like (a) replacing $(< \lambda)$-strategically complete" by "$(< \lambda)$-complete"

(d) if $\mathbb{P}$ is $(< \lambda)$-complete then $\mathbb{P}$ is λ-strategically complete.

Proof. <u>Proof of Lemma 1.3</u> 1) This is essentially by Laver [Lav78] using Laver's
{z26} diamond, see Definition 0.6, but for completeness we elaborate. By Laver [Lav78]
{z26} without loss of generality there is a Laver diamond $h : \lambda \to \mathscr{H}(\lambda)$, see Definition 0.6. Let $E = \{\theta : \theta < \lambda \text{ and } \alpha < \theta \Rightarrow h(\alpha) \in \mathscr{H}(\theta)\}$, clearly a club of λ and let $\langle \kappa_\varepsilon : \varepsilon < \lambda \rangle$ list $\{\theta \in E : \theta \text{ strongly inaccessible}\}$ in increasing order.

We now define $\mathbf{q}_\varepsilon$ and $\bar{\chi}^\varepsilon$ by induction on $\varepsilon \le \lambda$ such that:

$(*)$ (a) $\mathbf{q}_\varepsilon = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \le \varepsilon, \xi < \varepsilon \rangle$ is an Easton support iteration (so $\mathbb{P}_\zeta$. $\underset{\sim}{\mathbb{Q}}_\xi$ do not depend on ε, etc.)

(b) $\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_\zeta)$

(c) $\bar{\chi}^\varepsilon = \langle \chi_\xi : \zeta < \xi \rangle$ where each χ_ξ is a regular cardinal $\in [\kappa_\xi, \kappa_{\xi+1})$

(d) $\underset{\sim}{\mathbb{Q}}_\xi \in \mathscr{H}(\chi_{\xi+1})$ is a $\mathbb{P}_\xi$-name of a $(< \kappa_\xi)$-strategically complete forcing notion

(e) if $h(\xi) = (\underset{\sim}{\mathbb{Q}}, \chi)$ and the pair $(\underset{\sim}{\mathbb{Q}}, \chi)$ satisfies the requirements on $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\varepsilon)$ in clauses (c),(d) then $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\xi) = h(\xi)$.

Easily we can carry the induction so $\mathbf{q}_\lambda$ is well defined, $\mathbb{P}_\lambda \subseteq \mathscr{H}(\lambda)$ and "$\xi < \lambda \Rightarrow \mathbb{P}_\lambda/\mathbb{P}_\xi$ is $(< \kappa_\xi)$-strategically complete" hence $\mathbb{P}_\lambda/\mathbb{P}_\xi$ adds no new sequence of length $< \kappa_\xi$ of ordinals. Clearly it is enough to prove that in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $\boxdot_\lambda$.

Toward contradiction assume $\chi, \mathbb{P}, \mathscr{S} = \mathscr{S}_{\chi,\mathbb{P}}$ forms a counter-example in $\mathbf{V}^{\mathbb{P}_\lambda}$, hence there are $p_* \in \mathbb{P}_\lambda$ and $\mathbb{P}_\lambda$-name $\underset{\sim}{\chi}, \underset{\sim}{\mathbb{P}}, \underset{\sim}{\mathscr{S}}, \underset{\sim}{E}$ such that $p_* \Vdash_{\mathbb{P}_\lambda}$ "$\underset{\sim}{\chi} > \lambda$ is regular, $\underset{\sim}{\mathbb{P}} \in \mathscr{H}(\underset{\sim}{\chi})$ and $\underset{\sim}{\mathscr{S}}_{\underset{\sim}{\chi},\mathbb{P}}$ is defined as in $\boxdot_\lambda$ and $\underset{\sim}{E} \subseteq [\mathscr{H}(\chi)^{\mathbf{V}[\mathbb{P}_\lambda]}]^{<\lambda}$ is a club disjoint to $\underset{\sim}{\mathscr{S}}$".

As we can increase p_*, without loss of generality $\underset{\sim}{\chi} = \chi$; and as $\mathbf{V} \models$ "λ is supercompact and h is a Laver diamond" for some $(\mathbf{M}, \mathbf{j})$ we have

$(*)$ (a) $\mathbf{M}$ is a transitive class

(b) $\mathbf{M}$ is a model of ZFC

(c) ${}^{\chi}\mathbf{M} \subseteq M$

(d) $\mathbf{j}$ is an elementary embedding from $\mathbf{V}$ into $\mathbf{M}$

(e) $\mathbf{j}$ is with critical cardinal λ

(f) $\mathbf{j}(h)(\lambda) = (\underset{\sim}{\mathbb{P}}, \chi)$.

Let $\mathbf{q} = \mathbf{j}(\mathbf{q}_\lambda)$ so $\mathbf{q} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \le h(\lambda), \xi < h(\lambda) \rangle$ and $\zeta < \lambda \Rightarrow \mathbb{P}^{\mathbf{q}}_\zeta = \mathbb{P}_\zeta$, etc.

Clearly $\mathbf{M} \models$ "$\mathbf{j}(h)(\lambda)$ is a pair of the form $(\mathbb{P}', \chi')$ satisfying all relevant demands".

The rest should be clear.

(945) revision:2017-11-02 modified:2017-11-05

1A) Let $\mathbb{Q}$ be a forcing notion in $\mathbf{V}^{\mathbb{P}}$ which is $(< \lambda)$-strategically complete, $\emptyset \in \mathbb{Q}$ minimal, χ_1 large enough so that $\lambda, \underset{\sim}{\mathbb{Q}}, \underset{\sim}{E} \in \mathscr{H}(\chi_1)$ and we should prove that in $\mathbf{V}^{\mathbb{P}}$, the set $\mathscr{S}_{\chi_1}, \mathbb{Q}$ is stationary. So let $\underset{\sim}{\mathbb{Q}}, \underset{\sim}{E}$ be $\mathbb{P}$-names such that for some $p \in \mathbb{P}$ we have $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi_1)$ is $(< \lambda)$-strategically complete forcing notion, $\underset{\sim}{E}$ a club of $[\mathscr{H}(\chi_1)]^{<\lambda}$ disjoint to $\underset{\sim}{\mathscr{S}}_{\chi_1,\mathbb{P}}$", no need to use a name for χ_1 as we can increase p.

Let $\chi \gg \chi_1$; now $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi)$ is a $(< \lambda)$-strategically complete forcing notion and without loss of generality code (χ_1, E). As $\boxdot_\lambda$ holds in $\mathbf{V}$ we can apply it to the forcing $\mathbb{P}_{\geq p} * \underset{\sim}{\mathbb{Q}}$ so we can find a tuple $(N, \lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ witnessing it, in particular, $(p, \emptyset) \in \mathbf{G}_N, \mathbb{P} * \underset{\sim}{\mathbb{Q}} \in N$ so $\chi_1, \underset{\sim}{E} \in N$. Let $\mathbf{G}_{\mathbb{P}}$ be a subset of $\mathbb{P}$ generic over $\mathbf{V}$ which extends $\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N\}$, possible because $\mathbf{G}_N$ is in $\mathbf{V}$, a subset of $\mathbb{P}$ which has an upper bound. Next let $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_{\mathbb{P}}], N_1 = N[\mathbf{G}_{\mathbb{P}}], E_1 = \underset{\sim}{E}[\mathbf{G}_{\mathbb{P}}], \mathbb{A}_1 = \mathbb{A}[\mathbf{j}''_N(\mathbf{G}_{\mathbb{P}} \cap N)] = \mathbb{A}[\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N], \mathbf{G}_1 = \{\underset{\sim}{q}[\mathbf{j}''(G_{\mathbb{P}} \cap N)] : (p, \underset{\sim}{q}) \in \mathbf{G}_{\mathbb{P}}\}$.

Let $N_2 = N_1 {\restriction} \mathscr{H}(\chi_1)^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}, \mathscr{S} = \mathscr{S}_{\underset{\sim}{\mathbb{Q}}}[\mathbf{G}_{\mathbb{P}}], \mathbf{j}_1 =$ the lifting of $(\mathbf{j}{\restriction}(N \cap \mathscr{H}(\chi)))$, to mapping N_1 onto $\mathbb{A}_1$.

Now recalling p forces $\underset{\sim}{E}$ is disjoint to $\underset{\sim}{\mathscr{S}}$ clearly

$(*)$ $N_2 \in E$.

hence

$(*)$ $N_1 \notin \mathscr{S}_1$.

But easily in $\mathbf{V}_1$ we have: $(\lambda_N, \chi_N, \mathbf{j}_1, \mathbb{A}_1, M_1 = M, \mathbf{G}_1)$ witness $N_1 \in \mathscr{S} \cap E_1$, a contradiction to the choice of $\underset{\sim}{E}$.

2) Stage A: Without loss of generality $\mathbf{V} \models$ "$\mathfrak{b}_\lambda = \mu = \mathfrak{d}_\lambda$" as witnessed by $\langle f^*_\alpha : \alpha < \mu \rangle$.

No new point, still we elaborate recalling that composition of forcing notions e.g. $(\mathbb{P} * \underset{\sim}{\mathbb{Q}}^*)$ preserves "$(< \lambda)$-strategically complete and λ^+-c.c."

We use a $(< \lambda)$-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \mu, \beta < \mu \rangle$ with $\mathbb{P}_\mu$ the intended forcing such that:

(A) if $\alpha < \mu$ then $\underset{\sim}{\mathbb{Q}}_\alpha$ is the ($\mathbb{P}_\alpha$-name of the) dominating forcing, $\mathbb{Q}^{\rm dom}_\lambda$, i.e. $(\mathbb{Q}^{\rm dom}_\alpha)^{\mathbf{V}[\mathbb{P}_\alpha]}$ where the forcing $\mathbb{Q} = \mathbb{Q}^{\rm dom}_\alpha$ is from Definition 0.8. {z20}

Let $\underset{\sim}{f}^*_\alpha$ be the generic object for $\underset{\sim}{\mathbb{Q}}_\alpha$ for $\alpha < \mu$.

Now:

$(*)_1$ for $\alpha \leq \mu$ the forcing notion $\mathbb{P}_\alpha$ is $(< \lambda)$-complete because when $\alpha < \mu, \underset{\sim}{\mathbb{Q}}_\alpha$ is $(< \lambda)$-complete, i.e. $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is $(< \lambda)$-complete".

[Why? We prove this by induction on α; for $\underset{\sim}{\mathbb{Q}}_\alpha$, recall Definition 0.8.] {z20}

$(*)_2$ for each $\alpha \leq \mu$
- (a) the forcing notion $\mathbb{P}_\alpha$ and for $\alpha < \mu$ satisfy a strong form of the λ^+-c.c., (see [Sh:80] for definition, preservation and history; or fully [Sh:1036], [Sh:546, §1])
- (b) also the forcing notions $\underset{\sim}{\mathbb{Q}}_\alpha$ satisfies this

hence

modified:2017-11-05
(945) revision:2017-11-02

$(*)_3$ (a) forcing with $\mathbb{P}_\mu$ collapses no cardinal, changes no cofinality, and adds no sequence to ${}^{\lambda>}\mathbf{V}$;

(b) $({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\mu]} = \cup\{({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]} : \alpha < \mu\}$.

[Why? By $(*)_2 + (*)_1$ clause (a) holds, for clause (b) use also "$\mathbb{P}_\mu$ satisfies the λ^+-c.c." and the support in the iteration being $< \lambda$ recalling that μ is regular $> \lambda$. E.g. if $\Vdash_{\mathbb{P}_\mu}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$" then we can, for $\alpha < \lambda$ find maximal antichain $\langle p_{\alpha,i} : i < i_\alpha \le \lambda\rangle$ of $\mathbb{P}_\mu$ such that $p_{\alpha,i}$ forces a value to $\underset{\sim}{f}(\alpha)$; let $\alpha(*) = \sup(\cup\{\mathrm{dom}(p_{\alpha,i}) : \alpha < \lambda, i < i_\alpha\})$, so $\alpha(*) < \mu$ and $\underset{\sim}{f}$ is a $\mathbb{P}_{\alpha(*)}$-name.]

$(*)_4$ (a) in $\mathbf{V}^{\mathbb{P}_\mu}, \mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$ as witnessed by $\underset{\sim}{\bar{f}}^* = \langle \underset{\sim}{f}^*_\alpha : \alpha < \mu\rangle$, in fact $\Vdash_{\mathbb{P}_{\alpha+1}}$ "$\underset{\sim}{f}^*_\alpha \in {}^\lambda\lambda$ dominates $({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]}$ modulo J^{bd}_λ"

(b) in $\mathbf{V}^{\mathbb{P}_\mu}$ still $\boxdot_\lambda$ holds.

[Why Clause (a)? Easy using $(*)_3(b)$.
{a7} Why Clause (b)? Easy, by 1.3(1A).]

Stage B: In $\mathbf{V}$ (see Stage A) there are $\beta(*), \mathbf{q}, \bar{u}, \mathscr{U}_*, \ldots$ such that

$(*)_5(A)$ $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar{\theta},\beta}$ so in particular

(a) $\mathbf{q} = \langle \mathbb{P}_{0,\alpha}, \underset{\sim}{\mathbb{Q}}_{0,\beta} : \alpha \le \beta(*), \beta < \beta(*)\rangle$ is a $(< \lambda)$-support iteration

(b) $\bar{u} = \langle u_\beta : \beta < \beta(*)\rangle, \bar{\mathscr{P}} = \langle \mathscr{P}_\beta : \beta < \beta(*)\rangle$

(c) $u_\beta \subseteq \beta, \mathscr{P}_\beta \subseteq [u_\beta]^{\le\lambda}$ is closed under subsets

(d) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ has generic $\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$

{z32} (e) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ is as in 0.12(e) so is $\subseteq \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u_\beta\rangle]}$ and $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$" and $\bar{\underset{\sim}{\eta}} = \langle \underset{\sim}{\eta}_\beta : \beta < \beta(*)\rangle$

(f) $\mathscr{U}_* \subseteq \beta(*)$ has order type $\gamma(*) = \kappa$ and $\langle \beta^*_i : i \le \kappa\rangle$ lists $\mathscr{U}_* \cup \{\beta(*)\}$ in increasing order

(g) if $\beta \in \mathscr{U}_*$ then $[\mathscr{U}_* \cap \beta]^{\le\lambda} \subseteq \mathscr{P}_\beta$ hence $\subseteq u_\beta$ and $\Vdash_{\mathbb{P}_{0,\beta+1}}$ "if $\nu \in \mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}_* \cap \beta\rangle] \cap \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ then $\nu <_{J^{\mathrm{bd}}_\lambda} \underset{\sim}{\eta}_\beta$"

(h) if $\alpha \le \beta(*)$ then $\mathbb{P}_{0,\alpha}$ is $(< \lambda)$-strategically complete and λ^+-c.c.

(B) letting $\mathbb{P}'_i = \mathbb{P}_{\mathbf{q},\{\beta^*_j : j<i\}}$ for $i \le j$ we have

(a) The sequence $\langle \mathbb{P}'_i : i \le \gamma(*)\rangle$ of forcing notions is $\lessdot$-increasing, and is continuous for ordinals $i \le \gamma(*)$ of cofinality $\ge \lambda$

(b) $\mathbb{P}'_i$ is $(< \lambda)$-strategically complete for $i \le \gamma(*)$

(c) $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_{\gamma(*)}]} = \cup\{(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \gamma(*)\}$.

(d) The sequence $\langle \mathbb{P}_{1,\beta} : \beta \le \beta(*)\rangle$ is a sequence of forcing notions, $\lessdot$-increasing and if $\beta \le \beta(*)$ then $\mathbb{P}_{0,\beta} \lessdot \mathbb{P}_{1,\beta}$, in fact is dense in it and if $i \le \gamma(*)$ then $\mathbb{P}'_i \lessdot \mathbb{P}_{1,\beta^*_i}$.

We shall mention more properties later.
{z38} [Why are there such objects? We apply 0.14 and 0.12 and 0.13.
Also

$(*)_6$ (a) recall $\langle \beta^*_i : i \le \gamma(*)\rangle$ list $\mathscr{U}_* \cup \{\beta(*)\}$ in increasing order

(945) revision:2017-11-02 modified:2017-11-05

(b) for $i \le \gamma(*) = \kappa$, for $i < \gamma(*)$ let $\underset{\sim}{g}'_i$ be $\underset{\sim}{\eta}_{\beta^*_i}$ (to avoid excessive subscripts), see $(*)_5(A)(e)$ and **??** so $\mathbb{P}'_i \lessdot \mathbb{P}_{1,\beta^*_i}$ {c52}

(c) let $\underset{\sim}{\bar{g}}' = \langle \underset{\sim}{g}'_i : i < \kappa \rangle$

(d) let $\underset{\sim}{g}_\alpha = \underset{\sim}{\eta}_\alpha$ for $\alpha < \beta(*)$ and $\underset{\sim}{\bar{g}} = \langle \underset{\sim}{g}_\beta : \beta < \beta(*) \rangle$

(e) $\mathscr{P}_\alpha = \mathscr{P}_{\mathbf{q},\alpha}$ and without loss of generality $u_\alpha = \cup\{u : u \in \mathscr{P}_\alpha\}$ for $\alpha < \beta(*)$.

$(*)_7$ if $u \in \mathscr{P}_\alpha, \alpha < \beta(*)$ then $\Vdash_{\mathbb{P}_{0,\alpha+1}}$ "$\underset{\sim}{g}_\alpha \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ dominates $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\langle \underset{\sim}{g}_\beta : \beta \in u \rangle]}$", the order being modulo $J^{\rm bd}_\lambda$.]

[Why? As by 2.7(5).] {c35}

$(*)_8$ $\Vdash_{\mathbb{P}'_\kappa}$ "$\underset{\sim}{\bar{g}}' = \langle \underset{\sim}{g}'_i : i < \kappa \rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and cofinal in $(\prod_{\varepsilon<\lambda} \theta_\varepsilon, <_{J^{\rm bd}_\lambda})$".

[Why? By $(*)_7$ noting that $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup\{(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa\}$ which holds by $(*)_5(B)(c)$.]

Now

$(*)_9$ $\Vdash_{\mathbb{P}'_\kappa}$ "cov$_\lambda$(meagre) $\le \kappa$".

[Why? As we can look at $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ instead[2] of ${}^\lambda 2$ and for each $\varepsilon < \lambda, i < \kappa$ the set $B_{\varepsilon,i} = \{\eta \in \prod\limits_{\xi<\lambda} \theta_\xi$: for every $\zeta \in [\varepsilon,\lambda)$ we have $\eta(\zeta) \le \underset{\sim}{g}'_i(\zeta) < \theta_\zeta\}$ is closed nowhere dense, and by $(*)_8$ we have $\mathbf{V}^{\mathbb{P}'_\kappa} \models$ "$\prod_{\zeta<\lambda} \theta_\zeta = \cup\{B_{\varepsilon,i} : \varepsilon < \lambda, i < \kappa\}$". In fact, $\langle B_{0,i} : i < \kappa \rangle$ suffice.]

$(*)_{10}$ $\Vdash_{\mathbb{P}'_\kappa}$ "cov$_\lambda$(meagre) $\ge \kappa$".

[Why? Let us define the $\mathbb{P}'_{i+1}$-name $\underset{\sim}{\eta}'_i$ of a member of ${}^\lambda 2$ by $\underset{\sim}{\eta}'_i(\varepsilon) = 0$ iff $\underset{\sim}{g}'_i(\varepsilon)$ is even. Now clearly $\Vdash_{\mathbb{P}'_{i+1}}$ "$\underset{\sim}{\eta}'_i$ is a λ-Cohen sequence over $\mathbf{V}^{\mathbb{P}'_i}$". (Let us elaborate; $\underset{\sim}{\eta}'_i$ is also a $\mathbb{P}_{\beta^*_i+1}$-name and $\Vdash_{\mathbb{P}_{\beta^*_i+1}}$ "$\underset{\sim}{\eta}'_i$ is λ-Cohen over $\mathbf{V}^{\mathbb{P}_{\beta^*_i}}$ hence over $\mathbf{V}^{\mathbb{P}'_i}$"; the last hence because $\mathbb{P}'_i \lessdot \mathbb{P}_{\beta^*_i}$. As $\mathbb{P}_{\beta^*_i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ and $\mathbb{P}'_{i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ we are done.)

Also every closed nowhere dense subset of ${}^\lambda 2$ from $\mathbf{V}^{\mathbb{P}'_{\gamma(*)}}$ is from $\mathbf{V}^{\mathbb{P}'_i}$ for some $i < \gamma(*)$. So if $p \Vdash$ "cov$_\lambda$(meagre) $< \kappa$" then for some $\zeta < \kappa$ and $\underset{\sim}{A}_\varepsilon(\varepsilon < \zeta)$ we have $p \Vdash$ "$\underset{\sim}{A}_\varepsilon$ is a closed no-where dense set subset of ${}^\lambda 2$ for $i < j$" and $p \Vdash$ "$\bigcup\limits_i \underset{\sim}{A}_i$ is equal to the set of ${}^\lambda 2$". Without loss of generality each $\underset{\sim}{A}_\varepsilon$ is a $\mathbb{P}_{i(\varepsilon)}$-name, $i(\varepsilon) < \kappa$. Hence $i = \sup\{i(\varepsilon) : \varepsilon < \zeta\} < \kappa$ and $\underset{\sim}{\eta}'_i$ gives a contradiction to the choice of $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \zeta \rangle$; so $(*)_{10}$ holds instead.] $\square_{1.3}$

Discussion 1.6. 1) The reader may justly wonder why we use $\mathbf{V}' = \mathbf{V}[\underset{\sim}{\bar{g}}'] = \mathbf{V}[\underset{\sim}{\bar{g}} \restriction \mathscr{U}_*]$ rather than simply $\mathbf{V}[\underset{\sim}{\bar{g}}]$. Of course, nothing is lost by it, but why the extra complication?

[2]E.g. let $F : {}^\lambda 2 \to \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ be $F(\eta) = \rho$ iff $\eta \in {}^\lambda 2$ and for every $\varepsilon < \lambda, \rho(\varepsilon) = 0$ iff $(\forall i < \theta_\varepsilon)(\eta(\sum\limits_{\zeta<\varepsilon} \theta_\zeta + i) = 0)$ and $\rho(\varepsilon) = 1 + i$ iff $\eta(\sum\limits_{\zeta<\varepsilon} \theta_\zeta + i) = 1 \wedge (\forall j < i)(\eta(\sum\limits_{\zeta<\varepsilon} \theta_\zeta + j) = 0)$. Now if $\prod\limits_\varepsilon \theta_\varepsilon = \cup\{\mathscr{U}_i : i < \kappa\}$, each $\mathscr{U}_i$ closed nowhere dense then $\langle F^{-1}(\mathscr{U}_i) : i < \kappa \rangle$ witnesses cov$_\lambda$(meagre) $\le \kappa$.

(945) revision:2017-11-02 modified:2017-11-05

2) The answer is that during the proof we shall use: if $\zeta(i) \in \mathscr{U}_*$ is increasing with $i < \gamma(*)$ then also $\langle \underset{\sim}{g}_{\zeta(i)} : i < \kappa \rangle$ is generic over $\mathbf{V}$ for the subforcing of $\mathbb{P}_{1,\beta(*)}$ generated by $\underset{\sim}{\bar{g}} \restriction \mathscr{U}_*$; see $\circledast''_7$ inside the proof of $\circledast_6$. But using $\mathscr{U}_* = \beta(*)$, we do not know this.
3) Now in the parallel case for $\lambda = \aleph_0$ with FS iteration with full memory, such claim is true, see §0.
4) But we do not know the parallel of 3) for λ, so we use a substitute using $\mathscr{U}_*$, i.e. $\mathbb{P}'_\kappa$.

Proof. Continuation of the proof:

Now we come to the main and last point recalling $\langle f^*_\alpha : \alpha < \mu \rangle$ from Stage A

$(*)_{11}$ it is forced, i.e. $\Vdash_{\mathbb{P}'_{\gamma(*)}}$ that no $\underset{\sim}{f} \in ({}^\lambda\lambda)$ dominate $\{f^*_\alpha : \alpha < \mu\}$.

{a7} We shall show that it suffices to prove $(*)_{11}$ for proving 1.3(2), and that $(*)_{11}$ holds, thus finishing.

Why it suffices? As $\langle f^*_\alpha : \alpha < \mu \rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and $\mathrm{cf}(\mu) = \mu > \lambda$, this
{z35} implies $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda \geq \mu$". Also in $\mathbf{V}, \mu^\lambda = \mu > \kappa > \lambda$ and $|\mathbb{P}'_{\gamma(*)}| = \kappa^\lambda$ by (A)(g) of 0.13(4) which is $\leq \mu$ and $\mathbb{P}'_\kappa$ satisfies the λ^+-c.c. hence $\Vdash_{\mathbb{P}'_\kappa}$ "$2^\lambda = \mu$, hence together $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda = \mu$". Also by $(*)_5(B)(b)$, "$\mathbb{P}'_{\gamma(*)}$ is $(< \lambda)$-strategically complete $+\lambda^+$-c.c." and by $(*)_9 + (*)_{10}$ we know that "$\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$" so we are done; so $(*)_{11}$ is really the last piece missing. The rest of the proof is dedicated to proving that $(*)_{11}$ holds.

We shall use further nice properties of $\mathbb{P}'_j, \underset{\sim}{g}'_i (j \leq \gamma(*), i < \gamma(*))$ which holds by
{z38} $(*)_5 + (*)_6$ (and $(*)_7, (*)_8$) and their proof, i.e. 0.13, 0.14.

$\boxplus_1$ (a) (α) $\langle \underset{\sim}{g}'_\gamma : \gamma < \gamma(*) \rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$, i.e., if $\mathbf{G}$ is a subset of $\mathbb{P}'_{\gamma(*)}$ generic over $\mathbf{V}$ and $g'_i = \underset{\sim}{g}'_i[\mathbf{G}]$ then $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_i : i < \gamma(*) \rangle]$

(β) if $\nu \in ({}^\lambda\lambda)^{\mathbf{V}[\mathbf{G}]}$ then for some $\rho \in ({}^\lambda\gamma(*))^{\mathbf{V}}$ and λ-Borel function $\mathbf{B} \in \mathbf{V}$ we have $\nu = \mathbf{B}(\langle g'_{\rho(\varepsilon)} : \varepsilon < \lambda \rangle)$

(b) if in $\mathbf{V}[\mathbf{G}], g''_\gamma \in \prod_{\zeta<\lambda} \theta_\zeta$ for $\gamma < \gamma(*)$ and the set $\{(\gamma,\zeta) : \gamma < \gamma(*)$ and $\zeta < \lambda$ and $g''_\gamma(\zeta) \neq g'_\gamma(\zeta)\}$ has cardinality $< \lambda$ then $\bar{g}'' = \langle g''_\gamma : \gamma < \gamma(*) \rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$ and $\mathbf{V}[\bar{g}''] = \mathbf{V}[\bar{g}']$

(c) $\Vdash_{\mathbb{P}'_\gamma}$ "$\underset{\sim}{g}'_\gamma$ dominates $(\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\gamma]}$

(d) if $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$ is an increasing sequence of ordinals $< \gamma(*)$ (from $\mathbf{V}$!), then $\langle \underset{\sim}{g}'_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$

(e) if $\gamma < \gamma(*)$ then $\mathbb{P}'_\gamma$ is $(< \lambda)$-strategically complete and satisfies the λ^+-c.c.

We shall use $\boxplus_1$ freely.

To prove $(*)_{11}$ assume toward contradiction that this fails, for some condition $p^* \in \mathbb{P}'_{\gamma(*)}$ and $\mathbb{P}'_{\gamma(*)}$-name $\underset{\sim}{f}$ and λ-Borel function $\mathbf{B}$ and $\rho \in {}^\lambda\gamma(*)$ we have:

$\circledast_0$ $p^* \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$ and[3] dominates $({}^\lambda\lambda)^{\mathbf{V}}$" and $\underset{\sim}{f} = \mathbf{B}(\langle g'_{\rho(i)} : i < \lambda \rangle)$.

[3]The reader may wonder: is $\mathbf{B}$ used? Answer: here and the previous paragraph.

modified:2017-11-05
(945) revision:2017-11-02

Now let χ be regular large enough and we choose $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda \rangle$ such that

$\circledast_1$ (a) N_ε is as in $\boxdot_\lambda$ for the forcing notion $\mathbb{P}'_{\gamma(*)}, N_\varepsilon \in \mathscr{S}_{\chi,\mathbb{P}'_{\gamma(*)}}$, see $\boxdot_\lambda$ of 1.3 {a7}

(b) $\bar{N} \restriction \varepsilon \in N_\varepsilon$ hence $\bigcup\limits_{\zeta<\varepsilon} N_\zeta \subseteq N_\varepsilon$ and $\lambda_\varepsilon := N_\varepsilon \cap \lambda > \lambda_\varepsilon^- := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\} \geq \Sigma\{\lambda_\zeta : \zeta < \varepsilon\}$

(c) $\bar{\theta}, \mathbf{q}, p^*, \underset{\sim}{f}, \mathbf{B}, \rho$ belong to N_ε.

We can find $f^* \in {}^\lambda\lambda$, i.e. $\in ({}^\lambda\lambda)^{\mathbf{V}}$, such that

$\circledast_2$ for arbitrarily large $\varepsilon < \lambda$ for some $\zeta \in [\lambda_\varepsilon^-, \lambda_\varepsilon)$ we have $f^*(\zeta) > \lambda_\varepsilon$.

For $\varepsilon < \lambda$ let $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, \mathbb{A}_\varepsilon, \mathbf{G}_\varepsilon)$ be a witness for $(N_\varepsilon, \mathbb{P}'_{\gamma(*)}, \chi)$ recalling $\boxdot_\lambda$ of Definition 1.4 so $\lambda_\varepsilon \in (\varepsilon, \lambda)$ is strongly inaccessible and $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda_\zeta^- < \lambda_\zeta$, {a8}
recalling $\circledast_1$ and $\lambda_\varepsilon^- < \lambda$ is increasing continuous.

Let

$\circledast_3$ (a) $v_\varepsilon = N_\varepsilon \cap \gamma(*)$

(b) $i_\varepsilon = i(\varepsilon) = \mathrm{otp}(v_\varepsilon)$ and so $i(\varepsilon) = \mathbf{j}_\varepsilon(\gamma(*))$, etc.

(c) $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < i(\varepsilon) \rangle$ list v_ε in increasing order

(d) for $i < \mathrm{otp}(v_\varepsilon)$, equivalently $i < \mathbf{j}_\varepsilon(\gamma(*))$ let $\eta_i^\varepsilon = (\mathbf{j}_\varepsilon(\underset{\sim}{g}'_{\gamma_i(\varepsilon)}))^{\mathbb{A}'_\varepsilon[\mathbf{G}_\varepsilon]} \in \prod\limits_{\zeta<\lambda_\varepsilon} \theta_\zeta$ and let $\bar{\eta}^\varepsilon = \langle \eta_i^\varepsilon : i < i_\varepsilon \rangle$.

Note that clearly

$\circledast_4$ (a) $\bar{\eta}^\varepsilon$ is generic for $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$, moreover

(b) for each $\varepsilon < \lambda$, if we change $\eta_i^\varepsilon(\zeta)$ (legally, i.e. to an ordinal $< \theta_\zeta$) for $< \lambda_\varepsilon$ pairs $(i, \zeta) \in \mathrm{otp}(v_\varepsilon) \times \lambda_\varepsilon$ and get $\bar{\eta}'$, <u>then</u> also $\bar{\eta}'$ is generic for $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$ and $\mathbb{A}_\varepsilon[\bar{\eta}'] = M_\varepsilon$

(c) like $\boxplus_1$ with $\mathbf{V}, \mathbb{P}'_{\gamma(*)}, \lambda$ there standing for $\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$ here.

Hence clearly

$\circledast_4'$ for $\varepsilon < \lambda$, if $\bar{\nu} = \langle \nu_i : i < i(\varepsilon) \rangle$ (recalling $i(\varepsilon) = \mathrm{otp}(v_\varepsilon)$ is as in $\circledast_3(b)$), and $q \in \mathbb{P}'_{\gamma(*)}$ satisfy $i < i(\varepsilon) \Rightarrow q \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\underset{\sim}{g}'_{\gamma_i(\varepsilon)} \restriction \lambda_\varepsilon = \nu_i$" <u>then</u> q is $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$-generic naturally and $q \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\mathbf{j}_\varepsilon$ can be extended naturally to an isomorphism from $N_\varepsilon[\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle \underset{\sim}{g}'_\gamma : \gamma \in v_\varepsilon \rangle]$ onto $\mathbb{A}_\varepsilon[\bar{\eta}^\varepsilon]$".

[Why? Should be clear, see $\boxplus_1 + \circledast_4(c)$.]

By the assumption toward contradiction, $\circledast_0$, and $\mathbb{P}'_{\gamma(*)}$ being $(< \lambda)$-strategically complete, recalling $\boxplus_1$, there are $\zeta(*), p^{**}$ and p^+ such that (recall $p^* \in \mathbb{P}'_{\gamma(*)} \lessdot \mathbb{P}_{1,\beta(*)}$):

$\circledast_5$ (a) $p^* \leq p^{**} \in \mathbb{P}'_{\gamma(*)}$ and $p^+ \in \mathbb{P}_{0,\beta(*)}$ such that $\mathbb{P}_{1,\gamma(*)} \models$ "$p^{**} \leq p^+$"

(b) $\zeta(*) < \lambda$

(c) $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$f^*(\zeta) < \underset{\sim}{f}(\zeta)$ whenever $\zeta(*) \leq \zeta < \lambda$" where f^* is from $\circledast_2$

modified:2017-11-05

(945) revision:2017-11-02

(d) if $\gamma \in \mathrm{Dom}(p^+)$ then $\eta^{p^+(\gamma)}$ is an object (not just a $\mathbb{P}'_\gamma$-name) of length $\geq \zeta(*)$ (recall that $\eta^{p^+(\gamma)}$ is the trunk of the condition $p^+(\gamma)$, see clause $(\alpha)(b)$ of Definition 0.10(1)). {z23}

Note that possibly $\mathrm{Dom}(p^+) \nsubseteq \cup\{v_\varepsilon : \varepsilon < \lambda\}$. Choose $\varepsilon(*) < \lambda$ such that $\lambda_{\varepsilon(*)} > \zeta(*) + |\mathrm{Dom}(p^+)|$ and $\gamma \in \mathrm{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$ recalling clause (d) of $\circledast_5$ and $|\mathrm{Dom}(p^+)| < \lambda$ as $p^+ \in \mathbb{P}_{0,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is the limit of a $(< \lambda)$-support iteration.

By $\circledast_2$ we can add $(\exists\zeta)[\lambda^-_{\varepsilon(*)} \leq \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$. Our intention is to find $q \in \mathbb{P}_{0,\beta(*)}$ above p^+ which (in $\mathbb{P}_{1,\beta(*)}$) is above some $q' \in \mathbb{P}'_{\gamma(*)}$ which is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$-generic and forces $\underaccent{\tilde}{\mathbf{G}}_{\mathbb{P}'_{\gamma(*)}}$ to include a generic subset of $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$ which is induced by some $\bar\nu$ as in $\circledast'_4$, recalling $\circledast_4(b)$. Toward this in $\circledast_6$ below the intention is that $p^+_{i(*)}$ will serve as q.

Let $i(*) = i(\varepsilon(*))$ and γ_i for $i < i(*)$ be such that[4] $\langle \gamma_i : i < i(*)\rangle$ list $\{\beta^*_i : i \in v_{\varepsilon(*)}\} \subseteq \mathscr{U}_*$ in increasing order; recall $\mathscr{U}_* = \{\beta^*_i : i < \gamma(*)\}$ and $i < j < \gamma(*) \Rightarrow \beta^*_i < \beta^*_j$ and $v_{\varepsilon(*)} \subseteq \gamma(*)$ has order type $i(\varepsilon(*))$. Next let $\mathscr{U}_{**} = \{\gamma_i : i < i(*)\}$ and $\gamma_{i(*)} = \gamma(*)$ so $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in v_{\varepsilon(*)}\} = i(*) = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$. Recall $\gamma(*) = \kappa = \mathrm{cf}(\kappa) > \lambda, \mathrm{otp}(v_{\varepsilon(*)}) = \mathrm{otp}(N_{\varepsilon(*)} \cap \gamma(*)) = \mathrm{otp}(N_{\varepsilon(*)} \cap \kappa)$ hence $N_{\varepsilon(*)} \models$ "$i(*)$ is a regular cardinal $> \lambda_\varepsilon$" hence $i(*)$ is really a regular cardinal so call it σ. Now we define a game $\eth$ as follows[5]:

$\boxplus_2$ (A) each play lasts $i(*) + 1 = \sigma + 1$ moves and in the i-th move,
- (a) if $i = j + 1$ the antagonist player chooses $\xi_j = \xi(j) < \sigma$ such that $j_1 < j \Rightarrow \xi(j_1) < \xi(j)$
- (b) then, if $i = j + 1$ the protagonist chooses $\zeta_j = \zeta(j) \in (\xi(j), \sigma)$, <u>but</u> there are more restrictions implicit in $\boxplus_3$
- (c) in any case the protagonist also chooses $p^+_i, \bar\nu^i$ such that $\boxplus_3$ below holds;

(B) in the end of the play the protagonist wins the play <u>iff</u> he always has a legal move and in the end $\{\zeta(i) : i \leq i(*)\} \in \mathbb{A}_{\varepsilon(*)}$; note that trivially it belongs to $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar\nu^{\varepsilon(*)}]$

<u>where</u>

$\boxplus_3$ (a) $p^+_i \in \mathbb{P}_{0,\gamma_i}$

(b) if $j < i$ then $\mathbb{P}_{0,\gamma_i} \models$ "$p^+_j \leq p^+_i$"

(c) if $\gamma \in \cup\{\mathrm{Dom}(p^+_j) : j < i\}$ then $p^+_i \restriction \gamma \Vdash_{\mathbb{P}_{0,\gamma_i}}$ "$\underaccent{\tilde}{\eta}^{p^+_i(\gamma)}$ has length $\geq i(*)$ and $\geq \lambda_{\varepsilon(*)}$" moreover $\underaccent{\tilde}{\eta}^{p^+_i(\gamma)}$ is an object, $\eta^{p^+_i(\gamma)}$

(d) $\mathbb{P}_{0,\gamma_i} \models$ "$p^{**} \restriction \gamma_i \leq p^+_i$"

(e) $\bar\nu^i = \langle \nu_{\gamma_j} : j < i\rangle$ and $\nu_{\gamma_j} \in \prod\limits_{\iota < \lambda_{\varepsilon(*)}} \theta_\iota$

(f) for $j < i$ we have $\nu_{\gamma_j} \trianglelefteq \eta^{p^+_i(\gamma_j)}$ so $p^+_i \restriction \gamma_j \Vdash$ "$\nu_{\gamma_j} \triangleleft \underaccent{\tilde}{g}'_{\gamma_j}$" recalling $\boxplus_1$

(g) for $j < i$ we have (recall $\bar\eta^\varepsilon$ from $\circledast_3$)

[4]This is used in $\boxplus_3$ and the proof of $(*)_6$. Not to be confused with $\bar\gamma^\varepsilon$ of $\circledast_3(c)$.

[5]The idea is to scatter the $\eta^{\varepsilon(*)}_{\gamma_i}$'s. Why not use the original places? as then we have a problem in $\circledast_6$; the scattering is helpful because we are relying on §2-§4.

modified:2017-11-05

(945) revision:2017-11-02

(α) $\nu_{\gamma_j} = \eta^{\varepsilon(*)}_{\gamma_{\zeta(j)}}$ recalling $\eta^{\varepsilon(*)}_{\gamma_j}$ is from $\circledast_3(d)$ <u>or</u>

(β) $\gamma_j \in \mathrm{Dom}(p^{**})$ and $\{\iota < \lambda_{\varepsilon(*)} : \eta^{\varepsilon(*)}_{\zeta(j)}(\iota) \neq \nu_{\gamma_j}(\iota)\}$ is a bounded subset of $\lambda_{\varepsilon(*)}$.

We shall prove

$\circledast_6$ in the game $\Game$

(a) the antagonist has no winning strategy

(b) in any move the protagonist has a legal move, moreover for any $\zeta(i) \in (\xi(i), \sigma)$ large enough the protagonist can choose it.

<u>Why $\circledast_6$ suffice?</u>

By clause (a) of $\circledast_6$ we can choose a play $\langle (\xi(i), \zeta(i), p^+_i, \bar{\nu}^i) : i \leq \sigma\rangle$ in which the protagonist wins. Recalling $\mathbb{P}'_{\gamma(*)} \lessdot \mathbb{P}_{1,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is a dense subforcing of $\mathbb{P}_{1,\beta(*)}$, clearly

$\circledast_7$ there is p such that

(a) $p \in \mathbb{P}'_{\gamma(*)}$

(b) if $\mathbb{P}'_{\gamma(*)} \models$ "$p \leq p'$" and $p' \in \mathbb{P}_{0,\beta(*)}$ <u>then</u> p', p^+_σ are compatible in $\mathbb{P}_{i,\beta(*)}$

(c) p is above p^{**} and it forces $\underset{\sim}{g}'_{\gamma_i} {\restriction} \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$ for $i < i(*)$.

Then on the one hand

$\circledast'_7$ $p \in \mathbb{P}'_{\gamma(*)}$ being above p^{**} forces $f^* \restriction [\zeta(*), \lambda) < \underset{\sim}{f} \restriction [\zeta(*), \lambda)$ hence $f^* \restriction [\zeta(*), \lambda_{\varepsilon(*)}) < \underset{\sim}{f} \restriction [\zeta(*), \lambda_{\varepsilon(*)})$ recalling (see $\circledast_5$) that $\zeta(*) < \lambda_{\varepsilon(*)}$.

On the other hand,

$\circledast''_7$ p is $(N_{\varepsilon(*)}, \mathbb{P}'_{0,\gamma(*)})$-generic.

[Why? As it forces $\underset{\sim}{\eta}_{\gamma_i} \restriction \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$ for $i < i(*)$ and $\langle \nu_{\gamma_i} : i < i(*)\rangle$ is (see $\boxplus_3(g)$ recalling $\mathrm{Dom}(p^{**})$ has cardinality $< \lambda_{\varepsilon(*)}$) "almost equal" to $\langle \eta^{\varepsilon(*)}_{\zeta(i)} : i < i(*)\rangle$ which is a subsequence of the sequence from $\circledast_3$. That is $\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma$ and $\nu_{\gamma_i}(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(i)}(\iota)\} \subseteq \cup\{\{(i,\iota) : \iota < \lambda_{\varepsilon(*)}$ and $\nu_{\gamma_i}(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(i)}(\iota)\} : \gamma \in v_{\varepsilon(*)} \cap \mathrm{Dom}(p^{**})\}$ so is the union of $\leq |\mathrm{Dom}(p^+_\sigma)| < \lambda_{\varepsilon(*)}$ sets each of cardinality $< \lambda_{\varepsilon(*)}$ hence is of cardinality $< \lambda_{\varepsilon(*)}$. Hence by $\circledast_4(c) + \boxplus_1(d)$ the sequence $\bar{\nu}^{i(*)}$ is generic for $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$; this is a point where we rely on §2 - §4.]

As $\underset{\sim}{f} \in N_{\varepsilon(*)}$ it follows from $\circledast''_7$ that

$\circledast'''_7$ $p \Vdash$ "$\underset{\sim}{f} \restriction \lambda_{\varepsilon(*)}$ is a function from $\lambda_{\varepsilon(*)}$ to $\lambda_{\varepsilon(*)}$".

Together $\circledast'_7 + \circledast'''_7$ gives a contradiction by the choice of f^* in $\circledast_2$ and of $\varepsilon(*)$ above, hence $\circledast_6$ is enough.

<u>Why $\circledast_6$ is true?</u>

Let us prove $\circledast_6$; first, assuming clause (b) proved below, for clause (a) choose any strategy **st** for the antagonist and fix a partial strategy **st**$'$ for the protagonist choosing $(p^+_i, \bar{\nu}^i)$ depending on the previous choices and $\zeta(i) < i_{\varepsilon(*)}$ such that it is a legal move if relevant and possible. So the only freedom left for the protagonist

(945) revision:2017-11-02 modified:2017-11-05

is to choose the $\zeta(i)$. So (recalling $\boxplus_2(A)(a)$) we have in $\mathbf{V}$ a function $F : {}^{\sigma>}\sigma \to \sigma$ (so F uses **st** ...) such that:

$(*)_F$ playing the game such that the antagonist uses **st** and the protagonist uses $\mathbf{st}'$, arriving to the i-th move, $\bar\zeta = \langle \zeta(j) : j < i\rangle$ is well defined and for the protagonist any choice $\zeta_i \in (F(\bar\zeta), \sigma) \cap \mathscr{U}_{**}$ is legal.

Now we have to find an increasing sequence $\bar\zeta = \langle \zeta(i) : i < \sigma\rangle$ from $\mathbb{A}_{\varepsilon(*)}$ such that $F(\bar\zeta{\restriction}i) < \zeta(i) < \sigma$ and $\bar\zeta \in \mathbb{A}_{\varepsilon(*)}$. As $F \in \mathscr{H}(\chi_{\varepsilon(*)})$ and $\mathscr{H}(\chi_{\varepsilon(*)}) = \mathbb{A}_{\varepsilon(*)}[\mathbf{G}_{\varepsilon(*)}]$ where $\mathbf{G}_{\varepsilon(*)}$ is a subset of $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}'_{\gamma(*)}) \in \mathbb{A}_{\varepsilon(*)}$ generic over $\mathbb{A}_{\varepsilon(*)}$ and $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$ satisfies the $\lambda^+_{\varepsilon(*)}$-c.c. and $\sigma = \mathrm{cf}(\sigma) > \lambda_{\varepsilon(*)}$ this is possible. That is, there is a $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$-name $\underset{\sim}{F}_* \in \mathbb{A}_{\varepsilon(*)}$ such that $F = \underset{\sim}{F}_*[\mathbf{G}_{\varepsilon(*)}]$ and we define in $\mathbb{A}_{\varepsilon(*)}$ the function $F' : {}^{\sigma>}\sigma \to \sigma$ by $F'(\langle \zeta(j) : j < i\rangle) = \sup\{\xi + 1 : \xi \in \{\zeta(j) + 1 : j < i\}$ <u>or</u> $\xi < \sigma$ and $\nVdash_{\mathbf{j}(\mathbb{P}_{0,\beta(*)})}$ "$\underset{\sim}{F}(\langle \zeta(j) : j < i\rangle) \neq \xi\}$; clearly this is O.K.

We are left with proving $\circledast_6(b)$.

<u>Case 1</u>: $i = 0$.

Let $p^+_0 = p^{**} \restriction \gamma_0$.

<u>Case 2</u>: i limit.

By clauses (b) and (c) of $\boxplus_3$, there is $p^+_i \in \mathbb{P}_{0,\gamma_i}$ which is an upper bound (even l.u.b.) of $\{p^+_j : j < i\}$ and it is easily as required. Also $\bar\nu^i$ is well defined and as required.

<u>Case 3</u>: $i = j + 1$ and $\gamma_j \notin \mathrm{Dom}(p^{**})$.

Clearly γ_i is in $\mathscr{U}_*$ the successor of γ_j and $(\exists\iota)(\gamma_j = \beta^*_\iota \wedge \iota \in v_{\varepsilon(*)})$. As in case 4 below but easier by the properties of the iteration.

<u>Case 4</u>: $i = j + 1$ and $\gamma_j \in \mathrm{Dom}(p^{**})$

Again γ_i is in $\mathscr{U}_*$ the successor of γ_j and $(\exists\iota)(\gamma_j = \beta^*_\iota \wedge \iota \in v_{\varepsilon(*)})$.

First we find p'_j such that:

$\circledast_8$ (a) $p^+_j \le p'_j \in \mathbb{P}_{0,\gamma_j}$

(b) if $\gamma \in \mathrm{Dom}(p^+_j)$ then $p'_j \restriction \gamma \Vdash$ "$\ell g(\underset{\sim}{\eta}^{p'_j(\gamma)}) > i(*)$ (see $\boxplus_3(c)$)

(c) p'_j forces [6] a value to the pair $(\eta^{p^+(\gamma_j)}, \underset{\sim}{f}^{p^+(\gamma_j)} \restriction \lambda_{\varepsilon(*)})$; we call this pair $q_j = (\eta^{q_j}, f^{q_j})$.

This should be clear.

Second

$\circledast_9$ p^+_j hence p'_j is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$-generic and $\langle \nu_{\gamma_{j(1)}} : j(1) < j\rangle$ induces the generic.

[Why? As in the proof of $\circledast''_7$ above when we assume that we have carried the induction, by $\boxplus_2$, clause (g) and $\circledast_4$.]

Now

$\circledast_{10}$ (a) $f^{q_j} \in (\prod_{\zeta < \lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbb{A}_{\varepsilon(*)}[\bar\nu^j]}$

(b) for every large enough $\zeta \in (\xi(i), \sigma)$ we have

[6] recall that $\eta^{p^+(\gamma_j)}$ is an object, not a name and p^+_j is $(N_{\varepsilon(*)}, \mathbb{P}_{\gamma_j})$-generic

(945) revision:2017-11-02 modified:2017-11-05

$\bullet_1$ $f^{q_j} \le \eta_\zeta^{\varepsilon(*)} \mod J^{\rm bd}_{\lambda_\varepsilon}$

$\bullet_2$ $f^{q_j} \in \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)} \restriction \zeta]$

$\bullet_3$ $\langle \zeta(j_1) : j_1 < j\rangle \in \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)} \restriction \zeta]$.

(c) $\eta^{q_j} \triangleleft f^{q_j}$.

[Why? Clause (a) holds because $f^{q_j} \in (\prod\limits_{\zeta < \lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbf{V}}$, hence belongs to $\mathscr{H}(\chi_{\varepsilon(*)})$ which is the universe of $M_{\varepsilon(*)}$ so $f^{q_j} \in M_{\varepsilon(*)}$. But $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}]$; recalling $\bar\eta^{\varepsilon(*)}$ is a generic for $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$. Next as $\mathbb{P}'_{\gamma(*)}$ satisfies the λ^+-c.c. and $\lambda < \kappa = \mathrm{cf}(\gamma(*))$ so $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$ satisfies the $\lambda^+_{\varepsilon(*)}$-c.c. hence for some $\zeta_1 < \sigma, f^{q_j} \in \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)} \restriction \zeta_1]$. Similarly for some $\zeta_2 < \sigma$ we have $\langle \zeta(j_2) : j_2 < j\rangle$ belongs to $N'_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)} \restriction \zeta_2]$. Letting $\zeta \ge \max\{\zeta_1, \zeta_2\}$ clearly clauses $\bullet_2, \bullet_3$ of $\circledast_{10}(b)$ holds, also $\bullet_1$ there holds by $\boxplus_1(c)$ (and $\mathbf{j}_\varepsilon$, etc.).

Lastly, $\circledast_{10}(c)$ holds by $\circledast_8(c)$.]

Now we choose $\zeta(j)$ as in clause (b) of $\circledast_{10}$ and $\nu_j \in \prod\limits_{\varepsilon < \lambda_{\varepsilon(*)}} \theta_\varepsilon$ such that $\eta^{p^+(j)} \triangleleft \nu_j, f^{q_j} \le \nu_j$ and $\{\iota < \lambda_{\varepsilon(*)} : \nu_j(\iota) \ne \eta^{\varepsilon(*)}_{\zeta(j)}\}$ is a bounded subset of $\lambda_{\varepsilon(*)}$. Next choose $p^+_i \in \mathbb{P}'_{\gamma(*)}$ such that $p^+_i \restriction \gamma_j = p'_j, \eta^{p^+_i(\gamma_i)} = \nu_j$ and $f^{p^+_i(\gamma_i)} \restriction [\lambda_\varepsilon, \lambda) = f^{p^+(\gamma)} \restriction [\lambda_\varepsilon, \lambda)$.

So we have carried the induction hence proved $\circledast_6$ so we are done. $\square_{1.3}$

{a19} *Concluding Remark* 1.7. 1) Is the use of $\underset{\sim}{\bar g} \restriction \mathscr{U}_*$ rather than $\underset{\sim}{\bar g}$ in the proof necessary? See on this [Sh:F979].

modified:2017-11-05

(945) revision:2017-11-02

§ 2. On a relative to §2 having λ_0 and λ_1

{c0} **Hypothesis 2.1.** As in **??**, but now we have paramter $\bar{\lambda} = (\lambda_1, \lambda_0), \lambda_1 \geq \lambda_0 \geq \lambda$ and use $\mathbf{q}$ the usual $< \lambda$-support iteration.

Discussion 2.2. (2012.2.02) 1) λ_1 has the role of λ_0 in clause $(e)(\theta)$ of Definition
{c4} **??**.
{c29} 2) λ_0 has the role of λ_0 in Definition 2.4 and in **??**.
{c46} 3) At the moment, 2.11 seems unclear.
{c41} 4) Even with this separation to λ_0, λ_1 we can prove **??** but it requires more care ($\mathscr{X}$ more information).
{c30} 5) We may change Definition 2.5(1) to $(*)$ below, getting weaker results for $\mathbf{M}$ but ones which fit more general contents

- $(*)$ by induction on the ordinal γ we define the class $\mathbf{M}^{\rm ec}_\gamma$ as the class of $\mathbf{m} \in \mathbf{M}$ such that:
  - if $L \in \mathrm{Sub}(\mathbf{m})$ and $\beta = \mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$ <u>then</u> $m \leq_{\mathbf{M}} n \in M^{\rm ec}_\beta \Rightarrow P_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$.

6) We may assume $\chi = \chi^{\lambda_0}$ instead of $\chi = \chi^{\lambda_1}$ but using E in the proof more information $\mathscr{X}$ and $\mathscr{E}$.

Claim 2.3. *Continuing claim:*
10) If $L^+_{\mathbf{m}} \models$ "$s < t$" and $p \in \mathscr{I}_{s,t}$ and $\zeta = \ell g(\mathrm{tr}(p(s))$ <u>then</u> for every $i < \theta_\varepsilon$ large enough there is $q \in \mathbb{P}_t$ such that:

- $\mathbb{P}_t \models$ "$p \leq q$"
- *if $r \in \mathrm{Dom}(p) \backslash L_{\mathbf{m},\leq s}$ then $q(r) = p(r)$*
- *if $r \in \mathrm{Dom}(p) \cap L_{\mathbf{m},<s}$ then $\mathrm{tr}(p(r)) \triangleleft \mathrm{tr}(q(r))$*
- *$\mathrm{tr}(q(s))$ has length $\zeta + 1$*
- *$q(s)(\zeta) = i$.*

11) If $L^+_{\mathbf{m}} \models$ "$s < t$" and $p \in \mathscr{I}_{s,t}$ and $\zeta_0 < \zeta_1 < \lambda, \zeta_0 = \ell g(\mathrm{tr}(p(s)))$ and $\bar{u} = \langle u_\varepsilon : \zeta_0 \leq \varepsilon < \zeta_1 \rangle$ where u_ε is an unbounded subset of θ_ε for $\varepsilon \in [\zeta_0, \zeta_1)$ <u>then</u> there is $q \in \mathbb{P}_t$ above p such that $p \Vdash_{\mathbb{P}_t}$ "$\underset{\sim}{\eta}_s(\varepsilon) \in u_\varepsilon$ for $\varepsilon \in [\zeta_0, \zeta_1)$".
12) If $\xi \leq \lambda, r_\zeta \in u_t$ for $\zeta < \xi$ and $\mathbf{B}$ is a λ-Borel function from ${}^\xi(\Pi\bar{\theta})$ to $\mathbb{Q}_{\bar{\theta}}$ and $p \in \mathbb{P}_{\mathbf{m}}$ <u>then</u> for some q we have:

- $\mathbb{P}_{\mathbf{m}} \models$ "$p \leq q$"
- *$q \restriction L_{<t} \Vdash_{\mathbb{P}_t}$ "$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta<\xi} \leq_{\mathbb{Q}_{\bar{\theta}}} q(t)$ <u>or</u> $\mathbf{B}(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta<\varepsilon}, q(t)$ are incompatible in $\mathbb{Q}_t, p \in \mathbb{P}_{\mathbf{m}}$.*

13) If $t \in L_{\mathbf{m}}, j \leq \lambda, \xi_i \leq \lambda$ for $i < j, r_{i,\zeta} \in u_t$ for $i < j, \zeta < \xi_i$ and $p \restriction L_{<t} \Vdash_{\mathbb{P}_t}$ "$\langle \mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{r_{i,\zeta}}, \ldots)_{\zeta<\xi_i} : i < j \rangle$ is a maximal antichain of $\mathbb{Q}_t^{\mathbf{V}[\langle \underset{\sim}{\eta}_r : r \in u_t \rangle]}$" <u>then</u> for some q and $i < j$ we have:

- $\mathbb{P}_{\mathbf{m}} \models$ "$p \leq q$"
- *$q \restriction L_{<t} \Vdash_{\mathbb{P}_t}$ "$q(t)$ is above $\mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{r_{i,\zeta}}, \ldots)_{\zeta<\varepsilon_i}$ in $\mathbb{Q}_{\bar{\theta}}$".*

{c29} **Definition 2.4.** 1) For $\mathbf{m} \in \mathbf{M}$ let $\mathrm{cmp}(\mathbf{m})$ be the set of L such that:

- (a) $L \subseteq L_{\mathbf{m}}$

(945) revision:2017-11-02 modified:2017-11-05

(b) if $L \subseteq M_{\mathbf{m}}$ then $|L| \le 1$

(c) if $t \in L \backslash M_{\mathbf{m}}$ then $L = t/E'_{\mathbf{m}}$ hence:

(α) $L \backslash M_{\mathbf{m}} \ne \emptyset$ and $L \backslash M$ is an $E_{\mathbf{m}}$-equivalence class

(β) if $t \in L \backslash M_{\mathbf{m}}$ then $u_{\mathbf{m},t} \subseteq L$

(γ) $|L| \le \lambda_0$.

2) For $\mathbf{m} \in \mathbf{M}$

(a) let sub($\mathbf{m}$) be the set of L_1 such that for some $L_2 \in \mathrm{cmp}(\mathbf{m}), L_1$ is an initial segment of L_2, i.e. $L_2 \subseteq L_1 \wedge (\forall s,t)[s \in L_2 \wedge t \in L_1 \wedge s <_{\mathbf{m}} t \to s \in L_1]$

(b) let $\mathrm{sub}_*(\mathbf{m}) = \{L \in \mathrm{sub}(\mathbf{m})$: there are[7] $L' \in \mathrm{cmp}(\mathbf{m})$ and $Y_1, Y_2 \subseteq L'$ of cardinality $\le \lambda_0$ such that $L = \{s \in L' : (\exists t)(s \le_{\mathbf{m}} t \in \mathscr{Y}_1 \vee s <_{\mathbf{m}} \in \mathscr{Y}_2\}\}$.

3) Let

(a) $\mathrm{Cmp}(\mathbf{m}) = \{\bigcup_{\varepsilon<\zeta} L_\varepsilon : L_\varepsilon \in \mathrm{cmp}(\mathbf{m})$ for $\varepsilon < \zeta\}$

(b) $\mathrm{Cmp}_*(\mathbf{m}) = \{\bigcup_{\varepsilon<\zeta} L_\varepsilon : L_\varepsilon \in \mathrm{cmp}(\mathbf{m})$ for $\varepsilon < \zeta$ and $\zeta \le \lambda_0\}$

(c) $\mathrm{Sub}(\mathbf{m}) = \{L_1$: for some $L_2 \in \mathrm{Cmp}(\mathbf{m}), L_1$ is an initial segment of $L_2\}$

(d) $\mathrm{Sub}_*(\mathbf{m}) = \{\bigcup_{\varepsilon<\zeta} L_\varepsilon : \zeta \le \lambda_0$ and $L_\varepsilon \in \mathrm{sub}_*(\mathbf{m})$ for $\varepsilon < \zeta\}$.

Definition 2.5. 0) For $L \subseteq L_{\mathbf{m}}, \mathbf{m} \in \mathbf{M}$ let $\mathrm{dp}^*_{\mathbf{m}}(L) = \cup\{\mathrm{dp}_{M_{\mathbf{n}}}(t)+1 : t \in L \cap M_{\mathbf{m}}\}$. {c30}
1) For an ordinal γ let $\mathbf{M}^{\mathrm{ec}}_\gamma$ be the class of $\mathbf{m} \in \mathbf{M}$ such that:

($*$) if $L \in \mathrm{Sub}(\mathbf{m})$ and $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$ <u>then</u> $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$.

2) Let $\mathbf{M}^{\mathrm{ec}} = M^{\mathrm{ec}}_\infty$ be the class of $\mathbf{m}$ which $\in \mathbf{M}^{\mathrm{ec}}_\gamma$ for every γ.
3) Let $\mathbf{M}^{\mathrm{ec}}_{\chi,\gamma} = \{\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\gamma : |L_{\mathbf{m}}| \le \chi\}$, similarly $\mathbf{M}^{\mathrm{ec}}_{\chi,\infty}$.
4) Let $\mathbf{M}_{\mathrm{fc}}$ be the class of $\mathbf{m} \in \mathbf{M}$ satisfying $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$.

Observation 2.6. *1) $\le_{\mathbf{M}}$ is a partial order on $\mathbf{M}$.* {c32}
2) $\mathbf{M}^{\mathrm{ec}}_0 = \mathbf{M}$ and $\mathbf{M}^{\mathrm{ec}}_{\chi,0} = \mathbf{M}_\chi$.
3) If $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ and $L \in \mathrm{sub}(\mathbf{m})$ <u>then</u> $\mathbb{P}_{\mathbf{m}}(L), \mathbb{P}_{\mathbf{n}}(L)$ have the same set of elements.
4) If $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ <u>then</u> $\mathrm{cmp}(\mathbf{m}) \subseteq \mathrm{cmp}(\mathbf{n})$ moreover $L \in \mathrm{cmp}(\mathbf{m})$ <u>iff</u> $L \in \mathrm{cmp}(\mathbf{n}) \wedge L \subseteq L_{\mathbf{m}}$.
5) Like (4) for $\mathrm{sub}(-), \mathrm{Cmp}(-), \mathrm{Sub}(-), \mathrm{sub}_*(-), \mathrm{Cmp}_*(-), \mathrm{Sub}_*(-)$.
6) If $\chi = \chi^{\lambda_0}, \mathbf{m} \in \mathbf{M}_\chi, L \in \mathrm{Sub}(\mathbf{m}), \mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ and $\mathbb{P}_{\mathbf{m}}[L] \ne \mathbb{P}_{\mathbf{n}}[L]$ <u>then</u> we can find such $\mathbf{n}$ in $\mathbf{M}_\chi$.
7) If $\mathbf{m} \in \mathbf{M}$ and $t \in L_{\mathbf{m}}$ <u>then</u> $t \in L$ for some $L \in \mathrm{cmp}(\mathbf{m})$.
8) If $\mathbf{m} \in \mathbf{M}$ and $L_1 \subseteq L_{\mathbf{m}}$ has cardinality $\le \lambda_0$ then $L_1 \subseteq L_2$ for some $L_2 \in \mathrm{Cmp}_*(\mathbf{m})$.
9) If $L_1 \in \mathrm{sub}(\mathbf{m})$ then $L_1 = \cup\{L_1 \cap L_{\mathbf{m},\le t} : t \in L_1\}$ and $t \in L_1 \Rightarrow L_1 \cap L_{\mathbf{m} \le t} \in \mathrm{sub}_*(\mathbf{m})$.

Proof. Should be clear. $\square_{2.6}$

[7]version A: s_ε of $L_{\mathbf{m}}$ for $\varepsilon < \varepsilon(*) < \lambda$ and λ-Borel function $\mathbf{B}$ from $\varepsilon(*)_4$ to 2 and $L = \{t \in L' : \mathbf{B}(\ldots, \mathfrak{A}_I(s,t), \ldots) = 1$ where $\mathfrak{A}_I(s,t)$ is: 0 if $s <_I t$, 1 if $s = t$, 2 if $t <_J s$ and 3 otherwise.

(945) revision:2017-11-02 modified:2017-11-05

{c35}

Observation 2.7. *1) If* $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$ *is* $\le_{\mathbf{M}}$*-increasing* *then* $\mathbf{m}_\delta := \cup\{\mathbf{m}_\alpha : \alpha < \delta\}$ *is a* $\le_{\mathbf{M}}$*-upper bound of* $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$.
2) Moreover in part (1):

- (α)
  - $\mathrm{cmp}(\mathbf{m}_\delta) = \cup\{\mathrm{cmp}(\mathbf{m}_\alpha) : \alpha < \delta\}$
  - $\mathrm{sub}(\mathbf{m}_\delta) = \cup\{\mathrm{sub}(\mathbf{m}_\alpha) : \alpha < \delta\}$

  $\mathrm{sub}_*(\mathbf{m}_\delta) = \cup\{\mathrm{sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$
- (β) *if* $\mathrm{cf}(\lambda) \ge \lambda_0^+$ *then*
  - $\mathrm{Cmp}_*(\mathbf{m}_\delta) = \cup\{\mathrm{Cmp}(\mathbf{m}_\alpha) : \alpha < \delta\}$
  - $\mathrm{Sub}_*(\mathbf{m}_\delta) = \cup\{\mathrm{Sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$.

3) If $\gamma_1 < \gamma_2$ *then* $\mathbf{M}^{\mathrm{ec}}_{\gamma_1} \subseteq \mathbf{M}^{\mathrm{ec}}_{\gamma_2}$.

Proof. 1) Easy, remembering that $t \in \mathbf{m}_\alpha \wedge \alpha \le \beta < \delta \Rightarrow t/E'_{\mathbf{m}_\beta} = t/E'_{\mathbf{m}_\beta}$.
2) Easy.
3) Easy, too. $\square_{2.7}$

{c38}
{c30} **Claim 2.8.** *1) In Definition 2.5(1) we can consider only* $L \in \mathrm{Sub}_*(\mathbf{m})$ *such that* $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$.
2) So if $\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\infty$ *and* $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ *then* $\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$*, that is* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \lessdot \mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$ *and* $L \in \mathrm{Sub}(\mathbf{m}) \Rightarrow \mathbb{P}_{\mathbf{m}}[L] \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{n}}]$.
3) For $\mathbf{m} \in \mathbf{M}$ *we have:* $\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\infty$ *iff* $\mathbf{m} \in \mathbf{M}_{\mathrm{fc}}$*, i.e. for every* $\mathbf{n}, \mathbf{m} \le_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$.
4) $\mathrm{Cmp}_*(\mathbf{m})$ *has cardinality* $\le |L_{\mathbf{m}}|^{\le \lambda_0}$.
5) $\mathrm{Sub}_*(\mathbf{m})$ *has cardinality* $\le |L_{\mathbf{m}}|^{\lambda_0}$.
6) $|\mathrm{cmp}(\mathbf{m})| \le |L_{\mathbf{m}}| + 1$ *and* $|\mathrm{sub}_*(\mathbf{m})| \le |L_{\mathbf{m}}|^{\lambda_0}$.

Proof. 1) Let $\mathbf{m} \in \mathbf{M}$ satisfies the weaker version, and let $L \in \mathrm{Sub}(\mathbf{m})$ satisfies
{c29} $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$. By Definition 2.4(3)(c) and Observation 2.6(9) we can find $\langle L_\varepsilon, L^*_\varepsilon : \varepsilon < \zeta \rangle$ such that L be $\cup\{L_\varepsilon : \varepsilon < \zeta\}$ where $L_\varepsilon \subseteq L^*_\varepsilon \in \mathrm{Cmp}(\mathbf{m})$ and $L_\varepsilon \in \mathrm{sub}_*(\mathbf{m})$.

For every $u \in [\zeta]^{\le \lambda_0}$ let $L_u = \cup\{L_\varepsilon : \varepsilon \in u\}$ so clearly $\mathrm{dp}^*_{\mathbf{m}}(L_u) \le \mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$. Let $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$. For $u \in [\zeta]^{\le \lambda_0}$, for L_u the demand holds, i.e. $\mathbb{P}_{\mathbf{m}}[L_u] = \mathbb{P}_{\mathbf{n}}[L_u]$. Also if $u \subseteq v \in [\zeta]^{\le \lambda_0}$ then $\mathbb{P}_{\mathbf{m}}[L_u] \subseteq \mathbb{P}_{\mathbf{m}}[L_v]$.

{c30} Lastly, $\langle u : u \in [\zeta]^{\le \lambda_0} \rangle$ is λ_0^+-directed so by **??**(8) and the Definition 2.5(1) we are done.
{c4} 2) As $L_{\mathbf{m}} \in \mathrm{Sub}(\mathbf{m})$ and $\mathrm{dp}^*_{\mathbf{m}}(L_m) < \infty$ by clause (d) Definition **??**.
3) The implication $\Rightarrow$ holds by part (2). For the other implication assume $\mathbf{m} \notin \mathbf{M}^{\mathrm{ec}}_\infty$ so clause (*) there fails for some γ; this means that there are $L \in \mathrm{Sub}(\mathbf{m})$ and $\mathbf{n}$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ but $\mathbb{P}_{\mathbf{m}}[L] \ne \mathbb{P}_{\mathbf{n}}[L]$, by the definitions this implies $\neg(\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}})$ so we are done.
4) Note that $\{L \in \mathrm{cmp}(\mathbf{m}) : L \nsubseteq M_{\mathbf{m}}\} = \{t/E'_{\mathbf{m}} : t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}\}$ has cardinality $\le |L_{\mathbf{m}} \backslash M_{\mathbf{m}}| \le |L_{\mathbf{m}}|$ and $\{L \in \mathrm{cmp}(\mathbf{m}) : L \subseteq M_{\mathbf{m}}\} = L \subseteq M_{\mathbf{m}} : |L| \le \lambda_0\}$ has cardinality $\le |M_{\mathbf{m}}|^{\lambda_0} \subseteq |L_{\mathbf{m}}|^{\lambda_0}$.

5), 6) Similarly. $\square_{2.8}$

{c39}
Claim 2.9. *1) If* $\gamma \le \infty, \langle \mathbf{m}^*_\alpha : \alpha < \delta \rangle$ *is* $\le_{\mathbf{M}}$*-increasing,* $\mathrm{cf}(\delta) > \lambda_0$ *and* $\alpha < \delta \Rightarrow \mathbf{m}^*_\alpha \in \mathbf{M}^{\mathrm{ec}}_\gamma$ *then* $\mathbf{m}_* := \cup\{\mathbf{m}^*_\alpha : \alpha < \delta\} \in \mathbf{M}^{\mathrm{ec}}_\gamma$.
2) Similarly for λ_0^+*-directed union.*

{c38} *Proof.* 1) By 2.8(1), toward contradiction assume that $(\mathbf{n}, L)$ satisfies:

(945) revision:2017-11-02 modified:2017-11-05

$(*)$ (a) $\mathbf{m}_* \le_{\mathbf{M}} \mathbf{n}$

(b) $L \in \mathrm{Sub}_*(\mathbf{m}_*)$

(c) $\mathrm{dp}^*_{\mathbf{m}_*}(L) < \gamma$

(d) $\mathbb{P}_{\mathbf{m}_*}[L] \ne \mathbb{P}_{\mathbf{n}}[L]$.

As $L \in \mathrm{Sub}_*(\mathbf{m}_*)$, we have $L \subseteq \cup\{L_\varepsilon : \varepsilon < \varepsilon(*)\}$ where $\varepsilon(*) < \lambda^*_0$ and $L_\varepsilon \in \mathrm{sub}_*(\mathbf{m})$ for $\varepsilon < \varepsilon(*)$. But then $L_\varepsilon \in \mathrm{sub}_*(\mathbf{m}_{\alpha(\varepsilon)})$ for some $\alpha(\varepsilon) < \delta$ by 2.7(2)(α) {c35} but $\mathrm{cf}(\delta) > \lambda_0$ hence $\alpha(*) = \sup\{\alpha(\varepsilon) : \varepsilon < \varepsilon(*)\} < \delta$ so $L \in \mathrm{Sub}_*(\mathbf{m}_{\alpha(*)})$.

Now we have assumed $\mathrm{dp}^*_{\mathbf{m}_*}(L) < \gamma$ but $M_{\mathbf{m}_*} = M_{\mathbf{m}_{\alpha(*)}}$, so equivalently $\mathrm{dp}^*_{\mathbf{m}_{\alpha(*)}}(L) < \gamma$. By the claim assumption, $\mathbf{m}_\beta \in \mathbf{M}^{\mathrm{ec}}_\gamma$ and by 2.7(1) we have {c35} $\mathbf{m}_\beta \le_{\mathbf{M}} \mathbf{m}_*$.

As $\le_{\mathbf{M}}$ is a partial order on $\mathbf{M}$ and $\mathbf{m}_* \le_{\mathbf{M}} \mathbf{n}$ by $(*)(a)$ clearly $\mathbf{m}_{\alpha(*)} \le_{\mathbf{M}} \mathbf{n}$, but $\mathbf{m}_{\alpha(*)} \in M^{\mathrm{ec}}_\partial$ so together we deduce $\mathbb{P}_{\mathbf{m}_*}[L] = \mathbb{P}_{\mathbf{m}_\beta}[L] = \mathbb{P}_{\mathbf{n}}[L]$, so we are done. $\square_{2.9}$

{c41y}

Crucial Claim 2.10. *Let* $\chi = \chi^{\lambda_0}$.
1) If $\mathbf{m} \in \mathbf{M}$ *of cardinality* $\le \chi$ *there is* $\mathbf{n} \in \mathbf{M}^{\mathrm{ec}}_\infty$ *of cardinality* $\le \chi$ *such that* $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$.
2) If $\mathbf{m} \in \mathbf{M}_\chi$ *and* $\gamma < \chi^+$ *<u>then</u> there is* $\mathbf{n} \in \mathbf{M}^{\mathrm{ec}}_\gamma \cap \mathbf{M}_\chi$ *such that* $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$.

Proof. 1) By (2) using $\gamma = \mathrm{dp}^*_{\mathbf{m}}(M_{\mathbf{m}}) + 1 = \mathrm{dp}^*_{\mathbf{m}}(L_{\mathbf{m}}) + 1$ which necessarily is $< \|M_{\mathbf{m}}\|^+ \le \chi^+$.
2) We prove the statement by induction on γ. Recalling Definition 2.5(1), in each {c30} case consider $L \in \mathrm{Sub}(\mathbf{m})$ which satisfies $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$ and by 2.8(1) without loss {c38} of generality $L \in \mathrm{Sub}_*(\mathbf{m})$.

<u>Case 1</u>: $\gamma = 0$.
There is no such relevant L.

<u>Case 2</u>: $\gamma = 1$.
So $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$ means $\mathrm{dp}^*_{\mathbf{m}}(L) = 0$ which means $L \cap M_{\mathbf{m}} = \emptyset$. But then $\mathbf{n} = \mathbf{m}$ is as required because if $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}_1$ then $\{s \in L_{\mathbf{n}_1} : (\exists t \in L)(s \le_{\mathbf{n}_1} t)\} = L = \{s \in L_{\mathbf{m}} : (\exists t \in L)(s \le_{\mathbf{m}} t)\}$, i.e. L is an initial segment of both so $\mathbb{P}_{\mathbf{m},L} = \mathbb{P}_{\mathbf{n}_1,L}$ and the conclusion is obvious, or see **??**. {c13}

<u>Case 3</u>: $\gamma = \beta + 2$.
We try to choose $\mathbf{m}_\varepsilon$ by induction on $\varepsilon \le \chi^+$ such that:

$(*)$ (a) $\mathbf{m}_\varepsilon \in \mathbf{M}_\chi$

(b) $\langle \mathbf{m}_\zeta : \zeta \le \varepsilon \rangle$ is $\le_{\mathbf{M}}$-increasing continuous

(c) $\mathbf{m}_0 = \mathbf{m}$

(d) if $\varepsilon = 2\zeta + 1$ then for some $L \in \mathrm{Sub}_*(\mathbf{m}_{2\zeta})$ we have $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \ne \mathbb{P}_{\mathbf{m}_\varepsilon}[L]$ and $\mathrm{dp}^*_{\mathbf{m}_{2\zeta}}(L) < \gamma$

(e) if $\varepsilon = 2\zeta + 2$ then $\mathbf{m}_\varepsilon \in \mathbf{M}^{\mathrm{ec}}_{\beta+1}$.

<u>Subcase 3A</u>: We succeed to carry the induction.

modified:2017-11-05
(945) revision:2017-11-02

For each $\delta \in S^{\chi^+}_{\lambda_0^+} := \{\delta < \chi^+ : \mathrm{cf}(\delta) = \lambda_0^+\}$ (i.e. $\varepsilon = 2\delta + 1$ and note $2\delta = \delta$) there is L_δ as in $(*)(d)$. Recalling Definition 2.4(3)(d), as $\mathrm{cf}(\delta) = \lambda_0^+ > \lambda_0 \geq$ {c29}
$|\{t/E'_{\mathbf{m}_\delta} : t \in L_\delta \backslash M_{\mathbf{m}_\delta}\}|$ there is $\alpha(\delta) < \delta$ such that $L_\delta \subseteq L_{\mathbf{m}_{\alpha(\delta)}}$ hence by 2.7(2) + {c35}
2.8(5) for some stationary $S \subseteq S^{\chi^+}_{\lambda_0^+}$ and L_* we have $\delta \in S \Rightarrow L_\delta = L_* \wedge \alpha(\delta) = \alpha(*)$. {c38}
Without loss of generality $\alpha(*)$ is even and a successor ordinal. For $\delta \in S$, note that $\langle \mathbf{m}_{2\zeta+2} : \zeta < \delta\rangle$ is $\leq_{\mathbf{M}}$-increasing sequence of members of $\mathbf{M}^{\mathrm{ec}}_{\beta+1}$ with union $\mathbf{m}_\delta$ (by clauses (e) and (b) of $(*)$) hence $\mathbf{m}_\delta \in \mathbf{M}^{\mathrm{ec}}_{\beta+1}$ by 2.9; recall $\mathrm{cf}(\delta) > \lambda_0$ and {c39}
$M_{\mathbf{m}_\alpha} = M_{\mathbf{m}}$ for every α.

Now $\mathrm{dp}^*_{\mathbf{m}_\delta}(L_*) < \gamma$ and by the previous sentences necesarily $\mathrm{dp}^*_{\mathbf{m}_\delta}(L_*) = \beta + 1$ hence the set $\mathscr{Y} := \{t \in L_* \cap M_{\mathbf{m}} : \mathrm{dp}(t, M_{\mathbf{m}}) = \beta\}$ is not empty.

Let

$\boxplus_1$ (a) $L^*_0 = \{s \in L_* : \neg(\exists t \in \mathscr{Y})(t \leq s)\}$
(b) $L^*_1 = L^*_0 \cup \mathscr{Y}$
(c) $L^*_2 = L_*$.

Now

$\boxplus_2$ $\mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L^*_0]$ is equal to $\mathbb{P}_{\mathbf{m}_\alpha}[L^*_0]$ and to $\mathbb{P}_{\mathbf{m}_{\alpha+1}}[L^*_0]$ for $\alpha \in S \backslash \alpha(*)$ moreover for $\alpha \geq \alpha(*)$.

[Why? L^*_0 is an initial segment of L_* but $\delta \in S \Rightarrow L_* = L_\delta \in \mathrm{Sub}_*(\mathbf{m}_\delta)$, see above
{c29} hence, see Definition 2.4(2)(b),(3)(c) we have $L^*_0 \in \mathrm{Sub}_*(\mathbf{m}_\alpha)$ for $\alpha \geq \alpha(*)$. We
{c30} are done as $\mathbf{m}_{\alpha(*)} \in \mathbf{M}^{\mathrm{ec}}_{\beta+1}$ by $(*)(e)$ and 2.5(1).]

$\boxplus_3$ (a) $L^*_{0,\alpha} = \{s \in L_{\mathbf{m}_\alpha} : (\exists t \in L^*_0)(s \leq t)\}$ or $(\exists t \in \mathscr{Y})(s \leq t)$
(b) $L^*_{1,\alpha} = L^*_{0,\alpha} \cup Y$
(c) $L^*_{2,\alpha} = L^0_\alpha \cup L_*$

$\boxplus_4$ (a) $\mathrm{dp}^*_{\mathbf{m}_\alpha}(L^*_{0,\alpha}) < \beta + 1$
(b) $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L^*_{0,\alpha}] : \alpha \in [\alpha(*), \chi^+)$ is $\lessdot$ increasing.

$\boxplus_5$ (a) $L^*_{1,\alpha} \subseteq L_{\mathbf{m}_\alpha}$ is downward closed in $L_{\mathbf{m}_\alpha}$
(b) $\langle \mathbb{P}_{\mathbf{m}_\alpha}(L^*_{1,\alpha}) : \alpha \in [\alpha(*), \chi^+)\rangle$ is $\subseteq$-increasing
(c) moreover $\leq_{\mathrm{ic}}$-increasing.

[Why? Should be clear.]

$\boxplus_6$ there is a club E of χ^+ such that $\delta \in E \cap S^{\chi^+}_{\lambda_0^+} \backslash \alpha(*) \wedge \delta \leq \alpha < \chi^+ \Rightarrow P_{\mathbf{m}_\gamma}(L^*_{1,\delta}) \lessdot \mathbb{P}_{\mathbf{m}_\alpha}(L^*_{1,\alpha})$.

[Why? By $\boxplus_5$ as every $\mathbb{P}_{\mathbf{m}_\alpha}(L^*_{1,\alpha})$ satisfies the λ-c.c., $\lambda_0 \geq \lambda$ and $\chi^{\lambda_0} = \chi$.]

Now

$\boxplus_7$ (a) $L^*_{2,\alpha}$ is a downward closed subset of $L_{\mathbf{m}_\alpha}$ for $\alpha \in [\alpha(*), \chi^+]$
(b) if $\alpha(*) \leq \alpha \leq \beta < \chi^+$ and $\mathbb{P}_{\mathbf{m}_\alpha}(L^*_{1,\alpha}) \lessdot \mathbb{P}_{\mathbf{m}_\beta}(L^*_{1,\beta})$ then
(α) $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) = \mathbb{P}_{\mathbf{m}_\beta}(L_{2,\alpha}) \lessdot \mathbb{P}_{\mathbf{m}_\beta}$
(β) $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) \lessdot \mathbb{P}_{\mathbf{m}}(L_{2,\beta})$
(γ) $\mathbb{P}_{\mathbf{m}_\alpha}[L_{2,\alpha}] = \mathbb{P}_{\mathbf{m}_\beta}[L_{2,\beta}]$
(δ) $\mathbb{P}_{\mathbf{m}_\alpha}[L^*_1] = \mathbb{P}_{\mathbf{m}_\beta}[L^*_1]$.

modified:2017-11-05
(945) revision:2017-11-02

[Why? Clause (a) by inspection. Clause $(b)(\alpha)$, the equality is obvious, the $\lessdot \mathbb{P}_{\mathbf{m}_\beta}$ holds by clause (a). Now we prove clause $(b)(\beta)$, toward this

- $\bullet_1$ $t \in L_{2,\beta} \backslash L_{1,\alpha} \Leftrightarrow t \in Y \Leftrightarrow t \in L_{2,\alpha} \backslash L_{1,\alpha}$ (why? check)
- $\bullet_2$ $t \in L_{2,\alpha} \backslash L_{1,\alpha} \Rightarrow u_{\mathbf{m}_\beta,t} \subseteq L_{1,\alpha}$ (why? check)
- $\bullet_3$ $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) \subseteq \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\beta})$ (by assumption of (b)).

Next assume $p_* \in \mathbb{P}_{2,\beta}(L_{2,\beta})$ then $q_* = p_* \restriction L_{2,\alpha} \in \mathbb{P}_{2,\alpha}(L_{1,\alpha})$ and also $q' = p_\beta \restriction L_{1,\alpha} \in \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}), p' = p \restriction L_{1,\beta} \in \mathbb{P}_{\mathbf{m}_\beta})$. Also $q' \le q'' \in \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}) \Rightarrow q'' \cup (p \restriction (L_{1,\beta} \backslash L_{1,\alpha})$ is a common upper bound of q'', p' in $\mathbb{P}_{\mathbf{m}_\beta}(L_{1,\beta})$. Hence $q_* = q' \cup (p \restriction y)$ satisfies $q_* \in P_{\mathbf{m}_\alpha}(L_{2,\alpha})$ and $q_* \le q'_* \in P_{\mathbf{m}_\alpha}(L_{2,\alpha}) \Rightarrow q'_* \cup (p \restriction (L_{1,\beta} \backslash L_{1,\alpha})$ is a common upper bound of q'_*, p in $\mathbb{P}_{\mathbf{m}_\beta}(L_{2,\beta})$.

Together with $\bullet_3$ this paragraph shows

- $\bullet_4$ $\mathbb{P}_{\mathbf{m}_\alpha}[L_{2,\alpha}] \lessdot \mathbb{P}_{\mathbf{m}_\beta}[L_{2,\beta}]$ that is clause $(b)(\beta)$.

Clause $(b)(\gamma)$ holds by $(b)(\beta)$ and (a) and clause $(b)(\beta)$ follows as $L^*_2 \subseteq L^*_{2,\alpha} \subseteq L^*_{2,\alpha}$ so $\boxplus_7$ holds indeed.

Now choose $\delta \in E \cap S \backslash \alpha(*) \subseteq S^{\chi^+}_{\lambda_0^+}$ and we get a contradiction to the choice of $L_\delta = L_*$ so we are done with subcase 3A.

PAGE 21E NOT SENT

$(*)_2$ for $\alpha \in \chi^+ \backslash \alpha(*)$ and $t \in \mathscr{Y}$ let $\underset{\sim}{\mathbb{Q}}^\alpha_{\bar\theta,t}$ be the $\mathbb{P}^{\mathbf{m}_\alpha}$-name of $\mathbb{Q}_{\bar\theta}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\gamma : \gamma \in u(t) \cap L_* \rangle]}$ where $u(t) = u_{\mathbf{m}_{\alpha(*)},t}$

$(*)_3$ (a) $\underset{\sim}{\mathbb{Q}}^\alpha_{\bar\theta,t}$ does not depend on α, also the order on it does not depend on α

(b) $\Vdash_{\mathbb{P}_{\mathbf{m}_\alpha}}$ "$\underset{\sim}{\mathbb{Q}}^\alpha_{\bar\theta,t} \le_{\rm ic} \underset{\sim}{\mathbb{Q}}_{\bar\theta}$, i.e. $\underset{\sim}{\mathbb{Q}}^\alpha_{\bar\theta,t} \subseteq \mathbb{Q}_{\bar\theta}$ as quasi orders and any $p_1, p_2 \in \underset{\sim}{Q}^\alpha_{\bar\theta,t}$ compatible in $\mathbb{Q}_{\bar\theta}$ are compatible in $\mathbb{Q}^\alpha_{\bar\theta,t}$"

(c) $\mathbb{P}_{\mathbf{m}_\alpha}[L^*_1]/\mathbb{P}_{\mathbf{m}_\alpha}[L^*_0]$ is the same for $\alpha \in [\alpha(*), \chi^+)$

$(*)_4$ define $\mathbb{I}_\alpha = \{\underset{\sim}{\mathscr{I}} : \underset{\sim}{\mathscr{I}}$ is a $\mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L^*_0]$-name of a maximal antichain of $\mathbb{P}_{\mathbf{m}_\alpha}[L^*_1]/\mathbb{P}_{\mathbf{m}_\alpha}[L^*_0]\}$.

Let $L^*_{0,\alpha} = \{s \in L_{\mathbf{m}_\alpha} : (\exists t \in L^*_0)(s \le t)$ or $(\exists t \in \mathscr{Y})(s < t)\}$ and $S_* = \{\alpha < \chi^+ : (\exists \varepsilon)(\alpha = 2\varepsilon + 2)$ or $\mathrm{cf}(\alpha) = \lambda_0^+\}$

$(*)_5$ (a) $\mathrm{dp}^*_{\mathbf{m}_\alpha}(L^*_{0,\alpha}) < \beta + 1$

(b) $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L^*_{0,\alpha}] : \alpha \in S_* \backslash \alpha(*) \subseteq [\alpha(*), \chi^+)$ and $\alpha = (\exists \varepsilon)(\alpha = 2\varepsilon + 2)$ or $\alpha \in S\rangle$ is $\lessdot$ increasing.

[Why? For clause (a):

if $s \in L_{0,\alpha} \cap M_{\mathbf{m}_\alpha}$ then for some $t \in Y$ we have $s <_{L_\alpha} t$ hence $\mathrm{dp}(s, M_{\mathbf{m}}) < \mathrm{dp}(t, M_{\mathbf{m}}) = \beta$ hence $\mathrm{dp}(s, M_{\mathbf{m}}) \le \beta$, recalling Definition **??**(1)(β) hence we have {c2}
$\mathrm{dp}^*_{\mathbf{m}_\alpha}(L_{0,\alpha}) = \sup\{\mathrm{dp}(s, M_{\mathbf{m}}) + 1, s \in L \cap M_{\mathbf{m}}\} \le \beta$ as promised.

For clause (b), by the induction hypothesis for $\alpha_1 < \alpha_2$ from $\{2\varepsilon + 2 : \varepsilon < \chi^+\}$ we have $\mathbb{P}_{\mathbf{m}_{\alpha_1}}[L^*_{0,\alpha_1}] = \mathbb{P}_{\mathbf{m}_{\alpha_2}}[L^*_{0,\alpha_1}] \lessdot \mathbb{P}_{\mathbf{m}_{\alpha_2}}[L^*_{0,\alpha_2}]$. This implies the result also when we allow $\mathrm{cf}(\alpha_\ell) = \lambda$ by 2.9.] {c39}

So necessarily

$(*)_6$ $\mathbb{I}_\alpha$ is constant for $\alpha < \chi^+$ large enough.

modified:2017-11-05

(945) revision:2017-11-02

Hence recalling $(*)_3$ for some $\beta(*) \in (\alpha(*), \chi^+)$ we have

$(*)_7$ $\langle \mathbb{P}_{\mathbf{m}_\beta}[L_1^*] : \alpha < S_* \rangle$ is constant for $\beta \in [\beta(*), \chi^+)$.

But this implies the result for L_2^* as the $t \in L_2^* \backslash L_1^*$ does not "have memory outside L_1^*", i.e. $u_{\mathbf{m}_{\beta(*)},t} \subseteq L_1^*$ and is constant, so we are done.

Subcase 3B: We are stuck in ε, i.e. cannot define $\mathbf{m}_\varepsilon$.

Now $\varepsilon = 0$ is impossible. If ε is a limit, let $\mathbf{m}_\varepsilon = \cup\{\mathbf{m}_\zeta : \zeta < \varepsilon\}$ so $\zeta < \varepsilon \Rightarrow \mathbf{m}_\zeta \le_{\mathbf{N}} \mathbf{m}_\varepsilon \in \mathbf{M}_\chi$, contradiction. Also $\varepsilon = \zeta + 1, \zeta = 0$ is impossible.

If $\varepsilon = 2\zeta + 1$ so then clause (d) of $(*)$ is relevant so there is no $\mathbf{n} \in \mathbf{M}_\chi$ such that $\mathbf{m}_{2\zeta} \le_{\mathbf{N}} \mathbf{n}$ and $L \in \mathrm{Sub}_*(\mathbf{m}_{2\zeta})$ such that $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \ne \mathbb{P}_{\mathbf{m}_{2\zeta}}[L]$ and $\mathrm{dp}^*_{\mathbf{m}_{2\zeta}}(L) < \gamma$.
{c32} By 2.8(1) this applies also to $L \in \mathrm{Sub}(\mathbf{m}_{2\zeta})$. By claim 2.6(7) this applies also to
{c38} $\mathbf{n} \in \mathbf{M}$. Together so by 2.8(4) we can deduce that $\mathbf{m}_{2\zeta} \in \mathbf{M}^{\mathrm{ec}}_\gamma$ so we are done.

Lastly, if $\varepsilon = 2\zeta + 2$ so clause (e) of $(*)$ applies, then use the induction hypothesis for $\beta + 1$ to choose $\mathbf{m}_\varepsilon$.

Case 4: γ is limit.

Let $\delta < \chi^+$ be divisible by γ and of cofinality λ_0^+, recalling $\lambda_0^+ \le \chi$.

We choose $\mathbf{m}_\alpha$ for $\alpha \le \delta_0^+$ such that:

- $\mathbf{m}_\alpha \in \mathbf{M}_\chi$
- $\alpha = 0 \Rightarrow \mathbf{m}_\alpha = \mathbf{m}$
- $\mathbf{m}_\alpha$ is $\le_{\mathbf{M}}$-increasing continuous with α
- if $\varepsilon < \gamma$ and $\alpha = \varepsilon + 1 \mod \gamma$ then $\mathbf{m}_\alpha \in \mathbf{M}^{\mathrm{ec}}_\varepsilon$.

Clearly possible: if $\alpha = 0$ trivial, if α is a successor ordinal use the induction
{c39} hypothesis and if α is a limit recall $\delta < \chi^+$ and use 2.9. Now $\mathbf{m}_\delta$ is as required.

Why? Toward contradiction assume $\mathbf{m}_\delta \le_{\mathbf{M}} \mathbf{n}$ and $L \in \mathrm{Sub}_*(\mathbf{m}_{\lambda_0^+})$ satisfies $\gamma >$
{c35} $\mathrm{dp}_{\mathbf{m}_\delta}(L)$; let $\varepsilon := \mathrm{dp}_{\mathbf{m}_\alpha}(L)$. By 2.7(2)($\beta$) there is $\alpha < \delta$ such that $L \in \mathrm{Sub}_*(\mathbf{m}_\alpha)$ and without loss of generality $\alpha = \varepsilon + 1 \mod \gamma$, so we use the choice of $\mathbf{m}_{\alpha+1}$ to get contradiction.

{c39} So $\mathbf{m}_\delta$ is as required by 2.9.

Case 5: None of the above.

So $\gamma = \beta + 1, \beta$ a limit ordinal.

For $\alpha \le \beta$ let $L_\alpha = \{s \in L$: there is no $t \le s$ satisfying $t \in M_{\mathbf{m}}$ and $\mathrm{dp}(t, M_{\mathbf{m}}) \ge \alpha\}$. So $\langle L_\alpha : \alpha \le \beta \rangle$ is $\subseteq$-increasing continuous and $L \backslash L^*_\beta \subseteq L_{\mathbf{m}} \backslash M_{\mathbf{m}}$. As in Case 3 we first deal with L_β, and it is straight, and finish as there. $\square_{??}$

{c46}

Claim 2.11. *Assume $\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\infty, M \subseteq M_{\mathbf{m}}$ and $\mathbf{n}$ is defined just like $\mathbf{m}$ except that*
{c5} *$M_{\mathbf{n}} = M$ and $E_{\mathbf{n}} = \{(s,t) : s,t \in L_{\mathbf{m}}$ but $\{s,t\} \nsubseteq M_{\mathbf{n}}\}$; so by* **??***(5) clearly $\mathbf{n}$ is equivalent to $\mathbf{m}$ and $M_{\mathbf{n}} \subseteq M_{\mathbf{m}}$ but we have to change the context by replacing λ_0 by any $\lambda_0' \ge \lambda_0 + \|L_{\mathbf{m}}\|$.*

{c30} *Then $\mathbf{n} \in \mathbf{M}^{\mathrm{fc}}[\lambda_0']$, see Definition 2.5(1).*

Proof. Assume that

$(*)_1$ $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1$.

We now define $\mathbf{m}_1 \in \mathbf{M}$ by

$(*)_2$ (a) $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$

(945) revision:2017-11-02 modified:2017-11-05

(b) $\bar{u}_{\mathbf{m}_1} = \bar{u}_{\mathbf{n}_1}$
(c) $M_{\mathbf{m}_1} = M_{\mathbf{m}}$
(d) $E_{\mathbf{m}_1} = E_{\mathbf{m}} \cup (E_{\mathbf{n}_1} \restriction (L_{\mathbf{n}_1} \backslash (L_{\mathbf{n}} \backslash M))$.

Clearly

$(*)_3$ (a) indeed $\mathbf{m}_1 \in \mathbf{M}$
(b) $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1$
(c) $\mathbf{m}_1, \mathbf{n}_1$ are equivalent.

But $\mathbf{m} \in \mathbf{M}_{\rm fc}$ hence

$(*)_4$ $\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{m}_1}$.

But $\mathbb{P}_{\mathbf{n}} = \mathbb{P}_{\mathbf{m}}$ and $\mathbb{P}_{\mathbf{n}_1} = \mathbb{P}_{\mathbf{n}}$ by **??**(7) as $\mathbf{n}, \mathbf{m}$ are equivalent and $\mathbf{n}_1, \mathbf{m}_1$ are equivalent, so {c11}

$(*)_5$ $\mathbb{P}_{\mathbf{n}} \lessdot \mathbb{P}_{\mathbf{n}_1}$.

So we are done. $\square_{2.11}$

{c49y}

The Uniqueness Claim 2.12. *There is an isomorphism from $\mathbb{P}_{\mathbf{m}_1}[M_1]$ onto $\mathbb{P}_{\mathbf{m}_2}[M_2]$ which (recalling Definition* **??***) maps $p^*_{t,\eta}$ to $p^*_{h(t),\eta}$ for $t \in M_1, \eta \in \cup\{\prod\limits_{\varepsilon<\zeta} \theta_\varepsilon : \zeta < \lambda\}$ <u>when</u>:* {c19}

$\boxplus$ (a) *$\mathbf{m}_\ell \in \mathbf{M}^{\rm ec}_\infty$ for $\ell = 1, 2$*
(b) *$M_\ell \subseteq M_{\mathbf{m}_\ell}$ for $\ell = 1, 2$*
(c) *h is an isomorphism from $\mathbf{m}_1 \restriction M_1$ onto $\mathbf{m}_2 \restriction M_2$.*

Proof. By renaming without loss of generality $M_1 = M_2$ call it M and h is the identity and $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$. For $\ell = 1, 2$ let $\mathbf{m}'_\ell$ be like $\mathbf{n}$ in 2.11 with $\mathbf{m}_\ell, M_\ell$ here standing for $\mathbf{m}, M$ there so {c46}

$(*)_1$ $\mathbf{m}'_\ell \in \mathbf{M}_{\rm fc}[\lambda'_0]$.

We define $\mathbf{m}$ by:

$(*)_2$ (a) $s \in L_{\mathbf{m}}$ iff $s \in L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$
(b) $r \le_{\mathbf{m}} t$ <u>iff</u> $r \le_{\mathbf{m}_1} t$ or $r \le_{\mathbf{m}_2} t$ or $(\exists s \in M)(r \le_{\mathbf{m}_1} s \le_{\mathbf{m}_2} t)$ or $(\exists s \in M)(r \le_{\mathbf{m}_2} s \le_{\mathbf{m}_1} t)$
(c) $u_{\mathbf{m},t}$ is $u_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \backslash M$,
is $u_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_2} \backslash M$ and
is $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$ if $t \in M$
(d) $M_{\mathbf{m}} = M$
(e) $E_{\mathbf{m}} = E'_{\mathbf{m}_1} \cup E'_{\mathbf{m}_2}$.

For $\ell = 1, 2, \mathbb{P}_{\mathbf{m}_\ell}[M] = \mathbb{P}_{\mathbf{m}'_\ell}[M]$ by 2.11 and clearly $\mathbf{m}'_\ell \le_{\mathbf{M}} \mathbf{m}$, but $\mathbf{m}'_\ell \in \mathbf{M}_{\rm fc}[\lambda'_0]$, by 2.11, hence $\mathbb{P}_{\mathbf{m}'_\ell}[M] = \mathbb{P}_{\mathbf{m}}[M]$. Together $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}'_1}[M] = \mathbb{P}_{\mathbf{m}}[M] = \mathbb{P}_{\mathbf{m}'_2}[M] = \mathbb{P}_{\mathbf{m}_2}[M]$ as required. $\square_{2.11}$ {c46} {c46}

{c51y}

Conclusion 2.13. *For every ordinal δ_* there is $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\eta}_\alpha : \alpha \le \delta_* \rangle$ such that*

(A) (a) *$\langle \mathbb{P}_\alpha : \alpha \le \delta_* \rangle$ is $\lessdot$-increasing*

modified:2017-11-05

(945) revision:2017-11-02

(b) $\underset{\sim}{\eta}_\alpha$ *is a* $\mathbb{P}_{\alpha+1}$*-name of a member of* $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ *which dominates* $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_\alpha]}$

(c) $\underset{\sim}{\eta}_\alpha$ *is a generic for* $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$*, moreover* $\langle \underset{\sim}{\eta}_\beta : \beta < \alpha\rangle$ *is a generic for* $\mathbb{P}_\alpha$

{c19} (d) *every* $p \in \mathbb{P}_\alpha$ *is from* $\mathbb{L}_{\lambda^+}(Y_{<\alpha}, \mathbb{P}_\alpha)$ *when* $Y_{<\alpha}$ *is as in Definition* **??**

(e) $\mathbb{P}_\alpha$ *is* λ*-strategically complete and* λ^+*-c.c.*

(f) *if* $\delta \leq \delta_*$ *has cofinality* $> \lambda$ *<u>then</u>* $\mathbb{P}_\delta = \cup\{\mathbb{P}_\alpha : \alpha < \delta\}$

(g) $\mathbb{P}_{\delta_*}$ *has cardinality* $|\delta_*|^\lambda$.

(B) *if* $\mathscr{U} \subseteq \delta_*$ *<u>then</u> the complete subforcing generated by* $\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}\rangle$ *is isomorphic to* $\mathbb{P}_{\mathrm{otp}(\mathscr{U})}$

(C) *if* $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$ *is generic over* $\mathbf{V}$ *and* $\eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ *for* $\alpha < \delta_*$ *and* $\eta'_\alpha \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ *for* $\alpha < \delta_*$ *and* $\{(\alpha, \varepsilon) : \alpha < \delta_*, \varepsilon < \lambda$ *and* $\eta'_\alpha(\varepsilon) \neq \eta_\alpha(\varepsilon)\}$ *has cardinality* $< \lambda$ *<u>then</u> also* $\langle \eta'_\alpha : \alpha < \delta_*\rangle$ *is a generic for* $\mathbb{P}_{\delta_*}$*, determining a different* $\mathbf{G}'$ *but* $\mathbf{V}[G'] = \mathbf{V}[\mathbf{G}]$.

Proof. We define $\mathbf{m} \in \mathbf{M}$ by:

(∗) (a) $L_{\mathbf{m}} = \delta_*$

(b) $M_{\mathbf{m}} = \delta_*$

(c) $u_{\mathbf{m},\alpha} = \alpha$ for $\alpha < \delta_*$

(d) $E_{\mathbf{m}} = \emptyset$.

{c41} It is easy to check that indeed $\mathbf{m} \in \mathbf{M}$. So by **??**(1) there is $\mathbf{n} \in \mathbf{M}^{\mathrm{ec}}_\infty$ such that
{c33} $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$. Let $\mathbb{P}_\alpha := \mathbb{P}_{\mathbf{n}}[\{\beta : \beta < \alpha\}]$, so by **??** we are done. $\square_{??}$

Similarly we can prove

modified:2017-11-05

(945) revision:2017-11-02

modified:2017-11-05

(945) revision:2017-11-02

References

[Bar87] Tomek Bartoszyński, *Combinatorial aspects of measure and category*, Fundamenta Mathematicae **127** (1987), 225–239.

[Bar10] ______, *Invariants of Measure and Category*, Handbook of Set Theory, Vols. 1, 2, 3 (Matthew Foreman and Akihiro Kanamori, eds.), Springer, Dordrect, 2010, pp. 491–555.

[Bla] Andreas Blass, *Combinatorial Cardinal Characteristics of the Continuum*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), vol. 1, Springer, pp. 395–490.

[Lan92] A. Landver, *Baire numbers, uncountable cohen sets and perfect-set forcing*, Journal of Symbolic Logic **57** (1992), 1086–1107.

[Lav78] Richard Laver, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, Israel J. of Math. **29** (1978), 385–388.

[Lav78] ______, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, Israel J. of Math. **29** (1978), 385–388.

[Mil82] Arnold W. Miller, *A characterization of the least cardinal for which the baire category theorem fails*, Proceedings of the American Mathematical Society **86** (1982), 498–502.

[vD84] Eric K. van Douwen, *The integers and topology*, Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), Elsevier Science Publishers, 1984, pp. 111–167.

[Sh:80] Saharon Shelah, *A weak generalization of MA to higher cardinals*, Israel Journal of Mathematics **30** (1978), 297–306.

[JdSh:292] Jaime Ihoda (Haim Judah) and Saharon Shelah, *Souslin forcing*, The Journal of Symbolic Logic **53** (1988), 1188–1207.

[Sh:326] Saharon Shelah, *Vive la différence I: Nonisomorphism of ultrapowers of countable models*, Set Theory of the Continuum, Mathematical Sciences Research Institute Publications, vol. 26, Springer Verlag, 1992, arxiv:math.LO/9201245, pp. 357–405.

[CuSh:541] James Cummings and Saharon Shelah, *Cardinal invariants above the continuum*, Annals of Pure and Applied Logic **75** (1995), 251–268, arxiv:math.LO/9509228.

[Sh:546] Saharon Shelah, *Was Sierpiński right? IV*, Journal of Symbolic Logic **65** (2000), 1031–1054, arxiv:math.LO/9712282.

[MtSh:804] Pierre Matet and Saharon Shelah, *Positive partition relations for $P_\kappa(\lambda)$*, Preprint, arxiv:math.LO/0407440.

[Sh:F979] Saharon Shelah, *Iterating reasonable λ-complete definable forcing*.

[Sh:1004] ______, *A parallel to the null ideal for inaccessible λ*, Archive for Mathematical Logic **submitted**, arxiv:1202.5799.

[Sh:1036] ______, *Forcing axioms for λ-complete μ^+-c.c.*, preprint, arxiv:math.LO/1310.4042.

[Sh:1126] ______, *Corrected Iteration*, Sarajevo Journal of Mathematics **submitted**.

[Sh:F1199] ______, *The null ideal for uncountable cardinals*.

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

E-mail address: `shelah@math.huji.ac.il`

URL: `http://shelah.logic.at`