ON CON($\mathfrak{d}_\lambda >$ COV$_\lambda$(MEAGRE))
SH945

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Abstract. We prove the consistency of: for suitable strongly inaccessible cardinal λ the dominating number, i.e., the cofinality of ${}^\lambda\lambda$, is strictly bigger than cov_λ(meagre), i.e. the minimal number of nowhere dense subsets of ${}^\lambda 2$ needed to cover it. This answers a question of Matet.

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§ 0. Introduction

Cardinal invariants of the continuum have a long tradition of research. For a topologist, it can be viewed as investigating the space $\beta(\omega)$, the Stone-Čech compactification of ω. This point of view is taken, for example, in the celebrated paper of van Douwen [vD84]. For the set theoretic perspective see the recent excellent surveys, Blass [Bla], Bartoszyński [Bar10].

For set theorists, it is interesting to check the relationship between the relevant cardinal invariants. In this context, it is natural to generalize the problems to higher cardinals, above $\aleph_0$. One finds out, very soon, that for the class of (strongly) inaccessible cardinals, the generalizations are more reasonable and have more affinity to the $\aleph_0$ case. See Landver [Lan92], Cummings-Shelah [CuSh:541], Matet-Shelah [MtSh:804].

We shall define three cardinal invariants (but the paper deals, actually, just with two of them):

{z1}

Definition 0.1. The bounding and dominating numbers.

Let λ be an inaccessible cardinal.

Let $f, g \in {}^\lambda\lambda$

- (a) $f \leq^* g$ if $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$
- (b) $A \subseteq {}^\lambda\lambda$ is unbounded if there is no $h \in {}^\lambda\lambda$ so that $f \in A \Rightarrow f \leq^* h$
- (c) $A \subseteq {}^\lambda\lambda$ is dominating when for every $f \in {}^\lambda\lambda$ there exists $g \in A$ so that $f \leq^* g$
- (d) the bounding number for λ, denoted by $\mathfrak{b}_\lambda$, is $\min\{|A| : A$ is unbounded in ${}^\lambda\lambda\}$
- (e) the dominating number for λ, denoted by $\mathfrak{d}_\lambda$, is $\min\{|A| : A$ is dominating in ${}^\lambda\lambda\}$.

Notice that the usual definitions of $\mathfrak{b}$ and $\mathfrak{d}$ are $\mathfrak{b}_{\aleph_0}$ and $\mathfrak{d}_{\aleph_0}$ according to Definition 0.1. The definition of $\text{cov}_\lambda(\text{meagre})$ involves some topology.

{z1}
{cov.1}

Definition 0.2. The meagre covering number.

Let λ be a regular cardinal

- (a) ${}^\lambda 2$ is the space of functions from λ into 2
- (b) $({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$, for $\nu \in {}^{\lambda>}2 := \bigcup\limits_{\alpha<\lambda} {}^\alpha 2$
- (c) $\mathscr{U} \subseteq {}^\lambda 2$ is open in the topology $({}^\lambda 2)_{<\lambda}$, iff for every $\eta \in \mathscr{U}$ there exists $i < \lambda$ so that $({}^\lambda 2)^{[\eta \restriction i]} \subseteq \mathscr{U}$
- (d) $\text{cov}_\lambda(\text{meagre})$ is the minimal cardinality of a family of meagre subsets of $({}^\lambda 2)_{<\lambda}$, which covers this space.

This paper deals with the relationship between $\mathfrak{d}_\lambda$ and $\text{cov}_\lambda(\text{meagre})$. Matet asked (a personal communication) whether $\mathfrak{d}_\lambda \leq \text{cov}_\lambda(\text{meagre})$ is provable in ZFC. We give here a negative answer.

For λ a supercompact cardinal and $\lambda < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$, we force large $\mathfrak{d}_\lambda$ i.e., $\mathfrak{d}_\lambda = \mu$ and small covering number (i.e., $\text{cov}_\lambda(\text{meagre}) = \kappa$). A similar result should hold also for a wider class of cardinals and we intend to return elsewhere to this subject.

A point which in a previous version was just a step along the way the referee asked to justify fully, becomes a major point to which §2 - §4 are dedicated. A posteriori the point is that in the parallel case for $\lambda = \aleph_0$, such a claim is true. In fact, by Judah-Shelah [JdSh:292], if $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*)\rangle$ is FS iteration of Suslin-c.c.c. forcing notion, $\underset{\sim}{\mathbb{Q}}_\beta$ with the generic $\underset{\sim}{\eta}_\beta \in {}^\omega\omega$ and for notational transparency, its definition is with no parameter and $\zeta : \beta(*) \to \alpha(*)$ is increasing and $\mathbb{P} = \langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\beta : \alpha \leq \beta(*), \beta < \beta(*)\rangle$ is FS iteration, $\underset{\sim}{\mathbb{Q}}'_{\zeta(\beta)}$ defined exactly as $\underset{\sim}{\mathbb{Q}}_\beta$ but now in $\mathbf{V}^{\mathbb{P}'_\beta}$ then $\Vdash_{\mathbb{P}_{\alpha(*)}}$ "$\langle \underset{\sim}{\eta}_{\zeta(\beta)} : \beta < \beta(*)\rangle$ is generic for $\mathbb{P}'_{\beta(*)}$ over $\mathbf{V}$".

Now this is not clear to us for $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions. The solution is essentially to change the iteration: we use a "quite generic" $(< \lambda)$-support iteration which "includes" the one we like and use the complete subforcing it generates. Here we do only what is needed. On the general case this is continued in a work with H. Horowitz in preparation.

We try to use standard notation. We use $\theta, \kappa, \lambda, \mu, \chi$ for cardinals and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ for ordinals. We use also i and j as ordinals. We adopt the Cohen convention that $p \leq q$ means that q gives more information, in forcing notions. The symbol $\triangleleft$ is preserved for "being an initial segment". Also recall ${}^B A = \{f : f$ a function from B to $A\}$ and let ${}^{\alpha>}A = \cup\{{}^\beta A : \beta < \alpha\}$, some prefer ${}^{<\alpha}A$, but ${}^{\alpha>}A$ is used systematically in the author's papers. Lastly, $J^{\rm bd}_\lambda$ denotes the ideal of the bounded subsets of λ.

The picture of cardinal invariants related to uncountable λ is related but usually quite different than the one for $\aleph_0$, they are more similar if κ is "large" enough, mainly strongly inaccessible.

Let us sketch some known results. These results are related to the unequality number and the covering number for category. Recall:

{z14}

Definition 0.3. The unequality number.

Let κ be an infinite cardinal. The unequality number of $\kappa, \mathfrak{e}_\kappa$, is the minimal cardinal λ such that there is a set $\mathscr{F} \subseteq {}^\kappa\kappa$ of cardinality λ such that there is no $g \in {}^\lambda\lambda$ satisfying $(\forall f \in \mathscr{F})(\exists^\kappa \alpha < \lambda)(f(\alpha) = g(\alpha))$.

For $\kappa = \aleph_0, \mathfrak{e}_\kappa =$ cov$_{\aleph_0}$(meagre); see Bartoszyński (in [Bar87]) and Miller (in [Mil82]).

Now

(a) the statement $\mathfrak{e}_\kappa =$ cov$_\kappa$(meagre) is valid for $\kappa > \aleph_0$, in the case that κ is strongly inaccessible, by [Lan92]. But if κ is a successor cardinal, it may fail

(b) if $\kappa < \kappa^{<\kappa}$, then cov$_\kappa$(meagre) $= \kappa^+$. This is due to Landver (in [Lan92]).

We intend also to address:

{z15}

Problem 0.4. Can we replace "supercompact" by "strongly inaccessible"?

{z16}

Problem 0.5. 1) Can we prove the consistency of cov$_\lambda$(meagre) $< \mathfrak{b}_\lambda$?
2) For λ strongly inaccessible (or just Laver indestructible supercompact) is there a non-trivial λ^+-c.c. $(< \lambda)$-strategically complete forcing notion $\mathbb{Q}$ which is ${}^\lambda\lambda$-bounding?

We thank the referee, Shimoni Garti and Haim Horowitz for helpful comments. We say more in subsequent works [Sh:1004] and in preparation [Sh:F1199] and a work of Horowitz and the author on generalizing §2,§3,§4.

{z20}

Definition 0.6. 1) Fix $\lambda = \lambda^{<\lambda}$, the forcing $\mathbb{Q} = \mathbb{Q}^{\rm dom}_\lambda$ is defined by:

(α) $p \in \mathbb{Q}$ <u>iff</u>
- (a) $p = (\eta, f) = (\eta^p, f^p)$
- (b) $\eta \in {}^\varepsilon\lambda$ for some $\varepsilon < \lambda$, (η is called the trunk of p)
- (c) $f \in {}^\lambda\lambda$
- (d) $\eta \triangleleft f$

(β) $p \le_{\mathbb{Q}} q$ iff
- (a) $\eta^p \trianglelefteq \eta^q$
- (b) $f^p \le f^q$, i.e. $(\forall \varepsilon < \lambda) f^p(\varepsilon) \le f^q(\varepsilon)$
- (c) if[1] $\ell g(\eta^p) \le \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$.

2) The generic is $\underset{\sim}{\eta} = \cup\{\eta^p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}\}$.

{z23}

Definition 0.7. Let λ be inaccessible, $\bar\theta = \langle \theta_\varepsilon : \varepsilon < \lambda\rangle$ be a sequence of regular cardinals $< \lambda$ satisfying $\theta_\varepsilon > \varepsilon$.
1) We define the forcing notion $\mathbb{Q} = \mathbb{Q}_{\bar\theta}$ by

(α) $p \in \mathbb{Q}$ <u>iff</u>
- (a) $p = (\eta, f) = (\eta^p, f^p)$
- (b) $\eta \in \prod\limits_{\zeta < \varepsilon} \theta_\zeta$ for some $\varepsilon < \lambda$, (η is called the trunk of p)
- (c) $f \in \prod\limits_{\zeta < \lambda} \theta_\zeta$
- (d) $\eta \triangleleft f$

(β) $p \le_{\mathbb{Q}} q$ iff
- (a) $\eta^p \trianglelefteq \eta^q$
- (b) $f^p \le f^q$, i.e. $(\forall \varepsilon < \lambda) f^p(\varepsilon) \le f^q(\varepsilon)$
- (c) if $\ell g(\eta^p) \le \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$.

2) The generic is $\underset{\sim}{\eta} = \cup\{\eta^p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\bar\theta}}\}$.

{z25}

Remark 0.8. The forcing is parallel to the creature forcing from [Sh:326, §1,§2] which are ${}^\omega\omega$-bounding.

Recall

{z26}

Definition 0.9. Let κ be supercompact. We say $f : \kappa \to \mathscr{H}(\kappa)$ is a Laver diamond (for κ) <u>when</u> for every $x \in \mathbf{V}$ there are a normal fine ultrafilter on $I = [\lambda]^{<\kappa}$ for some λ such that the Mostowski collapse $\mathbf{j}$ on $\mathbf{V}^I/D$ maps $\langle f(\sup(u \cap \kappa)) : u \in I\rangle/D$ to x; (we can use elementary embeddings instead of an ultrafilter).

{z29}

Notation 0.10. If $\mathbb{P}$ is a forcing notion in $\mathbf{V}$ then $\mathbf{V}^{\mathbb{P}}$ denotes $\mathbf{V}[\mathbf{G}]$ for $\mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$; but in superscript we may write $\mathbf{V}[\mathbb{P}]$ instead.

[1]Actually this follows from $(\alpha) + (\beta)(a), (b)$; that is, if $\ell g(\eta^p) \le \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) = f^q(\varepsilon)$ by clause $(\alpha)(d)$ and $f^p(\varepsilon) \le f^q(\varepsilon)$ by clause $(\beta)(b)$.

§ 1. The forcing

We intend to prove here

Theorem 1.1. *Assume* {a2}

(a) λ *is supercompact*

(b) $\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mathrm{cf}(\mu) = \mu^\lambda$.

Then for some forcing notion $\mathbb{P}$ not collapsing cardinals $\geq \lambda, \lambda$ is still supercompact in $\mathbf{V}^{\mathbb{P}}$ and $\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa, \mathfrak{d}_\lambda = \mu$.

Proof. By 1.3 below. $\square_{1.1}$ {a7}

Recall

Definition 1.2. 1) We say that a forcing notion $\mathbb{P}$ is α-strategically complete <u>when</u> for each $p \in \mathbb{P}$ in the following game $\Game_\alpha(p, \mathbb{P})$ between the players COM and INC, the player COM has a winning strategy. {a5}

A play lasts α moves; in the β-th move, first the player COM chooses $p_\beta \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\beta$ and $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$ and second the player INC chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq_{\mathbb{P}} q_\beta$.

The player COM wins a play if he has a legal move for every $\beta < \alpha$.

2) We say that a forcing notion $\mathbb{P}$ is $(< \lambda)$-strategically complete <u>when</u> it is α-strategically complete for every $\alpha < \lambda$.

{a7}

Lemma 1.3. *1) If λ is supercompact <u>then</u> after some preliminary forcing of cardinality λ, λ is still supercompact and $\boxdot_\lambda$ below holds.*
1A) The statement $\boxdot_\lambda$ holds in $\mathbf{V}^{\mathbb{P}}$ <u>when</u> $\mathbf{V}$ satisfies $\boxdot_\lambda$ and $\mathbb{P}$ is a $(< \lambda)$-strategically complete forcing notion and in $\mathbb{P}$ any directed system of cardinality $< \lambda$ of conditions in $\mathbb{P}$ has an upper bound.
2) If λ is strongly inaccessible and $\boxdot_\lambda$ below holds and $\lambda^+ < \kappa = \mathrm{cf}(\kappa) < \mu = \mu^\lambda$ and μ is regular (for transparency), <u>then</u> for some λ^+-c.c., $(< \lambda)$-strategically complete forcing notion $\mathbb{P}$ we have $\Vdash_{\mathbb{P}}$ "$\mathfrak{d}_\lambda = \mu$, $\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$"

Definition 1.4. For λ supercompact we define $\boxdot_\lambda$ by: {a8}

$\boxdot_\lambda$ for any regular cardinal $\chi > \lambda$ and forcing notion $\mathbb{P} \in \mathscr{H}(\chi)$ which is $(< \lambda)$-strategically complete (see Definition 1.2(2)) the following set $\mathscr{S} = \mathscr{S}_{\mathbb{P}} = \mathscr{S}_{\chi, \mathbb{P}}$ is a stationary subset of $[\mathscr{H}(\chi)]^{< \lambda}$: {a5}

$\mathscr{S} = \mathscr{S}_{\mathbb{P}} = \mathscr{S}_{\chi, \mathbb{P}}$ is the set of N's such that for some $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N, \mathbb{A} = \mathbb{A}_N, M = M_N, \mathbf{G} = \mathbf{G}_N$ we have (and we may say $(\lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ is a witness for $N \in \mathscr{S}_{\chi, \mathbb{P}}$ or for $(N, \mathbb{P}, \chi)$):

(a) $N \prec (\mathscr{H}(\chi)^{\mathbf{V}}, \in)$ and $\mathbb{P} \in N$

(b) the Mostowski collapse of N is $\mathbb{A}'$ and let $\mathbf{j}_N : N \to \mathbb{A}$ be the unique isomorphism

(c) $N \cap \lambda = \lambda_N$ and ${}^{(\lambda_N)>}N \subseteq N$ and λ_N is strongly inaccessible

(d) $\mathbb{A} \subseteq M := (\mathscr{H}(\chi_N), \in)$ so both $\mathbb{A}$ and M are transitive

(e) $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$ is generic over N' for the forcing notion $\mathbf{j}_N(\mathbb{P})$

(f) $M = \mathbb{A}[\mathbf{G}]$.

Recall that:

{a7.3}

Fact 1.5.

(*a*) if $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is a $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions, then $\mathbb{P}_\delta$ is also $< \lambda$-strategically complete; (see e.g. [Sh:546]).

(*b*) If $\mathbb{P}$ is $(< \lambda)$-strategically complete forcing notion then $({}^{\lambda >}\mathrm{Ord})^{\mathbf{V}} = ({}^{\lambda >}\mathrm{Ord})^{\mathbf{V}^{\mathbb{P}}}$, and consequently λ is strongly inaccessible in $\mathbf{V}^{\mathbb{P}}$

(*c*) like (a) replacing $(< \lambda)$-strategically complete" by "$(< \lambda)$-complete"

(*d*) if $\mathbb{P}$ is $(< \lambda)$-complete then $\mathbb{P}$ is λ-strategically complete.

Proof. Proof of Lemma 1.3 1) This is essentially by Laver [Lav78] using Laver's {z26} diamond, see Definition 0.9, but for completeness we elaborate. By Laver [Lav78] {z26} without loss of generality there is a Laver diamond $h : \lambda \to \mathscr{H}(\lambda)$, see Definition 0.9. Let $E = \{\theta : \theta < \lambda \text{ and } \alpha < \theta \Rightarrow h(\alpha) \in \mathscr{H}(\theta)\}$, clearly a club of λ and let $\langle \kappa_\varepsilon : \varepsilon < \lambda \rangle$ list $\{\theta \in E : \theta \text{ strongly inaccessible}\}$ in increasing order.

We now define $\mathbf{q}_\varepsilon$ and $\bar{\chi}^\varepsilon$ by induction on $\varepsilon \leq \lambda$ such that:

$(*)$ (a) $\mathbf{q}_\varepsilon = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \leq \varepsilon, \xi < \varepsilon \rangle$ is an Easton support iteration (so $\mathbb{P}_\zeta$. $\underset{\sim}{\mathbb{Q}}_\xi$ do not depend on ε, etc.)

(b) $\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_\zeta)$

(c) $\bar{\chi}^\varepsilon = \langle \chi_\xi : \zeta < \xi \rangle$ where each χ_ξ is a regular cardinal $\in [\kappa_\xi, \kappa_{\xi+1})$

(d) $\underset{\sim}{\mathbb{Q}}_\xi \in \mathscr{H}(\chi_{\xi+1})$ is a $\mathbb{P}_\xi$-name of a κ_ξ^+-c.c. $(< \kappa_\xi)$-strategically complete forcing notion

(e) if $h(\xi) = (\underset{\sim}{\mathbb{Q}}, \chi)$ and the pair $(\underset{\sim}{\mathbb{Q}}, \chi)$ satisfies the requirements on $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\varepsilon)$ in clauses (c),(d) then $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\xi) = h(\xi)$.

Easily we can carry the induction so $\mathbf{q}_\lambda$ is well defined, $\mathbb{P}_\lambda \subseteq \mathscr{H}(\lambda)$ and "$\xi < \lambda \Rightarrow \mathbb{P}_\lambda / \mathbb{P}_\xi$ is $(< \kappa_\xi)$-strategically complete" hence $\mathbb{P}_\lambda / \mathbb{P}_\xi$ adds no new sequence of length $< \kappa_\xi$ of ordinals. Clearly it is enough to prove that in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $\boxdot_\lambda$.

Toward contradiction assume $\chi, \mathbb{P}, \mathscr{S} = \mathscr{S}, \chi, \mathbb{P}$ forms a counter-example in $\mathbf{V}^{\mathbb{P}_\lambda}$, hence there are $p_* \in \mathbb{P}_\lambda$ and $\mathbb{P}_\lambda$-name $\underset{\sim}{\chi}, \underset{\sim}{\mathbb{P}}, \underset{\sim}{\mathscr{S}}, \underset{\sim}{E}$ such that $p_* \Vdash_{\mathbb{P}_\lambda}$ "$\underset{\sim}{\chi} > \lambda$ is regular, $\underset{\sim}{\mathbb{P}} \in \mathscr{H}(\underset{\sim}{\chi})$ and $\underset{\sim}{\mathscr{S}}_{\underset{\sim}{\chi}, \mathbb{P}}$ is defined as in $\boxdot_\lambda$ and $\underset{\sim}{E} \subseteq [\mathscr{H}(\chi)^{\mathbf{V}[\mathbb{P}_\lambda]}]^{<\lambda}$ is a club disjoint to $\underset{\sim}{\mathscr{S}}$".

As we can increase p_*, without loss of generality $\underset{\sim}{\chi} = \chi$; and as $\mathbf{V} \models$ "λ is supercompact and h is a Laver diamond" for some $(\mathbf{M}, \mathbf{j})$ we have

$(*)$ (*a*) $\mathbf{M}$ is a transitive class

(*b*) $\mathbf{M}$ is a model of ZFC

(*c*) ${}^\chi\mathbf{M} \subseteq M$

(*d*) $\mathbf{j}$ is an elementary embedding from $\mathbf{V}$ into $\mathbf{M}$

(*e*) $\mathbf{j}$ is with critical cardinal λ

(*f*) $\mathbf{j}(h)(\lambda) = (\underset{\sim}{\mathbb{P}}, \chi)$.

Let $\mathbf{q} = \mathbf{j}(\mathbf{q}_\lambda)$ so $\mathbf{q} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \leq h(\lambda), \xi < h(\lambda) \rangle$ and $\zeta < \lambda \Rightarrow \mathbb{P}_\zeta^{\mathbf{q}} = \mathbb{P}_\zeta$, etc.

Clearly $\mathbf{M} \models$ "$\mathbf{j}(h)(\lambda)$ is a pair of the form $(\mathbb{P}', \chi')$ satisfying all relevant demands".

The rest should be clear.

1A) Let $\mathbb{Q}$ be a forcing notion in $\mathbf{V}^{\mathbb{P}}$ which is $(<\lambda)$-strategically complete, $\emptyset \in \mathbb{Q}$ minimal, χ_1 large enough so that $\lambda, \underset{\sim}{\mathbb{Q}}, \underset{\sim}{E} \in \mathscr{H}(\chi_1)$ and we should prove that in $\mathbf{V}^{\mathbb{P}}$, the set $\mathscr{S}_{\chi_1}, \mathbb{Q}$ is stationary. So let $\underset{\sim}{\mathbb{Q}}, \underset{\sim}{E}$ be $\mathbb{P}$-names such that for some $p \in \mathbb{P}$ we have $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi_1)$ is $(<\lambda)$-strategically complete forcing notion, $\underset{\sim}{E}$ a club of $[\mathscr{H}(\chi_1)]^{<\lambda}$ disjoint to $\underset{\sim}{\mathscr{S}}_{\chi_1,\mathbb{P}}$", no need to use a name for χ_1 as we can increase p.

Let $\chi \gg \chi_1$; now $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi)$ is a $(<\lambda)$-strategically complete forcing notion and without loss of generality code (χ_1, E). As $\boxdot_\lambda$ holds in $\mathbf{V}$ we can apply it to the forcing $\mathbb{P}_{\geq p} * \underset{\sim}{\mathbb{Q}}$ so we can find a tuple $(N, \lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ witnessing it, in particular, $(p, \emptyset) \in \mathbf{G}_N, \mathbb{P} * \underset{\sim}{\mathbb{Q}} \in N$ so $\chi_1, \underset{\sim}{E} \in N$. Let $\mathbf{G}_{\mathbb{P}}$ be a subset of $\mathbb{P}$ generic over $\mathbf{V}$ which extends $\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N\}$, possible because $\mathbf{G}_N$ is in $\mathbf{V}$, a subset of $\mathbb{P}$ which has an upper bound. Next let $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_{\mathbb{P}}], N_1 = N[\mathbf{G}_{\mathbb{P}}], E_1 = \underset{\sim}{E}[\mathbf{G}_{\mathbb{P}}], \mathbb{A}_1 = \mathbb{A}'[\mathbf{j}''_N(\mathbf{G}_{\mathbb{P}} \cap N)] = \mathbb{A}'[\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N], \mathbf{G}_1 = \{\underset{\sim}{q}[\mathbf{j}''(G_{\mathbb{P}} \cap N)] : (p, \underset{\sim}{q}) \in \mathbf{G}_{\mathbb{P}}\}$.

Let $N_2 = N_1 {\restriction} \mathscr{H}(\chi_1)^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}, \mathscr{S} = \mathscr{S}_{\underset{\sim}{\mathbb{Q}}}[\mathbf{G}_{\mathbb{P}}], \mathbf{j}_1 =$ the lifting of $(\mathbf{j}{\restriction}(N \cap \mathscr{H}(\chi)))$, to mapping N_1 onto $\mathbb{A}_1$.

Now recalling p forces $\underset{\sim}{E}$ is disjoint to $\underset{\sim}{\mathscr{S}}$ clearly

- $(*)$ $N_2 \in E$.

hence

- $(*)$ $N_1 \notin \mathscr{S}_1$.

But easily in $\mathbf{V}_1$ we have: $(\lambda_N, \chi_N, \mathbf{j}_1, \mathbb{A}_1, M_1 = M, \mathbf{G}_1)$ witness $N_1 \in \mathscr{S} \cap E_1$, a contradiction to the choice of $\underset{\sim}{E}$.

2) <u>Stage A</u>: Without loss of generality $\mathbf{V} \models$ "$\mathfrak{b}_\lambda = \mu = \mathfrak{d}_\lambda$" as witnessed by $\langle f^*_\alpha : \alpha < \mu \rangle$.

No new point, still we elaborate recalling that composition of forcing notions (e.g. $(\mathbb{P} * \underset{\sim}{\mathbb{Q}}^*)$ preserves "$(<\lambda)$-strategically complete and λ^+-c.c."

We use a $(<\lambda)$-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \mu, \beta < \mu \rangle$ with $\mathbb{P}_\mu$ the intended forcing such that:

- (A) if $\alpha < \mu$ <u>then</u> $\underset{\sim}{\mathbb{Q}}_\alpha$ is the ($\mathbb{P}_\alpha$-name of the) dominating forcing, $\mathbb{Q}^{\mathrm{dom}}_\lambda$, i.e. $(\mathbb{Q}^{\mathrm{dom}}_\alpha)^{\mathbf{V}[\mathbb{P}_\alpha]}$ where the forcing $\mathbb{Q} = \mathbb{Q}^{\mathrm{dom}}_\alpha$ is from Definition 0.6. {z20}

Let $\underset{\sim}{f}^*_\alpha$ be the generic object for $\underset{\sim}{\mathbb{Q}}_\alpha$ for $\alpha < \mu$.

Now:

- $(*)_1$ for $\alpha \leq \mu$ the forcing notion $\mathbb{P}_\alpha$ is $(<\lambda)$-complete because when $\alpha < \mu, \underset{\sim}{\mathbb{Q}}_\alpha$ is $(<\lambda)$-complete[2], i.e. $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is $(<\lambda)$-complete".

[Why? We prove this by induction on α; for $\mathbb{P}_\alpha$, recall 1.5(1)(c).] {a7.3}

- $(*)_2$ for each $\alpha \leq \mu$
  - (a) the forcing notion $\mathbb{P}_\alpha$ and for $\alpha < \mu$ satisfy a strong form of the λ^+-c.c., (see [Sh:80] for definition, preservation and history; or fully [Sh:1036], [Sh:546, §1])
  - (b) also the forcing notions $\underset{\sim}{\mathbb{Q}}_\alpha$ satisfies this

  hence

[2] for this, $\theta_\alpha > \alpha$ is enough

$(*)_3$ (a) forcing with $\mathbb{P}_\mu$ collapses no cardinal, changes no cofinality, and adds no sequence to ${}^{\lambda>}\mathbf{V}$;

(b) $({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\mu]} = \cup\{({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]} : \alpha < \mu\}$.

[Why? By $(*)_2 + (*)_1$ clause (a) holds, for clause (b) use also "$\mathbb{P}_\mu$ satisfies the λ^+-c.c." and the support in the iteration being $< \lambda$ recalling that μ is regular $> \lambda$. E.g. if $\Vdash_{\mathbb{P}_\mu}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$" then we can, for $\alpha < \lambda$ find maximal antichain $\langle p_{\alpha,i} : i < i_\alpha \leq \lambda\rangle$ of $\mathbb{P}_\mu$ such that $p_{\alpha,i}$ forces a value to $\underset{\sim}{f}(\alpha)$; let $\alpha(*) = \sup(\cup\{\mathrm{dom}(p_{\alpha,i}) : \alpha < \lambda, i < i_\alpha\})$, so $\alpha(*) < \mu$ and $\underset{\sim}{f}$ is a $\mathbb{P}_{\alpha(*)}$-name.]

$(*)_4$ (a) in $\mathbf{V}^{\mathbb{P}_\mu}, \mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$ as witnessed by $\underset{\sim}{\bar{f}}^* = \langle \underset{\sim}{f}^*_\alpha : \alpha < \mu\rangle$, in fact $\Vdash_{\mathbb{P}_{\alpha+1}}$ "$\underset{\sim}{f}^*_\alpha \in {}^\lambda\lambda$ dominates $({}^\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]}$ modulo J^{bd}_λ"

(b) in $\mathbf{V}^{\mathbb{P}_\mu}$ still $\boxdot_\lambda$ holds.

[Why Clause (a)? Easy using $(*)_3(b)$.

{a7} Why Clause (b)? Easy, by 1.3(1A).]

<u>Stage B</u>: In $\mathbf{V}$ (see Stage A) there are $\beta(*), \mathbf{q}, \bar{u}, \mathscr{U}_*, \ldots$ such that

$(*)_5(A)$ (a) $\mathbf{q} = \langle \mathbb{P}_{0,\alpha}, \underset{\sim}{\mathbb{Q}}_{0,\beta} : \alpha \leq \beta(*), \beta < \beta(*)\rangle$ is a $(< \lambda)$-support iteration

(b) $\bar{u} = \langle u_\beta : \beta < \beta(*)\rangle, \bar{\mathscr{P}} = \langle \mathscr{P}_\beta : \beta < \beta(*)\rangle$

(c) $u_\beta \subseteq \beta, \mathscr{P}_\beta \subseteq [u_\beta]^{\leq\lambda}$ is closed under subsets

(d) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ has generic $\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$

{c7} (e) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ is as in 2.10(4) so is $\subseteq \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u_\beta\rangle]}$ and $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$" and $\underset{\sim}{\bar{\eta}} = \langle \underset{\sim}{\eta}_\beta : \beta < \beta(*)\rangle$

(f) $\mathscr{U}_* \subseteq \beta(*)$ has order type $\gamma(*) = \kappa$ and $\langle \beta^*_i : i < \kappa\rangle$ lists $\mathscr{U}_*$ in increasing order
[explanation: when using §2,§4, $\mathscr{U}_*$ plays the role of $M_{\mathbf{m}}$ there]

(g) if $\beta \in \mathscr{U}_*$ then $\mathscr{U}_* \cap \beta \in \mathscr{P}_\beta$ hence $\subseteq u_\beta$ and $\Vdash_{\mathbb{P}_{0,\beta+1}}$ "if $\nu \in \mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}_* \cap \beta\rangle] \cap \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ then $\nu <_{J^{\mathrm{bd}}_\lambda} \underset{\sim}{\eta}_\beta$"

(h) if $\alpha \leq \beta(*)$ then $\mathbb{P}_{0,\alpha}$ is $(< \lambda)$-strategically complete and λ^+-c.c.

(B) (a) The sequence $\langle \mathbb{P}'_i : i \leq \gamma(*)\rangle$ of forcing notions is $\lessdot$-increasing, and is continuous for ordinals $i \leq \gamma(*)$ of cofinality $\geq \lambda$

(b) $\mathbb{P}'_i$ is $(< \lambda)$-strategically complete for $i \leq \gamma(*)$

(c) $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_{\gamma(*)}]} = \cup\{(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \gamma(*)\}$.

(d) The sequence $\langle \mathbb{P}_{1,\beta} : \beta \leq \beta(*)\rangle$ is a sequence of forcing notions, $\lessdot$-increasing and if $\beta \leq \beta(*)$ then $\mathbb{P}'_\beta \lessdot \mathbb{P}_{1,\beta}$ and $\mathbb{P}_{0,\beta} \lessdot \mathbb{P}_{1,\beta}$, in fact is dense in it.

We shall mention more properties later.

[Why are there such objects? Let M be a linear order isomorphic to $(\kappa, <)$, applying

{c52} 3.12 there is $\mathbf{m}$ as there. As $L_{\mathbf{m}}$ is a well founded partial order, we can let h be a one-to-one order preserving function from $L_{\mathbf{m}}$ onto some ordinal, call it $\beta(*)$.

So without loss of generality h is the identity, so let $L_{\mathbf{m}}$ be $(\beta(*), <_*), \mathscr{U}_* = M_{\mathbf{m}}, \mathbb{P}_{0,\alpha} = \mathbb{P}_{\mathbf{m}}[\{\beta : \beta < \alpha\}]$.

Also

$(*)_6$ (a) let $\langle \beta^*_i : i < \gamma(*)\rangle$ list $\mathscr{U}_*$ in increasing order
(b) for $i \le \kappa$, let $\mathbb{P}'_i$ be $\mathbb{P}_{\mathbf{m}}[\{\beta^*_j : j < i\}]$ and let $\underset{\sim}{g}'_i$ be $\underset{\sim}{\eta}_{\beta^*_i}$ (to avoid excessive subscript), see $(*)_5(A)(e)$ and 3.12 so $\mathbb{P}'_i \lessdot \mathbb{P}_{1,\beta^*_i}$ {c52}
(c) let $\underset{\sim}{\bar g}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$
(d) let $\underset{\sim}{g}_\alpha = \underset{\sim}{\eta}_\alpha$ for $\alpha < \beta(*)$ and $\underset{\sim}{\bar g} = \langle \underset{\sim}{g}_\beta : \beta < \beta(*)\rangle$
(e) $\mathscr{P}_\alpha = \mathscr{P}_{\mathbf{m},\alpha}$ and $u_\alpha = \cup\{u : u \in \mathscr{P}_\alpha\}$.

Now we are done proving $(*)_5$.]

$(*)_7$ if $u \in \mathscr{P}_\alpha, \alpha < \beta(*)$ then $\Vdash_{\mathbb{P}_{0,\alpha+1}}$ "$\underset{\sim}{g}_\alpha \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ dominates $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\langle \underset{\sim}{g}_\beta : \beta \in u\rangle]}$", the order being modulo $J^{\rm bd}_\lambda$.

[Why? As by 3.11 in $\mathbf{V}^{\mathbb{P}_{0,\alpha}}$ for each $g \in (\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\langle \underset{\sim}{\eta}_\beta : \beta \in u\rangle]}$ the set $\{(\eta, f) \in \underset{\sim}{\mathbb{Q}}_{0,\beta}$: for every $\varepsilon \in [\ell g(\eta), \lambda)$ we have $g(\varepsilon) \le f(\varepsilon)\}$ is a pre-dense open subset of $\underset{\sim}{\mathbb{Q}}_{0,\alpha}$.] {c51}

$(*)_8$ $\Vdash_{\mathbb{P}'_\kappa}$ "$\underset{\sim}{\bar g}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and cofinal in $(\prod_{\varepsilon<\lambda} \theta_\varepsilon, <_{J^{\rm bd}_\lambda})$".

[Why? By $(*)_7$ noting that $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup\{(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa\}$ which holds by $(*)_5(B)(c)$.]

Now

$(*)_9$ $\Vdash_{\mathbb{P}'_\kappa}$ "cov$_\lambda$(meagre) $\le \kappa$".

[Why? As we can look at $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ instead[3] of ${}^\lambda 2$ and for each $\varepsilon < \lambda, i < \kappa$ the set $B_{\varepsilon,i} = \{\eta \in \prod\limits_{\xi<\lambda} \theta_\xi$: for every $\zeta \in [\varepsilon, \lambda)$ we have $\eta(\zeta) \le \underset{\sim}{g}'_i(\zeta) < \theta_\zeta\}$ is closed nowhere dense, and by $(*)_8$ we have $\mathbf{V}^{\mathbb{P}'_\kappa} \models$ "$\prod_{\zeta<\lambda} \theta_\zeta = \cup\{B_{\varepsilon,i} : \varepsilon < \lambda, i < \kappa\}$". In fact, $\langle B_{0,i} : i < \kappa\rangle$ suffice.]

$(*)_{10}$ $\Vdash_{\mathbb{P}'_\kappa}$ "cov$_\lambda$(meagre) $\ge \kappa$".

[Why? Let us define the $\mathbb{P}'_{i+1}$-name $\underset{\sim}{\eta}'_i$ of a member of ${}^\lambda 2$ by $\underset{\sim}{\eta}'_i(\varepsilon) = 0$ iff $\underset{\sim}{g}'_i(\varepsilon)$ is even. Now clearly $\Vdash_{\mathbb{P}'_{i+1}}$ "$\underset{\sim}{\eta}'_i$ is a λ-Cohen sequence over $\mathbf{V}^{\mathbb{P}'_i}$". (Let us elaborate; $\underset{\sim}{\eta}'_i$ is also a $\mathbb{P}_{\beta^*_i+1}$-name and $\Vdash_{\mathbb{P}_{\beta^*_i+1}}$ "$\underset{\sim}{\eta}'_i$ is λ-Cohen over $\mathbf{V}^{\mathbb{P}_{\beta^*_i}}$ hence over $\mathbf{V}^{\mathbb{P}'_i}$"; the last hence because $\mathbb{P}'_i \lessdot \mathbb{P}_{\beta^*_i}$. As $\mathbb{P}_{\beta^*_i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ and $\mathbb{P}'_{i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ we are done.)

Also every closed nowhere dense subset of ${}^\lambda 2$ from $\mathbf{V}^{\mathbb{P}'_{\gamma(*)}}$ is from $\mathbf{V}^{\mathbb{P}'_i}$ for some $i < \gamma(*)$. So if $p \Vdash$ "cov$_\lambda$(meagre) $< \kappa$" then for some $\zeta < \kappa$ and $\underset{\sim}{A}_\varepsilon(\varepsilon < \zeta)$ we have $p \Vdash$ "$\underset{\sim}{A}_\varepsilon$ is a closed no-where dense set subset of ${}^\lambda 2$ for $i < j$" and $p \Vdash$ "$\bigcup\limits_i \underset{\sim}{A}_i$ is equal to the set of ${}^\lambda 2$". Without loss of generality each $\underset{\sim}{A}_\varepsilon$ is a $\mathbb{P}_{i(\varepsilon)}$-name,

[3]E.g. let $F : {}^\lambda 2 \to \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ be $F(\eta) = \rho$ iff $\eta \in {}^\lambda 2$ and for every $\varepsilon < \lambda, \rho(\varepsilon) = 0$ iff $(\forall i < \theta_\varepsilon)(\eta(\sum\limits_{\zeta<\varepsilon} \theta_\zeta + i) = 0)$ and $\rho(\varepsilon) = 1 + i$ iff $\eta(\sum\limits_{\zeta<\varepsilon} \theta_\zeta + i) = 1 \wedge (\forall j < i)(\eta(\sum\limits_{\zeta<\varepsilon} \theta_\zeta + j) = 0)$. Now if $\prod\limits_\varepsilon \theta_\varepsilon = \cup\{\mathscr{U}_i : i < \kappa\}$, each $\mathscr{U}_i$ closed nowhere dense then $\langle F^{-1}(\mathscr{U}_i) : i < \kappa\rangle$ witnesses cov$_\lambda$(meagre) $\le \kappa$.

$i(\varepsilon) < \kappa$. Hence $i = \sup\{i(\varepsilon) : \varepsilon < \zeta\} < \kappa$ and $\underset{\sim}{\eta}'_i$ gives a contradiction to the choice of $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \zeta\rangle$; so $(*)_{10}$ holds instead.] $\square_{1.3}$

Discussion 1.6. 1) The reader may justly wonder why we use $\mathbf{V}' = \mathbf{V}[\underset{\sim}{\bar{g}}'] = \mathbf{V}[\underset{\sim}{\bar{g}}\restriction \mathscr{U}_*]$ rather than simply $\mathbf{V}[\underset{\sim}{\bar{g}}]$. Of course, nothing is lost by it, but why the extra complication?
2) The answer is that during the proof we shall use: if $\zeta(i) \in \mathscr{U}_*$ is increasing with $i < \gamma(*)$ then also $\langle g_{\zeta(i)} : i < \kappa\rangle$ is generic over $\mathbf{V}$ for the subforcing of $\mathbb{P}_{1,\beta(*)}$ generated by $\underset{\sim}{\bar{g}}\restriction\mathscr{U}_*$; see $\circledast''_7$ inside the proof of $\circledast_6$. But using $\mathscr{U}_* = \beta(*)$, we do not know this.
3) Now in the parallel case for $\lambda = \aleph_0$ with FS iteration with full memory, such claim is true, see §0.
4) But we do not know the parallel of 3) for λ, so we use a substitute using $\mathscr{U}_*$, i.e. $\mathbb{P}'_\kappa$.

Proof. Continuation of the proof:

Now we come to the main and last point recalling $\langle f^*_\alpha : \alpha < \mu\rangle$ from Stage A

$(*)_{11}$ it is forced, i.e. $\Vdash_{\mathbb{P}'_{\gamma(*)}}$ that no $\underset{\sim}{f} \in ({}^\lambda\lambda)$ dominate $\{f^*_\alpha : \alpha < \mu\}$.

{a7} We shall show that it suffices to prove $(*)_{11}$ for proving 1.3(2), and that $(*)_{11}$ holds, thus finishing.

Why it suffices? As $\langle f^*_\alpha : \alpha < \mu\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and $\mathrm{cf}(\mu) = \mu > \lambda$, this implies $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda \ge \mu$". Also in $\mathbf{V}, 2^\lambda = \mu > \kappa > \lambda$ and $|\mathbb{P}'_{\gamma(*)}| = \kappa^\lambda$ by (A)(g)
{c51} of 3.11 which is $= \mu$ and $\mathbb{P}'_\kappa$ satisfies the λ^+-c.c. hence $\Vdash_{\mathbb{P}'_\kappa}$ "$2^\lambda = \mu$, hence $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda = \mu$". Also by $(*)_5(B)(b)$, "$\mathbb{P}'_{\gamma(*)}$ is $(<\lambda)$-strategically complete $+\lambda^+$-c.c." and by $(*)_9 + (*)_{10}$ we know that by "$\mathrm{cov}_\lambda(\text{meagre}) = \kappa$" so we are done; so $(*)_{11}$ is really the last piece missing. The rest of the proof is dedicated to proving that $(*)_{11}$ holds.

We shall use further nice properties of $\mathbb{P}'_j, \underset{\sim}{g}'_i (j \le \gamma(*), i < \gamma(*))$ which holds by
{c52} $(*)_5 + (*)_6$ (and $(*)_7, (*)_8$) and their proof, i.e. 3.11 or 3.12.

$\boxplus_1$ (a) (α) $\langle \underset{\sim}{g}'_\gamma : \gamma < \gamma(*)\rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$, i.e., if $\mathbf{G}$ is a subset of $\mathbb{P}'_{\gamma(*)}$ generic over $\mathbf{V}$ and $g'_i = \underset{\sim}{g}'_i[\mathbf{G}]$ <u>then</u> $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_i : i < \gamma(*)\rangle]$

(β) if $\nu \in ({}^\lambda\lambda)^{\mathbf{V}[\mathbf{G}]}$ then for some $\rho \in ({}^\lambda\gamma(*))^{\mathbf{V}}$ and λ-Borel function $\mathbf{B} \in \mathbf{V}$ we have $\nu = \mathbf{B}(\langle g'_{\rho(\varepsilon)} : \varepsilon < \lambda\rangle)$

(b) if in $\mathbf{V}[\mathbf{G}], g''_\gamma \in \prod\limits_{\zeta<\lambda} \theta_\zeta$ for $\gamma < \gamma(*)$ and the set $\{(\gamma,\zeta) : \gamma < \gamma(*)$ and $\zeta < \lambda$ and $g''_\gamma(\zeta) \ne g'_\gamma(\zeta)\}$ has cardinality $< \lambda$ then $\bar{g}'' = \langle g''_\gamma : \gamma < \gamma(*)\rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$ and $\mathbf{V}[\bar{g}''] = \mathbf{V}[\bar{g}']$

(c) $\Vdash_{\mathbb{P}'_\gamma}$ "$\underset{\sim}{g}'_\gamma$ dominates $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\gamma]}$

(d) if $\langle \zeta(\gamma) : \gamma < \gamma(*)\rangle$ is an increasing sequence of ordinals $< \gamma(*)$ (from $\mathbf{V}$!), then $\langle g'_{\zeta(\gamma)} : \gamma < \gamma(*)\rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$

(e) if $\gamma < \gamma(*)$ then $\mathbb{P}'_\gamma$ is $(<\lambda)$-strategically complete and satisfies the λ^+-c.c.

We shall use $\boxplus_1$ freely.

To prove $(*)_{11}$ assume toward contradiction that this fails, for some condition $p^* \in \mathbb{P}'_{\gamma(*)}$ and $\mathbb{P}'_{\gamma(*)}$-name $\underset{\sim}{f}$ and λ-Borel function $\mathbf{B}$ and $\rho \in {}^\lambda\gamma(*)$, we have (noting: the "moreover" holds as $f \in ({}^\lambda\lambda)^{\mathbf{V}[\underset{\sim}{\bar{g}} \restriction \mathscr{U}_*]}$)

$\circledast_0$ $p^* \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$ and[4] dominates $({}^\lambda\lambda)^{\mathbf{V}}$" and $\underset{\sim}{f} = \mathbf{B}(\langle g'_{\rho(i)} : i < \lambda\rangle)$.

Now let χ be regular large enough and we choose $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda\rangle$ such that

$\circledast_1$ (a) N_ε is as in $\boxdot_\lambda$ for the forcing notion $\mathbb{P}'_{\gamma(*)}, N_\varepsilon \in \mathscr{S}_{\chi,\mathbb{P}'_{\gamma(*)}}$, see $\boxdot_\lambda$ of 1.3 {a7}

(b) $\bar{N} \restriction \varepsilon \in N_\varepsilon$ hence $\bigcup\limits_{\zeta<\varepsilon} N_\zeta \subseteq N_\varepsilon$ and $\lambda_\varepsilon := N_\varepsilon \cap \lambda > \lambda_\varepsilon^- := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\} \geq \Sigma\{\lambda_\zeta : \zeta < \varepsilon\}$

(c) $\bar{\theta}, \mathbf{q}, p^*, \underset{\sim}{f}, \mathbf{B}, \rho$ belong to N_ε.

We can find $f^* \in {}^\lambda\lambda$, i.e. $\in ({}^\lambda\lambda)^{\mathbf{V}}$, such that

$\circledast_2$ for arbitrarily large $\varepsilon < \lambda$ for some $\zeta \in [\lambda_\varepsilon^-, \lambda_\varepsilon)$ we have $f^*(\zeta) > \lambda_\varepsilon$.

For $\varepsilon < \lambda$ let $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, \mathbb{A}_\varepsilon, \mathbf{G}_\varepsilon)$ be a witness for $(N_\varepsilon, \mathbb{P}'_{\gamma(*)}, \chi)$ recalling $\boxdot_\lambda$ of Definition 1.4 so $\lambda_\varepsilon \in (\varepsilon, \lambda)$ is strongly inaccessible and $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda_\zeta^- < \lambda_\zeta$, recalling $\circledast_1$. {a8}

Let

$\circledast_3$ (a) $v_\varepsilon = N_\varepsilon \cap \gamma(*)$

(b) $i_\varepsilon = i(\varepsilon) = \mathrm{otp}(v_\varepsilon)$ and so $i(\varepsilon) = \mathbf{j}_\varepsilon(\gamma(*))$, etc.

(c) $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < i(\varepsilon)\rangle$ list v_ε in increasing order

(d) for $i < \mathrm{otp}(v_\varepsilon)$, equivalently $i < \mathbf{j}_\varepsilon(\gamma(*))$ let $\eta_i^\varepsilon = (\mathbf{j}_\varepsilon(\underset{\sim}{g}'_i))^{N'_\varepsilon[\mathbf{G}_\varepsilon]} \in \prod\limits_{\zeta<\lambda_\varepsilon} \theta_\zeta$ and let $\bar{\eta}^\varepsilon = \langle \eta_i^\varepsilon : i < i_\varepsilon\rangle$.

Note that clearly

$\circledast_4$ (a) $\bar{\eta}^\varepsilon$ is generic for $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$, moreover

(b) for each $\varepsilon < \lambda$, if we change $\eta_i^\varepsilon(\zeta)$ (legally, i.e. $< \theta_\zeta$) for $< \lambda_\varepsilon$ pairs $(i, \zeta) \in \mathrm{otp}(v_\varepsilon) \times \lambda_\varepsilon$ and get $\bar{\eta}'$, <u>then</u> also $\bar{\eta}'$ is generic for $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$ and $N'_\varepsilon[\bar{\eta}'] = M_\varepsilon$

(c) like $\boxplus_1$ with $\mathbf{V}, \mathbb{P}'_{\gamma(*)}, \lambda$ there standing for $N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$ here.

Hence clearly

$\circledast_4'$ for $\varepsilon < \lambda$, if $\bar{\nu}' = \langle \nu_\varepsilon : i < i(\varepsilon)\rangle$ recalling $i(\varepsilon) = \mathrm{otp}(v_\varepsilon)$ is as in $\circledast_4(b)$, and $q \in \mathbb{P}'_{\gamma(*)}$ satisfies $i < i(\varepsilon) \Rightarrow q \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\underset{\sim}{g}'_{\gamma_i(\varepsilon)} \restriction \lambda_\varepsilon = \nu_i$" <u>then</u> q is $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$-generic naturally and $q \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\mathbf{j}_\varepsilon$ can be extended naturally to an isomorphism from $N_\varepsilon[\underset{\sim}{G}_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle \underset{\sim}{g}'_\gamma : \gamma \in v_\varepsilon\rangle]$ onto $N'_\varepsilon[\bar{\eta}']$".

[4]REFEREE asks: is $\mathbf{B}$ used? Answer: here and two lines above.

[Why? Should be clear, see $\boxplus_1 + \circledast_4(c)$.]

By the assumption toward contradiction, $\circledast_0$, and $\mathbb{P}'_{\gamma(*)}$ being $(<\lambda)$-strategically complete, recalling $\boxplus_1$, there are $\zeta(*), p^{**}$ and p^+ such that (recall $p^* \in \mathbb{P}'_{\gamma(*)} \lessdot \mathbb{P}_{1,\beta(*)}$):

$\circledast_5$ (a) $p^* \le p^{**} \in \mathbb{P}'_{\gamma(*)}$ and $p^+ \in \mathbb{P}_{0,\beta(*)}$ such that $\mathbb{P}_{1,\gamma(*)} \models$ "$p^{**} \le p^+$"

(b) $\zeta(*) < \lambda$

(c) $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$f^*(\zeta) < \underset{\sim}{f}(\zeta)$ whenever $\zeta(*) \le \zeta < \lambda$" where f^* is from $\circledast_2$

(d) if $\gamma \in \mathrm{Dom}(p^+)$ then $\eta^{p^+(\gamma)}$ is an object (not just a $\mathbb{P}'_\gamma$-name) of length $\ge \zeta(*)$ (recall that $\eta^{p^+(\gamma)}$ is the trunk of the condition $p^+(\gamma)$, see clause $(\alpha)(b)$ of Definition 0.7(1)). {z23}

Note that possibly $\mathrm{Dom}(p^+) \nsubseteq \cup\{v_\varepsilon : \varepsilon < \lambda\}$. Choose $\varepsilon(*) < \lambda$ such that $\lambda_{\varepsilon(*)} > \zeta(*) + |\mathrm{Dom}(p^+)|$ and $\gamma \in \mathrm{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$ recalling clause (d) of $\circledast_5$ and $|\mathrm{Dom}(p^+)| < \lambda$ as $p^+ \in \mathbb{P}_{0,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is the limit of a $(<\lambda)$-support iteration.

By $\circledast_2$ we can add $(\exists\zeta)[\lambda^-_{\varepsilon(*)} \le \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$. Our intention is to find $q \in \mathbb{P}_{0,\beta(*)}$ above p^+ which (in $\mathbb{P}_{1,\beta(*)}$) is above some $q' \in \mathbb{P}'_{\gamma(*)}$ which is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$-generic and forces $\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\gamma(*)}}$ to include a generic subset of $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$ which is induced by some $\bar\nu$ as in $\circledast'_4$, recalling $\circledast_4(b)$. Toward this in $\circledast_6$ below the intention is that $p^+_{i(*)}$ will serve as q.

Let $i(*) = i(\varepsilon(*))$ and γ_i for $i < i(*)$ be such that $\langle \gamma_i : i < i(*)\rangle$ list $\{\beta^*_i : i \in v_{\varepsilon(*)}\} \subseteq \mathscr{U}_*$ in increasing order; recall $\mathscr{U}_* = \{\beta^*_i : i < \gamma(*)\}$ and $i < j < \gamma(*) \Rightarrow \beta^*_i < \beta^*_j$ and $v_{\varepsilon(*)} \subseteq \gamma(*)$ has order type $i(\varepsilon(*))$. Next let $\mathscr{U}_{**} = \{\gamma_i : i < i(*)\}$ and $\gamma_{i(*)} = \gamma(*)$ so $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in v_{\varepsilon(*)}\} = i(*) = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$. Recall $\gamma(*) = \kappa = \mathrm{cf}(\kappa) > \lambda, \mathrm{otp}(v_{\varepsilon(*)}) = \mathrm{otp}(N_{\varepsilon(*)} \cap \gamma(*)) = \mathrm{otp}(N_{\varepsilon(*)} \cap \kappa)$ hence $N_{\varepsilon(*)} \models$ "$i(*)$ is a regular cardinal $> \lambda_\varepsilon$" hence $i(*)$ is really a regular cardinal so call it σ. Now we define a game $\eth$ as follows[5]:

$\boxplus_2$ (A) each play lasts $i(*) + 1 = \sigma + 1$ moves and in the i-th move,

(a) if $i = j + 1$ the antagonist player chooses $\xi_j = \xi(j) < \sigma$ such that $j_1 < j \Rightarrow \xi(j_1) < \xi(j)$

(b) then, if $i = j + 1$ the protagonist chooses $\zeta_j = \zeta(j) \in (\xi(j), \sigma)$, but there are more restrictions implicit in $\boxplus_3$

(c) in any case the protagonist also chooses $p^+_i, \bar\nu^i$ such that $\boxplus_3$ below holds;

(B) in the end of the play the protagonist wins the play iff he always has a legal move and in the end $\{\zeta(i) : i \le i(*)\} \in \mathbb{A}_{\varepsilon(*)}$; note that trivially it belongs to $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar\nu^{\varepsilon(*)}]$

where

$\boxplus_3$ (a) $p^+_i \in \mathbb{P}_{0,\gamma_i}$

(b) if $j < i$ then $\mathbb{P}_{0,\gamma_i} \models$ "$p^+_j \le p^+_i$"

[5] The idea is to scatter the $\eta^{\varepsilon(*)}_{\gamma_i}$'s. Why not use the original places? as then we have a problem in $\circledast_6$; the scattering is helpful because we are relying on §2-§4.

(c) if $\gamma \in \cup\{\mathrm{Dom}(p^+_j) : j < i\}$ then $p^+_i \restriction \gamma \Vdash_{\mathbb{P}_{0,\gamma_i}}$ "$\underset{\sim}{\eta}^{p^+_i(\gamma)}$ has length $\geq i(*)$ and $\geq \lambda_{\varepsilon(*)}$" moreover $\underset{\sim}{\eta}^{p^+_i(\gamma)}$ is an object, $\eta^{p^+_i(\gamma)}$

(d) $\mathbb{P}_{0,\gamma_i} \models$ "$p^{**} \restriction \gamma_i \leq p^+_i$"

(e) $\bar\nu^i = \langle \nu_{\gamma_j} : j < i\rangle$ and $\nu_{\gamma_j} \in \prod\limits_{\iota<\lambda_{\varepsilon(*)}} \theta_\iota$

(f) for $j < i$ we have $\nu_{\gamma_j} \trianglelefteq \eta^{p^+_i(\gamma_j)}$ so $p^+_i \restriction \gamma_j \Vdash$ "$\nu_{\gamma_j} \triangleleft \underset{\sim}{g}'_{\gamma_j}$" recalling $\boxplus_1$

(g) for $j < i$ we have (recall $\bar\eta^\varepsilon$ from $\circledast_3$)

 (α) $\nu_{\gamma_j} = \eta^{\varepsilon(*)}_{\gamma_{\zeta(j)}}$ recalling $\eta^{\varepsilon(*)}_{\gamma_j}$ is from $\circledast_3(d)$ <u>or</u>

 (β) $\gamma_j \in \mathrm{Dom}(p^{**})$ and $\{\iota < \lambda_{\varepsilon(*)} : \eta^{\varepsilon(*)}_{\zeta(j)}(\iota) \neq \nu_{\gamma_j}(\iota)\}$ is a bounded subset of $\lambda_{\varepsilon(*)}$.

We shall prove

$\circledast_6$ in the game $\mathfrak{D}$

(a) the antagonist has no winning strategy

(b) in any move the protagonist has a legal move, moreover for any $\zeta(i) \in (\xi(i), \sigma)$ large enough the protagonist can choose it.

<u>Why $\circledast_6$ suffice?</u>

By clause (a) of $\circledast_6$ we can choose a play $\langle(\xi(i), \zeta(i), p^+_i, \bar\nu^i) : i \leq \sigma\rangle$ in which the protagonist wins. Recalling $\mathbb{P}'_{\gamma(*)} \lessdot \mathbb{P}_{1,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is a dense subforcing of $\mathbb{P}_{1,\beta(*)}$, clearly

$\circledast_7$ there is p such that

(a) $p \in \mathbb{P}'_{\gamma(*)}$

(b) if $\mathbb{P}'_{\gamma(*)} \models$ "$p \leq p'$" and $p' \in \mathbb{P}_{0,\beta(*)}$ <u>then</u> p', p^+_σ are compatible in $\mathbb{P}_{i,\beta(*)}$

(c) p is above p^{**} and it forces $\underset{\sim}{g}'_{\gamma_i} \restriction \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$ for $i < i(*)$.

Then on the one hand

$\circledast'_7$ $p \in \mathbb{P}'_{\gamma(*)}$ being above p^{**} forces $f^* \restriction [\zeta(*), \lambda) < \underset{\sim}{f} \restriction [\zeta(*), \lambda)$ hence $f^* \restriction [\zeta(*), \lambda_{\varepsilon(*)}) < \underset{\sim}{f} \restriction [\zeta(*), \lambda_{\varepsilon(*)})$ recalling (see $\circledast_5$) that $\zeta(*) < \lambda_{\varepsilon(*)}$.

On the other hand,

$\circledast''_7$ p is $(N_{\varepsilon(*)}, \mathbb{P}'_{0,\gamma(*)})$-generic.

[Why? As it forces $\underset{\sim}{\eta}_{\gamma_i} \restriction \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$ for $i < i(*)$ and $\langle \nu_{\gamma_i} : i < i(*)\rangle$ is (see $\boxplus_3(g)$ recalling $\mathrm{Dom}(p^{**})$ has cardinality $< \lambda_{\varepsilon(*)}$) "almost equal" to $\langle \eta^{\varepsilon(*)}_{\zeta(i)} : i < i(*)\rangle$ which is a subsequence of the sequence from $\circledast_3$. That is $\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma$ and $\nu_{\gamma_i}(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(i)}(\iota)\} \subseteq \cup\{\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}$ and $\nu_{\gamma_i}(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(i)}(\iota)\} : \gamma \in v_{\varepsilon(*)} \cap \mathrm{Dom}(p^{**})\}$ so is the union of $\leq |\mathrm{Dom}(p^+_\sigma)| < \lambda_{\varepsilon(*)}$ sets each of cardinality $< \lambda_{\varepsilon(*)}$ hence is of cardinality $< \lambda_{\varepsilon(*)}$. Hence by $\circledast_4(c) + \boxplus_1(d)$ the sequence $\bar\nu^{i(*)}$ is generic for $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$; this is a point where we rely on §2 - §4.]

As $\underset{\sim}{f} \in N_{\varepsilon(*)}$ it follows from $\circledast''_7$ that

$\circledast'''_7$ $p \Vdash$ "$\underset{\sim}{f} \restriction \lambda_{\varepsilon(*)}$ is a function from $\lambda_{\varepsilon(*)}$ to $\lambda_{\varepsilon(*)}$".

Together $\circledast_7' + \circledast_7'''$ gives a contradiction by the choice of f^* in $\circledast_2$ and of $\varepsilon(*)$ above, hence $\circledast_6$ is enough.

Why $\circledast_6$ is true?

Let us prove $\circledast_6$; first, assuming clause (b) proved below, for clause (a) choose any strategy **st** for the antagonist and fix a partial strategy **st′** for the protagonist choosing $(p_i^+, \bar{\nu}^i)$ depending on the previous choices and $\zeta(i) < i_{\varepsilon(*)}$ such that it is a legal move if relevant and possible. So the only freedom left for the protagonist is to choose the $\zeta(i)$. So (recalling $\boxplus_2(A)(a)$) we have in $\mathbf{V}$ a function $F : {}^{\sigma >}\sigma \to \sigma$ (so F uses **st** ...) such that:

$(*)_F$ playing the game such that the antagonist uses **st** and the protagonist uses **st′**, arriving to the i-th move, $\bar{\zeta} = \langle \zeta(j) : j < i \rangle$ is well defined and for the protagonist any choice $\zeta_i \in (F(\bar{\zeta}), \sigma) \cap \mathscr{U}_{**}$ is legal.

Now we have to find an increasing sequence $\bar{\zeta} = \langle \zeta(i) : i < \sigma \rangle$ from $\mathbb{A}'_{\varepsilon(*)}$ such that $F(\bar{\zeta} \restriction i) < \zeta(i) < \sigma$ and $\bar{\zeta} \in \mathbb{A}'_{\varepsilon(*)}$. As $F \in \mathscr{H}(\chi_{\varepsilon(*)})$ and $\mathscr{H}(\chi_{\varepsilon(*)}) = N'_{\varepsilon(*)}[\mathbf{G}_{\varepsilon(*)}]$ where $\mathbf{G}_{\varepsilon(*)}$ is a subset of $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}'_{\gamma(*)}) \in \mathbb{A}'_{\varepsilon(*)}$ generic over $\mathbb{A}_{\varepsilon(*)}$ and $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$ satisfies the $\lambda^+_{\varepsilon(*)}$-c.c. and $\sigma = \mathrm{cf}(\sigma) > \lambda_{\varepsilon(*)}$ this is possible. That is, there is a $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$-name $\underset{\sim}{F}_* \in N'_{\varepsilon(*)}$ such that $F = \underset{\sim}{F}_*[\mathbf{G}_{\varepsilon(*)}]$ and we define in $N'_{\varepsilon(*)}$ the function $F' : {}^{\sigma>}\sigma \to \sigma$ by $F'(\langle \zeta(j) : j < i \rangle) = \sup\{\xi + 1 : \xi \in \{\zeta(j) + 1 : j < i\}$ <u>or</u> $\xi < \sigma$ and $\nVdash_{\mathbf{j}(\mathbb{P}_{0,\beta(*)})}$ "$\underset{\sim}{F}(\langle \zeta(j) : j < i \rangle) \neq \xi\}$; clearly this is O.K.

We are left with proving $\circledast_6(b)$.

<u>Case 1</u>: $i = 0$.

Let $p_0^+ = p^{**} \restriction \gamma_0$.

<u>Case 2</u>: i limit.

By clauses (b) and (c) of $\boxplus_3$, there is $p_i^+ \in \mathbb{P}_{0,\gamma_i}$ which is an upper bound (even l.u.b.) of $\{p_j^+ : j < i\}$ and it is easily as required. Also $\bar{\nu}^i$ is well defined and as required.

<u>Case 3</u>: $i = j + 1$ and $\gamma_j \notin \mathrm{Dom}(p^{**})$.

Clearly $\gamma_i = \gamma_j + 1$ and $\gamma_j \in v_{\varepsilon(*)}$. As in case 4 below but easier by the properties of the iteration.

<u>Case 4</u>: $i = j + 1$ and $\gamma_j \in \mathrm{Dom}(p^{**})$

Again $\gamma_i = \gamma_j + 1$ and $\gamma_j \in v_{\varepsilon(*)}$. First we find p'_j such that:

$\circledast_8$ (a) $p_j^+ \leq p'_j \in \mathbb{P}_{0,\gamma_j}$

(b) if $\gamma \in \mathrm{Dom}(p_j^+)$ then $p'_j \restriction \gamma \Vdash$ "$\ell g(\underset{\sim}{\eta}^{p'_j(\gamma)}) > i$"

(c) p'_j forces [6] a value to the pair $(\eta^{p^+(\gamma_j)}, \underset{\sim}{f}^{p^+(\gamma_j)} \restriction \lambda_{\varepsilon(*)})$; we call this pair q_j.

This should be clear.

Second

$\circledast_9$ p_j^+ hence p'_j is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$-generic and $\langle \nu_{\gamma_{j(1)}} : j(1) < j \rangle$ induces the generic.

[6] recall that $\eta^{p^*(\gamma)}$ is an object, not a name and p_j^+ is $(N_{\varepsilon(*)}, \mathbb{P}_{\gamma_j})$-generic

[Why? As in the proof of $\circledast''_7$ above when we assume that we have carried the induction, by $\boxplus_2$, clause (g) and $\circledast_4$.]

Now

$\circledast_{10}$ (a) $f^{q_j} \in (\prod_{\zeta<\lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbb{A}'_{\varepsilon(*)}[\bar\nu^j]}$

(b) for every $\zeta \in (\xi(i), \sigma)$ we have

$\bullet_1$ $f^{q_j} \le \eta^{\varepsilon(*)}_\zeta \mod J^{\mathrm{bd}}_{\lambda_\varepsilon}$

$\bullet_2$ $f^{q_j} \in \mathbb{A}'_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}\restriction\zeta]$

$\bullet_3$ $\langle \zeta(j_1) : j_1 < j\rangle \in \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}\restriction\zeta]$.

(c) $\eta^{q_j} \triangleleft f^{q_j}$.

[Why? Clause (a) holds because $f^{q_j}(\prod\limits_{\zeta<\lambda_{\varepsilon(\lambda)}} \theta_\zeta)^{\mathbf{V}}$, hence belongs to $\mathscr{H}(\chi_{\varepsilon(*)})$ which is the universe of $M_{\varepsilon(*)}$ so $f^{q_j} \in M_{\varepsilon(*)}$. But $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}]$; recalling $\bar\eta^{\varepsilon(*)}$ is a generic for $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$. Next as $\mathbb{P}'_{\gamma(*)}$ satisfies the λ^+-c.c. and $\lambda < \kappa = \mathrm{cf}(\gamma(*))$ so $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$ satisfies the $\lambda^+_{\varepsilon(*)}$-c.c. hence for some $\zeta_1 < \sigma, f^{q_j} \in \mathbb{A}'_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}\restriction\zeta_1]$. Similarly for some $\zeta_2 < \sigma$ we have $\langle\zeta(j_1) : j_2 < j\rangle$ belogns to $N'_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}\restriction\zeta_2]$. Letting $\zeta = \max\{\zeta_1,\zeta_2\}$ clearly clauses $\bullet_2, \bullet_3$ of $\circledast_{10}(b)$ holds, also $\bullet_1$ there holds by $\boxplus_1(c)$ (and $\mathbf{j}_\varepsilon$, etc.).

Lastly, $\circledast_{10}(c)$ holds by $\circledast_8(c)$.]

Now we choose $\zeta(j)$ as in clause (b) of $\circledast_{10}$ and $\nu_j \in \prod\limits_{\varepsilon<\lambda_{\varepsilon(*)}} \theta_\varepsilon$ such that $\eta^{p^+(j)} \triangleleft \nu_j, f^{q_j} \le \nu_j$ and $\{\iota < \lambda_{\varepsilon(*)} : \nu_j(\iota) \ne \eta^{\varepsilon(*)}_{\zeta(j)}\}$ is a bounded subset of $\lambda_{\varepsilon(*)}$. Next choose $p^+_i \in \mathbb{P}'_{\gamma(*)}$ such that $p^+_i\restriction\gamma_j = p'_j, \eta^{p^+_i(\gamma_i)} = \nu_j$ and $f^{p^+_i(\gamma_i)}\restriction[\lambda_\varepsilon,\lambda) = f^{p^+(\gamma)}\restriction[\lambda_\varepsilon,\lambda)$.

So we have carried the induction hence proved $\circledast_6$ so we are done. $\square_{1.3}$

Concluding Remark 1.7. 1) Is the use of $\underset{\sim}{\bar g}\restriction\mathscr{U}_*$ rather than $\underset{\sim}{\bar g}$ in the proof necessary? See on this [Sh:F979]. {a19}

§ 2. Iteration Parameters

{c0x}

{c4} **Explanation 2.1.** For $\mathbf{m} \in \mathbf{M}$ below (Definition 2.7):

(a) we use $L_\mathbf{m}$ as the index set for the iteration; always a well founded partial order

(b) $M_\mathbf{m}$ is the part of the index set we are really interested in, it may be $(\kappa, <)$ in §1

(c) the other part in the interesting case is "generic enough $\mathbf{m}$", more accurately enough existentially closed so that the iteration restricted to M will be "stabilize" under further extensions; inspite of $L_\mathbf{m}$ being required to be well founded this will be well defined.

{c0} **Hypothesis 2.2.** 1) $\lambda = \lambda^{<\lambda}$ strongly inaccessible.
2) $\bar\theta = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$.
3) θ_ε is an infinite regular cardinal $> \varepsilon$ and $< \lambda$.
4) Assume $\lambda_2 \geq \lambda_1 \geq \lambda$ are such that[7] $\lambda_2 = (\lambda_2)^\lambda \geq \beth_3(\lambda_1)$ so pendantically all notations should have the parameter $\bar\lambda = (\lambda_2, \lambda_1)$ and even $\bar\lambda = (\lambda_2, \lambda_1, \lambda, \bar\theta)$.

{c1} *Notation* 2.3. L, M denote partial orders, usually well founded.

Remark 2.4. Here no harm in adding

(a) $\theta_\varepsilon > \prod\limits_{\zeta<\varepsilon} 2^{\theta_\zeta} + 2^{\aleph_0}$ for $\varepsilon < \lambda$
or just

(b) $\bar\theta$ is increasing fast enough

(c) M a linear order, well founded (it suffices to assume even $M \cong (\kappa, <), \kappa$ regular $> \lambda$).

{c2} **Definition 2.5.** 1) For a partial order L let

(α) $\mathrm{dp}(L) = \cup\{\mathrm{dp}_L(t) + 1 : t \in L\}$, see below,

(β) $\mathrm{dp}_L(t) = \mathrm{dp}(t, L) \in \mathrm{Ord} \cup \{\infty\}$ be defined by $\mathrm{dp}_L(t) = \cup\{\mathrm{dp}_L(s) + 1$: $s <_L t\}$.

(γ) $L_{<t} = L{\restriction}\{s \in L : s <_L t\}$,

(ζ) $L_{\leq t} = L{\restriction}\{s \in L : s \leq_L t\}$.

2) Let $L^+ = L(+)$ be $L \cup \{\infty\}$ with the natural order (but we may write $t <_L \infty$ instead of $t <_{L(+)} \infty$).
3) We say the set L is an initial segment of the partial order L_* when

- $L \subseteq L_*$, i.e. $s \in L \Rightarrow s \in L_*$
- $s <_{L_*} t \wedge t \in L \Rightarrow s \in L$.

{c3} **Discussion 2.6.** Concerning the aim of the following choice, note the following.
{c4} 1) By the partial order we already can get partial memory, so why the u_s's (in 2.7)? Because $\bar u$ is not necessarily transitive, that is, $s \in u_t \not\Rightarrow u_s \subseteq u_t$. By partial order we cannot get it.
2) In [Sh:700] we use $\mathscr{P}_t$'s which are ideals, but here not necessarily: this makes a difference but it uses "$\mathbb{Q}_{\bar\theta}$ is close to being λ-centered", i.e. any subset of $\{p \in \mathbb{Q}_{\bar\theta} : \mathrm{tr}(p) = \eta\}$ of cardinality $< \theta_{\ell g(\eta)}$ has a lub in this subset.

{e23} [7] usually $\lambda_2 \geq \lambda_1$ suffice but see 4.11, 4.21

{c4} **Definition 2.7.** Let $\mathbf{M}$ be the class of objects $\mathbf{m}$, called iteration parameters, of the following form (so really $\mathbf{M} = \mathbf{M}[\bar\lambda]$ and if we omit clauses $(\theta), (\iota), (\lambda)$ we may write $\mathbf{M}[*]$):

- (a) L, a partial order,
- (b) $M \subseteq L$, as partial orders,
- (c) (α) $\bar u = \langle u_t : t \in L\rangle$ and $\bar{\mathscr{P}} = \langle \bar{\mathscr{P}}_t : t \in L\rangle$ and each $\mathscr{P}_t$ is closed under subsets,
  - (β) $u_t \subseteq \{s \in L : s <_L t\}$ and $u \in \mathscr{P}_t \Rightarrow u \subseteq u_t$,
- (d) $\mathrm{dp}(L) < \infty$, that is L is well founded,
- (e) (α) E' is a two-place relation (on L),
  - (β) $E'' := E' \restriction (L \backslash M)$ is an equivalence relation on $L \backslash M$
  - (γ) if $s, t \in L \backslash M$ are not E''-equivalent then $(s <_L t) \Leftrightarrow (\exists r \in M)(s < r < t)$
  - (δ) if $sE't$ then $s \notin M \vee t \notin M$
  - (ε) if $t \in L \backslash M$ then $\{s \in L : sE't\} = \{s \in L : tE's\}$; we call it t/E'; so E' is a symmetric relation
  - (ζ) if $s, t \in L \backslash M$ are E''-equivalent then $s/E' = t/E'$
  - (η) if $t \in L \backslash M$ then $u_t \subseteq t/E'$
  - (θ) if $t \in L \backslash M$ then t/E' has cardinality $\le \lambda_2$
  - (ι) $\|M\| \le \lambda_1$
  - (κ) if $t \in L$ and $u \in \mathscr{P}_t$ then $u \nsubseteq M \Rightarrow (\exists s)(s \in L \backslash M \wedge u \subseteq s/E')$
  - (λ) $\mathscr{P}_t$ has cardinality $\le \lambda_2$ for $t \in L \backslash M$ and for simplicity $\mathscr{P}_t \subseteq [u_t]^{\le \lambda}$ as only those sets matter.

{c5}

Notation 2.8. For $\mathbf{m} \in \mathbf{M}$.
0) In 2.7 we let $\mathbf{m} = (L_{\mathbf{m}}, M_{\mathbf{m}}, \bar u_{\mathbf{m}}, \bar{\mathscr{P}}_{\mathbf{m}}, E'_{\mathbf{m}})$ and $\bar u_{\mathbf{m}} = \langle u_{\mathbf{m},t} : t \in L_{\mathbf{m}}\rangle, \bar{\mathscr{P}}_{\mathbf{m}} =$ {c4}
$\langle \mathscr{P}_{\mathbf{m},t} : t \in L_{\mathbf{m}}\rangle$ and for $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ let $t/E_{\mathbf{m}} = (t/E'_{\mathbf{m}}) \cup M_{\mathbf{m}}$ and for $t \in M_{\mathbf{m}}$ let $t/E_{\mathbf{m}} = M_{\mathbf{m}}$; so there is no relation $E_{\mathbf{m}}$ but $t/E_{\mathbf{m}}$ for $t \in L_{\mathbf{m}}$ is well defined.
1) In 2.7, let $\mathrm{dp}_{\mathbf{m}}(t) = \mathrm{dp}_{L_{\mathbf{m}}}(t), \mathrm{dp}_{\mathbf{m}} = \mathrm{dp}(L_{\mathbf{m}})$ and $\le_{\mathbf{m}} = \le_{L_{\mathbf{m}}}$. {c4}
2) For $L \subseteq L_{\mathbf{m}}$:

- (a) let $\mathbf{n} = \mathbf{m} \restriction L$ means $\mathbf{n} \in \mathbf{M}, L_{\mathbf{n}} = L, \le_{\mathbf{n}} = \le_{\mathbf{m}} \restriction L_{\mathbf{n}}, u_{\mathbf{n},t} = u_{\mathbf{m},t} \cap L, \mathscr{P}_{\mathbf{n},t} = \mathscr{P}_{\mathbf{m},t} \cap [L]^{\le \lambda}$ for $t \in L$ and $M_{\mathbf{n}} = M_{\mathbf{m}} \cap L$;
- (b) let $\mathrm{dp}_{\mathbf{m}}(L) = \mathrm{dp}(L_{\mathbf{m}} \restriction L)$ and we may write $\mathrm{dp}(L)$ for $L \subseteq L_{\mathbf{m}}$.

3) For $t \in L_{\mathbf{m}}$, let $\mathbf{m}_{<t} = \mathbf{m}(<t) = \mathbf{m} \restriction L_{<t}$ where $L_{<t} = L_{\mathbf{m}(<t)} = L_{\mathbf{m},t} = \{s : s <_{\mathbf{m}} t\}$ so $u_{\mathbf{m}(<t),s} = u_{\mathbf{m},s}$ for $s \in L_{<t}$, etc.
3A) Also $\mathbf{m}_{\le t} = \mathbf{m}(\le t) = \mathbf{m} \restriction L_{\le t}$ where $L_{\le t} = L_{\mathbf{m}(\le t)} = L_{<t} \cup \{t\}$; let $L_{<\infty} = L, L_{\le \infty} = L^+$, etc.
4) $\mathbf{M}_{<\mu}$ is the class of $\mathbf{m} \in \mathbf{M}$ such that $L_{\mathbf{m}}$ has cardinality $< \mu$. Similarly $\mathbf{M}_{\le\mu}, \mathbf{M}_{=\mu}, \mathbf{M}_{>\mu}, \mathbf{M}_{\ge\mu}$; let $\mathbf{M}_\mu = \mathbf{M}_{=\mu}$.
5) For $\mathbf{m}, \mathbf{n} \in \mathbf{M}$ let $\mathbf{m} \approx \mathbf{n}$, and we may say $\mathbf{m}, \mathbf{n}$ are equivalent mean that $L_{\mathbf{m}} = L_{\mathbf{n}}$ and $t \in L_{\mathbf{n}} \Rightarrow u_{\mathbf{m},t} = u_{\mathbf{n},t} \wedge \mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$; note that there are no demands on M and E'.
6) We say f is an isomorphism from $\mathbf{m}_1 \in \mathbf{M}$ onto $\mathbf{m}_2 \in \mathbf{M}$ when:

(a) f is an isomorphism from the partial order $L_{\mathbf{m}_1}$ onto the partial order $L_{\mathbf{m}_2}$

(b) for $s,t \in L_{\mathbf{m}_1}$ we have $s \in u_{\mathbf{m}_1,t} \Leftrightarrow f(s) \in u_{\mathbf{m}_2,f(t)}$ and $\mathscr{P}_{\mathbf{m}_2,f(t)} = \{\{f(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{m}_1,t}\}$

(c) for $s,t \in L_{\mathbf{m}_1}$ we have $sE'_{\mathbf{m}_1}t \Leftrightarrow f(s)E'_{\mathbf{m}_2}f(t)$

(d) $M_{\mathbf{m}_2} = \{f(s) : s \in M_{\mathbf{m}_1}\}$.

7) We define weak isomorphisms as in part (6) omitting clauses (c),(d).

{c6}

Definition 2.9. For $\mathbf{m} \in \mathbf{M}$ let $L = L_{\mathbf{m}}$ and we define the iteration $\mathbf{q}_{\mathbf{m}}$ to consist of:

(a) a forcing notion $\mathbb{P}_t = \mathbb{P}_{\mathbf{m},t}$ for $t \in L^+$; we let $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_\infty$

(b) $\underset{\sim}{\mathbb{Q}}_t$ is the $\mathbb{P}_t$-name of a subforcing of $\mathbb{Q}_{\bar{\theta}}$ in the universe $\mathbf{V}^{\mathbb{P}_t}$, even $\le_{\rm ic}$ (i.e. incompatibility and compatibility are preserved)

(c) $p \in \mathbb{P}_t$ iff

  (α) p is a function

  (β) $\mathrm{Dom}(p) \subseteq L_{<t}$ has cardinality $< \lambda$

  (γ) if $s \in \mathrm{Dom}(p)$ then $p(s)$ consists of $\mathrm{tr}(p(s)) \in \prod\limits_{\varepsilon<\zeta(s)} \theta_\varepsilon$ for some $\zeta_s = \zeta(s) < \lambda$ and $\xi = \xi_{p(s)} = \xi(p(s)) \le \lambda$ and $\mathbf{B}_{p(s)}$ and $\bar{r} = \bar{r}_{p(s)} = \langle r(\zeta) : \zeta < \xi_{p(s)}\rangle = \langle r_{p(s)}(\zeta) : \zeta < \xi_{p(s)}\rangle \in {}^{\xi}(u_s)$ list the coordinates used in computing $p(s)$ and $\langle \mathbf{B}_{p(s),\iota}, \bar{r}_{p(s),\iota} : \iota < \iota(p(s))\rangle$ are such that:

- $\mathbf{B}_{p(s)}$ is a λ-Borel function[8], $\mathbf{B} = \mathbf{B}_{p(s)} : {}^{\xi}(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon) \to \Pi\bar{\theta}$ moreover into $(\Pi\bar{\theta})^{[\mathrm{tr}(p(s))]}$; and considering (d)($\alpha$) below less pedantically $p(s) = (\mathrm{tr}(p(s)), \underset{\sim}{f}_{p(s)})$, where $\underset{\sim}{f}_{p(s)} = \mathbf{B}_{p(s)}(\ldots, \underset{\sim}{\eta}_{r_{p(s)}(\zeta)}, \ldots)_{\zeta<\xi}$ which means: absolutely, i.e. in every $\mathbf{V}^{\mathbb{Q}}$, $\mathbb{Q}$ a $(<\lambda)$-strategically complete (which is λ^+-c.c.) forcing notion, $\mathbf{B}_{p(s)}$ is such a (λ-Borel) function; we may write $\xi_{p,s}$ instead of $\xi_{p(s)}$, etc.
- $\iota(p(s)) < \lambda$ moreover[9] $< \theta_{\ell g(\mathrm{tr}(p(s)))}$
- $\bar{r}_{p(s),\iota} = \bar{r}_{p(s)} \restriction w_{p(s),\iota}$ so $w_{p(s),\iota} = w(p(s),\iota) = \mathrm{dom}(\bar{r}_{p(s),\iota}) \subseteq \xi_{p(s)}$ and $\bar{r}_{p(s),\iota}$ is a subsequence of $\bar{r}_{p(s)}$
- $\mathbf{B}_{p(s),\iota}$ is a Borel function from ${}^{w(p(s),\iota)}(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)$ into $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{[\mathrm{tr}(p(s))]}$
- $\mathbf{B}_{p(s)}(\langle \underset{\sim}{\eta}_{r_{p(s)}(\zeta)} : \zeta < \xi_{p(s)}\rangle) = \sup\{\mathbf{B}_{p(s),\iota}(\langle \underset{\sim}{\eta}_{r_{p(s)}(\zeta)} : \zeta \in w_{p(s),\iota}\rangle) : \iota < \iota(p(s))\}$ so $f_{p(s)} = \sup\{\underset{\sim}{f}_{p(s),\iota} : \iota < \iota(p(s))\}, \underset{\sim}{f}_{p(s),\iota} = \mathbf{B}_{p(s),\iota}(\langle \underset{\sim}{\eta}_\zeta : \zeta \in w_{p(s),\iota}\rangle)\}$
- for each $\iota < \iota(p(s))$ for some $u \in \mathscr{P}_{\mathbf{m},s}$ we have $\{r_{p(s)}(\zeta) : \zeta < \xi_{p(s)}, \zeta \in w_{p(s),\iota}\} \subseteq u$ so is a subset of u_s
- if $\iota < \iota(p(s))$ and $\varepsilon \in w_{p(s),\iota}, r_{p(s)}(\varepsilon) \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $\{r_{p(s)}(\zeta) : \zeta \in w_{p(s),\iota}\} \subseteq r_{p(s)}(\varepsilon)/E_{\mathbf{m}}$, (follows)

---

[8]that is, a definition of one

{e31} [9]This and the rest of (c)(γ) are used in the proof of 4.17. The aim is that defining $\mathbf{B}_{p(s)}$ from $\langle \mathbf{B}_{p(s),\iota} : \iota\rangle$, the sup will not given in ε the value θ_ε.

(d) (α) $\underset{\sim}{\eta}_s$ is the $\mathbb{P}_t$-name, when $t \in L^+_{\mathbf{m}}, s \in L_{<t}$ defined by $\cup\{\mathrm{tr}(p(s)) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_t}\}$.

(β) For $p \in \mathbb{P}_t$ and $s \in \mathrm{Dom}(p)$ we interpret $p(s)$ as a $\mathbb{P}_s$-name $(\mathrm{tr}(p(s)), \mathbf{B}_{p,s}(\ldots, \underset{\sim}{\eta}_{r_{p,s}(\zeta)}, \ldots)_{\zeta < \xi_{p,s}})$

(e) $\mathbb{P}_t \models$ "$p \le q$" iff

(α) $p, q \in \mathbb{P}_t$

(β) $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$

(γ) if $t \in \mathrm{Dom}(p)$ then $(q {\restriction} L_{<t}) \Vdash_{\mathbb{P}_{\mathbf{m}(<t)}}$ "$p(t) \le_{\underset{\sim}{\mathbb{Q}}_{\bar\theta}} q(t)$".

Definition 2.10. 1) For $p \in \mathbb{P}_{\mathbf{m}}$ let {c7}

(a) fsupp(p), the full support of p be $\cup\{\{r_{p(s)}(\zeta) : \zeta < \xi_{p,s}\} \cup \{s\} : s \in \mathrm{Dom}(p)\}$

(b) wsupp(p), the wide support of p be $\cup\{t/E_{\mathbf{m}} : t \in \mathrm{fsupp}(p)\}$.

2) For $\mathbf{m} \in \mathbf{M}$ let $\mathbb{P}^{\mathbf{m}}_t = \mathbb{P}_{\mathbf{m},t}$, etc., in Definition 2.9. {c6}
3) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} {\restriction} \{p \in \mathbb{P}_{\mathbf{m}} : \mathrm{fsupp}(p) \subseteq L\}$.
4) For $\mathbf{m} \in \mathbf{M}$ and $t \in L_{\mathbf{m}}$ let[10] $\underset{\sim}{\mathbb{Q}}_t = \underset{\sim}{\mathbb{Q}}_{\mathbf{m},t}$ be the $\mathbb{P}_t$-name of $\mathbb{Q}_{\bar\theta} {\restriction} \{(\nu, f) \in \mathbb{Q}_{\bar\theta} : f = \sup\{f_\iota : \iota < \iota(*)\}$ where $\iota(*) < \theta_{\ell g(\nu)}$ and $f_\iota \in (\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\underset{\sim}{\eta}_s : s \in u]}$ for some $u \in \mathscr{P}_{\mathbf{m},t}\}$.

Claim 2.11. *For $\mathbf{m} \in \mathbf{M}$ (so $\mathbb{P}_t = \mathbb{P}_{\mathbf{m},t}$, etc.)* {c8}

(a) *the iteration $\mathbf{q}_{\mathbf{m}}$ is well defined, i.e. exist and is unique*

(b) (α) *if $t \in L^+_{\mathbf{m}}$ then $\mathbb{P}_t$ is indeed a forcing notion and is equal to $\mathbb{P}_{\mathbf{m}_{<t}}$,*

(β) *the $\mathbb{P}_t$-name $\underset{\sim}{\eta}_s$ does not depend on t as long as $s < t \in L^+_{\mathbf{m}}$,*

(γ) *$\underset{\sim}{\eta}_t$ is a $\mathbb{P}_{\mathbf{m}(\le t)}$-name*

(c) *if $s <_L t$ are from $L^+_{\mathbf{m}}$ then*

(α) *$p \in \mathbb{P}_s \Rightarrow p \in \mathbb{P}_t \wedge p {\restriction} L_{<s} = p$,*

(β) *if $p, q \in \mathbb{P}_s$ then $\mathbb{P}_t \models$ "$p \le q$" $\Leftrightarrow \mathbb{P}_s \models$ "$p \le q$",*

(γ) *if $p \in \mathbb{P}_t$ then $p {\restriction} L_{<s} \in \mathbb{P}_s$ and $\mathbb{P}_t \models$ "$(p {\restriction} L_{<s}) \le p$",*

(δ) *$\mathbb{P}_t \models$ "$p \le q$" $\Rightarrow \mathbb{P}_s \models$ "$p {\restriction} L_{<s} \le q {\restriction} L_{<s}$",*

(ε) *$\mathbb{P}_s \lessdot \mathbb{P}_t$ moreover*

(ζ) *$p \in \mathbb{P}_t \wedge (p {\restriction} L_{<s}) \le q \in \mathbb{P}_s \Rightarrow q \cup (p {\restriction} (L_{<t} \backslash L_{<s}) \in \mathbb{P}_t$ is a $\le$-lub of p, q*

(θ) *$\mathbb{P}_{\mathbf{m},t} = \mathbb{P}_{\mathbf{m} {\restriction} L_{<t}}$*

(d) *if L is an initial segment of $L_{\mathbf{m}}$ then $\mathbb{P}_{\mathbf{m} {\restriction} L} = \mathbb{P}_{\mathbf{m}} {\restriction} \{p \in \mathbb{P}_m : \mathrm{dom}(p) \subseteq L$, equivalently $\mathrm{fsupp}(p) \subseteq L\}$; this holds in particular for $L_{\le t}, \mathbf{m}_{\le t}$*

(e) *if $L_1 \subseteq L_2$ are initial segments of $L_{\mathbf{m}}$, then the parallel of clause (c) holds replacing $\mathbb{P}_{\mathbf{m},s}, \mathbb{P}_{\mathbf{m},t}$ by $\mathbb{P}_{\mathbf{m} {\restriction} L_1}, \mathbb{P}_{\mathbf{m} {\restriction} L_2}$, respectively.*

Proof. Straightforward. For $t \in L^+_{\mathbf{m}}$, by induction on $\mathrm{dp}_{\mathbf{m}}(t)$, define $\mathbb{P}_t$ and prove the relevant parts of (a),(b),(c),(d),(e). $\square_{2.11}$

Note

[10]not used, could have used it in 2.15 {c13}

{c10}

Observation 2.12. *If* $\mathbf{B}$ *is a* λ*-Borel function from* ${}^{\xi}(\Pi\bar{\theta})$ *to* $\mathscr{P}(\lambda)$ *or even* $\mathscr{H}(\lambda^+)$ *where* $\xi \le \lambda$ *<u>then</u> there is a* λ*-Borel function* $\mathbf{B}'$ *from* ${}^{\xi}(\Pi\bar{\theta})$ *to* $\mathbb{Q}_{\bar{\theta}}$ *(so absolutely to* $\mathbb{Q}_{\bar{\theta}}$*) such that for any* $\bar{\eta} \in {}^{\xi}(\Pi\bar{\theta})$ *we have, absolutely:*

- *if* $\mathbf{B}(\bar{\eta}) \in \mathbb{Q}_{\bar{\theta}}$ *then* $\mathbf{B}'(\bar{\eta}) = \mathbf{B}(\bar{\eta})$
- *if* $\mathbf{B}(\bar{\eta}) \notin \mathbb{Q}_{\bar{\theta}}$ *then* $\mathbf{B}'(\bar{\eta}) = (\emptyset, 0_\lambda)$*, the minimal member of* $\mathbb{Q}_{\bar{\theta}}$*.*

Proof. Easy. $\square_{2.12}$

{c11}

Claim 2.13. *Let* $\mathbf{m} \in \mathbf{M}$*.*
1) If $L^+_{\mathbf{m}} \models$ *"*$s < t$*" <u>then</u>*

(α) $\Vdash_{\mathbb{P}_{\mathbf{m},t}}$ "$\underset{\sim}{\eta}_s \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$"

(β) *if* $\mathbf{G} \subseteq \mathbb{P}_t$ *is generic over* $\mathbf{V}$ *and* $\eta_r = \underset{\sim}{\eta}_r[\mathbf{G}]$ *for* $r \in L_{\mathbf{m},<t}$ *and* $u \in \mathscr{P}_{\mathbf{m},s}$ *and* $\nu \in \Pi\bar{\theta}$ *is from* $\mathbf{V}[\langle \eta_r : r \in u\rangle] \subseteq \mathbf{V}[\mathbf{G}]$ *<u>then</u>* $\nu <_{J^{\mathrm{bd}}_\lambda} \eta_s$*.*

2) $\mathbb{P}_{\mathbf{m}}$ *satisfies the* λ^+*-c.c.*
3) $\mathbb{P}_{\mathbf{m}}$ *is* $(<\lambda)$*-strategically complete (even* λ*-strategically complete but not used).*
4) If $\bar{p} = \langle p_i : i < \delta\rangle$ *is* $\le_{\mathbb{P}_{\mathbf{m}}}$*-increasing,* $\delta < \lambda$ *and* $i < j < \delta \wedge t \in \mathrm{Dom}(p_i) \Rightarrow \mathrm{tr}(p_i(t)) \vartriangleleft \mathrm{tr}(p_j(t))$ *<u>then</u>*[11] $\bar{p}$ *has a* $\le_{\mathbb{P}_{\mathbf{m}}}$*-upper bound* p*. Moreover,* $\mathrm{Dom}(p) = \cup\{\mathrm{Dom}(p_i) : i < \delta\}$ *and* $s \in \mathrm{Dom}(p_i) \Rightarrow \mathrm{tr}(p(s)) = \cup\{\mathrm{tr}(p_j(s)) : j \in [i,\delta)\}$*; in fact also* $\mathrm{fsupp}(p) = \cup\{\mathrm{fsupp}(p_i) : i < \delta\}$ *and* p *is a lub. Also, we can weaken the demand above to* $i < \delta \wedge s \in \mathrm{Dom}(p_i) \Rightarrow \delta < \theta_{\varepsilon(s)}$ *where we let* $\varepsilon(s) = \sup\{\ell g(\mathrm{tr}(p_j(s))) : j \in [i,\delta)\}$*.*
5A) If $\zeta < \lambda$ *and* $L^+_{\mathbf{m}} \models$ *"*$s < t$*", <u>then</u> the following is a dense open subset of* $\mathbb{P}_t$*:* $\mathscr{I}_{s,t,\zeta} = \{p \in \mathbb{P}_t : s \in \mathrm{Dom}(p)$ *and* $\mathrm{tr}(p(s))$ *has length* $\ge \zeta\}$*.*
5B) If $p \in \mathbb{P}_{\mathbf{m}}$ *and* $\zeta < \lambda$ *<u>then</u> for some* $q \in \mathbb{P}_{\mathbf{m}}$ *we have* $p \le q$ *and* $t \in \mathrm{Dom}(p) \Rightarrow \mathrm{tr}(p(t)) \vartriangleleft \mathrm{tr}(q(t))$ *and* $t \in \mathrm{Dom}(q) \Rightarrow \ell g(\mathrm{tr}(q(t))) > \zeta$*.*
6) If $\underset{\sim}{x}$ *is a* $\mathbb{P}_{\mathbf{m}}$*-name of a member of* $\mathscr{H}(\lambda^+)$*, e.g. of* $\mathbb{Q}_{\bar{\theta}}$ *(in* $\mathbf{V}[\mathbb{P}_{\mathbf{m}}]$*) <u>then</u> for some* $\xi \le \lambda$ *and* λ*-Borel function* $\mathbf{B} : {}^{\xi}(\Pi\bar{\theta}) \to \mathscr{H}(\lambda^+)$ *and a sequence* $\langle r_\zeta : \zeta < \xi\rangle$ *of members of* $L_{\mathbf{m}}$ *we have* $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ *"*$\underset{\sim}{x} = \mathbf{B}(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta<\xi}$*".*
6A) If $t \in L^+_{\mathbf{m}}$ *and* $u \subseteq L_{<t}$ *and* $\Vdash_{\mathbb{P}_t}$ *"*$\underset{\sim}{y}$ *is a member of* $\mathbb{Q}_{\bar{\theta}}$ *from* $\mathbf{V}[\langle \underset{\sim}{\eta}_s : s \in u\rangle]$*",*
{c6} *<u>then</u> for some* $\xi \le \lambda$ *and* λ*-Borel functions as in 2.9(6)(*γ*),* $\mathbf{B}_i : {}^{\xi}(\Pi\bar{\theta}) \to \mathbb{Q}_{\bar{\theta}}$ *for* $i < \xi$ *and sequence* $\langle r_\zeta : \zeta < \xi\rangle$ *of members of* u *we have* $\Vdash_{\mathbb{P}_t}$ *"for some* $i < \xi$ *we have* $\underset{\sim}{y} = \mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta<\xi}$*".*
7) If $\mathbf{m}, \mathbf{n}$ *are equivalent <u>then</u>* $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{n}}$ *and* $\mathbb{P}_{\mathbf{m},t} = \mathbb{P}_{\mathbf{n},t}$ *for* $t \in L^+_{\mathbf{m}} = L^+_{\mathbf{n}}$*.*
8) Assume that $p, q \in \mathbb{P}_{\mathbf{m}}$ *are incompatible <u>then</u> there are* r *and* s *such that:*

(a) $r \in \mathbb{P}_{\mathbf{m},s}$

(b) $s \in \mathrm{Dom}(p) \cap \mathrm{Dom}(q)$

(c) $(q{\restriction}L_{\mathbf{m},<s}) \le_{\mathbb{P}_{\mathbf{m}}} r$

(d) $p{\restriction}L_{\mathbf{m},<s} \le_{\mathbb{P}_{\mathbf{m}}} r$

(e) $r \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p(s)$ *and* $q(s)$ *are incompatible in* $\mathbb{Q}_\theta$ *which means* $\mathrm{tr}(p(s)) \perp \mathrm{tr}(q(s))$*, i.e. are* $\trianglelefteq$*-incomparable <u>or</u>* $(\alpha) + (\beta) + (\gamma)$ *where:*

(α) $\ell g(\mathrm{tr}(q(s))) \ne \ell g(\mathrm{tr}(p(s)))$

[11] But $\mathrm{tr}(p_i(t)) \trianglelefteq \mathrm{tr}(p_j(t))$ does not suffice.

(β) *if* $\ell g(\mathrm{tr}(q(s))) < \ell g(\mathrm{tr}(p(s)))$ *then for some ordinal* $\varepsilon, \ell g(\mathrm{tr}(q(s))) \le \varepsilon < \ell g(\mathrm{tr}(p(s)))$ *and* $r{\restriction}L_{\mathbf{m}(<s)} \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}} \mathrm{tr}(p(s))(\varepsilon) < \underset{\sim}{f}_{q(s)}(\varepsilon)$"

(γ) *if* $\ell g(\mathrm{tr}(q(s))) > \ell g(\mathrm{tr}(p(s)))$ *then for some ordinal* $\varepsilon, \ell g(\mathrm{tr}(q(s))) > \varepsilon \ge \ell g(\mathrm{tr}(p(\alpha)))$ *and* $r{\restriction}L_{\mathbf{m}(<s)} \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}}$ "$\mathrm{tr}(q(s))(\varepsilon) < \underset{\sim}{f}_{p(s)}(\varepsilon)$".

9) $\Vdash_{\mathbb{P}_\mathbf{m}}$ "$\mathbf{V}[\langle \underset{\sim}{\eta}_s : s \in L_\mathbf{m}\rangle] = \mathbf{V}[\underset{\sim}{\mathbf{G}}]$".

Remark 2.14. What is the use of e.g. (6),(6A)? See 3.11(A)(b) and 2.15. {c53}

Proof. We prove all parts by induction on $\mathrm{dp}_\mathbf{m}$.
1) For clause (α) for each $\mathbf{m}$, using the induction hypothesis and 2.11(e), the problem is only when $\mathrm{dp}_\mathbf{m}(t) = \mathrm{dp}_\mathbf{m} - 1$ and use part (5A) proved below. For clause (β) use also part (6A) for $\mathbb{P}_{\mathbf{m}(<t)}$ proved below. In both cases the proof of the parts quoted does not rely on part (1). {c8}
2) If $p_\varepsilon \in \mathbb{P}_\mathbf{m}$ for $\varepsilon < \lambda^+$ then we by the Δ-system lemma can find u and unbounded $S \subseteq \lambda^+$ such that $\varepsilon \ne \zeta \in S \Rightarrow \mathrm{Dom}(p_\varepsilon) \cap \mathrm{Dom}(p_\zeta) = u$ and $\langle \mathrm{tr}(p_\varepsilon(\beta)) : \beta \in u\rangle$ is the same for all $\varepsilon \in S$. Now p_ε, p_ζ has a common upper bound for every $\varepsilon, \zeta \in u$, i.e. we define r by

- $\mathrm{Dom}(r) = \mathrm{Dom}(p_\varepsilon) \cup \mathrm{Dom}(p_\zeta)$
- $r(s) = p_\varepsilon(s)$ is $s \in \mathrm{Dom}(p_\varepsilon) \backslash \mathrm{Dom}(p_\zeta)$
- $r(s) = p_\zeta(s)$ if $s \in \mathrm{Dom}(p_\zeta) \backslash \mathrm{Dom}(p_\varepsilon)$
- if $s \in \mathrm{Dom}(p_\varepsilon) \cap \mathrm{Dom}(p_\zeta)$ <u>then</u> $r(s) = (\mathrm{tr}(p_\varepsilon(s)), \max\{\underset{\sim}{f}_{p_\varepsilon(s)}, \underset{\sim}{f}_{p_\zeta(s)}\})$.

3) By (4), the second sentence + (5B) below which use only the induction hypothesis.
4) We define p by:

- $\mathrm{Dom}(p) = \cup\{\mathrm{Dom}(p_i) : i < \delta\}$
- $\mathrm{tr}(p(s)) = \cup\{\mathrm{tr}(p_i(s)) : i < \delta$ satisfies $s \in \mathrm{Dom}(p_i)\}$
- $\underset{\sim}{f}_{p(s)} = \sup\{\underset{\sim}{f}_{p_i(s)} : i < \delta$ satisfies $s \in \mathrm{Dom}(p_i)\}$.

Note that here having to really start with $\langle \underset{\sim}{f}_{p_i(s),\iota} : \iota < \iota(p_i(s))\rangle$ and get $\langle \underset{\sim}{f}_{p(s),\iota} : \iota < \iota(p(s))\rangle$, see 2.9(c)($\gamma$) causes no problem, similarly in the proof of part (2) - just take the union. {c6}
5A) Obvious by the definition of $\mathbb{P}_\mathbf{m}$ and 2.11(c). {c8}
5B) The proof is split to cases.

<u>Case 1</u>: $\mathrm{dp}_\mathbf{m}$ is zero
So L_m is empty.

<u>Case 2</u>: $\mathrm{dp}_\mathbf{m} = \alpha + 1$
Hence $L_2 = \{s \in L : \mathrm{dp}_\mathbf{m}(s) = \alpha\}$ is non-empty and letting $L_1 = L_\mathbf{m} \backslash L_2$; clearly $s \in L_1 \Rightarrow \mathrm{dp}_\mathbf{m}(s) < \alpha$, so $\mathrm{dp}_{\mathbf{m}{\restriction}L_1} \le \alpha$. Let $\zeta_* = \sup(\{\ell g(\mathrm{tr}(p(s)) + 1 : s \in \mathrm{dom}(p)\} \cup \{\zeta + 1\})$. Hence applying (4) and (5B) to $\mathbf{m}{\restriction}L_1$, i.e. the induction hypothesis we can find q_1 such that $\mathbb{P}_{\mathbf{m}{\restriction}L_1} \models$ "$p{\restriction}L_1 \le q_1$" and $[s \in \mathrm{Dom}(q_1) \Rightarrow \ell g(\mathrm{tr}(q_1(s)) > \zeta_*]$ and q_1 forces a value to $\underset{\sim}{f}_{p(s),\iota}{\restriction}\zeta_*$, call it ρ_s for $s \in \mathrm{Dom}(p) \cap L_2$ and $\iota < \iota(p(s))$.
Define $q \in \mathbb{P}_\mathbf{m}$ by $\mathrm{Dom}(q) = \mathrm{Dom}(q_1) \cup (L_2 \cap \mathrm{dom}(p)), q{\restriction}L_1 = q_1$ and if $s \in L_2 \cap \mathrm{Dom}(p)$ then $q(s) = (\rho_s, \rho_s{}^\frown(\underset{\sim}{f}_{p(s)}{\restriction}[\zeta_*, \lambda))$, recalling 2.11. {c8}
Easily q is as required.

Case 3: $\delta = \mathrm{dp}_{\mathbf{m}}$ is a limit ordinal of cofinality $\geq \lambda$

So $\alpha = \sup\{\mathrm{dp}_{\mathbf{m}}(s) + 1 : s \in \mathrm{Dom}(p)\}$ is an ordinal $< \delta$ and let $L = \{s \in L_{\mathbf{m}} : \mathrm{dp}_{\mathbf{m}}(s) < \alpha\}$, so L is an initial segment of $L_{\mathbf{m}}$ and applying the induction hypothesis to $\mathbf{m}{\restriction}L, p$ we get q as required in $\mathbb{P}_{\mathbf{m}{\restriction}L}$ hence in $\mathbb{P}_{\mathbf{m}}$.

Case 4: $\delta = \mathrm{dp}_{\mathbf{m}}$ is a limit ordinal of cofinality $< \lambda$.

Let $\langle \alpha_i : i < \mathrm{cf}(\delta)\rangle$ be increasing continuous with limit δ, let $\alpha_{\mathrm{cf}(\delta)} = \delta$ and for $i \leq \mathrm{cf}(\delta)$ let $L_i := \{s \in L_{\mathbf{m}} : \mathrm{dp}_{\mathbf{m}}(s) < 1 + \alpha_i\}$.

Now we choose (p_i, ζ_i) by induction on $i < \mathrm{cf}(\delta)$ such that:

- (a) $p_i \in \mathbb{P}_{\mathbf{m}{\restriction}L_i}$
- (b) $\mathbb{P}_{\mathbf{m}{\restriction}L_i} \models$ "$p{\restriction}L_i \leq p_i$ and $p_j \leq p_i$" when $j < i$
- (c) if i is a limit ordinal then p_i is gotten from $\langle p_j : j < i\rangle$ as in part (4)
- (d) if $s \in \mathrm{Dom}(p_i)$ then $\ell g(\mathrm{tr}(p_i(s))) \geq \zeta_i$
- (e) $\langle \zeta_j : j < i\rangle$ is an increasing continuous sequence of ordinals $< \lambda$ and if i is non-limit then ζ_i is $> \zeta$ and $\geq \sum\limits_{j<i} |\mathrm{Dom}(p_j)| + |\mathrm{Dom}(p)|$ and $>$ $\sup(\{\ell g(\mathrm{tr}(p_j(s))) : j < i \text{ and } s \in p_j\} \cup \{\ell g(\mathrm{tr}(p(s))) : s \in \mathrm{Dom}(p)\})$.

{c8} Using 2.11 and the induction hypothesis this is easy.

6) For transparency assume $\Vdash$ "$\underset{\sim}{y} \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$" or just $\in {}^{\lambda}\mathbf{V}$. By parts (4) + (5B), i.e. part (3), for each $\zeta < \lambda$ the following subset of $\mathbb{P}_{\mathbf{m},t}$ is open and dense: $\mathscr{I}_\zeta = \{p \in \mathbb{P}_{\mathbf{m},t}$: for some $\nu \in \prod\limits_{\varepsilon<\zeta} \theta_\varepsilon$ or $\in {}^{\zeta}\mathbf{V}$ (from $\mathbf{V}$!) we have $p \Vdash_{\mathbb{P}_{\mathbf{m},t}}$ "$\underset{\sim}{y}{\restriction}\zeta = \nu$"$\}$.
Clearly there is a maximal antichain $\langle p_{\zeta,\varepsilon} : \varepsilon < \xi_\zeta\rangle$ of $\mathbb{P}_{\mathbf{m},t}$ included in $\mathscr{I}_\zeta$ and by part (2) without loss of generality $\xi_\zeta \leq \lambda$, the rest should be clear. In the general case we can code $\underset{\sim}{y}$ as a subset of λ, etc.

6A) This too should be clear as $\mathbb{P}_t$ satisfies the λ^+-c.c.

7) Look at the definitions.

8) Using parts (4) and (5B) and the definition this is easy.

9) Suppose toward contradiction that $\mathbf{G}_1 \neq \mathbf{G}_2$ are generic subsets of $\mathbb{P}_{\mathbf{m}}$ but $s \in L_{\mathbf{m}} \Rightarrow \underset{\sim}{\eta}_s[\mathbf{G}_1] = \eta_s = \underset{\sim}{\eta}_s[\mathbf{G}_2]$.

Let $p_1 \in \mathbf{G}_1 \backslash \mathbf{G}_2$ hence there is $p_2 \in \mathbf{G}_2$ such that $p_2 \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$p_1 \notin \underset{\sim}{\mathbf{G}}_2$" hence p_1, p_2 are incompatible. Let $L_* = \{s \in L_{\mathbf{m}} : \mathbf{G}_1 \cap \mathbb{P}_{\leq s} = \mathbf{G}_2 \cap \mathbb{P}_{\leq s}\}$ so L_* is an initial segment of $L_{\mathbf{m}}$. If $L_* = L_{\mathbf{m}}$ we can easily get a contradiction, so $L_* \neq L_{\mathbf{m}}$ and let $r \in L_{\mathbf{m}} \backslash L_*$ be such that $L_{<r} \subseteq L_*$. Now as in part (8) we can get a contradiction having found a common to upper bound to p_1, p_2.

Alternatively use part (6). $\square_{2.13}$

{c13}

Conclusion 2.15. *Let $\mathbf{m} \in \mathbf{M}$ and for notational transparency for some ordinal $\beta(*), t \in L_{\mathbf{m}} \Leftrightarrow t \in \beta(*)$ and $s <_{\mathbf{m}} t \Rightarrow s < t$. Then $\mathbf{q}$ is essentially a $(<\lambda)$-support iteration of length $\beta(*)$ with $\underset{\sim}{\mathbb{Q}}_\alpha = \{(\nu, f) \in \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\beta : \beta<\alpha\rangle]} : \nu \triangleleft f, f = \sup\{f_\iota : \iota < \iota(\alpha)\}, \iota(\alpha) < \lambda, \nu \triangleleft f_\iota$ and $\{f_\iota : \iota < \iota(\alpha)\} \subseteq \cup\{\mathbb{Q}_\theta^{\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u\rangle]} : u \in \mathscr{P}_{\mathbf{m},\alpha}\}\}$ with the natural order, i.e. the order of $\mathbb{Q}_\theta^{\mathbf{V}[\mathbb{P}_\alpha]}$ restricted to this set.*

{c11} *Proof.* Should be clear by 2.13. $\square_{2.15}$

Till now $(E_{\mathbf{m}}, M_{\mathbf{m}})$ have played no role and we could have omitted them.

{c26}

Definition 2.16. 1) We define the two-place relation $\le = \le_{\mathbf{M}}$ on $\mathbf{M}$ as follows: $\mathbf{m} \le \mathbf{n}$ <u>iff</u>

- (a) $L_{\mathbf{m}} \subseteq L_{\mathbf{n}}$, as partial orders of course,
- (b) $M_{\mathbf{m}} = M_{\mathbf{n}}$, yes! equal,
- (c) $u_{\mathbf{m},t} = u_{\mathbf{n},t} \cap L_{\mathbf{m}}$ and[12] $\mathscr{P}_{\mathbf{m},t} = \{u \cap L_{\mathbf{m}} : u \in \mathscr{P}_{\mathbf{n},t}\}$ for $t \in M_{\mathbf{m}}$,
- (d) $u_{\mathbf{m},t} = u_{\mathbf{n},t}$ and $\mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$ for $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$
- (e) if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $t/E'_{\mathbf{m}} = t/E'_{\mathbf{n}}$ hence $E'_{\mathbf{m}} = E'_{\mathbf{n}} \restriction L_{\mathbf{m}}$
- (f)
  - if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $\mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$
  - if $t \in M_{\mathbf{m}}$ and $s \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $\{u \in \mathscr{P}_{\mathbf{m},t} : u \subseteq s/E_{\mathbf{m}}\} = \{u \in \mathscr{P}_{\mathbf{n},t} : u \subseteq s/E_{\mathbf{n}}\}$
  - if $t \in M_{\mathbf{m}}$ then $\{u \in \mathscr{P}_{\mathbf{m},t} : u \subseteq M_{\mathbf{m}}\} = \{u \in \mathscr{P}_{\mathbf{n},t} : u \subseteq M_{\mathbf{n}}\}$.

2) We define the two-place relation $\le_* = \le^*_{\mathbf{M}}$ as in part (1) omitting clauses (b),(d),(e) and (f); natural but not used.

{c28}

Claim 2.17. *1) $\le_{\mathbf{M}}$ is a partial order.*
2) If $\langle \mathbf{m}_\alpha : \alpha < \delta\rangle$ is $\le_{\mathbf{M}}$-increasing, <u>then</u> its union $\mathbf{m}_\delta$ (naturally defined) is a $\le_{\mathbf{M}}$-lub and $|L_{\mathbf{m}_\delta}| \le \Sigma\{|L_{\mathbf{m}_\alpha}| : \alpha < \delta\}$.
3) If $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ and $L \subseteq L_{\mathbf{m}}$ <u>then</u> $p \in \mathbb{P}_{\mathbf{m}}(L) \Leftrightarrow p \in \mathbb{P}_{\mathbf{n}}(L)$ for every p.
4) If $m \le_M n$ and $\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$ and $L \subseteq L_{\mathbf{m}}$ <u>then</u> $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$ as quasi orders.

Proof. Easy, e.g.
1) Why is $L_{\mathbf{m}_\delta} := \cup\{L_{\mathbf{m}_\alpha} \alpha < \delta\}$ well founded? Toward contradiction assume $\bar{t} = \langle t_n : n < \omega\rangle$ is $<_{L_{\mathbf{m}_\delta}}$-decreasing. We can replace $\bar{t}$ by any infinite subsequence. So without loss of generality

- $(*)$ either (α) or (β) where
  - (α) for every $n < m$ there is $s_{n,m} \in M_{\mathbf{m}_0}$ such that $t_m <_{L_\delta} s_{n,m} <_{L_\delta} t_n$
  - (β) for no $n < m$ this holds.

If clause (α) holds, then $\langle s_{n,n+1} : n < \omega\rangle$ is a $<_{M_0}$-decreasing sequence contradiction. If (β) holds for any $n < m$, let $\alpha(n) = \min\{\alpha : t_n \in L_{\mathbf{m}_\alpha}\}$; without loss of generality it is monotonically increasing or constant so as $M_{\mathbf{m}_{\alpha(n)}} = M_{\mathbf{m}_0}$; by 2.16(1)(e) we get $t_n/E_{\mathbf{m}_{\alpha(n+1)}} = t_{n+1}/E_{\mathbf{m}_{\alpha(n+1)}}$ hence $t_{n+1} \in L_{\mathbf{m}_{\alpha(n)}}$ hence $\alpha(n+1) \le \alpha(n)$. As $L_{\mathbf{m}_{\alpha(n)}}$ is well founded we are done. {c26}
3) See the proof of $\boxplus_\alpha$ in the proof of 2.22. $\square_{2.17}$ {c33s} {c31}

Claim 2.18. $(\mathbf{M}, \le_{\mathbf{m}})$ *has amalgamation.*
That is, if $\mathbf{m}_0 \le_{\mathbf{M}} \mathbf{m}_1, \mathbf{m}_0 \le_{\mathbf{M}} \mathbf{m}_2$ and $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = L_{\mathbf{m}_0}$ <u>then</u> there is $\mathbf{m} \in \mathbf{M}$ such that $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}, \mathbf{m}_2 \le_{\mathbf{M}} \mathbf{m}$ and $L_{\mathbf{m}} = L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$.

Proof. Note that by clause $(e)(\gamma)$ of Definition 2.7 and clause (e) of Definition 2.16(1): {c26}

- $(*)$ assume $(s_1 \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}) \cap (s_3 \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0})$ and $s_2 \in L_{\mathbf{m}_0}$;
  - if $s_1 <_{\mathbf{m}_1} s_2 \wedge s_2 <_{\mathbf{m}_2} s_3$ <u>then</u> for some $s'_2, s''_2 \in M_{\mathbf{m}_0}$ we have $s'_2 \le_{\mathbf{m}_0} s_2 \le_{\mathbf{m}_0} s''_2, s_1 <_{\mathbf{m}_1} s'_2 \wedge s''_2 <_{\mathbf{m}_2} s_3$

[12]This is covered by clause (f) but see part (2).

- if $s_3 <_{\mathbf{m}_2} s_2 \wedge s_2 <_{\mathbf{m}_1} s_1$ then for some $s'_2, s''_2 \in M_{\mathbf{m}_0}$ we have $s'_2 \leq_{\mathbf{m}_0} s_2 \leq_{\mathbf{m}_0} s''_2$ and $s_3 <_{\mathbf{m}_2} s'_2 \wedge s''_2 <_{\mathbf{m}_1} s_1$.

We now define $\mathbf{m}$ by:

$(*)$ (a) (α) $t \in L_{\mathbf{m}}$ iff $t \in L_{\mathbf{m}_1} \vee t \in L_{\mathbf{m}_2}$
  (β) $M_{\mathbf{m}} = M_{\mathbf{m}_0}$
 (b) $s <_{\mathbf{m}} t$ iff one of the following occurs:
  (α) $s <_{\mathbf{m}_1} t$
  (β) $s <_{\mathbf{m}_2} t$
  (γ) $(s \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0})$ and $t \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$ and $(\exists r \in M_{\mathbf{m}_0})(s \leq_{\mathbf{m}_1} r \wedge r \leq_{\mathbf{m}_2} t)$
  (δ) $s \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$ and $t \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$ and for some $r \in M_{\mathbf{m}_0}, s \leq_{\mathbf{m}_2} r \wedge r \leq_{\mathbf{m}_1} t$
 (c) $u_{\mathbf{m},t}$ is
  (α) $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$ if[13] $t \in L_{\mathbf{m}_0}$
  (β) $u_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$
  (γ) $u_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$
 (d) $E'_{\mathbf{m}} = E'_{\mathbf{m}_1} \cup E'_{\mathbf{m}_2}$
 (e) $\mathscr{P}_{\mathbf{m},t}$ is
  (α) $\mathscr{P}_{\mathbf{m}_1,t} \cup \mathscr{P}_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_0}$
  (β) $\mathscr{P}_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$
  (γ) $\mathscr{P}_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$.

Clearly

$\odot$ $\mathbf{m} \in \mathbf{M}$ and $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$ and $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}$.

So we are done. $\square_{2.18}$

{c32n} **Observation 2.19.** *For $p, q \in \mathbb{P}_{\mathbf{m}}$ the following conditions are equivalent:*

(a) $q \Vdash$ "$p \in \mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{P}_{\mathbf{m}}}$"
(b) *if $s \in \mathrm{Dom}(p)$ then either $s \in \mathrm{Dom}(q)$ and $(q \restriction L_{\mathbf{m},<s}) \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p(s) \leq q(s)$" or $s \notin \mathrm{Dom}(q), \mathrm{tr}(p(s)) = \emptyset$ and $q \restriction L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p(s)$ is trivial, i.e. $\mathop{f}\limits_{\sim}{}_{p(s)}$ is constantly zero"*
(c) $\mathbb{P}_{\mathbf{m}} \models$ "$p \leq q^+$" *where* $\mathrm{Dom}(q^+) = \mathrm{Dom}(q) \cup \mathrm{Dom}(p)$ *and* $q^+(s)$ *is*
 (α) $q(s)$ *if* $s \in \mathrm{Dom}(q)$
 (β) *the trivial condition if* $s \in \mathrm{dom}(p) \backslash \mathrm{dom}(q)$*; note that* $\mathrm{fsupp}(q^+) = \mathrm{fsupp}(q) \cup \mathrm{Dom}(p)$.

Remark 2.20. We shall use this freely. Maybe better to change the order.

Proof. Obvious recalling the properties of $\mathbb{Q}_{\bar{\theta}}$. $\square_{2.19}$

{c33n} {c7} **Claim 2.21.** *For $\mathbf{m} \in \mathbf{M}$, recalling 2.10(3), we have $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$ when:*

$(*)$ (a) *$L_2 \subseteq L_3$ are initial segments of $L_{\mathbf{m}}$*

[13]but recall that $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}} \Rightarrow u_{\mathbf{m}_\ell,t} = u_{\mathbf{m}_0,t} \wedge \mathscr{P}_{\mathbf{m}_\ell,t} = \mathscr{P}_{\mathbf{m}_0,t}$

(b) $L_1 \subseteq L_3$ *and* $L_0 = L_1 \cap L_2$
(c) L_0 *is an initial segment of* L_1,
(d) $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$
(e) $L_1 \backslash L_0$ *is disjoint to* $M_{\mathbf{m}}$
(f) *if* $t \in L_1 \backslash L_0$ *then* $(t/E_{\mathbf{m}}) \cap L_{\mathbf{m},<t} \subseteq L_1$.

Proof. As $\mathrm{dp}_{\mathbf{m}}(L_1) < \infty$ it suffices to prove by induction on the ordinal γ that:

$\boxplus_\gamma$ if $\langle L_\ell : \ell \leq 3 \rangle$ satisfies $(*)$ of the claim and $\mathrm{dp}_{\mathbf{m}}(L_1) \leq \gamma$ <u>then</u>
- (a) $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$
- (b) we have $p_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$ and $p_1 \leq q_1 \in \mathbb{P}_{\mathbf{m}}(L_1) \Rightarrow p_3, q_1$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_3)$ <u>when</u>:
  - (α) $p_3 \in \mathbb{P}_{\mathbf{m}}(L_3)$
  - (β) $p_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$
  - (γ) if $p_0 \leq q_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ <u>then</u> $p_2 := p_3 {\restriction} L_2$ and q_0 are compatible in $P_{\mathbf{m}}(L_2)$
  - (δ) $p_1 = p_0 \cup (p_3 {\restriction} (L_1 \backslash L_0))$.

Why this holds? Assume we have arrived to γ.

<u>Clause (b)</u>: Recalling clause (f) of the assumption, indeed, $p_1 = p_0 \cup (p_3 {\restriction} (L_1 \backslash L_0)) \in \mathbb{P}_{\mathbf{m}}(L_1)$ by the definitions (clauses $(b)(\alpha), (\beta), (\delta)$ of $\boxplus_\gamma$), e.g. why $\mathrm{fsupp}(p_1) \subseteq L_1$? Note that if $s \in \mathrm{dom}(p_3 {\restriction} (L_1 \backslash L_0)$ then $s \in L_1 \backslash L_0 \subseteq L_1$ and $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\}$ is included in L_3 because $p \in \mathbb{P}_{\mathbf{m}}(L_3)$ and in $L_{<s}$ by Definition 2.9. As $s \in L_1 \backslash L_0$ {c6}
by $(*)(e)$ we have $s \notin M_{\mathbf{m}}$ hence by Definition 2.9 we have $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\} \subseteq$ {c6}
$u_s \subseteq s/E_{\mathbf{m}}$. By $(*)(f)$ we have $(s/E_{\mathbf{m}}) \cap L_{\mathbf{m}<t} \subseteq L_1$ hence together $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\} \subseteq L_1$, and we are done proving $\mathrm{fsupp}(p_1) \subseteq L_1$.

So the first statement in $\boxplus_\gamma(b)$ holds; what about the second? Toward contradiction assume q_1 contradicts the desired conclusion then by 2.13(8) there are s and {c11}
p_3^+ such that:

$\oplus$ (a) $s \in \mathrm{dom}(q_1) \cap \mathrm{dom}(p_3)$
(b) $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$
(c) p_3^+ is above $p_3 {\restriction} L_{\mathbf{m},<s}$ and above $q_1 {\restriction} L_{\mathbf{m},<s}$
(d) $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_3(s), q_1(s) \in \mathbb{Q}_{\bar{\theta}}$ are incompatible (in $\mathbb{Q}_{\bar{\theta}}$)".

So $s \in \mathrm{dom}(q_1) \subseteq L_1$ and as L_2 is an initial segment of $L_{\mathbf{m}}$ and clause (γ) of (b) (of $\boxplus_\gamma$), clearly $s \in L_0$ is impossible, so $s \in \mathrm{dom}(q_1) \backslash L_0 \subseteq L_1 \backslash L_0$. As $\mathbb{P}_{\mathbf{m}} \models$ "$p_1 \leq q_1$", necessarily $q_1 {\restriction} L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_1(s) \leq q_1(s)$", so as $q_1 {\restriction} L_{\mathbf{m},<s} \leq p_3^+ {\restriction} L_{\mathbf{m},<s}$ (by $\oplus(c)$), also $p_3^+ {\restriction} L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_1(s) \leq q_1(s)$". As $s \notin L_0$ clearly $p_1(s) = p_3(s)$ by clauses $\boxplus_\gamma(b)(\beta), (\delta)$, so $p_3^+ {\restriction} L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_3(s) \leq q_1(s)$" and again easy contradiction to $\oplus(d)$.

<u>Clause (a)</u>:

Clearly $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_3)$ as quasi orders. Next we shall prove $\mathbb{P}_{\mathbf{m}}(L_1) \leq_{\mathrm{ic}} \mathbb{P}_{\mathbf{m}}(L_3)$, so assume $q_1, q_2 \in \mathbb{P}_{\mathbf{m}}(L_1)$ has a common upper bound p_3 in $\mathbb{P}_{\mathbf{m}}(L_3)$, and we should find one in $\mathbb{P}_{\mathbf{m}}(L_1)$. Hence (see 2.9(e)($\beta$) we have $\mathrm{Dom}(q_1) \cup \mathrm{Dom}(q_2) \subseteq$ {c6}
$\mathrm{Dom}(p_3)$.

As $p_3 {\restriction} L_2 \in \mathbb{P}_{\mathbf{m}}(L_2)$ by $(*)(a)$ and we are assuming $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$, see $(*)(d)$ there is $p_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ such that $p_0 \leq q \in \mathbb{P}_{\mathbf{m}}(L_0) \Rightarrow q, p_3 {\restriction} L_2$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_2)$ and let $p_1 = p_0 \cup (p_3{\restriction}(L_1 \backslash L_0))$. By $\boxplus_\gamma(b)$, which we have proved noting that clauses $(\alpha) - (\delta)$ of $\boxplus_\gamma(b)$ holds, we know that $p_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$ and $p_1 \le p'_1 \in \mathbb{P}_{\mathbf{m}}(L_1) \Rightarrow p_3, p'_1$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_3)$. It suffices to prove that p_1 is a common upper bound of q_1, q_2.

We could have replaced p_0 by p'_0 whenever $p_0 \le p'_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$. So without loss of generality for $\ell = 1,2$ we have $\mathrm{dom}(q_\ell) \cap L_0 \subseteq \mathrm{dom}(p_0)$ hence $\subseteq \mathrm{dom}(p_1)$, also recall $\mathrm{dom}(q_\ell) \backslash L_0 \subseteq \mathrm{dom}(p_3) \cap L_1 \backslash L_0$ and by the choice of p_1 we have $\mathrm{dom}(p_3) \cap L_1 \backslash L_0 \subseteq \mathrm{dom}(p_1) \backslash L_0$.

So recalling $\mathrm{dom}(q_\ell) \subseteq L_1$ together $\mathrm{dom}(q_\ell) \subseteq \mathrm{dom}(p_1)$.

As we are assuming $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$ without loss of generality p_0 is above[14] $q_\ell{\restriction}L_0$. If toward contradiction we assume that $\ell \in \{1,2\}$ and $q_\ell \nleq p_1$ then for some $s \in \mathrm{Dom}(q_\ell)$ we have $(q_\ell{\restriction}L_{\mathbf{m},<s}) \le (p_1{\restriction}L_{\mathbf{m},<s})$ but $p_1{\restriction}L_{\mathbf{m},<s} \nVdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})}$ "$q_\ell(s) \le p_1(s)$". Clearly, $s \in L_0$ is impossible so $s \in L_1 \backslash L_0$ hence $s \notin M_{\mathbf{m}}$ by clause $(*)(e)$.

Let $L'_0 = L_0, L'_1 = L_0 \cup (L_1 \cap L_{\mathbf{m},<s}), L'_2 = L_2, L'_3 = L_3$ so (L'_0, L'_1, L'_2, L'_3) satisfies the assumptions of the present claim and $\mathrm{dp}_{\mathbf{m}}(L'_1) < \gamma$, hence by the induction hypothesis, $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}_{\mathbf{m}}(L'_3)$.

Recall $s \in L_1 \backslash L_0$ hence $(s/E_{\mathbf{m}}) \cap L_{\mathbf{m},<s} \subseteq L_1$ by clause (f) of the assumption of the claim, so $\mathrm{fsupp}(p_1{\restriction}\{s\}) \backslash \{s\}, \mathrm{fsupp}(q_\ell{\restriction}\{s\}) \backslash \{s\}$ are $\subseteq L'_1$ hence $p_1(s), q_\ell(s)$ are $\mathbb{P}_{\mathbf{m}}(L'_1)$-names. So recalling $p_1{\restriction}L_{\mathbf{m},<s} \nVdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})}$ "$q_\ell(s) \le p_1(s)$" and $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}_{\mathbf{m}}(L'_3)$ and $L_{\mathbf{m},<s} \subseteq L_3 = L'_3$ we have $p_1{\restriction}L'_1 \nVdash_{\mathbb{P}_{\mathbf{m}}(L'_1)}$ "$q_\ell(s) \le p_1(s)$". Hence there is p_1^+ such that $p_1{\restriction}L'_1 \le p_1^+ \in \mathbb{P}_{\mathbf{m}}(L'_1)$ such that $p_1^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_1)}$ "$q_\ell(s) \nleq p_1(s)$" so recalling $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}(L'_3)$ we have $p_1^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \nleq p_1(s)$".

But by $\boxplus_{\gamma_1}(b)$ for $\gamma_1 = \mathrm{dp}_{\mathbf{m}}(L'_1)$, we know that p_1^+ and $p_3{\restriction}L_{\mathbf{m},<s}$ are compatible (in $\mathbb{P}_{\mathbf{m}}$, equivalently $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$) so let $p_3^+ \in \mathbb{P}_m(L_{\mathbf{m},<s})$ be a common upper bound of $p_1^+, p_3{\restriction}L_{\mathbf{m},<s}$. Now $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \le p_1(s)$" because: $q_\ell \le p_3$ by the choice of p_3; $p_1(s) = p_3(s)$ by the choice of p_1 and $p_3 \le p_3^+$, see above. However, $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \nleq p_1(s)$" as $p_1^+ \le p_3^+$, see above.

So we have proved $\mathbb{P}_{\mathbf{m}}(L_1) \le_{\mathrm{ic}} \mathbb{P}_{\mathbf{m}}(L_3)$.

To finish proving clause $\boxplus_\gamma(a)$ that is $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$ note that clause $\boxplus_\gamma(b)$ does this as for every $p_3 \in \mathbb{P}_{\mathbf{m}}(L_3)$ there is p_0 as in $\boxplus_\gamma(\beta), (\gamma)$ by clause (d) of the claim's assumption and let p_1 be as defined in $\boxplus_\gamma(b)(\delta)$. $\square_{2.21}$

{c33s} **Claim 2.22.** *We have $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_1)$ (i.e. as quasi orders) and $\mathbb{P}_{\mathbf{m}_\ell}(L_1) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1,2$ <u>when</u>:*

$\boxdot$ (a) $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2$

(b) $L_0 \subseteq L_1 \subseteq L_{\mathbf{m}_1}$

(c) *L_0 is an initial segment of L_1*

(d) $\mathbb{P}_{\mathbf{m}_1}(L_0) = \mathbb{P}_{\mathbf{m}_2}(L_0)$

(e) *$\mathbb{P}_{\mathbf{m}_\ell}(L_0) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1,2$*

[14]Why? It suffices to prove that there is $p'_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ above p_0 and above $q_\ell{\restriction}L_0$. So toward contradiction assume this fails hence there is $p_0^+ \in \mathbb{P}_{\mathbf{m}}(L_0)$ above p_0 incompatible with $q_\ell{\restriction}L_0$. By the choice of p_0 we know that $p_0^+, (p_3{\restriction}L_2)$ are compatible, so let $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_2)$ be a common upper bound. Now L_2 is an initial segment of $L_{\mathbf{m}}$ by $(*)(a)$ and p_3 is above q_ℓ hence $p_3{\restriction}L_2$ is above $q_\ell{\restriction}L_2$ and as $q_\ell \in \mathbb{P}_{\mathbf{m}}(L_1), L_0 = L_1 \cap L_2$ we have $q_\ell{\restriction}L_2 = q_\ell{\restriction}L_0, p_3{\restriction}L_2$ is above $q_\ell{\restriction}L_0$ but p_3^+ is above $p_3{\restriction}L_2$ hence p_3^+ is above $q_\ell{\restriction}L_2$. Also p_3^+ is above p_0^+ which forces $q_\ell{\restriction}L_0 \notin \mathbf{G}_{\mathbb{P}_{\mathbf{m}}(L_0)}$, equivalently $q_\ell{\restriction}L_0 \notin \mathbf{G}_{\mathbb{P}_{\mathbf{m}}(L_2)}$, contradiction.

(f) *if* $t \in L_1 \backslash L_0$ *then* $t \notin M_{\mathbf{m}_2}$ *(but see present* $\boxplus_\alpha$*) and* $L_{\mathbf{m}_1,<t} \cap (t/E_{\mathbf{m}_1}) = L_{\mathbf{m}_2,<t} \cap (t/E_{\mathbf{m}_2}) \subseteq L_1$.

Remark 2.23. Used only in the proof of $\boxplus_{4.4}$ inside the proof of 4.19, so we could have used $M_\beta, \mathscr{E}$ from there. {e32}

Proof. For $\ell \in \{1,2\}$ let $\bar{L}_\ell = \langle L_{\ell,i} : i < 4\rangle$ be defined by:

$\oplus_1$ (a) $L_{\ell,0} = L_0$

(b) $L_{\ell,1} = L_1$

(c) $L_{\ell,2} = \{s \in L_{\mathbf{m}_\ell} : s \leq_{\mathbf{m}_\ell} t \text{ for some } t \in L_0\}$

(d) $L_{\ell,3} = L_{\mathbf{m}_\ell}$.

Clearly

$\oplus_2$ (a) $(\mathbf{m}_\ell, \bar{L}_\ell)$ satisfies the assumptions of 2.21 hence {c33n}

(b) $\mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,1}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,3})$ which means $\mathbb{P}_{\mathbf{m}_\ell}(L_1) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1,2$.

Why $\oplus_2$? Clearly it suffices to prove clause (a), so we just have to check clauses $(*)(a)-(f)$ of 2.22. {c33s}

Clause $(*)(a)$:

By $\oplus_1(d), L_{\ell,3} = L_{\mathbf{m}_\ell}$ hence is an initial segment of $L_{\mathbf{m}_\ell}$ and by $\oplus(c), L_{\ell,2}$ is an initial segment of $L_{\mathbf{m}_\ell}$ which is $L_{\ell,3}$ so $L_{\ell,2} \subseteq L_{\ell,3}$.

Clause $(*)(b)$:

For the first statement, $L_{\ell,1} \subseteq L_{\ell,3}$ is trivial by $\oplus_1(d) + \oplus_1(b) + \boxdot(a),(b)$. The second statement says $L_{\ell,0} = L_{\ell,1} \cap L_{\ell,2}$. Now $L_{\ell,0} \subseteq L_{\ell,1}$ by $\boxdot(a),(b)$ of the claim and $\oplus_1(a),(b)$. Also $L_{\ell,0} \subseteq L_{\ell,2}$ holds by $\oplus_1(c)$ (and $\oplus_1(a)$). Together $L_{\ell,0} \subseteq L_{\ell,1} \cap L_{\ell,2}$; to prove the inverse inclusion assume $s \in L_{\ell,2} \cap L_{\ell,1}$, so as $s \in L_{\ell,2}$ by $\oplus_1(c)$ there is $t \in L_0$ such that $s \leq_{\mathbf{m}_\ell} t$. But $s \in L_{\ell,1} = L_1$ so by $\boxdot(c)$ of the claim we have $s \in L_0$ as promised.

Clause $(*)(c)$:

Holds by $\boxdot(c)$ of the claim.

Clause $(*)(d)$:

By clause $\boxdot(f)$ of the claim and $\boxplus_2(c), L_{\ell,2}$ is an initial segment of $L_{\mathbf{m}_\ell}$, hence by 2.11(e) we have $\mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,2}) \lessdot \mathbb{P}_{\mathbf{m}_\ell} = \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,3})$. By $\boxdot(e)\, \mathbb{P}_{\mathbf{m}_0}(L_{\ell,0}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$; so together as $L_{\ell,0} \subseteq L_{\ell,2}$, we have $\mathbb{P}_{\mathbf{m}_\ell}(L_0) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,2})$. {c8}

Clause $(*)(e),(f)$:

Holds by $\boxdot(f)$ of the claim.

So $\oplus_2$ holds indeed. So now we deal with the other half.

Proof of: $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_1)$.

Let $\langle s_\alpha : \alpha < \alpha(*)\rangle$ list $L_1 \backslash L_0$ such that $s_\alpha \leq_{L_\mathbf{m}} s_\beta \Rightarrow \alpha \leq \beta$. This is possible as $L_{\mathbf{m}_2}$ is well founded.

Now

$\oplus_3$ for $\ell = 1,2$ and $\alpha \leq \alpha(*)$ let $\bar{L}^*_{\ell,\alpha} = \langle L^*_{\ell,\alpha,i} : i < 4\rangle$ be (so we can omit ℓ if $\ell = 0,1$)

(a) $L^*_{\ell,\alpha,0} = L_0$

(b) $L^*_{\ell,\alpha,1} = L_0 \cup \{s_\beta : \beta < \alpha\}$

(c) $L^*_{\ell,\alpha,2} = \{s \in L_{\mathbf{m}_\ell} : s \le_{\mathbf{m}_\ell} t \text{ for some } t \in L_0\}$

(d) $L^*_{\ell,\alpha,3} = L_{\mathbf{m}_\ell}$

{c33n} $\oplus_4$ (a) $(\bar{\mathbf{m}}_\ell, \bar{L}^*_{\ell,\alpha})$ satisfies the assumption of 2.21

(b) $\mathbb{P}_{\mathbf{m}_\ell}(L^*_{\ell,\alpha,1}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L^*_{\ell,\alpha,3})$.

{c33s} [Why? Note the $\mathbf{m}_\ell, \langle L^*_{\ell,\alpha,i} : i < 4\rangle$ satisfies the assumptions of 2.22, hence $\oplus_2$ holds for $\mathbf{m}_\ell, \bar{L}_{\ell,\alpha}$ for $\alpha \le \alpha(*)$. Now by induction on $\alpha \le \alpha(*)$ we prove that:

$\boxplus_\alpha$ $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) = \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$.

Case 1: $\alpha = 0$

As $L^*_{1,\alpha,1} = L_0 = L^*_{2,\alpha,1}$, clause $\boxdot(d)$ of the assumption gives $\boxplus_\alpha$ as promised.

Case 2: α a limit ordinal

Easy by the definition of the iteration. That is, first we know $p \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \Leftrightarrow \bigwedge\limits_{\beta<\alpha} [p{\restriction}L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})] \Leftrightarrow \bigwedge\limits_{\beta<\alpha} [p{\restriction}L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1})] \Leftrightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$; second, for $p,q \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1})$ by the definition of the order and the induction hypothesis, $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$" iff $\bigwedge\limits_{\beta<\alpha} [\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$p{\restriction}L^*_{\beta,1} \le q{\restriction}L^*_{\beta,1}$"$]$ iff $\bigwedge\limits_{\beta<\alpha} [\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \models$ "$p{\restriction}L^*_{\beta,1} \le q{\restriction}L^*_{\beta,1}$"$]$ iff $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$".

So $\boxplus_\alpha$ holds.

Case 3: $\alpha = \beta + 1$

Clearly

$(*)_1$ $p \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \Leftrightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$.

Next

$(*)_2$ assume $p,q \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1})$ and we shall prove that $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$" implies $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$".

[Why? If $s_\beta \notin \mathrm{dom}(p)$ this is obvious by the induction hypothesis.

Hence we can assume $s_\beta \in \mathrm{dom}(p)$, so as we are assuming $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$", clearly $s_\beta \in \mathrm{dom}(q)$ hence $s_\beta \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. First, similarly $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$(p{\restriction}L^*_{\beta,1}) \le (q{\restriction}L^*_{\beta,1})$" and $(q{\restriction}L^*_{\beta,1}) \Vdash_{\mathbb{P}_{\mathbf{m}_1,<s_\beta}}$ "$p(s_\beta) \le_{\mathbb{Q}_{\bar\theta}} q(s_\beta)$" by the definition of $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})$. Second, as $q{\restriction}L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) = \mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1})$ and $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \lessdot \mathbb{P}_{\mathbf{m}_2}$ by $\oplus_4$ and $\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \lessdot \mathbb{P}_{\mathbf{m}_2}$ by $\oplus_2$ and $p(s_\beta), q(s_\beta)$ are $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})$-names (as $\mathrm{fsupp}(p(s_\beta), \mathrm{fsupp}(q(s_\beta)) \subseteq L^*_{\beta,1})$ necessarily we have $q{\restriction}L^*_{\beta,1} \Vdash_{\mathbb{P}_{\mathbf{m}_2}}$ "$p(s_\beta) \le_{\mathbb{Q}_{\bar\theta}} q(s_\beta)$". Third, as $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$p{\restriction}L^*_{\beta,1} \le q{\restriction}L^*_{\beta,1}$", by the induction hypothesis $\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \models$ "$p{\restriction}L^*_{\beta,1} \le q{\restriction}L^*_{\beta,1}$. Fourth, by the last two sentence and the definition of the order in $\mathbb{P}_{\mathbf{m}_2}$ we have $\mathbb{P}_{\mathbf{m}_2} \models$ "$p \le q$" so the conclusion of $(*)_2$ holds also in this case.

Note that if $s_\beta \in \mathrm{dom}(p) \backslash \mathrm{dom}(q)$ then $p \nleq q$, so we are done proving $(*)_2$.]

$(*)_3$ if $p,q \in \mathbb{P}_{\mathbf{m},1}(L^*_{\alpha,1})$ and $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$" then $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$".

[Why? Similar to the proof of $(*)_2$.]

By $(*)_1, (*)_2, (*)_3$ clearly $\boxplus_\alpha$ holds. So we carried the induction so $\boxplus_\alpha$ holds for every $\alpha \le \alpha(*)$ and for $\alpha = \alpha(*)$ we get $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_2)$. Together with $\oplus_2(b)$ in the beginning of the proof we are done. $\square_{2.22}$

{c34}
Definition 2.24. 0) For $L \subseteq L_{\mathbf{m}}, \mathbf{m} \in \mathbf{M}$ let

(a) $\mathrm{dp}^*_{\mathbf{m}}(L) = \cup\{\mathrm{dp}_{M_{\mathbf{m}}}(t) + 1 : t \in L \cap M_{\mathbf{m}}\}$

(b) $L^{\mathrm{dp}}_{\mathbf{m},\gamma} = \{t \in L_{\mathbf{m}}$: if $s \le_{L_{\mathbf{m}}} t \wedge s \in M_{\mathbf{m}}$ then $\mathrm{dp}_{M_{\mathbf{m}}}(s) < \gamma$; moreover, $\sup\{\mathrm{dp}_{M_{\mathbf{m}}}(s) : s \in M_{\mathbf{m}}$ and $s <_{L_{\mathbf{m}}} t\} < \gamma\}$.

1) For an ordinal γ let $\mathbf{M}^{\mathrm{ec}}_\gamma$ be the class of $\mathbf{m} \in \mathbf{M}$ such that, recalling Definition 2.10(3): {c7}

(∗) if $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2$ <u>then</u> $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}(L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma})$ hence $L \subseteq L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}$ implies $\mathbb{P}_{\mathbf{m}_1}(L) = \mathbb{P}_{\mathbf{m}_2}(L)$ (by 2.17(4)). {c28}

2) Let $\mathbf{M}_{\mathrm{ec}} = M^{\mathrm{ec}}_\infty$ be the class of $\mathbf{m}$ which $\in \mathbf{M}^{\mathrm{ec}}_\gamma$ for every γ.
3) Let $\mathbf{M}^{\mathrm{ec}}_{\chi,\gamma} = \{\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\gamma : |L_{\mathbf{m}}| \le \chi\}$, similarly $\mathbf{M}^{\mathrm{ec}}_{\chi,\infty}$.

{c36d}
Observation 2.25. *1) Of course,* $\mathbf{M}^{\mathrm{ec}}_{\gamma_2} \subseteq \mathbf{M}^{\mathrm{ec}}_{\gamma_1}$ *and* $L^{\mathrm{dp}}_{\mathbf{m},\gamma_1} \subseteq L^{\mathrm{dp}}_{\mathbf{m},\gamma_2}$ *are initial segments of* $L_{\mathbf{m}}$ *when* $\gamma_1 \le \gamma_2$.
2) In 2.24(1), the following are equivalent: {c34}

(a) $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}(L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma})$ *for every* γ

(b) $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$.

Proof. 1) Easy.
2) First, concerning $(a) \Rightarrow (b)$, note that for γ large enough we have $\mathbb{P}_{\mathbf{m}_\ell}(L^{\mathrm{dp}}_{\mathbf{m}_\ell,\gamma}) = \mathbb{P}_{\mathbf{m}_\ell}$, so clear. Second, assume (b), note that $L^{\mathrm{dp}}_{\mathbf{m}_\ell,\gamma}$ is an initial segment of $L_{\mathbf{m}_\ell}$ hence $\mathbb{P}_{\mathbf{m}_\ell}(L^{\mathrm{dp}}_{\mathbf{m}_\ell,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1, 2$ by 2.11(c), hence we have $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$, but $\lessdot$ is transitive, hence $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}$. Also $\mathbb{P}_{\mathbf{m}_2}(L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}$ and $L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma} \subseteq L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma}$ by the definition hence by the definition $p \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \Rightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^{\mathrm{dp}}_{\mathbf{m}_2,\gamma})$; but lastly $(\mathbb{Q}_1 \lessdot \mathbb{P} \wedge \mathbb{Q}_2 \lessdot \mathbb{P} \wedge (\forall p)(p \in \mathbb{Q}_1 \to p \in \mathbb{Q}_2) \Rightarrow \mathbb{Q}_1 \lessdot \mathbb{Q}_2)$ so we are done. {c8} $\square_{2.25}$

{c41}
Crucial Claim 2.26. *If* $\chi \ge 2^{\lambda_2}$ *and* $\mathbf{m} \in \mathbf{M}_{\le\chi}$ <u>*then*</u> *for some* $\mathbf{n}$ *we have* $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$ *and* $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$.

Proof. Let $\mathscr{X} = \{\mathbf{n} : (\mathbf{m}{\restriction}M_{\mathbf{m}}) \le_{\mathbf{M}} \mathbf{n}$ and $L_{\mathbf{n}} \backslash M_{\mathbf{m}} = t/E''_{\mathbf{n}}$ for some t hence $\|L_{\mathbf{n}}\| \le \lambda_2\}$.

We define a two-place relation $\mathscr{E}$ on $\mathscr{X}$:

- $\mathbf{n}_1 \mathscr{E} \mathbf{n}_2$ <u>iff</u> ($\mathbf{n}_1, \mathbf{n}_2 \in \mathscr{X}$ and) there is an isomorphism h from $\mathbf{n}_1$ onto $\mathbf{n}_2$ over $\mathbf{m}{\restriction}M_{\mathbf{m}}$, that is: an isomorphism from $L_{\mathbf{n}_1}$ onto $L_{\mathbf{n}_2}$ over $M_{\mathbf{m}}$ such that $t \in L_{\mathbf{n}_1} \Rightarrow u_{\mathbf{n}_2,h(t)} = \{h(s) : s \in u_{\mathbf{n}_1,t}\}$ and $t \in L_{\mathbf{n}_1} \Rightarrow \mathscr{P}_{\mathbf{n}_2,h(t)} = \{\{h(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{n}_1,t}\}$ and $s, t \in L_{\mathbf{n}_1} \Rightarrow (sE'_{\mathbf{n}_1}t \Leftrightarrow h(s)E'_{\mathbf{n}_2}h(t))$.

Clearly $\mathscr{E}$ is an equivalence relation.

By our assumptions $\chi \ge 2^{\lambda_2}$ and $\mathbf{n} \in \mathscr{X} \Rightarrow |L_{\mathbf{n}}| \le \lambda_2 \wedge (\forall t \in L_{\mathbf{n}})(\mathscr{P}_{\mathbf{n},t} \subseteq [L_{\mathbf{n},<t}]^{\le\lambda_2})$ hence recalling $\lambda_2 = (\lambda_2)^\lambda$ clearly $\mathscr{E}$ has $\le 2^{\lambda_2}$ equivalence classes and let $\langle \mathbf{n}_\alpha : \alpha < 2^{\lambda_2}\rangle$ be a set of representatives (not necessary, but no harm in allowing repetitions).

By 2.17(2) and 2.18 we can find $\mathbf{n}$ such that: {c28}

$(*)_1$ (a) $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$

(b) for every $\alpha < 2^{\lambda_2}$ we can find $\langle t_{\alpha,i} : i < \chi\rangle$ such that

(α) $t_{\alpha,i} \in L_{\mathbf{n}} \backslash L_{\mathbf{m}}$

(β) $(\alpha \neq \beta) \vee (i \neq j) \Rightarrow t_{\alpha,i}/E_{\mathbf{n}} \neq t_{\beta,j}/E_{\mathbf{n}}$

{c5} (γ) $\mathbf{n} \restriction (t_{\alpha,i}/E_{\mathbf{n}})$ is $\mathscr{E}$-equivalent to $\mathbf{n}_\alpha$, see 2.8(0) on $t_{\alpha,i}/E_{\mathbf{n}}$.

Let us prove that $\mathbf{n}$ is as required. Let $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2$ and define $\mathscr{F}$ as the set of functions f such that some L_1, L_2:

$(*)_2$ (a) $L_\ell \subseteq L_{\mathbf{n}_2}$

(b) $M_{\mathbf{m}} = M_{\mathbf{n}} \subseteq L_1 \cap L_2$

(c) $L_\ell \backslash M_{\mathbf{m}}$ has cardinality $\le \lambda_2$

(d) L_ℓ is $E_{\mathbf{n}_2}$-closed, i.e. $t \in L_\ell \backslash M_{\mathbf{m}} \Rightarrow t/E_{\mathbf{n}_2} \subseteq L_\ell$

(e) f is an isomorphism from $\mathbf{n}_2 \restriction L_1$ onto $\mathbf{n}_2 \restriction L_2$ over $M_{\mathbf{m}}$, i.e.

- f is a one-to-one mapping from L_1 onto L_2
- $f \restriction M_{\mathbf{m}}$ is the identity
- f maps $\le_{\mathbf{n}_2} \restriction L_1$ onto $\le_{\mathbf{n}_2} \restriction L_2$
- for $s, t \in L_1$ we have $s E'_{\mathbf{n}_2} t \Leftrightarrow f(s) E'_{\mathbf{n}_2} f(t)$
- for $s, t \in L_1$ we have $s \in u_{\mathbf{n}_2,t} \Leftrightarrow f(s) \in u_{\mathbf{n}_2,f(t)}$
- for $t \in L_1$ we have $\mathscr{P}_{\mathbf{n}_2,f(t)} = \{\{f(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{n}_2,t}\}$.

Clearly

$(*)_3$ if $f \in \mathscr{F}$ and $L' \subseteq L_{\mathbf{n}_1}, L'' \subseteq L_{\mathbf{n}_2}$ and $|L'| + |L''| \le \lambda_2$ <u>then</u> for some $g \in \mathscr{F}$ extending f we have $L' \subseteq \mathrm{Dom}(g), L'' \subseteq \mathrm{Rang}(g)$ and $\mathrm{Rang}(g) \backslash (L'' \backslash \mathrm{Rang}(f)) \subseteq L_{\mathbf{n}_1}$.

We can finish as in the parallel of the Tarski-Vaught criterion for $\mathbb{L}_{\infty,\lambda_2^+}$. That is, first we can prove by induction on the ordinal $\gamma < |L_{\mathbf{n}_2}|^+$ and really just $\gamma < \|M_{\mathbf{n}_2}\|^+$ that:

$(*)_4$ letting $L_\gamma = L^{\mathrm{dp}}_{\mathbf{n}_2,\gamma}$, if $g \in \mathscr{F}$ <u>then</u>

(a) g maps $\mathrm{Dom}(g) \cap L_\gamma$ onto $\mathrm{Rang}(g) \cap L_\gamma$

(b) g induces an isomorphism $\hat{g}$ from $\mathbb{P}_{\mathbf{n}_2}(\mathrm{Dom}(g) \cap L_\gamma)$ onto $\mathbb{P}_{\mathbf{n}_2}(\mathrm{Rang}(g) \cap L_\gamma)$, that is: $\hat{g}(p) = q$ iff

(α) $p \in \mathbb{P}_{\mathbf{n}_2}(\mathrm{Dom}(g) \cap L_\gamma)$

(β) $q \in \mathbb{P}_{\mathbf{n}_2}(\mathrm{Rang}(g) \cap L_\gamma)$

(γ) g maps $\mathrm{dom}(p)$ onto $\mathrm{dom}(q)$ and $s \in \mathrm{dom}(p) \Rightarrow \mathrm{tr}(p(s)) = \mathrm{tr}(q(g(s)))$

(δ) if $s \in \mathrm{Dom}(g), g(s) = t \in \mathrm{Rang}(g)$ and $\underset{\sim}{f}_{p(s)} = \mathbf{B}_{p(s)}(\ldots, \underset{\sim}{\eta}_{r_{p(s)}(\zeta)}, \ldots)_{\zeta < \xi_{p(s)}}$ and $f_{q(t)} = \mathbf{B}_{q(t)}(\ldots, \underset{\sim}{\eta}_{r_{q(t)}(\zeta)}, \ldots)_{\zeta < \xi_{q(t)}}$ <u>then</u> $\xi_{q(t)} = \xi_{p(s)}, \mathbf{B}_{q(t)} = \mathbf{B}_{p(s)}$ and $\zeta < \xi_{p(s)} \Rightarrow r_{q(t)}(\zeta) = g(r_{p(s)}(\zeta))$

(ε) moreover in (δ) we have $\iota(s,p) = \iota(t,q)$ and if $\iota < \iota(s,p)$ then $w_{p,s,\iota} = w_{q,t,\iota}, \mathbf{B}_{p(s),\iota} = \mathbf{B}_{q(t),\iota}$.

Second,

$(*)_5$ $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \lessdot \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$.

[Why? By the definitions $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \subseteq \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ as quasi orders.

Also if $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ are compatible in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ let $q \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ be a common upper bound there. We can find an $E_{\mathbf{n}_2}$-closed $L' \subseteq L_\gamma \cap L_{\mathbf{n}_1}$ of cardinality $\le \lambda_2$ (recalling $\mathbf{n} \in \mathscr{X} \Rightarrow |L_{\mathbf{n}}| \le \lambda_2$) such that $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_1}(L')$ and $E_{\mathbf{n}_2}$-closed $L'' \subseteq L_\gamma$ of cardinality $\le \lambda_2$ such that $L' \subseteq L''$ and $q \in \mathbb{P}_{\mathbf{n}_2}(L'')$. Now we can find $f_1 \in \mathscr{F}$ such that $\mathrm{Dom}(f_1) = \cup\{t/E_{\mathbf{n}_2} : t \in L'\}$ recalling that $t/E_{\mathbf{m}} \supseteq M_{\mathbf{m}}$,
{c5} see 2.8(0) and f_1 is the identity. Then by $(*)_3$ we can find $f_2 \in \mathscr{F}$ extending f_1 with $\mathrm{Dom}(f_2) = \cup\{t/E_{\mathbf{n}_2} : t \in L''\}$ and $\mathrm{Rang}(f_2)\backslash\mathrm{Rang}(f_1) \subseteq L_{\mathbf{n}_1}$. So we have $\mathbb{P}_{\mathbf{n}_2} \models$ "$(p_1 \le \hat{f}_2(q)) \wedge p_2 \le \hat{f}_2(q)$" and $\hat{f}_2(q) \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ recalling $(*)_4$. So p_1, p_2 are compatible also in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$. Obviously, if $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ are compatible in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_2})$ say q witness then q is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{n}_1}(L_\gamma)$.

So every antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ is an antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$. Similarly to the above every maximal antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ is a maximal antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$; similarly for the other direction. So we are done.]

$(*)_6$ $\mathbb{P}_{\mathbf{n}_1}(L_\gamma \cap L_{\mathbf{n}_1}) = \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \lessdot \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$.

[Why? We prove this by induction on γ, as in proving the Tarski-Vaught criterion is sufficient (we shall later in the proof of 4.19, more specifically $\boxplus_4$ proves a similar statement in detail with weaker assumptions).] {e32}

Hence (using $\gamma = |L_{\mathbf{n}_2}|^+$)

$(*)_7$ $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$.

Hence for every $L \subseteq L_{\mathbf{n}_1}$ by 2.17(4) we have $\mathbb{P}_{\mathbf{n}_1}(L) = \mathbb{P}_{\mathbf{n}_2}(L)$ as required in {c28}
Definition 2.24. $\square_{2.26}$ {c34}

§ 3. The Corrected $\mathbb{P}_{\mathbf{m}}$

{z19} **Definition 3.1.** Let $\mathbb{P}$ be a forcing notion and $Y \subseteq \mathbb{P}$ and χ a regular cardinal.
1) Let $\mathbb{L}_\chi(Y)$ be the set of sentences formed from $\{p : p \in \mathbb{P}_Y\}$ closing under the operations $\neg p$ and $\bigwedge_{i<\alpha} p_i$, for $\alpha < \chi$; so propositional logic.
2) For $\mathbf{G} \subseteq \mathbb{P}$ and $\psi \in \mathbb{L}_\chi(Y)$ we define the truth value $\psi[\mathbf{G}]$ naturally (by induction on ψ starting with $p[\mathbf{G}] = \text{true} \Leftrightarrow p \in \mathbf{G}$).
3) Let $\mathbb{L}^+_\chi(Y,\mathbb{P})$, the $\mathbb{L}_\chi$-closure of Y for $\mathbb{P}$, ($Y \subseteq \mathbb{P}$; if $Y = \mathbb{P}$ we may omit Y) be the following partial order:

- set of elements $\{\psi \in \mathbb{L}_\chi(Y,\mathbb{P}) : \nVdash_{\mathbb{P}}$ "$\psi[\mathop{\mathbf{G}}\limits_{\sim}] = \text{false}$"$\}$
- the order $\psi_1 \le \psi_2$ iff $\Vdash_{\mathbb{P}}$ "if $\psi_2[\mathop{\mathbf{G}}\limits_{\sim}] = \text{true}$ then $\psi_1[\mathop{\mathbf{G}}\limits_{\sim}] = \text{true}$".

4) The completion of $\mathbb{P}$ is the $\mathbb{L}_\chi$-closure of $\mathbb{P}$ for $\mathbb{P}, \mathbb{L}_\chi(\mathbb{P})$ where χ is minimal such that $\mathbb{P}$ satisfies the χ-c.c.

{z44} **Claim 3.2.** *For a χ-c.c. forcing notion $\mathbb{P}$ and $Y \subseteq \mathbb{P}$ we have:*

(a) $\mathbb{L}^+_\chi(Y,\mathbb{P})$ *is a forcing notion*
(b) $\mathbb{P} \lessdot \mathbb{L}^+_\chi(\mathbb{P})$ *under the natural identification*[15]
(c) $\mathbb{L}^+_\chi(Y,\mathbb{P}) \lessdot \mathbb{L}^+_\chi(\mathbb{P})$
(d) $\mathbb{L}^+_{\chi_1}(Y,\mathbb{P}) \lessdot \mathbb{L}^+_{\chi_2}(Y,\mathbb{P})$ *when $\chi_1 \le \chi_2$ are regular (and $\chi_1 \ge \chi$)*
(e) *if $\mathbb{P}$ satisfies the χ_1-c.c. and $\chi_1 < \chi_2$ are regular <u>then</u> $\mathbb{L}^+_{\chi_1}(Y,\mathbb{P})$ is essentially equal to $\mathbb{L}^+_{\chi_2}(Y,P)$, i.e. up to the natural equivalence of elements in a quasi order*
(f) *if $Y = \mathbb{P}$ then $\mathbb{P}$ is a dense subset of $\mathbb{P}$.*

{z46} **Definition 3.3.** Let $\mathbf{m} \in \mathbf{M}$.
1) For $t \in L_{\mathbf{m}}, \varepsilon < \lambda$ and $\eta \in \prod_{i<\varepsilon} \theta_i$ let $p = p^*_{t,\eta} \in \mathbb{P}_{\mathbf{m}}$ be the function with domain $\{t\}$ such that $p(t) = (\eta, \eta\hat{\ } 0_\lambda)$, i.e. $f_{p(t)} \in \prod_{i<\lambda} \theta_i$ is defined by $f_{p(t)}(\varepsilon)$ is $\eta(\varepsilon)$ if $\varepsilon < \ell g(\eta)$ and is zero otherwise.
2) For $L \subseteq L_{\mathbf{m}}$ let $Y_L = Y_{\mathbf{m},L} = \{p^*_{t,\eta} : t \in L$ and $\eta \in \prod_{\varepsilon<\zeta} \theta_\varepsilon$ for some $\zeta < \lambda\}$.
{z19} 3) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}[L]$ be $\mathbb{L}_{\lambda^+}[Y_L, \mathbb{P}_{\mathbf{m}}]$, see Definition 3.1.
{c7} 4) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$, see Definition 2.10(1),
{c7} recalling 2.10(2),(3).
5) $\mathbb{P}'_{\mathbf{m}}$ is the partial order with the same set of elements as $\mathbb{P}_{\mathbf{m}}$ and $\le_{\mathbb{P}'_{\mathbf{m}}} = \{(p,q) : p,q \in \mathbb{P}_{\mathbf{m}}$ and no r above q is incompatible with $p\}$ and $\mathbb{P}'_{\mathbf{m}}(L) = \mathbb{P}'_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$, we may "forget" the distinction[16].
6) For quasi orders $\mathbb{Q}_1, \mathbb{Q}_2$ let $\mathbb{Q}_1 \subseteq' \mathbb{Q}_2$ mean that:

(a) $s \in \mathbb{Q}_1 \Rightarrow s \in \mathbb{Q}_2$
(b) $s \le_{\mathbb{Q}_1} t \Rightarrow s \le_{\mathbb{Q}_2} t$.

7) For quasi orders $\mathbb{Q}_1, \mathbb{Q}_2$ let $\mathbb{Q}_1 \subseteq'_{\text{ic}} \mathbb{Q}_2$ means that $\mathbb{Q}_1 \subseteq' \mathbb{Q}_2$ and

(c) if $s,t \in \mathbb{Q}_1$ are incompatible in $\mathbb{Q}_1$ then they are incompatible in $\mathbb{Q}_2$.

{z46} [15]Really $\mathbb{P} \lessdot' \mathbb{L}^+_\chi[\mathbb{P}]$ see 3.3, because $\mathbb{L}^+_\chi[\mathbb{P}] \models$ "$p \le q$" <u>iff</u> $q \Vdash_{\mathbb{P}}$ "$p \in \mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{P}}$".
{c32n} [16]Really the only difference is the possibility that $\text{dom}(p) \nsubseteq \text{dom}(q)$, see 2.19.

8) We define $\lessdot'$ similarly.

{z48}

Claim 3.4. *Let $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$.*
1) $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is equivalent to $\mathbb{P}_{\mathbf{m}}$ as forcing notions, in fact, $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{m}}(L_m) \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ and is a dense subset of it under the natural identification (see 3.1(1)), but we should pedantically use $\mathbb{P}'_{\mathbf{m}}(L_{\mathbf{m}})$ or use $\lessdot'$. {z19}
2) $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is $(<\lambda)$-strategically complete and is λ^+-c.c.
3) $\mathbb{P}_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}[L]$ as sets and $\mathbb{P}_{\mathbf{m}}[L] \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ and $\mathbb{P}_{\mathbf{m}}(L) \subseteq' \mathbb{P}_{\mathbf{m}}[L]$.
4) If $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}$ is generic over $\mathbf{V}$ and $\eta_t = \eta_t[\mathbf{G}]$ for $t \in L$ and $\mathbf{G}^+_L = \{\psi \in \mathbb{L}_{\lambda^+}(Y_L) : \psi[\mathbf{G}] = \text{true}\}$, see 3.1(3), <u>then</u> $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\mathbf{G}^+] = \mathbf{V}[\langle \eta_t : t \in L_{\mathbf{m}}\rangle]$. {z19}
5) In part (4), moreover $\mathbf{G}^+$ is a subset of $\mathbb{P}_{\mathbf{m}}[L]$ generic over $\mathbf{V}$.
6) $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_2)$ and $\mathbb{P}_{\mathbf{m}}[L_1] \lessdot \mathbb{P}_{\mathbf{m}}[L_2]$ <u>when</u> $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$.
7) If $\mathbf{m}, \mathbf{n} \in \mathbf{M}$ are equivalent <u>then</u> $\mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$ and $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$.
8) Assume I_* be a λ^+-directed partial order and $\bar{L} = \langle L_r : r \in I_*\rangle$ be such that $r \in I_* \Rightarrow L_r \subseteq L_{\mathbf{m}}$ and $r <_{I_*} s \Rightarrow L_r \subseteq L_s$ and $L = \cup\{L_r : r \in I_*\}$. <u>Then</u> $\mathbb{P}_{\mathbf{m}}[L] = \cup\{\mathbb{P}_{\mathbf{m}}[L_r] : r \in I_*\}$ and $\mathbb{P}_{\mathbf{m}}(L) = \cup\{\mathbb{P}_{\mathbf{m}}(L_r) : r \in I_*\}$.
9) If $\mathbf{m} \in \mathbf{M}_{\rm ec}$ and $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2$ then $\mathbb{P}_{\mathbf{m}_1}[L_{\mathbf{m}}] \lessdot \mathbb{P}_{\mathbf{m}_2}[L_{\mathbf{m}}]$.

{z50}

Remark 3.5. What about $\mathbb{P}_{\mathbf{m}}(L) \subseteq'_{\rm ic} \mathbb{P}_{\mathbf{m}}[L]$ and "$\mathbb{P}_{\mathbf{m}}(L) \lessdot' \mathbb{P}_{\mathbf{m}}[L]$?

The problem is the mapping $p \mapsto p \upharpoonleft L$ defined in 4.1(3) does not have the required properties of preserving order as the forcing appears there. {e4}

Proof. 1) Easy.
2) Follows by part (1) and 2.13. {c11}
3) The first statement by their definitions, the second statement by part (1).
4), 5), 6) Should be clear recalling 2.13(9). {c11}
7) Easy, recalling 2.13(7). {c11}
8),9) Easy. $\square_{3.4}$

{c43}

The Uniqueness Claim 3.6. *There is an isomorphism from $\mathbb{P}_{\mathbf{m}_1}[M_1]$ onto $\mathbb{P}_{\mathbf{m}_2}[M_2]$ which (recalling Definition 3.3(1)) maps $p^*_{t,\eta}$ to $p^*_{h(t),\eta}$ for $t \in M_1, \eta \in \cup\{\prod\limits_{\varepsilon<\zeta} \theta_\varepsilon : \zeta < \lambda\}$ <u>when</u>:* {z46}

$\boxplus$ (a) $\mathbf{m}_\ell \in \mathbf{M}^{\rm ec}_\infty$ *for* $\ell = 1,2$
(b) $M_\ell = M_{\mathbf{m}_\ell}$ *for* $\ell = 1,2$
(c) *h is an isomorphism from $\mathbf{m}_1 \restriction M_1$ onto $\mathbf{m}_2 \restriction M_2$.*

Proof. By renaming without loss of generality $M_1 = M_2$ call it M and h is the identity and $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$. Let $\mathbf{m}_0 = \mathbf{m}_1 \restriction M = \mathbf{m}_2 \restriction M$ so $\mathbf{m}_0 \le_{\mathbf{M}} \mathbf{m}_\ell$ for $\ell = 1,2$ and $L_{\mathbf{m}_0} = L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2}$.

By 2.18, there is $\mathbf{m}$ such that $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}$ and $\mathbf{m}_2 \le_{\mathbf{M}} \mathbf{m}$. As $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}^{\rm ec}_\infty$ {c31}
by 3.4(9) we have $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}}[M]$ and $\mathbb{P}_{\mathbf{m}_2}[M] = \mathbb{P}_{\mathbf{m}}[M]$ so together we get the desired conclusion. $\square_{3.6}$ {z48}

{c44}

Definition 3.7. 1) We call $\mathbf{m} \in \mathbf{M}$ reduced when $L_{\mathbf{m}} = M_{\mathbf{m}}$.
2) For $\mathbf{m} \in \mathbf{M}$ let $\mathbb{P}^{\rm cr}_{\mathbf{m}}$ be $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{m}}]$ and $\mathbb{P}^{\rm cr}_{\mathbf{m}}[L]$ be $\mathbb{P}_{\mathbf{n}}[L]$ for $L \subseteq L_{\mathbf{m}}$ <u>when</u> $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_{\rm ec}$.

{c47}

Remark 3.8. 1) Why is $\mathbb{P}^{\rm cr}_{\mathbf{m}}[L]$ well defined? see below.
2) Here "cr" stands for corrected.

The interest in the definition is because

{c48}

Claim 3.9. *1) If $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$ <u>then</u> $\mathbb{P}^{\rm cr}_{\mathbf{m}}[L]$ is well defined.*
2) $\mathbb{P}^{\rm cr}_{\mathbf{m}}[M_{\mathbf{m}}]$ is well defined and depend only on $\mathbf{m}{\restriction}M_{\mathbf{m}}$.
3) If $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ and $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$ <u>then</u> $\mathbb{P}^{\rm cr}_{\mathbf{m}}[L_1] = \mathbb{P}^{\rm cr}_{\mathbf{n}}[L_1] \lessdot \mathbb{P}^{\rm cr}_{\mathbf{n}}[L_2] \lessdot \mathbb{P}^{\rm cr}_{\mathbf{n}}$.

{c41} *Proof.* 1) By 2.26, $\mathbb{P}^{\rm cr}_{\mathbf{m}}[L]$ has at least one definition so it suffices to prove uniqueness. So assume $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_\ell \in \mathbf{M}_{\rm ec}$ for $\ell = 1,2$ and we should prove that $\mathbb{P}_{\mathbf{m}_1}[L] = \mathbb{P}_{\mathbf{m}_2}[L]$.
{c31} Without loss of generality $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = L_{\mathbf{m}}$. Now by 2.18 we can find $\mathbf{n} \in \mathbf{M}$ such
{c34} that $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{n}$ and $\mathbf{m}_2 \le_{\mathbf{M}} \mathbf{n}$; as $\mathbf{m}_\ell \in \mathbf{M}_{\rm ec}$ see Definition 2.24 we have $\mathbb{P}_{\mathbf{m}_\ell} \lessdot \mathbb{P}_{\mathbf{n}}$
{c43} for $\ell = 1,2$. As in the end of the proof of 3.6 we are done.
{c43} 2) By 3.6.
{c34}{c44} 3) Follows from Definition 2.24(2) and 3.7(2). $\square_{3.9}$

{c50}

Discussion 3.10. 1) But we like to prove for reduced $\mathbf{m} \in \mathbf{M}$ and $M \subseteq \mathbf{M}_{\mathbf{m}}$ that
{c80} $\mathbb{P}^{\rm cr}_{\mathbf{m}{\restriction}M} \lessdot \mathbb{P}^{\rm cr}_{\mathbf{m}}$. This is delayed to 4.26. We now prove it suffices.
{c51} 2) The reader may understand 3.11 without reading the rest of §2,§3,§4 by ignoring
{z14}{z44} clause (A)(d), <u>or</u> reading 0.3, 3.2.

{c51}

Conclusion 3.11. *For every ordinal δ_* there is $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\eta}_\alpha : \alpha \le \delta_* \rangle$ such that:*

(A) (a) *$\langle \mathbb{P}_\alpha : \alpha \le \delta_* \rangle$ is $\lessdot$-increasing*
(b) *$\underset{\sim}{\eta}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name of a member of $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ which dominates $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_\alpha]}$*
(c) *$\underset{\sim}{\eta}_\alpha$ is a generic for $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$, moreover $\langle \underset{\sim}{\eta}_\beta : \beta < \alpha \rangle$ is a generic for $\mathbb{P}_\alpha$*
{z46} (d) *$p \in \mathbb{P}_\alpha$ iff $p \in \mathbb{L}_{\lambda^+}(Y_\alpha, \mathbb{P}_\alpha)$ where Y_α is defined as in 3.3(2) with α*
{z19} *here standing for L there and see 3.1*
(e) *$\mathbb{P}_\alpha$ is $(<\lambda)$-strategically complete and λ^+-c.c.*
(f) *if $\delta \le \delta_*$ has cofinality $> \lambda$ (actually $\ge \lambda$ suffice) <u>then</u> $\mathbb{P}_\delta = \cup\{\mathbb{P}_\alpha : \alpha < \delta\}$*
(g) *$\mathbb{P}_{\delta_*}$ has cardinality $|\delta_*|^\lambda$*

(B) *if $\mathscr{U} \subseteq \delta_*$ <u>then</u> the complete subforcing generated by $\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U} \rangle$ is isomorphic to $\mathbb{P}_{{\rm otp}(\mathscr{U})}$*

(C) *if $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$ is generic over $\mathbf{V}$ and $\eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ for $\alpha < \delta_*$ and $\eta'_\alpha \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ for $\alpha < \delta_*$ and $\{(\alpha,\varepsilon) : \alpha < \delta_*, \varepsilon < \lambda$ and $\eta'_\alpha(\varepsilon) \ne \eta_\alpha(\varepsilon)\}$ has cardinality $< \lambda$ <u>then</u> also $\langle \eta'_\alpha : \alpha < \delta_* \rangle$ is a generic for $\mathbb{P}_{\delta_*}$, determining a different $\mathbf{G}'$ but $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\mathbf{G}]$*

(D) *in clause (C), moreover if $\mathscr{U} \subseteq \delta$ and $\langle \alpha_i : i < {\rm otp}(\mathscr{U}) \rangle$ list $\mathscr{U}$ in increasing order <u>then</u> for some unique $\mathbf{G}'' \subseteq \mathbb{P}_{{\rm otp}(\mathscr{U})}$ generic over $\mathbf{V}, i < {\rm otp}(\mathscr{U}) \Rightarrow \eta'_{\alpha_i} = \underset{\sim}{\eta}_i[\mathbf{G}'']$.*

Proof. Without loss of generality $\lambda_1 \ge |\delta_*|$.
We define $\mathbf{m} \in \mathbf{M}$ by:

$(*)$ (a) $L_{\mathbf{m}} = \delta_*$
(b) $M_{\mathbf{m}} = \delta_*$
(c) $u_{\mathbf{m},\alpha} = \alpha$ and $\mathscr{P}_{\mathbf{m},\alpha} = [\alpha]^{\le\lambda}$ for $\alpha < \delta_*$
(d) $E'_{\mathbf{m}} = \emptyset$.

It is easy to check that indeed $\mathbf{m} \in \mathbf{M}$ and let $\mathbf{n} \in \mathbf{M}_{\rm ec}$ be such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$, exists by the Crucial Claim 2.26 and let $\mathbb{P}_\alpha = \mathbb{P}_{\mathbf{n}}[\{i : i < \alpha\}]$ for $\alpha \le \delta_*$.

{c41} {c48} {c11} {z46} Now clearly clauses (A),(C) hold and $\mathbb{P}_\delta = \mathbb{P}^{\rm cr}_{\mathbf{m}}$ by 3.7(2), 3.9(1) and clause (A)(b) holds by 2.13(6A). As for clause (B), note that for every $L \subseteq \delta_*$, for $\mathbb{P}_{\mathbf{m}}[L]$ the sequence $\bar{\eta}_L = \langle \underset{\sim}{\eta}_\alpha : \alpha \in L\rangle$ is generic for $\mathbb{P}_{\mathbf{m}}[L]$ by Definition 3.3.

{c80} For $M \subseteq \delta_*$ let $\alpha = {\rm otp}(M)$ and $h : M \to \alpha$ be $h(i) = {\rm otp}(i \cap M)$ so h is an isomorphism from $\mathbf{m}{\restriction}M$ onto $\mathbf{m}{\restriction}\alpha$ hence by 4.26(2), with $\mathbf{m}, \mathbf{m}{\restriction}\alpha, M, \alpha$ here standing for $\mathbf{m}_1, \mathbf{m}_2, M_1, M_2$ there we have h induces an isomorphism from $\mathbb{P}^{\rm cr}_{\mathbf{m}}[M]$ onto $\mathbb{P}^{\rm cr}_{\mathbf{m}{\restriction}\alpha}[L_{\mathbf{m}{\restriction}\alpha}]$. Similarly, ${\rm id}_\alpha$ induces an isomorphism from $\mathbb{P}^{\rm cr}_{\mathbf{m}{\restriction}\alpha}$ onto $\mathbb{P}^{\rm cr}_{\mathbf{m}}[\alpha]$.

Together we get clause (B). Also Clause (D) follows so we are done. $\square_{3.11}$

Similarly we can deal with such iterations with partial memory and spell out how $\mathbb{P}^{\rm cr}_{\mathbf{m}}[L]$ is defined from a $(< \lambda)$-support iteration with partial memory.

{c52}

Conclusion 3.12. *Assume M is a well founded partial order and $\bar{u}' = \langle u'_t : t \in M\rangle, u_t \subseteq M_{<t}$ and $\bar{\mathscr{P}}' = \langle \mathscr{P}'_t : t \in M\rangle$ with $\mathscr{P}'_t \subseteq [u'_t]^{\le\lambda}$ is closed under subsets. <u>Then</u> we can find $\beta(*), h, \mathbb{P}_\beta = \mathbb{P}_{0,\beta}, \mathbb{P}_{1,\beta}, \underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{\eta}_\alpha, \underset{\sim}{\eta}'_s$ and $\mathbb{P}'_u$ (for $\beta \le \beta(*), \alpha < \beta(*), s \in M$ and $u \subseteq M$) and $\bar{u}, \bar{\mathscr{P}}$ such that:*

- (A) (a) $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha : \beta \le \beta(*), \alpha < \beta(*)\rangle$ *is $(< \lambda)$-support iteration*
  - (b) (α) $\bar{u} = \langle u_\beta : \beta < \beta(*)\rangle$ *such that* $u_\beta \subseteq \beta$
    - (β) $\bar{\mathscr{P}} = \langle \mathscr{P}_\beta : \beta < \beta(*)\rangle$ *such that* $\mathscr{P}_\beta \subseteq [u_\beta]^{\le\lambda}$
  - (c) $\underset{\sim}{\eta}_\alpha$ *is a $\mathbb{P}_{\alpha+1}$-name of a member of* $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$
  - (d) $\langle \underset{\sim}{\eta}_\alpha : \alpha < \beta\rangle$ *is generic for* $\mathbb{P}_\beta$
  - (e) $\underset{\sim}{\mathbb{Q}}_\alpha$ *is defined as in Definition 2.10(4)* {c7}
  - (f) $\Vdash_{\mathbb{P}_{\beta(*)}}$ "$\underset{\sim}{\eta}_\beta \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ *dominate every* $\nu \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ *from* $\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u\rangle]$ *when* $u \in \mathscr{P}_\beta$
- (B) (a) *h is a one-to-one function from M into*[17] $\beta(*)$*; stipulate* $h(\infty) = \beta(*)$
  - (b) $s <_M t \Leftrightarrow h(s) < h(t)$
  - (c) $u_{h(t)} \cap {\rm Rang}(h) = \{h(s) : s \in u'_t\}$
  - (d) $\mathscr{P}_{h(t)} \cap [{\rm Rang}(h)]^{\le\lambda} = \{\{h(s) : s \in u\} : u \in \mathscr{P}'_t\}$
- (C) (a) $\mathbb{P}_{1,\beta} = \mathbb{L}^+_{\lambda^+}(Y_\beta, \mathbb{P}_\beta)$ *where we let* $Y_\beta = \{p^*_{\alpha,\nu} : \alpha < \beta, \nu \in \prod\limits_{\varepsilon<\zeta} \theta_\varepsilon$ *for some* $\zeta < \lambda\}$*, see 3.1, 3.3(1)* {z40}
  - (b) $\mathbb{P}_{1,u} = \mathbb{L}^+_{\lambda^+}(Y_u, \mathbb{P}_\beta)$*, where Y_u is defined similarly when* $u \subseteq \beta(*)$
  - (c) $\mathbb{P}'_u$ *is a forcing notion for $u \subseteq M$ and $\underset{\sim}{\eta}'_s$ is a $\mathbb{P}'_{\{s\}}$-name for $s \in M$ sn*
  - (d) *h induces an isomorphism from $\mathbb{P}'_u$ onto $\mathbb{P}_{1,\{h(s):s\in u\}}$ for $u \subseteq M$ and $\underset{\sim}{\eta}'_s$ to $\underset{\sim}{\eta}_{h(s)}$ for* $s \in M$
  - (e) $\langle \underset{\sim}{\eta}_{h(s)} : s \in u\rangle$ *is generic for $\mathbb{P}'_u$ for* $u \subseteq M$
- (D) (a) $\mathbb{P}'_u \lessdot \mathbb{P}'_v$ *when* $u \subseteq v \subseteq M$
  - (b) $\mathbb{P}_\beta, \mathbb{P}_{1,u}, \mathbb{P}'_u$ *are $(< \lambda)$-strategically complete and λ^+-c.c.*

[17]In general not onto!

(c) *if* $M_1, M_2 \subseteq M$ *and* f *is an isomorphism from* M_1 *onto* M_2 *as partial orders such that* $t \in M_1 \Rightarrow u'_{h(t)} \cap M_2 = \{f(s) : s \in u'_t \cap M_1\}$ *and* $t \in M_1 \Rightarrow \mathscr{P}'_{h(t)} \cap [M_2]^{\leq \lambda} = \{\{f(s) : s \in u \cap M_1\} : u \in \mathscr{P}'_t\}$ *then the mapping* $h(s) \mapsto h(f(s))$ *induce an isomorphism from the forcing notion* $\mathbb{P}'_{M_1}$ *onto* $\mathbb{P}'_{M_2}$.

Proof. Similarly. $\square_{3.12}$

§ 4. The main conclusion

We have a debt from §3, i.e. see discussion 3.10. Toward this we explicate what appear in the proof of 2.26. {c50} {c41} {e4}

Definition 4.1. Let $\mathbf{m} \in \mathbf{M}$.
1) We say $\mathbf{m}$ is μ-wide <u>when</u> for every $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ there are $t_\alpha \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ for $\alpha < \mu$ such that:

(a) $\mathbf{m} \restriction (t_\alpha / E_{\mathbf{m}})$ is isomorphic to $\mathbf{m} \restriction (t/E_{\mathbf{m}})$ over $M_{\mathbf{m}}$
(b) $\beta < \gamma < \mu \Rightarrow t_\beta / E''_{\mathbf{m}} \neq t_\gamma / E''_{\mathbf{m}}$.

1A) We say $\mathbf{m}$ is wide <u>when</u> it is λ^+-wide. We say $\mathbf{m}$ is very wide when it is $|L_{\mathbf{m}}|$-wide.
2) We say $\mathbf{m}$ is full <u>when</u>: if $\mathbf{m} \restriction M_{\mathbf{m}} \leq_{\mathbf{M}} \mathbf{n}$ and $E''_{\mathbf{n}}$ has exactly one equivalence class <u>then</u> for some $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$, we have $\mathbf{n}$ is isomorphic to $\mathbf{m} \restriction (t/E_{\mathbf{m}})$ over $M_{\mathbf{m}}$.
3) For $L \subseteq L_{\mathbf{m}}$ we say $p \in \mathbb{P}_{\mathbf{m}}(L)$ is the projection (to L) of $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ and write $p = q \upharpoonright L$ <u>when</u>:

(a) $\mathrm{Dom}(p) = \mathrm{Dom}(q) \cap L$
(b) if $s \in \mathrm{Dom}(p)$ then
 (α) $\mathrm{tr}(p(s)) = \mathrm{tr}(q(s))$
 (β) $\{\underset{\sim}{f}_{p(s),\iota} : \iota < \iota(p(s))\} = \{\underset{\sim}{f}_{q(s),\iota} : \iota < \iota(q(s))$ and $\bar{r}_{p(s),\iota}$ is a sequence of members of $L\}$.

4) Let $\mathscr{F}_{\mathbf{m}}$ be the set of the functions f such that for some L_1, L_2:

(a) f is an isomorphism from $\mathbf{m} \restriction L_1$ onto $\mathbf{m} \restriction L_2$
(b) L_ℓ is a subset of $L_{\mathbf{m}}$ for $\ell = 1, 2$
(c) $M_{\mathbf{m}} \subseteq L_\ell$ for $\ell = 1, 2$ and $f \restriction M_{\mathbf{m}}$ is the identity
(d) L_ℓ is $E_{\mathbf{m}}$-closed, i.e. $M_{\mathbf{m}} \subseteq L_\ell$ and if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ and $t \in L_\ell$ then $t/E_{\mathbf{m}} \subseteq L_\ell$ for $\ell = 1, 2$
(e) $\{t/E''_{\mathbf{m}} : t \in L_\ell \backslash M_{\mathbf{m}}\}$ has cardinality $\leq \lambda$.

5) If $L_1, L_2 \subseteq L_{\mathbf{m}}$ and f is an isomorphism from $\mathbf{m} \restriction L_1$ onto $\mathbf{m} \restriction L_2$ <u>then</u> we let $\hat{f}$ be the one-to-one mapping[18] from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$ as in $(*)_4(b)$ of the proof of 2.26. {c41}
6) Let $\mathbb{P}^-_{\mathbf{m}}(L)$ be $\{p \in \mathbb{P}_{\mathbf{m}}(L) : \mathrm{fsupp}(p) \subseteq M_{\mathbf{m}}$ or $\mathrm{fsupp}(p) \subseteq t/E_{\mathbf{m}}$ for some $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}\}$ with the order inherited from $\mathbb{P}_{\mathbf{m}}$. {e5n}

Observation 4.2. *Let $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$.*
1) The projection of $q \in \mathbb{P}_{\mathbf{m}}$ to L is well defined and $\in \mathbb{P}_{\mathbf{m}}(L)$.
2) Moreover, it is unique.
3) If $p \in \mathbb{P}_{\mathbf{m}}(L)$ is the projection of $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ <u>then</u> $p \leq q$.
4) For every $p \in \mathbb{P}_{\mathbf{m}}, p$ is equivalent to $\mathscr{S}_p := \{p \upharpoonright L : L = t/E_{\mathbf{m}}$ for some $t \in \mathrm{fsupp}(p)\} \cup \{p \upharpoonright M_{\mathbf{m}}$: if $\mathrm{fsupp}(p) \subseteq M_{\mathbf{m}}\}$, i.e. $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}$ <u>iff</u> $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}$".
5) For every $p \in \mathbb{P}_{\mathbf{m}}, p$ is equivalent to $\mathscr{S}'_p := \{p^{[t,\iota]} : t \in \mathrm{dom}(p)$ and $\iota < \iota(p(t))\}$ where $p^{[t,\iota]} \in \mathbb{P}_{\mathbf{m}}$ has domain $\{t\}$ and $p(t) = (\mathrm{tr}(p_t), \mathbf{B}_{p(t),\iota}(\langle \eta_{r_{p(t)}(\zeta)} : \zeta \in w_{p(t),\iota}\rangle)$; recall Definition 2.9 for the meaning of $\iota(p(t)), \mathbf{B}_{p(t),\iota}$, etc. {c6}

[18] We have not said "order preserving"!

Remark 4.3. 1) Note that the choice in Definition 2.9$(c)(\gamma)$ to require such $\langle \underset{\sim}{f}_{p(t),\iota} :$ {e5p} {c6}
$\iota < \iota(p_t)\rangle$ exists, is necessary for 4.2(4), which is crucial in the proof of 4.26. {e80}
2) In Definition 4.1(1A) we can choose "wide means λ-wide" as when applying it, {e4}
if $X = \mathrm{fsupp}(p)$ then for some $Y \subseteq L_{\mathbf{m}}$ of cardinality $< \lambda, X \subseteq \cup\{t/E_{\mathbf{m}} : t \in Y\}$.

Proof. Easy, e.g.
4) If $\mathrm{fsupp}(p) \subseteq M_{\mathbf{m}}$ the statement says $\Vdash$ "$p \in \underset{\sim}{\mathbf{G}}$ iff $\{p\} \subseteq \underset{\sim}{\mathbf{G}}$", so trivial hence we assume $\mathrm{fsupp}(p) \nsubseteq M_{\mathbf{m}}$. Now if $t \in \mathrm{fsupp}(p)$ then trivially $p \restriction (t/E_{\mathbf{m}}) \le p$, hence $\Vdash$ "$p \in \underset{\sim}{\mathbf{G}}$ implies $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}}$".

For the other direction assume $q \in \mathbb{P}_{\mathbf{m}}$ forces $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}} \subseteq \mathbb{P}_{\mathbf{m}}$ and we shall prove that q is compatible with p, this suffices, so toward contradiction assume q, p are incompatible.

Without loss of generality $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$ and recalling $t \in \mathrm{fsupp}(p) \Rightarrow q \Vdash$ "$p \restriction (t/E_{\mathbf{m}}) \in \underset{\sim}{\mathbf{G}}$" clearly $s \in \mathrm{dom}(p) \Rightarrow q \Vdash$ "$\mathrm{tr}(p(s)) \subseteq \underset{\sim}{\eta}_s$" so necessarily
{c11} $s \in \mathrm{Dom}(p) \Rightarrow \mathrm{tr}(p(s)) \subseteq \mathrm{tr}(q(s))$. Recalling 2.13(8), as p, q are incompatible there are $s \in \mathrm{Dom}(p) \cap \mathrm{Dom}(q)$ and q_1 such that $q \restriction L_{\mathbf{m},<s} \le q_1 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$ and $q_1 \Vdash$ "$q(s), p(s)$ are incompatible in $\mathbb{Q}_{\bar{\theta}}$".

As $\mathrm{tr}(p(s)) \subseteq \mathrm{tr}(q(s))$ this implies $q_1 \Vdash$ "$\mathrm{tr}(q(s)), p(s)$ are incompatible, i.e.
{c6} $f_{p(s)} \restriction \ell g(\mathrm{tr}(q(s))) \nleq \mathrm{tr}(q(s))$". Recalling Definition 2.9$(c)(\gamma)$, $q_1 \Vdash$ "there is $\iota < \iota(s,p)$ such that $f_{p(s),\iota}, \mathrm{tr}(q(s))$ are incompatible". Possibly increasing q_1, we can fix ι. But letting $t \in \mathrm{fsupp}(p) \subseteq L_{\mathbf{m}}$ be such that $\bar{r}_{p(s),\iota} \subseteq t/E_{\mathbf{m}}$ this implies that $q_1 \Vdash$ "$p \restriction (t/E_{\mathbf{m}}) \notin \underset{\sim}{\mathbf{G}}$ or $\mathrm{tr}(q(s)) \nsubseteq \underset{\sim}{\eta}_s$". However, q_1, q are compatible and this contradicts the choice of q. $\square_{4.2}$

{e7}
{c41} **Claim 4.4.** *1) The* $\mathbf{n}$ *constructed in 2.26 satisfies: if* $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1$ *<u>then</u>* $\mathbf{n}_1$ *is wide, (if* $\mathbf{n}_1 \in \mathbf{M}_\chi$ *even very wide) and full.*
2) If $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$ *and* $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1$ *then* $\mathbf{n}_1 \in \mathbf{M}_{\mathrm{ec}}$.

{c41} *Proof.* 1) Holds by the proof of 2.26.
{c34} {e10} 2) Holds by Definition 2.24(1),(2). $\square_{4.4}$

Claim 4.5. *Assume* $\mathbf{m}$ *is wide.*
1) If $f \in \mathscr{F}_{\mathbf{m}}$ *and* $X \subseteq L_{\mathbf{m}}$ *has cardinality* $\le \lambda$ *<u>then</u> there is* g *such that:*

- (a) $g \in \mathscr{F}_{\mathbf{m}}$
- (b) $f \subseteq g$
- (c) $\mathrm{Dom}(g) = \mathrm{Rang}(g)$
- (d) $X \subseteq \mathrm{Dom}(g)$.

2) If $g \in \mathscr{F}_{\mathbf{m}}$ *and* $\mathrm{Dom}(g) = \mathrm{Rang}(g)$ *<u>then</u>* $g^{+\mathbf{m}} = g \cup \mathrm{id}_{L_{\mathbf{m}} \setminus \mathrm{Dom}(g)}$ *is an automorphism of* $\mathbf{m}$.
3) If f *is an automorphism of* $\mathbf{m}$ *<u>then</u> it naturally induces an automorphism* $\hat{f}$ *of*
{c41} $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ *similarly to* $\hat{f}$ *from* $(*)_4(b)$ *of the proof of 2.26.*
4) If $f \in \mathscr{F}_{\mathbf{m}}$ *<u>then</u> it induces an isomorphism* $\hat{f}$ *from* $\mathbb{P}_{\mathbf{m}}(\mathrm{Dom}(f))$ *onto* $\mathbb{P}_{\mathbf{m}}(\mathrm{Rang}(f))$.

{e4} *Proof.* 1) Easy by the definition of wide in 4.1(1) and of $\mathscr{F}_{\mathbf{m}}$ in 4.1(4).
2) Just read the definition of $\mathbf{m} \in \mathbf{M}$ and of $f \in \mathscr{F}_{\mathbf{m}}$, in particular:

- (a) if $t_1, t_2 \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ are not $E'_{\mathbf{m}}$-equivalent then $(t_1/E_{\mathbf{m}}) \cap (t_2/E_{\mathbf{m}}) = M_{\mathbf{m}}$ and $\le_{\mathbf{m}} \restriction (t_1/E_{\mathbf{m}} \cup t_2/E_{\mathbf{m}})$ is determined by $\le_{\mathbf{m}} \restriction (t_1/E_{\mathbf{m}}), \le_{\mathbf{m}} \restriction (t_2/E_{\mathbf{m}})$
- (b) $g \restriction M_{\mathbf{m}} = \mathrm{id}_{M_{\mathbf{m}}}$.

3) Naturally by the definition.
4) Let $g \in \mathscr{F}$ be as in part (1) and let $h = g^{+\mathbf{m}}$ so an automorphism of $\mathbf{m}$ which extends g as in part (2). So $\hat{h}$ is an automorphism of $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ and clearly $\hat{f} = \hat{h} \restriction \mathbb{P}_{\mathbf{m}}(\mathrm{Dom}(f))$ is as required. $\square_{4.5}$

{e16}

Claim 4.6. *Let $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$.*
If $f_1, f_2 \in \mathscr{F}_{\mathbf{m}}$ <u>then</u>:

(a) $f_1 \subseteq f_2 \Rightarrow \hat{f}_1 \subseteq \hat{f}_2$
(b) $f_1 = f_2^{-1} \Rightarrow \hat{f}_1 = (\hat{f}_2)^{-1}$.

Proof. Just consider the definition, see 4.1(5) and $(*)_4(b)$ of the proof of 2.26. $\square_{4.6}$ {e4}}

{e19}

Observation 4.7. *1) $\mathbb{P}^-_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}(L)$, see Definition 4.1(6).* {e4}
2) For every $p \in \mathbb{P}_{\mathbf{m}}$ there is a sequence $\langle p_i : i < i(*)\rangle$ of $\leq \lambda$ members of $\mathbb{P}^-_{\mathbf{m}}$ such that $\Vdash_{\mathbb{P}_{\mathbf{m}[L_{\mathbf{m}}]}}$ "$p \in \underset{\sim}{\mathbf{G}}$ iff $\{p_i : i < i(*)\} \subseteq \underset{\sim}{\mathbf{G}}$".

Proof. 1) By their definitions.
2) Should be clear, see Definition 4.1(6) and 4.2(4). $\square_{4.7}$ {e5h}

{e21}

Remark 4.8. 1) Observation 4.7 is not used. {e19}
2) Probably we can avoid using "wide" and prove the density of $\mathbf{M}_{\mathrm{ec}}$ with smaller cardinality but the present way seems more transparent.

{e24}

Definition 4.9. Assume $\mathbf{m} \in \mathbf{M}$.
1) Let $\mathscr{Y}_{\mathbf{m}}$ be the set of pairs $(t, \bar{s})$ such that $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ and $\bar{s} \in {}^{\zeta}(t/E''_{\mathbf{m}})$ for some $\zeta < \lambda^+$; we may write $\bar{s}$ instead of $(t, \bar{s})$ as usually $\bar{s}$ determines t.
2) By induction on the ordinal γ we define when $(t_1, \bar{s}_1), (t_2, \bar{s}_2)$ are γ-equivalent in $\mathbf{m}$ or are $(\mathbf{m}, \gamma)$-equivalent:

- (a) if $\gamma = 0$, letting $L_\ell = (M_{\mathbf{m}} \cup \mathrm{Rang}(\bar{s}_\ell))$ for $\ell = 1, 2$ there is h such that
  - (α) h is an isomorphism from $\mathbf{m} \restriction L_1$ onto $\mathbf{m} \restriction L_2$
  - (β) h maps $\bar{s}_1$ to $\bar{s}_2$
  - (γ) $h \restriction M_{\mathbf{m}}$ is the identity
  - (δ) h induces an isomorphism from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$ (as defined in 2.7$(*)_4(b)$) {c4}
  - (ε) moreover, h induces an isomorphism from $\mathbb{P}_{\mathbf{m}}[L_1]$ onto $\mathbb{P}_{\mathbf{m}}[L_2]$, as defined in 3.6, $p^*_{t,\eta} \mapsto p^*_{h(t),\eta}$, see 3.3(3)
- (b) if $\gamma = \beta + 1$ <u>then</u> for every $\varepsilon < \lambda^+$ and $\ell \in \{1, 2\}$ and $\bar{s}'_\ell \in {}^{\varepsilon}(t_\ell/E''_{\mathbf{m}})$ there is $\bar{s}'_{3-\ell} \in {}^{\varepsilon}(t_{3-\ell}/E''_{\mathbf{m}})$ such that $(t_1, \bar{s}_1{}^\frown\bar{s}'_1), (t_2, \bar{s}_2{}^\frown\bar{s}'_2)$ are β-equivalent {z48}
- (c) if γ is a limit ordinal <u>then</u> $(t_1, \bar{s}_1), (t_2, \bar{s}_2)$ are β-equivalent for every $\beta < \gamma$.

{e26}

Remark 4.10. 1) Note above that if $\bar{s}_\ell$ is the empty sequence <u>then</u> t_ℓ would not be determined by $\bar{s}_\ell$, still in those cases the equivalence just means $\bar{s}_1 = \bar{s}_2$.
2) We can use $t/E_{\mathbf{m}}$ or $t/E'_{\mathbf{m}}$ instead of $t/E''_{\mathbf{m}}$ as everything is over $M_{\mathbf{m}}$.

{e27}

Claim 4.11. *For $\mathbf{m} \in \mathbf{M}$ and ordinal α the number of equivalence classes of "being $(\mathbf{m}, \alpha)$-equivalent" is $\leq \beth_{1+\alpha+1}(\lambda_1)$.*

Proof. By induction on α.

Case 1: $\alpha = 0$

Note that the set of elements of $\mathbb{P}_{\mathbf{m}}(M_{\mathbf{m}} \cup \text{Rang}(\bar{s}))$ has cardinality $\le 2^{\lambda_1}$ (and even $\le (\lambda_1)^\lambda$) and depends just on $\mathbf{m}{\restriction}(M_{\mathbf{m}} \cup \text{Rang}(\bar{s}))$ but there are $\beth_2(\lambda_1)$ possibilities for the quasi order on $\mathbb{P}_{\mathbf{m}}(L_1)$ and even for $\mathbb{P}_{\mathbf{m}}[L_1]$.

Case 2: α is a limit ordinal

{e24} By clause (c) of Definition 4.9, the number of α-equivalence classes is $\le \prod\limits_{\beta<\alpha}$ (the number of β-equivalence classes) $\le \prod\limits_{\beta<\alpha} \beth_{1+\beta+1}(\lambda_1) \le (\beth_{1+\alpha+1}(\lambda_1))^{\beth_{1+\alpha}} = \beth_{1+\alpha+1}(\lambda_1)$.

Case 3: $\alpha = \beta + 1$

Clearly every α-equivalence class can be coded as a set of β-equivalence classes hence the number of α-equivalence classes is $\le 2^{\beth_{1+\beta+1}(\lambda_1)} = \beth_{1+\beta+2}(\lambda_1) = \beth_{1+\alpha+1}(\lambda_1)$, as promised. $\square_{4.11}$

{e28} **Definition 4.12.** For an ordinal β, let $\mathscr{F}_{\mathbf{m},\beta}$ be the set of function f such that for some $t^\ell_i, \bar{s}^\ell_i$ for $i < i(*)$ and $\ell \in \{1, 2\}$ we have:

- (a) $i(*) < \lambda^+$
- (b) $\langle t^\ell_i : i < i(*)\rangle$ is a sequence of pairwise non-$E''_{\mathbf{m}}$-equivalent members of $L_{\mathbf{m}} \backslash M_{\mathbf{m}}$
- (c) $\bar{s}^\ell_i \in {}^{\zeta(i)}(t^\ell_i/E''_{\mathbf{m}})$ where $\zeta(i) < \lambda^+$
- (d) $(t^1_i, \bar{s}^1_i), (t^2_i, \bar{s}^2_i)$ are β-equivalent (members of $\mathscr{Y}_m$)
- (e) f is an isomorphism from $\mathbf{m}{\restriction}L_1$ onto $\mathbf{m}{\restriction}L_2$ when $L_\ell = \cup\{\text{Rang}(\bar{s}^\ell_i) : i < i(*)\} \cup M_{\mathbf{m}}$
- (f) $f{\restriction}M_{\mathbf{m}} =$ the identity
- (g) f maps $\bar{s}^1_i$ to $\bar{s}^2_i$ for $i < i(*)$.

2) For $f \in \mathscr{F}_{\mathbf{m},0}$ we define $\hat{f}$ as the mapping from $\mathbb{P}_{\mathbf{m}}(\text{Dom}(f))$ onto $\mathbb{P}_{\mathbf{m}}(\text{Rang}(f))$ induced by f; see clause 4.9(2)(a)(ε).

{e24} {e29} **Claim 4.13.** *Assume* $\mathbf{m}$ *is wide. The conditions* $p, q \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ *are compatible* *when for some* ψ *the following condition holds:*

$(\text{st})_{p,q,\psi}$ (a) $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$

{c7} (b) $\text{wsupp}(p) \cap \text{wsupp}(q) \subseteq M_{\mathbf{m}}$, *see Definition 2.10(1)(b), equivalently* $s \in \text{fsupp}(p)\backslash M_{\mathbf{m}}, t \in \text{fsupp}(q)\backslash M_{\mathbf{m}} \Rightarrow \neg(sE''_{\mathbf{m}}t)$

(c) *if* $\psi \le \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ *then* φ, p *are compatible in* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$

(d) ψ, q *are compatible in* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, *equivalently* $q \nVdash_{\mathbb{P}_{\mathbf{m}}}$ "$\psi[\underset{\sim}{\mathbf{G}}] = false$".

Remark 4.14. 1) We can use $(\text{st})'_{p,q,\psi}$: omit clause (d) and add to clause (c): and φ, q are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$.
2) We use $\lambda > \aleph_0$ in the proof, to eliminate it we can immitate the completeness theorem for $\mathbb{L}_{\aleph_1,\aleph_0}$.

Proof. We choose (p_n, q_n, ψ_n) by induction on n such that:

$\boxplus_n$ $(a)(\alpha)$ $(\text{st})_{p_n,q_n,\psi_n}$ holds if n is even

(β) $(\text{st})_{q_n,p_n,\psi_n}$ holds if n is odd

(b) $(p_0, q_0, \psi_0) = (p, q, \psi)$

(c) if $n = 2m+1$ and $s \in \text{dom}(p_{2m}) \cap M_{\mathbf{m}}$ then $s \in \text{dom}(q_{2m+1})$ and $\text{tr}(p_{2m}(s)) \subseteq \text{tr}(q_{2m+1}(s))$

(d) if $n = 2m+2$ and $s \in \text{dom}(q_{2m+1}) \cap M_{\mathbf{m}}$ then $s \in \text{dom}(p_{2m+2})$ and $\text{tr}(q_{2m+1}(s)) \subseteq \text{tr}(p_{2m+2}(s))$

(e) if $n = m+1$ then $p_m \le p_n, q_m \le q_n$.

Case 1: For $n = 0$ use clause (b).

Case 2: $n = 2m+1$.

So the triple $(p_{2m}, q_{2m}, \psi_{2m})$ is well defined, let $u_{2m} = \text{Dom}(p_{2m}) \cap M_{\mathbf{m}}$ and let $\bar{\nu} = \langle \nu_s : s \in u_{2m}\rangle$ be defined by $\nu_s = \text{tr}(p_{2m}(s))$.

Clearly

$(*)_1$ $\psi_{2m} \vdash p^*_{s,\nu_s}$ for $s \in u_{2m}$.

[Why? Clearly $p_{2m} \vdash p^*_{s,\nu_s}$, i.e. $p^*_{s,\nu_s} \le p_{2m}$ in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, hence if $\psi_{2m} \nvdash p^*_{s,\nu_s}$ then $\psi' = \psi_{2m} \wedge \neg p^*_{s,\nu_s} \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ is $\ge \psi_{2m}$ hence compatible with p_{2m}, contradiction, see clause (c) in $(\text{st})_{p,q,\psi}$.]

$(*)_2$ there is $q'_{2m} \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ which is above q_{2m} and above ψ_{2m} hence $s \in u_{2m}$ implies $\nu_s \subseteq \text{tr}(q'_{2m}(s))$ and $s \in \text{Dom}(q'_{2m})$.

[Why? By clause (d) of $(\text{st})_{p_{2m},q_{2m},\psi_{2m}}$ which holds by $\boxplus_{2m}(a)(\alpha)$ recalling $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ is dense $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$; the "hence" by $(*)_1$.]

$(*)_3$ there is $\psi'_{2m} \in \mathbb{P}_{\mathbf{m}}[M_m]$ such that:

(a) if $\psi'_{2m} \le \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ then φ, q'_{2m} are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$

(b) if $s \in u_{2m}$ then $\psi'_{2m} \vdash p^*_{s,\nu_s}$

(c) $\psi_{2m} \le \psi'_{2m}$.

[Why? Obvious using the λ^+-c.c., i.e. $\psi'_{2m} = \psi_{2m} \wedge \neg(\vee\{\varphi : \varphi \in \mathscr{I}\})$ where $\mathscr{I}$ is a max antichain of members $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ satisfying $\psi \perp q'_{2m}$ in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$.]

$(*)_4$ without loss of generality $\text{wsupp}(q'_{2m}) \cap \text{wsupp}(p_{2m}) \subseteq M_{\mathbf{m}}$.

[Why? As $\mathbf{m}$ is wide using automorphisms of $\mathbf{m}$, i.e. by 4.5.] {e10}

$(*)_5$ there is $p'_{2m} \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ which is above p_{2m} and above ψ'_{2m}.

[Why? By the choice of ψ'_{2m} and clause (c) of $(\text{st})_{p_{2m},q_{2m},\psi_{2m}}$ which holds by $\boxplus_{2m}(a)(\alpha)$.]

$(*)_6$ without loss of generality $\text{fsupp}(p'_{2m}) \cap \text{fsupp}(q'_{2m}) \subseteq M_{\mathbf{m}}$.

[Why? As $\mathbf{m}$ is wide using 4.5.] {e10}

Lastly, let $p_n = p'_{2m}, q_n = q'_{2m}, \psi_n = \psi'_{2m}$ and check.

Case 3: $n = 2m+2$

Similar to case 2 the roles of the p's and the q's interchanged.

Having carried the induction we can find p_* the upper bound of $\{p_n : n < \omega\}$ as in 2.13(4), in particular: {c11}

$(*)_7$ (a) $\text{Dom}(p_*) = \bigcup_n \text{Dom}(p_n)$; in fact, also $\text{fsupp}(p_*) = \bigcup_n \text{fsupp}(p_n)$

(b) if $s \in \mathrm{Dom}(p_n)$ then $\mathrm{tr}(p_*(s)) = \bigcup_{k \geq n} \mathrm{tr}(p_k(s))$.

{c11} Similarly let q_* be the upper bound of $\{q_n : n < \omega\}$ as in 2.13(4), so again:

$(*)_8$ (a) $\mathrm{Dom}(q_*) = \bigcup_n \mathrm{Dom}(q_n)$, in fact also $\mathrm{fsupp}(q_*) = \bigcup_n \mathrm{fsupp}(q_n)$

(b) if $s \in \mathrm{Dom}(q_n)$ then $\mathrm{tr}(q_*(s)) = \bigcup_{k \geq n} \mathrm{tr}(q_k(s))$.

Hence

$(*)_9$ (a) $p_*, q_* \in \mathbb{P}_{\mathbf{m}}$

(b) $\mathrm{Dom}(p_*) \cap \mathrm{Dom}(q_*) \subseteq M_{\mathbf{m}}$, in fact, $\mathrm{fsupp}(p_*) \cap \mathrm{Dom}(q_*) \subseteq M_{\mathbf{m}}$

(c) $\mathrm{Dom}(p_*) \cap M_{\mathbf{m}} = \mathrm{Dom}(q_*) \cap M_{\mathbf{m}}$

(d) if $s \in \mathrm{Dom}(p_*) \cap M_{\mathbf{m}}$, equivalently, $s \in \mathrm{Dom}(p_*) \cap \mathrm{Dom}(q_*)$ then $\mathrm{tr}(p_*(s)) = \mathrm{tr}(q_*(s))$.

[Why? Clause (a) by properties of $\mathbb{P}_{\mathbf{m}}$ and $p_n \leq p_{n+1}, q_n \leq q_{n+1}$ see above, clause (b) as $\mathrm{Dom}(p_{2m}) \cap \mathrm{Dom}(q_{2m}) \subseteq M_{\mathbf{m}}$ as $(\mathrm{st})_{p_{2m},q_{2m},\psi_{2m}}$, clause (c) by $\boxplus_n(c),(d)$, the first conclusion and clause (d) by $\boxplus_n(c),(d)$, the second conclusion.]

It follows that p_*, q_* are compatible in $\mathbb{P}_{\mathbf{m}}$ but $p = p_0 \leq p_*, q = q_0 \leq q_*$, so p, q are compatible as promised. $\square_{4.13}$

{e30} **Claim 4.15.** *The set* $\{\psi_i : i < i(*)\} \cup \{\psi_*\}$ *has a common upper bound in* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ *<u>when</u>:*

$(*)$ (a) $\mathbf{m} \in \mathbf{M}$ *is wide*

(b) $i(*) < \lambda$

(c) $t_i \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ *for* $i < i(*)$

(d) t_i, t_j *are not* $E''_{\mathbf{m}}$*-equivalence for* $i < j < i(*)$

(e) $\psi_* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$

(f) $X_i = t_i / E_{\mathbf{m}}$

(g) $\psi_i \in \mathbb{P}_{\mathbf{m}}[X_i]$

(h) *if* $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \models$ "$\psi_* \leq \varphi$" *and* $i < i(*)$ *then* ψ_i, φ *are compatible in* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ *equivalently in* $\mathbb{P}_{\mathbf{m}}[X_i]$.

Remark 4.16. Note: λ-wide is enough.

Proof. As $\psi_* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$, there is $p \in \mathbb{P}_{\mathbf{m}}$ such that $p \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\psi_*[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}] = \text{true}$". As $\mathbf{m}$
{e10} is wide by 4.5 there is an automorphism f of $\mathbf{m}$ such that $i < i(*) \Rightarrow f''(\mathrm{wsupp}(p)) \cap X_i \subseteq M_{\mathbf{m}}$, hence without loss of generality $i < i(*) \Rightarrow \mathrm{wsupp}(p) \cap X_i \subseteq M_{\mathbf{m}}$. Now we choose p_i by induction on $i \leq i(*)$ such that:

$\boxplus$ (a) $p_i \in \mathbb{P}_{\mathbf{m}}$

(b) $\langle p_j : j \leq i \rangle$ is increasing

(c) if $s \in \mathrm{Dom}(p_i), i < i(*)$ <u>then</u> $\ell g(\mathrm{tr}(p_{i+1}(s)) > i(*)$

(d) $p_0 = p$

(e) if $i = j + 1$ then $p_i \Vdash$ "$\psi_j[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}] = \text{true}$

(f) $\mathrm{wsupp}(p_i)$ hence also $\mathrm{wsupp}(p_i)$ is disjoint to $\cup\{X_j \backslash M_{\mathbf{m}} : j \in [i, i(*))\}$.

This is sufficient for the claim as $p_{i(*)}$ is as required. So let us carry the induction. {c10} For $i = 0$ use clause (d), for i limit by 2.12(4) we know that $\langle p_j : j < i \rangle$ has a $\le_{\mathbb{P}_\mathbf{m}}$-upper bound p_i with domain $= \cup\{\mathrm{Dom}(p_j) : j < i\}$ and $\mathrm{wsupp}(p_i) \subseteq \cup\{\mathrm{wsupp}(p_j) :$ {c11} $j < i\}$ by 2.13(4), hence p_i is as required, in particular as in clause (f).

Lastly, assume $i = j + 1$, now there is $\varphi_j \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$ such that $\varphi_j \le \varphi \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}] \Rightarrow p_j, \varphi$ are compatible in $\mathbb{P}_\mathbf{m}[L_\mathbf{m}]$. By an assumption $p_j \Vdash$ "$\psi_*[\mathop{\mathbf{G}}_{\sim}{}_{\mathbb{P}_\mathbf{m}}] =$ true" as p_0 forces this hence $\psi_* \le \varphi_j$. As $\varphi_j \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$ by clause (h) of the assumption ψ_j, φ_j are compatible in $\mathbb{P}_\mathbf{m}[L_\mathbf{m}]$ hence have a common upper bound $\varphi_j^+ \in \mathbb{P}_\mathbf{m}[X_j]$, so there is $q_j^0 \in \mathbb{P}_\mathbf{m}$ above φ_j and ψ_j. As $\mathbf{m}$ is wide without loss {e29} of generality $\mathrm{wsupp}(q_j^0) \cap \mathrm{wsupp}(p_j) \subseteq M_\mathbf{m}$. Together (see 4.13) $(\mathrm{st})_{p_j, q_j^0, \varphi_j}$ holds {e29} hence by 4.13 p_j, q_j^0 has a common upper bound called p_i. As $\mathbf{m}$ is wide, without loss of generality $\mathrm{wsupp}(p_i) \cap X_i = M_\mathbf{m}$ for $j \in [i + 1, i(*))$.

Clearly p_i is as required so we have finished the induction. So we are done. $\square_{4.15}$

{e31}

Conclusion 4.17. *If $\mathbf{m}$ is wide and $f \in \mathscr{F}_{\mathbf{m},\beta}$ and L_1, L_2 its domain and range respectively <u>then</u> f induces an isomorphism $\hat{f}$ from $\mathbb{P}_\mathbf{m}(L_1)$ onto $\mathbb{P}_\mathbf{m}(L_2)$.*

{e31a}

Remark 4.18. 1) See Definition 4.1(5); note that this claim is not covered by Definition 4.1(4). {e4} {e4}

2) Here we use 4.2(4), so the choice in Definition 2.9$(c)(\gamma)$ is justified (see Remark 4.3(1)) used below in the proof. {e5p} {e5p}

3) We could have separated the definition of "analyze" and its properties.

4) Note that in Definition 4.9, we deal only with $L_1 \subseteq t/E_\mathbf{m}$ for some t. {e24}

5) How come even $\beta = 0$ is suitable for 4.17? The point is clause $(a)(\varepsilon)$ of Definition 4.9(2). But no real harm using larger β. {e31} {e24}

Proof. By the definitions, clearly $\hat{f}$ is a one-to-one function from $\mathbb{P}_\mathbf{m}(L_1)$ onto $\mathbb{P}_\mathbf{m}(L_2)$. Next assume $p_1, q_1 \in \mathbb{P}_\mathbf{m}(L_1), \mathrm{Dom}(p_1) \subseteq \mathrm{Dom}(q_1)$ and let $p_2 := \hat{f}(p_1), q_2 := \hat{f}(q_1)$; clearly they belong to $\mathbb{P}_\mathbf{m}(L_2)$. We shall prove that $\mathbb{P}_\mathbf{m} \models$ "$p_1 \le q_1$" iff $\mathbb{P}_\mathbf{m} \models$ "$p_2 \le q_2$".

Let $\langle t_i^1 : i < i(*) \rangle$ be such that:

$\oplus_1$ (a) $t_i^1 \in \mathrm{fsupp}(q_1) \backslash M_\mathbf{m} \subseteq L_1$ such that $\mathrm{fsupp}(q_1)$ is included in $\cup\{t_i^1/E_\mathbf{m} : i < i(*)\}$

(b) $\langle t_i^1 : i < i(*) \rangle$ are pairwise non $E''_\mathbf{m}$-equivalent.

Next let

$\oplus_2$ (c) $t_i^2 = f(t_i^1)$

(d) let $\bar{t}_\ell = \langle t_i^\ell : i < i(*) \rangle$ without loss of generality $\mathrm{fsupp}(p_\ell) \subseteq \cup\{t_i^\ell/E''_\mathbf{m} : i < j(*)\} \cup M_\mathbf{m}$, so $j(*) \le i(*)$.

For $i < i(*)$ let $\psi_{1,i}^* \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$ be such that: $\vartheta \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$ is compatible with $q_{1,i} := q_1 \restriction (t_i^1/E_\mathbf{m})$ (the projection!) <u>iff</u> $\vartheta \wedge \psi_{1,i}^* \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$; clearly exists as $\mathbb{P}_\mathbf{m}$ satisfies the λ^+-c.c. Let $\psi_1^* = \wedge\{\psi_{1,i}^* : i < i(*)\}$.

Now $\psi_1^* \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$ as $q_1 \Vdash$ "$\psi_1^*[\mathop{\mathbf{G}}_{\sim}{}_{\mathbb{P}_\mathbf{m}}] =$ true". We will say "$\psi_1^*, \bar{\psi}_1^* = \langle \psi_{1,i}^*, q_{1,i} : i < i(*) \rangle$ analyze q_1 or $(q_1, \bar{t}_1)$" <u>when</u> the above holds.

Next choose $\varphi_1^*, \langle \varphi_{1,i}^*, p_{1,i} : i < j(*) \rangle$ which analyze $p_1, \langle t_i^1 : i < j(*) \rangle$. Why possible? As above.

Lastly, let $\psi^*_{2,i} = \check{f}(\psi^*_{1,i}), p_{2,i} = \check{f}(p_{1,i}), \psi^*_2 = \check{f}(\psi^*_1), \varphi^*_{2,i} = \check{f}(\varphi^*_{1,i}), q_{2,i} = \check{f}(q_{1,i}), \varphi^*_2 = \hat{f}(\varphi^*_1)$ where $\check{f}$ is the function from $\mathbb{L}_{\lambda^+}(Y_{L_1}, \mathbb{P}_{\mathbf{m}})$ onto $\mathbb{L}_{\lambda^+}(Y_{L_2}, \mathbb{P}_{\mathbf{m}})$ induced by f, i.e. where $\check{f}$ is the one-to-one function with domain $\mathbb{L}_{\lambda^+}[Y_{L_1}]$ defined by $p^*_{t,\eta} \mapsto p^*_{f(t),\eta}$

- $(*)$ for $\ell = 1,2$ the sequence $(p_\ell, q_\ell, \bar{\psi}^*_\ell, \bar{\psi}^*_\ell, \varphi^*_\ell, \bar{\varphi}^*_\ell)$ where $\bar{\psi}^*_\ell = \langle \psi^*_{\ell,i}, q_{\ell,i} : i < i_\ell(*)\rangle, \bar{\varphi}^*_\ell = \langle \varphi^*_{\ell,i}, p_{\ell,i} : i < i(*)\rangle$ satisfy the same demands as listed above for $\ell = 1,2$, that is
  - (a) $(\psi^*_\ell, \bar{\psi}^*_\ell)$ analyze $(q_\ell, \bar{t}_\ell)$ for $\ell = 1,2$
  - (b) $(\varphi^*_\ell, \bar{\varphi}^*_\ell)$ analyze $(p_\ell, \bar{t}_\ell{\restriction}j(*))$ for $\ell = 1,2$.

[Why? Think, recalling $f{\restriction}(t^1_i/E_{\mathbf{m}})$ is an isomorphism from $\mathbf{m}{\restriction}((t^1_i/E_{\mathbf{m}}) \cap L_1)$ onto $\mathbf{m}{\restriction}((t^2_i/E_{\mathbf{m}}) \cap L_2)$, etc.]

Next

- $\boxplus$ for $\ell = 1,2$ we have $(A)_\ell \Leftrightarrow (B)_\ell$ where
  - $(A)_\ell$ $\mathbb{P}_{\mathbf{m}} \models$ "$p_\ell \le q_\ell$"
  - $(B)_\ell$ for every $i < j(*)$ we have $\mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}] \models$ "$(\varphi^*_\ell \wedge p_{\ell,i}) \le (\psi^*_\ell \wedge q_{\ell,i})$".

Why? First, assume that the condition $(B)_\ell$ fails, say for i, hence there is $\vartheta \in \mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}]$ such that $\mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}] \models$ "$(\psi^*_\ell \wedge q_{\ell,i}) \le \vartheta$" and $\varphi^*_\ell \wedge p_{\ell,i} \wedge \vartheta \notin \mathbb{P}_{\mathbf{m}}[t^\ell_i/E_m]$.
{e30} So by claim 4.15 there is $q^+_\ell \in \mathbb{P}_{\mathbf{m}}$ such that $q^+_\ell \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is above ϑ hence above ψ^*_ℓ and above $q_{\ell,j} = q_\ell \restriction (t^\ell_j/E_{\mathbf{m}})$ for $j < i(*)$. That is, first get $\psi \in \mathbb{P}_m[M_m]$ such that $\psi \ge \psi^*_\ell$ and $[\psi \le \psi' \in \mathbb{P}_m[M_m] \Rightarrow \psi', \vartheta$ are compatible] (using $\vartheta \ge \psi^*_\ell$). Then
{e30} apply 4.15 to $(\{q_{\ell,j} : j < j(*)\} \cup \{\vartheta\}) \cup \{\psi\}$ to get q^+_ℓ.

{e5n} Hence by 4.2(4) the condition q^+_ℓ is above q_ℓ but $q^+_\ell \Vdash$ "$\varphi^*_\ell \wedge p_{\ell,i}[\underset{\sim}{\mathbf{G}}] = \text{false}$" as q^+_ℓ is above ϑ. However, $p_\ell \Vdash_{\mathbb{P}_{\mathbf{m}[L_{\mathbf{m}}]}}$ "$p_{\ell,i} \in \underset{\sim}{\mathbf{G}}$ and $\varphi^*_\ell \in \mathbf{G}$". By the last two sentences q^+_ℓ, p_ℓ are incompatible, in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ equivalently in $\mathbb{P}_{\mathbf{m}}$. So indeed $\neg(B)_\ell \Rightarrow \neg(A)_\ell$.

For the other direction assume condition $(B)_\ell$ holds, but condition $(A)_\ell$ fails and we shall get a contradiction. So there is $q^+_\ell \in \mathbb{P}_{\mathbf{m}}$ above q_ℓ incompatible with p_ℓ.

For each $i < i(*)$ as $(\psi^*_\ell, \langle \psi^*_{\ell,j}, q_{\ell,j} : j < i(*)\rangle)$ analyze q_ℓ, clearly $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\psi^*_\ell \wedge q_{\ell,i}) \le q_\ell$" but $q_\ell \le q^+_\ell$ hence $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\psi^*_\ell \wedge q_{\ell,i}) \le q^+_\ell$", and as we are
{e5n} assuming clause $(B)_\ell$ we have $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\varphi^*_\ell \wedge p_{\ell,i}) \le q^+_\ell$". Hence by 4.2(4), q^+_ℓ is above p_ℓ, contradiction. So indeed $(B)_\ell \Rightarrow (A)_\ell$.

{e28} Together, $\boxplus$ holds. Now clearly $(B)_1 \Leftrightarrow (B)_2$, see Definition 4.9, 4.12; so by $\boxplus$ we have $(A)_1 \Leftrightarrow (A)_2$ which is the desired conclusion. $\square_{4.17}$

{e32}

Claim 4.19. *We have $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}}$ __when__:*

- (a) $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}$
- (b) *if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}_1}$ and $\bar{s} \in {}^\zeta(t/E''_{\mathbf{m}}), \zeta < \lambda^+$ __then__ we can find $t_i, \bar{s}_i$ for $i < \lambda^+$ such that:*
  - (α) $t_i \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}_1}$
  - (β) $t_i/E''_{\mathbf{m}_1} \ne t_j/E''_{\mathbf{m}_1}$ *when* $i \ne j < \lambda^+$
  - (γ) $\bar{s}_i \in {}^\zeta(t_i/E''_{\mathbf{m}_1})$
  - (δ) $(t_i, \bar{s}_i)$ *is ξ-equivalent to $(t, \bar{s})$ in $\mathbf{m}$ where*[19] $\xi = 1$.
- (c) $\mathbf{m}$ *is wide.*

[19] no real harm in using larger ξ

{e31} *Remark* 4.20. In the proof we use conclusion 4.17 but not clause $(a)(\varepsilon)$ of Definition
{e24} 4.9(2).

Proof.

$\boxplus_1$ for $f \in \mathscr{F}_{\mathbf{m},\beta}$

(a) $\hat{f}$ preserves "p_2 is above p_1 in $\mathbb{P}_{\mathbf{m}}$", and its negations

(b) if $\beta > 0$ then $\hat{f}$ preserves also incompatibility in $\mathbb{P}_{\mathbf{m}}$.

[Why? Clause (a) holds by 4.17. For clause (b) use clause (a) and Definitions 4.9 {e34}
and 4.12 or see the proof of $\boxplus_2$.] {e28}

$\boxplus_2$ if $p_i \in \mathbb{P}_{\mathbf{m}_1}$ for $i < i(*) < \lambda^+$ and $p \in \mathbb{P}_{\mathbf{m}}$ <u>then</u> there is p^* such that:

(a) $p^* \in \mathbb{P}_{\mathbf{m}_1}$, equivalently $p^* \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1})$

(b) $\mathbb{P}_{\mathbf{m}_1} \models$ "$p_i \le p^*$" iff $\mathbb{P}_{\mathbf{m}} \models$ "$p_i \le p$"

(c) $\mathbb{P}_{\mathbf{m}_1} \models$ "p_i, p^* are compatible" iff $\mathbb{P}_{\mathbf{m}} \models$ "p_i, p are compatible".

[Why? Let $q_i \in \mathbb{P}_{\mathbf{m}}$ be such that: if p_i, p are compatible in $\mathbb{P}_{\mathbf{m}}$ then $p_i \le q_i \wedge p \le q_i$. We can find $L_1 \subseteq L_2$ such that

- $M_{\mathbf{m}} \subseteq L_1 \subseteq L_{\mathbf{m}_1}, |L_1 \backslash M_{\mathbf{m}}| \le \lambda$
- $\{p_i : i < i(*)\} \subseteq \mathbb{P}_{\mathbf{m}}(L_1)$
- $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}, |L_2 \backslash M_{\mathbf{m}}| \le \lambda$ and $p, q_i \in \mathbb{P}_{\mathbf{m}}(L_2)$ for $i < i(*)$.

By the assumption of the claim there is $f \in \mathscr{F}_{\mathbf{m},1}$ such that:

- $\mathrm{Dom}(f) \subseteq \cup\{(t/E''_{\mathbf{m}}) \cap L_2 : t \in L_2\} \cup M_{\mathbf{m}}$
- $t \in L_1 \Rightarrow f{\restriction}(t/E_{\mathbf{m}} \cap L_2) = \mathrm{id}_{(t/E_{\mathbf{m}}) \cap L_2}$
- if $q \in \{q_i : i < i(*)\} \cup \{p\} \cup \{p_i : i < i(*)\}$ and $t \in \mathrm{Dom}(q) \backslash M_{\mathbf{m}}$ then $\mathrm{fsupp}(q(t)) \subseteq \mathrm{Dom}(f)$
- $\mathrm{Rang}(f) \subseteq L_{\mathbf{m}_1}$.

Let $p^* = \hat{f}(p)$: by $\boxplus_1(a)$ clearly clauses (a),(b) of $\boxplus_2$ holds; and the choice of the q_i's also the implication "if" of clause (c). The "only if" of clause (c) holds by $\boxplus_1(b)$ so we are done.]

$\boxplus_3$ if $p \in \mathbb{P}_{\mathbf{m}}$ then $p \in \mathbb{P}_{\mathbf{m}_1}$ <u>iff</u> $\mathrm{fsupp}(p) \subseteq L_{\mathbf{m}_1}$.

[Why? Obvious.]

Recalling Definition 2.24(0)(b) {c34}

$\boxplus_4$ for every ordinal γ, we have $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$.

[Why? We shall prove this by induction on γ using $\boxplus_2 + \boxplus_3$.

Note that

$\boxplus_{4.1}$ (a) $L^{\mathrm{dp}}_{\mathbf{m},\gamma} \cap L_{\mathbf{m}_1} = L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}$

(b) if $f \in \mathscr{F}_{\mathbf{m},\beta}, s \in \mathrm{Dom}(f)$ and β is an ordinal then

- $s \in L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma} \Leftrightarrow f(s) \in L^{\mathrm{dp}}_{\mathbf{m},\gamma}$

(c) the parallel of $\boxplus_2$ holds for $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},\gamma})$ so $p^* \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1,\gamma})$

(d) $L^{\mathrm{dp}}_{\mathbf{m},\gamma}$ is an initial segment of $L_{\mathbf{m}}$

(e) $L^{\rm dp}_{\mathbf{m}_1,\gamma}$ is an initial segment of $L_{\mathbf{m}_1}$

(f) $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1})$, similarly for $\mathbf{m}$.

We shall use this freely. The inductive proof on γ splits to three cases.

Case 1: $\gamma = 0$

So

- $E = E''_{\mathbf{m}} \restriction L^{\rm dp}_{\mathbf{m},\gamma}$ is an equivalence relation on $L^{\rm dp}_{\mathbf{m},\gamma}$
- $E \restriction L^{\rm dp}_{\mathbf{m}_1,\gamma} = E''_{\mathbf{m}_1} \restriction L^{\rm dp}_{\mathbf{m}_1,\gamma}$
- if $t \in L^{\rm dp}_{\mathbf{m}_1,\gamma}$ then $t \notin M_{\mathbf{m}_1}, t/E'_{\mathbf{m}_1} = t/E'_{\mathbf{m}}, (t/E'_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma} = (t/E_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma} = (t/E''_{\mathbf{m}}) \cap L^{\rm dp}_{\mathbf{m},\gamma}$ initial segment of $L_{\mathbf{m}_1}$ and of $L_{\mathbf{m}}$ and $\mathbb{P}_{\mathbf{m}}((t/E_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma}) = \mathbb{P}_{\mathbf{m}_1}((t/E_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma})$
- $\mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\gamma})$ is the product with $(< \lambda)$-support of $\{\mathbb{P}_{\mathbf{m}}((t/E_{\mathbf{m}_1}) \cap L^{\rm dp}_{\mathbf{m}_1,\gamma}) : t \in L^{\rm dp}_{\mathbf{m},\gamma}\}$
- similarly for $\mathbf{m}_1$.

So the result should be clear.

Case 2: $\gamma = \beta + 1$

Let $M_\beta = \{s \in M_{\mathbf{m}} : {\rm dp}_{\mathbf{m}}(s) = \beta\}$, clearly

$\boxplus_{4.2}$ (a) M_β is a set of pairwise incomparable elements

(b) (α) $s \in M_\beta \Rightarrow L_{\mathbf{m}_1,<s} \subseteq L^{\rm dp}_{\mathbf{m}_1,\beta} \wedge L_{\mathbf{m},<s} \subseteq L^{\rm dp}_{\mathbf{m}_1,\beta}$

(β) similarly for $\mathbf{m}$

(c) M_β is disjoint to $L^{\rm dp}_{\mathbf{m}_1,\beta}, L^{\rm dp}_{\mathbf{m},\beta}$

(d) $M_\beta \subseteq L^{\rm dp}_{\mathbf{m}_1,\gamma}$

(e) $L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta$ is an initial segment of $L_{\mathbf{m}}$

(f) $L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta$ is an initial segment of $L_{\mathbf{m}_1}$.

As first half we prove

$\boxplus_{4.3}$ $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta) \lessdot \mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta)$.

Why? Recalling $\boxplus_{4.1}(a)$, note

(a) for $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta)$ we have $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma} \cup M_\beta) \models$ "$p \le q$" iff $\mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m}} \cup M_\beta) \models$ "$p \le q$".

[Why? Immediate by the definition of the order and the induction hypothesis.]

(b) for $p_1, p_2 \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta)$ <u>then</u> p_1, p_2 are compatible in $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\beta} \cup M_\beta)$ iff they are compatible in $\mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta)$.

[Why? The implication $\Rightarrow$ holds by clause (a). So assume $p_3 \in \mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta)$ is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{m}}(L^{\rm dp}_{\mathbf{m},\beta} \cup M_\beta)$ equivalently in $\mathbb{P}_{\mathbf{m}}$.

Now there is $f \in \mathscr{F}_{\mathbf{m},1}$ such that

- $f \restriction ({\rm fsupp}(p_1) \cup {\rm fsupp}(p_2))$ is the identity, moreover
- $s \in {\rm wsupp}(p_1) \cup {\rm wsupp}(p_2) \wedge s \in {\rm dom}(f) \Rightarrow f(s) = s$,

- $\mathrm{Dom}(f) = \cup\{\mathrm{fsupp}(p_\ell) : \ell = 1, 2, 3\}$
- $\mathrm{Rang}(f) \subseteq L_{\mathbf{m}_1}$.

Hence clearly $f{\restriction}M_\beta = \mathrm{id}_{M_\beta}$ so by $\boxplus_{4.1}(b)$ we have $\mathrm{Rang}(f) \subseteq L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta$ so $\hat{f}(p_3) \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta)$.

By $\boxplus_1$ the condition $\hat{f}(p_3)$ is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{m}}$ and by the previous sentence also in $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta)$, so by clause (a) the conclusion of (b) holds.]

(c) if $\mathscr{I}$ is a maximal antichain in $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta)$ <u>then</u> $\mathscr{I}$ is a maximal antichain of $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\beta} \cup M_\beta)$.

[Why? As in the proof of (b) and of $\boxplus_2$.]

So we are done proving $\boxplus_{4.3}$.

Now we return to proving $\boxplus_4$, note

$\boxplus_{4.4}$ let $\mathscr{E} = \{(s_1, s_2) : s_1, s_2 \in L_*$ and $s_1/E_{\mathbf{m}} = s_2/E_{\mathbf{m}}\}$ where $L_* = L^{\mathrm{dp}}_{\mathbf{m},\gamma} \backslash (L^{\mathrm{dp}}_{\mathbf{m},\beta} \cup M_\beta)\}$ <u>then</u>

(a) $\mathscr{E}$ is an equivalence relation on L_*

(b) if $s_1, s_2 \in L_*$ and $s_1 \le_{L_{\mathbf{m}}} s_2$ then $s_1 \mathscr{E} s_2$

(c) if $s_1, s_2 \in L_*$ and $s_1 \mathscr{E} s_2$ then $s_1 \in L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma} \Leftrightarrow s_2 \in L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}$ (and both $\notin M_\beta$)

(d) if $s \in L_*$ then $L_{\mathbf{m},<s} \subseteq L^{\mathrm{dp}}_{\mathbf{m},\beta} \cup M_\beta \cup (s/\mathscr{E})$

(e) if $s \in L_* \cap L_{\mathbf{m}_1}$ then $L_{\mathbf{m}_1,<s} \subseteq L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta \cup (s/\mathscr{E})$.

Hence let $L_0 = L^{\mathrm{dp}}_{\mathbf{m}_1,\beta} \cup M_\beta$ and $L_1 = L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma} = L^{\mathrm{dp}}_{\mathbf{m}_1} \cup M_\beta$ they satisfy all the assumptions of 2.22 hence its conclusion, so we are done easily proving Case 2 of $\boxplus_4$. {c33s}

<u>Case 3</u>: γ is a limit ordinal

Clearly $p \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ iff $p \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$; also each of them implies $p \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$ by the induction hypothesis. Also for $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ we have $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \models$ "$p \le q$" <u>iff</u> $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma}) \models$ "$p \le q$" by the definition of the order and the induction hypothesis. Together $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma}) \subseteq \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$, (as partial orders).

Next assume that $q_1, q_2 \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ and p_3 is a common upper bound of q_1, q_2 in $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$.

We shall find $p_1 \in \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ such that:

$(*)_1$ (a) p_1 is above q_1, q_2 (in $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ or equivalently in $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$),

(b) if $p_1 \le p_1' \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ then p_1', p_3 are compatible in $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$.

This clearly suffices; why? e.g. if $\{r_i : i < i(*)\} \subseteq \mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ is a maximal antichain of $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma})$ but not of $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$, let $q_1 = q_2 = \emptyset$ and $p_3 \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m},\gamma})$ be incompatible with every r_i; let p_1 be as in $(*)_1$, it gives a contradiction.

If $\mathrm{cf}(\gamma) \ge \lambda$ then for some $\gamma_1 < \gamma$ we have $q_1, q_2 \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma_1})$ and $\mathrm{fsupp}(p_3) \cap L^{\mathrm{dp}}_{\mathbf{m},\gamma} \subseteq L^{\mathrm{dp}}_{\mathbf{m},\gamma_1}$ and use the induction hypothesis on γ_1 for clause (a) of $(*)_1$; for clause (b) of $(*)_1$ we also recall 2.13(8); (alternatively immitate the case $\mathrm{cf}(\gamma) < \lambda$, choosing "changing our minds" $\gamma_\varepsilon < \gamma$ with the induction). So assume $\aleph_0 \le \mathrm{cf}(\gamma) < \lambda$ and let $\langle \gamma_\varepsilon : \varepsilon < \mathrm{cf}(\gamma)\rangle$ be increasing continuous with limit γ. {c11}

Now we choose $p_{1,\varepsilon}$ by induction on $\varepsilon \le \mathrm{cf}(\gamma)$ such that:

$(*)_2$ (a) $p_{1,\varepsilon} \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dp}}_{\mathbf{m}_1,\gamma_\varepsilon})$

(b) $(\gamma_\varepsilon, q_1 {\restriction} L^{\mathrm{dp}}_{\mathbf{m},\gamma_\varepsilon}, q_2 {\restriction} L^{\mathrm{dp}}_{\mathbf{m},\gamma_\varepsilon}, p_3 {\restriction} L^{\mathrm{dp}}_{\mathbf{m},\gamma_\varepsilon}, p_{1,\varepsilon})$ are like $(\gamma, q_1, q_2, p_3, p_1)$ in $(*)_1$

(c) $p_{1,\zeta} \le p_{1,\varepsilon}$ for $\zeta < \varepsilon$

(d) if $\varepsilon = \zeta + 1$ and $s \in \mathrm{dom}(p_{1,\zeta})$ then $\ell g(\mathrm{tr}(p_\varepsilon(s))) > \mathrm{cf}(\gamma)$.

So we are done proving $\boxplus_4$.]

$\boxplus_5$ $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}}$.

[Why? By $\boxplus_4$ for γ large enough.]

So we are done. $\square_{4.19}$

{c73}

Claim 4.21. *If $\mathbf{m} \in \mathbf{M}$ is reduced or just $L_{\mathbf{m}}$ has cardinality $\le \lambda_2$ <u>then</u> there is $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$ of cardinality $\le \lambda_2$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$.*

{c76}

Remark 4.22. By this we may restrict ourselves to $\mathbf{M}_{\le \lambda_2}$ (but then similarly in the end of §3).

Proof. We choose χ large enough and $\mathbf{m}_* \in \mathbf{M}_\chi$ which is wide, belongs to $\mathbf{M}_{\mathrm{ec}}$ and
{c41} $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_*$; moreover is full and very wide (as constructed in 2.26).

We can choose $\mathbf{n}$ such that:

$(*)$ (a) $\mathbf{n} \in \mathbf{M}$ and $\mathbf{n}$ is wide and $|L_{\mathbf{n}}| = \lambda_2$

(b) $\mathbf{m} \le \mathbf{n} \le_{\mathbf{M}} \mathbf{m}_*$

{e32} (c) $(\mathbf{n}, \mathbf{m}_*)$ satisfies the criterion from 4.19, with $\mathbf{m}_1, \mathbf{m}$ there standing for $\mathbf{n}, \mathbf{m}_*$ here.

{e24} [Why? Let $\xi = 1$ and recalling Definition 4.9(1) choose $\langle (t_\alpha, \bar{s}_\alpha) : \alpha < \lambda_2)\rangle$ such that $(t_\alpha, \bar{s}_\alpha) \in \mathscr{Y}_{\mathbf{m}_*}, t_\alpha \in L_{\mathbf{m}_*} \backslash M_{\mathbf{m}_*}, \langle t_\alpha / E_{\mathbf{m}} : \alpha < \lambda_2\rangle$ are pairwise distinct and for every $(t, \bar{s}) \in \mathscr{Y}_{\mathbf{m}_*}$ there are λ^+ ordinals $\alpha < \lambda_2$ such that $(t, \bar{s}), (t_\alpha, \bar{s}_\alpha)$ are
{e27} ξ-equivalent, possible by 4.11 recalling $\lambda_2 \ge \beth_3(\lambda_1)$. Let $L' = \cup\{t_\alpha / E_{\mathbf{m}_*} : \alpha < \lambda_2\} \cup L_{\mathbf{m}}$ and for each $t \in L' \backslash M_{\mathbf{m}_*}$ let $\langle s_{t,\alpha} : \alpha < \lambda^+\rangle$ be such that $s_{t,\alpha} \in L_{\mathbf{m}_*} \backslash M_{\mathbf{m}_*}$ and $\mathbf{m}_* {\restriction} (s_{t,\alpha}/E_{\mathbf{m}_*})$ is isomorphic to $\mathbf{m}_* {\restriction} (t/E_{\mathbf{m}_*})$ over $M_{\mathbf{m}}$. Let $L = L' \cup \{s_{t,\alpha} : \alpha < \lambda^+, t \in L' \backslash M_{\mathbf{m}_*}\}$ and $\mathbf{n} = \mathbf{m}_* {\restriction} L$. Now it is easy to check that $\mathbf{n}$ is as required.]

It suffices to prove that $\mathbf{n}$ belongs to $\mathbf{M}_{\mathrm{ec}}$, let $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2$.

Without loss of generality $L_{\mathbf{n}_2}$ has cardinality $\le 2^{\lambda_2}$, by the LST argument (even $\le \lambda_2$, as we are assuming $\lambda_2 = (\lambda_2)^\lambda$), and as $\mathbf{m}_*$ is very wide and full without
{e32} loss of generality $\mathbf{n}_2 \le_{\mathbf{M}} \mathbf{m}_*$. Now $(\mathbf{n}_1, \mathbf{m}_*)$ satisfies the criterion from 4.19 hence $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{m}_*}$.

{e32} Also the pair $(\mathbf{n}_2, \mathbf{m}_*)$ satisfies the criterion from 4.19 looking at the criterion.
{e32} Hence by 4.19 we have $\mathbb{P}_{\mathbf{n}_2} \lessdot \mathbb{P}_{\mathbf{m}_*}$.

As $\mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2 \le_{\mathbf{M}} \mathbf{m}_*$ from the last two sentences it easily follows that $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$, so we are done. $\square_{4.21}$

Discussion 4.23. In what way does this proof help? Will it not be simpler to
{c6} omit in Definition 2.9 clause (c) the $\iota_{p(s)}, \mathbf{B}_{p(s),\iota}$, etc.?

{e4} In this case in 4.1 we cannot define the projection directly hence we should look for projection as in general forcing, but then we run into problems of absoluteness.
{e32} More specifically, 4.19 seems to be problematic; anyhow this does not matter.

Definition 4.24. For $\mathbf{m} \in \mathbf{M}$ and $M \subseteq L_{\mathbf{m}}$ of cardinality $\leq \lambda_1$ we define $\mathbf{n} := \mathbf{m}\langle M\rangle \in \mathbf{M}$ as follows: {e37}

- (a) $L_{\mathbf{n}} = L_{\mathbf{m}}$ even as a partial order
- (b) $\bar{u}_{\mathbf{n}} = \bar{u}_{\mathbf{m}}$ and $\bar{\mathscr{P}}_{\mathbf{n}} = \bar{\mathscr{P}}_{\mathbf{m}}$
- (c) $M_{\mathbf{n}} = M$; not $M_{\mathbf{m}}$!
- (d) $E'_{\mathbf{n}} = \{(s,t) : s,t \in L_{\mathbf{m}}$ and $\{s,t\} \nsubseteq M\}$.

Claim 4.25. *Assume* $\mathbf{m} \in \mathbf{M}_{\leq\lambda_2}$ *and* $M \subseteq M_{\mathbf{m}}$. {e39}
1) $\mathbf{n} := \mathbf{m}\langle M\rangle$ *indeed belongs to* $\mathbf{M}$ *and is equivalent to* $\mathbf{m}$ *hence* $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}}) = \mathbb{P}_{\mathbf{n}}(L_{\mathbf{m}})$.
2) If $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1$ *<u>then</u> for some* $\mathbf{m}_1$ *we have* $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1$ *and* $\mathbf{m}_1, \mathbf{n}_1$ *are equivalent.*
3) If $\mathbf{m} \in \mathbf{M}_{\rm ec}$ *and* $\mathbf{n} = \mathbf{m}\langle M\rangle$ *<u>then</u>* $\mathbf{n} \in \mathbf{M}_{\rm ec}$.

Proof. 1) Check, noting that $t \in L_{\mathbf{n}} \backslash M_{\mathbf{n}} \Rightarrow t \in L_{\mathbf{m}} \backslash M \Rightarrow |t/E'_{\mathbf{m}}| \leq |L_{\mathbf{n}}| = |L_{\mathbf{m}}| \leq \lambda_2$ and $|M_{\mathbf{m}}| = |M| \leq |M_{\mathbf{m}}| \leq \lambda_1$.
2) Given such $\mathbf{n}_1$ we now define $\mathbf{m}_1 \in \mathbf{M}$ by:

$(*)_1$
- (a) $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$,
- (b) $\bar{u}_{\mathbf{m}_1} = \bar{u}_{\mathbf{n}_1}$ and $\bar{\mathscr{P}}_{\mathbf{m}_1} = \bar{\mathscr{P}}_{\mathbf{n}_1}$
- (c) $M_{\mathbf{m}_1} = M_{\mathbf{m}}$,
- (d) $E'_{\mathbf{m}_1} = \{(s,t) : sE'_{\mathbf{m}}t$ <u>or</u> $\{s,t\} \nsubseteq L_{\mathbf{m}} \backslash M$ and $sE'_{\mathbf{n}_1}t\}$.

Clearly

$(*)_2$
- (a) $\langle M_{\mathbf{m}}\rangle\hat{}\langle s/E''_{\mathbf{m}} : s \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}}\rangle\hat{}\langle t/E''_{\mathbf{n}_1} : t \in L_{\mathbf{n}_1} \backslash L_{\mathbf{n}}\rangle$ is a partition of $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$
- (b) $E''_{\mathbf{m}_1} = E'_{\mathbf{m}_1} {\restriction} \{(s,t) \in E'_{\mathbf{m}_1}$ and $s,t \notin M_{\mathbf{m}}\}$ is an equivalence relation, its equivalence classes being the sets listed in clause (a) except $M_{\mathbf{m}}$
- (c) $\mathbf{m}_1$ satisfies clasue $(e)(\gamma)$ of Definition 2.7 {c4}

$(*)_3$
- (a) if $s \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then
  - (α) $s \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}_1}$
  - (β) $s/E'_{\mathbf{m}_1} = s/E'_{\mathbf{m}}$
  - (γ) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1,s} = u_{\mathbf{n},s} = u_{\mathbf{m},s}$
  - (δ) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{m},s}$
- (b) if $s \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}}$ then
  - (α) $s \in L_{\mathbf{n}_1} \backslash L_{\mathbf{n}}$
  - (β) $s/E'_{\mathbf{m}_1} = s/E'_{\mathbf{n}_1}$
  - (γ) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1}, s$
  - (δ) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{n}_1,s}$
- (c) if $s \in M_{\mathbf{m}_1}$, i.e. $s \in M_{\mathbf{m}}$ then
  - (α) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1,s}$

(β) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{n}_1,s}$

and

$(*)_4$ (a) indeed $\mathbf{m}_1 \in \mathbf{M}$,
(b) $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1$,
(c) $\mathbf{m}_1, \mathbf{n}_1$ are equivalent.

So we are done.
3) Assume $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2$, as in the proof of part (2) there are $\mathbf{m}_1, \mathbf{m}_2$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2$ and $\mathbf{m}_\ell, \mathbf{n}_\ell$ are equivalent for $\ell = 1, 2$. As $\mathbf{m} \in \mathbf{M}_{\rm ec}$ we have $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$ but this means $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$, as required. $\square_{4.25}$

{c80}

Conclusion 4.26. *1) If $\mathbf{m} \in \mathbf{M}, M \subseteq M_{\mathbf{m}}$ and $\mathbf{n} = \mathbf{m} \restriction M$ <u>then</u> $\mathbb{P}^{\rm cr}_{\mathbf{n}} \lessdot \mathbb{P}^{\rm cr}_{\mathbf{m}}$.*
2) If $\mathbf{m}_\ell \in \mathbf{M}$ and $M_\ell \subseteq M_{\mathbf{m}_\ell}$ for $\ell = 1, 2$ and h is an isomorphism from $\mathbf{m}_1 \restriction M_1$ onto $\mathbf{m}_2 \restriction M_2$ <u>then</u> h induces an isomorphism from $\mathbb{P}^{\rm cr}_{\mathbf{m}_1}[M_1]$ onto $\mathbb{P}^{\rm cr}_{\mathbf{m}_2}[M_2]$.

{c73} *Proof.* 1) As in the proof of 4.21, without loss of generality $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\le \lambda_2}$. By 4.21 there is $\mathbf{m}_* \in \mathbf{M}^{\rm ec}_{\lambda_2}$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_*$ hence $\mathbb{P}^{\rm cr}_{\mathbf{m}} = \mathbb{P}_{\mathbf{m}_*}[L_{\mathbf{m}}]$.

{e39} Let $\mathbf{n}_* = \mathbf{m}_*\langle M \rangle$, see 4.24, so $\mathbf{n}_* \restriction M = \mathbf{n}$ and by 4.25(3) we have $\mathbf{n}_* \in \mathbf{M}_{\rm ec}$, hence $\mathbb{P}_{\mathbf{n}_*}[L_{\mathbf{n}}] = \mathbb{P}^{\rm cr}_{\mathbf{n}}$. But $\mathbf{n}_*, \mathbf{m}_*$ are equivalent, hence $\mathbb{P}_{\mathbf{n}_*} = \mathbb{P}_{\mathbf{m}_*}$ hence $\mathbb{P}_{\mathbf{n}_*}[L] = \mathbb{P}_{\mathbf{m}_*}[L]$
{c48} for every $L \subseteq L_{\mathbf{m}_*}$ hence by 3.9(3) $\mathbb{P}^{\rm cr}_{\mathbf{n}} = \mathbb{P}_{\mathbf{n}_*}[L_{\mathbf{n}}] \lessdot \mathbb{P}_{\mathbf{n}_*}[L_{\mathbf{m}}] = \mathbb{P}_{\mathbf{m}_*}[L_{\mathbf{m}}] = \mathbb{P}^{\rm cr}_{\mathbf{m}}$. So the conclusion holds.
2) Easy, too. $\square_{4.26}$

§ 5. On a relative to §2 having λ_0 and λ_1

Hypothesis 5.1. As in 2.2, but now we have paramter $\bar{\lambda} = (\lambda_1, \lambda_0), \lambda_1 \geq \lambda_0 \geq \lambda$ and use $\mathbf{q}$ the usual $< \lambda$-support iteration. {c0}

Discussion 5.2. (2012.2.02) 1) λ_1 has the role of λ_0 in clause $(e)(\theta)$ of Definition 2.7. {c4}
2) λ_0 has the role of λ_0 in Definition 5.4 and in 2.7. {c29}
3) At the moment, 5.11 seems unclear. {c46}
4) Even with this separation to λ_0, λ_1 we can prove 2.26 but it requires more care ($\mathscr{X}$ more information). {c41}
5) We may change Definition 5.5(1) to $(*)$ below, getting weaker results for $\mathbf{M}$ but ones which fit more general contents {c30}

- $(*)$ by induction on the ordinal γ we define the class $\mathbf{M}^{\rm ec}_\gamma$ as the class of $\mathbf{m} \in \mathbf{M}$ such that:
  - if $L \in \mathrm{Sub}(\mathbf{m})$ and $\beta = \mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$ <u>then</u> $m \leq_{\mathbf{M}} n \in M^{\rm ec}_\beta \Rightarrow P_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$.

6) We may assume $\chi = \chi^{\lambda_0}$ instead of $\chi = \chi^{\lambda_1}$ but using E in the proof more information $\mathscr{X}$ and $\mathscr{E}$.

Claim 5.3. *Continuing claim:*
10) If $L^+_{\mathbf{m}} \models$ "$s < t$" and $p \in \mathscr{I}_{s,t}$ and $\zeta = \ell g(\mathrm{tr}(p(s))$ <u>then</u> for every $i < \theta_\varepsilon$ large enough there is $q \in \mathbb{P}_t$ such that:

- $\mathbb{P}_t \models$ "$p \leq q$"
- *if $r \in \mathrm{Dom}(p) \backslash L_{\mathbf{m}, \leq s}$ then $q(r) = p(r)$*
- *if $r \in \mathrm{Dom}(p) \cap L_{\mathbf{m}, < s}$ then $\mathrm{tr}(p(r)) \triangleleft \mathrm{tr}(q(r))$*
- *$\mathrm{tr}(q(s))$ has length $\zeta + 1$*
- *$q(s)(\zeta) = i$.*

11) If $L^+_{\mathbf{m}} \models$ "$s < t$" and $p \in \mathscr{I}_{s,t}$ and $\zeta_0 < \zeta_1 < \lambda, \zeta_0 = \ell g(\mathrm{tr}(p(s)))$ and $\bar{u} = \langle u_\varepsilon : \zeta_0 \leq \varepsilon < \zeta_1 \rangle$ where u_ε is an unbounded subset of θ_ε for $\varepsilon \in [\zeta_0, \zeta_1)$ <u>then</u> there is $q \in \mathbb{P}_t$ above p such that $p \Vdash_{\mathbb{P}_t}$ "$\underset{\sim}{\eta}_s(\varepsilon) \in u_\varepsilon$ for $\varepsilon \in [\zeta_0, \zeta_1)$".
12) If $\xi \leq \lambda, r_\zeta \in u_t$ for $\zeta < \xi$ and $\mathbf{B}$ is a λ-Borel function from ${}^\xi(\Pi \bar{\theta})$ to $\mathbb{Q}_{\bar{\theta}}$ and $p \in \mathbb{P}_{\mathbf{m}}$ <u>then</u> for some q we have:

- $\mathbb{P}_{\mathbf{m}} \models$ "$p \leq q$"
- *$q \restriction L_{<t} \Vdash_{\mathbb{P}_t}$ "$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta < \xi} \leq_{\mathbb{Q}_{\bar{\theta}}} q(t)$ <u>or</u> $\mathbf{B}(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta < \varepsilon}, q(t)$ are incompatible in $\mathbb{Q}_t, p \in \mathbb{P}_{\mathbf{m}}$.*

13) If $t \in L_{\mathbf{m}}, j \leq \lambda, \xi_i \leq \lambda$ for $i < j, r_{i,\zeta} \in u_t$ for $i < j, \zeta < \xi_i$ and $p \restriction L_{<t} \Vdash_{\mathbb{P}_t}$ "$\langle \mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{r_{i,\zeta}}, \ldots)_{\zeta < \xi_i} : i < j \rangle$ is a maximal antichain of $\mathbb{Q}_t^{\mathbf{V}[\langle \underset{\sim}{\eta}_r : r \in u_t \rangle]}$" <u>then</u> for some q and $i < j$ we have:

- $\mathbb{P}_{\mathbf{m}} \models$ "$p \leq q$"
- *$q \restriction L_{<t} \Vdash_{\mathbb{P}_t}$ "$q(t)$ is above $\mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{r_{i,\zeta}}, \ldots)_{\zeta < \varepsilon_i}$ in $\mathbb{Q}_{\bar{\theta}}$".*

Definition 5.4. 1) For $\mathbf{m} \in \mathbf{M}$ let $\mathrm{cmp}(\mathbf{m})$ be the set of L such that: {c29}

(a) $L \subseteq L_{\mathbf{m}}$

(*b*) if $L \subseteq M_{\mathbf{m}}$ then $|L| \leq 1$

(*c*) if $t \in L \backslash M_{\mathbf{m}}$ then $L = t/E'_{\mathbf{m}}$ hence:

(*α*) $L \backslash M_{\mathbf{m}} \neq \emptyset$ and $L \backslash M$ is an $E_{\mathbf{m}}$-equivalence class

(*β*) if $t \in L \backslash M_{\mathbf{m}}$ then $u_{\mathbf{m},t} \subseteq L$

(*γ*) $|L| \leq \lambda_0$.

2) For $\mathbf{m} \in \mathbf{M}$

(*a*) let sub($\mathbf{m}$) be the set of L_1 such that for some $L_2 \in \text{cmp}(\mathbf{m}), L_1$ is an initial segment of L_2, i.e. $L_2 \subseteq L_1 \wedge (\forall s,t)[s \in L_2 \wedge t \in L_1 \wedge s <_{\mathbf{m}} t \rightarrow s \in L_1]$

(*b*) let $\text{sub}_*(\mathbf{m}) = \{L \in \text{sub}(\mathbf{m})$: there are[20] $L' \in \text{cmp}(\mathbf{m})$ and $Y_1, Y_2 \subseteq L'$ of cardinality $\leq \lambda_0$ such that $L = \{s \in L' : (\exists t)(s \leq_{\mathbf{m}} t \in \mathscr{Y}_1 \vee s <_{\mathbf{m}} \in \mathscr{Y}_2\}\}$.

3) Let

(*a*) $\text{Cmp}(\mathbf{m}) = \{\bigcup\limits_{\varepsilon<\zeta} L_\varepsilon : L_\varepsilon \in \text{cmp}(\mathbf{m}) \text{ for } \varepsilon < \zeta\}$

(*b*) $\text{Cmp}_*(\mathbf{m}) = \{\bigcup\limits_{\varepsilon<\zeta} L_\varepsilon : L_\varepsilon \in \text{cmp}(\mathbf{m}) \text{ for } \varepsilon < \zeta \text{ and } \zeta \leq \lambda_0\}$

(*c*) $\text{Sub}(\mathbf{m}) = \{L_1$: for some $L_2 \in \text{Cmp}(\mathbf{m}), L_1$ is an initial segment of $L_2\}$

(*d*) $\text{Sub}_*(\mathbf{m}) = \{\bigcup\limits_{\varepsilon<\zeta} L_\varepsilon : \zeta \leq \lambda_0 \text{ and } L_\varepsilon \in \text{sub}_*(\mathbf{m}) \text{ for } \varepsilon < \zeta\}$.

{c30}

Definition 5.5. 0) For $L \subseteq L_{\mathbf{m}}, \mathbf{m} \in \mathbf{M}$ let $\text{dp}^*_{\mathbf{m}}(L) = \cup\{\text{dp}_{M_{\mathbf{n}}}(t)+1 : t \in L \cap M_{\mathbf{m}}\}$.
1) For an ordinal γ let $\mathbf{M}^{\text{ec}}_\gamma$ be the class of $\mathbf{m} \in \mathbf{M}$ such that:

(∗) if $L \in \text{Sub}(\mathbf{m})$ and $\text{dp}^*_{\mathbf{m}}(L) < \gamma$ <u>then</u> $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$.

2) Let $\mathbf{M}^{\text{ec}} = M^{\text{ec}}_\infty$ be the class of $\mathbf{m}$ which $\in \mathbf{M}^{\text{ec}}_\gamma$ for every γ.
3) Let $\mathbf{M}^{\text{ec}}_{\chi,\gamma} = \{\mathbf{m} \in \mathbf{M}^{\text{ec}}_\gamma : |L_{\mathbf{m}}| \leq \chi\}$, similarly $\mathbf{M}^{\text{ec}}_{\chi,\infty}$.
4) Let $\mathbf{M}_{\text{fc}}$ be the class of $\mathbf{m} \in \mathbf{M}$ satisfying $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$.

{c32}

Observation 5.6. *1) $\leq_{\mathbf{M}}$ is a partial order on $\mathbf{M}$.
2) $\mathbf{M}^{\text{ec}}_0 = \mathbf{M}$ and $\mathbf{M}^{\text{ec}}_{\chi,0} = \mathbf{M}_\chi$.
3) If $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ and $L \in \text{sub}(\mathbf{m})$ <u>then</u> $\mathbb{P}_{\mathbf{m}}(L), \mathbb{P}_{\mathbf{n}}(L)$ have the same set of elements.
4) If $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ <u>then</u> $\text{cmp}(\mathbf{m}) \subseteq \text{cmp}(\mathbf{n})$ moreover $L \in \text{cmp}(\mathbf{m})$ <u>iff</u> $L \in \text{cmp}(\mathbf{n}) \wedge L \subseteq L_{\mathbf{m}}$.
5) Like (4) for $\text{sub}(-), \text{Cmp}(-), \text{Sub}(-), \text{sub}_*(-), \text{Cmp}_*(-), \text{Sub}_*(-)$.
6) If $\chi = \chi^{\lambda_0}, \mathbf{m} \in \mathbf{M}_\chi, L \in \text{Sub}(\mathbf{m}), \mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ and $\mathbb{P}_{\mathbf{m}}[L] \neq \mathbb{P}_{\mathbf{n}}[L]$ <u>then</u> we can find such $\mathbf{n}$ in $\mathbf{M}_\chi$.
7) If $\mathbf{m} \in \mathbf{M}$ and $t \in L_{\mathbf{m}}$ <u>then</u> $t \in L$ for some $L \in \text{cmp}(\mathbf{m})$.
8) If $\mathbf{m} \in \mathbf{M}$ and $L_1 \subseteq L_{\mathbf{m}}$ has cardinality $\leq \lambda_0$ then $L_1 \subseteq L_2$ for some $L_2 \in \text{Cmp}_*(\mathbf{m})$.
9) If $L_1 \in \text{sub}(\mathbf{m})$ then $L_1 = \cup\{L_1 \cap L_{\mathbf{m},\leq t} : t \in L_1\}$ and $t \in L_1 \Rightarrow L_1 \cap L_{\mathbf{m} \leq t} \in \text{sub}_*(\mathbf{m})$.*

Proof. Should be clear. $\square_{5.6}$

[20]version A: s_ε of $L_{\mathbf{m}}$ for $\varepsilon < \varepsilon(*) < \lambda$ and λ-Borel function $\mathbf{B}$ from $\varepsilon(*)_4$ to 2 and $L = \{t \in L' : \mathbf{B}(\ldots, \mathfrak{A}_I(s,t), \ldots) = 1$ where $\mathfrak{A}_I(s,t)$ is: 0 if $s <_I t, 1$ if $s = t, 2$ if $t <_J s$ and 3 otherwise.

{c35}

Observation 5.7. *1) If $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$ is $\le_{\mathbf{M}}$-increasing <u>then</u> $\mathbf{m}_\delta := \cup\{\mathbf{m}_\alpha : \alpha < \delta\}$ is a $\le_{\mathbf{M}}$-upper bound of $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$.*
2) Moreover in part (1):

- (α) • $\mathrm{cmp}(\mathbf{m}_\delta) = \cup\{\mathrm{cmp}(\mathbf{m}_\alpha) : \alpha < \delta\}$
  - • $\mathrm{sub}(\mathbf{m}_\delta) = \cup\{\mathrm{sub}(\mathbf{m}_\alpha) : \alpha < \delta\}$
  - $\mathrm{sub}_*(\mathbf{m}_\delta) = \cup\{\mathrm{sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$
- (β) *if* $\mathrm{cf}(\lambda) \ge \lambda_0^+$ *then*
  - • $\mathrm{Cmp}_*(\mathbf{m}_\delta) = \cup\{\mathrm{Cmp}(\mathbf{m}_\alpha) : \alpha < \delta\}$
  - • $\mathrm{Sub}_*(\mathbf{m}_\delta) = \cup\{\mathrm{Sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$.

3) If $\gamma_1 < \gamma_2$ then $\mathbf{M}^{\mathrm{ec}}_{\gamma_1} \subseteq \mathbf{M}^{\mathrm{ec}}_{\gamma_2}$.

Proof. 1) Easy, remembering that $t \in \mathbf{m}_\alpha \wedge \alpha \le \beta < \delta \Rightarrow t/E'_{\mathbf{m}_\beta} = t/E'_{\mathbf{m}_\beta}$.
2) Easy.
3) Easy, too. $\square_{5.7}$

{c38}

Claim 5.8. *1) In Definition 5.5(1) we can consider only $L \in \mathrm{Sub}_*(\mathbf{m})$ such that $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$.* {c30}
2) So if $\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\infty$ and $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ <u>then</u> $\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$, that is $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \lessdot \mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$ and $L \in \mathrm{Sub}(\mathbf{m}) \Rightarrow \mathbb{P}_{\mathbf{m}}[L] \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{n}}]$.
3) For $\mathbf{m} \in \mathbf{M}$ we have: $\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\infty$ iff $\mathbf{m} \in \mathbf{M}_{\mathrm{fc}}$, i.e. for every $\mathbf{n}, \mathbf{m} \le_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$.
4) $\mathrm{Cmp}_*(\mathbf{m})$ has cardinality $\le |L_{\mathbf{m}}|^{\le \lambda_0}$.
5) $\mathrm{Sub}_*(\mathbf{m})$ has cardinality $\le |L_{\mathbf{m}}|^{\lambda_0}$.
6) $|\mathrm{cmp}(\mathbf{m})| \le |L_{\mathbf{m}}| + 1$ and $|\mathrm{sub}_*(\mathbf{m})| \le |L_{\mathbf{m}}|^{\lambda_0}$.

Proof. 1) Let $\mathbf{m} \in \mathbf{M}$ satisfies the weaker version, and let $L \in \mathrm{Sub}(\mathbf{m})$ satisfies $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$. By Definition 5.4(3)(c) and Observation 5.6(9) we can find $\langle L_\varepsilon, L^*_\varepsilon : \varepsilon < \zeta \rangle$ such that L be $\cup\{L_\varepsilon : \varepsilon < \zeta\}$ where $L_\varepsilon \subseteq L^*_\varepsilon \in \mathrm{Cmp}(\mathbf{m})$ and $L_\varepsilon \in \mathrm{sub}_*(\mathbf{m})$. {c32}

For every $u \in [\zeta]^{\le \lambda_0}$ let $L_u = \cup\{L_\varepsilon : \varepsilon \in u\}$ so clearly $\mathrm{dp}^*_{\mathbf{m}}(L_u) \le \mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$. Let $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$. For $u \in [\zeta]^{\le \lambda_0}$, for L_u the demand holds, i.e. $\mathbb{P}_{\mathbf{m}}[L_u] = \mathbb{P}_{\mathbf{n}}[L_u]$. Also if $u \subseteq v \in [\zeta]^{\le \lambda_0}$ then $\mathbb{P}_{\mathbf{m}}[L_u] \subseteq \mathbb{P}_{\mathbf{m}}[L_v]$.

Lastly, $\langle u : u \in [\zeta]^{\le \lambda_0} \rangle$ is λ_0^+-directed so by **??**(8) and the Definition 5.5(1) we are done. {c30}
2) As $L_{\mathbf{m}} \in \mathrm{Sub}(\mathbf{m})$ and $\mathrm{dp}^*_{\mathbf{m}}(L_m) < \infty$ by clause (d) Definition 2.7. {c4}
3) The implication $\Rightarrow$ holds by part (2). For the other implication assume $\mathbf{m} \notin \mathbf{M}^{\mathrm{ec}}_\infty$ so clause (*) there fails for some γ; this means that there are $L \in \mathrm{Sub}(\mathbf{m})$ and $\mathbf{n}$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ but $\mathbb{P}_{\mathbf{m}}[L] \ne \mathbb{P}_{\mathbf{n}}[L]$, by the definitions this implies $\neg(\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}})$ so we are done.
4) Note that $\{L \in \mathrm{cmp}(\mathbf{m}) : L \nsubseteq M_{\mathbf{m}}\} = \{t/E'_{\mathbf{m}} : t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}\}$ has cardinality $\le |L_{\mathbf{m}} \backslash M_{\mathbf{m}}| \le |L_{\mathbf{m}}|$ and $\{L \in \mathrm{cmp}(\mathbf{m}) : L \subseteq M_{\mathbf{m}}\} = L \subseteq M_{\mathbf{m}} : |L| \le \lambda_0\}$ has cardinality $\le |M_{\mathbf{m}}|^{\lambda_0} \subseteq |L_{\mathbf{m}}|^{\lambda_0}$.

5), 6) Similarly. $\square_{5.8}$

{c39}

Claim 5.9. *1) If $\gamma \le \infty, \langle \mathbf{m}^*_\alpha : \alpha < \delta \rangle$ is $\le_{\mathbf{M}}$-increasing, $\mathrm{cf}(\delta) > \lambda_0$ and $\alpha < \delta \Rightarrow \mathbf{m}^*_\alpha \in \mathbf{M}^{\mathrm{ec}}_\gamma$ <u>then</u> $\mathbf{m}_* := \cup\{\mathbf{m}^*_\alpha : \alpha < \delta\} \in \mathbf{M}^{\mathrm{ec}}_\gamma$.*
2) Similarly for λ_0^+-directed union.

Proof. 1) By 5.8(1), toward contradiction assume that $(\mathbf{n}, L)$ satisfies: {c38}

$(*)$ (a) $\mathbf{m}_* \le_{\mathbf{M}} \mathbf{n}$

(b) $L \in \mathrm{Sub}_*(\mathbf{m}_*)$

(c) $\mathrm{dp}^*_{\mathbf{m}_*}(L) < \gamma$

(d) $\mathbb{P}_{\mathbf{m}_*}[L] \ne \mathbb{P}_{\mathbf{n}}[L]$.

As $L \in \mathrm{Sub}_*(\mathbf{m}_*)$, we have $L \subseteq \cup\{L_\varepsilon : \varepsilon < \varepsilon(*)\}$ where $\varepsilon(*) < \lambda^*_0$ and $L_\varepsilon \in$ {c35} $\mathrm{sub}_*(\mathbf{m})$ for $\varepsilon < \varepsilon(*)$. But then $L_\varepsilon \in \mathrm{sub}_*(\mathbf{m}_{\alpha(\varepsilon)})$ for some $\alpha(\varepsilon) < \delta$ by 5.7(2)(α) but $\mathrm{cf}(\delta) > \lambda_0$ hence $\alpha(*) = \sup\{\alpha(\varepsilon) : \varepsilon < \varepsilon(*)\} < \delta$ so $L \in \mathrm{Sub}_*(\mathbf{m}_{\alpha(*)})$.

Now we have assumed $\mathrm{dp}^*_{\mathbf{m}_*}(L) < \gamma$ but $M_{\mathbf{m}_*} = M_{\mathbf{m}_{\alpha(*)}}$, so equivalently {c35} $\mathrm{dp}^*_{\mathbf{m}_{\alpha(*)}}(L) < \gamma$. By the claim assumption, $\mathbf{m}_\beta \in \mathbf{M}^{\mathrm{ec}}_\gamma$ and by 5.7(1) we have $\mathbf{m}_\beta \le_{\mathbf{M}} \mathbf{m}_*$.

As $\le_{\mathbf{M}}$ is a partial order on $\mathbf{M}$ and $\mathbf{m}_* \le_{\mathbf{M}} \mathbf{n}$ by $(*)(a)$ clearly $\mathbf{m}_{\alpha(*)} \le_{\mathbf{M}} \mathbf{n}$, but $\mathbf{m}_{\alpha(*)} \in M^{\mathrm{ec}}_\partial$ so together we deduce $\mathbb{P}_{\mathbf{m}_*}[L] = \mathbb{P}_{\mathbf{m}_\beta}[L] = \mathbb{P}_{\mathbf{n}}[L]$, so we are done. $\square_{5.9}$

{c41y}

Crucial Claim 5.10. *Let $\chi = \chi^{\lambda_0}$.*
1) If $\mathbf{m} \in \mathbf{M}$ of cardinality $\le \chi$ there is $\mathbf{n} \in \mathbf{M}^{\mathrm{ec}}_\infty$ of cardinality $\le \chi$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$.
2) If $\mathbf{m} \in \mathbf{M}_\chi$ and $\gamma < \chi^+$ <u>then</u> there is $\mathbf{n} \in \mathbf{M}^{\mathrm{ec}}_\gamma \cap \mathbf{M}_\chi$ such that $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$.

Proof. 1) By (2) using $\gamma = \mathrm{dp}^*_{\mathbf{m}}(M_{\mathbf{m}}) + 1 = \mathrm{dp}^*_{\mathbf{m}}(L_{\mathbf{m}}) + 1$ which necessarily is $< \|M_{\mathbf{m}}\|^+ \le \chi^+$.
{c30} 2) We prove the statement by induction on γ. Recalling Definition 5.5(1), in each {c38} case consider $L \in \mathrm{Sub}(\mathbf{m})$ which satisfies $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$ and by 5.8(1) without loss of generality $L \in \mathrm{Sub}_*(\mathbf{m})$.

<u>Case 1</u>: $\gamma = 0$.
There is no such relevant L.

<u>Case 2</u>: $\gamma = 1$.
So $\mathrm{dp}^*_{\mathbf{m}}(L) < \gamma$ means $\mathrm{dp}^*_{\mathbf{m}}(L) = 0$ which means $L \cap M_{\mathbf{m}} = \emptyset$. But then $\mathbf{n} = \mathbf{m}$ is as required because if $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}_1$ then $\{s \in L_{\mathbf{n}_1} : (\exists t \in L)(s \le_{\mathbf{n}_1} t)\} = L = \{s \in L_{\mathbf{m}} : (\exists t \in L)(s \le_{\mathbf{m}} t)\}$, i.e. L is an initial segment of both so $\mathbb{P}_{\mathbf{m},L} = \mathbb{P}_{\mathbf{n}_1,L}$ and {c13} the conclusion is obvious, or see 2.15.

<u>Case 3</u>: $\gamma = \beta + 2$.
We try to choose $\mathbf{m}_\varepsilon$ by induction on $\varepsilon \le \chi^+$ such that:

$(*)$ (a) $\mathbf{m}_\varepsilon \in \mathbf{M}_\chi$

(b) $\langle \mathbf{m}_\zeta : \zeta \le \varepsilon \rangle$ is $\le_{\mathbf{M}}$-increasing continuous

(c) $\mathbf{m}_0 = \mathbf{m}$

(d) if $\varepsilon = 2\zeta + 1$ then for some $L \in \mathrm{Sub}_*(\mathbf{m}_{2\zeta})$ we have
$\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \ne \mathbb{P}_{\mathbf{m}_\varepsilon}[L]$ and $\mathrm{dp}^*_{\mathbf{m}_{2\zeta}}(L) < \gamma$

(e) if $\varepsilon = 2\zeta + 2$ then $\mathbf{m}_\varepsilon \in \mathbf{M}^{\mathrm{ec}}_{\beta+1}$.

<u>Subcase 3A</u>: We succeed to carry the induction.

For each $\delta \in S^{\chi^+}_{\lambda_0^+} := \{\delta < \chi^+ : \mathrm{cf}(\delta) = \lambda_0^+\}$ (i.e. $\varepsilon = 2\delta + 1$ and note $2\delta = \delta$)
{c29} there is L_δ as in $(*)(d)$. Recalling Definition 5.4(3)(d), as $\mathrm{cf}(\delta) = \lambda_0^+ > \lambda_0 \geq$
{c35} $|\{t/E'_{\mathbf{m}_\delta} : t \in L_\delta \backslash M_{\mathbf{m}_\delta}\}|$ there is $\alpha(\delta) < \delta$ such that $L_\delta \subseteq L_{\mathbf{m}_{\alpha(\delta)}}$ hence by 5.7(2) + 5.8(5) for some stationary $S \subseteq S^{\chi^+}_{\lambda_0^+}$ and L_* we have $\delta \in S \Rightarrow L_\delta = L_* \wedge \alpha(\delta) = \alpha(*)$.
{c38} Without loss of generality $\alpha(*)$ is even and a successor ordinal. For $\delta \in S$, note that $\langle \mathbf{m}_{2\zeta+2} : \zeta < \delta\rangle$ is $\leq_{\mathbf{M}}$-increasing sequence of members of $\mathbf{M}^{\mathrm{ec}}_{\beta+1}$ with union
{c39} $\mathbf{m}_\delta$ (by clauses (e) and (b) of $(*)$) hence $\mathbf{m}_\delta \in \mathbf{M}^{\mathrm{ec}}_{\beta+1}$ by 5.9; recall $\mathrm{cf}(\delta) > \lambda_0$ and $M_{\mathbf{m}_\alpha} = M_{\mathbf{m}}$ for every α.

Now $\mathrm{dp}^*_{\mathbf{m}_\delta}(L_*) < \gamma$ and by the previous sentences necesarily $\mathrm{dp}^*_{\mathbf{m}_\delta}(L_*) = \beta + 1$ hence the set $\mathscr{Y} := \{t \in L_* \cap M_{\mathbf{m}} : \mathrm{dp}(t, M_{\mathbf{m}}) = \beta\}$ is not empty.

Let

$\boxplus_1$ (a) $L^*_0 = \{s \in L_* : \neg(\exists t \in \mathscr{Y})(t \leq s)\}$
(b) $L^*_1 = L^*_0 \cup \mathscr{Y}$
(c) $L^*_2 = L_*$.

Now

$\boxplus_2$ $\mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L^*_0]$ is equal to $\mathbb{P}_{\mathbf{m}_\alpha}[L^*_0]$ and to $\mathbb{P}_{\mathbf{m}_{\alpha+1}}[L^*_0]$ for $\alpha \in S \backslash \alpha(*)$ moreover for $\alpha \geq \alpha(*)$.

[Why? L^*_0 is an initial segment of L_* but $\delta \in S \Rightarrow L_* = L_\delta \in \mathrm{Sub}_*(\mathbf{m}_\delta)$, see above hence, see Definition 5.4(2)(b),(3)(c) we have $L^*_0 \in \mathrm{Sub}_*(\mathbf{m}_\alpha)$ for $\alpha \geq \alpha(*)$. We {c29}
are done as $\mathbf{m}_{\alpha(*)} \in \mathbf{M}^{\mathrm{ec}}_{\beta+1}$ by $(*)(e)$ and 5.5(1).] {c30}

$\boxplus_3$ (a) $L^*_{0,\alpha} = \{s \in L_{\mathbf{m}_\alpha} : (\exists t \in L^*_0)(s \leq t)\}$ or $(\exists t \in \mathscr{Y})(s \leq t)$
(b) $L^*_{1,\alpha} = L^*_{0,\alpha} \cup Y$
(c) $L^*_{2,\alpha} = L^0_\alpha \cup L_*$

$\boxplus_4$ (a) $\mathrm{dp}^*_{\mathbf{m}_\alpha}(L^*_{0,\alpha}) < \beta + 1$
(b) $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L^*_{0,\alpha}] : \alpha \in [\alpha(*), \chi^+)$ is $\lessdot$ increasing.

$\boxplus_5$ (a) $L^*_{1,\alpha} \subseteq L_{\mathbf{m}_\alpha}$ is downward closed in $L_{\mathbf{m}_\alpha}$
(b) $\langle \mathbb{P}_{\mathbf{m}_\alpha}(L^*_{1,\alpha}) : \alpha \in [\alpha(*), \chi^+)\rangle$ is $\subseteq$-increasing
(c) moreover $\leq_{\mathrm{ic}}$-increasing.

[Why? Should be clear.]

$\boxplus_6$ there is a club E of χ^+ such that $\delta \in E \cap S^{\chi^+}_{\lambda_0^+} \backslash \alpha(*) \wedge \delta \leq \alpha < \chi^+ \Rightarrow P_{\mathbf{m}_\gamma}(L^*_{1,\delta}) \lessdot \mathbb{P}_{\mathbf{m}_\alpha}(L^*_{1,\alpha})$.

[Why? By $\boxplus_5$ as every $\mathbb{P}_{\mathbf{m}_\alpha}(L^*_{1,\alpha})$ satisfies the λ-c.c., $\lambda_0 \geq \lambda$ and $\chi^{\lambda_0} = \chi$.]

Now

$\boxplus_7$ (a) $L^*_{2,\alpha}$ is a downward closed subset of $L_{\mathbf{m}_\alpha}$ for $\alpha \in [\alpha(*), \chi^+]$
(b) if $\alpha(*) \leq \alpha \leq \beta < \chi^+$ and $\mathbb{P}_{\mathbf{m}_\alpha}(L^*_{1,\alpha}) \lessdot \mathbb{P}_{\mathbf{m}_\beta}(L^*_{1,\beta})$ then
(α) $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) = \mathbb{P}_{\mathbf{m}_\beta}(L_{2,\alpha}) \lessdot \mathbb{P}_{\mathbf{m}_\beta}$
(β) $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) \lessdot \mathbb{P}_{\mathbf{m}}(L_{2,\beta})$
(γ) $\mathbb{P}_{\mathbf{m}_\alpha}[L_{2,\alpha}] = \mathbb{P}_{\mathbf{m}_\beta}[L_{2,\beta}]$
(δ) $\mathbb{P}_{\mathbf{m}_\alpha}[L^*_1] = \mathbb{P}_{\mathbf{m}_\beta}[L^*_1]$.

[Why? Clause (a) by inspection. Clause $(b)(\alpha)$, the equality is obvious, the $\lessdot \mathbb{P}_{\mathbf{m}_\beta}$ holds by clause (a). Now we prove clause $(b)(\beta)$, toward this

- $\bullet_1$ $t \in L_{2,\beta} \backslash L_{1,\alpha} \Leftrightarrow t \in Y \Leftrightarrow t \in L_{2,\alpha} \backslash L_{1,\alpha}$ (why? check)
- $\bullet_2$ $t \in L_{2,\alpha} \backslash L_{1,\alpha} \Rightarrow u_{\mathbf{m}_\beta,t} \subseteq L_{1,\alpha}$ (why? check)
- $\bullet_3$ $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) \subseteq \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\beta})$ (by assumption of (b)).

Next assume $p_* \in \mathbb{P}_{2,\beta}(L_{2,\beta})$ then $q_* = p_* \restriction L_{2,\alpha} \in \mathbb{P}_{2,\alpha}(L_{1,\alpha})$ and also $q' = p_\beta \restriction L_{1,\alpha} \in \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}), p' = p \restriction L_{1,\beta} \in \mathbb{P}_{\mathbf{m}_\beta})$. Also $q' \le q'' \in \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}) \Rightarrow q'' \cup (p \restriction (L_{1,\beta} \backslash L_{1,\alpha})$ is a common upper bound of q'', p' in $\mathbb{P}_{\mathbf{m}_\beta}(L_{1,\beta})$. Hence $q_* = q' \cup (p \restriction y)$ satisfies $q_* \in P_{\mathbf{m}_\alpha}(L_{2,\alpha})$ and $q_* \le q'_* \in P_{\mathbf{m}_\alpha}(L_{2,\alpha}) \Rightarrow q'_* \cup (p \restriction (L_{1,\beta} \backslash L_{1,\alpha})$ is a common upper bound of q'_*, p in $\mathbb{P}_{\mathbf{m}_\beta}(L_{2,\beta})$.

Together with $\bullet_3$ this paragraph shows

- $\bullet_4$ $\mathbb{P}_{\mathbf{m}_\alpha}[L_{2,\alpha}] \lessdot \mathbb{P}_{\mathbf{m}_\beta}[L_{2,\beta}]$ that is clause $(b)(\beta)$.

Clause $(b)(\gamma)$ holds by $(b)(\beta)$ and (a) and clause $(b)(\beta)$ follows as $L^*_2 \subseteq L^*_{2,\alpha} \subseteq L^*_{2,\alpha}$ so $\boxplus_7$ holds indeed.

Now choose $\delta \in E \cap S \backslash \alpha(*) \subseteq S^{\chi^+}_{\lambda_0^+}$ and we get a contradiction to the choice of $L_\delta = L_*$ so we are done with subcase 3A.

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$(*)_2$ for $\alpha \in \chi^+ \backslash \alpha(*)$ and $t \in \mathscr{Y}$ let $\underset{\sim}{\mathbb{Q}}^\alpha_{\bar\theta,t}$ be the $\mathbb{P}^{\mathbf{m}_\alpha}$-name of $\mathbb{Q}_{\bar\theta}^{\mathbf{V}[\langle \eta_\gamma : \gamma \in u(t) \cap L_* \rangle]}$ where $u(t) = u_{\mathbf{m}_{\alpha(*)},t}$

$(*)_3$ (a) $\underset{\sim}{\mathbb{Q}}^\alpha_{\bar\theta,t}$ does not depend on α, also the order on it does not depend on α

(b) $\Vdash_{\mathbb{P}_{\mathbf{m}_\alpha}}$ "$\underset{\sim}{\mathbb{Q}}^\alpha_{\bar\theta,t} \le_{\rm ic} \underset{\sim}{\mathbb{Q}}_{\bar\theta}$, i.e. $\underset{\sim}{\mathbb{Q}}^\alpha_{\bar\theta,t} \subseteq \mathbb{Q}_{\bar\theta}$ as quasi orders and any $p_1, p_2 \in \underset{\sim}{Q}^\alpha_{\bar\theta,t}$ compatible in $\mathbb{Q}_{\bar\theta}$ are compatible in $\mathbb{Q}^\alpha_{\bar\theta,t}$"

(c) $\mathbb{P}_{\mathbf{m}_\alpha}[L^*_1]/\mathbb{P}_{\mathbf{m}_\alpha}[L^*_0]$ is the same for $\alpha \in [\alpha(*), \chi^+)$

$(*)_4$ define $\mathbb{I}_\alpha = \{\underset{\sim}{\mathscr{I}} : \underset{\sim}{\mathscr{I}}$ is a $\mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L^*_0]$-name of a maximal antichain of $\mathbb{P}_{\mathbf{m}_\alpha}[L^*_1]/\mathbb{P}_{\mathbf{m}_\alpha}[L^*_0]\}$.

Let $L^*_{0,\alpha} = \{s \in L_{\mathbf{m}_\alpha} : (\exists t \in L^*_0)(s \le t)$ or $(\exists t \in \mathscr{Y})(s < t)\}$ and $S_* = \{\alpha < \chi^+ : (\exists \varepsilon)(\alpha = 2\varepsilon + 2)$ or $\mathrm{cf}(\alpha) = \lambda^+_0\}$

$(*)_5$ (a) $\mathrm{dp}^*_{\mathbf{m}_\alpha}(L^*_{0,\alpha}) < \beta + 1$

(b) $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L^*_{0,\alpha}] : \alpha \in S_* \backslash \alpha(*) \subseteq [\alpha(*), \chi^+)$ and $\alpha = (\exists \varepsilon)(\alpha = 2\varepsilon + 2)$ or $\alpha \in S\rangle$ is $\lessdot$ increasing.

[Why? For clause (a):

if $s \in L_{0,\alpha} \cap M_{\mathbf{m}_\alpha}$ then for some $t \in Y$ we have $s <_{L_\alpha} t$ hence $\mathrm{dp}(s, M_{\mathbf{m}}) <$
{c2} $\mathrm{dp}(t, M_{\mathbf{m}}) = \beta$ hence $\mathrm{dp}(s, M_{\mathbf{m}}) \le \beta$, recalling Definition 2.5(1)(β) hence we have $\mathrm{dp}^*_{\mathbf{m}_\alpha}(L_{0,\alpha}) = \sup\{\mathrm{dp}(s, M_{\mathbf{m}}) + 1, s \in L \cap M_{\mathbf{m}}\} \le \beta$ as promised.

For clause (b), by the induction hypothesis for $\alpha_1 < \alpha_2$ from $\{2\varepsilon + 2 : \varepsilon < \chi^+\}$ we have $\mathbb{P}_{\mathbf{m}_{\alpha_1}}[L^*_{0,\alpha_1}] = \mathbb{P}_{\mathbf{m}_{\alpha_2}}[L^*_{0,\alpha_1}] \lessdot \mathbb{P}_{\mathbf{m}_{\alpha_2}}[L^*_{0,\alpha_2}]$. This implies the result also
{c39} when we allow $\mathrm{cf}(\alpha_\ell) = \lambda$ by 5.9.]

So necessarily

$(*)_6$ $\mathbb{I}_\alpha$ is constant for $\alpha < \chi^+$ large enough.

Hence recalling $(*)_3$ for some $\beta(*) \in (\alpha(*), \chi^+)$ we have

$(*)_7$ $\langle \mathbb{P}_{\mathbf{m}_\beta}[L^*_1] : \alpha < S_* \rangle$ is constant for $\beta \in [\beta(*), \chi^+)$.

But this implies the result for L^*_2 as the $t \in L^*_2 \backslash L^*_1$ does not "have memory outside L^*_1", i.e. $u_{\mathbf{m}_{\beta(*)},t} \subseteq L^*_1$ and is constant, so we are done.

<u>Subcase 3B</u>: We are stuck in ε, i.e. cannot define $\mathbf{m}_\varepsilon$.

Now $\varepsilon = 0$ is impossible. If ε is a limit, let $\mathbf{m}_\varepsilon = \cup\{\mathbf{m}_\zeta : \zeta < \varepsilon\}$ so $\zeta < \varepsilon \Rightarrow \mathbf{m}_\zeta \leq_{\mathbf{N}} \mathbf{m}_\varepsilon \in \mathbf{M}_\chi$, contradiction. Also $\varepsilon = \zeta + 1, \zeta = 0$ is impossible.

If $\varepsilon = 2\zeta + 1$ so then clause (d) of $(*)$ is relevant so there is no $\mathbf{n} \in \mathbf{M}_\chi$ such that $\mathbf{m}_{2\zeta} \leq_{\mathbf{N}} \mathbf{n}$ and $L \in \mathrm{Sub}_*(\mathbf{m}_{2\zeta})$ such that $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \neq \mathbb{P}_{\mathbf{m}_{2\zeta}}[L]$ and $\mathrm{dp}^*_{\mathbf{m}_{2\zeta}}(L) < \gamma$. By 5.8(1) this applies also to $L \in \mathrm{Sub}(\mathbf{m}_{2\zeta})$. By claim 5.6(7) this applies also to $\mathbf{n} \in \mathbf{M}$. Together so by 5.8(4) we can deduce that $\mathbf{m}_{2\zeta} \in \mathbf{M}^{\mathrm{ec}}_\gamma$ so we are done. {c38} {c38}

Lastly, if $\varepsilon = 2\zeta + 2$ so clause (e) of $(*)$ applies, then use the induction hypothesis for $\beta + 1$ to choose $\mathbf{m}_\varepsilon$.

<u>Case 4</u>: γ is limit.

Let $\delta < \chi^+$ be divisible by γ and of cofinality λ^+_0, recalling $\lambda^+_0 \leq \chi$.

We choose $\mathbf{m}_\alpha$ for $\alpha \leq \delta^+_0$ such that:

- $\mathbf{m}_\alpha \in \mathbf{M}_\chi$
- $\alpha = 0 \Rightarrow \mathbf{m}_\alpha = \mathbf{m}$
- $\mathbf{m}_\alpha$ is $\leq_{\mathbf{M}}$-increasing continuous with α
- if $\varepsilon < \gamma$ and $\alpha = \varepsilon + 1 \mod \gamma$ then $\mathbf{m}_\alpha \in \mathbf{M}^{\mathrm{ec}}_\varepsilon$.

Clearly possible: if $\alpha = 0$ trivial, if α is a successor ordinal use the induction hypothesis and if α is a limit recall $\delta < \chi^+$ and use 5.9. Now $\mathbf{m}_\delta$ is as required. {c39}

Why? Toward contradiction assume $\mathbf{m}_\delta \leq_{\mathbf{M}} \mathbf{n}$ and $L \in \mathrm{Sub}_*(\mathbf{m}_{\lambda^+_0})$ satisfies $\gamma > \mathrm{dp}_{\mathbf{m}_\delta}(L)$; let $\varepsilon := \mathrm{dp}_{\mathbf{m}_\alpha}(L)$. By 5.7(2)($\beta$) there is $\alpha < \delta$ such that $L \in \mathrm{Sub}_*(\mathbf{m}_\alpha)$ and without loss of generality $\alpha = \varepsilon + 1 \mod \gamma$, so we use the choice of $\mathbf{m}_{\alpha+1}$ to get contradiction. {c35}

So $\mathbf{m}_\delta$ is as required by 5.9. {c39}

<u>Case 5</u>: None of the above.

So $\gamma = \beta + 1, \beta$ a limit ordinal.

For $\alpha \leq \beta$ let $L_\alpha = \{s \in L$: there is no $t \leq s$ satisfying $t \in M_{\mathbf{m}}$ and $\mathrm{dp}(t, M_{\mathbf{m}}) \geq \alpha\}$. So $\langle L_\alpha : \alpha \leq \beta \rangle$ is $\subseteq$-increasing continuous and $L \backslash L^*_\beta \subseteq L_{\mathbf{m}} \backslash M_{\mathbf{m}}$. As in Case 3 we first deal with L_β, and it is straight, and finish as there. $\square_{2.26}$

{c46}

Claim 5.11. *Assume* $\mathbf{m} \in \mathbf{M}^{\mathrm{ec}}_\infty, M \subseteq M_{\mathbf{m}}$ *and* $\mathbf{n}$ *is defined just like* $\mathbf{m}$ *except that* $M_{\mathbf{n}} = M$ *and* $E_{\mathbf{n}} = \{(s,t) : s,t \in L_{\mathbf{m}}$ *but* $\{s,t\} \nsubseteq M_{\mathbf{n}}\}$*; so by 2.8(5) clearly* $\mathbf{n}$ *is equivalent to* $\mathbf{m}$ *and* $M_{\mathbf{n}} \subseteq M_{\mathbf{m}}$ *but we have to change the context by replacing* λ_0 *by any* $\lambda'_0 \geq \lambda_0 + \|L_{\mathbf{m}}\|$. {c5}

<u>*Then*</u> $\mathbf{n} \in \mathbf{M}^{\mathrm{fc}}[\lambda'_0]$, *see Definition 5.5(1).* {c30}

Proof. Assume that

$(*)_1$ $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1$.

We now define $\mathbf{m}_1 \in \mathbf{M}$ by

$(*)_2$ (a) $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$

(b) $\bar u_{\mathbf{m}_1} = \bar u_{\mathbf{n}_1}$

(c) $M_{\mathbf{m}_1} = M_{\mathbf{m}}$

(d) $E_{\mathbf{m}_1} = E_{\mathbf{m}} \cup (E_{\mathbf{n}_1} \restriction (L_{\mathbf{n}_1} \backslash (L_{\mathbf{n}} \backslash M))$.

Clearly

$(*)_3$ (a) indeed $\mathbf{m}_1 \in \mathbf{M}$

(b) $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1$

(c) $\mathbf{m}_1, \mathbf{n}_1$ are equivalent.

But $\mathbf{m} \in \mathbf{M}_{\rm fc}$ hence

$(*)_4$ $\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{m}_1}$.

{c11} But $\mathbb{P}_{\mathbf{n}} = \mathbb{P}_{\mathbf{m}}$ and $\mathbb{P}_{\mathbf{n}_1} = \mathbb{P}_{\mathbf{n}}$ by 2.13(7) as $\mathbf{n}, \mathbf{m}$ are equivalent and $\mathbf{n}_1, \mathbf{m}_1$ are equivalent, so

$(*)_5$ $\mathbb{P}_{\mathbf{n}} \lessdot \mathbb{P}_{\mathbf{n}_1}$.

So we are done. $\square_{5.11}$

{c49y} **The Uniqueness Claim 5.12.** *There is an isomorphism from $\mathbb{P}_{\mathbf{m}_1}[M_1]$ onto*
{c19} *$\mathbb{P}_{\mathbf{m}_2}[M_2]$ which (recalling Definition* **??***) maps $p^*_{t,\eta}$ to $p^*_{h(t),\eta}$ for $t \in M_1, \eta \in \cup\{\prod\limits_{\varepsilon<\zeta} \theta_\varepsilon : \zeta < \lambda\}$ <u>when</u>:*

$\boxplus$ (a) *$\mathbf{m}_\ell \in \mathbf{M}^{\rm ec}_\infty$ for $\ell = 1,2$*

(b) *$M_\ell \subseteq M_{\mathbf{m}_\ell}$ for $\ell = 1,2$*

(c) *h is an isomorphism from $\mathbf{m}_1 \restriction M_1$ onto $\mathbf{m}_2 \restriction M_2$.*

Proof. By renaming without loss of generality $M_1 = M_2$ call it M and h is the
{c46} identity and $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$. For $\ell = 1,2$ let $\mathbf{m}'_\ell$ be like $\mathbf{n}$ in 5.11 with $\mathbf{m}_\ell, M_\ell$ here standing for $\mathbf{m}, M$ there so

$(*)_1$ $\mathbf{m}'_\ell \in \mathbf{M}_{\rm fc}[\lambda'_0]$.

We define $\mathbf{m}$ by:

$(*)_2$ (a) $s \in L_{\mathbf{m}}$ iff $s \in L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$

(b) $r \le_{\mathbf{m}} t$ <u>iff</u> $r \le_{\mathbf{m}_1} t$ or $r \le_{\mathbf{m}_2} t$ or $(\exists s \in M)(r \le_{\mathbf{m}_1} s \le_{\mathbf{m}_2} t)$ or $(\exists s \in M)(r \le_{\mathbf{m}_2} s \le_{\mathbf{m}_1} t)$

(c) $u_{\mathbf{m},t}$ is $u_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \backslash M$,
is $u_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_2} \backslash M$ and
is $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$ if $t \in M$

(d) $M_{\mathbf{m}} = M$

(e) $E_{\mathbf{m}} = E'_{\mathbf{m}_1} \cup E'_{\mathbf{m}_2}$.

{c46} For $\ell = 1,2, \mathbb{P}_{\mathbf{m}_\ell}[M] = \mathbb{P}_{\mathbf{m}'_\ell}[M]$ by 5.11 and clearly $\mathbf{m}'_\ell \le_{\mathbf{M}} \mathbf{m}$, but $\mathbf{m}'_\ell \in \mathbf{M}_{\rm fc}[\lambda'_0]$,
{c46} by 5.11, hence $\mathbb{P}_{\mathbf{m}'_\ell}[M] = \mathbb{P}_{\mathbf{m}}[M]$. Together $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}'_1}[M] = \mathbb{P}_{\mathbf{m}}[M] = \mathbb{P}_{\mathbf{m}'_2}[M] = \mathbb{P}_{\mathbf{m}_2}[M]$ as required. $\square_{5.11}$

{c51y} **Conclusion 5.13.** *For every ordinal δ_* there is $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\eta}_\alpha : \alpha \le \delta_* \rangle$ such that*

(A) (a) *$\langle \mathbb{P}_\alpha : \alpha \le \delta_* \rangle$ is $\lessdot$-increasing*

(b) $\underset{\sim}{\eta}_\alpha$ *is a* $\mathbb{P}_{\alpha+1}$*-name of a member of* $\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ *which dominates* $(\prod\limits_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_\alpha]}$

(c) $\underset{\sim}{\eta}_\alpha$ *is a generic for* $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$*, moreover* $\langle \underset{\sim}{\eta}_\beta : \beta < \alpha\rangle$ *is a generic for* $\mathbb{P}_\alpha$

(d) *every* $p \in \mathbb{P}_\alpha$ *is from* $\mathbb{L}_{\lambda^+}(Y_{<\alpha}, \mathbb{P}_\alpha)$ *when* $Y_{<\alpha}$ *is as in Definition* **??** {c19}

(e) $\mathbb{P}_\alpha$ *is* λ*-strategically complete and* λ^+*-c.c.*

(f) *if* $\delta \le \delta_*$ *has cofinality* $> \lambda$ *<u>then</u>* $\mathbb{P}_\delta = \cup\{\mathbb{P}_\alpha : \alpha < \delta\}$

(g) $\mathbb{P}_{\delta_*}$ *has cardinality* $|\delta_*|^\lambda$.

(B) *if* $\mathscr{U} \subseteq \delta_*$ *<u>then</u> the complete subforcing generated by* $\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}\rangle$ *is isomorphic to* $\mathbb{P}_{\mathrm{otp}(\mathscr{U})}$

(C) *if* $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$ *is generic over* $\mathbf{V}$ *and* $\eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ *for* $\alpha < \delta_*$ *and* $\eta'_\alpha \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$ *for* $\alpha < \delta_*$ *and* $\{(\alpha, \varepsilon) : \alpha < \delta_*, \varepsilon < \lambda$ *and* $\eta'_\alpha(\varepsilon) \neq \eta_\alpha(\varepsilon)\}$ *has cardinality* $< \lambda$ *<u>then</u> also* $\langle \eta'_\alpha : \alpha < \delta_*\rangle$ *is a generic for* $\mathbb{P}_{\delta_*}$*, determining a different* $\mathbf{G}'$ *but* $\mathbf{V}[G'] = \mathbf{V}[\mathbf{G}]$.

Proof. We define $\mathbf{m} \in \mathbf{M}$ by:

$(*)$ (a) $L_{\mathbf{m}} = \delta_*$

(b) $M_{\mathbf{m}} = \delta_*$

(c) $u_{\mathbf{m},\alpha} = \alpha$ for $\alpha < \delta_*$

(d) $E_{\mathbf{m}} = \emptyset$.

It is easy to check that indeed $\mathbf{m} \in \mathbf{M}$. So by 2.26(1) there is $\mathbf{n} \in \mathbf{M}^{\mathrm{ec}}_\infty$ such that {c41}
$\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$. Let $\mathbb{P}_\alpha := \mathbb{P}_{\mathbf{n}}[\{\beta : \beta < \alpha\}]$, so by **??** we are done. $\square_{3.12}$ {c33}

Similarly we can prove

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