

ON  $\text{CON}(\mathfrak{d}_\lambda > \text{COV}_\lambda(\text{MEAGRE}))$   
SH945

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ABSTRACT. We prove the consistency of: for suitable strongly inaccessible cardinal  $\lambda$  the dominating number, i.e., the cofinality of  ${}^\lambda\lambda$ , is strictly bigger than  $\text{cov}_\lambda(\text{meagre})$ , i.e. the minimal number of nowhere dense subsets of  ${}^\lambda 2$  needed to cover it. This answers a question of Matet.

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## § 0. INTRODUCTION

Cardinal invariants of the continuum have a long tradition of research. For a topologist, it can be viewed as investigating the space  $\beta(\omega)$ , the Stone-Čech compactification of  $\omega$ . This point of view is taken, for example, in the celebrated paper of van Douwen [vD84]. For the set theoretic perspective see the recent excellent surveys, Blass [Bla], Bartoszyński [Bar10].

For set theorists, it is interesting to check the relationship between the relevant cardinal invariants. In this context, it is natural to generalize the problems to higher cardinals, above  $\aleph_0$ . One finds out, very soon, that for the class of (strongly) inaccessible cardinals, the generalizations are more reasonable and have more affinity to the  $\aleph_0$  case. See Landver [Lan92], Cummings-Shelah [CuSh:541], Matet-Shelah [MtSh:804].

We shall define three cardinal invariants (but the paper deals, actually, just with two of them):

{z1}

**Definition 0.1.** The bounding and dominating numbers.

Let  $\lambda$  be an inaccessible cardinal.

Let  $f, g \in {}^\lambda\lambda$

- (a)  $f \leq^* g$  if  $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$
- (b)  $A \subseteq {}^\lambda\lambda$  is unbounded if there is no  $h \in {}^\lambda\lambda$  so that  $f \in A \Rightarrow f \leq^* h$
- (c)  $A \subseteq {}^\lambda\lambda$  is dominating when for every  $f \in {}^\lambda\lambda$  there exists  $g \in A$  so that  $f \leq^* g$
- (d) the bounding number for  $\lambda$ , denoted by  $\mathfrak{b}_\lambda$ , is  $\min\{|A| : A \text{ is unbounded in } {}^\lambda\lambda\}$
- (e) the dominating number for  $\lambda$ , denoted by  $\mathfrak{d}_\lambda$ , is  $\min\{|A| : A \text{ is dominating in } {}^\lambda\lambda\}$ .

Notice that the usual definitions of  $\mathfrak{b}$  and  $\mathfrak{d}$  are  $\mathfrak{b}_{\aleph_0}$  and  $\mathfrak{d}_{\aleph_0}$  according to Definition 0.1. The definition of  $\text{cov}_\lambda(\text{meagre})$  involves some topology.

{cov.1}

**Definition 0.2.** The meagre covering number.

Let  $\lambda$  be a regular cardinal

- (a)  ${}^\lambda 2$  is the space of functions from  $\lambda$  into 2
- (b)  $({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$ , for  $\nu \in {}^{\lambda >} 2 := \bigcup_{\alpha < \lambda} {}^\alpha 2$
- (c)  $\mathcal{U} \subseteq {}^\lambda 2$  is open in the topology  $({}^\lambda 2)_{< \lambda}$ , iff for every  $\eta \in \mathcal{U}$  there exists  $i < \lambda$  so that  $({}^\lambda 2)^{[\eta \upharpoonright i]} \subseteq \mathcal{U}$
- (d)  $\text{cov}_\lambda(\text{meagre})$  is the minimal cardinality of a family of meagre subsets of  $({}^\lambda 2)_{< \lambda}$ , which covers this space.

This paper deals with the relationship between  $\mathfrak{d}_\lambda$  and  $\text{cov}_\lambda(\text{meagre})$ . Matet asked (a personal communication) whether  $\mathfrak{d}_\lambda \leq \text{cov}_\lambda(\text{meagre})$  is provable in ZFC. We give here a negative answer.

For  $\lambda$  a supercompact cardinal and  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , we force large  $\mathfrak{d}_\lambda$  i.e.,  $\mathfrak{d}_\lambda = \mu$  and small covering number (i.e.,  $\text{cov}_\lambda(\text{meagre}) = \kappa$ ). A similar result should hold also for a wider class of cardinals and we intend to return elsewhere to this subject.

A point which in a previous version was just a step along the way, the referee asked to justify fully. This was done but eventually is separated to [?]. A posteriori the point is that in the parallel case for  $\lambda = \aleph_0$ , for full memory FS iteration such a claim is true. In fact, by Judah-Shelah [JdSh:292], if  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$  is FS iteration of Suslin-c.c.c. forcing notion,  $\mathbb{Q}_\beta$  with the generic  $\eta_\beta \in {}^\omega\omega$  and for notational transparency, its definition is with no parameter and  $\zeta : \beta(*) \rightarrow \alpha(*)$  is increasing and  $\mathbb{P} = \langle \mathbb{P}'_\alpha, \mathbb{Q}'_\beta : \alpha \leq \beta(*), \beta < \beta(*) \rangle$  is FS iteration, but  $\mathbb{Q}'_\beta$  is defined exactly as  $\mathbb{Q}_{\zeta(\beta)}$  but now in  $\mathbf{V}^{\mathbb{P}'_\beta}$  rather than in  $\mathbf{V}^{\mathbb{P}_{\zeta(\beta)}}$  then  $\Vdash_{\mathbb{P}_{\alpha(*)}} \langle \eta_{\zeta(\beta)} : \beta < \beta(*) \rangle$  is generic for  $\mathbb{P}'_{\beta(*)}$  over  $\mathbf{V}$ .

Now this is not clear to us for  $(< \lambda)$ -support iteration of  $(< \lambda)$ -strategically complete forcing notions. The solution is essentially to change the iteration: to use a “quite generic”  $(< \lambda)$ -support iteration which “includes” the one we like and use the complete subforcing it generates; see [Sh:1126].

We try to use standard notation. We use  $\theta, \kappa, \lambda, \mu, \chi$  for cardinals and  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$  for ordinals. We use also  $i$  and  $j$  as ordinals. We adopt the Cohen convention that  $p \leq q$  means that  $q$  gives more information, in forcing notions. The symbol  $\triangleleft$  is preserved for “being an initial segment”. Also recall  ${}^B A = \{f : f \text{ a function from } B \text{ to } A\}$  and let  ${}^{\alpha>} A = \cup \{{}^\beta A : \beta < \alpha\}$ , some prefer  ${}^{<\alpha} A$ , but  ${}^{\alpha>} A$  is used systematically in the author’s papers. Lastly,  $J_\lambda^{\text{bd}}$  denotes the ideal of the bounded subsets of  $\lambda$ .

The picture of cardinal invariants related to uncountable  $\lambda$  is related but usually quite different than the one for  $\aleph_0$ , they are more similar if  $\kappa$  is “large” enough, mainly strongly inaccessible.

Let us sketch some known results. These results are related to the inequality number and the covering number for category. Recall:

{z14}

**Definition 0.3.** The inequality number.

Let  $\kappa$  be an infinite cardinal. The inequality number of  $\kappa, \mathfrak{c}_\kappa$ , is the minimal cardinal  $\lambda$  such that there is a set  $\mathcal{F} \subseteq {}^\kappa \kappa$  of cardinality  $\lambda$  such that there is no  $g \in {}^\kappa \kappa$  satisfying  $(\forall f \in \mathcal{F})(\exists^\kappa \alpha < \kappa)(f(\alpha) = g(\alpha))$ .

For  $\kappa = \aleph_0, \mathfrak{c}_\kappa = \text{cov}_{\aleph_0}(\text{meagre})$ ; see Bartoszyński (in [Bar87]) and Miller (in [Mil82]).

Now

- (a) the statement  $\mathfrak{c}_\kappa = \text{cov}_\kappa(\text{meagre})$  is valid for  $\kappa > \aleph_0$ , in the case that  $\kappa$  is strongly inaccessible, by [Lan92]. But if  $\kappa$  is a successor cardinal, it may fail
- (b) if  $\kappa < \kappa^{<\kappa}$ , then  $\text{cov}_\kappa(\text{meagre}) = \kappa^+$ . This is due to Landver (in [Lan92]).

We intend also to address:

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**Problem 0.4.** Can we replace “supercompact” by “strongly inaccessible”?

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**Problem 0.5.** 1) Can we prove the consistency of  $\text{cov}_\lambda(\text{meagre}) < \mathfrak{b}_\lambda$ ?

2) For  $\lambda$  strongly inaccessible (or just Laver indestructible supercompact) is there a non-trivial  $\lambda^+$ -c.c.  $(< \lambda)$ -strategically complete forcing notion  $\mathbb{Q}$  which is  ${}^\lambda \lambda$ -bounding?

We thank the referee, Shimoni Garti and Haim Horowitz for helpful comments. We say more in subsequent works [Sh:1004] and in preparation [Sh:F1199].

Recall

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**Definition 0.6.** Let  $\kappa$  be supercompact. We say  $f : \kappa \rightarrow \mathcal{H}(\kappa)$  is a Laver diamond (for  $\kappa$ ) when for every  $x \in \mathbf{V}$  there are a normal fine ultrafilter on  $I = [\lambda]^{<\kappa}$  for some  $\lambda$  such that the Mostowski collapse  $\mathbf{j}$  on  $\mathbf{V}^I/D$  maps  $\langle f(\sup(u \cap \kappa)) : u \in I \rangle / D$  to  $x$ ; (we can use elementary embeddings instead of an ultrafilter).

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*Notation 0.7.* If  $\mathbb{P}$  is a forcing notion in  $\mathbf{V}$  then  $\mathbf{V}^{\mathbb{P}}$  denotes  $\mathbf{V}[\mathbf{G}]$  for  $\mathbf{G} \subseteq \mathbb{P}$  generic over  $\mathbf{V}$ ; but in superscript we may write  $\mathbf{V}[\mathbb{P}]$  instead.

{z20}

The parallel to dominating = Hechler forcing is:

**Definition 0.8.** 1) Fix  $\lambda = \lambda^{<\lambda}$ , the forcing  $\mathbb{Q} = \mathbb{Q}_\lambda^{\text{dom}}$  is defined by:

- ( $\alpha$ )  $p \in \mathbb{Q}$  iff
  - (a)  $p = (\eta, f) = (\eta^p, f^p)$
  - (b)  $\eta \in {}^\varepsilon \lambda$  for some  $\varepsilon < \lambda$ , ( $\eta$  is called the trunk of  $p$ )
  - (c)  $f \in {}^\lambda \lambda$
  - (d)  $\eta \triangleleft f$
- ( $\beta$ )  $p \leq_{\mathbb{Q}} q$  iff
  - (a)  $\eta^p \leq \eta^q$
  - (b)  $f^p \leq f^q$ , i.e.  $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$
  - (c) if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$ .

2) The generic is  $\eta = \cup \{\eta^p : p \in \mathbf{G}_{\mathbb{Q}}\}$ .

For transparency

{z23x}

{z23}

**Convention 0.9.** Below  $\lambda, \bar{\theta}$  are as in 0.10 below.

A parallel to the forcing in [Sh:326], really a c.c.c. version of it is.

{z23}

**Definition 0.10.** Let  $\lambda$  be inaccessible,  $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$  be a sequence of regular cardinals  $< \lambda$  satisfying  $\theta_\varepsilon > \varepsilon$ .

1) We define the forcing notion  $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}$  by

- ( $\alpha$ )  $p \in \mathbb{Q}$  iff
  - (a)  $p = (\eta, f) = (\eta^p, f^p)$
  - (b)  $\eta \in \prod_{\zeta < \varepsilon} \theta_\zeta$  for some  $\varepsilon < \lambda$ , ( $\eta$  is called the trunk of  $p$ )
  - (c)  $f \in \prod_{\zeta < \lambda} \theta_\zeta$
  - (d)  $\eta \triangleleft f$
- ( $\beta$ )  $p \leq_{\mathbb{Q}} q$  iff
  - (a)  $\eta^p \leq \eta^q$
  - (b)  $f^p \leq f^q$ , i.e.  $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$
  - (c) if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$ .

2) The generic is  $\eta = \cup \{\eta^p : p \in \mathbf{G}_{\mathbb{Q}_{\bar{\theta}}}\}$ .

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*Remark 0.11.* The forcing is parallel to the creature forcing from [Sh:326, §1,§2] which are  ${}^\omega \omega$ -bounding.

<sup>1</sup>Actually this follows from ( $\alpha$ ) + ( $\beta$ )(a), (b); that is, if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^q(\varepsilon) = f^q(\varepsilon)$  by clause ( $\alpha$ )(d) and  $f^p(\varepsilon) \leq f^q(\varepsilon)$  by clause ( $\beta$ )(b).

{z32}

**Definition 0.12.** For an ordinal  $\alpha_* = \alpha(*)$  let  $\mathbf{Q}_{\lambda, \bar{\theta}, \alpha(*)}$  be the class of objects  $\mathbf{q}$  consisting of (omitting  $\alpha_*$  means for some  $\alpha_*$  and  $\ell g(\mathbf{q}) = \alpha_{\mathbf{q}} = \alpha_*$ ):

- (a)  $\bar{u} = \langle u_\alpha : \alpha < \alpha_* \rangle$  and  $\mathcal{P} = \langle \mathcal{P}_\alpha : \alpha < \alpha_* \rangle$  where  $\mathcal{P}_\alpha \subseteq [u_\alpha]^{\leq \lambda}$ ,  $u_\alpha \subseteq \alpha$ , without loss of generality  $\mathcal{P}_\alpha$  is closed under subsets (but is not necessarily an ideal)
- (b)  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha_*, \beta < \alpha_* \rangle$  is a  $(< \lambda)$ -support iteration let  $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\mathbf{q}, \alpha(\mathbf{q})}$  and  $\mathbb{P}_{0, \alpha} = \mathbb{P}_\alpha, \mathbb{Q}_{0, \alpha} = \mathbb{Q}_\alpha$
- (c) each of  $\mathbb{P}_\alpha$  is strategically  $(< \lambda)$ -complete and  $\lambda^+$ -c.c.
- (d)  $\eta_\beta \in \Pi \bar{\theta}$  is the generic of  $\mathbb{Q}_\beta$  where  $\eta_\beta$ , the generic of  $\mathbb{Q}_p$  (defined in clause (e) below) is  $\cup \{ \eta_p : p \in \mathbf{G}_{\mathbb{Q}_\beta} \}$
- (e) if  $\mathbf{G} \subseteq \mathbb{P}_\beta$  is generic over  $\mathbf{V}$  then  $\eta_\alpha[\mathbf{G}]$  in  $(\Pi \bar{\theta}, <_{J_\lambda^{\text{bd}}})$  dominate every  $\nu \in \mathbf{V}[\langle \eta_\gamma : \gamma \in u \rangle]$  when  $u \in \mathcal{P}_\alpha$ ; moreover, in  $\mathbf{V}[\mathbf{G}]$ ,  $\mathbb{Q}_\beta[\mathbf{G}]$  is the subforcing of  $\mathbb{Q}_\beta$  consisting of the  $p \in \mathbb{Q}_\beta$  such that: for some  $\bar{s}, \bar{f}, \eta_p$  (so  $\eta_p = \eta$ , etc.) we have
  - ( $\alpha$ )  $p = (\eta, f) = (\eta_p, f_p)$  so  $\eta \in \prod_{\varepsilon < \zeta} \theta_\varepsilon$  for some  $\zeta < \lambda$
  - ( $\beta$ )  $\bar{s} = \langle (u_i, f_i) : i < i_* \rangle$
  - ( $\gamma$ )  $i_* < \lambda, u_i \in \mathcal{P}_\beta, \eta \triangleleft f_i \in \Pi \bar{\theta}$  and  $f_i \in \mathbf{V}[\langle \eta_\gamma[\mathbf{G}] : \gamma \in u_i \rangle]$
  - ( $\delta$ )  $f = \sup \{ f_i : i < i_* \}$ , i.e.  $\varepsilon < \lambda \Rightarrow f(\varepsilon) = \cup \{ f_i(\varepsilon) : i < i_* \}$
- (f) for  $\alpha \leq \alpha_*$ ,  $\mathbb{P}_{2, \alpha}$  is the completion of  $\mathbb{P}_\alpha$ ; we can express it via transforming  $\mathbb{P}_\alpha$  to a complete Boolean Algebra, or say:
  - (\*)<sub>1</sub> the elements of  $\mathbb{P}_{2, \alpha}$  are of the form  $\mathbf{B}(\dots, \eta_\gamma, \dots)_{i < \ell(*)}$  where:
    - ( $\alpha$ )  $i(*) \leq \lambda$
    - ( $\beta$ )  $\gamma_i \in \mathcal{U}$  for  $i < i_*$
    - ( $\gamma$ )  $\mathbf{B}$  is a  $\lambda$ -Borel function from  ${}^{i(*)}(\Pi \bar{\theta})$  into  $\{0, 1\} = \{\text{false}, \text{true}\}$ ;  
 $\mathbf{B}$  is from  $\mathbf{V}$ , of course, such that  $\Vdash_{\mathbb{P}_{\mathbf{q}}} \text{“} \mathbf{B}(\dots, \eta_{\gamma_i}, \dots)_{i < i(*)} = 0 \text{”}$
  - (\*)<sub>2</sub> the order is natural:  $\mathbb{P}_{2, \alpha} \models \text{“} \mathbf{B}_1(\dots, \eta_{\gamma(i,1)}, \dots)_{i < i(*)} \leq \mathbf{B}_2(\dots, \eta_{\gamma(i,2)}, \dots)_{i < i(2)} \text{”}$   
 iff  $\Vdash_{\mathbb{P}_\alpha} \text{“} \text{if } \mathbf{B}_2(\dots, \eta_{\gamma(i,2)}[\mathbf{G}, \dots]_{i < i(1)} \text{ is equal to 1 then so is } \mathbf{B}_1(\dots, \eta_{\gamma(i,1)}, \dots)_{i < i(1)} \text{”}$
- (g) for  $\mathcal{U} \subseteq \alpha_*$  let  $\mathbb{P}_{\mathcal{U}}$  be the subforcing of  $\mathbb{P}_{2, \alpha(\mathbf{q})}$  consist of  $\{ \mathbf{B}(\dots, \eta_{\gamma(i)}, \dots)_{i < i(*)} \in \mathbb{P}_{\alpha(\mathbf{q})} : i(*) \leq \lambda \text{ and } \gamma_i \in \mathcal{U} \text{ for every } i < i(*) \}$ .

{z35}

**Claim 0.13.** 1) For any sequence  $\langle u_\alpha, \mathcal{P}_\alpha : \alpha < \alpha_* \rangle$  as above, i.e. as in clause (a) of Definition 0.12, there is one and only one  $\mathbf{q}$  as above and the  $\mathbb{P}_{\mathbf{q}, \mathcal{U}}$ 's are as demanded.

{z32}

2) For every  $\alpha \leq \alpha_*$  the set  $\mathbb{P}_\alpha^\bullet$  of  $p \in \mathbb{P}_\alpha$  satisfying the following is dense:

- (a)  $\eta_p, i_p, \langle u_{p,i} : i < i_p \rangle$  are objects (not just  $\mathbb{P}_\alpha$ -names)
- (b) each  $f_i$  has the form  $\mathbf{B}(\dots, \eta_{\gamma(i,1)}, \dots)_{j < j(*) \leq \lambda}$  where  $\{ \gamma(i, j) : j < j(i) \} \subseteq u_{p,i}$ .

3) Above for every  $v \subseteq \alpha$  and  $j_* < \lambda$  the set of  $p \in \mathbb{P}_\alpha^\bullet$  such that  $v \subseteq \text{dom}(p) \wedge (\forall \beta \in \text{dom}(p)) (\ell g(\eta_{p(\beta)}) > j_*)$  is dense.

4)  $\mathbb{P}_{\mathbf{q}, 1, \alpha} \triangleleft \mathbb{P}_{\mathbf{q}, 2, \alpha}$  moreover  $\mathbb{P}_{\mathbf{q}, 1, \alpha}$  is dense in  $\mathbb{P}_{\mathbf{q}, 2, \alpha}^\bullet$  and  $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \alpha_{\mathbf{q}} \Rightarrow \mathbb{P}_{\mathbf{q}, \mathcal{U}_1} \triangleleft \mathbb{P}_{\mathbf{q}, \mathcal{U}_2} \triangleleft \mathbb{P}_{\mathbf{q}, \alpha}$  so  $\mathbb{P}_{\mathbf{q}, \{ \beta : \beta < \alpha \}} = \mathbb{P}_{\mathbf{q}, 2, \alpha}$  and  $|\mathbb{P}_{\mathbf{q}, \mathcal{U}}| \leq |\mathcal{U}|^\lambda$ .

5) If  $\alpha < \alpha_*$  and  $u \in \mathcal{P}_\alpha$  then  $\eta_\alpha \in \Pi \bar{\theta}$  dominate every  $\nu \in (\Pi \bar{\theta})^{\mathbf{V}[\bar{\eta} \upharpoonright u]}$ .

6) Assume  $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q}}$  is generic over  $\mathbf{V}$ ,  $\eta_\alpha = \eta_\alpha[\mathbf{G}]$  and  $\eta'_\alpha \in (\Pi\bar{\theta})^{\mathbf{V}[\mathbf{G}]}$  for  $\alpha < \alpha_*$  and  $\{(\alpha, \varepsilon) : \alpha < \alpha_*, \varepsilon < \alpha \text{ and } \eta_\alpha(\varepsilon) \neq \eta'_\alpha(\varepsilon)\}$  has cardinality  $< \lambda$ . Then for some (really unique)  $\mathbf{G}'$  we have  $\mathbf{G}' \subseteq \mathbb{P}_{\mathbf{q}}$  is generic over  $\mathbf{V}$  and  $\alpha$ .

*Proof.* See [?].

□<sub>0.13</sub>

{z38}

**Theorem 0.14.** For any ordinal  $\alpha_*$  there is a quadruple  $(\mathbf{q}, \delta_*, \mathcal{U}, h)$  such that:

- (A) (a)  $\mathbf{q} \in \mathbf{Q}_{\lambda, \bar{\theta}}$  and let  $\delta_* = \text{lg}(\mathbf{q})$
- (b)  $\mathcal{U} \subseteq \delta_*$  has order type  $\alpha_*$
- (c)  $h$  is the order preserving function from  $\alpha_*$  onto  $\mathcal{U}$
- (d) if  $\alpha \in \mathcal{U}$  then  $\mathcal{U} \cap \alpha \in \mathcal{P}_{\mathbf{Q}, \alpha}$
- (B) if  $\mathcal{U}_1 \subseteq \mathcal{U}$ ,  $\mathcal{U}_2 \subseteq \mathcal{U}$  is an initial segment of  $\mathcal{U}$ ,  $\text{otp}(\mathcal{U}_1) = \text{otp}(\mathcal{U}_2)$  and  $g$  is the order preserving function from  $\mathcal{U}_1$  onto  $\mathcal{U}_2$ , then  $g$  induces an isomorphism  $\hat{g}$  from  $\mathbb{P}_{\mathbf{q}, \mathcal{U}_2}$  onto  $\mathbb{P}_{\mathbf{q}, \mathcal{U}_1}$  mapping  $\eta_\beta$  to  $\eta_{g(\beta)}$  for  $\beta \in \mathcal{U}_2$ .

*Proof.* By [Sh:1126].

□<sub>0.14</sub>

## § 1. THE FORCING

We intend to prove here

**Theorem 1.1.** *Assume*

- (a)  $\lambda$  is supercompact
- (b)  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \text{cf}(\mu) = \mu^\lambda$ .

Then for some forcing notion  $\mathbb{P}$  not collapsing cardinals  $\geq \lambda$ ,  $\lambda$  is still supercompact in  $\mathbf{V}^{\mathbb{P}}$  and  $\text{cov}_\lambda(\text{meagre}) = \kappa, \mathfrak{d}_\lambda = \mu$ .

*Proof.* By 1.3 below.

Recall

**Definition 1.2.** 1) We say that a forcing notion  $\mathbb{P}$  is  $\alpha$ -strategically complete when for each  $p \in \mathbb{P}$  in the following game  $\mathfrak{D}_\alpha(p, \mathbb{P})$  between the players COM and INC, the player COM has a winning strategy.

A play lasts  $\alpha$  moves; in the  $\beta$ -th move, first the player COM chooses  $p_\beta \in \mathbb{P}$  such that  $p \leq_{\mathbb{P}} p_\beta$  and  $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$  and second the player INC chooses  $q_\beta \in \mathbb{P}$  such that  $p_\beta \leq_{\mathbb{P}} q_\beta$ .

The player COM wins a play if he has a legal move for every  $\beta < \alpha$ .

2) We say that a forcing notion  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete when it is  $\alpha$ -strategically complete for every  $\alpha < \lambda$ .

**Lemma 1.3.** 1) If  $\lambda$  is supercompact then after some preliminary forcing of cardinality  $\lambda$ ,  $\lambda$  is still supercompact and  $\square_\lambda$  below holds.

1A) The statement  $\square_\lambda$  holds in  $\mathbf{V}^{\mathbb{P}}$  when  $\mathbf{V}$  satisfies  $\square_\lambda$  and  $\mathbb{P}$  is a  $(< \lambda)$ -strategically complete forcing notion and in  $\mathbb{P}$  any directed system of cardinality  $< \lambda$  of conditions in  $\mathbb{P}$  has an upper bound.

2) If  $\lambda$  is strongly inaccessible and  $\square_\lambda$  below holds and  $\lambda^+ < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$  and  $\mu$  is regular (for transparency), then for some  $\lambda^+$ -c.c.,  $(< \lambda)$ -strategically complete forcing notion  $\mathbb{P}$  we have  $\Vdash_{\mathbb{P}} \text{“}\mathfrak{d}_\lambda = \mu, \text{cov}_\lambda(\text{meagre}) = \kappa\text{”}$

**Definition 1.4.** For  $\lambda$  supercompact we define  $\square_\lambda$  by:

$\square_\lambda$  for any regular cardinal  $\chi > \lambda$  and forcing notion  $\mathbb{P} \in \mathcal{H}(\chi)$  which is  $(< \lambda)$ -strategically complete (see Definition 1.2(2)) the following set  $\mathcal{S} = \mathcal{S}_{\mathbb{P}} = \mathcal{S}_{\chi, \mathbb{P}}$  is a stationary subset of  $[\mathcal{H}(\chi)]^{< \lambda}$ :

- $\mathcal{S} = \mathcal{S}_{\mathbb{P}} = \mathcal{S}_{\chi, \mathbb{P}}$  is the set of  $N$ 's such that for some  $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N, \mathbb{A} = \mathbb{A}_N, M = M_N, \mathbf{G} = \mathbf{G}_N$  we have (and we may say  $(\lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$  is a witness for  $N \in \mathcal{S}_{\chi, \mathbb{P}}$  or for  $(N, \mathbb{P}, \chi)$ ):
- (a)  $N \prec (\mathcal{H}(\chi)^{\mathbf{V}}, \in)$  and  $\mathbb{P} \in N$
- (b) the Mostowski collapse of  $N$  is  $\mathbb{A}$  and let  $\mathbf{j}_N : N \rightarrow \mathbb{A}$  be the unique isomorphism
- (c)  $N \cap \lambda = \lambda_N$  and  $(\lambda_N)^{> N} \subseteq N$  and  $\lambda_N$  is strongly inaccessible
- (d)  $\mathbb{A} \subseteq M := (\mathcal{H}(\chi_N), \in)$  so both  $\mathbb{A}$  and  $M$  are transitive
- (e)  $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$  is generic over  $\mathbb{A}$  for the forcing notion  $\mathbf{j}_N(\mathbb{P})$
- (f)  $M = \mathbb{A}[\mathbf{G}]$ .

Recall that:

{a7.3}

**Fact 1.5.**

- (a) if  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \delta, \beta < \delta \rangle$  is a  $(< \lambda)$ -support iteration of  $(< \lambda)$ -strategically complete forcing notions, then  $\mathbb{P}_\delta$  is also  $< \lambda$ -strategically complete; (see e.g. [Sh:546]).
- (b) If  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete forcing notion then  $({}^\lambda \text{Ord})^{\mathbf{V}} = ({}^\lambda \text{Ord})^{\mathbf{V}^{\mathbb{P}}}$ , and consequently  $\lambda$  is strongly inaccessible in  $\mathbf{V}^{\mathbb{P}}$ .
- (c) like (a) replacing  $(< \lambda)$ -strategically complete" by " $(< \lambda)$ -complete"
- (d) if  $\mathbb{P}$  is  $(< \lambda)$ -complete then  $\mathbb{P}$  is  $\lambda$ -strategically complete.

*Proof.* Proof of Lemma 1.3 1) This is essentially by Laver [Lav78] using Laver's diamond, see Definition 0.6, but for completeness we elaborate. By Laver [Lav78] without loss of generality there is a Laver diamond  $h : \lambda \rightarrow \mathcal{H}(\lambda)$ , see Definition 0.6. Let  $E = \{\theta : \theta < \lambda \text{ and } \alpha < \theta \Rightarrow h(\alpha) \in \mathcal{H}(\theta)\}$ , clearly a club of  $\lambda$  and let  $\langle \kappa_\varepsilon : \varepsilon < \lambda \rangle$  list  $\{\theta \in E : \theta \text{ strongly inaccessible}\}$  in increasing order.

We now define  $\mathbf{q}_\varepsilon$  and  $\bar{\chi}^\varepsilon$  by induction on  $\varepsilon \leq \lambda$  such that:

- (\*) (a)  $\mathbf{q}_\varepsilon = \langle \mathbb{P}_\zeta, \mathbb{Q}_\xi : \zeta \leq \varepsilon, \xi < \varepsilon \rangle$  is an Easton support iteration (so  $\mathbb{P}_\zeta, \mathbb{Q}_\xi$  do not depend on  $\varepsilon$ , etc.)
- (b)  $\mathbb{P}_\zeta \subseteq \mathcal{H}(\kappa_\zeta)$
- (c)  $\bar{\chi}^\varepsilon = \langle \chi_\xi : \zeta < \xi \rangle$  where each  $\chi_\xi$  is a regular cardinal  $\in [\kappa_\xi, \kappa_{\xi+1})$
- (d)  $\mathbb{Q}_\xi \in \mathcal{H}(\chi_{\xi+1})$  is a  $\mathbb{P}_\xi$ -name of a  $(< \kappa_\xi)$ -strategically complete forcing notion
- (e) if  $h(\xi) = (\mathbb{Q}, \chi)$  and the pair  $(\mathbb{Q}, \chi)$  satisfies the requirements on  $(\mathbb{Q}_\xi, \chi_\xi)$  in clauses (c),(d) then  $(\mathbb{Q}_\xi, \chi_\xi) = h(\xi)$ .

Easily we can carry the induction so  $\mathbf{q}_\lambda$  is well defined,  $\mathbb{P}_\lambda \subseteq \mathcal{H}(\lambda)$  and " $\xi < \lambda \Rightarrow \mathbb{P}_\lambda / \mathbb{P}_\xi$  is  $(< \kappa_\xi)$ -strategically complete" hence  $\mathbb{P}_\lambda / \mathbb{P}_\xi$  adds no new sequence of length  $< \kappa_\xi$  of ordinals. Clearly it is enough to prove that in  $\mathbf{V}^{\mathbb{P}_\lambda}$  we have  $\square_\lambda$ .

Toward contradiction assume  $\chi, \mathbb{P}, \mathcal{S} = \mathcal{S}_{\chi, \mathbb{P}}$  forms a counter-example in  $\mathbf{V}^{\mathbb{P}_\lambda}$ , hence there are  $p_* \in \mathbb{P}_\lambda$  and  $\mathbb{P}_\lambda$ -name  $\chi, \mathbb{P}, \mathcal{S}, \bar{E}$  such that  $p_* \Vdash_{\mathbb{P}_\lambda}$  " $\chi > \lambda$  is regular,  $\mathbb{P} \in \mathcal{H}(\chi)$  and  $\mathcal{S}_{\chi, \mathbb{P}}$  is defined as in  $\square_\lambda$  and  $\bar{E} \subseteq [\mathcal{H}(\chi)^{\mathbf{V}^{\mathbb{P}_\lambda}}]^{< \lambda}$  is a club disjoint to  $\mathcal{S}$ ".

As we can increase  $p_*$ , without loss of generality  $\chi = \chi$ ; and as  $\mathbf{V} \models$  " $\lambda$  is supercompact and  $h$  is a Laver diamond" for some  $(\mathbf{M}, \mathbf{j})$  we have

- (\*) (a)  $\mathbf{M}$  is a transitive class
- (b)  $\mathbf{M}$  is a model of ZFC
- (c)  ${}^x \mathbf{M} \subseteq M$
- (d)  $\mathbf{j}$  is an elementary embedding from  $\mathbf{V}$  into  $\mathbf{M}$
- (e)  $\mathbf{j}$  is with critical cardinal  $\lambda$
- (f)  $\mathbf{j}(h)(\lambda) = (\mathbb{P}, \chi)$ .

Let  $\mathbf{q} = \mathbf{j}(\mathbf{q}_\lambda)$  so  $\mathbf{q} = \langle \mathbb{P}_\zeta, \mathbb{Q}_\xi : \zeta \leq h(\lambda), \xi < h(\lambda) \rangle$  and  $\zeta < \lambda \Rightarrow \mathbb{P}_\zeta^{\mathbf{q}} = \mathbb{P}_\zeta$ , etc.

Clearly  $\mathbf{M} \models$  " $\mathbf{j}(h)(\lambda)$  is a pair of the form  $(\mathbb{P}', \chi')$  satisfying all relevant demands".

The rest should be clear.



1A) Let  $\mathbb{Q}$  be a forcing notion in  $\mathbf{V}^{\mathbb{P}}$  which is  $(< \lambda)$ -strategically complete,  $\emptyset \in \mathbb{Q}$  minimal,  $\chi_1$  large enough so that  $\lambda, \mathbb{Q}, \underline{E} \in \mathcal{H}(\chi_1)$  and we should prove that in  $\mathbf{V}^{\mathbb{P}}$ , the set  $\mathcal{S}_{\chi_1}, \mathbb{Q}$  is stationary. So let  $\mathbb{Q}, \underline{E}$  be  $\mathbb{P}$ -names such that for some  $p \in \mathbb{P}$  we have  $p \Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{H}(\chi_1) \text{ is } (< \lambda)\text{-strategically complete forcing notion, } \underline{E} \text{ a club of } [\mathcal{H}(\chi_1)]^{< \lambda} \text{ disjoint to } \mathcal{S}_{\chi_1, \mathbb{P}}\text{”}$ , no need to use a name for  $\chi_1$  as we can increase  $p$ .

Let  $\chi \gg \chi_1$ ; now  $\mathbb{P} * \mathbb{Q} \in \mathcal{H}(\chi)$  is a  $(< \lambda)$ -strategically complete forcing notion and without loss of generality code  $(\chi_1, E)$ . As  $\square_\lambda$  holds in  $\mathbf{V}$  we can apply it to the forcing  $\mathbb{P}_{\geq p} * \mathbb{Q}$  so we can find a tuple  $(N, \lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$  witnessing it, in particular,  $(p, \emptyset) \in \mathbf{G}_N, \mathbb{P} * \mathbb{Q} \in N$  so  $\chi_1, \underline{E} \in N$ . Let  $\mathbf{G}_{\mathbb{P}}$  be a subset of  $\mathbb{P}$  generic over  $\mathbf{V}$  which extends  $\{p' : (p', q') \in \mathbf{G}_N\}$ , possible because  $\mathbf{G}_N$  is in  $\mathbf{V}$ , a subset of  $\mathbb{P}$  which has an upper bound. Next let  $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_{\mathbb{P}}], N_1 = N[\mathbf{G}_{\mathbb{P}}], E_1 = \underline{E}[\mathbf{G}_{\mathbb{P}}], \mathbb{A}_1 = \mathbb{A}[\mathbf{j}''_N(\mathbf{G}_{\mathbb{P}} \cap N)] = \mathbb{A}[\{p' : (p', q') \in \mathbf{G}_N\}], \mathbf{G}_1 = \{q[\mathbf{j}''(G_{\mathbb{P}} \cap N)] : (p, q) \in \mathbf{G}_{\mathbb{P}}\}$ .

Let  $N_2 = N_1 \upharpoonright \mathcal{H}(\chi_1)^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}, \mathcal{S} = \mathcal{S}_{\mathbb{Q}}[\mathbf{G}_{\mathbb{P}}], \mathbf{j}_1$  = the lifting of  $(\mathbf{j} \upharpoonright (N \cap \mathcal{H}(\chi)))$ , to mapping  $N_1$  onto  $\mathbb{A}_1$ .

Now recalling  $p$  forces  $\underline{E}$  is disjoint to  $\mathcal{S}$  clearly

$$(*) N_2 \in E.$$

hence

$$(*) N_1 \notin \mathcal{S}_1.$$

But easily in  $\mathbf{V}_1$  we have:  $(\lambda_N, \chi_N, \mathbf{j}_1, \mathbb{A}_1, M_1 = M, \mathbf{G}_1)$  witness  $N_1 \in \mathcal{S} \cap E_1$ , a contradiction to the choice of  $\underline{E}$ .

2) Stage A: Without loss of generality  $\mathbf{V} \models \text{“}\mathfrak{b}_\lambda = \mu = \mathfrak{d}_\lambda\text{”}$  as witnessed by  $\langle f_\alpha^* : \alpha < \mu \rangle$ .

No new point, still we elaborate recalling that composition of forcing notions e.g.  $(\mathbb{P} * \mathbb{Q}^*)$  preserves “ $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c.”

We use a  $(< \lambda)$ -support iteration  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu, \beta < \mu \rangle$  with  $\mathbb{P}_\mu$  the intended forcing such that:

$$(A) \text{ if } \alpha < \mu \text{ then } \mathbb{Q}_\alpha \text{ is the } (\mathbb{P}_\alpha\text{-name of the) dominating forcing, } \mathbb{Q}_\lambda^{\text{dom}}, \text{ i.e. } (\mathbb{Q}_\alpha^{\text{dom}})^{\mathbf{V}[\mathbb{P}_\alpha]} \text{ where the forcing } \mathbb{Q} = \mathbb{Q}_\alpha^{\text{dom}} \text{ is from Definition 0.8.} \quad \{\mathbf{z20}\}$$

Let  $f_\alpha^*$  be the generic object for  $\mathbb{Q}_\alpha$  for  $\alpha < \mu$ .

Now:

$$(*)_1 \text{ for } \alpha \leq \mu \text{ the forcing notion } \mathbb{P}_\alpha \text{ is } (< \lambda)\text{-complete because when } \alpha < \mu, \mathbb{Q}_\alpha \text{ is } (< \lambda)\text{-complete, i.e. } \Vdash_{\mathbb{P}_\alpha} \text{“}\mathbb{Q}_\alpha \text{ is } (< \lambda)\text{-complete”}.$$

[Why? We prove this by induction on  $\alpha$ ; for  $\mathbb{Q}_\alpha$ , recall Definition 0.8.] {\mathbf{z20}}

$$(*)_2 \text{ for each } \alpha \leq \mu$$

- (a) the forcing notion  $\mathbb{P}_\alpha$  and for  $\alpha < \mu$  satisfy a strong form of the  $\lambda^+$ -c.c., (see [Sh:80] for definition, preservation and history; or fully [Sh:1036], [Sh:546, §1])
- (b) also the forcing notions  $\mathbb{Q}_\alpha$  satisfies this

hence

- (\*)<sub>3</sub> (a) forcing with  $\mathbb{P}_\mu$  collapses no cardinal, changes no cofinality, and adds no sequence to  ${}^{\lambda>}\mathbf{V}$ ;  
 (b)  $({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\mu]} = \cup\{({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]} : \alpha < \mu\}$ .

[Why? By (\*)<sub>2</sub> + (\*)<sub>1</sub> clause (a) holds, for clause (b) use also “ $\mathbb{P}_\mu$  satisfies the  $\lambda^+$ -c.c.” and the support in the iteration being  $< \lambda$  recalling that  $\mu$  is regular  $> \lambda$ . E.g. if  $\Vdash_{\mathbb{P}_\mu} \check{f} \in {}^\lambda\lambda$ ” then we can, for  $\alpha < \lambda$  find maximal antichain  $\langle p_{\alpha,i} : i < i_\alpha \leq \lambda \rangle$  of  $\mathbb{P}_\mu$  such that  $p_{\alpha,i}$  forces a value to  $\check{f}(\alpha)$ ; let  $\alpha(*) = \sup(\cup\{\text{dom}(p_{\alpha,i}) : \alpha < \lambda, i < i_\alpha\})$ , so  $\alpha(*) < \mu$  and  $\check{f}$  is a  $\mathbb{P}_{\alpha(*)}$ -name.]

- (\*)<sub>4</sub> (a) in  $\mathbf{V}^{\mathbb{P}_\mu}$ ,  $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$  as witnessed by  $\check{f}^* = \langle \check{f}_\alpha^* : \alpha < \mu \rangle$ , in fact  $\Vdash_{\mathbb{P}_{\alpha+1}} \check{f}_\alpha^* \in {}^\lambda\lambda$  dominates  $({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]}$  modulo  $J_\lambda^{\text{bd}}$   
 (b) in  $\mathbf{V}^{\mathbb{P}_\mu}$  still  $\square_\lambda$  holds.

[Why Clause (a)? Easy using (\*)<sub>3</sub>(b).]

{a7} Why Clause (b)? Easy, by 1.3(1A).]

Stage B: In  $\mathbf{V}$  (see Stage A) there are  $\beta(*), \mathbf{q}, \bar{u}, \mathcal{U}_*, \dots$  such that

(\*)<sub>5</sub>(A)  $\mathbf{q} \in \mathbf{Q}_{\lambda, \bar{\theta}, \beta}$  so in particular

(a)  $\mathbf{q} = \langle \mathbb{P}_{0,\alpha}, \mathbb{Q}_{0,\beta} : \alpha \leq \beta(*), \beta < \beta(*) \rangle$  is a  $(< \lambda)$ -support iteration

(b)  $\bar{u} = \langle u_\beta : \beta < \beta(*) \rangle, \bar{\mathcal{P}} = \langle \mathcal{P}_\beta : \beta < \beta(*) \rangle$

(c)  $u_\beta \subseteq \beta, \mathcal{P}_\beta \subseteq [u_\beta]^{\leq \lambda}$  is closed under subsets

(d)  $\mathbb{Q}_{0,\beta}$  has generic  $\eta_\beta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$

{z32} (e)  $\mathbb{Q}_{0,\beta}$  is as in 0.12(e) so is  $\subseteq \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \eta_\alpha : \alpha \in u_\beta \rangle]}$  and  $\Vdash_{\mathbb{P}_{\beta+1}} \check{\eta}_\beta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$ ” and  
 $\check{\eta} = \langle \eta_\beta : \beta < \beta(*) \rangle$

(f)  $\mathcal{U}_* \subseteq \beta(*)$  has order type  $\gamma(*) = \kappa$  and  $\langle \beta_i^* : i \leq \kappa \rangle$  lists  $\mathcal{U}_* \cup \{\beta(*)\}$  in increasing order

(g) if  $\beta \in \mathcal{U}_*$  then  $[\mathcal{U}_* \cap \beta]^{\leq \lambda} \subseteq \mathcal{P}_\beta$  hence  $\subseteq u_\beta$  and  $\Vdash_{\mathbb{P}_{0,\beta+1}}$  “if  $\nu \in \mathbf{V}[\langle \eta_\alpha : \alpha \in \mathcal{U}_* \cap \beta \rangle] \cap \prod_{\varepsilon < \lambda} \theta_\varepsilon$  then  $\nu <_{J_\lambda^{\text{bd}}} \eta_\beta$ ”

(h) if  $\alpha \leq \beta(*)$  then  $\mathbb{P}_{0,\alpha}$  is  $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c.

(B) letting  $\mathbb{P}'_i = \mathbb{P}_{\mathbf{q}, \{\beta_j^* : j < i\}}$  for  $i \leq j$  we have

(a) The sequence  $\langle \mathbb{P}'_i : i \leq \gamma(*) \rangle$  of forcing notions is  $\triangleleft$ -increasing, and is continuous for ordinals  $i \leq \gamma(*)$  of cofinality  $\geq \lambda$

(b)  $\mathbb{P}'_i$  is  $(< \lambda)$ -strategically complete for  $i \leq \gamma(*)$

(c)  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_{\gamma(*)}]} = \cup\{(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \gamma(*)\}$ .

(d) The sequence  $\langle \mathbb{P}_{1,\beta} : \beta \leq \beta(*) \rangle$  is a sequence of forcing notions,  $\triangleleft$ -increasing and if  $\beta \leq \beta(*)$  then  $\mathbb{P}_{0,\beta} \triangleleft \mathbb{P}_{1,\beta}$ , in fact is dense in it and if  $i \leq \gamma(*)$  then  $\mathbb{P}'_i \triangleleft \mathbb{P}_{1,\beta_i^*}$ .

We shall mention more properties later.

{z32} [Why are there such objects? We apply 0.14 and 0.12 and 0.13.

Also

- (\*)<sub>6</sub> (a) recall  $\langle \beta_i^* : i \leq \gamma(*) \rangle$  list  $\mathcal{U}_* \cup \{\beta(*)\}$  in increasing order

- {c52}
- (b) for  $i \leq \gamma(*) = \kappa$ , for  $i < \gamma(*)$  let  $g'_i$  be  $\eta_{\beta_i^*}$  (to avoid excessive subscripts), see  $(*)_5(A)(e)$  and ?? so  $\mathbb{P}'_i \triangleleft \mathbb{P}_{1, \beta_i^*}$
  - (c) let  $\bar{g}' = \langle g'_i : i < \kappa \rangle$
  - (d) let  $g_\alpha = \eta_\alpha$  for  $\alpha < \beta(*)$  and  $\bar{g} = \langle g_\beta : \beta < \beta(*) \rangle$
  - (e)  $\mathcal{P}_\alpha = \mathcal{P}_{\mathbf{q}, \alpha}$  and without loss of generality  $u_\alpha = \cup\{u : u \in \mathcal{P}_\alpha\}$  for  $\alpha < \beta(*)$ .
- $(*)_7$  if  $u \in \mathcal{P}_\alpha, \alpha < \beta(*)$  then  $\Vdash_{\mathbb{P}_{0, \alpha+1}}$  “ $g_\alpha \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\langle g_\beta : \beta \in u \rangle]}$ ”, the order being modulo  $J_\lambda^{\text{bd}}$ .]

[Why? As by 2.7(5).]

{c35}

- $(*)_8$   $\Vdash_{\mathbb{P}'_\kappa}$  “ $\bar{g}' = \langle g'_i : i < \kappa \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing and cofinal in  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_\lambda^{\text{bd}}})$ ”.

[Why? By  $(*)_7$  noting that  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup\{(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa\}$  which holds by

$(*)_5(B)(c)$ .]

Now

- $(*)_9$   $\Vdash_{\mathbb{P}'_\kappa}$  “ $\text{cov}_\lambda(\text{meagre}) \leq \kappa$ ”.

[Why? As we can look at  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  instead<sup>2</sup> of  ${}^\lambda 2$  and for each  $\varepsilon < \lambda, i < \kappa$  the set  $B_{\varepsilon, i} = \{\eta \in \prod_{\xi < \lambda} \theta_\xi : \text{for every } \zeta \in [\varepsilon, \lambda) \text{ we have } \eta(\zeta) \leq g'_i(\zeta) < \theta_\zeta\}$  is closed nowhere dense, and by  $(*)_8$  we have  $\mathbf{V}^{\mathbb{P}'_\kappa} \models$  “ $\prod_{\zeta < \lambda} \theta_\zeta = \cup\{B_{\varepsilon, i} : \varepsilon < \lambda, i < \kappa\}$ ”. In fact,  $\langle B_{0, i} : i < \kappa \rangle$  suffice.]

- $(*)_10$   $\Vdash_{\mathbb{P}'_\kappa}$  “ $\text{cov}_\lambda(\text{meagre}) \geq \kappa$ ”.

[Why? Let us define the  $\mathbb{P}'_{i+1}$ -name  $\eta'_i$  of a member of  ${}^\lambda 2$  by  $\eta'_i(\varepsilon) = 0$  iff  $g'_i(\varepsilon)$  is even. Now clearly  $\Vdash_{\mathbb{P}'_{i+1}}$  “ $\eta'_i$  is a  $\lambda$ -Cohen sequence over  $\mathbf{V}^{\mathbb{P}'_i}$ ”. (Let us elaborate;  $\eta'_i$  is also a  $\mathbb{P}_{\beta_i^*+1}$ -name and  $\Vdash_{\mathbb{P}_{\beta_i^*+1}}$  “ $\eta'_i$  is  $\lambda$ -Cohen over  $\mathbf{V}^{\mathbb{P}_{\beta_i^*}}$  hence over  $\mathbf{V}^{\mathbb{P}'_i}$ ”; the last hence because  $\mathbb{P}'_i \triangleleft \mathbb{P}_{\beta_i^*}$ . As  $\mathbb{P}_{\beta_i^*+1} \triangleleft \mathbb{P}_{\beta_{i+1}^*}$  and  $\mathbb{P}'_{i+1} \triangleleft \mathbb{P}_{\beta_{i+1}^*}$  we are done.)

Also every closed nowhere dense subset of  ${}^\lambda 2$  from  $\mathbf{V}^{\mathbb{P}'_{\gamma(*)}}$  is from  $\mathbf{V}^{\mathbb{P}'_i}$  for some  $i < \gamma(*)$ . So if  $p \Vdash$  “ $\text{cov}_\lambda(\text{meagre}) < \kappa$ ” then for some  $\zeta < \kappa$  and  $\underline{A}_\varepsilon (\varepsilon < \zeta)$  we have  $p \Vdash$  “ $\underline{A}_\varepsilon$  is a closed no-where dense set subset of  ${}^\lambda 2$  for  $i < j$ ” and  $p \Vdash$  “ $\bigcup_i \underline{A}_i$

is equal to the set of  ${}^\lambda 2$ ”. Without loss of generality each  $\underline{A}_\varepsilon$  is a  $\mathbb{P}_{i(\varepsilon)}$ -name,  $i(\varepsilon) < \kappa$ . Hence  $i = \sup\{i(\varepsilon) : \varepsilon < \zeta\} < \kappa$  and  $\eta'_i$  gives a contradiction to the choice of  $\langle \underline{A}_\varepsilon : \varepsilon < \zeta \rangle$ ; so  $(*)_10$  holds instead.  $\square_{1.3}$

**Discussion 1.6.** 1) The reader may justly wonder why we use  $\mathbf{V}' = \mathbf{V}[\bar{g}'] = \mathbf{V}[\bar{g} \upharpoonright \mathcal{U}_*]$  rather than simply  $\mathbf{V}[\bar{g}]$ . Of course, nothing is lost by it, but why the extra complication?

<sup>2</sup>E.g. let  $F : {}^\lambda 2 \rightarrow \prod_{\varepsilon < \lambda} \theta_\varepsilon$  be  $F(\eta) = \rho$  iff  $\eta \in {}^\lambda 2$  and for every  $\varepsilon < \lambda, \rho(\varepsilon) = 0$  iff  $(\forall i < \theta_\varepsilon)(\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + i) = 0)$  and  $\rho(\varepsilon) = 1 + i$  iff  $\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + i) = 1 \wedge (\forall j < i)(\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + j) = 0)$ . Now if  $\prod_{\varepsilon} \theta_\varepsilon = \cup\{\mathcal{U}_i : i < \kappa\}$ , each  $\mathcal{U}_i$  closed nowhere dense then  $\langle F^{-1}(\mathcal{U}_i) : i < \kappa \rangle$  witnesses  $\text{cov}_\lambda(\text{meagre}) \leq \kappa$ .

- 2) The answer is that during the proof we shall use: if  $\zeta(i) \in \mathcal{U}_*$  is increasing with  $i < \gamma(*)$  then also  $\langle g_{\zeta(i)} : i < \kappa \rangle$  is generic over  $\mathbf{V}$  for the subforcing of  $\mathbb{P}_{1,\beta(*)}$  generated by  $\bar{g} \upharpoonright \mathcal{U}_*$ ; see  $\textcircled{6}''$  inside the proof of  $\textcircled{6}$ . But using  $\mathcal{U}_* = \beta(*)$ , we do not know this.
- 3) Now in the parallel case for  $\lambda = \aleph_0$  with FS iteration with full memory, such claim is true, see §0.
- 4) But we do not know the parallel of 3) for  $\lambda$ , so we use a substitute using  $\mathcal{U}_*$ , i.e.  $\mathbb{P}'_\kappa$ .

*Proof.* Continuation of the proof:

Now we come to the main and last point recalling  $\langle f_\alpha^* : \alpha < \mu \rangle$  from Stage A

(\*)<sub>11</sub> it is forced, i.e.  $\Vdash_{\mathbb{P}'_{\gamma(*)}}$  that no  $\underline{f} \in {}^\lambda \lambda$  dominate  $\{f_\alpha^* : \alpha < \mu\}$ .

{a7} We shall show that it suffices to prove (\*)<sub>11</sub> for proving 1.3(2), and that (\*)<sub>11</sub> holds, thus finishing.

{z35} Why it suffices? As  $\langle f_\alpha^* : \alpha < \mu \rangle$  is  $< J_\lambda^{\text{bd}}$ -increasing and  $\text{cf}(\mu) = \mu > \lambda$ , this implies  $\Vdash_{\mathbb{P}'_\kappa}$  “ $\mathfrak{d}_\lambda \geq \mu$ ”. Also in  $\mathbf{V}$ ,  $\mu^\lambda = \mu > \kappa > \lambda$  and  $|\mathbb{P}'_{\gamma(*)}| = \kappa^\lambda$  by (A)(g) of 0.13(4) which is  $\leq \mu$  and  $\mathbb{P}'_\kappa$  satisfies the  $\lambda^+$ -c.c. hence  $\Vdash_{\mathbb{P}'_\kappa}$  “ $2^\lambda = \mu$ , hence together  $\Vdash_{\mathbb{P}'_\kappa}$  “ $\mathfrak{d}_\lambda = \mu$ ”. Also by (\*)<sub>5</sub>(B)(b), “ $\mathbb{P}'_{\gamma(*)}$  is  $(< \lambda)$ -strategically complete  $+\lambda^+$ -c.c.” and by (\*)<sub>9</sub> + (\*)<sub>10</sub> we know that “ $\text{cov}_\lambda(\text{meagre}) = \kappa$ ” so we are done; so (\*)<sub>11</sub> is really the last piece missing. The rest of the proof is dedicated to proving that (\*)<sub>11</sub> holds.

{z38} We shall use further nice properties of  $\mathbb{P}'_j, g'_i (j \leq \gamma(*), i < \gamma(*))$  which holds by (\*)<sub>5</sub> + (\*)<sub>6</sub> (and (\*)<sub>7</sub>, (\*)<sub>8</sub>) and their proof, i.e. 0.13, 0.14.

- $\boxplus_1$  (a) (α)  $\langle g'_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$ , i.e., if  $\mathbf{G}$  is a subset of  $\mathbb{P}'_{\gamma(*)}$  generic over  $\mathbf{V}$  and  $g'_i = g'_i[\mathbf{G}]$  then  $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_i : i < \gamma(*) \rangle]$
- (β) if  $\nu \in {}^\lambda \lambda \mathbf{V}[\mathbf{G}]$  then for some  $\rho \in {}^\lambda \gamma(*) \mathbf{V}$  and  $\lambda$ -Borel function  $\mathbf{B} \in \mathbf{V}$  we have  $\nu = \mathbf{B}(\langle g'_{\rho(\varepsilon)} : \varepsilon < \lambda \rangle)$
- (b) if in  $\mathbf{V}[\mathbf{G}]$ ,  $g''_\gamma \in \prod_{\zeta < \lambda} \theta_\zeta$  for  $\gamma < \gamma(*)$  and the set  $\{(\gamma, \zeta) : \gamma < \gamma(*) \text{ and } \zeta < \lambda \text{ and } g''_\gamma(\zeta) \neq g'_\gamma(\zeta)\}$  has cardinality  $< \lambda$  then  $\bar{g}'' = \langle g''_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$  and  $\mathbf{V}[\bar{g}''] = \mathbf{V}[\bar{g}']$
- (c)  $\Vdash_{\mathbb{P}'_\gamma}$  “ $\bar{g}'_\gamma$  dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon) \mathbf{V}[\mathbb{P}'_\gamma]$ ”
- (d) if  $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$  is an increasing sequence of ordinals  $< \gamma(*)$  (from  $\mathbf{V}$ !), then  $\langle g'_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$
- (e) if  $\gamma < \gamma(*)$  then  $\mathbb{P}'_\gamma$  is  $(< \lambda)$ -strategically complete and satisfies the  $\lambda^+$ -c.c.

We shall use  $\boxplus_1$  freely.

To prove (\*)<sub>11</sub> assume toward contradiction that this fails, for some condition  $p^* \in \mathbb{P}'_{\gamma(*)}$  and  $\mathbb{P}'_{\gamma(*)}$ -name  $\underline{f}$  and  $\lambda$ -Borel function  $\mathbf{B}$  and  $\rho \in {}^\lambda \gamma(*)$  we have:

$\textcircled{0}$   $p^* \Vdash_{\mathbb{P}'_{\gamma(*)}}$  “ $\underline{f} \in {}^\lambda \lambda$  and<sup>3</sup> dominates  $({}^\lambda \lambda) \mathbf{V}$ ” and  $\underline{f} = \mathbf{B}(\langle g'_{\rho(i)} : i < \lambda \rangle)$ .

<sup>3</sup>The reader may wonder: is  $\mathbf{B}$  used? Answer: here and the previous paragraph.

Now let  $\chi$  be regular large enough and we choose  $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda \rangle$  such that

- ⊗<sub>1</sub> (a)  $N_\varepsilon$  is as in  $\square_\lambda$  for the forcing notion  $\mathbb{P}'_{\gamma(*)}$ ,  $N_\varepsilon \in \mathcal{S}_{\chi, \mathbb{P}'_{\gamma(*)}}$ , see  $\square_\lambda$  of 1.3 {a7}
- (b)  $\bar{N} \upharpoonright \varepsilon \in N_\varepsilon$  hence  $\bigcup_{\zeta < \varepsilon} N_\zeta \subseteq N_\varepsilon$  and  $\lambda_\varepsilon := N_\varepsilon \cap \lambda > \lambda_\varepsilon^- := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\} \geq \Sigma\{\lambda_\zeta : \zeta < \varepsilon\}$
- (c)  $\bar{\theta}, \mathbf{q}, p^*, \underline{f}, \mathbf{B}, \rho$  belong to  $N_\varepsilon$ .

We can find  $f^* \in {}^\lambda \lambda$ , i.e.  $\in (\lambda \lambda)^{\mathbf{V}}$ , such that

- ⊗<sub>2</sub> for arbitrarily large  $\varepsilon < \lambda$  for some  $\zeta \in [\lambda_\varepsilon^-, \lambda_\varepsilon)$  we have  $f^*(\zeta) > \lambda_\varepsilon$ .

For  $\varepsilon < \lambda$  let  $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, \mathbb{A}_\varepsilon, \mathbf{G}_\varepsilon)$  be a witness for  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)}, \chi)$  recalling  $\square_\lambda$  of Definition 1.4 so  $\lambda_\varepsilon \in (\varepsilon, \lambda)$  is strongly inaccessible and  $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda_\zeta^- < \lambda_\zeta$ , recalling ⊗<sub>1</sub> and  $\lambda_\varepsilon^- < \lambda$  is increasing continuous. {a8}

Let

- ⊗<sub>3</sub> (a)  $v_\varepsilon = N_\varepsilon \cap \gamma(*)$
- (b)  $i_\varepsilon = i(\varepsilon) = \text{otp}(v_\varepsilon)$  and so  $i(\varepsilon) = \mathbf{j}_\varepsilon(\gamma(*))$ , etc.
- (c)  $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < i(\varepsilon) \rangle$  list  $v_\varepsilon$  in increasing order
- (d) for  $i < \text{otp}(v_\varepsilon)$ , equivalently  $i < \mathbf{j}_\varepsilon(\gamma(*))$  let  $\eta_i^\varepsilon = (\mathbf{j}_\varepsilon(g'_{\gamma_i(\varepsilon)}))^{A'_\varepsilon[\mathbf{G}_\varepsilon]} \in \prod_{\zeta < \lambda_\varepsilon} \theta_\zeta$  and let  $\bar{\eta}^\varepsilon = \langle \eta_i^\varepsilon : i < i_\varepsilon \rangle$ .

Note that clearly

- ⊗<sub>4</sub> (a)  $\bar{\eta}^\varepsilon$  is generic for  $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$ , moreover
- (b) for each  $\varepsilon < \lambda$ , if we change  $\eta_i^\varepsilon(\zeta)$  (legally, i.e. to an ordinal  $< \theta_\zeta$ ) for  $< \lambda_\varepsilon$  pairs  $(i, \zeta) \in \text{otp}(v_\varepsilon) \times \lambda_\varepsilon$  and get  $\bar{\eta}'$ , then also  $\bar{\eta}'$  is generic for  $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$  and  $\mathbb{A}_\varepsilon[\bar{\eta}'] = M_\varepsilon$
- (c) like  $\boxplus_1$  with  $\mathbf{V}, \mathbb{P}'_{\gamma(*)}, \lambda$  there standing for  $\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$  here.

Hence clearly

- ⊗'<sub>4</sub> for  $\varepsilon < \lambda$ , if  $\bar{\nu} = \langle \nu_i : i < i(\varepsilon) \rangle$  (recalling  $i(\varepsilon) = \text{otp}(v_\varepsilon)$  is as in ⊗<sub>3</sub>(b)), and  $q \in \mathbb{P}'_{\gamma(*)}$  satisfy  $i < i(\varepsilon) \Rightarrow q \Vdash_{\mathbb{P}'_{\gamma(*)}} "g'_{\gamma_i(\varepsilon)} \upharpoonright \lambda_\varepsilon = \nu_i"$  then  $q$  is  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$ -generic naturally and  $q \Vdash_{\mathbb{P}'_{\gamma(*)}} "\mathbf{j}_\varepsilon$  can be extended naturally to an isomorphism from  $N_\varepsilon[\mathbf{G}_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle g'_\gamma : \gamma \in v_\varepsilon \rangle]$  onto  $\mathbb{A}_\varepsilon[\bar{\eta}^\varepsilon]"$ .

[Why? Should be clear, see  $\boxplus_1 + \boxplus_4(c)$ .]

By the assumption toward contradiction, ⊗<sub>0</sub>, and  $\mathbb{P}'_{\gamma(*)}$  being  $(< \lambda)$ -strategically complete, recalling  $\boxplus_1$ , there are  $\zeta(*), p^{**}$  and  $p^+$  such that (recall  $p^* \in \mathbb{P}'_{\gamma(*)} < \mathbb{P}_{1, \beta(*)}$ ):

- ⊗<sub>5</sub> (a)  $p^* \leq p^{**} \in \mathbb{P}'_{\gamma(*)}$  and  $p^+ \in \mathbb{P}_{0, \beta(*)}$  such that  $\mathbb{P}_{1, \gamma(*)} \Vdash "p^{**} \leq p^+"$
- (b)  $\zeta(*) < \lambda$
- (c)  $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}} "f^*(\zeta) < \underline{f}(\zeta)$  whenever  $\zeta(*) \leq \zeta < \lambda"$  where  $f^*$  is from ⊗<sub>2</sub>

- (d) if  $\gamma \in \text{Dom}(p^+)$  then  $\eta^{p^+(\gamma)}$  is an object (not just a  $\mathbb{P}'_\gamma$ -name) of length  $\geq \zeta(*)$  (recall that  $\eta^{p^+(\gamma)}$  is the trunk of the condition  $p^+(\gamma)$ , see clause  $(\alpha)(b)$  of Definition 0.10(1)). {z23}

Note that possibly  $\text{Dom}(p^+) \not\subseteq \cup\{v_\varepsilon : \varepsilon < \lambda\}$ . Choose  $\varepsilon(*) < \lambda$  such that  $\lambda_{\varepsilon(*)} > \zeta(*) + |\text{Dom}(p^+)|$  and  $\gamma \in \text{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$  recalling clause (d) of  $\otimes_5$  and  $|\text{Dom}(p^+)| < \lambda$  as  $p^+ \in \mathbb{P}_{0,\beta(*)}$  and  $\mathbb{P}_{0,\beta(*)}$  is the limit of a  $(< \lambda)$ -support iteration.

By  $\otimes_2$  we can add  $(\exists \zeta)[\lambda_{\varepsilon(*)}^- \leq \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$ . Our intention is to find  $q \in \mathbb{P}_{0,\beta(*)}$  above  $p^+$  which (in  $\mathbb{P}_{1,\beta(*)}$ ) is above some  $q' \in \mathbb{P}'_{\gamma(*)}$  which is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ -generic and forces  $\mathbf{G}_{\mathbb{P}'_{\gamma(*)}}$  to include a generic subset of  $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$  which is induced by some  $\bar{\nu}$  as in  $\otimes'_4$ , recalling  $\otimes_4(b)$ . Toward this in  $\otimes_6$  below the intention is that  $p_{i(*)}^+$  will serve as  $q$ .

Let  $i(*) = i(\varepsilon(*))$  and  $\gamma_i$  for  $i < i(*)$  be such that<sup>4</sup>  $\langle \gamma_i : i < i(*) \rangle$  list  $\{\beta_i^* : i \in v_{\varepsilon(*)}\} \subseteq \mathcal{U}_*$  in increasing order; recall  $\mathcal{U}_* = \{\beta_i^* : i < \gamma(*)\}$  and  $i < j < \gamma(*) \Rightarrow \beta_i^* < \beta_j^*$  and  $v_{\varepsilon(*)} \subseteq \gamma(*)$  has order type  $i(\varepsilon(*))$ . Next let  $\mathcal{U}_{**} = \{\gamma_i : i < i(*)\}$  and  $\gamma_{i(*)} = \gamma(*)$  so  $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in v_{\varepsilon(*)}\} = i(*) = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$ . Recall  $\gamma(*) = \kappa = \text{cf}(\kappa) > \lambda$ ,  $\text{otp}(v_{\varepsilon(*)}) = \text{otp}(N_{\varepsilon(*)} \cap \gamma(*)) = \text{otp}(N_{\varepsilon(*)} \cap \kappa)$  hence  $N_{\varepsilon(*)} \models "i(*)$  is a regular cardinal  $> \lambda_\varepsilon"$  hence  $i(*)$  is really a regular cardinal so call it  $\sigma$ . Now we define a game  $\boxplus$  as follows<sup>5</sup>:

- $\boxplus_2$  (A) each play lasts  $i(*) + 1 = \sigma + 1$  moves and in the  $i$ -th move,  
 (a) if  $i = j + 1$  the antagonist player chooses  $\xi_j = \xi(j) < \sigma$  such that  $j_1 < j \Rightarrow \xi(j_1) < \xi(j)$   
 (b) then, if  $i = j + 1$  the protagonist chooses  $\zeta_j = \zeta(j) \in (\xi(j), \sigma)$ , but there are more restrictions implicit in  $\boxplus_3$   
 (c) in any case the protagonist also chooses  $p_i^+, \bar{\nu}^i$  such that  $\boxplus_3$  below holds;  
 (B) in the end of the play the protagonist wins the play iff he always has a legal move and in the end  $\{\zeta(i) : i \leq i(*)\} \in \mathbb{A}_{\varepsilon(*)}$ ; note that trivially it belongs to  $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar{\nu}^{\varepsilon(*)}]$

where

- $\boxplus_3$  (a)  $p_i^+ \in \mathbb{P}_{0,\gamma_i}$   
 (b) if  $j < i$  then  $\mathbb{P}_{0,\gamma_i} \models "p_j^+ \leq p_i^+"$   
 (c) if  $\gamma \in \cup\{\text{Dom}(p_j^+) : j < i\}$  then  $p_i^+ \upharpoonright \gamma \Vdash_{\mathbb{P}_{0,\gamma_i}} "\eta^{p_i^+(\gamma)}$  has length  $\geq i(*)$  and  $\geq \lambda_{\varepsilon(*)}"$  moreover  $\eta^{p_i^+(\gamma)}$  is an object,  $\eta^{p_i^+(\gamma)}$   
 (d)  $\mathbb{P}_{0,\gamma_i} \models "p^{**} \upharpoonright \gamma_i \leq p_i^+"$   
 (e)  $\bar{\nu}^i = \langle \nu_{\gamma_j} : j < i \rangle$  and  $\nu_{\gamma_j} \in \prod_{\iota < \lambda_{\varepsilon(*)}} \theta_\iota$   
 (f) for  $j < i$  we have  $\nu_{\gamma_j} \leq \eta^{p_i^+(\gamma_j)}$  so  $p_i^+ \upharpoonright \gamma_j \Vdash "\nu_{\gamma_j} \triangleleft g'_{\gamma_j}"$  recalling  $\boxplus_1$   
 (g) for  $j < i$  we have (recall  $\bar{\eta}^\varepsilon$  from  $\otimes_3$ )

<sup>4</sup>This is used in  $\boxplus_3$  and the proof of  $(*)_6$ . Not to be confused with  $\bar{\gamma}^\varepsilon$  of  $\otimes_3(c)$ .

<sup>5</sup>The idea is to scatter the  $\eta_{\gamma_i}^{\varepsilon(*)}$ 's. Why not use the original places? as then we have a problem in  $\otimes_6$ ; the scattering is helpful because we are relying on  $\S 2$ - $\S 4$ .

- ( $\alpha$ )  $\nu_{\gamma_j} = \eta_{\zeta(j)}^{\varepsilon(*)}$  recalling  $\eta_{\gamma_j}^{\varepsilon(*)}$  is from  $\textcircled{3}(d)$  or  
 ( $\beta$ )  $\gamma_j \in \text{Dom}(p^{**})$  and  $\{\iota < \lambda_{\varepsilon(*)} : \eta_{\zeta(j)}^{\varepsilon(*)}(\iota) \neq \nu_{\gamma_j}(\iota)\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ .

We shall prove

- $\textcircled{6}$  in the game  $\mathcal{D}$
- (a) the antagonist has no winning strategy
  - (b) in any move the protagonist has a legal move, moreover for any  $\zeta(i) \in (\xi(i), \sigma)$  large enough the protagonist can choose it.

Why  $\textcircled{6}$  suffice?

By clause (a) of  $\textcircled{6}$  we can choose a play  $\langle (\xi(i), \zeta(i), p_i^+, \bar{\nu}^i) : i \leq \sigma \rangle$  in which the protagonist wins. Recalling  $\mathbb{P}'_{\gamma(*)} \triangleleft \mathbb{P}_{1,\beta(*)}$  and  $\mathbb{P}_{0,\beta(*)}$  is a dense subforcing of  $\mathbb{P}_{1,\beta(*)}$ , clearly

- $\textcircled{7}$  there is  $p$  such that
- (a)  $p \in \mathbb{P}'_{\gamma(*)}$
  - (b) if  $\mathbb{P}'_{\gamma(*)} \Vdash "p \leq p'"$  and  $p' \in \mathbb{P}_{0,\beta(*)}$  then  $p', p_\sigma^+$  are compatible in  $\mathbb{P}_{i,\beta(*)}$
  - (c)  $p$  is above  $p^{**}$  and it forces  $g'_{\gamma_i} \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < i(*)$ .

Then on the one hand

- $\textcircled{7}'$   $p \in \mathbb{P}'_{\gamma(*)}$  being above  $p^{**}$  forces  $f^* \upharpoonright [\zeta(*), \lambda] < \underline{f} \upharpoonright [\zeta(*), \lambda]$  hence  $f^* \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)}] < \underline{f} \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)}]$  recalling (see  $\textcircled{5}$ ) that  $\zeta(*) < \lambda_{\varepsilon(*)}$ .

On the other hand,

- $\textcircled{7}''$   $p$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{0,\gamma(*)})$ -generic.

[Why? As it forces  $\eta_{\gamma_i} \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < i(*)$  and  $\langle \nu_{\gamma_i} : i < i(*) \rangle$  is (see  $\textcircled{3}(g)$ ) recalling  $\text{Dom}(p^{**})$  has cardinality  $< \lambda_{\varepsilon(*)}$  “almost equal” to  $\langle \eta_{\zeta(i)}^{\varepsilon(*)} : i < i(*) \rangle$  which is a subsequence of the sequence from  $\textcircled{3}$ . That is  $\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} \subseteq \cup\{(i, \iota) : \iota < \lambda_{\varepsilon(*)} \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} : \gamma \in v_{\varepsilon(*)} \cap \text{Dom}(p^{**})\}$  so is the union of  $\leq |\text{Dom}(p_\sigma^+)| < \lambda_{\varepsilon(*)}$  sets each of cardinality  $< \lambda_{\varepsilon(*)}$  hence is of cardinality  $< \lambda_{\varepsilon(*)}$ . Hence by  $\textcircled{4}(c) + \textcircled{1}(d)$  the sequence  $\bar{\nu}^{i(*)}$  is generic for  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ ; this is a point where we rely on §2 - §4.]

As  $\underline{f} \in N_{\varepsilon(*)}$  it follows from  $\textcircled{7}''$  that

- $\textcircled{7}'''$   $p \Vdash "f \upharpoonright \lambda_{\varepsilon(*)} \text{ is a function from } \lambda_{\varepsilon(*)} \text{ to } \lambda_{\varepsilon(*)}"$ .

Together  $\textcircled{7}' + \textcircled{7}'''$  gives a contradiction by the choice of  $f^*$  in  $\textcircled{2}$  and of  $\varepsilon(*)$  above, hence  $\textcircled{6}$  is enough.

Why  $\textcircled{6}$  is true?

Let us prove  $\textcircled{6}$ ; first, assuming clause (b) proved below, for clause (a) choose any strategy  $\mathbf{st}$  for the antagonist and fix a partial strategy  $\mathbf{st}'$  for the protagonist choosing  $(p_i^+, \bar{\nu}^i)$  depending on the previous choices and  $\zeta(i) < i_{\varepsilon(*)}$  such that it is a legal move if relevant and possible. So the only freedom left for the protagonist

is to choose the  $\zeta(i)$ . So (recalling  $\boxplus_2(A)(a)$ ) we have in  $\mathbf{V}$  a function  $F : \sigma > \sigma \rightarrow \sigma$  (so  $F$  uses  $\mathbf{st} \dots$ ) such that:

- (\*) $_F$  playing the game such that the antagonist uses  $\mathbf{st}$  and the protagonist uses  $\mathbf{st}'$ , arriving to the  $i$ -th move,  $\bar{\zeta} = \langle \zeta(j) : j < i \rangle$  is well defined and for the protagonist any choice  $\zeta_i \in (F(\bar{\zeta}), \sigma) \cap \mathcal{U}_{**}$  is legal.

Now we have to find an increasing sequence  $\bar{\zeta} = \langle \zeta(i) : i < \sigma \rangle$  from  $\mathbb{A}_{\varepsilon(*)}$  such that  $F(\bar{\zeta} \upharpoonright i) < \zeta(i) < \sigma$  and  $\bar{\zeta} \in \mathbb{A}_{\varepsilon(*)}$ . As  $F \in \mathcal{H}(\chi_{\varepsilon(*)})$  and  $\mathcal{H}(\chi_{\varepsilon(*)}) = \mathbb{A}_{\varepsilon(*)}[\mathbf{G}_{\varepsilon(*)}]$  where  $\mathbf{G}_{\varepsilon(*)}$  is a subset of  $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}'_{\gamma(*)}) \in \mathbb{A}_{\varepsilon(*)}$  generic over  $\mathbb{A}_{\varepsilon(*)}$  and  $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$  satisfies the  $\lambda_{\varepsilon(*)}^+$ -c.c. and  $\sigma = \text{cf}(\sigma) > \lambda_{\varepsilon(*)}$  this is possible. That is, there is a  $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$ -name  $\bar{F}_* \in \mathbb{A}_{\varepsilon(*)}$  such that  $F = \bar{F}_*[\mathbf{G}_{\varepsilon(*)}]$  and we define in  $\mathbb{A}_{\varepsilon(*)}$  the function  $F' : \sigma > \sigma \rightarrow \sigma$  by  $F'(\langle \zeta(j) : j < i \rangle) = \sup\{\xi + 1 : \xi \in \{\zeta(j) + 1 : j < i\} \text{ or } \xi < \sigma \text{ and } \mathcal{K}_{\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})} \text{ “}\bar{F}_*(\langle \zeta(j) : j < i \rangle) \neq \xi\} \}$ ; clearly this is O.K.

We are left with proving  $\otimes_6(b)$ .

Case 1:  $i = 0$ .

Let  $p_0^+ = p^{**} \upharpoonright \gamma_0$ .

Case 2:  $i$  limit.

By clauses (b) and (c) of  $\boxplus_3$ , there is  $p_i^+ \in \mathbb{P}_{0,\gamma_i}$  which is an upper bound (even l.u.b.) of  $\{p_j^+ : j < i\}$  and it is easily as required. Also  $\bar{v}^i$  is well defined and as required.

Case 3:  $i = j + 1$  and  $\gamma_j \notin \text{Dom}(p^{**})$ .

Clearly  $\gamma_i$  is in  $\mathcal{U}_*$  the successor of  $\gamma_j$  and  $(\exists \iota)(\gamma_j = \beta_\iota^* \wedge \iota \in v_{\varepsilon(*)})$ . As in case 4 below but easier by the properties of the iteration.

Case 4:  $i = j + 1$  and  $\gamma_j \in \text{Dom}(p^{**})$

Again  $\gamma_i$  is in  $\mathcal{U}_*$  the successor of  $\gamma_j$  and  $(\exists \iota)(\gamma_j = \beta_\iota^* \wedge \iota \in v_{\varepsilon(*)})$ .

First we find  $p'_j$  such that:

- $\otimes_8$  (a)  $p_j^+ \leq p'_j \in \mathbb{P}_{0,\gamma_j}$   
 (b) if  $\gamma \in \text{Dom}(p_j^+)$  then  $p'_j \upharpoonright \gamma \Vdash \text{“} \ell g(\eta^{p'_j(\gamma)}) > i(*) \text{”}$  (see  $\boxplus_3(c)$ )  
 (c)  $p'_j$  forces  $^6$  a value to the pair  $(\eta^{p^+(\gamma_j)}, f^{p^+(\gamma_j)} \upharpoonright \lambda_{\varepsilon(*)})$ ; we call this pair  $q_j = (\eta^{q_j}, f^{q_j})$ .

This should be clear.

Second

- $\otimes_9$   $p_j^+$  hence  $p'_j$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic and  $\langle \nu_{\gamma_j(1)} : j(1) < j \rangle$  induces the generic.

[Why? As in the proof of  $\otimes''_4$  above when we assume that we have carried the induction, by  $\boxplus_2$ , clause (g) and  $\otimes_4$ .]

Now

- $\otimes_{10}$  (a)  $f^{q_j} \in (\prod_{\zeta < \lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbb{A}_{\varepsilon(*)}[\bar{v}^j]}$   
 (b) for every large enough  $\zeta \in (\xi(i), \sigma)$  we have

<sup>6</sup>recall that  $\eta^{p^+(\gamma_j)}$  is an object, not a name and  $p_j^+$  is  $(N_{\varepsilon(*)}, \mathbb{P}_{\gamma_j})$ -generic



- <sub>1</sub>  $f^{q_j} \leq \eta_\zeta^{\varepsilon(*)} \bmod J_{\lambda_\varepsilon}^{\text{bd}}$
  - <sub>2</sub>  $f^{q_j} \in \mathbb{A}_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$
  - <sub>3</sub>  $\langle \zeta(j_1) : j_1 < j \rangle \in \mathbb{A}_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$ .
- (c)  $\eta^{q_j} \triangleleft f^{q_j}$ .

[Why? Clause (a) holds because  $f^{q_j} \in (\prod_{\zeta < \lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbf{V}}$ , hence belongs to  $\mathcal{H}(\chi_{\varepsilon(*)})$

which is the universe of  $M_{\varepsilon(*)}$  so  $f^{q_j} \in M_{\varepsilon(*)}$ . But  $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)}]$ ; recalling  $\bar{\eta}^{\varepsilon(*)}$  is a generic for  $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$ . Next as  $\mathbb{P}'_{\gamma(*)}$  satisfies the  $\lambda^+$ -c.c. and  $\lambda < \kappa = \text{cf}(\gamma(*))$  so  $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$  satisfies the  $\lambda_{\varepsilon(*)}^+$ -c.c. hence for some  $\zeta_1 < \sigma$ ,  $f^{q_j} \in \mathbb{A}_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta_1]$ . Similarly for some  $\zeta_2 < \sigma$  we have  $\langle \zeta(j_2) : j_2 < j \rangle$  belongs to  $N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta_2]$ . Letting  $\zeta \geq \max\{\zeta_1, \zeta_2\}$  clearly clauses •<sub>2</sub>, •<sub>3</sub> of  $\textcircled{*}_{10}(b)$  holds, also •<sub>1</sub> there holds by  $\textcircled{*}_{11}(c)$  (and  $\mathbf{j}_\varepsilon$ , etc.).

Lastly,  $\textcircled{*}_{10}(c)$  holds by  $\textcircled{*}_{8}(c)$ .]

Now we choose  $\zeta(j)$  as in clause (b) of  $\textcircled{*}_{10}$  and  $\nu_j \in \prod_{\varepsilon < \lambda_{\varepsilon(*)}} \theta_\varepsilon$  such that  $\eta^{p^+(j)} \triangleleft$

$\nu_j$ ,  $f^{q_j} \leq \nu_j$  and  $\{\iota < \lambda_{\varepsilon(*)} : \nu_j(\iota) \neq \eta_{\zeta(j)}^{\varepsilon(*)}\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ . Next choose  $p_i^+ \in \mathbb{P}'_{\gamma(*)}$  such that  $p_i^+ \upharpoonright \gamma_j = p_j^+$ ,  $\eta^{p_i^+(\gamma_i)} = \nu_j$  and  $f^{p_i^+(\gamma_i)} \upharpoonright [\lambda_\varepsilon, \lambda] = f^{p^+(\gamma)} \upharpoonright [\lambda_\varepsilon, \lambda]$ .

So we have carried the induction hence proved  $\textcircled{*}_6$  so we are done.  $\square_{1.3}$

{a19}

*Concluding Remark 1.7.* 1) Is the use of  $\bar{g} \upharpoonright \mathcal{U}_*$  rather than  $\bar{g}$  in the proof necessary? See on this [Sh:F979].

§ 2. ON A RELATIVE TO §2 HAVING  $\lambda_0$  AND  $\lambda_1$ 

{c0} **Hypothesis 2.1.** As in ??, but now we have paramter  $\bar{\lambda} = (\lambda_1, \lambda_0), \lambda_1 \geq \lambda_0 \geq \lambda$  and use  $\mathbf{q}$  the usual  $< \lambda$ -support iteration.

{c4} **Discussion 2.2.** (2012.2.02) 1)  $\lambda_1$  has the role of  $\lambda_0$  in clause (e)( $\theta$ ) of Definition ??.

{c20} 2)  $\lambda_0$  has the role of  $\lambda_0$  in Definition 2.4 and in ??.

{c46} 3) At the moment, 2.11 seems unclear.

{c41} 4) Even with this separation to  $\lambda_0, \lambda_1$  we can prove ?? but it requires more care ( $\mathcal{X}$  more information).

{c30} 5) We may change Definition 2.5(1) to (\*) below, getting weaker results for  $\mathbf{M}$  but ones which fit more general contents

(\*) by induction on the ordinal  $\gamma$  we define the class  $\mathbf{M}_\gamma^{\text{ec}}$  as the class of  $\mathbf{m} \in \mathbf{M}$  such that:

- if  $L \in \text{Sub}(\mathbf{m})$  and  $\beta = \text{dp}_\mathbf{m}^*(L) < \gamma$  then  $m \leq_\mathbf{M} n \in M_\beta^{\text{ec}} \Rightarrow P_\mathbf{m}[L] = \mathbb{P}_\mathbf{n}[L]$ .

6) We may assume  $\chi = \chi^{\lambda_0}$  instead of  $\chi = \chi^{\lambda_1}$  but using  $E$  in the proof more information  $\mathcal{X}$  and  $\mathcal{E}$ .

**Claim 2.3.** *Continuing claim:*

10) If  $L_\mathbf{m}^+ \models "s < t"$  and  $p \in \mathcal{I}_{s,t}$  and  $\zeta = \text{lg}(\text{tr}(p(s)))$  then for every  $i < \theta_\varepsilon$  large enough there is  $q \in \mathbb{P}_t$  such that:

- $\mathbb{P}_t \models "p \leq q"$
- if  $r \in \text{Dom}(p) \setminus L_{\mathbf{m}, \leq s}$  then  $q(r) = p(r)$
- if  $r \in \text{Dom}(p) \cap L_{\mathbf{m}, < s}$  then  $\text{tr}(p(r)) \triangleleft \text{tr}(q(r))$
- $\text{tr}(q(s))$  has length  $\zeta + 1$
- $q(s)(\zeta) = i$ .

11) If  $L_\mathbf{m}^+ \models "s < t"$  and  $p \in \mathcal{I}_{s,t}$  and  $\zeta_0 < \zeta_1 < \lambda, \zeta_0 = \text{lg}(\text{tr}(p(s)))$  and  $\bar{u} = \langle u_\varepsilon : \zeta_0 \leq \varepsilon < \zeta_1 \rangle$  where  $u_\varepsilon$  is an unbounded subset of  $\theta_\varepsilon$  for  $\varepsilon \in [\zeta_0, \zeta_1)$  then there is  $q \in \mathbb{P}_t$  above  $p$  such that  $p \Vdash_{\mathbb{P}_t} " \eta_s(\varepsilon) \in u_\varepsilon \text{ for } \varepsilon \in [\zeta_0, \zeta_1) "$ .

12) If  $\xi \leq \lambda, r_\zeta \in u_t$  for  $\zeta < \xi$  and  $\mathbf{B}$  is a  $\lambda$ -Borel function from  ${}^\xi(\Pi\bar{\theta})$  to  $\mathbb{Q}_{\bar{\theta}}$  and  $p \in \mathbb{P}_\mathbf{m}$  then for some  $q$  we have:

- $\mathbb{P}_\mathbf{m} \models "p \leq q"$
- $q \upharpoonright L_{< t} \Vdash_{\mathbb{P}_t} " \mathbf{B}(\dots, \eta_{r_\zeta}, \dots)_{\zeta < \xi} \leq_{\mathbb{Q}_{\bar{\theta}}} q(t) \text{ or } \mathbf{B}(\dots, \eta_{r_\zeta}, \dots)_{\zeta < \xi}, q(t) \text{ are incompatible in } \mathbb{Q}_{\bar{\theta}}, p \in \mathbb{P}_\mathbf{m} "$ .

13) If  $t \in L_\mathbf{m}, j \leq \lambda, \xi_i \leq \lambda$  for  $i < j, r_{i,\zeta} \in u_t$  for  $i < j, \zeta < \xi_i$  and  $p \upharpoonright L_{< t} \Vdash_{\mathbb{P}_t} " \langle \mathbf{B}_i(\dots, \eta_{r_{i,\zeta}}, \dots)_{\zeta < \xi_i} : i < j \rangle \text{ is a maximal antichain of } \mathbb{Q}_t^{\mathbf{V}[(\eta_r : r \in u_t)]} "$  then for some  $q$  and  $i < j$  we have:

- $\mathbb{P}_\mathbf{m} \models "p \leq q"$
- $q \upharpoonright L_{< t} \Vdash_{\mathbb{P}_t} "q(t) \text{ is above } \mathbf{B}_i(\dots, \eta_{r_{i,\zeta}}, \dots)_{\zeta < \xi_i} \text{ in } \mathbb{Q}_{\bar{\theta}} "$ .

{c29} **Definition 2.4.** 1) For  $\mathbf{m} \in \mathbf{M}$  let  $\text{cmp}(\mathbf{m})$  be the set of  $L$  such that:

- (a)  $L \subseteq L_\mathbf{m}$

- (b) if  $L \subseteq M_{\mathbf{m}}$  then  $|L| \leq 1$
- (c) if  $t \in L \setminus M_{\mathbf{m}}$  then  $L = t/E'_{\mathbf{m}}$  hence:
  - ( $\alpha$ )  $L \setminus M_{\mathbf{m}} \neq \emptyset$  and  $L \setminus M$  is an  $E_{\mathbf{m}}$ -equivalence class
  - ( $\beta$ ) if  $t \in L \setminus M_{\mathbf{m}}$  then  $u_{\mathbf{m},t} \subseteq L$
  - ( $\gamma$ )  $|L| \leq \lambda_0$ .

2) For  $\mathbf{m} \in \mathbf{M}$

- (a) let  $\text{sub}(\mathbf{m})$  be the set of  $L_1$  such that for some  $L_2 \in \text{cmp}(\mathbf{m})$ ,  $L_1$  is an initial segment of  $L_2$ , i.e.  $L_2 \subseteq L_1 \wedge (\forall s, t)[s \in L_2 \wedge t \in L_1 \wedge s <_{\mathbf{m}} t \rightarrow s \in L_1]$
- (b) let  $\text{sub}_*(\mathbf{m}) = \{L \in \text{sub}(\mathbf{m}) : \text{there are}^7 L' \in \text{cmp}(\mathbf{m}) \text{ and } Y_1, Y_2 \subseteq L' \text{ of cardinality } \leq \lambda_0 \text{ such that } L = \{s \in L' : (\exists t)(s \leq_{\mathbf{m}} t \in Y_1 \vee s <_{\mathbf{m}} Y_2)\}\}$ .

3) Let

- (a)  $\text{Cmp}(\mathbf{m}) = \{\bigcup_{\varepsilon < \zeta} L_\varepsilon : L_\varepsilon \in \text{cmp}(\mathbf{m}) \text{ for } \varepsilon < \zeta\}$
- (b)  $\text{Cmp}_*(\mathbf{m}) = \{\bigcup_{\varepsilon < \zeta} L_\varepsilon : L_\varepsilon \in \text{cmp}(\mathbf{m}) \text{ for } \varepsilon < \zeta \text{ and } \zeta \leq \lambda_0\}$
- (c)  $\text{Sub}(\mathbf{m}) = \{L_1 : \text{for some } L_2 \in \text{Cmp}(\mathbf{m}), L_1 \text{ is an initial segment of } L_2\}$
- (d)  $\text{Sub}_*(\mathbf{m}) = \{\bigcup_{\varepsilon < \zeta} L_\varepsilon : \zeta \leq \lambda_0 \text{ and } L_\varepsilon \in \text{sub}_*(\mathbf{m}) \text{ for } \varepsilon < \zeta\}$ .

**Definition 2.5.** 0) For  $L \subseteq L_{\mathbf{m}}$ ,  $\mathbf{m} \in \mathbf{M}$  let  $\text{dp}_{\mathbf{m}}^*(L) = \cup\{\text{dp}_{M_{\mathbf{m}}}(t)+1 : t \in L \cap M_{\mathbf{m}}\}$ . {c30}

1) For an ordinal  $\gamma$  let  $\mathbf{M}_\gamma^{\text{ec}}$  be the class of  $\mathbf{m} \in \mathbf{M}$  such that:

- (\*) if  $L \in \text{Sub}(\mathbf{m})$  and  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$  then  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$ .

2) Let  $\mathbf{M}^{\text{ec}} = M_\infty^{\text{ec}}$  be the class of  $\mathbf{m}$  which  $\in \mathbf{M}_\gamma^{\text{ec}}$  for every  $\gamma$ .

3) Let  $\mathbf{M}_{\chi, \gamma}^{\text{ec}} = \{\mathbf{m} \in \mathbf{M}_\gamma^{\text{ec}} : |L_{\mathbf{m}}| \leq \chi\}$ , similarly  $\mathbf{M}_{\chi, \infty}^{\text{ec}}$ .

4) Let  $\mathbf{M}_{\text{fc}}$  be the class of  $\mathbf{m} \in \mathbf{M}$  satisfying  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} < \mathbb{P}_{\mathbf{n}}$ . {c32}

**Observation 2.6.** 1)  $\leq_{\mathbf{M}}$  is a partial order on  $\mathbf{M}$ .

2)  $\mathbf{M}_0^{\text{ec}} = \mathbf{M}$  and  $\mathbf{M}_{\chi, 0}^{\text{ec}} = \mathbf{M}_\chi$ .

3) If  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $L \in \text{sub}(\mathbf{m})$  then  $\mathbb{P}_{\mathbf{m}}(L), \mathbb{P}_{\mathbf{n}}(L)$  have the same set of elements.

4) If  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  then  $\text{cmp}(\mathbf{m}) \subseteq \text{cmp}(\mathbf{n})$  moreover  $L \in \text{cmp}(\mathbf{m})$  iff  $L \in \text{cmp}(\mathbf{n}) \wedge L \subseteq L_{\mathbf{m}}$ .

5) Like (4) for  $\text{sub}(-), \text{Cmp}(-), \text{Sub}(-), \text{sub}_*(-), \text{Cmp}_*(-), \text{Sub}_*(-)$ .

6) If  $\chi = \chi^{\lambda_0}$ ,  $\mathbf{m} \in \mathbf{M}_\chi$ ,  $L \in \text{Sub}(\mathbf{m})$ ,  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $\mathbb{P}_{\mathbf{m}}[L] \neq \mathbb{P}_{\mathbf{n}}[L]$  then we can find such  $\mathbf{n}$  in  $\mathbf{M}_\chi$ .

7) If  $\mathbf{m} \in \mathbf{M}$  and  $t \in L_{\mathbf{m}}$  then  $t \in L$  for some  $L \in \text{cmp}(\mathbf{m})$ .

8) If  $\mathbf{m} \in \mathbf{M}$  and  $L_1 \subseteq L_{\mathbf{m}}$  has cardinality  $\leq \lambda_0$  then  $L_1 \subseteq L_2$  for some  $L_2 \in \text{Cmp}_*(\mathbf{m})$ .

9) If  $L_1 \in \text{sub}(\mathbf{m})$  then  $L_1 = \cup\{L_1 \cap L_{\mathbf{m}, \leq t} : t \in L_1\}$  and  $t \in L_1 \Rightarrow L_1 \cap L_{\mathbf{m}, \leq t} \in \text{sub}_*(\mathbf{m})$ .

*Proof.* Should be clear. □<sub>2.6</sub>

<sup>7</sup>version A:  $s_\varepsilon$  of  $L_{\mathbf{m}}$  for  $\varepsilon < \varepsilon(*) < \lambda$  and  $\lambda$ -Borel function  $\mathbf{B}$  from  $\varepsilon(*)_4$  to 2 and  $L = \{t \in L' : \mathbf{B}(\dots, \mathfrak{A}_I(s, t), \dots) = 1 \text{ where } \mathfrak{A}_I(s, t) \text{ is: } 0 \text{ if } s <_I t, 1 \text{ if } s = t, 2 \text{ if } t <_I s \text{ and } 3 \text{ otherwise.}\}$

{c35}

**Observation 2.7.** 1) If  $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing then  $\mathbf{m}_\delta := \cup\{\mathbf{m}_\alpha : \alpha < \delta\}$  is a  $\leq_{\mathbf{M}}$ -upper bound of  $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$ .

2) Moreover in part (1):

- ( $\alpha$ ) •  $\text{cmp}(\mathbf{m}_\delta) = \cup\{\text{cmp}(\mathbf{m}_\alpha) : \alpha < \delta\}$
- $\text{sub}(\mathbf{m}_\delta) = \cup\{\text{sub}(\mathbf{m}_\alpha) : \alpha < \delta\}$
- $\text{sub}_*(\mathbf{m}_\delta) = \cup\{\text{sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$
- ( $\beta$ ) if  $\text{cf}(\lambda) \geq \lambda_0^+$  then
  - $\text{Cmp}_*(\mathbf{m}_\delta) = \cup\{\text{Cmp}(\mathbf{m}_\alpha) : \alpha < \delta\}$
  - $\text{Sub}_*(\mathbf{m}_\delta) = \cup\{\text{Sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$ .

3) If  $\gamma_1 < \gamma_2$  then  $\mathbf{M}_{\gamma_1}^{\text{ec}} \subseteq \mathbf{M}_{\gamma_2}^{\text{ec}}$ .

*Proof.* 1) Easy, remembering that  $t \in \mathbf{m}_\alpha \wedge \alpha \leq \beta < \delta \Rightarrow t/E'_{\mathbf{m}_\beta} = t/E'_{\mathbf{m}_\alpha}$ .

2) Easy.

3) Easy, too. □<sub>2.7</sub>

{c38}

{c30}

**Claim 2.8.** 1) In Definition 2.5(1) we can consider only  $L \in \text{Sub}_*(\mathbf{m})$  such that  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$ .

2) So if  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}$  and  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  then  $\mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{n}}$ , that is  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \triangleleft \mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$  and  $L \in \text{Sub}(\mathbf{m}) \Rightarrow \mathbb{P}_{\mathbf{m}}[L] \triangleleft \mathbb{P}_{\mathbf{n}}[L]$ .

3) For  $\mathbf{m} \in \mathbf{M}$  we have:  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}$  iff  $\mathbf{m} \in \mathbf{M}_{\text{fc}}$ , i.e. for every  $\mathbf{n}, \mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{n}}$ .

4)  $\text{Cmp}_*(\mathbf{m})$  has cardinality  $\leq |L_{\mathbf{m}}|^{\leq \lambda_0}$ .

5)  $\text{Sub}_*(\mathbf{m})$  has cardinality  $\leq |L_{\mathbf{m}}|^{\lambda_0}$ .

6)  $|\text{cmp}(\mathbf{m})| \leq |L_{\mathbf{m}}| + 1$  and  $|\text{sub}_*(\mathbf{m})| \leq |L_{\mathbf{m}}|^{\lambda_0}$ .

{c29}

*Proof.* 1) Let  $\mathbf{m} \in \mathbf{M}$  satisfies the weaker version, and let  $L \in \text{Sub}(\mathbf{m})$  satisfies  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$ . By Definition 2.4(3)(c) and Observation 2.6(9) we can find  $\langle L_\varepsilon, L_\varepsilon^* : \varepsilon < \zeta \rangle$  such that  $L \in \cup\{L_\varepsilon : \varepsilon < \zeta\}$  where  $L_\varepsilon \subseteq L_\varepsilon^* \in \text{Cmp}(\mathbf{m})$  and  $L_\varepsilon \in \text{sub}_*(\mathbf{m})$ .

For every  $u \in [\zeta]^{\leq \lambda_0}$  let  $L_u = \cup\{L_\varepsilon : \varepsilon \in u\}$  so clearly  $\text{dp}_{\mathbf{m}}^*(L_u) \leq \text{dp}_{\mathbf{m}}^*(L) < \gamma$ . Let  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ . For  $u \in [\zeta]^{\leq \lambda_0}$ , for  $L_u$  the demand holds, i.e.  $\mathbb{P}_{\mathbf{m}}[L_u] = \mathbb{P}_{\mathbf{n}}[L_u]$ . Also if  $u \subseteq v \in [\zeta]^{\leq \lambda_0}$  then  $\mathbb{P}_{\mathbf{m}}[L_u] \subseteq \mathbb{P}_{\mathbf{m}}[L_v]$ .

{c20}

Lastly,  $\langle u : u \in [\zeta]^{\leq \lambda_0} \rangle$  is  $\lambda_0^+$ -directed so by ??(8) and the Definition 2.5(1) we are done.

{c4}

2) As  $L_{\mathbf{m}} \in \text{Sub}(\mathbf{m})$  and  $\text{dp}_{\mathbf{m}}^*(L_{\mathbf{m}}) < \infty$  by clause (d) Definition ??.

3) The implication  $\Rightarrow$  holds by part (2). For the other implication assume  $\mathbf{m} \notin \mathbf{M}_\infty^{\text{ec}}$  so clause (\*) there fails for some  $\gamma$ ; this means that there are  $L \in \text{Sub}(\mathbf{m})$  and  $\mathbf{n}$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  but  $\mathbb{P}_{\mathbf{m}}[L] \not\triangleleft \mathbb{P}_{\mathbf{n}}[L]$ , by the definitions this implies  $\neg(\mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{n}})$  so we are done.

4) Note that  $\{L \in \text{cmp}(\mathbf{m}) : L \not\subseteq M_{\mathbf{m}}\} = \{t/E'_{\mathbf{m}} : t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}\}$  has cardinality  $\leq |L_{\mathbf{m}} \setminus M_{\mathbf{m}}| \leq |L_{\mathbf{m}}|$  and  $\{L \in \text{cmp}(\mathbf{m}) : L \subseteq M_{\mathbf{m}}\} = L \subseteq M_{\mathbf{m}} : |L| \leq \lambda_0$  has cardinality  $\leq |M_{\mathbf{m}}|^{\lambda_0} \subseteq |L_{\mathbf{m}}|^{\lambda_0}$ .

{c39}

5), 6) Similarly. □<sub>2.8</sub>

**Claim 2.9.** 1) If  $\gamma \leq \infty$ ,  $\langle \mathbf{m}_\alpha^* : \alpha < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing,  $\text{cf}(\delta) > \lambda_0$  and  $\alpha < \delta \Rightarrow \mathbf{m}_\alpha^* \in \mathbf{M}_\gamma^{\text{ec}}$  then  $\mathbf{m}_* := \cup\{\mathbf{m}_\alpha^* : \alpha < \delta\} \in \mathbf{M}_\gamma^{\text{ec}}$ .

2) Similarly for  $\lambda_0^+$ -directed union.

{c38}

*Proof.* 1) By 2.8(1), toward contradiction assume that  $(\mathbf{n}, L)$  satisfies:

- (\*) (a)  $\mathbf{m}_* \leq_{\mathbf{M}} \mathbf{n}$
- (b)  $L \in \text{Sub}_*(\mathbf{m}_*)$
- (c)  $\text{dp}_{\mathbf{m}_*}^*(L) < \gamma$
- (d)  $\mathbb{P}_{\mathbf{m}_*}[L] \neq \mathbb{P}_{\mathbf{n}}[L]$ .

As  $L \in \text{Sub}_*(\mathbf{m}_*)$ , we have  $L \subseteq \cup\{L_\varepsilon : \varepsilon < \varepsilon(*)\}$  where  $\varepsilon(*) < \lambda_0^*$  and  $L_\varepsilon \in \text{sub}_*(\mathbf{m})$  for  $\varepsilon < \varepsilon(*)$ . But then  $L_\varepsilon \in \text{sub}_*(\mathbf{m}_{\alpha(\varepsilon)})$  for some  $\alpha(\varepsilon) < \delta$  by 2.7(2)( $\alpha$ ) but  $\text{cf}(\delta) > \lambda_0$  hence  $\alpha(*) = \sup\{\alpha(\varepsilon) : \varepsilon < \varepsilon(*)\} < \delta$  so  $L \in \text{Sub}_*(\mathbf{m}_{\alpha(*)})$ . {c35}

Now we have assumed  $\text{dp}_{\mathbf{m}_*}^*(L) < \gamma$  but  $M_{\mathbf{m}_*} = M_{\mathbf{m}_{\alpha(*)}}$ , so equivalently  $\text{dp}_{\mathbf{m}_{\alpha(*)}}^*(L) < \gamma$ . By the claim assumption,  $\mathbf{m}_\beta \in \mathbf{M}_\gamma^{\text{ec}}$  and by 2.7(1) we have  $\mathbf{m}_\beta \leq_{\mathbf{M}} \mathbf{m}_*$ . {c35}

As  $\leq_{\mathbf{M}}$  is a partial order on  $\mathbf{M}$  and  $\mathbf{m}_* \leq_{\mathbf{M}} \mathbf{n}$  by (\*) (a) clearly  $\mathbf{m}_{\alpha(*)} \leq_{\mathbf{M}} \mathbf{n}$ , but  $\mathbf{m}_{\alpha(*)} \in M_\beta^{\text{ec}}$  so together we deduce  $\mathbb{P}_{\mathbf{m}_*}[L] = \mathbb{P}_{\mathbf{m}_\beta}[L] = \mathbb{P}_{\mathbf{n}}[L]$ , so we are done. □<sub>2.9</sub>

**Crucial Claim 2.10.** *Let  $\chi = \chi^{\lambda_0}$ .* {c41y}

- 1) If  $\mathbf{m} \in \mathbf{M}$  of cardinality  $\leq \chi$  there is  $\mathbf{n} \in \mathbf{M}_\infty^{\text{ec}}$  of cardinality  $\leq \chi$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ .
- 2) If  $\mathbf{m} \in \mathbf{M}_\chi$  and  $\gamma < \chi^+$  then there is  $\mathbf{n} \in \mathbf{M}_\gamma^{\text{ec}} \cap \mathbf{M}_\chi$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ .

*Proof.* 1) By (2) using  $\gamma = \text{dp}_{\mathbf{m}}^*(M_{\mathbf{m}}) + 1 = \text{dp}_{\mathbf{m}}^*(L_{\mathbf{m}}) + 1$  which necessarily is  $< \|M_{\mathbf{m}}\|^+ \leq \chi^+$ .

2) We prove the statement by induction on  $\gamma$ . Recalling Definition 2.5(1), in each case consider  $L \in \text{Sub}(\mathbf{m})$  which satisfies  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$  and by 2.8(1) without loss of generality  $L \in \text{Sub}_*(\mathbf{m})$ . {c30}  
{c38}

Case 1:  $\gamma = 0$ .

There is no such relevant  $L$ .

Case 2:  $\gamma = 1$ .

So  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$  means  $\text{dp}_{\mathbf{m}}^*(L) = 0$  which means  $L \cap M_{\mathbf{m}} = \emptyset$ . But then  $\mathbf{n} = \mathbf{m}$  is as required because if  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}_1$  then  $\{s \in L_{\mathbf{n}_1} : (\exists t \in L)(s \leq_{\mathbf{n}_1} t)\} = L = \{s \in L_{\mathbf{m}} : (\exists t \in L)(s \leq_{\mathbf{m}} t)\}$ , i.e.  $L$  is an initial segment of both so  $\mathbb{P}_{\mathbf{m},L} = \mathbb{P}_{\mathbf{n}_1,L}$  and the conclusion is obvious, or see ???. {c13}

Case 3:  $\gamma = \beta + 2$ .

We try to choose  $\mathbf{m}_\varepsilon$  by induction on  $\varepsilon \leq \chi^+$  such that:

- (\*) (a)  $\mathbf{m}_\varepsilon \in \mathbf{M}_\chi$
- (b)  $\langle \mathbf{m}_\zeta : \zeta \leq \varepsilon \rangle$  is  $\leq_{\mathbf{M}}$ -increasing continuous
- (c)  $\mathbf{m}_0 = \mathbf{m}$
- (d) if  $\varepsilon = 2\zeta + 1$  then for some  $L \in \text{Sub}_*(\mathbf{m}_{2\zeta})$  we have  $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \neq \mathbb{P}_{\mathbf{m}_\varepsilon}[L]$  and  $\text{dp}_{\mathbf{m}_{2\zeta}}^*(L) < \gamma$
- (e) if  $\varepsilon = 2\zeta + 2$  then  $\mathbf{m}_\varepsilon \in \mathbf{M}_{\beta+1}^{\text{ec}}$ .

Subcase 3A: We succeed to carry the induction.

For each  $\delta \in S_{\lambda_0^+}^{\chi^+} := \{\delta < \chi^+ : \text{cf}(\delta) = \lambda_0^+\}$  (i.e.  $\varepsilon = 2\delta + 1$  and note  $2\delta = \delta$ ) there is  $L_\delta$  as in  $(*)$ (d). Recalling Definition 2.4(3)(d), as  $\text{cf}(\delta) = \lambda_0^+ > \lambda_0 \geq$  {c29}  
 $|\{t/E'_{\mathbf{m}_\delta} : t \in L_\delta \setminus M_{\mathbf{m}_\delta}\}|$  there is  $\alpha(\delta) < \delta$  such that  $L_\delta \subseteq L_{\mathbf{m}_{\alpha(\delta)}}$  hence by 2.7(2) + {c35}  
 2.8(5) for some stationary  $S \subseteq S_{\lambda_0^+}^{\chi^+}$  and  $L_*$  we have  $\delta \in S \Rightarrow L_\delta = L_* \wedge \alpha(\delta) = \alpha(*)$ . {c38}  
 Without loss of generality  $\alpha(*)$  is even and a successor ordinal. For  $\delta \in S$ , note that  $\langle \mathbf{m}_{2\zeta+2} : \zeta < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing sequence of members of  $\mathbf{M}_{\beta+1}^{\text{ec}}$  with union  $\mathbf{m}_\delta$  (by clauses (e) and (b) of  $(*)$ ) hence  $\mathbf{m}_\delta \in \mathbf{M}_{\beta+1}^{\text{ec}}$  by 2.9; recall  $\text{cf}(\delta) > \lambda_0$  and  $M_{\mathbf{m}_\alpha} = M_{\mathbf{m}}$  for every  $\alpha$ . {c39}

Now  $\text{dp}_{\mathbf{m}_\delta}^*(L_*) < \gamma$  and by the previous sentences necessarily  $\text{dp}_{\mathbf{m}_\delta}^*(L_*) = \beta + 1$  hence the set  $\mathcal{Y} := \{t \in L_* \cap M_{\mathbf{m}} : \text{dp}(t, M_{\mathbf{m}}) = \beta\}$  is not empty.

Let

- ⊞<sub>1</sub> (a)  $L_0^* = \{s \in L_* : \neg(\exists t \in \mathcal{Y})(t \leq s)\}$
- (b)  $L_1^* = L_0^* \cup \mathcal{Y}$
- (c)  $L_2^* = L_*$ .

Now

- ⊞<sub>2</sub>  $\mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L_0^*]$  is equal to  $\mathbb{P}_{\mathbf{m}_\alpha}[L_0^*]$  and to  $\mathbb{P}_{\mathbf{m}_{\alpha+1}}[L_0^*]$  for  $\alpha \in S \setminus \alpha(*)$  moreover for  $\alpha \geq \alpha(*)$ .

{c29} [Why?  $L_0^*$  is an initial segment of  $L_*$  but  $\delta \in S \Rightarrow L_* = L_\delta \in \text{Sub}_*(\mathbf{m}_\delta)$ , see above  
 hence, see Definition 2.4(2)(b),(3)(c) we have  $L_0^* \in \text{Sub}_*(\mathbf{m}_\alpha)$  for  $\alpha \geq \alpha(*)$ . We  
 {c30} are done as  $\mathbf{m}_{\alpha(*)} \in \mathbf{M}_{\beta+1}^{\text{ec}}$  by  $(*)$ (e) and 2.5(1).]

- ⊞<sub>3</sub> (a)  $L_{0,\alpha}^* = \{s \in L_{\mathbf{m}_\alpha} : (\exists t \in L_0^*)(s \leq t)\}$  or  $(\exists t \in \mathcal{Y})(s \leq t)$
- (b)  $L_{1,\alpha}^* = L_{0,\alpha}^* \cup Y$
- (c)  $L_{2,\alpha}^* = L_\alpha^0 \cup L_*$
- ⊞<sub>4</sub> (a)  $\text{dp}_{\mathbf{m}_\alpha}^*(L_{0,\alpha}^*) < \beta + 1$
- (b)  $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L_{0,\alpha}^*] : \alpha \in [\alpha(*), \chi^+] \rangle$  is  $\triangleleft$ -increasing.
- ⊞<sub>5</sub> (a)  $L_{1,\alpha}^* \subseteq L_{\mathbf{m}_\alpha}$  is downward closed in  $L_{\mathbf{m}_\alpha}$
- (b)  $\langle \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}^*) : \alpha \in [\alpha(*), \chi^+] \rangle$  is  $\subseteq$ -increasing
- (c) moreover  $\leq_{\text{ic}}$ -increasing.

[Why? Should be clear.]

- ⊞<sub>6</sub> there is a club  $E$  of  $\chi^+$  such that  $\delta \in E \cap S_{\lambda_0^+}^{\chi^+} \setminus \alpha(*) \wedge \delta \leq \alpha < \chi^+ \Rightarrow P_{\mathbf{m}_\gamma}(L_{1,\delta}^*) \triangleleft \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}^*)$ .

[Why? By ⊞<sub>5</sub> as every  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}^*)$  satisfies the  $\lambda$ -c.c.,  $\lambda_0 \geq \lambda$  and  $\chi^{\lambda_0} = \chi$ .]

Now

- ⊞<sub>7</sub> (a)  $L_{2,\alpha}^*$  is a downward closed subset of  $L_{\mathbf{m}_\alpha}$  for  $\alpha \in [\alpha(*), \chi^+]$
- (b) if  $\alpha(*) \leq \alpha \leq \beta < \chi^+$  and  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}^*) \triangleleft \mathbb{P}_{\mathbf{m}_\beta}(L_{1,\beta}^*)$  then
  - ( $\alpha$ )  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) = \mathbb{P}_{\mathbf{m}_\beta}(L_{2,\alpha}) \triangleleft \mathbb{P}_{\mathbf{m}_\beta}$
  - ( $\beta$ )  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) \triangleleft \mathbb{P}_{\mathbf{m}}(L_{2,\beta})$
  - ( $\gamma$ )  $\mathbb{P}_{\mathbf{m}_\alpha}[L_{2,\alpha}] = \mathbb{P}_{\mathbf{m}_\beta}[L_{2,\beta}]$
  - ( $\delta$ )  $\mathbb{P}_{\mathbf{m}_\alpha}[L_1^*] = \mathbb{P}_{\mathbf{m}_\beta}[L_1^*]$ .

[Why? Clause (a) by inspection. Clause (b)( $\alpha$ ), the equality is obvious, the  $\ll \mathbb{P}_{\mathbf{m}_\beta}$  holds by clause (a). Now we prove clause (b)( $\beta$ ), toward this

- <sub>1</sub>  $t \in L_{2,\beta} \setminus L_{1,\alpha} \Leftrightarrow t \in Y \Leftrightarrow t \in L_{2,\alpha} \setminus L_{1,\alpha}$  (why? check)
- <sub>2</sub>  $t \in L_{2,\alpha} \setminus L_{1,\alpha} \Rightarrow u_{\mathbf{m}_\beta, t} \subseteq L_{1,\alpha}$  (why? check)
- <sub>3</sub>  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) \subseteq \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\beta})$  (by assumption of (b)).

Next assume  $p_* \in \mathbb{P}_{2,\beta}(L_{2,\beta})$  then  $q_* = p_* \upharpoonright L_{2,\alpha} \in \mathbb{P}_{2,\alpha}(L_{1,\alpha})$  and also  $q' = p_\beta \upharpoonright L_{1,\alpha} \in \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha})$ ,  $p' = p \upharpoonright L_{1,\beta} \in \mathbb{P}_{\mathbf{m}_\beta}$ . Also  $q' \leq q'' \in \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}) \Rightarrow q'' \cup (p \upharpoonright (L_{1,\beta} \setminus L_{1,\alpha}))$  is a common upper bound of  $q'', p'$  in  $\mathbb{P}_{\mathbf{m}_\beta}(L_{1,\beta})$ . Hence  $q_* = q' \cup (p \upharpoonright Y)$  satisfies  $q_* \in P_{\mathbf{m}_\alpha}(L_{2,\alpha})$  and  $q_* \leq q'_* \in P_{\mathbf{m}_\alpha}(L_{2,\alpha}) \Rightarrow q'_* \cup (p \upharpoonright (L_{1,\beta} \setminus L_{1,\alpha}))$  is a common upper bound of  $q'_*, p$  in  $\mathbb{P}_{\mathbf{m}_\beta}(L_{2,\beta})$ .

Together with •<sub>3</sub> this paragraph shows

- <sub>4</sub>  $\mathbb{P}_{\mathbf{m}_\alpha}[L_{2,\alpha}] \ll \mathbb{P}_{\mathbf{m}_\beta}[L_{2,\beta}]$  that is clause (b)( $\beta$ ).

Clause (b)( $\gamma$ ) holds by (b)( $\beta$ ) and (a) and clause (b)( $\beta$ ) follows as  $L_2^* \subseteq L_{2,\alpha}^* \subseteq L_{2,\alpha}^*$  so  $\boxplus_7$  holds indeed.

Now choose  $\delta \in E \cap S \setminus \alpha(*) \subseteq S_{\lambda_0^+}^{\chi^+}$  and we get a contradiction to the choice of  $L_\delta = L_*$  so we are done with subcase 3A.

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(\*)<sub>2</sub> for  $\alpha \in \chi^+ \setminus \alpha(*)$  and  $t \in \mathcal{Y}$  let  $\mathbb{Q}_{\bar{\theta}, t}^\alpha$  be the  $\mathbb{P}^{\mathbf{m}_\alpha}$ -name of  $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \eta_\gamma : \gamma \in u(t) \cap L_* \rangle]}$  where  $u(t) = u_{\mathbf{m}_{\alpha(*)}, t}$

(\*)<sub>3</sub> (a)  $\mathbb{Q}_{\bar{\theta}, t}^\alpha$  does not depend on  $\alpha$ , also the order on it does not depend on  $\alpha$

(b)  $\Vdash_{\mathbb{P}_{\mathbf{m}_\alpha}} \text{“} \mathbb{Q}_{\bar{\theta}, t}^\alpha \leq_{\text{ic}} \mathbb{Q}_{\bar{\theta}}, \text{i.e. } \mathbb{Q}_{\bar{\theta}, t}^\alpha \subseteq \mathbb{Q}_{\bar{\theta}} \text{ as quasi orders and any } p_1, p_2 \in \mathbb{Q}_{\bar{\theta}, t}^\alpha \text{ compatible in } \mathbb{Q}_{\bar{\theta}} \text{ are compatible in } \mathbb{Q}_{\bar{\theta}, t}^\alpha \text{”}$

(c)  $\mathbb{P}_{\mathbf{m}_\alpha}[L_1^*] / \mathbb{P}_{\mathbf{m}_\alpha}[L_0^*]$  is the same for  $\alpha \in [\alpha(*), \chi^+]$

(\*)<sub>4</sub> define  $\mathbb{I}_\alpha = \{ \mathcal{J} : \mathcal{J} \text{ is a } \mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L_0^*]\text{-name of a maximal antichain of } \mathbb{P}_{\mathbf{m}_\alpha}[L_1^*] / \mathbb{P}_{\mathbf{m}_\alpha}[L_0^*] \}$ .

Let  $L_{0,\alpha}^* = \{ s \in L_{\mathbf{m}_\alpha} : (\exists t \in L_0^*)(s \leq t) \text{ or } (\exists t \in \mathcal{Y})(s < t) \}$  and  $S_* = \{ \alpha < \chi^+ : (\exists \varepsilon)(\alpha = 2\varepsilon + 2) \text{ or } \text{cf}(\alpha) = \lambda_0^+ \}$

(\*)<sub>5</sub> (a)  $\text{dp}_{\mathbf{m}_\alpha}^*(L_{0,\alpha}^*) < \beta + 1$

(b)  $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L_{0,\alpha}^*] : \alpha \in S_* \setminus \alpha(*) \subseteq [\alpha(*), \chi^+] \text{ and } \alpha = (\exists \varepsilon)(\alpha = 2\varepsilon + 2) \text{ or } \alpha \in S \rangle$  is  $\ll$  increasing.

[Why? For clause (a):

if  $s \in L_{0,\alpha} \cap M_{\mathbf{m}_\alpha}$  then for some  $t \in Y$  we have  $s <_{L_\alpha} t$  hence  $\text{dp}(s, M_{\mathbf{m}}) < \text{dp}(t, M_{\mathbf{m}}) = \beta$  hence  $\text{dp}(s, M_{\mathbf{m}}) \leq \beta$ , recalling Definition ??(1)( $\beta$ ) hence we have  $\text{dp}_{\mathbf{m}_\alpha}^*(L_{0,\alpha}) = \sup\{\text{dp}(s, M_{\mathbf{m}}) + 1, s \in L \cap M_{\mathbf{m}}\} \leq \beta$  as promised. {c2}

For clause (b), by the induction hypothesis for  $\alpha_1 < \alpha_2$  from  $\{2\varepsilon + 2 : \varepsilon < \chi^+\}$  we have  $\mathbb{P}_{\mathbf{m}_{\alpha_1}}[L_{0,\alpha_1}^*] = \mathbb{P}_{\mathbf{m}_{\alpha_2}}[L_{0,\alpha_1}^*] \ll \mathbb{P}_{\mathbf{m}_{\alpha_2}}[L_{0,\alpha_2}^*]$ . This implies the result also when we allow  $\text{cf}(\alpha_\ell) = \lambda$  by 2.9.] {c39}

So necessarily

(\*)<sub>6</sub>  $\mathbb{I}_\alpha$  is constant for  $\alpha < \chi^+$  large enough.

Hence recalling  $(*)_3$  for some  $\beta(*) \in (\alpha(*), \chi^+)$  we have

$$(*)_7 \langle \mathbb{P}_{\mathbf{m}_\beta}[L_1^*] : \alpha < S_* \rangle \text{ is constant for } \beta \in [\beta(*), \chi^+).$$

But this implies the result for  $L_2^*$  as the  $t \in L_2^* \setminus L_1^*$  does not “have memory outside  $L_1^*$ ”, i.e.  $u_{\mathbf{m}_{\beta(*)}, t} \subseteq L_1^*$  and is constant, so we are done.

Subcase 3B: We are stuck in  $\varepsilon$ , i.e. cannot define  $\mathbf{m}_\varepsilon$ .

Now  $\varepsilon = 0$  is impossible. If  $\varepsilon$  is a limit, let  $\mathbf{m}_\varepsilon = \cup\{\mathbf{m}_\zeta : \zeta < \varepsilon\}$  so  $\zeta < \varepsilon \Rightarrow \mathbf{m}_\zeta \leq_{\mathbf{N}} \mathbf{m}_\varepsilon \in \mathbf{M}_\chi$ , contradiction. Also  $\varepsilon = \zeta + 1, \zeta = 0$  is impossible.

If  $\varepsilon = 2\zeta + 1$  so then clause (d) of  $(*)$  is relevant so there is no  $\mathbf{n} \in \mathbf{M}_\chi$  such that  $\mathbf{m}_{2\zeta} \leq_{\mathbf{N}} \mathbf{n}$  and  $L \in \text{Sub}_*(\mathbf{m}_{2\zeta})$  such that  $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \neq \mathbb{P}_{\mathbf{m}_{2\zeta}}[L]$  and  $\text{dp}_{\mathbf{m}_{2\zeta}}^*(L) < \gamma$ .  
 {c32} By 2.8(1) this applies also to  $L \in \text{Sub}(\mathbf{m}_{2\zeta})$ . By claim 2.6(7) this applies also to  
 {c38}  $\mathbf{n} \in \mathbf{M}$ . Together so by 2.8(4) we can deduce that  $\mathbf{m}_{2\zeta} \in \mathbf{M}_\gamma^{\text{ec}}$  so we are done.

Lastly, if  $\varepsilon = 2\zeta + 2$  so clause (e) of  $(*)$  applies, then use the induction hypothesis for  $\beta + 1$  to choose  $\mathbf{m}_\varepsilon$ .

Case 4:  $\gamma$  is limit.

Let  $\delta < \chi^+$  be divisible by  $\gamma$  and of cofinality  $\lambda_0^+$ , recalling  $\lambda_0^+ \leq \chi$ .

We choose  $\mathbf{m}_\alpha$  for  $\alpha \leq \delta_0^+$  such that:

- $\mathbf{m}_\alpha \in \mathbf{M}_\chi$
- $\alpha = 0 \Rightarrow \mathbf{m}_\alpha = \mathbf{m}$
- $\mathbf{m}_\alpha$  is  $\leq_{\mathbf{M}}$ -increasing continuous with  $\alpha$
- if  $\varepsilon < \gamma$  and  $\alpha = \varepsilon + 1 \pmod{\gamma}$  then  $\mathbf{m}_\alpha \in \mathbf{M}_\varepsilon^{\text{ec}}$ .

{c39} Clearly possible: if  $\alpha = 0$  trivial, if  $\alpha$  is a successor ordinal use the induction hypothesis and if  $\alpha$  is a limit recall  $\delta < \chi^+$  and use 2.9. Now  $\mathbf{m}_\delta$  is as required.

{c35} Why? Toward contradiction assume  $\mathbf{m}_\delta \leq_{\mathbf{M}} \mathbf{n}$  and  $L \in \text{Sub}_*(\mathbf{m}_{\lambda_0^+})$  satisfies  $\gamma > \text{dp}_{\mathbf{m}_\delta}(L)$ ; let  $\varepsilon := \text{dp}_{\mathbf{m}_\alpha}(L)$ . By 2.7(2)( $\beta$ ) there is  $\alpha < \delta$  such that  $L \in \text{Sub}_*(\mathbf{m}_\alpha)$  and without loss of generality  $\alpha = \varepsilon + 1 \pmod{\gamma}$ , so we use the choice of  $\mathbf{m}_{\alpha+1}$  to get contradiction.

{c39} So  $\mathbf{m}_\delta$  is as required by 2.9.

Case 5: None of the above.

So  $\gamma = \beta + 1, \beta$  a limit ordinal.

{c46} For  $\alpha \leq \beta$  let  $L_\alpha = \{s \in L : \text{there is no } t \leq s \text{ satisfying } t \in M_{\mathbf{m}} \text{ and } \text{dp}(t, M_{\mathbf{m}}) \geq \alpha\}$ . So  $\langle L_\alpha : \alpha \leq \beta \rangle$  is  $\subseteq$ -increasing continuous and  $L \setminus L_\beta^* \subseteq L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ . As in Case 3 we first deal with  $L_\beta$ , and it is straight, and finish as there.  $\square??$

{c5} **Claim 2.11.** Assume  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}, M \subseteq M_{\mathbf{m}}$  and  $\mathbf{n}$  is defined just like  $\mathbf{m}$  except that  $M_{\mathbf{n}} = M$  and  $E_{\mathbf{n}} = \{(s, t) : s, t \in L_{\mathbf{m}} \text{ but } \{s, t\} \not\subseteq M_{\mathbf{n}}\}$ ; so by ??(5) clearly  $\mathbf{n}$  is equivalent to  $\mathbf{m}$  and  $M_{\mathbf{n}} \subseteq M_{\mathbf{m}}$  but we have to change the context by replacing  $\lambda_0$  by any  $\lambda'_0 \geq \lambda_0 + \|L_{\mathbf{m}}\|$ .

{c30} Then  $\mathbf{n} \in \mathbf{M}^{\text{fc}}[\lambda'_0]$ , see Definition 2.5(1).

*Proof.* Assume that

$$(*)_1 \mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1.$$

We now define  $\mathbf{m}_1 \in \mathbf{M}$  by

$$(*)_2 (a) \quad L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$$



- (b)  $\bar{u}_{\mathbf{m}_1} = \bar{u}_{\mathbf{n}_1}$
- (c)  $M_{\mathbf{m}_1} = M_{\mathbf{m}}$
- (d)  $E_{\mathbf{m}_1} = E_{\mathbf{m}} \cup (E_{\mathbf{n}_1} \upharpoonright (L_{\mathbf{n}_1} \setminus (L_{\mathbf{n}} \setminus M)))$ .

Clearly

- (\*)<sub>3</sub> (a) indeed  $\mathbf{m}_1 \in \mathbf{M}$
- (b)  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1$
- (c)  $\mathbf{m}_1, \mathbf{n}_1$  are equivalent.

But  $\mathbf{m} \in \mathbf{M}_{\text{fc}}$  hence

- (\*)<sub>4</sub>  $\mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{m}_1}$ .

But  $\mathbb{P}_{\mathbf{n}} = \mathbb{P}_{\mathbf{m}}$  and  $\mathbb{P}_{\mathbf{n}_1} = \mathbb{P}_{\mathbf{n}}$  by ??(7) as  $\mathbf{n}, \mathbf{m}$  are equivalent and  $\mathbf{n}_1, \mathbf{m}_1$  are equivalent, so {c11}

- (\*)<sub>5</sub>  $\mathbb{P}_{\mathbf{n}} \triangleleft \mathbb{P}_{\mathbf{n}_1}$ .

So we are done. □<sub>2.11</sub>

**The Uniqueness Claim 2.12.** *There is an isomorphism from  $\mathbb{P}_{\mathbf{m}_1}[M_1]$  onto  $\mathbb{P}_{\mathbf{m}_2}[M_2]$  which (recalling Definition ??) maps  $p_{t,\eta}^*$  to  $p_{h(t),\eta}^*$  for  $t \in M_1, \eta \in \cup\{\prod_{\varepsilon < \zeta} \theta_\varepsilon : \zeta < \lambda\}$  when :* {c49y} {c19}

- ⊕ (a)  $\mathbf{m}_\ell \in \mathbf{M}_\infty^{\text{ec}}$  for  $\ell = 1, 2$
- (b)  $M_\ell \subseteq M_{\mathbf{m}_\ell}$  for  $\ell = 1, 2$
- (c)  $h$  is an isomorphism from  $\mathbf{m}_1 \upharpoonright M_1$  onto  $\mathbf{m}_2 \upharpoonright M_2$ .

*Proof.* By renaming without loss of generality  $M_1 = M_2$  call it  $M$  and  $h$  is the identity and  $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$ . For  $\ell = 1, 2$  let  $\mathbf{m}'_\ell$  be like  $\mathbf{n}$  in 2.11 with  $\mathbf{m}_\ell, M_\ell$  here standing for  $\mathbf{m}, M$  there so {c46}

- (\*)<sub>1</sub>  $\mathbf{m}'_\ell \in \mathbf{M}_{\text{fc}}[\lambda'_0]$ .

We define  $\mathbf{m}$  by:

- (\*)<sub>2</sub> (a)  $s \in L_{\mathbf{m}}$  iff  $s \in L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$
- (b)  $r \leq_{\mathbf{m}} t$  iff  $r \leq_{\mathbf{m}_1} t$  or  $r \leq_{\mathbf{m}_2} t$  or  $(\exists s \in M)(r \leq_{\mathbf{m}_1} s \leq_{\mathbf{m}_2} t)$  or  $(\exists s \in M)(r \leq_{\mathbf{m}_2} s \leq_{\mathbf{m}_1} t)$
- (c)  $u_{\mathbf{m},t}$  is  $u_{\mathbf{m}_1,t}$  if  $t \in L_{\mathbf{m}_1} \setminus M$ ,  
is  $u_{\mathbf{m}_2,t}$  if  $t \in L_{\mathbf{m}_2} \setminus M$  and  
is  $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$  if  $t \in M$
- (d)  $M_{\mathbf{m}} = M$
- (e)  $E_{\mathbf{m}} = E'_{\mathbf{m}_1} \cup E'_{\mathbf{m}_2}$ .

For  $\ell = 1, 2, \mathbb{P}_{\mathbf{m}_\ell}[M] = \mathbb{P}_{\mathbf{m}'_\ell}[M]$  by 2.11 and clearly  $\mathbf{m}'_\ell \leq_{\mathbf{M}} \mathbf{m}$ , but  $\mathbf{m}'_\ell \in \mathbf{M}_{\text{fc}}[\lambda'_0]$ , {c46}  
by 2.11, hence  $\mathbb{P}_{\mathbf{m}'_\ell}[M] = \mathbb{P}_{\mathbf{m}}[M]$ . Together  $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}'_1}[M] = \mathbb{P}_{\mathbf{m}}[M] = \mathbb{P}_{\mathbf{m}_2}[M] = \mathbb{P}_{\mathbf{m}'_2}[M] = \mathbb{P}_{\mathbf{m}_2}[M]$  as required. □<sub>2.11</sub> {c46}

**Conclusion 2.13.** *For every ordinal  $\delta_*$  there is  $\mathbf{q} = \langle \mathbb{P}_\alpha, \eta_\alpha : \alpha \leq \delta_* \rangle$  such that* {c51y}

- (A) (a)  $\langle \mathbb{P}_\alpha : \alpha \leq \delta_* \rangle$  is  $\triangleleft$ -increasing

- {c19} (b)  $\eta_\alpha$  is a  $\mathbb{P}_{\alpha+1}$ -name of a member of  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  which dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_\alpha]}$
- (c)  $\eta_\alpha$  is a generic for  $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$ , moreover  $\langle \eta_\beta : \beta < \alpha \rangle$  is a generic for  $\mathbb{P}_\alpha$
- (d) every  $p \in \mathbb{P}_\alpha$  is from  $\mathbb{L}_{\lambda^+}(Y_{<\alpha}, \mathbb{P}_\alpha)$  when  $Y_{<\alpha}$  is as in Definition ??
- (e)  $\mathbb{P}_\alpha$  is  $\lambda$ -strategically complete and  $\lambda^+$ -c.c.
- (f) if  $\delta \leq \delta_*$  has cofinality  $> \lambda$  then  $\mathbb{P}_\delta = \cup \{\mathbb{P}_\alpha : \alpha < \delta\}$
- (g)  $\mathbb{P}_{\delta_*}$  has cardinality  $|\delta_*|^\lambda$ .
- (B) if  $\mathcal{U} \subseteq \delta_*$  then the complete subforcing generated by  $\langle \eta_\alpha : \alpha \in \mathcal{U} \rangle$  is isomorphic to  $\mathbb{P}_{\text{otp}(\mathcal{U})}$
- (C) if  $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$  is generic over  $\mathbf{V}$  and  $\eta_\alpha = \eta_\alpha[\mathbf{G}]$  for  $\alpha < \delta_*$  and  $\eta'_\alpha \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  for  $\alpha < \delta_*$  and  $\{(\alpha, \varepsilon) : \alpha < \delta_*, \varepsilon < \lambda \text{ and } \eta'_\alpha(\varepsilon) \neq \eta_\alpha(\varepsilon)\}$  has cardinality  $< \lambda$  then also  $\langle \eta'_\alpha : \alpha < \delta_* \rangle$  is a generic for  $\mathbb{P}_{\delta_*}$ , determining a different  $\mathbf{G}'$  but  $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\mathbf{G}]$ .

*Proof.* We define  $\mathbf{m} \in \mathbf{M}$  by:

- (\*) (a)  $L_{\mathbf{m}} = \delta_*$   
 (b)  $M_{\mathbf{m}} = \delta_*$   
 (c)  $u_{\mathbf{m}, \alpha} = \alpha$  for  $\alpha < \delta_*$   
 (d)  $E_{\mathbf{m}} = \emptyset$ .

{c41} It is easy to check that indeed  $\mathbf{m} \in \mathbf{M}$ . So by ??(1) there is  $\mathbf{n} \in \mathbf{M}_\infty^{\text{cc}}$  such that  
 {c33}  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ . Let  $\mathbb{P}_\alpha := \mathbb{P}_{\mathbf{n}}[\{\beta : \beta < \alpha\}]$ , so by ?? we are done.  $\square_{??}$

Similarly we can prove

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