

ON  $\text{CON}(\mathfrak{d}_\lambda > \text{COV}_\lambda(\text{MEAGRE}))$   
SH945

SAHARON SHELAH

ABSTRACT. We prove the consistency of: for suitable strongly inaccessible cardinal  $\lambda$  the dominating number, i.e., the cofinality of  ${}^\lambda\lambda$ , is strictly bigger than  $\text{cov}_\lambda(\text{meagre})$ , i.e. the minimal number of nowhere dense subsets of  ${}^\lambda 2$  needed to cover it. This answers a question of Matet.

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## § 0. INTRODUCTION

Cardinal invariants on the continuum have a long tradition of research. For a topologist, it can be viewed as investigating the space  $\beta(\omega)$ , the Stone Čech compactification of  $\omega$ . This point of view is taken, for example, in the celebrated paper of van Douwen [vD84]. From the set theoretic perspective see the recent excellent surveys, Blass [Bla], Bartoszyński [Bar10].

For set theorists, it is interesting to check the relationship between the relevant cardinal invariants. In this context, it is natural to generalize the problems to higher cardinals, above  $\aleph_0$ . One finds out, very soon, that for the class of (strongly) inaccessible cardinals, the generalizations are more reasonable and have more affinity to the  $\aleph_0$  case. See Landver [Lan92], Cummings-Shelah [CuSh:541]

We shall define three cardinal invariants (but the paper deals, actually, just with two of them):

{z1}

**Definition 0.1.** The bounding and dominating numbers.

Let  $\lambda$  be an inaccessible cardinal.

Let  $f, g \in {}^\lambda\lambda$

- (a)  $f \leq^* g$  if  $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$
- (b)  $A \subseteq {}^\lambda\lambda$  is unbounded if there is no  $h \in {}^\lambda\lambda$  so that  $f \in A \Rightarrow f \leq^* h$
- (c)  $A \subseteq {}^\lambda\lambda$  is dominating when for every  $f \in {}^\lambda\lambda$  there exists  $g \in A$  so that  $f \leq^* g$
- (d) the bounding number for  $\lambda$ , denoted by  $\mathfrak{b}_\lambda$ , is  $\min\{|A| : A \text{ is unbounded in } {}^\lambda\lambda\}$
- (e) the dominating number for  $\lambda$ , denoted by  $\mathfrak{d}_\lambda$ , is  $\min\{|A| : A \text{ is dominating in } {}^\lambda\lambda\}$ .

Notice that the usual definitions of  $\mathfrak{b}$  and  $\mathfrak{d}$  are  $\mathfrak{b}_{\aleph_0}$  and  $\mathfrak{d}_{\aleph_0}$  according to Definition 0.1. The definition of  $\text{cov}_\lambda(\text{meagre})$  involves some topology.

{cov.1}

**Definition 0.2.** The meagre covering number.

Let  $\lambda$  be a regular cardinal

- (a)  ${}^\lambda 2$  is the space of functions from  $\lambda$  into 2
- (b)  $({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$ , for  $\nu \in \bigcup_{\alpha < \lambda} {}^\alpha 2$
- (c)  $\mathcal{U} \subseteq {}^\lambda 2$  is open in the topology  $({}^\lambda 2)_{< \lambda}$ , iff for every  $\eta \in \mathcal{U}$  there exists  $i < \lambda$  so that  $({}^\lambda 2)^{[\eta \upharpoonright i]} \subseteq \mathcal{U}$
- (d)  $\text{cov}_\lambda(\text{meagre})$  is the minimal cardinality of a family of meagre subsets of  $({}^\lambda 2)_{< \lambda}$ , which covers this space.

The paper deals with the relationship between  $\mathfrak{d}_\lambda$  and  $\text{cov}_\lambda(\text{meagre})$ . Matet asked (a personal communication) whether  $\mathfrak{d}_\lambda \leq \text{cov}_\lambda(\text{meagre})$  is provable in ZFC. We give here a negative answer.

For  $\lambda$  a supercompact cardinal and  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , we force large  $\mathfrak{d}_\lambda$  i.e.,  $\mathfrak{d}_\lambda = \mu$  and small covering number (i.e.,  $\text{cov}_\lambda(\text{meagre}) = \kappa$ ). A similar result should hold also for a wider class of cardinals and we intend to return to this subject.

A point which in a previous version was just a step along the way the referee asked to justify fully, becomes a major point to which §2 - §4 are dedicated. A

posteriori the point is that the parallel case for  $\lambda = \aleph_0$ , such claim is true. In fact, by Judah-Shelah [JdSh:292], if  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$  is FS iteration of Suslin-c.c.c. forcing notion,  $\mathbb{Q}_\beta$  with the generic  $\eta_\beta \in {}^\omega\omega$  and for notational transparency, its definition is with no parameter and  $\zeta : \beta(*) \rightarrow \alpha(*)$  is increasing and  $\mathbb{P} = \langle \mathbb{P}'_\alpha, \mathbb{Q}'_\beta : \alpha \leq \beta(*), \beta < \beta(*) \rangle$  is FS iteration,  $\mathbb{Q}'_{\zeta(\beta)}$  defined exactly as  $\mathbb{Q}_\beta$  but now in  $\mathbf{V}^{\mathbb{P}'_\beta}$  then  $\Vdash_{\mathbb{P}'_{\alpha(*)}} \langle \eta_{\zeta(\beta)} : \beta < \beta(*) \rangle$  is generic for  $\mathbb{P}_{\beta(*)}$  over  $\mathbf{V}$ .

Now this is not clear to us for  $(< \lambda)$ -support iteration of  $(< \lambda)$ -strategically complete forcing notions. The solution is essentially to change the iteration: we use a “quite generic”  $(< \lambda)$ -support iteration which “include” the one we like and use the complete subforcing it generates. Here we do only what is needed. On the general case we intend to continue in a work with H. Horowitz.

We try to use standard notation. We use  $\theta, \kappa, \lambda, \mu, \chi$  for cardinals and  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$  for ordinals. We use also  $i$  and  $j$  as ordinals. We adopt the Cohen convention that  $p \leq q$  means that  $q$  gives more information, in forcing notions. The symbol  $\triangleleft$  is preserved for “being an initial segment”. Also recall  ${}^B A = \{f : f \text{ a function from } B \text{ to } A\}$  and let  ${}^{\alpha>} A = \cup \{{}^\beta A : \beta < \alpha\}$ , some prefer  ${}^{<\alpha} A$ , but  ${}^{\alpha>} A$  is used systematically in the author’s papers. Lastly,  $J_\lambda^{\text{bd}}$  denotes the ideal of the bounded subsets of  $\lambda$ .

The picture of cardinal invariants related to uncountable  $\lambda$  is related but usually quite different than the one for  $\aleph_0$ , they are more similar if  $\kappa$  is “large” enough, mainly strongly inaccessible.

Let us sketch some known results. These results are related to the inequality number and the covering number for category. Recall:

{z14}

**Definition 0.3.** The inequality number.

Let  $\kappa$  be an infinite cardinal. The inequality number of  $\kappa, \mathfrak{e}_\kappa$ , is the minimal cardinal  $\lambda$  such that there is a set  $\mathcal{F} \subseteq {}^\kappa \kappa$  of cardinality  $\lambda$  such that there is no  $g \in {}^\lambda \lambda$  satisfying  $(\forall f \in \mathcal{F})(\exists^\kappa \alpha < \lambda)(f(\alpha) = g(\alpha))$ .

For  $\kappa = \aleph_0, \mathfrak{e}_\kappa = \text{cov}_{\aleph_0}(\text{meagre})$ ; see Bartoszyński (in [Bar87]) and Miller (in [Mil82]).

Now

- (a) the statement  $\mathfrak{e}_\kappa = \text{cov}_\kappa(\text{meagre})$  is valid for  $\kappa > \aleph_0$ , in the case that  $\kappa$  is strongly inaccessible, by [Lan92]. But if  $\kappa$  is a successor cardinal, it may fail
- (b) if  $\mathfrak{d}_\kappa$  is only finitely many cardinals away from  $\kappa$ , then  $\mathfrak{e}_\kappa = \mathfrak{d}_\kappa$ . This can be found in Matet-Shelah [MtSh:804]
- (c) if  $\kappa < \kappa^{<\kappa}$ , then  $\text{cov}_\kappa(\mathcal{M}) = \kappa^+$ . This is due to Landver (in [Lan92])
- (d) it is consistent to get (a) and (b) together, so that  $\text{cov}_\kappa(\text{meagre}) < \mathfrak{e}_\kappa$ . This follows from Cummings-Shelah (in [CuSh:541]).

We intend also to address:

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**Problem 0.4.** Can we replace “super-compact” by “strongly inaccessible”?

{z16}

**Problem 0.5.** 1) Can we prove the consistency of  $\text{cov}_\lambda(\text{meagre}) < \mathfrak{b}_\lambda$ ?

2) For  $\lambda$  strongly inaccessible (or just Laver indestructible super-compact) is there a non-trivial  $\lambda^+$ -c.c.  $(< \lambda)$ -strategically complete forcing notion  $\mathbb{Q}$  which is  ${}^\lambda \lambda$ -bounding?

We thank the referee, Shimoni Garti and Haim Horowitz for helpful comments. We say more in subsequent works [Sh:1004] and in preparation [Sh:F1199] and a work of Horowitz and the author on generalizing §2,§3.

{z20}

**Definition 0.6.** Fix  $\lambda = \lambda^{<\lambda}$ , the forcing  $\mathbb{Q} = \mathbb{Q}_\lambda^{\text{dom}}$  is defined by:

- ( $\alpha$ )  $p \in \mathbb{Q}$  iff
  - (a)  $p = (\eta, f) = (\eta^p, f^p)$
  - (b)  $\eta \in {}^\varepsilon\lambda$  for some  $\varepsilon < \lambda$ , ( $\eta$  is called the trunk of  $p$ )
  - (c)  $f \in {}^\lambda\lambda$
  - (d)  $\eta \triangleleft f$
- ( $\beta$ )  $p \leq_{\mathbb{Q}} q$  iff
  - (a)  $\eta^p \leq \eta^q$
  - (b)  $f^p \leq f^q$ , i.e.  $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$
  - (c) if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$ ; this follows.

{z23}

**Definition 0.7.** Let  $\lambda$  be inaccessible,  $\langle \theta_\varepsilon : \varepsilon < \lambda \rangle$  be a sequence of regular cardinals  $< \lambda$  satisfying  $\theta_\varepsilon > \varepsilon$ .

1) We define the forcing notion  $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}$  by

- ( $\alpha$ )  $p \in \mathbb{Q}$  iff
  - (a)  $p = (\eta, f) = (\eta^p, f^p)$
  - (b)  $\eta \in \prod_{\zeta < \varepsilon} \theta_\zeta$  for some  $\varepsilon < \lambda$ , ( $\eta$  is called the trunk of  $p$ )
  - (c)  $f \in \prod_{\zeta < \lambda} \theta_\zeta$
  - (d)  $\eta \triangleleft f$
- ( $\beta$ )  $p \leq_{\mathbb{Q}} q$  iff
  - (a)  $\eta^p \leq \eta^q$
  - (b)  $f^p \leq f^q$ , i.e.  $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$
  - (c) if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$ ; this follows

2) The generic is  $\eta = \cup \{ \eta^p : p \in \mathbf{G}_{\mathbb{Q}_{\bar{\theta}}} \}$ .

3) We define  $\mathbb{Q}_\lambda^{\text{dom}}$  similarly with  $\varepsilon < \lambda \Rightarrow \theta_\varepsilon = \lambda$ .

{z25}

*Remark 0.8.* The forcing is parallel to the creature forcing from [Sh:326, §1,§2] which are  ${}^\omega\omega$ -bounding.

{z26}

Recall

**Definition 0.9.** Let  $\kappa$  be supercompact. We say  $f : \kappa \rightarrow \mathcal{H}(\kappa)$  is a Laver diamond (for  $\kappa$ ) when for every  $x \in \mathbf{V}$  there are a normal fine ultrafilter on  $I = [\lambda]^{<\kappa}$  for some  $\lambda$  such that the Mostowski collapse  $\mathbf{j}$  on  $\mathbf{V}^I/D$  maps  $\langle f(\sup(u \cap \kappa)) : u \in I \rangle / D$  to  $x$ ; (we can use elementary embeddings instead of an ultrafilter).

## § 1. THE FORCING

{a2}

**Theorem 1.1.** *Assume*

- (a)  $\lambda$  is supercompact
- (b)  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \text{cf}(\mu) = \mu^\lambda$ .

Then for some forcing notion  $\mathbb{P}$  not collapsing cardinals  $\geq \lambda$ ,  $\lambda$  is still supercompact in  $\mathbf{V}^{\mathbb{P}}$  and  $\text{cov}_\lambda(\text{meagre}) = \kappa$ ,  $\mathfrak{d}_\lambda = \mu$ .

*Proof.* By 1.3 below. □<sub>1.1</sub> {a7}

Recall {a5}

**Definition 1.2.** 1) We say that a forcing notion  $\mathbb{P}$  is  $\alpha$ -strategically complete when for each  $p \in \mathbb{P}$  in the following game  $\mathfrak{D}_\alpha(p, \mathbb{P})$  between the players COM and INC, the player COM has a winning strategy.

A play lasts  $\alpha$  moves; in the  $\beta$ -th move, first the player COM chooses  $p_\beta \in \mathbb{P}$  such that  $p \leq_{\mathbb{P}} p_\beta$  and  $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$  and second the player INC chooses  $q_\beta \in \mathbb{P}$  such that  $p_\beta \leq_{\mathbb{P}} q_\beta$ .

The player COM wins a play if he has a legal move for every  $\beta < \alpha$ .

2) We say that a forcing notion  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete when it is  $\alpha$ -strategically complete for every  $\alpha < \lambda$ . {a7}

**Lemma 1.3.** 1) If  $\lambda$  is supercompact then after some preliminary forcing of cardinality  $\lambda$ ,  $\lambda$  is still supercompact and  $\square_\lambda$  below holds. {a7}

2) If  $\lambda$  is strongly inaccessible and  $\square_\lambda$  below holds and  $\lambda^+ < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , then for some  $\lambda^+$ -c.c.,  $(< \lambda)$ -strategically complete forcing notion  $\mathbb{P}$  we have  $\Vdash_{\mathbb{P}} \mathfrak{d}_\lambda = \mu$ ,  $\text{cov}_\lambda(\text{meagre}) = \kappa$ "

where

$\square_\lambda$  for any regular cardinal  $\chi > \lambda$  and forcing notion  $\mathbb{P} \in \mathcal{H}(\chi)$  which is  $(< \lambda)$ -strategically complete (see Definition 1.2(2)) the following set  $\mathcal{S} = \mathcal{S}_{\mathbb{P}}$  is a stationary subset of  $[\mathcal{H}(\chi)]^{< \lambda}$ : {a5}

$\mathcal{S} = \mathcal{S}_{\mathbb{P}} = \mathcal{S}_{\chi, \mathbb{P}}$  is the set of  $N$ 's such that for some  $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N, N' = N'_N, M = M_N, \mathbf{G} = \mathbf{G}_N, P_N$  we have (and say  $(\lambda_N, \chi_N, \dots)$  is a witness for  $N \in \mathcal{S}_{\chi, \mathbb{P}}$  or for  $(N, \mathbb{P}, \chi)$ ):

- (a)  $N \prec (\mathcal{H}(\chi)^{\mathbf{V}}, \in)$  and  $\mathbb{P} \in N$
- (b) the Mostowski collapse  $N'$  of  $N$  is  $\subseteq \mathcal{H}(\chi_N)$ , and let  $\mathbf{j}_N : N \rightarrow N'$  be the unique isomorphism
- (c)  $N \cap \lambda = \lambda_N$  and  $(^{\lambda_N})^> N \subseteq N$  and  $\lambda_N$  is strongly inaccessible
- (d)  $N' \subseteq M := (\mathcal{H}(\chi_N), \in)$  so both  $N'$  and  $M$  are transitive
- (e)  $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$  is generic over  $N'$  for the forcing notion  $\mathbf{j}_N(\mathbb{P})$
- (f)  $M = N'[\mathbf{G}]$ . {a7.3}

*Remark 1.4.* Recall that:

- (a) if  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \delta, \beta < \delta \rangle$  be a  $(< \lambda)$ -support iteration of  $(< \lambda)$ -strategically complete forcing notions, then  $\mathbb{P}_\delta$  is also  $< \lambda$ -strategically complete; (see e.g. [Sh:546]).

- (b) If  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete forcing notion then  $(^{\lambda} \text{Ord})^{\mathbf{V}} = (^{\lambda} \text{Ord})^{\mathbf{V}^{\mathbb{P}}}$ ,  
and consequently  $\lambda$  is strongly inaccessible in  $\mathbf{V}^{\mathbb{P}}$
- (c) like (a) replacing “ $(< \lambda)$ -strategically complete” by “ $(< \lambda)$ -complete”
- (d) if  $\mathbb{P}$  is  $(< \lambda)$ -complete then  $\mathbb{P}$  is  $\lambda$ -strategically complete.

{a8} **Definition 1.5.** We may say  $(N, \lambda_N, \chi_N, \mathbf{j}_N, N'_N, M_N, \mathbf{G}_N)$  is a witness for  $(N, \mathbb{P})$   
{a7} when clauses (a)-(f) from 1.3(2) hold.

{z26} *Proof. Proof of Lemma 1.3* 1) This is essentially by Laver [Lav78] using Laver’s diamond, see Definition 0.9.

2) Stage A: Without loss of generality  $\mathbf{V} \models “\mathfrak{b}_\lambda = \mu = \mathfrak{d}_\lambda”$  as witnessed by  $\langle \check{f}_\alpha^* : \alpha < \mu \rangle$ .

No new point, still we elaborate recalling that composition  $(\mathbb{P} * \mathbb{Q}^*)$  preserve “ $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c.”

We use a  $(< \lambda)$ -support iteration  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu, \beta < \mu \rangle$  such that:

- {z20} (A) if  $\alpha < \mu$  then  $\mathbb{Q}_\alpha$  is the  $(\mathbb{P}_\alpha$ -name of the) dominating forcing,  $\mathbb{Q}_\alpha^{\text{dom}}$ , i.e.  $(\mathbb{Q}_\alpha^{\text{dom}})^{\mathbf{V}^{\mathbb{P}_\alpha}}$  where the forcing  $\mathbb{Q} = \mathbb{Q}_\lambda^{\text{dom}}$  is from Definition 0.6.

Let  $\check{f}_\alpha^*$  be the generic object for  $\mathbb{Q}_\alpha$  for  $\alpha < \mu$ .

Now:

- (\*)<sub>1</sub> for  $\alpha \leq \mu$  the forcing notion  $\mathbb{P}_\alpha$  is  $(< \lambda)$ -complete because when  $\alpha < \mu$ ,  $\mathbb{Q}_\alpha$  is  $(< \lambda)$ -complete<sup>1</sup>, i.e.  $\Vdash_{\mathbb{P}_\alpha} “\mathbb{Q}_\alpha \text{ is } (< \lambda)\text{-complete}”$ .

{a7.3} [Why? We prove this by induction on  $\alpha$ ; for  $\mathbb{P}_\alpha$ , recall 1.4(1)(c).]

- (\*)<sub>2</sub> for each  $\alpha \leq \mu$ ,  $\mathbb{P}_\alpha$  and for  $\alpha < \mu$ , the forcing notions  $\mathbb{Q}_\alpha$  satisfy a strong form of the  $\lambda^+$ -c.c., (see [Sh:80] for definition, preservation and history; or fully [Sh:546, §1])

hence

- (\*)<sub>3</sub> (a) forcing with  $\mathbb{P}_\mu$  collapses no cardinal, changes no cofinality, and adds no sequence to  $^{\lambda} \mathbf{V}$ ;  
(b)  $(^{\lambda} \lambda)^{\mathbf{V}^{\mathbb{P}_\mu}} = \cup \{ (^{\lambda} \lambda)^{\mathbf{V}^{\mathbb{P}_\alpha}} : \alpha < \mu \}$ .

[Why? By (\*)<sub>2</sub> + (\*)<sub>1</sub> clause (a) holds, for clause (b) use also the support in the iteration being  $< \lambda$  recalling that  $\mu$  is regular  $> \lambda$ . E.g. if  $\Vdash_{\mathbb{P}_\mu} “\check{f} \in ^{\lambda} \lambda”$  then we can, for  $\alpha < \lambda$  find maximal antichain  $\langle p_{\alpha, i} : i < i_\alpha \leq \lambda \rangle$  of  $\mathbb{P}_\mu$  such that  $p_{\alpha, i}$  forces a value to  $\check{f}(\alpha)$ ; let  $\alpha(*) = \sup(\cup \{ \text{dom}(p_{\alpha, i}) : \alpha < \lambda, i < i_\alpha \})$ , so  $\alpha(*) < \mu$  and  $\check{f}$  is a  $\mathbb{P}_{\alpha(*)}$ -name.]

- (\*)<sub>4</sub> (a) in  $\mathbf{V}^{\mathbb{P}_\mu}$ ,  $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$  as witnessed by  $\bar{f}^* = \langle \check{f}_\alpha^* : \alpha < \mu \rangle$ , in fact  $\Vdash_{\mathbb{P}_{\alpha+1}} “\check{f}_\alpha^* \in ^{\lambda} \lambda \text{ dominates } (^{\lambda} \lambda)^{\mathbf{V}^{\mathbb{P}_\alpha}} \text{ modulo } J_\lambda^{\text{bd}}”$   
(b) in  $\mathbf{V}^{\mathbb{P}_\mu}$  still  $\square_\lambda$  holds.

[Why? Clause (a) easy using (\*)<sub>3</sub>(b).

{a19} Why? Easy, see details in 1.7(2).]

Stage B: In  $\mathbf{V}$  (see Stage A) there are  $\beta(*), \mathbf{q}, \bar{u}, \mathcal{U}_*, \dots$  such that

<sup>1</sup>for this,  $\theta_\alpha > \alpha$  is enough

- (\*)<sub>5</sub>(A) (a)  $\mathbf{q} = \langle \mathbb{P}_{0,\alpha}, \mathbb{Q}_{0,\beta} : \alpha \leq \beta(*), \beta < \beta(*) \rangle$  is a  $(< \lambda)$ -support iteration
- (b)  $\bar{u} = \langle u_\beta : \beta < \beta(*) \rangle, \bar{\mathcal{P}} = \langle \mathcal{P}_\beta : \beta < \beta(*) \rangle$
- (c)  $u_\beta \subseteq \beta, \mathcal{U}_* \subseteq \beta(*), \mathcal{P}_\beta \subseteq [u_\beta]^{\leq \lambda}$  is closed under subsets
- (d)  $\mathbb{Q}_{0,\beta}$  has generic  $\eta_\beta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$
- (e)  $\mathbb{Q}_{0,\beta}$  is as in 3.11 so is  $\subseteq \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[(\eta_\alpha : \alpha \in u_\beta)]}$  and  $\eta_\beta$  is its generic so  $\Vdash_{\mathbb{P}_{\beta+1}}$  “ $\eta_\beta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$ ” and  $\bar{\eta} = \langle \eta_\beta : \beta < \beta(*) \rangle$  {c51}
- (f)  $\mathcal{U}_* \subseteq \gamma(*)$  has order type  $\gamma(*) = \kappa$  and  $\langle \beta_i^* : i < \kappa \rangle$  list  $\mathcal{U}_*$  in increasing order
- (g) if  $\beta \in \mathcal{U}_*$  then  $\mathcal{U}_* \cap \beta \in \mathcal{P}_\beta$  hence  $\subseteq u_\beta$  and  $\Vdash_{\mathbb{P}_{\beta+1}}$  “if  $\nu \in \mathbf{V}[(\eta_\alpha : \alpha \in \mathcal{U}_* \cap \beta)] \cap \prod_{\varepsilon < \lambda} \theta_\varepsilon$  then  $\nu <_{J_\lambda^{\text{bd}}} \eta_\beta$ ”
- (h)  $\mathbb{P}_{0,\alpha}$  are  $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c.
- (B) (a)  $\langle \mathbb{P}'_i : i \leq \kappa \rangle$  is  $\ll$ -increasing, continuous for ordinals  $i \leq \kappa$  of cofinality  $\geq \lambda$
- (b)  $\mathbb{P}'_i$  is  $(< \lambda)$ -strategically complete for  $i \leq \kappa$
- (c)  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup \{ (\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa \}$
- (d)  $\mathbb{P}'_i < \mathbb{P}_{1,\beta(*)}$  and  $\mathbb{P}_{0,\beta} < \mathbb{P}_{1,\beta}$ , in fact is dense in it.

We shall mention more properties later.

[Why? Let  $M$  be a linear order isomorphic to  $(\kappa, <)$ , applying 3.12 there is  $\mathbf{m}$  as there. Let  $h$  be a one-to-one order preserving function from  $L_{\mathbf{m}}$  onto some ordinal,  $\beta(*)$ .] {c52}

So without loss of generality  $h$  is the identity, so let  $L_{\mathbf{m}}$  be  $(\beta(*), <_*)$ ,  $\mathcal{U}_* = M_{\mathbf{m}}, \mathbb{P}_{0,\alpha} = \mathbb{P}_{\mathbf{m}}[\{\beta : \beta < \alpha\}]$ ,  $u_\alpha = u_{\mathbf{m},\alpha}$

- (\*)<sub>6</sub> (a) let  $\langle \beta_i^* : i < \gamma(*) \rangle$  list  $\mathcal{U}_*$  in increasing order so  $\gamma(*) = \kappa$
- (b) for  $i \leq \kappa$ , let  $\mathbb{P}'_i$  is  $\mathbb{P}_{\mathbf{m}}[\{\beta_j^* : j < i\}]$  and let  $\underline{g}'_i$  be  $\eta_{\beta_i^*}$ , see (c) and 3.12 so  $\mathbb{P}'_i < \mathbb{P}_{1,\beta_i^*}$  {c52}
- (c) let  $\underline{g}' = \langle \underline{g}'_i : i < \kappa \rangle$
- (d) let  $\underline{g}_\alpha = \eta_\alpha$  for  $\alpha < \beta(*)$  and  $\bar{g} = \langle g_\beta : \beta < \beta(*) \rangle$
- (\*)<sub>7</sub> if  $u \in \mathcal{P}_\alpha, \alpha < \beta(*)$  then  $\Vdash_{\mathbb{P}_{0,\alpha+1}}$  “ $\underline{g}_\alpha \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{[g_\beta : \beta \in u]}$ ”, the order being modulo  $J_\lambda^{\text{bd}}$ .

[Why? As by 3.11 in  $\mathbf{V}^{\mathbb{P}_{0,\alpha}}$  for each  $g \in (\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[(\eta_\beta : \beta \in u)]}$  the set  $\{(\eta, f) \in \mathbb{Q}_{0,\beta} : \text{for every } \varepsilon \in [\ell g(\eta), \lambda) \text{ we have } g(\varepsilon) \leq f(\varepsilon)\}$  is a pre-dense open subset of  $\mathbb{Q}_{0,\alpha}$ .] {c51}

- (\*)<sub>8</sub>  $\Vdash_{\mathbb{P}'_\kappa}$  “ $\bar{g}' = \langle \underline{g}'_i : i < \kappa \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing and cofinal in  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_\lambda^{\text{bd}}})$ ”.

[Why? By (\*)<sub>7</sub> noting that  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup \{ (\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa \}$  which holds by

(\*)<sub>5</sub>(B)(c).]

Now

- (\*)<sub>9</sub>  $\Vdash_{\mathbb{P}'_\kappa}$  “ $\text{cov}_\lambda(\text{meagre}) \leq \kappa$ ”.

[Why? As we can look at  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  instead<sup>2</sup> of  ${}^\lambda 2$  and for each  $\varepsilon < \lambda, i < \kappa$  the set  $B_{\varepsilon, i} = \{\eta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon : \text{for every } \zeta \in [\varepsilon, \lambda) \text{ we have } \eta(\zeta) \leq g'_i(\zeta) < \theta_\zeta\}$  is closed nowhere dense, and by  $(*)_8$  we have  $\mathbf{V}^{\mathbb{P}^\kappa} \models \text{“}\prod_{\zeta < \lambda} \theta_\zeta = \cup\{B_{\varepsilon, i} : \varepsilon < \lambda, i < \kappa\}$ ”. In fact,  $\langle B_{0, i} : i < \kappa \rangle$  suffice.]

$(*)_{10} \Vdash_{\mathbb{P}'_\kappa} \text{“cov}_\lambda(\text{meagre}) \geq \kappa$ ”.

[Why? Let us define the  $\mathbb{P}'_{i+1}$ -name  $\eta'_i$  of a member of  ${}^\lambda 2$  by  $\eta'_i(\varepsilon) = 0$  iff  $g'_i(\varepsilon)$  is even. Now clearly  $\Vdash_{\mathbb{P}'_{i+1}} \text{“}\eta'_i \text{ is a } \lambda\text{-Cohen sequence over } \mathbf{V}^{\mathbb{P}^i}$ ”.

Also every closed nowhere dense subset of  ${}^\lambda 2$  from  $\mathbf{V}^{\mathbb{P}^{\gamma(*)}}$  is from  $\mathbf{V}^{\mathbb{P}^i}$  for some  $i < \gamma(*)$ , so we are done.]  $\square_{1.3}$

**Discussion 1.6.** 1) The reader may justly wonder why we use  $\mathbf{V}' = \mathbf{V}[\bar{g}'] = \mathbf{V}[\bar{g} \upharpoonright \mathcal{U}_*]$  rather than simply  $\mathbf{V}[\bar{g}]$ . Of course, nothing is lost by it, but why the extra complication?

2) The answer is that during the proof we shall use: if  $\zeta(i) \in \mathcal{U}_*$  is increasing with  $i < \gamma(*)$  then also  $\langle g_{\zeta(i)} : i < \kappa \rangle$  is generic over  $\mathbf{V}$  for the subforcing of  $\mathbb{P}_{1, \beta(*)}$  generated by  $\bar{g} \upharpoonright \mathcal{U}_*$ ; see  $\otimes'_7$  inside the proof of  $\otimes_6$ . But using  $\mathcal{U}_* = \beta(*)$ , we do not know this.

3) Now in the parallel case for  $\lambda = \aleph_0$ , such claim is true, see  $\S 0$ .

4) But we do not know the parallel of 3) for  $\lambda$ , so we use a substitute using  $\mathcal{U}_*$ , i.e.  $\mathbb{P}'_\kappa$ .

*Proof.* Continuation of the proof:

Now we come to the main and last point recalling  $\langle f_\alpha^* : \alpha < \mu \rangle$  from Stage A

$(*)_{11}$  it is forced, i.e.  $\Vdash_{\mathbb{P}'_\kappa}$  that no  $\underline{f} \in ({}^\lambda \lambda)$  dominate  $\{f_\alpha^* : \alpha < \mu\}$ .

{a7} We shall show that it suffices to prove  $(*)_{11}$  for proving 1.3(2), and that  $(*)_{11}$  holds, thus finishing.

{c52} Why it suffices? As  $\langle f_\alpha^* : \alpha < \mu \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing and  $\text{cf}(\mu) = \mu > \lambda$ , this implies  $\Vdash_{\mathbb{P}'_\kappa} \text{“}\mathfrak{d}_\lambda \geq \mu$ ”. Also in  $\mathbf{V}, 2^\lambda = \mu \geq \kappa > \lambda$  and  $|\mathbb{P}'_\kappa| = \kappa^\lambda$  by (g) of 3.12 which is  $= \mu$  and  $\mathbb{P}'_\kappa$  satisfies the  $\lambda^+$ -c.c. hence  $\Vdash_{\mathbb{P}'_\kappa} \text{“}2^\lambda = \mu$ , hence  $\Vdash_{\mathbb{P}'_\kappa} \text{“}\mathfrak{d}_\lambda = \mu$ ”. Also by  $(*)_5(B)(b)$ , “ $\mathbb{P}'_\kappa$  is  $(< \lambda)$ -strategically complete  $+ \lambda^+$ -c.c.” and  $(*)_9 + (*)_{10}$  for “ $\text{cov}_\lambda(\text{meagre}) = \kappa$ ” so we are done; so  $(*)_{11}$  is really the last piece missing. The rest of the proof is dedicated to proving that  $(*)_{11}$  holds.

{c52} We shall use further nice properties of  $\mathbb{P}'_j, g'_i(j \leq \gamma(*), i < \gamma(*))$  which holds by  $(*)_5 + (*)_6$  (and  $(*)_7, (*)_8$ ) and their proof, i.e. 3.11 or 3.12.

$\boxplus_1$  (a)( $\alpha$ )  $\langle g'_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$ , i.e., if  $\mathbf{G}$  is a subset of  $\mathbb{P}'_{\gamma(*)}$  generic over  $\mathbf{V}$  and  $g'_i = g'_i[\mathbf{G}]$  then  $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_i : i < \gamma(*) \rangle]$

<sup>2</sup>E.g. let  $F : {}^\lambda 2 \rightarrow \prod_{\varepsilon < \lambda} \theta_\varepsilon$  be  $F(\eta) = \rho$  iff  $\eta \in {}^\lambda 2$  and for every  $\varepsilon < \lambda, \rho(\varepsilon) = 0$  iff  $(\forall i < \theta_\varepsilon)(\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + i) = 0)$  and  $\rho(\varepsilon) = 1 + i$  iff  $\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + i) = 1 \wedge (\forall j < i)(\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + j) = 0)$ . Now if  $\prod_{\varepsilon} \theta_\varepsilon = \cup\{\mathcal{U}_i : i < \kappa\}$ , each  $\mathcal{U}_i$  closed nowhere dense then  $\langle F^{-1}(\mathcal{U}_i) : i < \kappa \rangle$  witnesses  $\text{cov}_\lambda(\text{meagre}) \leq \kappa$ .



- ( $\beta$ ) if  $\nu \in {}^{(\lambda)\lambda}\mathbf{V}[\mathbf{G}]$  then for some  $\rho \in {}^{(\lambda)\gamma(*)}\mathbf{V}$  and Borel function  $\mathbf{B} \in \mathbf{V}$  we have  $\nu = \mathbf{B}(\langle g'_{\rho(\varepsilon)} : \varepsilon < \lambda \rangle)$
- (b) if in  $\mathbf{V}[\mathbf{G}]$ ,  $g''_\gamma \in \prod_{\zeta < \lambda} \theta_\zeta$  for  $\gamma < \gamma(*)$  and the set  $\{(\gamma, \zeta) : \gamma < \gamma(*) \text{ and } \zeta < \lambda \text{ and } g''_\gamma(\zeta) \neq g'_\gamma(\zeta)\}$  has cardinality  $< \lambda$  then  $\bar{g}'' = \langle g''_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$  and  $\mathbf{V}[\bar{g}''] = \mathbf{V}[g']$
- (c)  $\Vdash_{\mathbb{P}'_\gamma}$  “ $g'_\gamma$  dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\gamma]}$ ”
- (d) if  $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$  is an increasing sequence of ordinals  $< \gamma(*)$ , then  $\langle g'_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$
- (e)  $\mathbb{P}'_\gamma$  is  $(< \lambda)$ -strategically complete and satisfies the  $\lambda^+$ -c.c.

We shall use  $\boxplus_1$  freely.

To prove  $(*)_{11}$  assume toward contradiction that this fails, so  $\mathbb{P}'_{\gamma(*)}$  satisfies the  $\lambda^+$ -c.c. and for some condition  $p^* \in \mathbb{P}'_{\gamma(*)}$  and  $\mathbb{P}'_{\gamma(*)}$ -name  $\underline{f}$  and  $\lambda$ -Borel function  $\mathbf{B}$  and  $\rho \in {}^\lambda\beta(*)$ , moreover  $\rho \in {}^\lambda(\mathcal{U}_*)$  we have (noting: the “moreover” holds as  $f \in {}^{(\lambda)\lambda}\mathbf{V}[\bar{g}'\upharpoonright\mathcal{U}_*]$ )

$$\otimes_0 p^* \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\underline{f} \in {}^\lambda\lambda \text{ and dominates } ({}^\lambda\lambda)^{\mathbf{V}}\text{” and } \underline{f} = \mathbf{B}(\langle g_{\rho(i)} : i < \lambda \rangle).$$

Now let  $\chi$  be regular large enough and we choose  $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda \rangle$  such that

- $\otimes_1$  (a)  $N_\varepsilon$  is as in  $\boxplus_\lambda$  for the forcing notion  $\mathbb{P}'_{\gamma(*)}$ ,  
 $N_\varepsilon \in \mathcal{S}_{\chi, \mathbb{P}'_{\gamma(*)}}$ , see  $\boxplus$  of 1.3 {a7}
- (b)  $\bar{N} \upharpoonright \varepsilon \in N_\varepsilon$  hence  $\bigcup_{\zeta < \varepsilon} N_\zeta \subseteq N_\varepsilon$  and  $\lambda_\varepsilon := N_\varepsilon \cap \lambda > \lambda_\varepsilon^- := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\} \geq \Sigma\{\lambda_\zeta : \zeta < \varepsilon\}$
- (c)  $\bar{\theta}, \mathbf{q}, p^*, \underline{f}, \mathbf{B}, \rho$  belong to  $N_\varepsilon$
- (d) let  $\delta(\varepsilon) = \text{otp}(\gamma(*) \cap N_\varepsilon)$ .

We can find  $f^* \in {}^\lambda\lambda$ , i.e.  $\in ({}^\lambda\lambda)^{\mathbf{V}}$ , such that

$$\otimes_2 \text{ for arbitrarily large } \varepsilon < \lambda \text{ for some } \zeta \in [\lambda_\varepsilon^-, \lambda_\varepsilon) \text{ we have } f^*(\zeta) > \lambda_\varepsilon.$$

For  $\varepsilon < \lambda$  let  $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, N'_\varepsilon, \mathbf{G}_\varepsilon)$  be a witness for  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)}, \chi)$  recalling  $\boxplus_\lambda$  of Definition 1.5 so  $\lambda_\varepsilon \in (\varepsilon, \lambda)$  is strongly inaccessible and  $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda_\zeta^- < \lambda_\zeta$ , recalling  $\otimes_1$  and so  $\delta(\varepsilon) = \mathbf{j}_\varepsilon(\delta_{\gamma(*)})$ , etc. {a8}

Let

- $\otimes_3$  (a)  $v_\varepsilon = N_\varepsilon \cap \gamma(*)$
- (b)  $i(\varepsilon) = \text{otp}(v_\varepsilon)$
- (c)  $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < i(\varepsilon) \rangle$  list  $v_\varepsilon$  in increasing order
- (d) for  $i < \text{otp}(v_\varepsilon)$ , equivalently  $i < \mathbf{j}_\varepsilon(\gamma(*))$  let  $\eta_i^\varepsilon = (\mathbf{j}_\varepsilon(g'_i))^{N'_\varepsilon[\mathbf{G}_\varepsilon]} \in \prod_{\zeta < \lambda_\varepsilon} \theta_\zeta$  and let  $\bar{\eta}^\varepsilon = \langle \eta_i^\varepsilon : i < \text{otp}(v_\varepsilon) \rangle$ .

Note that clearly

$$\otimes_4 (a) \bar{\eta}^\varepsilon \text{ is generic for } (N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})), \text{ moreover}$$

- (b) for each  $\varepsilon < \lambda$ , if we change  $\eta_i^\varepsilon(\zeta)$  (legally, i.e.  $< \theta_\zeta$ ) for  $< \lambda_\varepsilon$  pairs  $(i, \zeta) \in \text{otp}(u_\varepsilon) \times \lambda_\varepsilon$  and get  $\bar{\eta}'$ , then also  $\bar{\eta}'$  is generic for  $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$  and  $N'_\varepsilon[\bar{\eta}'] = M_\varepsilon$
- (c) like  $\boxplus_1$  with  $\mathbf{V}, \mathbb{P}'_{\gamma(*)}, \lambda$  there standing for  $N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$  here.

Hence clearly

- $\boxplus'_4$  for  $\varepsilon < \lambda$ , if  $\bar{\eta}' = \langle \nu_i : i < i(\varepsilon) \rangle$  recalling  $i(\varepsilon) = \text{otp}(u_\varepsilon)$  is as in  $\boxplus_4(b)$ , and  $q \in \mathbb{P}'_{\gamma(*)}$  satisfies  $i < i(\varepsilon) \Rightarrow q \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“} \check{g}'_{\gamma_i(\varepsilon)} \upharpoonright \lambda_\varepsilon = \nu_i \text{”}$  then  $q$  is  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$ -generic naturally and  $q \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“} \mathbf{j}_\varepsilon \text{ can be extended naturally to an isomorphism from } N_\varepsilon[G_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle \check{g}_\gamma : \gamma \in v_\varepsilon \rangle] \text{ onto } N'_\varepsilon[\bar{\eta}'] \text{”}$ .

[Why? Should be clear, see  $\boxplus_1 + \boxplus_4(c)$ .]

By the assumption toward contradiction,  $\boxplus_0$ , and  $\mathbb{P}'_{\gamma(*)}$  being  $(< \lambda)$ -strategically complete recalling  $\boxplus_1$ , there are  $\zeta(*), p^{**}$  and  $p^+$  such that (recall  $p^* \in \mathbb{P}'_{\gamma(*)} = \ll \mathbb{P}_{1, \beta(*)}$ ):

- $\boxplus_5$  (a)  $p^* \leq p^{**} \in \mathbb{P}'_{\beta(*)}$  and  $p^+ \in \mathbb{P}_{0, \gamma(*)}$  such that  $\mathbb{P}_{1, \gamma(*)} \Vdash \text{“} p^{**} \leq p^+ \text{”}$
- (b)  $\zeta(*) < \lambda$
- (c)  $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“} f^*(\zeta) < \check{f}(\zeta) \text{ whenever } \zeta(*) \leq \zeta < \lambda \text{”}$  where  $f^*$  is from  $\boxplus_2$
- (d) if  $\gamma \in \text{Dom}(p^+)$  then  $\eta^{p^+(\gamma)}$  is an object (not just a  $\mathbb{P}'_\gamma$ -name) of length  $\geq \zeta(*)$  (recall that  $\eta^{p^+(\gamma)}$  is the trunk of the condition, see clause  $(\alpha)(b)$  of Definition 0.7(1)).

{z23}

Note that possibly  $\text{Dom}(p^+) \not\subseteq \cup \{v_\varepsilon : \varepsilon < \lambda\}$ . Choose  $\varepsilon(*) < \lambda$  such that  $\lambda_{\varepsilon(*)} > \zeta(*) + |\text{Dom}(p^+)|$  and  $\gamma \in \text{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$  recalling clause (d) of  $\boxplus_5$  and  $|\text{Dom}(p^+)| < \lambda$  as  $p^+ \in \mathbb{P}_{0, \beta(*)}$  and  $\mathbb{P}_{0, \beta(*)}$  is the limit of a  $(< \lambda)$ -support iteration.

By  $\boxplus_2$  we can add  $(\exists \zeta)[\lambda_{\varepsilon(*)}^- \leq \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$ . Our intention is to find  $q \in \mathbb{P}_{0, \beta(*)}$  above  $p^+$  which is above some  $q' \in \mathbb{P}'_{\gamma(*)}$  which is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ -generic and forces  $\mathbf{G}_{\mathbb{P}'_{\gamma(*)}}$  to include a generic subset of  $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$  which is induced by some  $\bar{\eta}'$  as in  $\boxplus_4(b)$ . Toward this in  $\boxplus_6$  below the intention is that  $p_{i(*)}^+$  will serve as  $q$ .

Let  $i(*) = i(\varepsilon(*))$  and  $\gamma_i$  for  $i < i(*)$  be such that  $\langle \gamma_i : i < i(*) \rangle$  list  $\{\beta_i^* : i \in v_{\varepsilon(*)}\} \subseteq \mathcal{U}_*$  in increasing order; recall  $\mathcal{U}_* = \{\beta_i^* : i < \gamma(*)\}$  and  $i < j < \gamma(*) \Rightarrow \beta_i^* < \beta_j^*$  and  $v_{\varepsilon(*)} \subseteq \gamma(*)$  has order type  $i(\varepsilon(*))$ . Next let  $\mathcal{U}_{**} = \{\gamma_i : i < i(*)\}$  and  $\gamma_{i(*)} = \gamma(*)$  so  $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in v_{\varepsilon(*)}\} = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$ . Recall  $\gamma(*) = \kappa = \text{cf}(\kappa) > \lambda$ ,  $\text{otp}(v_{\varepsilon(*)}) = \text{otp}(N_{\varepsilon(*)} \cap \gamma(*)) = \text{otp}(N_{\varepsilon(*)} \cap \kappa)$  hence  $N_{\varepsilon(*)} \Vdash \text{“} i(*) \text{ is a regular cardinal } > \lambda_\varepsilon \text{”}$  hence  $i(*)$  is really a regular cardinal so call it  $\sigma$ . Now we define a game  $\boxplus$  as follows<sup>3</sup>:

- $\boxplus_2$  (A) each play lasts  $i(*) + 1 = \sigma + 1$  moves and in the  $i$ -th move,
- (a) if  $i = j + 1$  the antagonist player chooses  $\xi_j = \xi(j) < \sigma$  such that  $j_1 < j \Rightarrow \xi(j_1) < \xi(j)$

<sup>3</sup>The idea is to scatter the  $\eta_{\gamma_i}^{\varepsilon(*)}$ 's. Why not use the original places? as then we have a problem in  $\boxplus_6$ .

- (b) then, if  $i = j + 1$  the protagonist chooses  $\zeta_j = \zeta(j) \in (\xi(j), \sigma)$ , but there are more restrictions implicit in  $\boxplus_3$
- (c) in any case the protagonist chooses  $p_i^+, \bar{\nu}^i$  such that  $\boxplus_3$  below holds;
- (B) in the end of the play the protagonist wins the play iff he always has a legal move and in the end  $\{\zeta(i) : i \leq i(*)\} \in N'_{\varepsilon(*)}$ ; note that trivially it belongs to  $M_{\varepsilon(*)} = N'_{\varepsilon(*)}[\bar{\nu}^{\varepsilon(*)}]$

where

- $\boxplus_3$  (a)  $p_i^+ \in \mathbb{P}_{0, \gamma_i}$
- (b) if  $j < i$  then  $\mathbb{P}_{0, \gamma_i} \models "p_j^+ \leq p_i^+"$
- (c) if  $\gamma \in \cup\{\text{Dom}(p_j^+) : j < i\}$  then
  - $p_i^+ \upharpoonright \gamma \Vdash_{\mathbb{P}_{0, \gamma_i}} " \eta^{p_i^+(\gamma)} \text{ has length } \geq i(*) \text{ and } \geq \lambda_{\varepsilon(*)} "$
  - moreover  $\eta^{p_i^+(\gamma)}$  is an object,  $\eta^{p_i^+(\gamma)}$
- (d)  $\mathbb{P}_{0, \gamma_i} \models "p^{**} \upharpoonright \gamma_i \leq p_i^+"$
- (e)  $\bar{\nu}^i = \langle \nu_{\gamma_j} : j < i \rangle$  and  $\nu_{\gamma_j} \in \prod_{\iota < \lambda_{\varepsilon(*)}} \theta_\iota$
- (f) for  $j < i$  we have  $\nu_{\gamma_j} \leq \eta^{p_i^+(\gamma_j)}$  so  $p_i^+ \upharpoonright \gamma_j \Vdash " \nu_{\gamma_j} \triangleleft \underline{g}'_{\gamma_j} "$  recalling  $\boxplus_1$
- (g) for  $j < i$  we have (recall  $\bar{\eta}^\varepsilon$  from  $\otimes_3$ )
  - ( $\alpha$ )  $\nu_{\gamma_j} = \eta_{\gamma_{\zeta(j)}}^{\varepsilon(*)}$  recalling  $\eta_{\gamma_j}^{\varepsilon(*)}$  is from  $\otimes_3(d)$  or
  - ( $\beta$ )  $\gamma_j \in \text{Dom}(p^{**})$  and  $\{\iota < \lambda_{\varepsilon(*)} : \eta_{\zeta(j)}^{\varepsilon(*)}(\iota) \neq \nu_{\gamma_j}(\iota)\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ .

We shall prove

- $\otimes_6$  in the game  $\mathcal{D}$ 
  - (a) the antagonist has no winning strategy
  - (b) in any move the protagonist has a legal move, moreover for any  $\zeta(i) \in (\xi(i), \sigma)$  large enough the protagonist can choose it.

Why  $\otimes_6$  suffice?

By clause (a) of  $\otimes_6$  we can choose a play  $\langle (\xi(i), \zeta(i), p_i^+, \bar{\nu}^i) : i \leq \sigma \rangle$  in which the protagonist wins. Recalling  $\mathbb{P}'_{\gamma(*)} \triangleleft \mathbb{P}_{1, \beta(*)}$  and  $\mathbb{P}_{0, \beta(*)}$  is a dense subforcing of  $\mathbb{P}_{1, \beta(*)}$ , clearly

- $\otimes_7$  there is  $p$  such that
  - (a)  $p \in \mathbb{P}'_{\gamma(*)}$
  - (b) if  $\mathbb{P}'_{\gamma(*)} \models "p \leq p'"$  then  $p', p_\sigma^+$  are compatible in  $\mathbb{P}_{0, \beta(*)}$
  - (c)  $p$  is above  $p^{**}$  and it forces  $\underline{g}'_i \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < \gamma(*)$ .

Then on the one hand

- $\otimes'_7$   $p \in \mathbb{P}'_{\gamma(*)}$  being above  $p^{**}$  forces  $f^* \upharpoonright [\zeta(*), \lambda] < \underline{f} \upharpoonright [\zeta(*), \lambda]$  hence  $f^* \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)}] < \underline{f} \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)}]$  recalling that  $\zeta(*) < \lambda_{\varepsilon(*)}$ .

On the other hand,

- $\otimes''_7$   $p$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{0, \gamma(*)})$ -generic.

[Why? As it forces  $\eta_{\gamma_i} \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < i(*)$  and  $\langle \nu_{\gamma_i} : i < i(*) \rangle$  is (see  $\boxplus_3(g)$  recalling  $\text{Dom}(p^{**})$  has cardinality  $< \lambda_{\varepsilon(*)}$ ) “almost equal” to  $\langle \eta_{\zeta(i)}^{\varepsilon(*)} : i < i(*) \rangle$  which is a subsequence of the sequence from  $\boxplus_3$ . That is  $\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} \subseteq \cup\{(i, \iota) : \iota < \lambda_{\varepsilon(*)} \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} : \gamma \in v_{\varepsilon(*)} \cap \text{Dom}(p^{**})\}$  so is the union of  $\leq |\text{Dom}(p^+)| < \lambda_{\varepsilon(*)}$  sets each of cardinality  $< \lambda_{\varepsilon(*)}$  hence is of cardinality  $< \lambda_{\varepsilon(*)}$ . Hence by  $\boxplus_4(c) + \boxplus_1(d)$  the sequence  $\bar{\nu}^{i(*)}$  is generic for  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ .]

As  $f \in N_{\varepsilon(*)}$  it follows from  $\boxplus_7''$  that

$$\boxplus_7''' \quad p \Vdash \text{“} f \upharpoonright \lambda_{\varepsilon(*)} \text{ is a function from } \lambda_{\varepsilon(*)} \text{ to } \lambda_{\varepsilon(*)}\text{”}.$$

Together  $\boxplus_7' + \boxplus_7'''$  gives a contradiction by the choice of  $f^*$  in  $\boxplus_2$  and of  $\varepsilon(*)$  above, hence  $\boxplus_6$  is enough.

Why  $\boxplus_6$  is true?

Let us prove  $\boxplus_6$ ; first for clause (a) choose any strategy  $\mathbf{st}$  for the antagonist and fix a partial strategy  $\mathbf{st}'$  for the protagonist choosing  $(p_i^+, \bar{\nu}^i)$  from: the previous choices and  $\zeta(i)$  if relevant and possible. So the only freedom left is for the protagonist to choose the  $\zeta(i)$ . So (recalling  $\boxplus_{12}(A)(a)$ ) we have in  $\mathbf{V}$  a function  $F : \sigma > \sigma \rightarrow \sigma$  such that:

(\*)<sub>F</sub> playing the game such that the antagonist uses  $\mathbf{st}$  and the protagonist uses  $\mathbf{st}'$ , arriving to the  $i$ -th move,  $\bar{\zeta} = \langle \zeta(j) : j < i \rangle$  is well defined and for the protagonist any choice  $\zeta_i \in (F(\bar{\zeta}), \sigma) \cap \mathcal{U}_{**}$  is legal.

Now we have to find an increasing sequence  $\bar{\zeta} = \langle \zeta(i) : i < \sigma \rangle$  from  $N'_{\varepsilon(*)}$  such that  $F(\bar{\zeta} \upharpoonright i) < \zeta(i) < \sigma$  and  $\bar{\zeta} \in N'_{\varepsilon(*)}$ . As  $F \in \mathcal{H}(\chi_{\varepsilon(*)})$  and  $\mathcal{H}(\chi_{\varepsilon(*)}) = N'_{\varepsilon(*)}[\mathbf{G}_{\varepsilon(*)}]$  where  $\mathbf{G}_{\varepsilon(*)}$  is a subset of  $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}'_{\gamma(*)}) \in N'_{\varepsilon(*)}$  generic over  $N'_{\varepsilon(*)}$  and  $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$  satisfies the  $\lambda_{\varepsilon(*)}^+$ -c.c. and  $\sigma = \text{cf}(\sigma) > \lambda_{\varepsilon(*)}$  this is possible. That is, there is a  $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$ -name  $\bar{F}_* \in N'_{\varepsilon(*)}$  such that  $F = \bar{F}_*[\mathbf{G}_{\varepsilon(*)}]$  and we define in  $N'_{\varepsilon(*)}$  the function  $F' : \sigma > \sigma \rightarrow \sigma$  by  $F'(\langle \zeta(j) : j < i \rangle) = \sup\{\xi + 1 : \xi \in \{\zeta(j) + 1 : j < i\} \text{ or } \xi < \sigma \text{ and } \not\Vdash_{\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})} \text{“} \bar{F}(\langle \zeta(j) : j < i \rangle) \neq \xi \text{”}\}$ ; clearly this is O.K.

We are left with proving  $\boxplus_6(b)$ .

Case 1:  $i = 0$ .

Let  $p_0^+ = p^{**} \upharpoonright \gamma_0$ .

Case 2:  $i$  limit.

By clauses (a) and (b) of  $\boxplus_3$ , there is  $p_i^+ \in \mathbb{P}_{0,\gamma_i}$  which is an upper bound (even l.u.b.) of  $\{p_j^+ : j < i\}$  and it is easily as required. Also  $\bar{\nu}^i$  is well defined and as required.

Case 3:  $i = j + 1$  and  $\gamma_j \notin \text{Dom}(p^{**})$ .

Clearly  $\gamma_i = \gamma_j + 1$  and  $\gamma_j \in v_{\varepsilon(*)}$ . As in case 4 below but easier by the properties of the iteration.

Case 4:  $i = j + 1$  and  $\gamma_j \in \text{Dom}(p^{**})$

Again  $\gamma_i = \gamma_j + \delta(*)$  and  $\gamma_j \in v_{\varepsilon(*)}$ . First we find  $p'_j$  such that:

$$\boxplus_8 \quad (a) \quad p_j^+ \leq p'_j \in \mathbb{P}_{0,\gamma_j}$$

- (b) if  $\gamma \in \text{Dom}(p_j^+)$  then  $p_j^+ \upharpoonright \gamma \Vdash \text{“}\ell g(\eta^{p_j^+(\gamma)}) > i\text{”}$   
(c)  $p_j^+$  forces <sup>4</sup> a value to the pair  $(\eta^{p^+(\gamma_i)}, f^{p^+(\gamma_j)} \upharpoonright \lambda_{\varepsilon(*)})$ ; we call this pair  $q_j$ .

This should be clear.

Second

$\otimes_9$   $p_j^+$  hence  $p_j^+$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic and  $\langle \nu_{\gamma_j(1)} : j(1) < j \rangle$  induces the generic.

[Why? As in the proof of  $\otimes_7''$  above when we assume that we have carried the induction, by  $\boxplus_2$ , clause (g) and  $\otimes_4$ .]

Now

- $\otimes_{10}$  (a)  $f^{q_j} \in (\prod_{\zeta < \lambda_{\varepsilon(*)}} \theta_\zeta)^{N'_{\varepsilon(*)}[\bar{\nu}^j]}$   
(b) for every  $\zeta \in (\xi(i), \sigma)$  we have
- $f^{q_j} \leq \eta_\zeta^{\varepsilon(*)} \text{ mod } J_{\lambda_\varepsilon}^{\text{bd}}$
  - $f^{q_j} \in N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$
  - $\langle \zeta(j_1) : j_1 < j \rangle \in N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$ .
- (c)  $\eta^{q_j} \triangleleft f^{q_j}$ .

[Why? Clause (a) follows from clause (b) and clause (b) should be clear by  $\otimes_9$  as we can choose  $\zeta(i)$  large enough recalling  $\otimes_6$ . Also clause (c) follows from (b).]

Now we choose  $\zeta(j)$  as in clause (b) of  $\otimes_{10}$  and  $\nu_j \in \prod_{\varepsilon < \lambda_{\varepsilon(*)}} \theta_\varepsilon$  such that  $\eta^{p^+(j)} \triangleleft$

$\nu_j, f^{q_j} \leq \nu_j$  and  $\{\iota < \lambda_{\varepsilon(*)} : \nu_j(\iota) \neq \eta_{\zeta(j)}^{\varepsilon(*)}\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ . Next choose  $p_i^+ \in \mathbb{P}'_{\gamma(*)}$  such that  $p_i^+ \upharpoonright \gamma_j = p_j^+, \eta^{p_i^+(\gamma_i)} = \nu_j$  and  $f^{p_i^+(\gamma_i)} \upharpoonright [\lambda_\varepsilon, \lambda] = f^{p^+(\gamma)} \upharpoonright [\lambda_\varepsilon, \lambda]$ .

So we have carried the induction hence proved  $\otimes_6$  so we are done.  $\square_{1.3}$

{a19}

*Concluding Remark 1.7.* 1) Is the use of  $\bar{g} \upharpoonright \mathcal{U}_*$  rather than  $\bar{g}$  in the proof necessary? See on this [Sh:F979].

2) Why is  $\square_\lambda$  still true in  $\mathbf{V}^\mathbb{P}$  when  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete when in  $\mathbb{P}$  any directed increasing system of cardinality  $< \lambda$  of conditions in  $\mathbb{P}$  has an upper bound?

Let  $\mathbb{Q}$  be a forcing notion in  $\mathbf{V}^\mathbb{P}$  which is  $(< \lambda)$ -strategically complete  $\chi_1$  large enough and we should prove that in  $\mathbf{V}^\mathbb{P}$ , the set  $\mathcal{S}_\mathbb{Q}$  is stationary. So let  $\mathbb{Q}, \bar{E}$  be  $\mathbb{P}$ -names such that for some  $p \in \mathbb{P}$  we have  $p \Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{H}(\chi_1)$  is  $(< \lambda)$ -strategically complete forcing notion,  $\bar{E}$  a club of  $[\mathcal{H}(\chi_1)]^{< \lambda}$  disjoint to  $\mathcal{S}_\mathbb{P}$ ” (no need to use a name for  $\chi_1$  as we can increase it).

Let  $\chi \gg \chi_1$ ; now  $\mathbb{P} * \mathbb{Q} \in \mathcal{H}(\chi)$  is a  $(< \lambda)$ -strategically complete forcing notion and without loss of generality code  $(\chi_1, \bar{E})$ . As  $\square_\lambda$  holds in  $\mathbf{V}$  we can find  $N, \lambda_N, \chi_N, \mathbf{j}_N, N'_N, M_N, \mathbf{G}_N$  such that  $P * \mathbb{Q} \in N$  so  $\chi_1, \bar{E} \in N$ . Let  $p_N = (p_1, p_2)$  so  $p_2 \in \mathbb{P}$  and  $\Vdash_{\mathbb{P}} \text{“}p_2 \in \mathbb{Q}\text{”}$ . Let  $\mathbf{G}_\mathbb{P}$  be a subset of  $\mathbb{P}$  generic over  $\mathbf{V}$  and  $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_\mathbb{P}], N_1 = N[\mathbf{G}_\mathbb{P}], \bar{E}_1 = \bar{E}[\mathbf{G}_\mathbb{P}, N'_1 = N'[\mathbf{j}''(\mathbf{G}_\mathbb{P} \cap N)] = N'[\{p' : (p', q') \in \mathbf{G}\}], \mathbf{G}_1 = \{q[\mathbf{j}''(\mathbf{G}_\mathbb{P} \cap N)] : (p, q) \in \mathbf{G}\}$ .

Let  $N_2 = N_2 \upharpoonright \mathcal{H}(\chi_1)^{\mathbf{V}[\mathbf{G}]}, \mathcal{S} = \mathcal{S}_\mathbb{Q}, \mathbf{j}_1 =$  the lifting of  $(\mathbf{j} \upharpoonright (N \cap \mathcal{H}(\chi)))$ , to mapping  $N_1$  onto  $N'_1$ .

Now clearly

<sup>4</sup>recall that  $\eta^{p^+(\gamma)}$  is an object, not a name and  $p_j^+$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic

(\*)  $N_2 \in E$ .

hence

(\*)  $N_1 \notin \mathcal{S}_1$ .

But easily  $\lambda_N, \chi_N, \mathbf{j}_1, N'_1, M_1 = M, \mathbf{G}_1 = \underline{g}[\mathbf{G}_{\mathbb{P}}]$  witness  $N_1 \in \mathcal{S} \cap E_1$ , a contradiction to the choice of  $E$ .

{z26} 3) We may elaborate the proof of 0.9(1).

{z26} *Proof.* 1) For completeness we elaborate. Without loss of generality there is a Laver diamond  $h : \lambda \rightarrow \mathcal{H}(\lambda)$ , see Definition 0.9. Let  $E = \{\theta : \theta < \lambda \text{ and } \alpha < \theta \Rightarrow h(\alpha) \in \mathcal{H}(\theta)\}$ , clearly a club of  $\lambda$  and let  $\langle \lambda_\varepsilon : \varepsilon < \lambda \rangle$  list  $\{\theta \in E : \theta \text{ strongly inaccessible}\}$  in increasing order.

We now define  $\mathbf{q}_\varepsilon$  by induction on  $\varepsilon \leq \lambda$  such that:

- (\*) (a)  $\mathbf{q}_\varepsilon = \langle \mathbb{P}_\zeta, \mathbb{Q}_\xi : \zeta \leq \varepsilon, \xi < \varepsilon \rangle$  is an Easton support iteration (so  $\mathbb{P}_\zeta, \mathbb{Q}_\xi$  do not depend on  $\varepsilon$ , etc.)
- (b)  $\mathbb{P}_\zeta \subseteq \mathcal{H}(\kappa_\zeta)$
- (c)  $\chi_\xi$  is a regular cardinal  $\in [\kappa_\xi, \kappa_{\xi+1})$
- (d)  $\mathbb{Q}_\xi \in \mathcal{H}(\chi_{\xi+1})$  is a  $\mathbb{P}_\xi$ -name of a  $\kappa_\xi^+$ -c.c. ( $< \kappa_\xi$ )-strategically complete forcing notion
- (e) if  $h(\xi) = (\mathbb{Q}, \xi)$  and  $(\mathbb{Q}, \chi)$  satisfies the requirements in clauses (c),(d) then  $(\mathbb{Q}_\xi, \chi_\xi) = h(\xi)$ .

Easily we can carry the induction so  $\mathbf{q}_\lambda$  is well defined,  $\mathbb{P}_\lambda \subseteq \mathcal{H}(\lambda)$  and “ $\xi < \lambda \Rightarrow \mathbb{P}_\lambda/\mathbb{P}_\xi$  is ( $< \kappa_\xi$ )-strategically complete” hence  $\mathbb{P}_\lambda/\mathbb{P}_\xi$  adds no new sequence of length  $< \kappa_\xi$  of ordinals. Clearly it is enough to prove that in  $\mathbf{V}^{\mathbb{P}_\lambda}$  we have  $\sqsupseteq_\lambda$ .

Toward contradiction assume  $\chi, \mathbb{P}, \mathcal{S} = \mathcal{S}_{\mathbb{P}}$  forms a counter-example, hence there are  $p_* \in \mathbb{P}_\lambda$  and  $\mathbb{P}_\lambda$ -name  $\underline{\chi}, \underline{\mathbb{P}}, \underline{\mathcal{S}}, \underline{E}$  such that  $p_* \Vdash_{\mathbb{P}_\lambda}$  “ $\underline{\chi} > \lambda$  is regular,  $\underline{\mathbb{P}} \in \mathcal{H}(\underline{\chi}), \underline{\mathcal{S}} = \mathcal{S}_{\underline{\mathbb{P}}}$  and  $\underline{E} \subseteq [\mathcal{H}(\underline{\chi})^{\mathbf{V}^{\mathbb{P}_\lambda}}]^{< \lambda}$  is a club disjoint to  $\underline{\mathcal{S}}$ ”.

As we can increase  $p_*$ , without loss of generality  $\underline{\chi} = \chi$ ; and as  $\mathbf{V} \models$  “ $\lambda$  is supercompact” for some  $(\mathbf{M}, \mathbf{j})$  we have

- (\*) (a)  $\mathbf{M}$  is a transitive class
- (b)  $\mathbf{M}$  is a model of ZFC
- (c)  ${}^x\mathbf{M} \subseteq M$
- (d)  $\mathbf{j}$  is an elementary embedding from  $\mathbf{V}$  into  $\mathbf{M}$
- (e)  $\mathbf{j}$  is with critical cardinal  $\lambda$
- (f)  $\mathbf{j}(h)(\lambda) = (\underline{\mathbb{P}}, \underline{\chi})$ .

Let  $\mathbf{q} = \mathbf{j}(\mathbf{q}_\lambda)$  so  $\mathbf{q} = \langle \mathbb{P}_\zeta, \mathbb{Q}_\xi : \zeta \leq \mathbf{h}(\lambda), \xi < \mathbf{h}(\lambda) \rangle$  and  $\zeta < \lambda \Rightarrow \mathbb{P}_\zeta^{\mathbf{q}} = \mathbb{P}_\zeta$ , etc.

Clearly  $\mathbf{M} \models$  “ $\mathbf{j}(h)(\lambda)$  is a pair of the form  $(\mathbb{P}', \chi')$  satisfying all relevant demands”.

The rest should be clear. □<sub>1.3</sub>

## § 2. ITERATION PARAMETERS

{c0x}

**Explanation 2.1.** For  $\mathbf{m} \in \mathbf{M}$  below:

- (a) we use  $L_{\mathbf{m}}$  as the index set for the iteration; always a well founded partial order
- (b)  $M_{\mathbf{m}}$  is the part of the index set we are really interested in, it may be  $(\kappa, <)$  in §1
- (c) the other part in the interesting case is “generic enough  $\mathbf{m}$ ”, more accurately enough existentially closed so that the iteration restricted to  $M$  will be “stabilize” under further extensions; inspite of  $L_{\mathbf{m}}$  being required to be well founded this will be well defined.

{c0}

**Hypothesis 2.2.** 1)  $\lambda = \lambda^{<\lambda}$  strongly inaccessible.2)  $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ .3)  $\theta_\varepsilon$  is an infinite regular cardinal  $> \varepsilon$  and  $< \lambda$ .4) Assume  $\lambda_2 \geq \lambda_1 \geq \lambda$  such that<sup>5</sup>  $\lambda_2 = (\lambda_2)^\lambda \geq \beth_3(\lambda_1)$  so pendentically all notations should have the parameter  $\bar{\lambda} = (\lambda_2, \lambda_1)$  and even  $\bar{\lambda} = (\lambda_2, \lambda_1, \lambda)$ .*Notation 2.3.*  $L, M$  denote partial orders, usually well founded.

{c1}

*Remark 2.4.* Here no harm in adding(a)  $\theta_\varepsilon > \prod_{\zeta < \varepsilon} 2^{\theta_\zeta} + 2^{\aleph_0}$  for  $\varepsilon < \lambda$   
or just(b)  $\bar{\theta}$  is increasing fast enough(c)  $M$  a linear order, well founded (and even  $M \cong (\kappa, <), \kappa$  regular  $> \lambda$ ).

{c2}

**Definition 2.5.** 1) For a partial order  $L$  let(α)  $\text{dp}(L) = \cup\{\text{dp}_L(t) + 1 : t \in L\}$ , see below,(β)  $\text{dp}_L(t) = \text{dp}(t, L) \in \text{Ord} \cup \{\infty\}$  be defined by  $\text{dp}_L(t) = \cup\{\text{dp}_L(s) + 1 : s <_L t\}$ .(γ)  $L_{<t} = L \upharpoonright \{s \in L : s <_L t\}$ ,(ζ)  $L_{\leq t} = L \upharpoonright \{s \in L : s \leq_L t\}$ .2) Let  $L^+ = L(+)$  be  $L \cup \{\infty\}$  with the natural order (but we may write  $t <_L \infty$  instead of  $t <_{L(+)} \infty$ ).3) We say the set  $L$  is an initial segment of the partial order  $L_*$  when  $s \in L \Rightarrow s \in L_*$  and  $s <_{L_*} t \wedge t \in L \Rightarrow s \in L$ .

{c3}

**Discussion 2.6.** Concerning the aim of the following choice, note the following.1) By the partial order we already can get partial memory, so why the  $u_s$ 's (in 2.7). Because  $\bar{u}$  is not necessarily transitive  $s \in u_t \not\Rightarrow u_s \subseteq u_t$ . By partial order we cannot get it.

{c4}

2) In [Sh:700] we use  $\mathcal{P}_t$  - an ideal, but here not necessary: this makes a difference but it uses “ $\mathbb{Q}_{\bar{\theta}}$  is close to being  $\lambda$ -centered, i.e. any subset  $a \{p \in \mathbb{Q}_{\bar{\theta}} : \text{tr}(p) = \eta\}$  of cardinality  $< \theta_{\ell g(\eta)}$  has a lub in this subset.<sup>5</sup>usually  $\lambda_2 \geq \lambda_1$  suffice but see 4.11, 4.21

{e23}

**Definition 2.7.** Let  $\mathbf{M}$  be the class of objects  $\mathbf{m}$ , called iteration parameters, of the following form (so really  $\mathbf{M} = \mathbf{M}[\bar{\lambda}]$  and if we omit clauses  $(\theta), (\iota), (\lambda)$  we may write  $\mathbf{M}[*]$ ): {c4}

- (a)  $L$ , a partial order,
- (b)  $M \subseteq L$ , as partial orders,
- (c)  $(\alpha)$   $\bar{u} = \langle u_t : t \in L \rangle$  and  $\bar{\mathcal{P}} = \langle \bar{\mathcal{P}}_t : t \in L \rangle$  and each  $\mathcal{P}_t$  is closed under subsets,
  - ( $\beta$ )  $u_t \subseteq \{s \in L : s <_L t\}$  and  $u \in \mathcal{P}_t \Rightarrow u \subseteq u_t$ ,
- (d)  $\text{dp}(L) < \infty$ , that is  $L$  is well founded,
- (e)  $(\alpha)$   $E'$  is a two-place relation (on  $L$ ),
  - ( $\beta$ )  $E'' := E' \upharpoonright (L \setminus M)$  is an equivalence relation on  $L \setminus M$
  - ( $\gamma$ ) if  $s, t \in L \setminus M$  are not  $E''$ -equivalent then  $(s <_L t) \Leftrightarrow (\exists r \in M)(s < r < t)$
  - ( $\delta$ ) if  $sE't$  then  $s \notin M \vee t \notin M$
  - ( $\varepsilon$ ) if  $t \in L \setminus M$  then  $\{s \in L : sE't\} = \{s \in L : tE's\}$ ; we call it  $t/E'$ ; so  $E'$  is a symmetric relation
  - ( $\zeta$ ) if  $s, t \in L \setminus M$  are  $E''$ -equivalent then  $s/E' = t/E'$
  - ( $\eta$ ) if  $t \in L \setminus M$  then  $u_t \subseteq t/E'$
  - ( $\theta$ ) if  $t \in L \setminus M$  then  $t/E'$  has cardinality  $\leq \lambda_2$
  - ( $\iota$ )  $\|M\| \leq \lambda_1$
  - ( $\kappa$ ) if  $t \in L$  and  $u \in \mathcal{P}_t$  then  $u \not\subseteq M \Rightarrow (\exists s)(s \in L \setminus M \wedge u \subseteq s/E')$
  - ( $\lambda$ )  $\mathcal{P}_t$  has cardinality  $\leq \lambda_2$  for  $t \in L \setminus M$  and for simplicity  $\mathcal{P}_t \subseteq [u_t]^{\leq \lambda}$  as only those sets matter.

{c5}

*Notation 2.8.* For  $\mathbf{m} \in \mathbf{M}$ .

{c4} 0) In 2.7 we let  $\mathbf{m} = (L_{\mathbf{m}}, M_{\mathbf{m}}, \bar{u}_{\mathbf{m}}, \bar{\mathcal{P}}_{\mathbf{m}}, E'_{\mathbf{m}})$  and  $\bar{u}_{\mathbf{m}} = \langle u_{\mathbf{m},t} : t \in L_{\mathbf{m}} \rangle$ ,  $\bar{\mathcal{P}}_{\mathbf{m}} = \langle \mathcal{P}_{\mathbf{m},t} : t \in L_{\mathbf{m}} \rangle$  and for  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  let  $t/E_{\mathbf{m}} = (t/E'_{\mathbf{m}}) \cup M_{\mathbf{m}}$  and for  $t \in M_{\mathbf{m}}$  let  $t/E_{\mathbf{m}} = M_{\mathbf{m}}$ .

{c4} 1) In 2.7, let  $\text{dp}_{\mathbf{m}}(t) = \text{dp}_{L_{\mathbf{m}}}(t)$ ,  $\text{dp}_{\mathbf{m}} = \text{dp}(L_{\mathbf{m}})$  and  $\leq_{\mathbf{m}} = \leq_{L_{\mathbf{m}}}$ .

2) For  $L \subseteq L_{\mathbf{m}}$ :

(a) let  $\mathbf{n} = \mathbf{m} \upharpoonright L$  means  $\mathbf{n} \in \mathbf{M}$ ,  $L_{\mathbf{n}} = L$ ,  $\leq_{\mathbf{n}} = \leq_{\mathbf{m}} \upharpoonright L_{\mathbf{n}}$ ,  $u_{\mathbf{n},t} = u_{\mathbf{m},t} \cap L$ ,  $\mathcal{P}_{\mathbf{n},t} = \mathcal{P}_{\mathbf{m},t} \cap [L]^{\leq \lambda}$  for  $t \in L$  and  $M_{\mathbf{n}} = M_{\mathbf{m}} \cap L$ ;

(b) let  $\text{dp}_{\mathbf{m}}(L) = \text{dp}(L_{\mathbf{m}} \upharpoonright L)$  or  $\text{dp}(L)$  for  $L \subseteq L_{\mathbf{m}}$ .

3) For  $t \in L_{\mathbf{m}}$ , let  $\mathbf{m}_{<t} = \mathbf{m}(\leq t) = \mathbf{m} \upharpoonright L_{<t}$  where  $L_{<t} = L_{\mathbf{m}(\leq t)} = L_{\mathbf{m},t} = \{s : s <_{\mathbf{m}} t\}$  so  $u_{\mathbf{m}(\leq t),s} = u_{\mathbf{m},s}$  for  $s \in L_{<t}$ , etc.

3A) Also  $\mathbf{m}_{\leq t} = \mathbf{m}(\leq t) = \mathbf{m} \upharpoonright L_{\leq t}$  where  $L_{\leq t} = L_{\mathbf{m}(\leq t)} = L_{<t} \cup \{t\}$ ; let  $L_{<\infty} = L$ ,  $L_{\leq \infty} = L^+$ , etc.

4)  $\mathbf{M}_{<\mu}$  is the class of  $\mathbf{m} \in \mathbf{M}$  such that  $L_{\mathbf{m}}$  has cardinality  $< \mu$ . Similarly  $\mathbf{M}_{\leq \mu}$ ,  $\mathbf{M}_{=\mu}$ ,  $\mathbf{M}_{>\mu}$ ,  $\mathbf{M}_{\geq \mu}$ ; let  $\mathbf{M}_{\mu} = \mathbf{M}_{=\mu}$ .

5) For  $\mathbf{m}, \mathbf{n} \in \mathbf{M}$  let  $\mathbf{m} \approx \mathbf{n}$ , and we may say  $\mathbf{m}, \mathbf{n}$  are equivalent mean  $L_{\mathbf{m}} = L_{\mathbf{n}}$  and  $t \in L_{\mathbf{n}} \Rightarrow u_{\mathbf{m},t} = u_{\mathbf{n},t} \wedge \mathcal{P}_{\mathbf{m},t} = \mathcal{P}_{\mathbf{n},t}$ .

6) We say  $f$  is an isomorphism from  $\mathbf{m}_1 \in \mathbf{M}$  onto  $\mathbf{m}_2 \in \mathbf{M}$  when:



- (a)  $f$  is an isomorphism from the partial order  $L_{\mathbf{m}_1}$  onto the partial order  $L_{\mathbf{m}_2}$
- (b) for  $s, t \in L_{\mathbf{m}_1}$  we have  $s \in u_{\mathbf{m}_1, t} \Leftrightarrow f(s) \in u_{\mathbf{m}_2, f(t)}$  and  $\mathcal{P}_{\mathbf{m}_2, f(t)} = \{\{f(s) : s \in u\} : u \in \mathcal{P}_{\mathbf{m}_1, t}\}$
- (c) for  $s, t \in L_{\mathbf{m}_1}$  we have  $sE'_{\mathbf{m}_1} t \Leftrightarrow f(s)E'_{\mathbf{m}_2} f(t)$
- (d)  $M_{\mathbf{m}_2} = \{f(s) : s \in M_{\mathbf{m}_1}\}$ .

7) We define weak isomorphisms as in part (6) omitting clauses (c),(d).

{c6}

**Definition 2.9.** For  $\mathbf{m} \in \mathbf{M}$  let  $L = L_{\mathbf{m}}$  and we define the iteration  $\mathbf{q}_{\mathbf{m}}$  to consist of:

- (a) a forcing notion  $\mathbb{P}_t = \mathbb{P}_{\mathbf{m}, t}$  for  $t \in L^+$ ; we let  $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_\infty$
- (b)  $\mathbb{Q}_t$  is the  $\mathbb{P}_t$ -name of a subforcing of  $\mathbb{Q}_{\bar{\theta}}$  in the universe  $\mathbf{V}^{\mathbb{P}_t}$ , even  $\subseteq_{\text{ic}}$
- (c)  $p \in \mathbb{P}_t$  iff
- ( $\alpha$ )  $p$  is a function
  - ( $\beta$ )  $\text{Dom}(p) \subseteq L_{<t}$  has cardinality  $< \lambda$
  - ( $\gamma$ ) if  $s \in \text{Dom}(p)$  then  $p(s)$  consists of  $\text{tr}(p(s)) \in \prod_{\varepsilon < \zeta(s)} \theta_\varepsilon$  for some  $\zeta_s = \zeta(s) < \lambda$  and  $\xi = \xi_{p(s)} \leq \lambda$  and  $\mathbf{B}_{p(s)}$  and  $\bar{r} = \bar{r}_{p(s)} = \langle r(\zeta) : \zeta < \xi_{p(s)} \rangle = \langle r_{p(s)}(\zeta) : \zeta < \xi_{p(s)} \rangle \in {}^\xi u_s$  the coordinates used in computing  $p(s)$  and  $\langle \mathbf{B}_{p(s), \iota}, \bar{r}_{p(s), \iota} : \iota < \iota(p(s)) \rangle$  such that:
    - $\mathbf{B}_{p(s)}$  is a  $\lambda$ -Borel function<sup>6</sup>,  $\mathbf{B} = \mathbf{B}_{p(s)} : {}^\xi (\prod_{\varepsilon < \lambda} \theta_\varepsilon) \rightarrow \Pi \bar{\theta}$  moreover into  $(\Pi \bar{\theta})^{[\text{tr}(p(s))]}$ ; and considering (d)( $\alpha$ ) below less pedantically  $p(s) = (\text{tr}(p(s)), \underline{f}_{p(s)})$ , where  $\underline{f}_{p(s)} = \mathbf{B}_{p(s)}(\dots, \eta_{r_{p(s)}(\zeta)}, \dots)_{\zeta < \xi}$  which means: absolutely, i.e. in every  $\mathbf{V}^{\mathbb{Q}}, \mathbb{Q}$  a  $(< \lambda)$ -strategically complete (which is  $\lambda^+$ -c.c.) forcing notion,  $\mathbf{B}_{p(s)}$  is such a  $(\lambda$ -Borel) function; we may write  $\xi_{p, s}$  instead of  $\xi_{p(s)}$ , etc.
    - $\iota(p(s)) < \lambda$  moreover<sup>7</sup>  $< \theta_{\ell g(\text{tr}(p(s)))}$
    - $\bar{r}_{p(s), \iota} = \bar{r}_{p(s)} \upharpoonright w_{p(s), \iota}$  so  $w_{p(s), \iota} = w(p(s), \iota) = \text{dom}(\bar{r}_{p(s), \iota}) \subseteq \xi_{p(s)}$  and  $\bar{r}_{p(s), \iota}$  is a subsequence of  $\bar{r}_{p(s)}$
    - $\mathbf{B}_{p(s), \iota}$  is a Borel function from  $w(p(s), \iota) (\prod_{\varepsilon < \lambda} \theta_\varepsilon)$  into  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{[\text{tr}(p(s))]}$
    - $\mathbf{B}_{p(s)}(\langle \eta_{r_{p(s)}(\zeta)} : \zeta < \xi_{p(s)} \rangle) = \sup\{\mathbf{B}_{p(s), \iota}(\langle \eta_{r_{p(s)}(\zeta)} : \zeta \in w_{p(s), \iota} \rangle) : \iota < \iota(p(s))\}$  so  $\underline{f}_{p(s)} = \sup\{\underline{f}_{p(s), \iota} : \iota < \iota(p(s))\}$ ,  $\underline{f}_{p(s), \iota} = \mathbf{B}_{p(s), \iota}(\langle \eta_\zeta : \zeta \in w_{p(s), \iota} \rangle)$
    - for each  $\iota < \iota(p(s))$  for some  $u \in \mathcal{P}_{\mathbf{m}, s}$  we have  $\{r_{p(s)}(\zeta) : \zeta < \xi_{p(s)}, \zeta \in w_{p(s), \iota}\} \subseteq u$  so is a subset of  $u_s$
    - if  $\iota < \iota(p(s))$  and  $\xi \in w_{p(s), \iota}, r_{p(s)}(\xi) \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  then  $\{r_{p(s)}(\zeta) : \zeta \in w_{p(s), \iota}\} \subseteq r_{p(s)}(\xi)/E_{\mathbf{m}}$ , (follows)
- (d) ( $\alpha$ )  $\eta_s$  is the  $\mathbb{P}_t$ -name, when  $t \in L_{\mathbf{m}}^+, s \in L_{<t}$  defined by  $\cup\{\text{tr}(p(s)) : p \in \mathbf{G}_{\mathbb{P}_t}\}$ .

<sup>6</sup>that is a definition of one

<sup>7</sup>This and the rest of (c)( $\gamma$ ) are used in the proof of 4.17. The aim is that defining  $B_{p(s)}$  from  $\{B_{p(s), \iota} : \iota\}$ , the sup will not given in  $\varepsilon$  the value  $\theta_\varepsilon$ . {e31}

- ( $\beta$ ) For  $p \in \mathbb{P}_t$  and  $s \in \text{Dom}(p)$  we interpret  $p(s)$  as a  $\mathbb{P}_s$ -name  
 $(\text{tr}(p(s)), \mathbf{B}_{p,s}(\dots, \eta_{r_{p,s}(\zeta)}, \dots)_{\zeta < \xi_{p,s}})$
- (e)  $\mathbb{P}_t \models "p \leq q"$  iff
- ( $\alpha$ )  $p, q \in \mathbb{P}_t$
  - ( $\beta$ )  $\text{Dom}(p) \subseteq \text{Dom}(q)$
  - ( $\gamma$ ) if  $t \in \text{Dom}(p)$  then  $(q \upharpoonright L_{<t}) \Vdash_{\mathbb{P}_{\mathbf{m}(<t)}} "p(t) \leq_{\mathbb{Q}_{\bar{\theta}}} q(t)"$ .
- {c7} **Definition 2.10.** 1) For  $p \in \mathbb{P}_{\mathbf{m}}$  let
- (a)  $\text{fsupp}(p)$ , the full support of  $p$  be  $\cup\{r_{p(s)}(\zeta) : \zeta < \xi_{p,s}\} \cup \{s : s \in \text{Dom}(p)\}$
  - (b)  $\text{wsupp}(p)$ , the wide support of  $p$  be  $\cup\{t/E_{\mathbf{m}} : t \in \text{fsupp}(p)\}$ .
- {c6} 2) For  $\mathbf{m} \in \mathbf{M}$  let  $\mathbb{P}_{\mathbf{m}}^{\mathbf{m}} = \mathbb{P}_{\mathbf{m},t}$ , etc., in Definition 2.9.
- 3) For  $L \subseteq L_{\mathbf{m}}$  let  $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} \upharpoonright \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$ .
- 4) For  $\mathbf{m} \in \mathbf{M}$  and  $t \in L_{\mathbf{m}}$  let<sup>8</sup>  $\mathbb{Q}_t = \mathbb{Q}_{\mathbf{m},t}$  be the  $\mathbb{P}_t$ -name of  $\mathbb{Q}_{\bar{\theta}} \upharpoonright \{(\nu, f) \in \mathbb{Q}_{\bar{\theta}} : f = \sup\{f_\iota : \iota < \iota(*)\}$  where  $\iota(*) < \theta_{\ell g(\nu)}$  and  $f_\iota \in (\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\eta_s : s \in u]}$  for some  $u \in \mathcal{P}_{\mathbf{m},t}\}$ .
- {c8} **Claim 2.11.** For  $\mathbf{m} \in \mathbf{M}$  (so  $\mathbb{P}_t = \mathbb{P}_{\mathbf{m},t}$ , etc.)

- (a) the iteration  $\mathbf{q}_{\mathbf{m}}$  is well defined,
- (b) ( $\alpha$ ) if  $t \in L_{\mathbf{m}}^+$  then  $\mathbb{P}_t$  is indeed a forcing notion and is equal to  $\mathbb{P}_{\mathbf{m}(<t)}$ ,  
 ( $\beta$ ) the  $\mathbb{P}_t$ -name  $\eta_s$  does not depend on  $t$  as long as  $s < t \in L_{\mathbf{m}}^+$ ,
- (c) if  $s <_L t$  are from  $L_{\mathbf{m}}^+$  then
  - ( $\alpha$ )  $p \in \mathbb{P}_s \Rightarrow p \in \mathbb{P}_t \wedge p \upharpoonright L_{<s} = p$ ,
  - ( $\beta$ ) if  $p, q \in \mathbb{P}_s$  then  $\mathbb{P}_t \models "p \leq q" \Leftrightarrow \mathbb{P}_s \models "p \leq q"$ ,
  - ( $\gamma$ ) if  $p \in \mathbb{P}_t$  then  $p \upharpoonright L_{<s} \in \mathbb{P}_s$  and  $\mathbb{P}_t \models "(p \upharpoonright L_{<s}) \leq p"$ ,
  - ( $\delta$ )  $\mathbb{P}_t \models "p \leq q" \Rightarrow \mathbb{P}_s \models "p \upharpoonright L_{<s} \leq q \upharpoonright L_{<s}"$ ,
  - ( $\varepsilon$ )  $\mathbb{P}_s < \mathbb{P}_t$  moreover
  - ( $\zeta$ )  $p \in \mathbb{P}_t \wedge (p \upharpoonright L_{<s}) \leq q \in \mathbb{P}_s \Rightarrow q \cup (p \upharpoonright (L_{<t} \setminus L_{<s})) \in \mathbb{P}_t$  is a  $\leq$ -lub of  $p, q$
  - ( $\theta$ )  $\mathbb{P}_{\mathbf{m},t} = \mathbb{P}_{\mathbf{m} \upharpoonright L_{<t}}$
- (d) if  $L$  is an initial segment of  $L_{\mathbf{m}}$  then  $\mathbb{P}_{\mathbf{m} \upharpoonright L} = \mathbb{P}_{\mathbf{m}} \upharpoonright \{p \in \mathbb{P}_{\mathbf{m}} : \text{dom}(p) \subseteq L, \text{equivalently } \text{fsupp}(p) \subseteq L\}$
- (e) if  $L_1 \subseteq L_2$  are initial segments of  $L_{\mathbf{m}}$ , then the parallel of clause (c) holds replacing  $\mathbb{P}_{\mathbf{m},s}, \mathbb{P}_{\mathbf{m},t}$  by  $\mathbb{P}_{\mathbf{m} \upharpoonright L_1}, \mathbb{P}_{\mathbf{m} \upharpoonright L_2}$ , respectively.

*Proof.* Straightforward. For  $t \in L_{\mathbf{m}}^+$ , by induction on  $\text{dp}_{\mathbf{m}}(t)$ , define  $\mathbb{P}_t$  and prove the relevant parts of (a),(b),(c). □<sub>2.11</sub>

{c10} Note

**Observation 2.12.** If  $\mathbf{B}$  is a  $\lambda$ -Borel function from  ${}^\xi(\Pi\bar{\theta})$  to  $\mathcal{P}(\lambda)$  or even  $\mathcal{H}(\lambda^+)$  where  $\xi \leq \lambda$  then there is a  $\lambda$ -Borel function  $\mathbf{B}'$  from  ${}^\xi(\Pi\bar{\theta})$  to  $\mathbb{Q}_{\bar{\theta}}$  (so absolutely to  $\mathbb{Q}_{\bar{\theta}}$ ) such that for any  $\bar{\eta} \in {}^\xi(\Pi\bar{\theta})$  we have, absolutely:

- if  $\mathbf{B}(\bar{\eta}) \in \mathbb{Q}_{\bar{\theta}}$  then  $\mathbf{B}'(\bar{\eta}) = \mathbf{B}(\bar{\eta})$

{c13} <sup>8</sup>not used, could have used it in 2.15

- {c11} • if  $\mathbf{B}(\bar{\eta}) \notin \mathbb{Q}_{\bar{\theta}}$  then  $\mathbf{B}'(\bar{\eta}) = (\emptyset, 0_\lambda)$ , the minimal member of  $\mathbb{Q}_{\bar{\theta}}$ .

**Claim 2.13.** Let  $\mathbf{m} \in \mathbf{M}$ .

1) If  $L_{\mathbf{m}}^+ \models$  “ $s < t$ ” then

$$(\alpha) \Vdash_{\mathbb{P}_{\mathbf{m},t}} \text{“}\eta_s \in \prod_{\varepsilon < \lambda} \theta_\varepsilon\text{”}$$

( $\beta$ ) if  $\mathbf{G} \subseteq \mathbb{P}_t$  is generic over  $\mathbf{V}$  and  $\eta_r = \eta_r[\mathbf{G}]$  for  $r \in L_{\mathbf{m},<t}$  and  $s \in L_{<t}, u \in \mathcal{P}_{\mathbf{m},s}$  and  $\nu \in \Pi\bar{\theta}$  is from  $\mathbf{V}[\langle \eta_r : r \in u \rangle] \subseteq \mathbf{V}[\mathbf{G}]$  then  $\nu <_{J_\lambda^{\text{bd}}} \eta_s$ .

2)  $\mathbb{P}_{\mathbf{m}}$  satisfies the  $\lambda^+$ -c.c.

3)  $\mathbb{P}_{\mathbf{m}}$  is  $(< \lambda)$ -strategically complete (even  $\lambda$ -strategically complete but not used).

4) If  $\bar{p} = \langle p_i : i < \delta \rangle$  is  $\leq_{\mathbb{P}_{\mathbf{m}}}$ -increasing,  $\delta < \lambda$  and  $i < j < \delta \wedge t \in \text{Dom}(p_i) \Rightarrow \text{tr}(p_i(t)) < \text{tr}(p_j(t))$  then  $\bar{p}$  has a  $\leq_{\mathbb{P}_{\mathbf{m}}}$ -upper bound  $p$ . Moreover,  $\text{Dom}(p) = \cup\{\text{Dom}(p_i) : i < \delta\}$  and  $s \in \text{Dom}(p_i) \Rightarrow \text{tr}(p(s)) = \cup\{\text{tr}(p_j(s)) : j \in [i, \delta]\}$ ; in fact also  $\text{fsupp}(p) = \cup\{\text{fsupp}(p_i) : i < \delta\}$  and  $p$  is a lub. Also, we can weaken the demand above to  $i < \delta \wedge s \in \text{Dom}(p_i) \Rightarrow \delta < \theta_{\varepsilon(s)}$  where we let  $\varepsilon(s) = \sup\{\text{lg}(\text{tr}(p_j(s))) : j \in [i, \delta]\}$ .

5A) If  $\zeta < \lambda$  and  $L_{\mathbf{m}}^+ \models$  “ $s < t$ ”, then the following is a dense open subset of  $\mathbb{P}_t$ :  $\mathcal{S}_{s,t,\zeta} = \{p \in \mathbb{P}_t : s \in \text{Dom}(p) \text{ and } \text{tr}(p(s)) \text{ has length } \geq \zeta\}$ .

5B) If  $p \in \mathbb{P}_{\mathbf{m}}$  and  $\zeta < \lambda$  then for some  $q \in \mathbb{P}_{\mathbf{m}}$  we have  $p \leq q$  and  $t \in \text{Dom}(p) \Rightarrow \text{tr}(p(t)) < \text{tr}(q(t))$  and  $t \in \text{Dom}(q) \Rightarrow \text{lg}(\text{tr}(q(t))) > \zeta$ .

6) If  $q$  is a  $\mathbb{P}_{\mathbf{m}}$ -name of a member of  $\mathcal{H}(\lambda^+)$ , e.g. of  $\mathbb{Q}_{\bar{\theta}}$  (in  $\mathbf{V}[\mathbb{P}_{\mathbf{m}}]$ ) then for some  $\xi \leq \lambda$  and  $\lambda$ -Borel function  $\mathbf{B} : \xi(\Pi\bar{\theta}) \rightarrow \mathcal{H}(\lambda^+)$  and a sequence  $\langle r_\zeta : \zeta < \xi \rangle$  of members of  $L_{\mathbf{m}}$  we have  $\Vdash_{\mathbb{P}_{\mathbf{m}}} \text{“}q = \mathbf{B}(\dots, \eta_{r_\zeta}, \dots)_{\zeta < \xi}\text{”}$ .

6A) If  $t \in L_{\mathbf{m}}^+$  and  $u \subseteq L_{<t}$  and  $\Vdash_{\mathbb{P}_t}$  “ $y$  is a member of  $\mathbb{Q}_{\bar{\theta}}$  from  $\mathbf{V}[\langle \eta_s : s \in u \rangle]$ ”, then for some  $\xi \leq \lambda$  and  $\lambda$ -Borel functions as in 2.9(6)( $\gamma$ ),  $\mathbf{B}_i : \xi(\Pi\bar{\theta}) \rightarrow \mathbb{Q}_{\bar{\theta}}$  for  $i < \xi$  and sequence  $\langle r_\zeta : \zeta < \xi \rangle$  of members of  $u_{\mathbf{m},t}$  we have  $\Vdash_{\mathbb{P}_t}$  “for some  $i < \xi$  we have  $y = \mathbf{B}_i(\dots, \eta_{r_\zeta}, \dots)_{\zeta < \xi}$ ”. Similarly when  $\Vdash$  “ $y \in \mathcal{H}(\lambda^+)$ ”. {c6}

7) If  $\mathbf{m}, \mathbf{n}$  are equivalent then  $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{n}}$  and  $\mathbb{P}_{\mathbf{m},t} = \mathbb{P}_{\mathbf{n},t}$  for  $t \in L_{\mathbf{m}} = L_{\mathbf{n}}$ .

8) Assume that  $p, q \in \mathbb{P}_{\mathbf{m}}$  are incompatible then there are  $r$  and  $s$  such that:

$$(a) q \leq_{\mathbb{P}_{\mathbf{m}}} r$$

$$(b) s \in \text{Dom}(p) \cap \text{Dom}(q)$$

$$(c) q(s) = r(s) \text{ moreover } \text{Dom}(q) \setminus L_{<s} = \text{Dom}(r) \setminus L_{<s} \text{ and } q \upharpoonright (L_{\mathbf{m}} \setminus L_{<s}) \leq r \upharpoonright (L_{\mathbf{m}} \setminus L_{<s})$$

$$(d) p \upharpoonright L_{\mathbf{m},<s} \leq_{\mathbb{P}_{\mathbf{m}}} r \upharpoonright L_{\mathbf{m},<s}$$

(e)  $r \upharpoonright L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$  “ $p(s)$  and  $q(s)$  are incompatible in  $\mathbb{Q}_{\bar{\theta}}$  which means:

$$(\alpha) \text{lg}(\text{tr}(q(s))) \neq \text{lg}(\text{tr}(p(s)))$$

( $\beta$ ) if  $\text{lg}(\text{tr}(q(s))) < \text{lg}(\text{tr}(p(s)))$  then for some ordinal  $\varepsilon$ ,  $\text{lg}(\text{tr}(q(s))) \leq \varepsilon < \text{lg}(\text{tr}(p(s)))$  and  $r \upharpoonright L_{\mathbf{m}(<s)} \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}} \text{tr}(p(s))(\varepsilon) < \underline{f}_{q(s)}(\varepsilon)$ ”

( $\gamma$ ) if  $\text{lg}(\text{tr}(q(s))) > \text{lg}(\text{tr}(p(s)))$  then for some ordinal  $\varepsilon$ ,  $\text{lg}(\text{tr}(q(s))) > \varepsilon \geq \text{lg}(\text{tr}(p(s)))$  and  $r \upharpoonright L_{\mathbf{m}(<s)} \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}} \text{“tr}(q(s))(\varepsilon) < \underline{f}_{p(s)}(\varepsilon)\text{”}$ .

9)  $\Vdash_{\mathbb{P}_{\mathbf{m}}} \text{“}\mathbf{V}[\langle \eta_s : s \in L_{\mathbf{m}} \rangle] = \mathbf{V}[\mathbf{G}]\text{”}$ .

Remark 2.14. What is the use of e.g. (6), (6A)? See 3.11(A)(b) and 2.15.

{c53}

*Proof.* We prove all parts by induction on  $\text{dp}_{\mathbf{m}}$ .

1) For clause  $(\alpha)$  for each  $\mathbf{m}$ , using the induction hypothesis and 2.11(e), the problem is only when  $\text{dp}_{\mathbf{m}}(t) = \text{dp}_{\mathbf{m}} - 1$  and use part (5A) proved below. For clause  $(\beta)$  use also part (6A) for  $\mathbb{P}_{\mathbf{m}(\langle t \rangle)}$  proved below. In both cases the proof of the parts quoted does not rely on part (1). {c8}

2) If  $p_\varepsilon \in \mathbb{P}_{\mathbf{m}}$  for  $\varepsilon < \lambda^+$  then we by the  $\Delta$ -system lemma can find  $u$  and unbounded  $S \subseteq \lambda^+$  such that  $\varepsilon \neq \zeta \in S \Rightarrow \text{Dom}(p_\varepsilon) \cap \text{Dom}(p_\zeta) = u$  and  $\langle \text{tr}(p_\varepsilon(\beta)) : \beta \in u \rangle$  is the same for all  $\varepsilon \in S$ . Now  $p_\varepsilon, p_\zeta$  has a common upper bound for every  $\varepsilon, \zeta \in u$ , i.e. we define  $r$  by

- $\text{Dom}(r) = \text{Dom}(p_\varepsilon) \cup \text{Dom}(p_\zeta)$
- $r(s) = p_\varepsilon(s)$  if  $s \in \text{Dom}(p_\varepsilon) \setminus \text{Dom}(p_\zeta)$
- $r(s) = p_\zeta(s)$  if  $s \in \text{Dom}(p_\zeta) \setminus \text{Dom}(p_\varepsilon)$
- if  $s \in \text{Dom}(p_\varepsilon) \cap \text{Dom}(p_\zeta)$  then  $r(s) = (\text{tr}(p_\varepsilon(s)), \max\{\underline{f}_{p_\varepsilon(s)}, \underline{f}_{p_\zeta(s)}\})$ .

3) By (4), the second sentence + (5B) below which use only the induction hypothesis.

4) We define  $p$  by:

- $\text{Dom}(p) = \cup\{\text{Dom}(p_i) : i < \delta\}$
- $\text{tr}(p(s)) = \cup\{\text{tr}(p_i(s)) : i < \delta \text{ satisfies } s \in \text{Dom}(p_i)\}$
- $\underline{f}_{p(s)} = \sup\{\underline{f}_{p_i(s)} : i < \delta \text{ satisfies } s \in \text{Dom}(p_i)\}$ .

{c6} Note that here having to really start with  $\langle \underline{f}_{p_i(s), \iota} : \iota < \iota(p_i(s)) \rangle$  and get  $\langle \underline{f}_{p(s), \iota} : \iota < \iota(p(s)) \rangle$ , see 2.9(c)( $\gamma$ ) causes no problem, similarly in the proof of part (2) - just take the union.

{c8} 5A) Obvious by the definition of  $\mathbb{P}_{\mathbf{m}}$  and 2.11(c).

5B) The proof is split to cases.

Case 1:  $\text{dp}_{\mathbf{m}}$  is zero

So  $L_m$  is empty.

Case 2:  $\text{dp}_{\mathbf{m}} = \alpha + 1$

Hence  $L_2 = \{s \in L : \text{dp}_{\mathbf{m}}(s) = \alpha\}$  is non-empty and letting  $L_1 = L_m \setminus L_2$ ; clearly  $s \in L_1 \Rightarrow \text{dp}_{\mathbf{m}}(s) < \alpha$ , so  $\text{dp}_{\mathbf{m} \upharpoonright L_1} \leq \alpha$ . Let  $\zeta_* = \sup\{\ell g(\text{tr}(p(s)) + 1 : s \in \text{dom}(p)) \cup \{\zeta + 1\}\}$ . Hence applying (4) and (5B) to  $\mathbf{m} \upharpoonright L_1$ , i.e. the induction hypothesis we can find  $q_1$  such that  $\mathbb{P}_{\mathbf{m} \upharpoonright L_1} \models "p \upharpoonright L_1 \leq q_1"$  and  $[s \in \text{Dom}(q_1) \Rightarrow \ell g(\text{tr}(q_1(s)) > \zeta_*]$  and  $q_1$  forces a value to  $\underline{f}_{p(s), \iota} \upharpoonright \zeta_*$ , call it  $\rho_s$  for  $s \in \text{Dom}(p) \cap L_2$  and  $\iota < \iota(p(s))$ .

{c8} Define  $q \in \mathbb{P}_{\mathbf{m}}$  by  $\text{Dom}(q) = \text{Dom}(q_1) \cup (L_2 \cap \text{dom}(p))$ ,  $q \upharpoonright L_1 = q_1$  and if  $s \in L_2 \cap \text{Dom}(p)$  then  $q(s) = (\rho_s, \rho_s \hat{\ } (\underline{f}_{p(s)} \upharpoonright [\zeta_*, \lambda]))$ , recalling 2.11.

Easily  $q$  is as required.

Case 3:  $\delta = \text{dp}_{\mathbf{m}}$  is a limit ordinal of cofinality  $\geq \lambda$

So  $\alpha = \sup\{\text{dp}_{\mathbf{m}}(s) + 1 : s \in \text{Dom}(p)\}$  is an ordinal  $< \delta$  and let  $L = \{s \in L_m : \text{dp}_{\mathbf{m}}(s) < \alpha\}$ , so  $L$  is an initial segment of  $L_m$  and applying the induction hypothesis to  $\mathbf{m} \upharpoonright L$ ,  $p$  we get  $q$  as required in  $\mathbb{P}_{\mathbf{m} \upharpoonright L}$  hence in  $\mathbb{P}_{\mathbf{m}}$ .

Case 4:  $\delta = \text{dp}_{\mathbf{m}}$  is a limit ordinal of cofinality  $< \lambda$ .

Let  $\langle \alpha_i : i < \text{cf}(\delta) \rangle$  be increasing continuous with limit  $\delta$ , let  $\alpha_{\text{cf}(\delta)} = \delta$  and for  $i \leq \text{cf}(\delta)$  let  $L_i := \{s \in L_m : \text{dp}_{\mathbf{m}}(s) < 1 + \alpha_i\}$ .

Now we choose  $(p_i, \zeta_i)$  by induction on  $i < \text{cf}(\delta)$  such that:

- (a)  $p_i \in \mathbb{P}_{\mathbf{m} \upharpoonright L_i}$
- (b)  $\mathbb{P}_{\mathbf{m} \upharpoonright L_i} \models "p \upharpoonright L_i \leq p_i \text{ and } p_j \leq p_i"$  when  $j < i$
- (c) if  $i$  is a limit ordinal then  $p_i$  is gotten from  $\langle p_j : j < i \rangle$  as in part (4)
- (d) if  $s \in \text{Dom}(p_i)$  then  $\ell g(\text{tr}(p_i(s))) \geq \zeta_i$
- (e)  $\langle \zeta_j : j < i \rangle$  is an increasing continuous sequence of ordinals  $< \lambda$  and if  $i$  is non-limit then  $\zeta_i$  is  $> \zeta$  and  $\geq \sum_{j < i} |\text{Dom}(p_j)| + |\text{Dom}(p)|$  and  $> \sup(\{\ell g(\text{tr}(p_j(s))) : j < i \text{ and } s \in p_j\} \cup \{\ell g(\text{tr}(p(s))) : s \in \text{Dom}(p)\})$ .

Using 2.11 and the induction hypothesis this is easy. {c8}

6) For transparency assume  $\Vdash "y \in \prod_{\varepsilon < \lambda} \theta_\varepsilon"$  or just  $\in {}^\lambda \mathbf{V}$ . By parts (4) + (5B),

i.e. part (3), for each  $\zeta < \lambda$  the following subset of  $\mathbb{P}_{\mathbf{m},t}$  is open and dense:  
 $\mathcal{S}_\zeta = \{p \in \mathbb{P}_{\mathbf{m},t} : \text{for some } \nu \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ (from } \mathbf{V}!) \text{ we have } p \Vdash_{\mathbb{P}_{\mathbf{m},t}} "y \upharpoonright \zeta = \nu"\}$ .

Clearly there is a maximal antichain  $\langle p_{\zeta,\varepsilon} : \varepsilon < \zeta_\zeta \rangle$  of  $\mathbb{P}_{\mathbf{m},t}$  included in  $\mathcal{S}_\zeta$  and by part (2) without loss of generality  $\zeta_\zeta \leq \lambda$ , the rest should be clear. In the general case we can code  $y$  as a subset of  $\lambda$ , etc.

6A) This too should be clear as  $\mathbb{P}_t$  satisfies the  $\lambda^+$ -c.c.

7) Look at the definitions.

8) Using parts (4) and (5B) and the definition this is easy.

9) Suppose toward contradiction that  $\mathbf{G}_1 \neq \mathbf{G}_2$  are generic subsets of  $\mathbb{P}_{\mathbf{m}}$  but  $s \in L_{\mathbf{m}} \Rightarrow \eta_s[\mathbf{G}_1] = \eta_s = \eta_s[\mathbf{G}_2]$ .

Let  $p_1 \in \mathbf{G}_1 \setminus \mathbf{G}_2$  hence there is  $p_2 \in \mathbf{G}_2$  such that  $p_2 \Vdash_{\mathbb{P}_{\mathbf{m}}} "p_1 \notin \mathbf{G}_2"$  hence  $p_1, p_2$  are incompatible. Let  $L_* = \{s \in L_{\mathbf{m}} : \mathbf{G}_1 \cap \mathbb{P}_{\leq s} = \mathbf{G}_2 \cap \mathbb{P}_{\leq s}\}$  so  $L_*$  is an initial segment of  $L_{\mathbf{m}}$ . If  $L_* = L_{\mathbf{m}}$  we can easily get a contradiction, so  $L_* \neq L_{\mathbf{m}}$  and let  $r \in L_{\mathbf{m}} \setminus L_*$  be such that  $L_{<r} \subseteq L_*$ . Now in part (8) we can get a contradiction having found a common to upper bound to  $p_1, p_2$ . □<sub>2.13</sub> {c13}

**Conclusion 2.15.** *Let  $\mathbf{m} \in \mathbf{M}$  and for notational transparency for some ordinal  $\beta(*), t \in L_{\mathbf{m}} \Leftrightarrow t \in \beta(*)$  and  $s <_{\mathbf{m}} t \Rightarrow s < t$ . Then  $\mathbf{q}$  is essentially a  $(< \lambda)$ -support iteration with  $\mathbb{Q}_\alpha = \{(\nu, f) \in \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[(\eta_\beta: \beta < \alpha)]} : \nu \triangleleft f, f = \sup\{f_\iota : \iota < \iota(\alpha)\}, \iota(\alpha) < \lambda, \nu \triangleleft f_\iota \text{ and } \{f_\iota : \iota < \iota(\alpha)\} \subseteq \cup\{\mathbb{Q}_\theta^{\mathbf{V}[(\eta_\alpha: \alpha \in u)]} : u \in \mathcal{P}_{\mathbf{m},\alpha}\}\}$ .*

*Proof.* Should be clear by 2.13. □<sub>2.15</sub> {c11}

Till now  $(E_{\mathbf{m}}, M_{\mathbf{m}})$  have played no role and we could have omitted them. {c26}

**Definition 2.16.** 1) We define the two-place relation  $\leq = \leq_{\mathbf{M}}$  on  $\mathbf{M}$  as follows:  
 $\mathbf{m} \leq \mathbf{n}$  iff

- (a)  $L_{\mathbf{m}} \subseteq L_{\mathbf{n}}$ , as partial orders of course,
- (b)  $M_{\mathbf{m}} = M_{\mathbf{n}}$ , yes! equal,
- (c)  $u_{\mathbf{m},t} = u_{\mathbf{n},t} \cap L_{\mathbf{m}}$  and<sup>9</sup>  $\mathcal{P}_{\mathbf{m},t} = \{u \cap L_{\mathbf{m}} : u \in \mathcal{P}_{\mathbf{n},t}\}$  for  $t \in M_{\mathbf{m}}$ ,
- (d)  $u_{\mathbf{m},t} = u_{\mathbf{n},t}$  and  $\mathcal{P}_{\mathbf{m},t} = \mathcal{P}_{\mathbf{n},t}$  for  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$
- (e) if  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  then  $t/E'_{\mathbf{m}} = t/E'_{\mathbf{n}}$  hence  $E'_{\mathbf{m}} = E'_{\mathbf{n}} \upharpoonright L_{\mathbf{m}}$
- (f) • if  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  then  $\mathcal{P}_{\mathbf{m},t} = \mathcal{P}_{\mathbf{n},t}$

<sup>9</sup>This is covered by clause (f) but see part (2).

- if  $t \in M_{\mathbf{m}}$  and  $s \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  then  $\{u \in \mathcal{P}_{\mathbf{m},t} : u \subseteq s/E_{\mathbf{m}}\} = \{u \in \mathcal{P}_{\mathbf{n},t} : u \subseteq s/E_{\mathbf{n}}\}$
- if  $t \in M_{\mathbf{m}}$  then  $\{u \in \mathcal{P}_{\mathbf{m},t} : u \subseteq M_{\mathbf{m}}\} = \{u \in \mathcal{P}_{\mathbf{n},t} : u \subseteq M_{\mathbf{n}}\}$ .

2) We define the two-place relation  $\leq_* = \leq_{\mathbf{M}}^*$  as in part (1) omitting clauses (b),(e) and (f).

{c28}

**Claim 2.17.** 1)  $\leq_{\mathbf{M}}$  is a partial order.

2) If  $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing, then its union  $\mathbf{m}_\delta$  (naturally defined) is a  $\leq_{\mathbf{M}}$ -lub and  $|L_{\mathbf{m}_\delta}| \leq \Sigma\{|L_{\mathbf{m}_\alpha}| : \alpha < \delta\}$ .

3) If  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $L \subseteq L_{\mathbf{m}}$  then  $p \in \mathbb{P}_{\mathbf{m}}(L) \Leftrightarrow p \in \mathbb{P}_{\mathbf{n}}(L)$  for every  $p$ .

4) If  $\mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{n}}$  and  $L \subseteq L_{\mathbf{m}}$  then  $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$  as quasi orders.

*Proof.* Easy. □<sub>2.17</sub>

{c31}

**Claim 2.18.**  $(\mathbf{M}, \leq_{\mathbf{m}})$  has amalgamation.

That is, if  $\mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}_1, \mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}_2$  and  $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = L_{\mathbf{m}_0}$  then there is  $\mathbf{m} \in \mathbf{M}$  such that  $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}, \mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}$  and  $L_{\mathbf{m}} = L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$ .

{c4} *Proof.* Note that by clause (e)( $\gamma$ ) of Definition 2.7:

- (\*) assume  $(s_1 \in L_{\mathbf{m}_1} \setminus L_{\mathbf{m}_0}) \cap (s_3 \in L_{\mathbf{m}_2} \setminus L_{\mathbf{m}_0})$  and  $s_2 \in L_{\mathbf{m}_0}$ ;
- if  $s_1 <_{\mathbf{m}_1} s_2 \wedge s_2 <_{\mathbf{m}_2} s_3$  then  $s_2 \in M_{\mathbf{m}_0}$
  - if  $s_3 <_{\mathbf{m}_2} s_2 \wedge s_2 <_{\mathbf{m}_1} s_1$  then  $s_2 \in M_{\mathbf{m}_0}$ .

We now define  $\mathbf{m}$  by:

- (\*) (a)
- ( $\alpha$ )  $t \in L_{\mathbf{m}}$  iff  $t \in L_{\mathbf{m}_1} \vee t \in L_{\mathbf{m}_2}$
  - ( $\beta$ )  $M_{\mathbf{m}} = M_{\mathbf{m}_0}$
- (b)  $s <_{\mathbf{m}} t$  iff one of the following occurs:
- ( $\alpha$ )  $s <_{\mathbf{m}_1} t$
  - ( $\beta$ )  $s <_{\mathbf{m}_2} t$
  - ( $\gamma$ )  $(s \in L_{\mathbf{m}_1} \setminus L_{\mathbf{m}_0})$  and  $t \in L_{\mathbf{m}_2} \setminus L_{\mathbf{m}_0}$  and  $(\exists r \in M_{\mathbf{m}_0})(s \leq_{\mathbf{m}_1} r \wedge r \leq_{\mathbf{m}_2} t)$
  - ( $\delta$ )  $s \in L_{\mathbf{m}_2} \setminus L_{\mathbf{m}_0}$  and  $t \in L_{\mathbf{m}_1} \setminus L_{\mathbf{m}_0}$  and for some  $r \in M_{\mathbf{m}_0}$ ,  $s \leq_{\mathbf{m}_2} r \wedge r \leq_{\mathbf{m}_1} t$
- (c)  $u_{\mathbf{m},t}$  is
- ( $\alpha$ )  $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$  if<sup>10</sup>  $t \in L_{\mathbf{m}_0}$
  - ( $\beta$ )  $u_{\mathbf{m}_1,t}$  if  $t \in L_{\mathbf{m}_1} \setminus L_{\mathbf{m}_0}$
  - ( $\gamma$ )  $u_{\mathbf{m}_2,t}$  if  $t \in L_{\mathbf{m}_2} \setminus L_{\mathbf{m}_0}$
- (d)  $E'_{\mathbf{m}} = E'_{\mathbf{m}_1} \cup E'_{\mathbf{m}_2}$
- (e)  $\mathcal{P}_{\mathbf{m},t}$  is
- ( $\alpha$ )  $\mathcal{P}_{\mathbf{m}_1,t} \cup \mathcal{P}_{\mathbf{m}_2,t}$  if  $t \in L_{\mathbf{m}_0}$
  - ( $\beta$ )  $\mathcal{P}_{\mathbf{m}_1,t}$  if  $t \in L_{\mathbf{m}_1} \setminus L_{\mathbf{m}_0}$
  - ( $\gamma$ )  $\mathcal{P}_{\mathbf{m}_2,t}$  if  $t \in L_{\mathbf{m}_2} \setminus L_{\mathbf{m}_0}$ .

Clearly

<sup>10</sup>but recall that  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}} \Rightarrow u_{\mathbf{m},t} = u_{\mathbf{m}_0,t} \wedge \mathcal{P}_{\mathbf{m},t} = \mathcal{P}_{\mathbf{m}_0,t}$

⊙  $\mathfrak{m} \in \mathbf{M}$  and  $\mathfrak{m}_1 \leq_{\mathbf{M}} \mathfrak{m}$  and  $\mathfrak{m}_2 \leq_{\mathbf{M}} \mathfrak{m}$ .

So we are done. □<sub>2.18</sub>

{c32n}

**Observation 2.19.** For  $p, q \in \mathbb{P}_{\mathfrak{m}}$  the following conditions are equivalent:

- (a)  $q \Vdash "p \in \mathfrak{G}_{\mathbb{P}_{\mathfrak{m}}}"$
- (b) if  $s \in \text{Dom}(p)$  then either  $s \in \text{Dom}(q)$  and  $(q \upharpoonright L_{\mathfrak{m}, < s}) \Vdash_{\mathbb{P}_{\mathfrak{m}, < s}} "p(s) \leq q(s)"$  or  $s \notin \text{Dom}(q)$ ,  $\text{tr}(p(s)) = \emptyset$  and  $q \upharpoonright L_{\mathfrak{m}, < s} \Vdash_{\mathbb{P}_{\mathfrak{m}, < s}} "p(s) \text{ is trivial, i.e. } \dot{f}_{p(s)} \text{ is constantly zero}"$
- (c)  $\mathbb{P}_{\mathfrak{m}} \models "p \leq q^+"$  where  $\text{Dom}(q^+) = \text{Dom}(q) \cup \text{Dom}(p)$  and  $q^+(s)$  is
  - (α)  $q(s)$  if  $s \in \text{Dom}(q)$
  - (β) the trivial condition if  $s \in \text{dom}(p) \setminus \text{dom}(q)$ ; note that  $\text{fsupp}(q^+) = \text{fsupp}(q) \cup \text{Dom}(p)$ .

*Remark 2.20.* We shall use this freely. Maybe better to change the order.

*Proof.* Obvious. □<sub>2.19</sub>

{c33n}

**Claim 2.21.** For  $\mathfrak{m} \in \mathbf{M}$  we have  $\mathbb{P}_{\mathfrak{m}}(L_1) \triangleleft \mathbb{P}_{\mathfrak{m}}(L_3)$  when:

- (\*) (a)  $L_2 \subseteq L_3$  are initial segments of  $L_{\mathfrak{m}}$
- (b)  $L_1 \subseteq L_3$  and  $L_0 = L_1 \cap L_2$
- (c)  $L_0$  is an initial segment of  $L_1$ , really follows
- (d)  $\mathbb{P}_{\mathfrak{m}}(L_0) \triangleleft \mathbb{P}_{\mathfrak{m}}(L_2)$
- (e)  $L_1 \setminus L_0$  is disjoint to  $M_{\mathfrak{m}}$
- (f) if  $t \in L_1 \setminus L_0$  then  $(t/E_{\mathfrak{m}}) \cap L_{\mathfrak{m}, < t} \subseteq L_1$ .

*Proof.* As  $\text{dp}_{\mathfrak{m}}(L_1) < \infty$  it suffices to prove by induction on the ordinal  $\gamma$  that:

- ⊕<sub>γ</sub> if  $\langle L_\ell : \ell \leq 3 \rangle$  satisfies (\*) of the claim and  $\text{dp}_{\mathfrak{m}}(L_1) \leq \gamma$  then
  - (a)  $\mathbb{P}_{\mathfrak{m}}(L_1) \triangleleft \mathbb{P}_{\mathfrak{m}}(L_3)$
  - (b) we have  $p_1 \in \mathbb{P}_{\mathfrak{m}}(L_1)$  and  $p_1 \leq q_1 \in \mathbb{P}_{\mathfrak{m}}(L_1) \Rightarrow p_3, q_1$  are compatible in  $\mathbb{P}_{\mathfrak{m}}(L_3)$  when:
    - (α)  $p_3 \in \mathbb{P}_{\mathfrak{m}}(L_3)$
    - (β)  $p_0 \in \mathbb{P}_{\mathfrak{m}}(L_0)$
    - (γ) if  $p_0 \leq q_0 \in \mathbb{P}_{\mathfrak{m}}(L_0)$  then  $p_2 := p_3 \upharpoonright L_2$  and  $q_0$  are compatible in  $\mathbb{P}_{\mathfrak{m}}(L_2)$
    - (δ)  $p_1 = p_0 \cup (p_3 \upharpoonright (L_1 \setminus L_0))$ .

Why this holds? Assume we have arrived to  $\gamma$ .

Clause (b): Recalling clause (f) of the assumption, indeed,  $p_1 \in \mathbb{P}_{\mathfrak{m}}(L_1)$  by the definitions (and clause (c)). If  $q_1$  contradicts the desired conclusion then by 2.13(8) there are  $s$  and  $p_3^+$  such that: {c11}

- ⊕ (a)  $s \in \text{dom}(q_1) \cap \text{dom}(p_3)$
- (b)  $p_3^+ \in \mathbb{P}_{\mathfrak{m}}(L_{\mathfrak{m}, < s})$
- (c)  $p_3^+$  is above  $p_3 \upharpoonright L_{\mathfrak{m}, < s}$  and above  $q_1 \upharpoonright L_{\mathfrak{m}, < s}$
- (d)  $p_3^+ \Vdash_{\mathbb{P}_{\mathfrak{m}, < s}} "p_3(s), q_1(s) \in \mathbb{Q}_{\bar{\theta}} \text{ are incompatible (in } \mathbb{Q}_{\bar{\theta}})"$ .

So  $s \in \text{dom}(q_1) \subseteq L_1$  and as  $L_2$  is an initial segment and clause  $(\gamma)$  of (b), clearly  $s \in L_0$  is impossible, so  $s \in L_1 \setminus L_0$ . As  $\mathbb{P}_{\mathbf{m}} \models "p_1 \leq q_1"$ , necessarily  $q_1 \upharpoonright L_{\mathbf{m}, < s} \Vdash_{\mathbb{P}_{\mathbf{m}, < s}} "p_1(s) \leq q_1(s)"$ , so as  $q_1 \upharpoonright L_{\mathbf{m}, < s} \leq p_3^+ \upharpoonright L_{\mathbf{m}, < s}$  (by  $\oplus(c)$ ), also  $p_3^+ \upharpoonright L_{\mathbf{m}, < s} \Vdash_{\mathbb{P}_{\mathbf{m}, < s}} "p_1(s) \leq q_1(s)"$ . As  $s \notin L_0$  clearly  $p_1(s) = p_3(s)$  by clause  $\boxplus_{\gamma}(b)(\delta)$ , so  $p_3^+ \upharpoonright L_{\mathbf{m}, < s} \Vdash_{\mathbb{P}_{\mathbf{m}, < s}} "p_3(s) \leq q_1(s)"$  and again easy contradiction to  $\oplus(d)$ .

Clause (a):

Clearly  $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_3)$  as quasi orders. Next we shall prove  $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq_{\text{ic}} \mathbb{P}_{\mathbf{m}}(L_3)$ , so assume  $q_1, q_2 \in \mathbb{P}_{\mathbf{m}}(L_1)$  has a common upper bound  $p_3$  in  $\mathbb{P}_{\mathbf{m}}(L_3)$ , and we should find one in  $\mathbb{P}_{\mathbf{m}}(L_1)$ . Without loss of generality  $\text{Dom}(q_1) \cup \text{Dom}(q_2) \subseteq \text{Dom}(p_3)$ .

As we are assuming  $\mathbb{P}_{\mathbf{m}}(L_0) \triangleleft \mathbb{P}_{\mathbf{m}}(L_2)$ , see  $(*) (d)$  there is  $p_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$  such that  $p_0 \leq q \in \mathbb{P}_{\mathbf{m}}(L_0) \Rightarrow q, p_3 \upharpoonright L_2$  are compatible in  $\mathbb{P}_{\mathbf{m}}(L_2)$  and let  $p_1 = p_0 \cup (p_3 \upharpoonright (L_1 \setminus L_0))$ . By  $\boxplus_{\gamma}(b)$  we know that  $p_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$  and  $p_1 \leq p'_1 \in \mathbb{P}_{\mathbf{m}}(L_1) \Rightarrow p_3, p'_1$  are compatible in  $\mathbb{P}_{\mathbf{m}}(L_2)$ . It suffices to prove that  $p_1$  is a common upper bound of  $q_1, q_2$ .

We may replace  $p_0$  by  $p'_0$  if  $p_0 \leq p'_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ . So without loss of generality for  $\ell = 1, 2$  we have  $\text{dom}(q_{\ell}) \cap L_0 \subseteq \text{dom}(p_0)$  hence  $\subseteq \text{dom}(p_1)$ , also recall  $\text{dom}(q_{\ell}) \setminus L_0 \subseteq \text{dom}(p_3) \cap L_1 \setminus L_0$  and by the choice of  $p_1$  we have  $\text{dom}(p_3) \cap L_1 \setminus L_0 \subseteq \text{dom}(p_1) \setminus L_0$ .

So together  $\text{dom}(q_{\ell}) \subseteq \text{dom}(p_1)$ .

As we are assuming  $\mathbb{P}_{\mathbf{m}}(L_0) \triangleleft \mathbb{P}_{\mathbf{m}}(L_2) \triangleleft \mathbb{P}_{\mathbf{m}}$  without loss of generality  $p_0$  is above<sup>11</sup>  $q_{\ell} \upharpoonright L_0$ . If toward contradiction assume that  $\ell \in \{1, 2\}$  and  $q_{\ell} \not\leq p_1$  then for some  $s \in \text{Dom}(q_{\ell})$  we have  $(q_{\ell} \upharpoonright L_{\mathbf{m}, < s}) \leq (p_1 \upharpoonright L_{\mathbf{m}, < s})$  but  $p_1 \upharpoonright L_{\mathbf{m}, < s} \not\Vdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, < s})} "q_{\ell}(s) \leq p_1(s)"$ . Clearly,  $s \in L_0$  is impossible so  $s \in L_1 \setminus L_0$  hence  $s \notin M_{\mathbf{m}}$ .

Let  $L'_0 = L_0, L'_1 = L_0 \cup (L_1 \cap L_{\mathbf{m}, < s}), L'_2 = L_2, L'_3 = L_3$  so  $(L'_0, L'_1, L'_2, L'_3)$  satisfies the assumptions of the present claim and  $\text{dp}_{\mathbf{m}}(L'_1) \leq \text{dp}_{\mathbf{m}}(s) < \gamma$ , hence by the induction hypothesis,  $\mathbb{P}_{\mathbf{m}}(L'_1) \triangleleft \mathbb{P}_{\mathbf{m}}(L'_3)$ .

Recall  $s \in L_1 \setminus L_0$  hence  $(s/E_{\mathbf{m}}) \cap L_{\mathbf{m}, < s} \subseteq L_1$  by clause (f) of the assumption of the claim, so  $\text{fsupp}(p_1 \upharpoonright \{s\}), \text{fsupp}(q_{\ell} \upharpoonright \{s\})$  are  $\subseteq L'_1$  hence  $p_1(s), q_{\ell}(s)$  are  $\mathbb{P}_{\mathbf{m}}(L'_1)$ -names. So recalling  $p_1 \upharpoonright L_{\mathbf{m}, < s} \not\Vdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, < s})} "q_{\ell}(s) \leq p_1(s)"$  and  $\mathbb{P}_{\mathbf{m}}(L'_1) \triangleleft \mathbb{P}_{\mathbf{m}}(L'_3)$  and  $L_{\mathbf{m}, < s} \subseteq L_3 = L'_3$  we have  $p_1 \upharpoonright L'_1 \not\Vdash_{\mathbb{P}_{\mathbf{m}}(L'_1)} "q_{\ell}(s) \leq p_1(s)"$ . Hence there is  $p_1^+$  such that  $p_1 \upharpoonright L'_1 \leq p_1^+ \in \mathbb{P}_{\mathbf{m}}(L'_1)$  such that  $p_1^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_1)} "q_{\ell}(s) \not\leq p_1(s)"$  so recalling  $\mathbb{P}_{\mathbf{m}}(L'_1) \triangleleft \mathbb{P}_{\mathbf{m}}(L'_3)$  we have  $p_1^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)} "q_{\ell}(s) \not\leq p_1(s)"$ .

But by  $\boxplus_{\gamma_1}(b)$  for  $\gamma_1 = \text{dp}_{\mathbf{m}}(s)$ , we know that  $p_1^+$  and  $p_3 \upharpoonright L_{\mathbf{m}, < s}$  are compatible (in  $\mathbb{P}_{\mathbf{m}}$ , equivalently  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, < s})$ ) so let  $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, < s})$  be a common upper bound of  $p_1^+, p_3 \upharpoonright L_{\mathbf{m}, < s}$ . Now  $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)} "q_{\ell}(s) \leq p_1(s)"$  as  $p_3 \leq p_3^+$ , see above, and  $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)} "q_{\ell}(s) \not\leq p_1(s)"$  as  $p_1^+ \leq p_3^+$ , see above.

Together a contradiction. So we have proved  $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq_{\text{ic}} \mathbb{P}_{\mathbf{m}}(L_3)$ .

To finish proving clause  $\boxplus_{\gamma}(a)$  that is  $\mathbb{P}_{\mathbf{m}}(L_1) \triangleleft \mathbb{P}_{\mathbf{m}}(L_3)$  note that clause  $\boxplus_{\gamma}(b)$  does this as for every  $p_3 \in \mathbb{P}_{\mathbf{m}}(L_3)$  there is  $p_0$  as in  $\boxplus_{\gamma}(\beta), (\gamma)$  by clause (d) of the claim's assumption and let  $p_1$  be as defined in  $\boxplus_{\gamma}(b)(\delta)$ .  $\square_{2.21}$

<sup>11</sup>Why? It suffices to prove that there is  $p'_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$  above  $p_0$  and above  $q_{\ell} \upharpoonright L_0$ . So toward contradiction assume this fails hence there is  $p_0^+ \in \mathbb{P}_{\mathbf{m}}(L_0)$  above  $p_0$  incompatible with  $q_{\ell} \upharpoonright L_0$ . By the choice of  $p_0$  we know that  $p_0^+, (p_3 \upharpoonright L_2)$  are compatible, so let  $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_2)$  be a common upper bound. Hence  $p_3^+$  is above  $q_{\ell} \upharpoonright L_2$  as  $p_3 \upharpoonright L_2$  is. Also  $p_3^+$  is above  $p_0^+$  which forces  $q_{\ell} \upharpoonright L_0 \notin \mathbf{G}_{\mathbb{P}_{\mathbf{m}}(L_0)}$ , equivalently  $q_{\ell} \upharpoonright L_0 \notin \mathbf{G}_{\mathbb{P}_{\mathbf{m}}(L_2)}$ , contradiction.



{c33s}

**Claim 2.22.** We have  $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_1)$  and  $\mathbb{P}_{\mathbf{m}_\ell}(L_1) \triangleleft \mathbb{P}_{\mathbf{m}_\ell}$  for  $\ell = 1, 2$  when:

- (\*) (a)  $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$
- (b)  $L_0 \subseteq L_1 \subseteq L_{\mathbf{m}_1}$
- (c)  $L_0$  is an initial segment of  $L_1$
- (d)  $\mathbb{P}_{\mathbf{m}_1}(L_0) = \mathbb{P}_{\mathbf{m}_2}(L_0)$
- (e)  $\mathbb{P}_{\mathbf{m}_\ell}(L_0) \triangleleft \mathbb{P}_{\mathbf{m}_\ell}$  for  $\ell = 1, 2$
- (f) if  $t \in L_1 \setminus L_0$  then  $t \notin M_{\mathbf{m}_2}$  and  $L_{\mathbf{m}_1, < t} \cap (t/E_{\mathbf{m}_1}) = L_{\mathbf{m}_2, < t} \cap (t/E_{\mathbf{m}_1}) \subseteq L_1$ .

*Remark 2.23.* Used only in the proof of  $\boxplus_{4.4}$  inside the proof of 4.19, so we could have used  $M_\beta, \mathcal{E}$  from there. {e32}

*Proof.* For  $\ell \in \{1, 2\}$  let  $\bar{L}_\ell = \langle L_{\ell, i} : i < 4 \rangle$  be defined by:

- $\oplus_1$  (a)  $L_{\ell, 0} = L_0$
- (b)  $L_{\ell, 1} = L_1$
- (c)  $L_{\ell, 2} = \{s \in L_{\mathbf{m}_\ell} : s \leq_{\mathbf{m}_\ell} t \text{ for some } t \in L_0\}$
- (d)  $L_{\ell, 3} = L_{\mathbf{m}_\ell}$ .

Clearly

- $\oplus_2$  (a)  $(\mathbf{m}_\ell, \bar{L}_\ell)$  satisfies the assumptions of 2.21 hence {c33n}
- (b)  $\mathbb{P}_{\mathbf{m}_\ell}(L_{\ell, 1}) \triangleleft \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell, 3})$  which means  $\mathbb{P}_{\mathbf{m}_\ell}(L_1) \triangleleft \mathbb{P}_{\mathbf{m}_\ell}$  for  $\ell = 1, 2$ .

Let  $\langle s_\alpha : \alpha < \alpha(*) \rangle$  list  $L_1 \setminus L_0$  such that  $s_\alpha \leq_{L_{\mathbf{m}}} s_\beta \Rightarrow \alpha \leq \beta$ . Let  $L_{0, \alpha} = L_0 \cup \{s_\beta : \beta < \alpha\}$  for  $\alpha \leq \alpha(*)$ . Now by induction on  $\alpha \leq \alpha(*)$  we prove that

$$\boxplus_\alpha \mathbb{P}_{\mathbf{m}_1}(L_{0, \alpha}) = \mathbb{P}_{\mathbf{m}_2}(L_{0, \alpha}).$$

Case 1:  $\alpha = 0$

As  $L_{0, \alpha} = L_0$ , clause (d) of the assumption gives  $\boxplus_\alpha$  as promised.

Case 2:  $\alpha$  a limit ordinal

Easy by the definition of the iteration. That is, first we know  $p \in \mathbb{P}_{\mathbf{m}_1}(L_{0, \alpha}) \Leftrightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L_{0, \alpha})$ ; second, for  $p, q \in \mathbb{P}_{\mathbf{m}_1}(L_{0, \alpha})$  by the definition of the order and the induction hypothesis,  $\mathbb{P}_{\mathbf{m}_1}(L_{0, \alpha}) \models "p \leq q"$  iff  $\mathbb{P}_{\mathbf{m}_2}(L_{0, \alpha}) \models "p \leq q"$ .

So  $\boxplus_\alpha$  holds.

Case 3:  $\alpha = \beta + 1$

Clearly

$$(*)_1 \quad p \in \mathbb{P}_{\mathbf{m}_1}(L_{0, \alpha}) \Leftrightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L_{0, \alpha}).$$

Next

(\*)<sub>2</sub> assume  $p, q \in \mathbb{P}_{\mathbf{m}_1}(L_{0, \alpha})$  and we shall prove that  $\mathbb{P}_{\mathbf{m}_1}(L_{0, \alpha}) \models "p \leq q"$  implies  $\mathbb{P}_{\mathbf{m}_2}(L_{0, \alpha}) \models "p \leq q"$ .

[Why? If  $s_\beta \notin \text{dom}(p)$  this is obvious by the induction hypothesis.

Next assume  $s_\beta \in \text{dom}(p) \cap \text{dom}(q)$ . First,  $\mathbb{P}_{\mathbf{m}_1}(L_{0,\beta}) \Vdash "(p \restriction L_{0,\beta}) \leq (q \restriction L_{0,\beta})"$  and  $(q \restriction L_{0,\beta}) \Vdash_{\mathbb{P}_{\mathbf{m}_1, < s_\beta}} "p(s_\beta) \leq_{\mathbb{Q}_{\bar{\theta}}} q(s_\beta)"$  by the definition of  $\mathbb{P}_{\mathbf{m}_1}(L_{0,\beta})$ . Second, as  $q \restriction L_{0,\beta} \in \mathbb{P}_{\mathbf{m}_1}(L_{0,\beta}) = \mathbb{P}_{\mathbf{m}_2}(L_{0,\beta})$  and  $\mathbb{P}_{\mathbf{m}_1}(L_{0,\beta}) \triangleleft \mathbb{P}_{\mathbf{m}_2}$  and  $p(s_\beta), q(s_\beta)$  are  $\mathbb{P}_{\mathbf{m}_1}(L_{0,\beta})$ -names (as  $\text{fsupp}(p(s_\beta), \text{fsupp}(q(s_\beta))) \subseteq L_{0,\beta}$ ) also  $q \restriction L_{0,\beta} \Vdash_{\mathbb{P}_{\mathbf{m}_2}} "p(s_\beta) \leq_{\mathbb{Q}_{\bar{\theta}}} q(s_\beta)"$ . Third, as  $\mathbb{P}_{\mathbf{m}_1}(L_{0,\beta}) \Vdash "p \restriction L_{0,\beta} \leq q \restriction L_{0,\beta}"$ , by the induction hypothesis  $\mathbb{P}_{\mathbf{m}_2}(L_{0,\beta}) \Vdash "p \restriction L_{0,\beta} \leq q \restriction L_{0,\beta}"$ . Fourth, by the last two sentence and the definition of the order in  $\mathbb{P}_{\mathbf{m}_2}$  we have  $\mathbb{P}_{\mathbf{m}_2} \Vdash "p \leq q"$  so the conclusion of  $(*)_2$  holds also in this case.

If  $s_\beta \in \text{dom}(p) \setminus \text{dom}(q)$  then  $p \not\leq q$ , so we are done proving  $(*)_2$ .

$(*)_3$  if  $p, q \in \mathbb{P}_{\mathbf{m}_1}(L_{0,\alpha})$  and  $\mathbb{P}_{\mathbf{m}_2}(L_{0,\alpha}) \Vdash "p \leq q"$  then  $\mathbb{P}_{\mathbf{m}_1}(L_{0,\alpha}) \Vdash "p \leq q"$ .

[Why? Similar to the proof of  $(*)_2$ .]

By  $(*)_1, (*)_2, (*)_3$  clearly  $\boxplus_\alpha$  holds. So we carried the induction so  $\boxplus_\alpha$  holds for every  $\alpha < \alpha^*$  and for  $\alpha = \alpha^*$  we get  $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_2)$ . Together with  $\boxplus_2(b)$  in the beginning of the proof we are done.  $\square_{2.22}$

{c34}

**Definition 2.24.** 0) For  $L \subseteq L_{\mathbf{m}}, \mathbf{m} \in \mathbf{M}$  let

- (a)  $\text{dp}_{\mathbf{m}}^*(L) = \cup \{ \text{dp}_{M_{\mathbf{m}}}(t) + 1 : t \in L \cap M_{\mathbf{m}} \}$
- (b)  $L_{\mathbf{m}, \gamma}^{\text{dp}} = \{ t \in L_{\mathbf{m}} : \text{if } s \leq_{L_{\mathbf{m}}} t \wedge s \in M_{\mathbf{m}} \text{ then } \text{dp}_{M_{\mathbf{m}}}(s) < \gamma; \text{ moreover, } \text{sup} \{ \text{dp}_{M_{\mathbf{m}}}(s) : s \in M_{\mathbf{m}} \text{ and } s <_{L_{\mathbf{m}}} t \} < \gamma \}$ .

{c7} 1) For an ordinal  $\gamma$  let  $\mathbf{M}_\gamma^{\text{ec}}$  be the class of  $\mathbf{m} \in \mathbf{M}$  such that, recalling Definition 2.10(3):

- (\*) if  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$  then  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) \triangleleft \mathbb{P}_{\mathbf{m}_2}(L_{\mathbf{m}_2, \gamma}^{\text{dp}})$  hence  $L \subseteq L_{\mathbf{m}_1, \gamma}^{\text{dp}}$  implies  $\mathbb{P}_{\mathbf{m}_1}(L) = \mathbb{P}_{\mathbf{m}_2}(L)$ .

2) Let  $\mathbf{M}_{\text{ec}} = M_\infty^{\text{ec}}$  be the class of  $\mathbf{m}$  which  $\in \mathbf{M}_\gamma^{\text{ec}}$  for every  $\gamma$ .

3) Let  $\mathbf{M}_{\chi, \gamma}^{\text{ec}} = \{ \mathbf{m} \in \mathbf{M}_\gamma^{\text{ec}} : |L_{\mathbf{m}}| \leq \chi \}$ , similarly  $\mathbf{M}_{\chi, \infty}^{\text{ec}}$ .

{c36d}

**Observation 2.25.** 1) Of course,  $\mathbf{M}_{\gamma_2}^{\text{ec}} \subseteq \mathbf{M}_{\gamma_1}^{\text{ec}}$  and  $L_{\mathbf{m}, \gamma_1}^{\text{dp}} \subseteq L_{\mathbf{m}, \gamma_2}^{\text{dp}}$  are initial segments of  $L_{\mathbf{m}}$  when  $\gamma_1 \leq \gamma_2$ .

{c34} 2) In 2.24(1), the following are equivalent:

- (a)  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) \triangleleft \mathbb{P}_{\mathbf{m}_2}(L_{\mathbf{m}_2, \gamma}^{\text{dp}})$  for every  $\gamma$
- (b)  $\mathbb{P}_{\mathbf{m}_1} \triangleleft \mathbb{P}_{\mathbf{m}_2}$ .

*Proof.* 1) Easy.

{c8} 2) First, (a)  $\Rightarrow$  (b), note that for  $\gamma$  large enough we have  $\mathbb{P}_{\mathbf{m}_\ell}(L_{\mathbf{m}_\ell, \gamma}^{\text{dp}}) = \mathbb{P}_{\mathbf{m}_\ell}$ , so clear. Second, assume (b), note that  $\mathbb{P}_{\mathbf{m}_\ell}(L_{\mathbf{m}_\ell, \gamma}^{\text{dp}}) \triangleleft \mathbb{P}_{\mathbf{m}_\ell}$  for  $\ell = 1, 2$  by 2.11(c), hence we have  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) \triangleleft \mathbb{P}_{\mathbf{m}_1} \triangleleft \mathbb{P}_{\mathbf{m}_2}$ , but  $\triangleleft$  is transitive, hence  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) \triangleleft \mathbb{P}_{\mathbf{m}_2}$ . Also  $\mathbb{P}_{\mathbf{m}_2}(L_{\mathbf{m}_2, \gamma}^{\text{dp}}) \triangleleft \mathbb{P}_{\mathbf{m}_2}$  and  $L_{\mathbf{m}_1, \gamma}^{\text{dp}} \subseteq L_{\mathbf{m}_2, \gamma}^{\text{dp}}$  by the definition hence by the definition  $p \in \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) \Rightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L_{\mathbf{m}_2, \gamma}^{\text{dp}})$ ; but lastly  $(\mathbb{Q}_1 \triangleleft \mathbb{P} \wedge \mathbb{Q}_2 \triangleleft \mathbb{P} \wedge (\forall p)(p \in \mathbb{Q}_1 \rightarrow p \in \mathbb{Q}_2)) \Rightarrow \mathbb{Q}_1 \triangleleft \mathbb{Q}_2$  so we are done.  $\square$

{c41}

**Crucial Claim 2.26.** If  $\chi \geq 2^{\lambda_2}$  and  $\mathbf{m} \in \mathbf{M}_{\leq \chi}$  then for some  $\mathbf{n}$  we have  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$  and  $\mathbf{n} \in \mathbf{M}_{\text{ec}}$ .

*Proof.* Let  $\mathcal{X} = \{\mathbf{n} : (\mathbf{m} \upharpoonright M_{\mathbf{m}}) \leq_{\mathbf{M}} \mathbf{n} \text{ and } L_{\mathbf{n}} \setminus M_{\mathbf{m}} = t/E_{\mathbf{n}}'' \text{ for some } t \text{ hence } \|L_{\mathbf{n}}\| \leq \lambda_2\}$ .

We define a two-place relation  $\mathcal{E}$  on  $\mathcal{X}$ :

- $\mathbf{n}_1 \mathcal{E} \mathbf{n}_2$  iff  $(\mathbf{n}_1, \mathbf{n}_2 \in \mathcal{X} \text{ and})$  there is an isomorphism  $h$  from  $\mathbf{n}_1$  onto  $\mathbf{n}_2$  over  $\mathbf{m} \upharpoonright M_{\mathbf{m}}$ , that is: an isomorphism from  $L_{\mathbf{n}_1}$  onto  $L_{\mathbf{n}_2}$  over  $M_{\mathbf{m}}$  such that  $t \in L_{\mathbf{n}_1} \Rightarrow u_{\mathbf{n}_2, h(t)} = \{h(s) : s \in u_{\mathbf{n}_1, t}\}$  and  $t \in L_{\mathbf{n}_1} \Rightarrow \mathcal{P}_{\mathbf{n}_2, h(t)} = \{\{h(s) : s \in u\} : u \in \mathcal{P}_{\mathbf{n}_1, t}\}$  and  $s, t \in L_{\mathbf{n}_1} \Rightarrow (sE_{\mathbf{n}_1}' t \Leftrightarrow h(s)E_{\mathbf{n}_2}' h(t))$ .

Clearly  $\mathcal{E}$  is an equivalence relation.

By our assumptions  $\chi \geq 2^{\lambda_2}$  and  $\mathbf{n} \in \mathcal{X} \Rightarrow |L_{\mathbf{n}}| \leq \lambda_2 \wedge (\forall t \in L_{\mathbf{n}})(\mathcal{P}_{\mathbf{n}, t} \subseteq [L_{\mathbf{n}, < t}]^{\leq \lambda})$  hence recalling  $\lambda_2 = (\lambda_n)^\lambda$  clearly  $\mathcal{E}$  has  $\leq 2^{\lambda_2}$  equivalence classes and let  $\langle \mathbf{n}_\alpha : \alpha < 2^{\lambda_2} \rangle$  be a set of representatives (not necessary, but no harm in allowing repetitions).

We can find  $\mathbf{n}$  such that:

- (\*)<sub>1</sub> (a)  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$
- (b) for every  $\alpha < 2^{\lambda_2}$  we can find  $\langle t_{\alpha, i} : i < \chi \rangle$  such that
  - (\alpha)  $t_{\alpha, i} \in L_{\mathbf{n}} \setminus L_{\mathbf{m}}$
  - (\beta)  $(\alpha \neq \beta) \vee (i \neq j) \Rightarrow t_{\alpha, i}/E_{\mathbf{n}} \neq t_{\beta, j}/E_{\mathbf{n}}$
  - (\gamma)  $\mathbf{n} \upharpoonright (t_{\alpha, i}/E_{\mathbf{n}})$  is  $\mathcal{E}$ -equivalent to  $\mathbf{n}_\alpha$ .

Let us prove that  $\mathbf{n}$  is as required. Let  $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$  and define  $\mathcal{F}$  as the set of functions  $f$  such that some  $L_1, L_2$ :

- (\*)<sub>2</sub> (a)  $L_\ell \subseteq L_{\mathbf{n}_2}$
- (b)  $M_{\mathbf{m}} = M_{\mathbf{n}} \subseteq L_1 \cap L_2$
- (c)  $L_\ell \setminus M_{\mathbf{m}}$  has cardinality  $\leq \lambda_2$
- (d)  $L_\ell$  is  $E_{\mathbf{n}_2}$ -closed, i.e.  $t \in L_\ell \setminus M_{\mathbf{m}} \Rightarrow t/E_{\mathbf{n}_2} \subseteq L_\ell$
- (e)  $f$  is an isomorphism from  $\mathbf{n}_2 \upharpoonright L_1$  onto  $\mathbf{n}_2 \upharpoonright L_2$  over  $M_{\mathbf{m}}$ , i.e.
  - $f$  is a one-to-one mapping from  $L_1$  onto  $L_2$
  - $f \upharpoonright M_{\mathbf{m}}$  is the identity
  - $f$  maps  $\leq_{\mathbf{n}_2} \upharpoonright L_1$  onto  $\leq_{\mathbf{n}_2} \upharpoonright L_2$
  - for  $s, t \in L_1$  we have  $sE_{\mathbf{n}_2}' t \Leftrightarrow f(s)E_{\mathbf{n}_2}' f(t)$
  - for  $s, t \in L_1$  we have  $s \in u_{\mathbf{n}_2, t} \Leftrightarrow f(s) \in u_{\mathbf{n}_2, f(t)}$
  - for  $t \in L_1$  we have  $\mathcal{P}_{\mathbf{n}_2, f(t)} = \{\{f(s) : s \in u\} : u \in \mathcal{P}_{\mathbf{n}_2, t}\}$ .

Clearly

- (\*)<sub>3</sub> if  $f \in \mathcal{F}$  and  $L' \subseteq L_{\mathbf{n}_1}, L'' \subseteq L_{\mathbf{n}_2}$  and  $|L'| + |L''| \leq \lambda_2$  then for some  $g \in \mathcal{F}$  extending  $f$  we have  $L' \subseteq \text{Dom}(g), L'' \subseteq \text{Rang}(g)$  and  $\text{Rang}(g) \setminus L'' \setminus \text{Rang}(f) \subseteq L_{\mathbf{n}_1}$ .

We can finish as in the parallel of the Tarski-Vaught criterion for  $\mathbb{L}_{\infty, \lambda_2^+}$ . That is, first we can prove by induction on the ordinal  $\gamma < |L_{\mathbf{n}_2}|^+$  and really just  $\gamma < \|M_{\mathbf{n}_2}\|^+$  that

- (\*)<sub>4</sub> letting  $L_\gamma = L_{\mathbf{n}_2, \gamma}^{\text{dp}}$ , if  $g \in \mathcal{F}$  then
  - (a)  $g$  maps  $\text{Dom}(g) \cap L_\gamma$  onto  $\text{Rang}(g) \cap L_\gamma$

- (b)  $g$  induces an isomorphism  $\hat{g}$  from  $\mathbb{P}_{\mathbf{n}_2}(\text{Dom}(g) \cap L_\gamma)$  onto  $\mathbb{P}_{\mathbf{n}_2}(\text{Rang}(g) \cap L_\gamma)$ , that is:  $\hat{g}(p) = q$  iff
- ( $\alpha$ )  $p \in \mathbb{P}_{\mathbf{n}_2}(\text{Dom}(g) \cap L_\gamma)$
  - ( $\beta$ )  $q \in \mathbb{P}_{\mathbf{n}_2}(\text{Rang}(g) \cap L_\gamma)$
  - ( $\gamma$ )  $g$  maps  $\text{dom}(p)$  onto  $\text{dom}(q)$  and  $s \in \text{dom}(p) \Rightarrow \text{tr}(p(s)) = \text{tr}(q(g(s)))$
  - ( $\delta$ ) if  $s \in \text{Dom}(g)$ ,  $g(s) = t \in \text{Rang}(g)$  and  $f_{p(s)} = \mathbf{B}_{p(s)}(\dots, \eta_{r_{p(s)}(\zeta)}, \dots)_{\zeta < \xi_{p(s)}}$  and  $f_{q(t)} = \mathbf{B}_{q(t)}(\dots, \eta_{r_{q(t)}(\zeta)}, \dots)_{\zeta < \xi_{q(t)}}$  then  $\xi_{q(t)} = \xi_{p(s)}$ ,  $\mathbf{B}_{q(t)} = \mathbf{B}_{p(s)}$  and  $\zeta < \xi_{p(s)} \Rightarrow r_{q(t)}(\zeta) = g(r_{p(s)}(\zeta))$
  - ( $\varepsilon$ ) moreover in ( $\delta$ ) we have  $\iota(s, p) = \iota(t, q)$  and if  $\iota < \iota(s, p)$  then  $w_{p,s,\iota} = w_{q,t,\iota}$ ,  $\mathbf{B}_{p(s),\iota} = \mathbf{B}_{q(t),\iota}$ .

Second,

$$(*)_5 \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \triangleleft \mathbb{P}_{\mathbf{n}_2}(L_\gamma).$$

[Why? By the definitions  $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \subseteq \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$  as partial orders.

Also if  $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$  are compatible in  $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$  let  $q \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$  be a common upper bound there. We can find an  $E_{\mathbf{n}_2}$ -closed  $L' \subseteq L_\gamma \cap L_{\mathbf{n}_1}$  of cardinality  $\leq \lambda_2$  such that  $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_1}(L')$  and  $E_{\mathbf{n}_2}$ -closed  $L'' \subseteq L_\gamma$  of cardinality  $\leq \lambda_2$  such that  $L' \subseteq L''$  and  $q \in \mathbb{P}_{\mathbf{n}_2}(L'')$ . Now we can find  $f_1 \in \mathcal{F}$  such that  $\text{Dom}(f_1) = \cup\{t/E_{\mathbf{n}_2} : t \in L'\}$  recalling that  $t/E_{\mathbf{m}} \supseteq M_{\mathbf{m}}$ , see 2.8(0) and  $f_1$  is the identity. Then by  $(*)_3$  we can find  $f_2 \in \mathcal{F}$  extending  $f_1$  with  $\text{Dom}(f_2) = \cup\{t/E_{\mathbf{n}_2} : t \in L''\}$  and  $\text{Rang}(f_2) \setminus \text{Rang}(f_1) \subseteq L_{\mathbf{n}_1}$ . So we have  $\mathbb{P}_{\mathbf{n}_2} \models "(p_1 \leq \hat{f}_2(q)) \wedge p_2 \leq \hat{f}_2(q)"$  and  $\hat{f}_2(q) \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$  recalling  $(*)_4$ . So  $p_1, p_2$  are compatible also in  $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ .

{c5}

So every antichain of  $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$  is an antichain of  $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ . Similarly to the above every maximal antichain of  $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$  is a maximal antichain of  $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ , so we are done.]

$$(*)_6 \mathbb{P}_{\mathbf{n}_1}(L_\gamma \cap L_{\mathbf{n}_1}) = \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \triangleleft \mathbb{P}_{\mathbf{n}_2}(L_\gamma).$$

[Why? We prove this by induction on  $\gamma$ , as in proving the Tarski-Vaught criterion is sufficient (we shall later in the proof of 4.19, more specifically  $\boxplus_4$  proves a similar statement in detail with weaker assumptions).]

{e32}

Hence (using  $\gamma = |L_{\mathbf{n}_2}|^+$ )

$$(*)_7 \mathbb{P}_{\mathbf{n}_1} \triangleleft \mathbb{P}_{\mathbf{n}_2}.$$

{c28} Hence for every  $L \subseteq L_{\mathbf{n}_1}$  by 2.17(4) we have  $\mathbb{P}_{\mathbf{n}_1}(L) = \mathbb{P}_{\mathbf{n}_2}(L)$  as required in  
 {c34} Definition 2.24. □<sub>2.26</sub>

§ 3. THE CORRECTED  $\mathbb{P}_m$ 

{z19}

**Definition 3.1.** Let  $\mathbb{P}$  be a forcing notion and  $Y \subseteq \mathbb{P}$  and  $\chi$  a regular cardinal.1) Let  $\mathbb{L}_\chi(Y)$  be the set of sentences formed from  $\{p : p \in \mathbb{P}_Y\}$  closing under the operations  $\neg p$  and  $\bigwedge_{i < \alpha} p_i$ , for  $\alpha < \chi$ ; so propositional logic.2) For  $\mathbf{G} \subseteq \mathbb{P}$  and  $\psi \in \mathbb{L}_\chi(Y)$  we define the truth value  $\psi[\mathbf{G}]$  naturally (by induction on  $\psi$  starting with  $p[\mathbf{G}] = \text{true} \Leftrightarrow p \in \mathbf{G}$ ).3) Let  $\mathbb{L}_\chi^+(Y, \mathbb{P})$ , the  $\mathbb{L}_\chi$ -closure of  $Y$  for  $\mathbb{P}$ , ( $Y \subseteq \mathbb{P}$ ; if  $Y = \mathbb{P}$  we may omit  $Y$ ) be the following partial order:

- set of elements  $\{\psi \in \mathbb{L}_\chi(Y) : \not\vdash_{\mathbb{P}} \text{“}\psi[\mathbf{G}] = \text{false”}\}$
- the order  $\psi_1 \leq \psi_2$  iff  $\Vdash_{\mathbb{P}} \text{“if } \psi_2[\mathbf{G}] = \text{true then } \psi_1[\mathbf{G}] = \text{true”}$ .

4) The completion of  $\mathbb{P}$  is the  $\mathbb{L}_\chi$ -closure of  $\mathbb{P}$  for  $\mathbb{P}$ ,  $\mathbb{L}_\chi(\mathbb{P})$  where  $\chi$  is minimal such that  $\mathbb{P}$  satisfies the  $\chi$ -c.c.

{z44}

**Claim 3.2.** For a forcing notion  $\mathbb{P}$  and  $Y \subseteq \mathbb{P}$  we have:

- (a)  $\mathbb{L}_\chi^+(Y, \mathbb{P})$  is a forcing notion
- (b)  $\mathbb{P} \leq \mathbb{L}_\chi^+(\mathbb{P})$  under the natural identification<sup>12</sup>
- (c)  $\mathbb{L}_\chi^+(Y, \mathbb{P}) \leq \mathbb{L}_\chi^+(\mathbb{P})$
- (d)  $\mathbb{L}_{\chi_1}^+(Y, \mathbb{P}) \leq \mathbb{L}_{\chi_2}^+(Y, \mathbb{P})$  when  $\chi_1 \leq \chi_2$  are regular
- (e) if  $\mathbb{P}$  satisfies the  $\chi_1$ -c.c. and  $\chi_1 < \chi_2$  are regular then  $\mathbb{L}_{\chi_1}^+(Y, \mathbb{P})$  is essentially equal to  $\mathbb{L}_{\chi_2}^+(Y, \mathbb{P})$ , i.e. up to the natural equivalence of elements in a quasi order.

{z46}

**Definition 3.3.** Let  $\mathbf{m} \in \mathbf{M}$ .1) For  $t \in L_{\mathbf{m}}$ ,  $\varepsilon < \lambda$  and  $\eta \in \prod_{i < \varepsilon} \theta_i$  let  $p = p_{t, \eta}^* \in \mathbb{P}_{\mathbf{m}}$  be the function with domain  $\{t\}$  such that  $p(t) = (\eta, \eta \hat{\ } 0_\lambda)$ , i.e.  $f_{p(t)} \in \prod_{i < \lambda} \theta_i$  is defined by  $f_{p(t)}(\varepsilon)$  is  $\eta(\varepsilon)$  if  $\varepsilon < \text{lg}(\eta)$  and is zero otherwise.2) For  $L \subseteq L_{\mathbf{m}}$  let  $Y_L = Y_{\mathbf{m}, L} = \{p_{t, \eta}^* : t \in L \text{ and } \eta \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ for some } \varepsilon < \lambda\}$ .3) For  $L \subseteq L_{\mathbf{m}}$  let  $\mathbb{P}_{\mathbf{m}}[L]$  be  $\mathbb{L}_{\lambda^+}[Y_L, \mathbb{P}_{\mathbf{m}}]$ , see Definition 3.1.

{z19}

4) For  $L \subseteq L_{\mathbf{m}}$  let  $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} \upharpoonright \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$ , see Definition 2.10(1), recalling 2.10(2),(3).

{c7}

5)  $\mathbb{P}'_{\mathbf{m}}$  is the partial order with the same set of elements as  $\mathbb{P}_{\mathbf{m}}$  and  $\leq_{\mathbb{P}'_{\mathbf{m}}} = \{(p, q) : p, q \in \mathbb{P}_{\mathbf{m}} \text{ and no } r \text{ above } q \text{ is incompatible with } p\}$  and  $\mathbb{P}'_{\mathbf{m}}(L) = \mathbb{P}'_{\mathbf{m}} \upharpoonright \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$ , we may “forget” the distinction<sup>13</sup>.

{c7}

6) For quasi orders  $\mathbb{Q}_1, \mathbb{Q}_2$  let  $\mathbb{Q}_1 \leq' \mathbb{Q}_2$  mean that:

- (a)  $s \in \mathbb{Q}_1 \Rightarrow s \in \mathbb{Q}_2$
- (b)  $s \leq_{\mathbb{Q}_1} t \Rightarrow s \leq_{\mathbb{Q}_2} t$ .

7) For quasi orders  $\mathbb{Q}_1, \mathbb{Q}_2$  let  $\mathbb{Q}_1 \leq'_{\text{ic}} \mathbb{Q}_2$  means that  $\mathbb{Q}_1 \leq' \mathbb{Q}_2$  and

- (c) if  $s, t \in \mathbb{Q}_1$  are incompatible in  $\mathbb{Q}_1$  then they are incompatible in  $\mathbb{Q}_2$ .

8) We define  $\leq'$  similarly.<sup>12</sup>Really  $\mathbb{P} \leq' \mathbb{L}_\chi^+(\mathbb{P})$  see 3.3, because  $\mathbb{L}_\chi^+(\mathbb{P}) \Vdash \text{“}p \leq q \text{” iff } q \Vdash_{\mathbb{P}} \text{“}p \in \mathbf{G}_{\mathbb{P}} \text{”}$ .

{z46}

<sup>13</sup>Really the only difference is the possibility that  $\text{dom}(p) \not\subseteq \text{dom}(q)$ , see 2.19.

{c32n}

{z48}

**Claim 3.4.** Let  $\mathbf{m} \in \mathbf{M}$  and  $L \subseteq L_{\mathbf{m}}$ .

- {z19} 1)  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  is equivalent to  $\mathbb{P}_{\mathbf{m}}$  as forcing notions, in fact,  $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}}) \triangleleft \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  and is a dense subset of it under the natural identification (see 3.1(1)), but we should pedantically use  $\mathbb{P}'_{\mathbf{m}}(L_{\mathbf{m}})$  or use  $\triangleleft'$ .
- 2)  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  is  $(\lt \lambda)$ -strategically complete and is  $\lambda^+$ -c.c.
- 3)  $\mathbb{P}_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}[L]$  as sets and  $\mathbb{P}_{\mathbf{m}}[L] \triangleleft \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  and  $\mathbb{P}_{\mathbf{m}}(L) \subseteq' \mathbb{P}_{\mathbf{m}}[L]$ .
- {z19} 4) If  $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}$  is generic over  $\mathbf{V}$  and  $\eta_t = \eta_t[\mathbf{G}]$  for  $t \in L$  and  $\mathbf{G}_L^+ = \{\psi \in \mathbb{L}_{\lambda^+}(Y_L) : \psi[\mathbf{G}] = \text{true}\}$ , see 3.1(3), then  $\mathbf{V}[\mathbf{G}] = \tilde{\mathbf{V}}[\mathbf{G}^+] = \mathbf{V}[\langle \eta_t : t \in L_{\mathbf{m}} \rangle]$ .
- 5) In part (4), moreover  $\mathbf{G}^+$  is a subset of  $\mathbb{P}_{\mathbf{m}}[L]$  generic over  $\mathbf{V}$ .
- 6)  $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_2)$  and  $\mathbb{P}_{\mathbf{m}}[L_1] \triangleleft \mathbb{P}_{\mathbf{m}}[L_2]$  when  $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$ .
- 7) If  $\mathbf{m}, \mathbf{n} \in \mathbf{M}$  are equivalent then  $\mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$  and  $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$ .
- 8) Assume  $I_*$  be a  $\lambda_2^+$ -directed partial order and  $\bar{L} = \langle L_r : r \in I_* \rangle$  be such that  $r \in I_* \Rightarrow L_{t_r} \subseteq L_{\mathbf{m}}$  and  $r \triangleleft_{I_*} s \Rightarrow L_r \subseteq L_s$  and  $L = \cup\{L_r : r \in I_*\}$ . Then  $\mathbb{P}_{\mathbf{m}}[L] = \cup\{\mathbb{P}_{\mathbf{m}}[L_r] : r \in I_*\}$  and  $\mathbb{P}_{\mathbf{m}}(L) = \cup\{\mathbb{P}_{\mathbf{m}}(L_r) : r \in I_*\}$ .

{z50} *Remark 3.5.* What about  $\mathbb{P}_{\mathbf{m}}(L) \subseteq'_{\text{ic}} \mathbb{P}_{\mathbf{m}}[L]$  and “ $\mathbb{P}_{\mathbf{m}}(L) \triangleleft' \mathbb{P}_{\mathbf{m}}[L]$ ”?

{e4} The problem is the mapping  $p \mapsto p \upharpoonright L$  defined in 4.1(3) does not have the required properties of preserving order as the forcing appears there.

*Proof.* 1) Easy.

{c11} 2) Follows by part (1) and 2.13.

3) The first statement by their definitions, the second statement by part (1).

{c11} 4), 5), 6) Should be clear recalling 2.13(9).

{c11} 7) Easy, recalling 2.13(7).

8) Easy. □<sub>3.4</sub>

{c43} **The Uniqueness Claim 3.6.** There is an isomorphism from  $\mathbb{P}_{\mathbf{m}_1}[M_1]$  onto  $\mathbb{P}_{\mathbf{m}_2}[M_2]$  which (recalling Definition 3.3(1)) maps  $p_{t,\eta}^*$  to  $p_{h(t),\eta}^*$  for  $t \in M_1, \eta \in \cup\{\prod_{\varepsilon < \zeta} \theta_\varepsilon :$

{z46}  $\zeta < \lambda\}$  when :

⊕ (a)  $\mathbf{m}_\ell \in \mathbf{M}_\infty^{\text{ec}}$  for  $\ell = 1, 2$

(b)  $M_\ell = M_{\mathbf{m}_\ell}$  for  $\ell = 1, 2$

(c)  $h$  is an isomorphism from  $\mathbf{m}_1 \upharpoonright M_1$  onto  $\mathbf{m}_2 \upharpoonright M_2$ .

*Proof.* By renaming without loss of generality  $M_1 = M_2$  call it  $M$  and  $h$  is the identity and  $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$ . Let  $\mathbf{m}_0 = \mathbf{m}_1 \upharpoonright M = \mathbf{m}_2 \upharpoonright M$  so  $\mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}_\ell$  for  $\ell = 1, 2$  and  $L_{\mathbf{m}_0} = L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2}$ .

{c31} By 2.18, there is  $\mathbf{m}$  such that  $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$  and  $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}$ . As  $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}_\infty^{\text{ec}}$  we have  $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}}[M]$  and  $\mathbb{P}_{\mathbf{m}_2}[M] = \mathbb{P}_{\mathbf{m}}[M]$  so together we get the desired conclusion. □<sub>3.6</sub>

{c44} **Definition 3.7.** 1) We call  $\mathbf{m} \in \mathbf{M}$  reduced when  $L_{\mathbf{m}} = M_{\mathbf{m}}$ .

2) For  $\mathbf{m} \in \mathbf{M}$  let  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}$  be  $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{m}}]$  and  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}[L]$  be  $\mathbb{P}_{\mathbf{n}}[L]$  for  $L \subseteq L_{\mathbf{m}}$  when  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_{\text{ec}}$ .

{c47} *Remark 3.8.* 1) Why is  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}[L]$  well defined? see below.

2) Here “cr” stands for corrected.

The interest in the definition is because

{c48}

**Claim 3.9.** 1) If  $\mathbf{m} \in \mathbf{M}$  and  $L \subseteq L_{\mathbf{m}}$  then  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}[L]$  is well defined.

2)  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}[M_{\mathbf{m}}]$  is well defined and depend only on  $\mathbf{m}[M_{\mathbf{m}}]$ .

3) If  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$  then  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}[L_1] = \mathbb{P}_{\mathbf{n}}^{\text{cr}}[L_1] \triangleleft \mathbb{P}_{\mathbf{n}}^{\text{cr}}[L_2] \triangleleft \mathbb{P}_{\mathbf{n}}^{\text{cr}}$ .

*Proof.* 1) By 2.26,  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}[L]$  has at least one definition so it suffices to prove uniqueness. {c41}

So assume  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_\ell \in \mathbf{M}_{\text{ec}}$  for  $\ell = 1, 2$  and we should prove that  $\mathbb{P}_{\mathbf{m}_1}[L] = \mathbb{P}_{\mathbf{m}_2}[L]$ .

Without loss of generality  $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = L_{\mathbf{m}}$ . Now by 2.18 we can find  $\mathbf{n} \in \mathbf{M}$  such {c31}

that  $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{n}$  and  $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{n}$ ; as  $\mathbf{m}_\ell \in \mathbf{M}_{\text{ec}}$  see Definition 2.24 we have  $\mathbb{P}_{\mathbf{m}_\ell} \triangleleft \mathbb{P}_{\mathbf{n}}$  {c34}

for  $\ell = 1, 2$ . As in the end of the proof of 3.6 we are done. {c43}

2) By 3.6. {c43}

3) Follows from Definition 2.24(2) and 3.7(2).  $\square_{3.9}$  {c44}

**Discussion 3.10.** But we like to prove for reduced  $\mathbf{m} \in \mathbf{M}$  and  $M \subseteq M_{\mathbf{m}}$  that  $\mathbb{P}_{\mathbf{m}|M}^{\text{cr}} \triangleleft \mathbb{P}_{\mathbf{m}}^{\text{cr}}$ . This is delayed to 4.26. We now prove it suffices. {c80}

**Conclusion 3.11.** For every ordinal  $\delta_*$  there is  $\mathbf{q} = \langle \mathbb{P}_\alpha, \eta_\alpha : \alpha \leq \delta_* \rangle$  such that: {c51}

- (A) (a)  $\langle \mathbb{P}_\alpha : \alpha \leq \delta_* \rangle$  is  $\triangleleft$ -increasing
- (b)  $\eta_\alpha$  is a  $\mathbb{P}_{\alpha+1}$ -name of a member of  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  which dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_\alpha]}$
- (c)  $\eta_\alpha$  is a generic for  $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$ , moreover  $\langle \eta_\beta : \beta < \alpha \rangle$  is a generic for  $\mathbb{P}_\alpha$
- (d)  $p \in \mathbb{P}_\alpha$  iff  $p \in \mathbb{L}_{\lambda^+}(Y_\alpha, \mathbb{P}_\alpha)$  on  $Y_\alpha$  defined as in 3.3(2) with  $\alpha$  here standing for  $L$  there and see 3.1 {z46}
- (e)  $\mathbb{P}_\alpha$  is  $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c. {z19}
- (f) if  $\delta \leq \delta_*$  has cofinality  $> \lambda$  then  $\mathbb{P}_\delta = \cup \{ \mathbb{P}_\alpha : \alpha < \delta \}$
- (g)  $\mathbb{P}_{\delta_*}$  has cardinality  $|\delta_*|^\lambda$ .
- (B) if  $\mathcal{U} \subseteq \delta_*$  then the complete subforcing generated by  $\langle \eta_\alpha : \alpha \in \mathcal{U} \rangle$  is isomorphic to  $\mathbb{P}_{\text{otp}(\mathcal{U})}$
- (C) if  $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$  is generic over  $\mathbf{V}$  and  $\eta_\alpha = \eta_\alpha[\mathbf{G}]$  for  $\alpha < \delta_*$  and  $\eta'_\alpha \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  for  $\alpha < \delta_*$  and  $\{(\alpha, \varepsilon) : \alpha < \delta_*, \varepsilon < \lambda \text{ and } \eta'_\alpha(\varepsilon) \neq \eta_\alpha(\varepsilon)\}$  has cardinality  $< \lambda$  then also  $\langle \eta'_\alpha : \alpha < \delta_* \rangle$  is a generic for  $\mathbb{P}_{\delta_*}$ , determining a different  $\mathbf{G}'$  but  $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\mathbf{G}]$
- (D) in clause (C), moreover if  $\mathcal{U} \subseteq \delta$  and  $\langle \alpha_i : i < \text{otp}(\mathcal{U}) \rangle$  list  $\mathcal{U}$  in increasing order then for some unique  $\mathbf{G}'' \subseteq \mathbb{P}_{\text{otp}(\mathcal{U})}$  generic over  $\mathbf{V}$ ,  $i < \text{otp}(\mathcal{U}) \Rightarrow \eta'_{\alpha_i} = \eta_i[\mathbf{G}'']$ .

*Proof.* Without loss of generality  $\lambda_1 \geq |\delta_*|$ .

We define  $\mathbf{m} \in \mathbf{M}$  by:

- (\*) (a)  $L_{\mathbf{m}} = \delta_*$
- (b)  $M_{\mathbf{m}} = \delta_*$
- (c)  $u_{\mathbf{m}, \alpha} = \alpha$  and  $\mathcal{P}_{\mathbf{m}, \alpha} = [\alpha]^{\leq \lambda}$  for  $\alpha < \delta_*$
- (d)  $E'_{\mathbf{m}} = \emptyset$ .

It is easy to check that indeed  $\mathbf{m} \in \mathbf{M}$  and let  $\mathbf{n} \in \mathbf{M}_{\text{ec}}$  be such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ , exists by the Crucial Claim 2.26 and let  $\mathbb{P}_\alpha = \mathbb{P}_{\mathbf{n}}[\{i : i < \alpha\}]$ . {c41}

Now clearly clauses (A),(C) hold and  $\mathbb{P}_\delta = \mathbb{P}_{\mathbf{m}}^{\text{cr}}$  by 3.7(2), 3.9(1) and clause (A)(b) holds by 2.13(6A). As for clause (B), note that for every  $L \subseteq \delta_*$ , for  $\mathbb{P}_{\mathbf{m}}[L]$  the sequence  $\bar{\eta}_L = \langle \eta_\alpha : \alpha \in L \rangle$  is generic for  $\mathbb{P}_{\mathbf{m}}[L]$  by Definition 3.3. {c48}  
{c11}  
{z46}

For  $M \subseteq \delta_*$  let  $\alpha = \text{otp}(M)$  and  $h : M \rightarrow \alpha$  be  $h(i) = \text{otp}(i \cap M)$  so  $h$  is an isomorphism from  $\mathbf{m} \upharpoonright M$  onto  $\mathbf{m} \upharpoonright \alpha$  hence by 4.26(2), with  $\mathbf{m}, \mathbf{m} \upharpoonright \alpha, M, \alpha$  here standing for  $\mathbf{m}_1, \mathbf{m}_2, M_1, M_2$  there we have  $h$  induces an isomorphism from  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}[M]$  onto  $\mathbb{P}_{\mathbf{m} \upharpoonright \alpha}^{\text{cr}}[L_{\mathbf{m} \upharpoonright \alpha}]$ . Similarly,  $\text{id}_\alpha$  induces an isomorphism from  $\mathbb{P}_{\mathbf{m} \upharpoonright \alpha}^{\text{cr}}$  onto  $\mathbb{P}_{\mathbf{m}}^{\text{cr}}[\alpha]$ . {c80}

Together we get clause (B). Also Clause (D) follows so we are done.  $\square_{3.11}$

Similarly we can deal with such iterations with partial memory.

{c52}

**Conclusion 3.12.** *Assume  $M$  is a well founded partial order and  $\bar{u}' = \langle u'_t : t \in M \rangle, u_t \subseteq M_{< t}$  and  $\bar{\mathcal{P}}' = \langle \mathcal{P}'_t : t \in M \rangle$  with  $\mathcal{P}'_t \subseteq [u'_t]^{\leq \lambda}$  is closed under subsets. Then we can find  $\beta(*), h, \mathbb{P}_\beta = \mathbb{P}_{0,\beta}, \mathbb{P}_{1,\beta}, \mathbb{Q}_\alpha, \eta_\alpha, \eta'_s$  and  $\mathbb{P}'_u$  (for  $\beta \leq \beta(*), \alpha < \beta(*), s \in M$  and  $u \subseteq M$ ) and  $\bar{u}, \bar{\mathcal{P}}$  such that:*

- (A) (a)  $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \beta \leq \beta(*), \alpha < \beta(*) \rangle$  is  $(< \lambda)$ -support iteration
- (b)( $\alpha$ )  $\bar{u} = \langle u_\beta : \beta < \beta(*) \rangle$  such that  $u_\beta \subseteq \beta$
- ( $\beta$ )  $\bar{\mathcal{P}} = \langle \mathcal{P}_\beta : \beta < \beta(*) \rangle$  such that  $\mathcal{P}_\beta \subseteq [u_\beta]^{\leq \lambda}$
- (c)  $\eta_\alpha$  is a  $\mathbb{P}_{\alpha+1}$ -name of a member of  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$
- (d)  $\langle \eta_\alpha : \alpha < \beta \rangle$  is generic for  $\mathbb{P}_\beta$
- (e)  $\mathbb{Q}_\alpha$  is defined as in Definition 2.10(4)
- (f)  $\Vdash_{\mathbb{P}_{\beta(*)}} \text{“} \eta_\beta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon \text{ dominate every } \nu \in \prod_{\varepsilon < \lambda} \theta_\varepsilon \text{ from } \mathbf{V}[\langle \eta_\alpha : \alpha \in u \rangle] \text{ when } u \in \mathcal{P}_\beta \text{”}$
- (B) (a)  $h$  is a one-to-one function from  $M$  into<sup>14</sup>  $\beta(*)$ ; stipulate  $h(\infty) = \beta(*)$
- (b)  $s <_M t \Leftrightarrow h(s) < h(t)$
- (c)  $u_{h(t)} \cap \text{Rang}(h) = \{h(s) : s \in u'_t\}$
- (d)  $\mathcal{P}_{h(t)} \cap [\text{Rang}(h)]^{\leq \lambda} = \{\{h(s) : s \in u\} : u \in \mathcal{P}'_t\}$
- (C) (a)  $\mathbb{P}_{1,\beta} = \mathbb{L}_{\lambda^+}^+(Y_\beta, \mathbb{P}_\beta)$  where we let  $Y_\beta = \{p_{\alpha,\nu}^* : \alpha < \beta, \nu \in \prod_{\varepsilon < \zeta} \theta_\varepsilon$  for some  $\zeta < \lambda\}$ , see 3.1, 3.3(1)
- (b)  $\mathbb{P}_{1,u} = \mathbb{L}_{\lambda^+}^+(Y_u, \mathbb{P}_\beta)$ , where  $Y_u$  is defined similarly when  $u \subseteq \beta(*)$
- (c)  $\mathbb{P}'_u$  is a forcing notion for  $u \subseteq M$  and  $\eta'_s$  is a  $\mathbb{P}'_{\{s\}}$ -name for  $s \in M$
- (d)  $h$  induces an isomorphism from  $\mathbb{P}'_u$  onto  $\mathbb{P}_{1,\{h(s):s \in u\}}$  for  $u \subseteq M$  and  $\eta'_s$  to  $\eta_{h(s)}$  for  $s \in M$
- (e)  $\langle \eta_{h(s)} : s \in u \rangle$  is generic for  $\mathbb{P}'_u$  for  $u \subseteq M$
- (D) (a)  $\mathbb{P}'_u < \mathbb{P}'_v$  when  $u \subseteq v \subseteq M$
- (b)  $\mathbb{P}_\beta, \mathbb{P}_{1,u}, \mathbb{P}'_u$  are  $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c.

{c7}

{z49}

<sup>14</sup>In general not onto!



- (c) if  $M_1, M_2 \subseteq M$  and  $f$  is an isomorphism from  $M_1$  onto  $M_2$  as partial orders such that  $t \in M_1 \Rightarrow u'_{h(t)} \cap M_2 = \{f(s) : s \in u'_t \cap M_1\}$  and  $t \in M_1 \Rightarrow \mathcal{P}'_{h(t)} \cap [M_2]^{\leq \lambda} = \{\{f(s) : s \in u \cap M_1\} : u \in \mathcal{P}'_t\}$  then the mapping  $h(s) \mapsto h(f(s))$  induce an isomorphism from the forcing notion  $\mathbb{P}'_{M_1}$  onto  $\mathbb{P}'_{M_2}$ .

## § 4. THE MAIN CONCLUSION

{c50} We have a debt from §3, i.e. see discussion 3.10. Toward this we explicate what  
 {c41} appear in the proof of 2.26.  
 {e4}

**Definition 4.1.** Let  $\mathbf{m} \in \mathbf{M}$ .

1) We say  $\mathbf{m}$  is  $\mu$ -wide when for every  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  there are  $t_{\alpha} \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  for  $\alpha < \mu$  such that:

- (a)  $\mathbf{m} \upharpoonright (t_{\alpha}/E_{\mathbf{m}})$  is isomorphic to  $\mathbf{m} \upharpoonright (t/E_{\mathbf{m}})$  over  $M_{\mathbf{m}}$
- (b)  $\beta < \gamma < \mu \Rightarrow t_{\beta}/E_{\mathbf{m}}'' \neq t_{\gamma}/E_{\mathbf{m}}''$ .

1A) We say  $\mathbf{m}$  is wide when it is  $\lambda^+$ -wide. We say  $\mathbf{m}$  is very wide when it is  $|L_{\mathbf{m}}|$ -wide.

2) We say  $\mathbf{m}$  is full when: if  $\mathbf{m} \upharpoonright M_{\mathbf{m}} \leq_{\mathbf{M}} \mathbf{n}$  and  $E_{\mathbf{n}}''$  has exactly one equivalence class then for some  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ , we have  $\mathbf{n}$  is isomorphic to  $\mathbf{m} \upharpoonright (t/E_{\mathbf{m}})$  over  $M_{\mathbf{m}}$ .

3) For  $L \subseteq L_{\mathbf{m}}$  we say  $p \in \mathbb{P}_{\mathbf{m}}(L)$  is the projection (to  $L$ ) of  $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$  and write  $p = q \upharpoonright L$  when:

- (a)  $\text{Dom}(p) = \text{Dom}(q) \cap L$
- (b) if  $s \in \text{Dom}(p)$  then
  - ( $\alpha$ )  $\text{tr}(p(s)) = \text{tr}(q(s))$
  - ( $\beta$ )  $\{f_{p(s),\iota} : \iota < \iota(p(s))\} = \{f_{q(s),\iota} : \iota < \iota(q(s))\}$  and  $\bar{r}_{p(s),\iota}$  is a sequence of members of  $L$ .

4) Let  $\mathcal{F}_{\mathbf{m}}$  be the set of the functions  $f$  such that for some  $L_1, L_2$ :

- (a)  $f$  is an isomorphism from  $\mathbf{m} \upharpoonright L_1$  onto  $\mathbf{m} \upharpoonright L_2$
- (b)  $L_{\ell}$  is a subset of  $L_{\mathbf{m}}$  for  $\ell = 1, 2$
- (c)  $M_{\mathbf{m}} \subseteq L_{\ell}$  for  $\ell = 1, 2$
- (d)  $L_{\ell}$  is  $E_{\mathbf{m}}$ -closed, i.e.  $M_{\mathbf{m}} \subseteq L_{\ell}$  and if  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  and  $t \in L_{\ell}$  then  $t/E_{\mathbf{m}} \subseteq L_{\ell}$  for  $\ell = 1, 2$
- (e)  $\{t/E_{\mathbf{m}}'' : t \in L_{\ell} \setminus M_{\mathbf{m}}\}$  has cardinality  $\leq \lambda$ .

{c41} 5) If  $L_1, L_2 \subseteq L_{\mathbf{m}}$  and  $f$  is an isomorphism from  $\mathbf{m} \upharpoonright L_1$  onto  $\mathbf{m} \upharpoonright L_2$  then we let  $\hat{f}$  be the one-to-one mapping<sup>15</sup> from  $\mathbb{P}_{\mathbf{m}}(L_1)$  onto  $\mathbb{P}_{\mathbf{m}}(L_2)$  as in  $(*)_4(b)$  of the proof of 2.26; check the definition.

{e5n} 6) Let  $\mathbb{P}_{\mathbf{m}}^-(L)$  be  $\{p \in \mathbb{P}_{\mathbf{m}}(L) : \text{fsupp}(p) \subseteq M_{\mathbf{m}} \text{ or } \text{fsupp}(p) \subseteq t/E_{\mathbf{m}} \text{ for some } t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}\}$  with the order inherited from  $\mathbb{P}_{\mathbf{m}}$ .

**Observation 4.2.** Let  $\mathbf{m} \in \mathbf{M}$  and  $L \subseteq L_{\mathbf{m}}$ .

- 1) The projection of  $q \in \mathbb{P}_{\mathbf{m}}$  to  $L$  is well defined and  $\in \mathbb{P}_{\mathbf{m}}(L)$ .
- 2) Moreover, it is unique.
- 3) If  $p \in \mathbb{P}_{\mathbf{m}}(L)$  is the projection of  $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$  then  $p \leq q$ .
- 4) For every  $p \in \mathbb{P}_{\mathbf{m}}$ ,  $p$  is equivalent to  $\mathcal{S}_p := \{p \upharpoonright L : L = t/E_{\mathbf{m}} \text{ for some } t \in \text{fsupp}(p)\} \cup \{p \upharpoonright M_{\mathbf{m}} : \text{if } \text{fsupp}(p) \subseteq M_{\mathbf{m}}\}$ , i.e.  $\Vdash_{\mathbb{P}_{\mathbf{m}}} "p \in \mathbf{G}_{\mathbb{P}_{\mathbf{m}}} \text{ iff } \mathcal{S}_p \subseteq \mathbf{G}_{\mathbb{P}_{\mathbf{m}}}"$ .
- 5) For every  $p \in \mathbb{P}_{\mathbf{m}}$ ,  $p$  is equivalent to  $\mathcal{S}'_p := \{p^{[t,\iota]} : t \in \text{dom}(p) \text{ and } \iota < \iota(p_t)\}$  where  $p^{[t,\iota]} \in \mathbb{P}_{\mathbf{m}}$  has domain  $\{t\}$  and  $p(t) = (\text{tr}(p_t), \mathbf{B}_{p(t),\iota}(\langle \eta_{r_{p(t)}(\zeta)}, \zeta \in w_{p(t),\iota} \rangle))$ .

<sup>15</sup>We have not said "order preserving".

$\{c6\}$  *Remark 4.3.* 1) Note that the choice in Definition 2.9(c)( $\gamma$ ) to require such  $\langle f_{p(t),\iota} : \iota < \iota(p_t) \rangle$  exists, is necessary for 4.2(4), which is crucial in the proof of 4.26.  
 $\{e8n\}$  2) In Definition 4.1(1A) we can choose “wide means  $\lambda$ -wide” as when applying it, if  $X = \text{fsupp}(p)$  then for some  $Y \subseteq L_{\mathbf{m}}$  of cardinality  $< \lambda$ ,  $X = \bigcup \{t/E_{\mathbf{m}} : t \in Y\}$ .  
 $\{e4\}$

*Proof.* Easy, e.g.

4) If  $\text{fsupp}(p) \subseteq M_{\mathbf{m}}$  the statement says  $\Vdash “p \in \mathbf{G} \text{ iff } \{p\} \subseteq \mathbf{G}”$ , so trivial hence we assume  $\text{fsupp}(p) \not\subseteq M_{\mathbf{m}}$ . Now if  $t \in \text{fsupp}(p)$  then trivially  $p \upharpoonright (t/E_{\mathbf{m}}) \leq q_p$ , hence  $\Vdash “p \in \mathbf{G} \text{ implies } \mathcal{S}_p \subseteq \mathbf{G}”$ .

For the other direction assume  $q \in \mathbb{P}_{\mathbf{m}}$  forces  $\mathcal{S}_p \subseteq \mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}$  and we shall prove that  $q$  is compatible with  $p$ , this suffices, so toward contradiction assume  $q, p$  are incompatible.

Without loss of generality  $\text{Dom}(p) \subseteq \text{Dom}(q)$  and recalling  $t \in \text{fsupp}(p) \Rightarrow q \Vdash “p \upharpoonright (t/E_{\mathbf{m}}) \in \mathbf{G}”$  clearly  $s \in \text{dom}(p) \Rightarrow q \Vdash “\text{tr}(p(s)) \subseteq \eta_s”$  so  $s \in \text{Dom}(p) \Rightarrow \text{tr}(p(s)) \subseteq \text{tr}(q(s))$ . As  $p, q$  are incompatible there are  $s \in \text{Dom}(p) \cap \text{Dom}(q)$  and  $q_1$  such that  $q \upharpoonright L_{\mathbf{m}, < s} \leq q_1 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, < s})$  and  $q_1 \Vdash “q(s), p(s) \text{ are incompatible in } \mathbb{Q}_{\delta}”$ .

As  $\text{tr}(p(s)) \subseteq \text{tr}(q(s))$  this implies  $q_1 \Vdash “\text{tr}(q(s)), p(s) \text{ are incompatible, i.e. } f_{p(s)} \upharpoonright \ell g(\text{tr}(q(s))) \not\subseteq \text{tr}(q(s))”$ . Recalling Definition 2.9(c)( $\gamma$ ),  $q_1 \Vdash “\text{there is } \iota < \iota(s, p) \text{ such that } f_{p(s), \iota}, \text{tr}(q(s)) \text{ are incompatible}”$ . Possibly increasing  $q_1$ , we can fix  $\iota$ . But letting  $t \in \text{fsupp}(p) \subseteq L_{\mathbf{m}}$  be such that  $\bar{r}_{p(s), \iota} \subseteq t/E_{\mathbf{m}}$  this implies that  $q_1 \Vdash “p \upharpoonright (t/E_{\mathbf{m}}) \notin \mathbf{G} \text{ or } \text{tr}(q(s)) \not\subseteq \eta_s”$ . However,  $q_1, q$  are compatible and this contradicts the choice of  $q$ .  $\square_{4.2}$

**Claim 4.4.** 1) *The  $\mathbf{n}$  constructed in 2.26 satisfies: if  $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1$  then  $\mathbf{n}_1$  is wide,*  $\{c6\}$   
*(if  $\mathbf{n}_1 \in \mathbf{M}_\chi$  even very wide) and full.*  $\{e7\}$

2) *If  $\mathbf{n} \in \mathbf{M}_{ec}$  and  $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1$  then  $\mathbf{n}_1 \in \mathbf{M}_{ec}$ .*  $\{c41\}$

*Proof.* 1) Holds by the proof of 2.26.  $\square_{4.2}$

2) Holds by Definition 2.24(1),(2).  $\square_{4.4}$

**Claim 4.5.** *Assume  $\mathbf{m}$  is wide.*  $\{c41\}$

1) *If  $f \in \mathcal{F}_{\mathbf{m}}$  and  $X \subseteq L_{\mathbf{m}}$  has cardinality  $\leq \lambda$  then there is  $g$  such that:*  $\{c34\}$

- (a)  $g \in \mathcal{F}_{\mathbf{m}}$
- (b)  $f \subseteq g$
- (c)  $\text{Dom}(g) = \text{Rang}(g)$
- (d)  $X \subseteq \text{Dom}(g)$ .

2) *If  $g \in \mathcal{F}_{\mathbf{m}}$  and  $\text{Dom}(g) = \text{Rang}(g)$  then  $g^{+\mathbf{m}} = g \cup \text{id}_{L_{\mathbf{m}} \setminus \text{Dom}(g)}$  is an automorphism of  $\mathbf{m}$ .*  $\{e10\}$

3) *If  $f$  is an automorphism of  $\mathbf{m}$  then it naturally induces an automorphism  $\hat{f}$  of  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$  similarly to  $\hat{f}$  from  $(*)_4(b)$  of the proof of 2.26.*  $\{c41\}$

4) *If  $f \in \mathcal{F}_{\mathbf{m}}$  then it induces an isomorphism  $\hat{f}$  from  $\mathbb{P}_{\mathbf{m}}(\text{Dom}(f))$  onto  $\mathbb{P}_{\mathbf{m}}(\text{Rang}(f))$ .*

*Proof.* 1) Easy by the definition of wide in 4.1(1) and of  $\mathcal{F}_{\mathbf{m}}$  in 4.1(4).  $\{e4\}$

2) Check.

3) Naturally by the definition.

4) Let  $g \in \mathcal{F}$  be as in part (1) and let  $h = g^{+\mathbf{m}}$  so an automorphism of  $\mathbf{m}$  which extends  $g$  as in part (2). So  $\hat{h}$  is an automorphism of  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$  and clearly  $\hat{f} = \hat{h} \upharpoonright \mathbb{P}_{\mathbf{m}}(\text{Dom}(f))$  is as required.  $\square_{4.5}$

{e16}

**Claim 4.6.** Let  $\mathbf{m} \in \mathbf{M}$  and  $L \subseteq L_{\mathbf{m}}$ .

If  $f_1, f_2 \in \mathcal{F}_{\mathbf{m}}$  then  $\hat{f}_1 \subseteq \hat{f}_2$  and  $f_1 = f_2^{-1} \Rightarrow \hat{f}_1 = (\hat{f}_2)^{-1}$ .

~~{e44}~~ *Proof.* Just consider the definition, see 4.1(5) and  $(*)_4(b)$  of the proof of 2.26.  $\square_{4.6}$

{e19}

**Observation 4.7.** 1)  $\mathbb{P}_{\mathbf{m}}^-(L) \subseteq \mathbb{P}_{\mathbf{m}}(L)$ , see Definition 4.1(6).

2) For every  $p \in \mathbb{P}_{\mathbf{m}}^-$  there is a sequence  $\langle p_i : i < i(*) \rangle$  of  $\leq \lambda$  members of  $\mathbb{P}_{\mathbf{m}}^-$  such that  $\Vdash_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]} "p \in \mathbf{G} \text{ iff } \{p_i : i < i(*)\} \subseteq \mathbf{G}"$ .

*Proof.* 1) By their definitions.

~~{e64}~~

2) Should be clear, see Definition 4.1(6) and 4.2(4).  $\square_{4.7}$

{e21}

*Remark 4.8.* 1) Observation 4.7 is not used.

2) Probably we can avoid using "wide" and prove the density of  $\mathbf{M}_{\text{ec}}$  with smaller cardinality but the present way seems more transparent.

{e24}

**Definition 4.9.** Assume  $\mathbf{m} \in \mathbf{M}$ .

1) Let  $\mathcal{Z}_{\mathbf{m}}$  be the set of pairs  $(t, \bar{s})$  such that  $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  and  $\bar{s} \in {}^\zeta(t/E_{\mathbf{m}}'')$  for some  $\zeta < \lambda^+$ ; we may write  $\bar{s}$  instead of  $(t, \bar{s})$  as usually  $\bar{s}$  determines  $t$ .

2) By induction on the ordinal  $\gamma$  we define when  $(t_1, \bar{s}_1), (t_2, \bar{s}_2)$  are  $\gamma$ -equivalent in  $\mathbf{m}$  or are  $(\mathbf{m}, \gamma)$ -equivalent:

(a) if  $\gamma = 0$ , letting  $L_\ell = (M_{\mathbf{m}} \cup \text{Rang}(\bar{s}_\ell))$  for  $\ell = 1, 2$  there is  $h$  such that

( $\alpha$ )  $h$  is an isomorphism from  $\mathbf{m} \upharpoonright L_1$  onto  $\mathbf{m} \upharpoonright L_2$

( $\beta$ )  $h_1$  maps  $\bar{s}_1$  to  $\bar{s}_2$

( $\gamma$ )  $h \upharpoonright M_{\mathbf{m}}$  is the identity

( $\delta$ )  $h$  induces an isomorphism from  $\mathbb{P}_{\mathbf{m}}(L_1)$  onto  $\mathbb{P}_{\mathbf{m}}(L_2)$

( $\varepsilon$ ) moreover,  $h$  induces an isomorphism from  $\mathbb{P}_{\mathbf{m}}[L_1]$  onto  $\mathbb{P}_{\mathbf{m}}[L_2]$

(b) if  $\gamma = \beta + 1$  then for every  $\varepsilon < \lambda^+$  and  $\ell \in \{1, 2\}$  and  $\bar{s}'_\ell \in {}^\varepsilon(t_\ell/E_{\mathbf{m}}'')$  there is  $\bar{s}'_{3-\ell} \in {}^\varepsilon(t_{3-\ell}/E_{\mathbf{m}}'')$  such that  $(t_1, \bar{s}_1 \hat{\wedge} \bar{s}'_1), (t_2, \bar{s}_2 \hat{\wedge} \bar{s}'_2)$  are  $\beta$ -equivalent

(c) if  $\gamma$  is a limit ordinal then  $(t_1, \bar{s}_1), (t_2, \bar{s}_2)$  are  $\beta$ -equivalent for every  $\beta < \gamma$ .

{e26}

*Remark 4.10.* 1) Note above that if  $\bar{s}_\ell$  is the empty sequence then  $t_\ell$  would not be determined by  $\bar{s}_\ell$ , still in those cases the equivalence just means  $\bar{s}_1 = \bar{s}_2$ .

2) We can use  $t/E_{\mathbf{m}}$  or  $t/E'_{\mathbf{m}}$  instead of  $t/E''_{\mathbf{m}}$  as everything is over  $M_{\mathbf{m}}$ .

{e27}

**Claim 4.11.** For  $\mathbf{m} \in \mathbf{M}$  and ordinal  $\alpha$  the number of equivalence classes of "being  $(\mathbf{m}, \alpha)$ -equivalent" is  $\leq \beth_{1+\alpha+1}(\lambda_1)$ .

*Proof.* By induction on  $\alpha$ .

Case 1:  $\alpha = 0$

Note that the set of elements of  $\mathbb{P}_{\mathbf{m}}(M_{\mathbf{m}} \cup \text{Rang}(\bar{s}))$  has cardinality  $\leq 2^{\lambda_1}$  (and even  $\leq (\lambda_1)^\lambda$ ) and depends just on  $\mathbf{m} \upharpoonright (M_{\mathbf{m}} \cup \text{Rang}(\bar{s}))$  but there are  $\beth_2(\lambda_1)$  possibilities for the quasi order on  $\mathbb{P}_{\mathbf{m}}(L_1)$  and even for  $\mathbb{P}_{\mathbf{m}}[L_1]$ .

Case 2:  $\alpha$  is a limit ordinal

{e24}

By clause (c) of Definition 4.9, the number of  $\alpha$ -equivalence classes is  $\leq \prod_{\beta < \alpha} \beth_{1+\beta+1}(\lambda_1)$   
 (the number of  $\beta$ -equivalence classes)  $\leq \prod_{\beta < \alpha} \beth_{1+\beta+1}(\lambda_1) \leq (\beth_{1+\alpha+1}(\lambda_1))^{\beth_{1+\alpha}} = \beth_{1+\alpha+1}(\lambda_1)$ .

Case 3:  $\alpha = \beta + 1$

Clearly every  $\alpha$ -equivalence class can be coded as a set of  $\beta$ -equivalence classes hence the number of  $\alpha$ -equivalence classes is  $\leq 2^{\beth_{1+\beta+1}(\lambda_1)} = \beth_{1+\beta+2}(\lambda_1) = \beth_{1+\alpha+1}(\lambda_1)$ , as promised.  $\square_{4.11}$  {e28}

**Definition 4.12.** For an ordinal  $\beta$ , let  $\mathcal{F}_{\mathbf{m},\beta}$  be the set of function  $f$  such that for some  $t_i^\ell, \bar{s}_i^\ell$  for  $i < i(*)$  and  $\ell \in \{1, 2\}$  we have:

- (a)  $i(*) < \lambda^+$
- (b)  $\langle t_i^\ell : i < i(*) \rangle$  is a sequence of pairwise non- $E_{\mathbf{m}}''$ -equivalent members of  $L_{\mathbf{m}} \setminus M_{\mathbf{m}}$
- (c)  $\bar{s}_i^\ell \in {}^{\zeta(i)}(t_i^\ell / E_{\mathbf{m}}'')$  where  $\zeta(i) < \lambda^+$
- (d)  $(t_i^1, \bar{s}_i^1), (t_i^2, \bar{s}_i^2)$  are  $\beta$ -equivalent (members of  $\mathcal{Y}_{\mathbf{m}}$ )
- (e)  $f$  is an isomorphism from  $\mathbf{m} \upharpoonright L_1$  onto  $\mathbf{m} \upharpoonright L_2$  when  $L_\ell = \cup \{ \text{Rang}(\bar{s}_i^\ell) : i < i(*) \} \cup M_{\mathbf{m}}$
- (f)  $f \upharpoonright M_{\mathbf{m}} =$  the identity
- (g)  $f$  maps  $\bar{s}_i^1$  to  $\bar{s}_i^2$  for  $i < i(*)$ .

2) For  $f \in \mathcal{F}_{\mathbf{m},0}$  we define  $\hat{f}$  as the mapping from  $\mathbb{P}_{\mathbf{m}}(\text{Dom}(f))$  onto  $\mathbb{P}_{\mathbf{m}}(\text{Rang}(f))$  induced by  $f$ ; see clause 4.9(2)(a)( $\varepsilon$ ). {e24} {e29}

**Claim 4.13.** Assume  $\mathbf{m}$  is wide. The conditions  $p, q \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  are compatible when for some  $\psi$  the following condition holds:

- (st) <sub>$p,q,\psi$</sub>  (a)  $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$
- (b)  $\text{wsupp}(p) \cap \text{wsupp}(q) \subseteq M_{\mathbf{m}}$ , see Definition 2.10(1)(b), equivalently  $s \in \text{fsupp}(p) \setminus M_{\mathbf{m}}, t \in \text{fsupp}(q) \setminus M_{\mathbf{m}} \Rightarrow \neg(s E_{\mathbf{m}}'' t)$  {c7}
- (c) if  $\psi \leq \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$  then  $\varphi, p$  are compatible in  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$
- (d)  $\psi, q$  are compatible in  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ , equivalently  $q \not\mathcal{K}_{\mathbb{P}_{\mathbf{m}}} \psi$  “ $\psi[\mathbf{G}] = \text{false}$ ”.

*Remark 4.14.* 1) We can use (st) <sub>$p,q,\psi$</sub> ' : omit clause (d) and add to clause (c): and  $\varphi, q$  are compatible in  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ .

2) We use  $\lambda > \aleph_0$  in the proof, to eliminate it we can immitate the completeness theorem for  $\mathbb{L}_{\aleph_1, \aleph_0}$ .

*Proof.* We choose  $(p_n, q_n, \psi_n)$  by induction on  $n$  such that:

- $\boxplus_n$  (a)( $\alpha$ ) (st) <sub>$p_n, q_n, \psi_n$</sub>  holds if  $n$  is even
- ( $\beta$ ) (st) <sub>$q_n, p_n, \psi_n$</sub>  holds if  $n$  is odd
- (b)  $(p_0, q_0, \psi_0) = (p, q, \psi)$
- (c) if  $n = 2m + 1$  and  $s \in \text{dom}(p_{2m}) \cap M_{\mathbf{m}}$  then  $s \in \text{dom}(q_{2m+1})$  and  $\text{tr}(p_{2m}(s)) \subseteq \text{tr}(q_{2m+1}(s))$
- (d) if  $n = 2m + 2$  and  $s \in \text{dom}(q_{2m+1}) \cap M_{\mathbf{m}}$  then  $s \in \text{dom}(p_{2m+2})$  and  $\text{tr}(q_{2m+1}(s)) \subseteq \text{tr}(p_{2m+2}(s))$
- (e) if  $n = m + 1$  then  $p_m \leq p_n, q_m \leq q_n$ .

Case 1: For  $n = 0$  use clause (b).

Case 2:  $n = 2m + 1$ .

So the triple  $(p_{2m}, q_{2m}, \psi_{2m})$  is well defined, let  $u_{2m} = \text{Dom}(p_{2m}) \cap M_{\mathbf{m}}$  and let  $\bar{\nu} = \langle \nu_s : s \in u_{2m} \rangle$  be defined by  $\nu_s = \text{tr}(p_{2m}(s))$ .

Clearly

(\*)<sub>1</sub>  $\psi_{2m} \vdash p_{s,\nu_s}^*$  for  $s \in u_{2m}$ .

[Why? Clearly  $p_{2m} \vdash p_{s,\nu_s}^*$ , i.e.  $p_{s,\nu_s}^* \leq p_{2m}$  in  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ , hence if  $\psi_{2m} \not\vdash p_{s,\nu_s}^*$  then  $\psi' = \psi_{2m} \wedge \neg p_{s,\nu_s}^* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$  is  $\geq \psi_{2m}$  hence compatible with  $p_{2m}$ , contradiction, see clause (c) in  $(st)_{p,q,\psi}$ .]

(\*)<sub>2</sub> there is  $q'_{2m} \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$  which is above  $q_{2m}$  and above  $\psi_{2m}$  hence  $s \in u_{2m}$  implies  $\nu_s \subseteq \text{tr}(q'_{2m}(s))$  and  $s \in \text{Dom}(q'_{2m})$ .

[Why? By clause (d) of  $(st)_{p_{2m},q_{2m},\psi_{2m}}$  which holds by  $\boxplus_{2m}(a)(\alpha)$  recalling  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$  is dense  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ ; the “hence” by  $(*)_1$ .]

(\*)<sub>3</sub> there is  $\psi'_{2m} \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  such that:

- (a) if  $\psi'_{2m} \leq \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$  then  $\varphi, q'_{2m}$  are compatible in  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$
- (b) if  $s \in u_{2m}$  then  $\psi'_{2m} \vdash p_{s,\nu_s}^*$
- (c)  $\psi_{2m} \leq \psi'_{2m}$ .

[Why? Obvious using the  $\lambda^+$ -c.c.]

(\*)<sub>4</sub> without loss of generality  $\text{wsupp}(q'_{2m}) \cap \text{wsupp}(p_{2m}) \subseteq M_{\mathbf{m}}$ .

{e10} [Why? As  $\mathbf{m}$  is wide using automorphisms of  $\mathbf{m}$ , i.e. by 4.5.]

(\*)<sub>5</sub> there is  $p'_{2m} \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  which is above  $p_{2m}$  and above  $\psi'_{2m}$ .

[Why? By the choice of  $\psi'_{2m}$  and clause (c) of  $(st)_{p_{2m},q_{2m},\psi_{2m}}$  which holds by  $\boxplus_{2m}(a)(\alpha)$ .]

(\*)<sub>6</sub> without loss of generality  $\text{fsupp}(p'_{2m}) \cap \text{fsupp}(q'_{2m}) \subseteq M_{\mathbf{m}}$ .

{e10} [Why? As  $\mathbf{m}$  is wide using 4.5.]

Lastly, let  $p_n = p'_{2m}, q_n = q'_{2m}, \psi_n = \psi'_{2m}$  and check.

Case 3:  $n = 2m + 2$

Similar to case 2 the roles of the  $p$ 's and the  $q$ 's interchanged.

{c11} Having carried the induction we can find  $p_*$  the upper bound of  $\{p_n : n < \omega\}$  as in 2.13(4), in particular:

- (\*)<sub>7</sub> (a)  $\text{Dom}(p_*) = \bigcup_n \text{Dom}(p_n)$ ; in fact, also  $\text{fsupp}(p_*) = \bigcup_n \text{fsupp}(p_n)$
- (b) if  $s \in \text{Dom}(p_n)$  then  $\text{tr}(p_*(s)) = \bigcup_{k \geq n} \text{tr}(p_k(s))$ .

{c11} Similarly let  $q_*$  be the upper bound of  $\{q_n : n < \omega\}$  as in 2.13(4), so again:

- (\*)<sub>8</sub> (a)  $\text{Dom}(q_*) = \bigcup_n \text{Dom}(q_n)$ , in fact also  $\text{fsupp}(q_*) = \bigcup_n \text{fsupp}(q_n)$
- (b) if  $s \in \text{Dom}(q_n)$  then  $\text{tr}(q_*(s)) = \bigcup_{k \geq n} \text{tr}(q_k(s))$ .

Hence

- (\*)<sub>9</sub> (a)  $p_*, q_* \in \mathbb{P}_{\mathbf{m}}$
- (b)  $\text{Dom}(p_*) \cap \text{Dom}(q_*) \subseteq M_{\mathbf{m}}$ , in fact,  $\text{fsupp}(p_*) \cap \text{Dom}(q_*) \subseteq M_{\mathbf{m}}$
- (c)  $\text{Dom}(p_*) \cap M_{\mathbf{m}} = \text{Dom}(q_*) \cap M_{\mathbf{m}}$
- (d) if  $s \in \text{Dom}(p_*) \cap M_{\mathbf{m}}$ , equivalently,  $s \in \text{Dom}(p_*) \cap \text{Dom}(q_*)$  then  $\text{tr}(p_*(s)) = \text{tr}(q_*(s))$ .

[Why? Clause (a) by properties of  $\mathbb{P}_\mathbf{m}$  and  $p_n \leq p_{n+1}, q_n \leq q_{n+1}$  see above, clause (b) as  $\text{Dom}(p_{2m}) \cap \text{Dom}(q_{2m}) \subseteq M_\mathbf{m}$  as  $(\text{st})_{p_{2m}, q_{2m}, \psi_{2m}}$ , clause (c) by  $\boxplus_n(c), (d)$ , the first conclusion and clause (d) by  $\boxplus_n(c), (d)$ , the second conclusion.]

It follows that  $p_*, q_*$  are compatible in  $\mathbb{P}_\mathbf{m}$  but  $p = p_0 \leq p_*, q = q_0 \leq q_*$ , so  $p, q$  are compatible as promised.  $\square_{4.13}$

{e30}

**Claim 4.15.** *The set  $\{\psi_i : i < i(*)\} \cup \{\psi_*\}$  has a common upper bound in  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}]$  when:*

- (\*) (a)  $\mathbf{m} \in \mathbf{M}$  is wide
- (b)  $i(*) < \lambda$
- (c)  $t_i \in L_\mathbf{m} \setminus M_\mathbf{m}$  for  $i < i(*)$
- (d)  $t_i, t_j$  are not  $E_\mathbf{m}''$ -equivalence for  $i < j < i(*)$
- (e)  $\psi_* \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$
- (f)  $X_i = t_i/E_\mathbf{m}$
- (g)  $\psi_i \in \mathbb{P}_\mathbf{m}[X_i]$
- (h) if  $\mathbb{P}_\mathbf{m}[M_\mathbf{m}] \models \text{“}\psi_* \leq \varphi\text{”}$  and  $i < i(*)$  then  $\psi_i, \varphi$  are compatible in  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}]$  equivalently in  $\mathbb{P}_\mathbf{m}[X_i]$ .

*Remark 4.16.* Note:  $\lambda$ -wide is enough.

*Proof.* As  $\psi_* \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$ , there is  $p \in \mathbb{P}_\mathbf{m}$  such that  $p \Vdash_{\mathbb{P}_\mathbf{m}} \text{“}\psi_*[\mathbf{G}_{\mathbb{P}_\mathbf{m}}] = \text{true”}$ . As  $\mathbf{m}$  is wide by 4.5 there is an automorphism  $f$  of  $\mathbf{m}$  such that  $i < i(*) \Rightarrow f''(\text{fsupp}(p)) \cap X_i \subseteq M_\mathbf{m}$ , hence without loss of generality  $i < i(*) \Rightarrow \text{fsupp}(p) \cap X_i \subseteq M_\mathbf{m}$ . Now we choose  $p_i$  by induction on  $i \leq i(*)$  such that:

{e10}

- $\boxplus$  (a)  $p_i \in \mathbb{P}_\mathbf{m}$
- (b)  $\langle p_j : j \leq i \rangle$  is increasing
- (c) if  $s \in \text{Dom}(p_i), i < i(*)$  then  $\ell g(\text{tr}(p_{i+1}(s))) > i(*)$
- (d)  $p_0 = p$
- (e) if  $i = j + 1$  then  $p_i \Vdash \text{“}\psi_j[\mathbf{G}_{\mathbb{P}_\mathbf{m}}] = \text{true”}$
- (f)  $\text{fsupp}(p_i)$  hence also  $\text{wsupp}(p_i)$  is disjoint to  $\cup\{X_j \setminus M_\mathbf{m} : j \in [i, i(*)]\}$ .

This is sufficient for the claim as  $p_{i(*)}$  is as required. So let us carry the induction. For  $i = 0$  use clause (d), for  $i$  limit by 2.12(4) we know that  $\langle p_j : j < i \rangle$  has a  $\leq_{\mathbb{P}_\mathbf{m}}$ -upper bound  $p_i$  with domain  $\subseteq \cup\{\text{Dom}(p_j) : j < i\}$  and  $\text{wsupp}(p_i) \subseteq \cup\{\text{wsupp}(p_j) : j < i\}$  by 2.13(4), hence  $p_i$  is as required, in particular as in clause (f).

{c10}

{c11}

Lastly, assume  $i = j + 1$ , now there is  $\varphi_j \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$  such that  $\varphi_j \leq \varphi \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}] \Rightarrow p_j, \varphi$  are compatible in  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}]$ . By an assumption  $p_j \Vdash \text{“}\psi_*[\mathbf{G}_{\mathbb{P}_\mathbf{m}}] = \text{true”}$  as  $p_0$  forces this hence  $\psi_* \leq \varphi_j$ . As  $\varphi_j \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$  by clause (h) of the assumption  $\psi_j, \varphi_j$  are compatible in  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}]$  hence have a common upper bound  $\varphi_j^+ \in \mathbb{P}_\mathbf{m}[X_j]$ , so there is  $q_j^0 \in \mathbb{P}_\mathbf{m}$  above  $\varphi_j$  and  $\psi_j$ . As  $\mathbf{m}$  is wide without loss of generality  $\text{wsupp}(q_j^0) \cap \text{wsupp}(p_j) \subseteq M_\mathbf{m}$ . Together (see 4.13)  $(\text{st})_{p_j, q_j^0, \varphi_j}$  holds hence by 4.13  $p_j, q_j^0$  has a common upper bound called  $p_i$ . As  $\mathbf{m}$  is wide, without loss of generality  $\text{fsupp}(p_i) \cap X_i = M_\mathbf{m}$  for  $j \in [i + 1, i(*)]$ .

{e29}

{e29}

Clearly  $p_i$  is as required so we have finished the induction. So we are done.  $\square_{4.15}$

{e31}

**Conclusion 4.17.** *If  $f \in \mathcal{F}_{\mathbf{m}, \beta}$  and  $L_1, L_2$  its domain and range respectively then  $f$  induces an isomorphism  $\hat{f}$  from  $\mathbb{P}_\mathbf{m}(L_1)$  onto  $\mathbb{P}_\mathbf{m}(L_2)$ .*

- Remark 4.18.* 1) See Definition 4.1(5); note that this claim is not covered by Definition 4.1(4). {e4}  
 2) Here we use 4.2(4), so the choice in Definition 2.9(c)( $\gamma$ ). {e4}  
 3) We could have separated the definition of “analyze” and its properties. {e6}  
 4) Note that in Definition 4.9, we deal only with  $L_1 \subseteq t/E_{\mathbf{m}}$  for some  $t$ . {e24}

*Proof.* By the definitions, clearly  $\hat{f}$  is a one-to-one function from  $\mathbb{P}_{\mathbf{m}}(L_1)$  onto  $\mathbb{P}_{\mathbf{m}}(L_2)$ . Next assume  $p_1, q_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$ ,  $\text{Dom}(p_1) \subseteq \text{Dom}(q_1)$  and let  $p_2 := \hat{f}(p_1)$ ,  $q_2 := \hat{f}(q_1)$ ; clearly they belong to  $\mathbb{P}_{\mathbf{m}}(L_2)$ . We shall prove that  $\mathbb{P}_{\mathbf{m}} \models “p_1 \leq q_1”$  iff  $\mathbb{P}_{\mathbf{m}} \models “p_2 \leq q_2”$ .

Let  $\langle t_i^1 : i < i(*) \rangle$  be such that:

- $\oplus_1$  (a)  $t_i^1 \in \text{Dom}(q_1) \setminus M_{\mathbf{m}} \subseteq L_1$  such that  $\text{fsupp}(q_1)$  is included in  $\cup \{t_i^1/E_{\mathbf{m}} : i < i(*)\}$   
 (b)  $\langle t_i^1 : i < i(*) \rangle$  are pairwise non  $E_{\mathbf{m}}''$ -equivalent.

Next let

- $\oplus_2$  (c)  $t_i^2 = f(t_i^1)$   
 (d) let  $\bar{t}_\ell = \langle t_i^\ell : i < i(*) \rangle$  without loss of generality  $\text{fsupp}(p_\ell) \subseteq \cup \{t_i^\ell/E_{\mathbf{m}}'' : i < j(*)\} \cup M_{\mathbf{m}}$ , so  $j(*) \leq i(*)$ .

For  $i < i(*)$  let  $\psi_{1,i}^* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$  be such that:  $\vartheta \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$  is compatible with  $q_{1,i} := q_1 \upharpoonright (t_i^1/E_{\mathbf{m}})$  (projection!) iff  $\vartheta \wedge \psi_{1,i}^* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ ; clearly exists as  $\mathbb{P}_{\mathbf{m}}$  satisfies the  $\lambda^+$ -c.c. Let  $\psi_1^* = \wedge \{\psi_{1,i}^* : i < i(*)\}$ .

Now  $\psi_1^* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$  as  $q_1 \Vdash “\psi_1^*[\mathbf{G}_{\mathbb{P}_{\mathbf{m}}}] = \text{true}”$ . We will say “ $\psi_1^*, \bar{\psi}_1^* = \langle \psi_{1,i}^*, q_{1,i} : i < i(*) \rangle$  analyze  $q_1$  or  $(q_1, \bar{t}_1)$ ” when the above holds.

Next choose  $\varphi_1^*, \langle \varphi_{1,i}^*, p_{1,i} : i < j(*) \rangle$  which analyze  $p_1, \langle t_i^1 : i < j(*) \rangle$ . Why possible? As above.

Lastly, let  $\psi_{2,i}^* = \hat{f}(\psi_{1,i}^*), p_{2,i} = \hat{f}(p_{1,i}), \psi_2^* = \hat{f}(\psi_1^*), \varphi_{2,i}^* = \hat{f}(\varphi_{1,i}^*), q_{2,i} = \hat{f}(q_{1,i}), \varphi_2^* = \hat{f}(\varphi_1^*)$ .

Now

- (\*) for  $\ell = 1, 2$  the sequence  $(p_\ell, q_\ell, \bar{\psi}_\ell^* = \langle \psi_{\ell,i}^*, q_{\ell,i} : i < i_\ell(*) \rangle, \psi_\ell^*, \bar{\varphi}_\ell^* = \langle \varphi_{\ell,i}^*, p_{\ell,i} : i < i(*) \rangle, \varphi_\ell^*)$  satisfies the same demands as listed above for  $\ell = 1, 2$ , that is  
 (a)  $(\psi_\ell^*, \bar{\psi}_\ell^*)$  analyze  $(q_\ell, \bar{t}_\ell)$  for  $\ell = 1, 2$   
 (b)  $(\varphi_1^*, \bar{\varphi}_1^*)$  analyze  $(p_\ell, \bar{t}_\ell \upharpoonright j(*))$  for  $\ell = 1, 2$ .

[Why? Think, recalling  $f \upharpoonright (t_i^1/E_{\mathbf{m}})$  is an isomorphism from  $\mathbf{m} \upharpoonright (t_i^1/E_{\mathbf{m}})$  onto  $\mathbf{m} \upharpoonright (t_i^2/E_{\mathbf{m}})$ , etc.]

Next

$\boxplus$  for  $\ell = 1, 2$  we have  $(A)_\ell \Leftrightarrow (B)_\ell$  where

(A) $_\ell$   $\mathbb{P}_{\mathbf{m}} \models “p_\ell \leq q_\ell”$

(B) $_\ell$  for every  $i < j(*)$  we have  $\mathbb{P}_{\mathbf{m}}[t_i^\ell/E_{\mathbf{m}}] \models “(\varphi_\ell^* \wedge p_{\ell,i}) \leq (\psi_\ell^* \wedge q_{\ell,i})”$ .

Why? First, assume the (B) $_\ell$  condition fails, say for  $i$ , hence there is  $\vartheta \in \mathbb{P}_{\mathbf{m}}[t_i^\ell/E_{\mathbf{m}}]$  such that  $\mathbb{P}_{\mathbf{m}}[t_i^\ell/E_{\mathbf{m}}] \models “(\psi_\ell^* \wedge q_{\ell,i}) \leq \vartheta”$  and  $\varphi_\ell^* \wedge p_{\ell,i} \wedge \vartheta \notin \mathbb{P}_{\mathbf{m}}[t_i^\ell/E_{\mathbf{m}}]$ . So by claim



{e30} 4.15 there is  $q_\ell^+ \in \mathbb{P}_\mathbf{m}$  such that  $q_\ell^+ \in \mathbb{P}_\mathbf{m}[L_\mathbf{m}]$  is above  $\vartheta$  hence above  $\psi_\ell^*$  and above  
 {e5n}  $q_{\ell,j} = q_\ell \upharpoonright (t_j^\ell/E_\mathbf{m})$  for  $j < i(*)$ . Hence by 4.2(4) the condition  $q_\ell^+$  is above  $q_\ell$  but  
 $q_\ell^+ \Vdash \text{“}\varphi_\ell^* \wedge p_{\ell,i}[\mathbf{G}] = \text{false”}$  as  $q_\ell^+$  is above  $\vartheta$ . However,  $p_\ell \Vdash_{\mathbb{P}_\mathbf{m}[L_\mathbf{m}]} \text{“}p_{\ell,i} \in \mathbf{G} \text{ and } \varphi_\ell^* \in \mathbf{G}\text{”}$ . By the last two sentences  $q_\ell^+, p_\ell$  are incompatible, in  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}]$  equivalently  
 in  $\mathbb{P}_\mathbf{m}$ . So indeed  $\neg(B)_\ell \Rightarrow \neg(A)_\ell$ .

For the other direction assume condition  $(B)_\ell$  holds, but condition  $(A)_\ell$  fails and we shall get a contradiction. So there is  $q_\ell^+ \in \mathbb{P}_\mathbf{m}$  above  $q_\ell$  incompatible with  $p_\ell$ .

For each  $i < i(*)$  as  $(\psi_\ell^*, \langle \psi_{\ell,j}^*, q_{\ell,j} : j < i(*) \rangle)$  analyze  $q_\ell$ , clearly  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}] \models \text{“}(\psi_\ell^* \wedge q_{\ell,i}) \leq q_\ell\text{”}$  but  $q_\ell \leq q_\ell^+$  hence  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}] \models \text{“}(\psi_\ell^* \wedge q_{\ell,i}) \leq q_\ell^+\text{”}$ , and as we are assuming clause  $(B)_\ell$  we have  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}] \models \text{“}(\varphi_\ell^* \wedge p_{\ell,i}) \leq q_\ell^+\text{”}$ . Hence by 4.2(4),  $q_\ell^+$  is above  $p_\ell$ , contradiction. So indeed  $(B)_\ell \Rightarrow (A)_\ell$ . {e5n}

Together,  $\boxplus$  holds. Now clearly  $(B)_1 \Leftrightarrow (B)_2$ , see Definition 4.9, 4.12; so by  $\boxplus$  we have  $(A)_1 \Leftrightarrow (A)_2$  which is the desired conclusion.  $\square_{4.15}$  {e28}

**Claim 4.19.** *We have  $\mathbb{P}_{\mathbf{m}_1} \triangleleft \mathbb{P}_\mathbf{m}$  when:* {e32}

- (a)  $\mathbf{m}_1 \leq \mathbf{M} \mathbf{m}$
- (b) if  $t \in L_{\mathbf{m}_1} \setminus M_{\mathbf{m}_1}$  and  $\bar{s} \in \zeta(t/E''_{\mathbf{m}_1}), \zeta < \lambda^+$  then we can find  $t_i, \bar{s}_i$  for  $i < \lambda^+$  such that
  - ( $\alpha$ )  $t_i \in L_{\mathbf{m}_1} \setminus M_{\mathbf{m}_1}$
  - ( $\beta$ )  $t_i/E''_{\mathbf{m}_1} \neq t_j/E''_{\mathbf{m}_1}$  when  $i \neq j < \lambda^+$
  - ( $\gamma$ )  $\bar{s}_i \in \zeta(t_i/E''_{\mathbf{m}_1})$
  - ( $\delta$ )  $(t_i, \bar{s}_i)$  is  $\xi$ -equivalent to  $(t, \bar{s})$  in  $\mathbf{m}$  where<sup>16</sup>  $\xi = 1$ .
- (c)  $\mathbf{m}$  is wide.

*Remark 4.20.* In the proof we use conclusion 4.17 but not clause (a)( $\varepsilon$ ) of Definition 4.9. {e31} {e24}

*Proof.*

- $\boxplus_1$  for  $f \in \mathcal{F}_{\mathbf{m},\beta}$ 
  - (a)  $\hat{f}$  preserves “ $p_2$  is above  $p_1$  in  $\mathbb{P}_\mathbf{m}$ ”, and its negations
  - (b) if  $\beta > 0$  then  $\hat{f}$  preserves also incompatibility in  $\mathbb{P}_\mathbf{m}$ .

[Why? Clause (a) holds by 4.17. For clause (b) use clause (a) and Definitions 4.9 and 4.12 or see the proof of  $\boxplus_2$ .] {e34} {e28}

$\boxplus_2$  if  $p_i \in \mathbb{P}_{\mathbf{m}_1}$  for  $i < i(*) < \lambda^+$  and  $p \in \mathbb{P}_\mathbf{m}$  then there is  $p^*$  such that:

- (a)  $p^* \in \mathbb{P}_{\mathbf{m}_1}$ , equivalently  $p^* \in \mathbb{P}_\mathbf{m}(L_{\mathbf{m}_1})$
- (b)  $\mathbb{P}_{\mathbf{m}_1} \models \text{“}p_i \leq p^*\text{”}$  iff  $\mathbb{P}_\mathbf{m} \models \text{“}p_i \leq p\text{”}$
- (c)  $\mathbb{P}_{\mathbf{m}_1} \models \text{“}p_i, p^* \text{ are compatible”}$  iff  $\mathbb{P}_\mathbf{m} \models \text{“}p_i, p \text{ are compatible”}$ .

[Why? Let  $q_i \in \mathbb{P}_\mathbf{m}$  be such that: if  $p_i, p$  are compatible in  $\mathbb{P}_\mathbf{m}$  then  $p_i \leq q_i \wedge p \leq q_i$ . We can find  $L_1 \subseteq L_2$  such that

- $M_\mathbf{m} \subseteq L_1 \subseteq L_{\mathbf{m}_1}, |L_1 \setminus M_\mathbf{m}| \leq \lambda$
- $\{p_i : i < i(*)\} \subseteq \mathbb{P}_\mathbf{m}(L_1)$
- $L_1 \subseteq L_2 \subseteq L_\mathbf{m}, |L_2 \setminus M_\mathbf{m}| \leq \lambda$  and  $p, q_i \in \mathbb{P}_\mathbf{m}(L_2)$  for  $i < i(*)$ .

<sup>16</sup>no real harm in using larger  $\xi$

By the assumption of the claim there is  $f \in \mathcal{F}_{\mathbf{m},1}$  such that

- $\text{Dom}(f) \subseteq \cup\{(t/E''_{\mathbf{m}}) \cap L_2 : t \in L_2\} \cup M_{\mathbf{m}}$
- $t \in L_1 \Rightarrow f \upharpoonright (t/E_{\mathbf{m}} \cap L_2) = \text{id}_{t/E_{\mathbf{m}} \cap L_2}$
- if  $q \in \{q_i : i < i(*)\} \cup \{p\} \cup \{p_i : i < i(*)\}$  and  $t \in \text{Dom}(q) \setminus M_{\mathbf{m}}$  then  $\text{fsupp}(q(t)) \subseteq \text{Dom}(f)$
- $\text{Rang}(f) \subseteq L_{\mathbf{m}_1}$ .

Let  $p^* = \hat{f}(p)$ : by  $\boxplus_1(a)$  clearly clauses (a),(b) of  $\boxplus_2$  holds; also the implication “if” of clause (c). The “only if” of clause (c) holds by  $\boxplus_1(b)$  so we are done.]

$\boxplus_3$  if  $p \in \mathbb{P}_{\mathbf{m}}$  then  $p \in \mathbb{P}_{\mathbf{m}_1}$  iff  $\text{fsupp}(p) \subseteq L_{\mathbf{m}_1}$ .

[Why? Obvious.]

{c34} Recalling Definition 2.24(0)(b)

$\boxplus_4$  for every ordinal  $\gamma$ , we have  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) < \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$ .

[Why? We shall prove this by induction on  $\gamma$  using  $\boxplus_2 + \boxplus_3$ .

Note that

- $\boxplus_{4.1}$  (a)  $L_{\mathbf{m}, \gamma}^{\text{dp}} \cap L_{\mathbf{m}_1} = L_{\mathbf{m}_1, \gamma}^{\text{dp}}$   
 (b) if  $f \in \mathcal{F}_{\mathbf{m}, \beta}$ ,  $s \in \text{Dom}(f)$  and  $\beta$  is an ordinal then
- $s \in L_{\mathbf{m}_1, \gamma}^{\text{dp}} \Leftrightarrow f(s) \in L_{\mathbf{m}, \gamma}^{\text{dp}}$
- (c) the parallel of  $\boxplus_2$  holds for  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma})$  so  $p^* \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1, \gamma})$   
 (d)  $L_{\mathbf{m}, \gamma}^{\text{dp}}$  is an initial segment of  $L_{\mathbf{m}}$   
 (e)  $L_{\mathbf{m}_1, \gamma}^{\text{dp}}$  is an initial segment of  $L_{\mathbf{m}_1}$   
 (f)  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) < \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1})$ , similarly for  $\mathbf{m}$ .

We shall use this freely. The inductive proof on  $\gamma$  splits to three cases.

Case 1:  $\gamma = 0$

So

- $E = E''_{\mathbf{m}} \upharpoonright L_{\mathbf{m}, \gamma}^{\text{dp}}$  is an equivalence relation on  $L_{\mathbf{m}, \gamma}^{\text{dp}}$
- $E \upharpoonright L_{\mathbf{m}_1, \gamma}^{\text{dp}} = E''_{\mathbf{m}_1} \upharpoonright L_{\mathbf{m}_1, \gamma}^{\text{dp}}$
- if  $t \in L_{\mathbf{m}_1, \gamma}^{\text{dp}}$  then  $(t/E) \subseteq L_{\mathbf{m}_1, \gamma}^{\text{dp}}$  and  $\mathbb{P}_{\mathbf{m}}(t/E) = \mathbb{P}_{\mathbf{m}_1}(t/E)$
- $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$  is the product with  $(< \lambda)$ -support of  $\{\mathbb{P}_{\mathbf{m}}(t/E) : t \in L_{\mathbf{m}, \gamma}^{\text{dp}}\}$
- similarly for  $\mathbf{m}_1$ .

So the result should be clear.

Case 2:  $\gamma = \beta + 1$

Let  $M_{\beta} = \{s \in M_{\mathbf{m}} : \text{dp}_{\mathbf{m}}(s) = \beta\}$ , clearly

- $\boxplus_{4.2}$  (a)  $M_{\beta}$  is a set of pairwise incomparable elements  
 (b)( $\alpha$ )  $s \in M_{\beta} \Rightarrow L_{\mathbf{m}_1, < s} \subseteq L_{\mathbf{m}_1, \beta}^{\text{dp}} \cap L_{\mathbf{m}, < s} \subseteq L_{\mathbf{m}_1, \beta}^{\text{dp}}$   
 ( $\beta$ ) similarly for  $\mathbf{m}$   
 (c)  $M_{\beta}$  is disjoint to  $L_{\mathbf{m}_1, \beta}^{\text{dp}}, L_{\mathbf{m}, \beta}^{\text{dp}}$   
 (d)  $M_{\beta} \subseteq L_{\mathbf{m}_1, \gamma}^{\text{dp}}$

- (e)  $L_{\mathbf{m},\beta}^{\text{dp}} \cup M_\beta$  is an initial segment of  $L_{\mathbf{m}}$   
(f)  $L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta$  is an initial segment of  $L_{\mathbf{m}_1}$ .

As first half we prove

$$\boxplus_{4.3} \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta) \triangleleft \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},\beta}^{\text{dp}} \cup M_\beta).$$

Why? Recalling  $\boxplus_{4.1}(a)$ , note

- (a) for  $p, q \in \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta)$  we have  $\mathbb{P}_{\mathbf{m}_1,\gamma} \models "p \leq q"$  iff  $\mathbb{P}_{\mathbf{m},\gamma} \models "p \leq q"$ .

[Why? Immediate by the definition of the order and the induction hypothesis.]

- (b) for  $p_1, p_2 \in \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta)$  then  $p_1, p_2$  are compatible in  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta)$  iff they are compatible in  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},\beta}^{\text{dp}} \cup M_\beta)$ .

[Why? The implication  $\Rightarrow$  holds by clause (a). So assume  $p_3 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},\beta}^{\text{dp}} \cup M_\beta)$  is a common upper bound of  $p_1, p_2$  in  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},\beta}^{\text{dp}} \cup M_\beta)$  equivalently in  $\mathbb{P}_{\mathbf{m}}$ .

Now there is  $f \in \mathcal{F}_{\mathbf{m},1}$  such that

- $f \upharpoonright (\text{fsupp}(p_1) \cup \text{fsupp}(p_2))$  is the identity, moreover
- $s \in \text{wsupp}(p_1) \cup \text{wsupp}(p_2) \wedge s \in \text{dom}(f) \Rightarrow f(s) = s$ ,
- $\text{Dom}(f) = \cup \{\text{fsupp}(p_\ell) : \ell = 1, 2, 3\}$
- $\text{Rang}(f) \subseteq L_{\mathbf{m}_1}$ .

Hence clearly  $f \upharpoonright M_\beta = \text{id}_{M_\beta}$  so by  $\boxplus_{4.1}(b)$  we have  $\text{Rang}(f) \subseteq L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta$  so  $\hat{f}(p_3) \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta)$ .

By  $\boxplus_1$  the condition  $\hat{f}(p_3)$  is a common upper bound of  $p_1, p_2$  in  $\mathbb{P}_{\mathbf{m}}$  and by the previous sentence also in  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta)$ , so by clause (a) the conclusion of (b) holds.]

- (c) if  $\mathcal{S}$  is a maximal antichain in  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta)$  then  $\mathcal{S}$  is a maximal antichain of  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},\beta}^{\text{dp}} \cup M_\beta)$ .

[Why? As in the proof of (b) and of  $\boxplus_2$ .]

So we are done proving  $\boxplus_{4.3}$ .

Now we return to proving  $\boxplus_4$ , note

$$\boxplus_{4.4} \text{ let } \mathcal{E} = \{(s_1, s_2) : s_1, s_2 \in L_* \text{ and } s_1/E_{\mathbf{m}} = s_2/E_{\mathbf{m}}\} \text{ where } L_* = L_{\mathbf{m},\gamma}^{\text{dp}} \setminus (L_{\mathbf{m},\beta}^{\text{dp}} \cup M_\beta) \} \text{ then}$$

- (a)  $\mathcal{E}$  is an equivalence relation on  $L_*$   
(b) if  $s_1, s_2 \in L_*$  and  $s_1 \leq_{L_{\mathbf{m}}} s_2$  then  $s_1 \mathcal{E} s_2$   
(c) if  $s_1, s_2 \in L_*$  and  $s_1 \mathcal{E} s_2$  then  $s_1 \in L_{\mathbf{m}_1,\gamma}^{\text{dp}} \Leftrightarrow s_2 \in L_{\mathbf{m}_1,\gamma}^{\text{dp}}$  (and both  $\notin M_\beta$ )  
(d) if  $s \in L_*$  then  $L_{\mathbf{m},<s} \subseteq L_{\mathbf{m},\beta}^{\text{dp}} \cup M_\beta \cup (s/\mathcal{E})$   
(e) if  $s \in L_* \cap L_{\mathbf{m}_1}$  then  $L_{\mathbf{m}_1,<s} \subseteq L_{\mathbf{m}_1,\beta}^{\text{dp}} \cup M_\beta \cup (s/\mathcal{E})$ .

Hence let  $L_0 = L_{\mathbf{m}_1, \beta}^{\text{dp}} \cup M_\beta$  and  $L_1 = L_{\mathbf{m}_1, \gamma}^{\text{dp}} \cup M_\beta$  they satisfy all the assumptions of 2.22 hence its conclusion, so we are done easily proving Case 2 of  $\boxplus_4$ . {c33s}

Case 3:  $\gamma$  is a limit ordinal

Clearly  $p \in \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$  iff  $p \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$ ; also each of them implies  $p \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$ . Also for  $p, q \in \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$  we have  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) \models "p \leq q"$  iff  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}}) \models "p \leq q"$  by the definition of the order and the induction hypothesis. Together  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}}) \subseteq \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$ , (as partial orders).

Next assume that  $q_1, q_2 \in \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$  and  $p_3$  is a common upper bound of  $q_1, q_2$  in  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$ .

We shall find  $p_1 \in \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$  such that:

- (\*)<sub>1</sub> (a)  $p_1$  is above  $q_1, q_2$  (in  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$  or equivalently in  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$ ),
- (b) if  $p_1 \leq p'_1 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$  then  $p'_1, p_3$  are compatible in  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$ .

This clearly suffices; (e.g. if  $\{r_i : i < i(*)\} \subseteq \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$  is a maximal antichain of  $\mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1, \gamma}^{\text{dp}})$  but not of  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$ , let  $q_1 = q_2 = \emptyset$  and  $p_3 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}, \gamma}^{\text{dp}})$  be incompatible with every  $r_i$ ; let  $p_1$  be as in (\*)<sub>1</sub>, it gives a contradiction.

If  $\text{cf}(\gamma) \geq \lambda$  then for some  $\gamma_1 < \gamma$  we have  $q_1, q_2 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1, \gamma_1}^{\text{dp}})$  and use the induction hypothesis on  $\gamma_1$ . So assume  $\aleph_0 \leq \text{cf}(\gamma) < \lambda$  and let  $\langle \gamma_\varepsilon : \varepsilon < \text{cf}(\gamma) \rangle$  be increasing continuous with limit  $\gamma$ .

Now we choose  $p_{1, \varepsilon}$  by induction on  $\varepsilon \leq \text{cf}(\gamma)$  such that:

- (\*)<sub>2</sub> (a)  $p_{1, \varepsilon} \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1, \gamma_\varepsilon}^{\text{dp}})$
- (b)  $(\gamma_\varepsilon, q_1 \upharpoonright L_{\mathbf{m}, \gamma_\varepsilon}^{\text{dp}}, q_2 \upharpoonright L_{\mathbf{m}, \gamma_\varepsilon}^{\text{dp}}, p_3 \upharpoonright L_{\mathbf{m}, \gamma_\varepsilon}^{\text{dp}}, p_{1, \varepsilon})$  are like  $(\gamma, q_1, q_2, p_3, p_1)$  in (\*)<sub>1</sub>
- (c)  $p_{1, \zeta} \leq p_{1, \varepsilon}$  for  $\zeta < \varepsilon$
- (d) if  $\varepsilon = \zeta + 1$  and  $s \in \text{dom}(p_{1, \zeta})$  then  $\ell g(\text{tr}(p_\varepsilon(s))) > \text{cf}(\gamma)$ .

So we are done proving  $\boxplus_4$ .]

$$\boxplus_5 \mathbb{P}_{\mathbf{m}_1} \triangleleft \mathbb{P}_{\mathbf{m}}.$$

[Why? By  $\boxplus_4$  for  $\gamma$  large enough.]

So we are done. □<sub>4.19</sub>

{c73}

**Claim 4.21.** *If  $\mathbf{m} \in \mathbf{M}$  is reduced or just  $L_{\mathbf{m}}$  has cardinality  $\leq \lambda_2$  then there is  $\mathbf{n} \in \mathbf{M}_{\text{ec}}$  of cardinality  $\leq \lambda_2$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ .*

{c76}

*Remark 4.22.* By this we may restrict ourselves to  $\mathbf{M}_{\leq \lambda_2}$  (but then similarly in the end of §3).

{c41}

*Proof.* We choose  $\chi$  large enough and  $\mathbf{m}_* \in \mathbf{M}_\chi$  which is wide, belongs to  $\mathbf{M}_{\text{ec}}$  and  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_*$ ; moreover is full and very wide (as constructed in 2.26).

We can choose  $\mathbf{n}$  such that:

{e32}

- (\*) (a)  $\mathbf{n} \in \mathbf{M}$  and  $\mathbf{n}$  is wide and  $|L_{\mathbf{n}}| = \lambda_2$
- (b)  $\mathbf{m} \leq \mathbf{n} \leq_{\mathbf{M}} \mathbf{m}_*$
- (c)  $(\mathbf{n}, \mathbf{m}_*)$  satisfies the criterion from 4.19, with  $\mathbf{m}_1, \mathbf{m}$  there standing for  $\mathbf{n}, \mathbf{m}_*$  here.

[Why? Let  $\xi = 1, \langle (t_\alpha, \bar{s}_\alpha) : \alpha < \lambda_2 \rangle$  be such that  $(t_\alpha, \bar{s}_\alpha) \in \mathcal{Y}_{\mathbf{m}_*}, t_\alpha \in L_{\mathbf{m}_*} \setminus M_{\mathbf{m}_*}, \langle t_\alpha / E_{\mathbf{m}} : \alpha < \lambda_2 \rangle$  are pairwise distinct and for every  $(t, \bar{s}) \in \mathcal{Y}_{\mathbf{m}}$  there are  $\lambda^+$  ordinals  $\alpha < \lambda_2$  such that  $(t, \bar{s}), (t_\alpha, \bar{s}_\alpha)$  are  $\xi$ -equivalent, possible by 4.11. Let  $L' = \cup \{t_\alpha / E_{\mathbf{m}} : \alpha < \lambda_2\} \cup L_{\mathbf{m}}$  and for each  $t \in L' \setminus M_{\mathbf{m}_*}$  let  $\langle s_{t,\alpha} : \alpha < \lambda^+ \rangle$  be such that  $s_{t,\alpha} \in L_{\mathbf{m}_*} \setminus M_{\mathbf{m}_*}$  and  $\mathbf{m}_* \upharpoonright (s_{t,\alpha} / E_{\mathbf{m}_*})$  is isomorphic to  $\mathbf{m}_* \upharpoonright (t / E_{\mathbf{m}_*})$  over  $M_{\mathbf{m}}$ . Let  $L' = L \cup \{s_{t,\alpha} : \alpha < \lambda^+, t \in L' \setminus M_{\mathbf{m}_*}\}$  and  $\mathbf{n} = \mathbf{m}_* \upharpoonright L$ . Now it is easy to check that  $\mathbf{n}$  is as required.]

It suffices to prove  $\mathbf{n}$  belongs to  $\mathbf{M}_{\text{ec}}$ , let  $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$ .

Without loss of generality  $L_{\mathbf{n}_2}$  has cardinality  $\leq 2^{\lambda_2}$ , by the LST argument (even  $\leq \lambda_2$ , as we are assuming  $\lambda_2 = (\lambda_2)^\lambda$ ), and as  $\mathbf{m}_*$  is very wide and full without loss of generality  $\mathbf{n}_2 \leq_{\mathbf{M}} \mathbf{m}_*$ . Now  $(\mathbf{n}_1, \mathbf{m}_*)$  satisfies the criterion from 4.19 hence  $\mathbb{P}_{\mathbf{n}_1} \triangleleft \mathbb{P}_{\mathbf{m}_*}$ . {e32}

Also the pair  $(\mathbf{n}_2, \mathbf{m}_*)$  satisfies the criterion from 4.19 looking at the criterion. {e32}  
Hence by 4.19 we have  $\mathbb{P}_{\mathbf{n}_2} \triangleleft \mathbb{P}_{\mathbf{m}_*}$ . {e32}

As  $\mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2 \leq_{\mathbf{M}} \mathbf{m}_*$  from the last two sentences it easily follows that  $\mathbb{P}_{\mathbf{n}_1} \triangleleft \mathbb{P}_{\mathbf{n}_2}$ , so we are done.  $\square_{4.21}$

**Discussion 4.23.** In what way does this proof help? Will it not be simpler to omit in Definition 2.9 clause (c) the  $\iota_{p(s)}, \mathbf{B}_{p(s),\iota}$ , etc.?

In this case in 4.1 we cannot define the projection directly hence we should look for projection as in general forcing, but then we run into problems of absoluteness. More specifically, 4.19 seems to be problematic; anyhow this does not matter.

**Definition 4.24.** For  $\mathbf{m} \in \mathbf{M}$  and  $M \subseteq L_{\mathbf{m}}$  we define  $\mathbf{n} := \mathbf{m} \langle M \rangle \in \mathbf{M}$  as follows:

- (a)  $L_{\mathbf{n}} = L_{\mathbf{m}}$  even as a partial order
- (b)  $\bar{u}_{\mathbf{n}} = \bar{u}_{\mathbf{m}}$  and  $\bar{\mathcal{P}}_{\mathbf{n}} = \bar{\mathcal{P}}_{\mathbf{m}}$
- (c)  $M_{\mathbf{n}} = M$ ; not  $M_{\mathbf{m}}$ !
- (d)  $E'_{\mathbf{n}} = \{(s, t) : s, t \in L_{\mathbf{m}} \text{ and } \{s, t\} \not\subseteq M\}$ .

**Claim 4.25.** Assume  $\mathbf{m} \in \mathbf{M}_{\leq \lambda_2}$  and  $M \subseteq M_{\mathbf{m}}$ .

1)  $\mathbf{n} := \mathbf{m} \langle M \rangle$  indeed belongs to  $\mathbf{M}$  and is equivalent to  $\mathbf{m}$  hence  $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}}) = \mathbb{P}_{\mathbf{n}}(L_{\mathbf{m}})$ .

2) If  $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1$  then for some  $\mathbf{m}_1$  we have  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1$  and  $\mathbf{m}_1, \mathbf{n}_1$  are equivalent.

3) If  $\mathbf{m} \in \mathbf{M}_{\text{ec}}$  and  $\mathbf{n} = \mathbf{m} \langle M \rangle$  then  $\mathbf{n} \in \mathbf{M}_{\text{ec}}$ .

*Proof.* 1) Check, noting that  $t \in L_{\mathbf{n}} \setminus M_{\mathbf{n}} \Rightarrow t \in L_{\mathbf{m}} \setminus M \Rightarrow |t/E'_{\mathbf{m}}| \leq |L_{\mathbf{n}}| = |L_{\mathbf{m}}| \leq \lambda_2$ .

2) Given such  $\mathbf{n}_1$  we now define  $\mathbf{m}_1 \in \mathbf{M}$  by:

- (\*)<sub>1</sub> (a)  $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$ ,
- (b)  $\bar{u}_{\mathbf{m}_1} = \bar{u}_{\mathbf{n}_1}$  and  $\bar{\mathcal{P}}_{\mathbf{m}_1} = \bar{\mathcal{P}}_{\mathbf{n}_1}$
- (c)  $M_{\mathbf{m}_1} = M_{\mathbf{m}}$ ,
- (d)  $E'_{\mathbf{m}_1} = \{(s, t) : sE'_{\mathbf{m}}t \text{ or } \{s, t\} \not\subseteq L_{\mathbf{m}} \text{ and } sE'_{\mathbf{n}_1}t\}$ .

Clearly

- (\*)<sub>2</sub> (a) if  $s \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$  then
  - ( $\alpha$ )  $s \in L_{\mathbf{m}_1} \setminus M_{\mathbf{m}_1}$
  - ( $\beta$ )  $s/E'_{\mathbf{m}_1} = s/E'_{\mathbf{m}}$

- ( $\gamma$ )  $u_{\mathbf{m}_1, s} = u_{\mathbf{n}_1, s} = u_{\mathbf{n}, s} = u_{\mathbf{m}, s}$
- ( $\delta$ )  $\mathcal{P}_{\mathbf{m}_1, s} = \mathcal{P}_{\mathbf{m}, s}$
- (b) if  $s \in L_{\mathbf{m}_1} \setminus L_{\mathbf{m}}$  then
  - ( $\alpha$ )  $s \in L_{\mathbf{n}_1} \setminus L_{\mathbf{n}}$
  - ( $\beta$ )  $s/E'_{\mathbf{m}_1} = s/E'_{\mathbf{n}_1}$
  - ( $\gamma$ )  $u_{\mathbf{m}_1, s} = u_{\mathbf{n}_1, s}$
  - ( $\delta$ )  $\mathcal{P}_{\mathbf{m}_1, s} = \mathcal{P}_{\mathbf{n}_1, s}$
- (c) if  $s \in M_{\mathbf{m}_1}$ , i.e.  $s \in M_{\mathbf{m}}$  then
  - ( $\alpha$ )  $u_{\mathbf{m}_1, s} = u_{\mathbf{n}_1, s}$
  - ( $\beta$ )  $\mathcal{P}_{\mathbf{m}_1, s} = \mathcal{P}_{\mathbf{n}_1, s}$

and

- (\*)<sub>3</sub> (a) indeed  $\mathbf{m}_1 \in \mathbf{M}$ ,
- (b)  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1$ ,
- (c)  $\mathbf{m}_1, \mathbf{n}_1$  are equivalent.

So we are done.

3) Assume  $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$ , as in the proof of part (2) there are  $\mathbf{m}_1, \mathbf{m}_2$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$  and  $\mathbf{m}_\ell, \mathbf{n}_\ell$  are equivalent for  $\ell = 1, 2$ . As  $\mathbf{m} \in \mathbf{M}_{\text{ec}}$  we have  $\mathbb{P}_{\mathbf{m}_1} \triangleleft \mathbb{P}_{\mathbf{m}_2}$  but this means  $\mathbb{P}_{\mathbf{n}_1} \triangleleft \mathbb{P}_{\mathbf{n}_2}$ , as required.  $\square_{4.25}$

{c80}

**Conclusion 4.26.** 1) If  $\mathbf{m} \in \mathbf{M}$ ,  $M \subseteq M_{\mathbf{m}}$  and  $\mathbf{n} = \mathbf{m} \upharpoonright M$  then  $\mathbb{P}_{\mathbf{n}}^{\text{cr}} \triangleleft \mathbb{P}_{\mathbf{m}}^{\text{cr}}$ .

2) If  $\mathbf{m}_\ell \in \mathbf{M}$  and  $M_\ell \subseteq M_{\mathbf{m}_\ell}$  for  $\ell = 1, 2$  and  $h$  is an isomorphism from  $\mathbf{m}_1 \upharpoonright M_1$  onto  $\mathbf{m}_2 \upharpoonright M_2$  then  $h$  induces an isomorphism from  $\mathbb{P}_{\mathbf{m}_1}^{\text{cr}}[M_1]$  onto  $\mathbb{P}_{\mathbf{m}_2}^{\text{cr}}[M_2]$ .

{c73}

*Proof.* 1) As in the proof of 4.21, without loss of generality  $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\leq \lambda_2}$ . By 4.21 there is  $\mathbf{m}_* \in \mathbf{M}_{\lambda_2}^{\text{ec}}$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_*$  hence  $\mathbb{P}_{\mathbf{m}}^{\text{cr}} = \mathbb{P}_{\mathbf{m}_*}^{\text{cr}}[L_{\mathbf{m}}]$ .

{e39}

Let  $\mathbf{n}_* = \mathbf{m}_* \upharpoonright M$ , see 4.24, so  $\mathbf{n}_* \upharpoonright M = \mathbf{n}$  and by 4.25(3) we have  $\mathbf{n}_* \in \mathbf{M}_{\text{ec}}$ , hence  $\mathbb{P}_{\mathbf{n}_*}^{\text{cr}}[L_{\mathbf{n}}] = \mathbb{P}_{\mathbf{n}}^{\text{cr}}$ . But  $\mathbf{n}_*, \mathbf{m}_*$  are equivalent, hence  $\mathbb{P}_{\mathbf{n}_*} = \mathbb{P}_{\mathbf{m}_*}$  hence  $\mathbb{P}_{\mathbf{n}_*}^{\text{cr}}[L] = \mathbb{P}_{\mathbf{m}_*}^{\text{cr}}[L]$  for every  $L \subseteq L_{\mathbf{m}_*}$  hence  $\mathbb{P}_{\mathbf{n}}^{\text{cr}} = \mathbb{P}_{\mathbf{n}_*}^{\text{cr}}[L_{\mathbf{n}}] \triangleleft \mathbb{P}_{\mathbf{n}_*}^{\text{cr}}[L_{\mathbf{m}}] = \mathbb{P}_{\mathbf{m}_*}^{\text{cr}}[L_{\mathbf{m}}] = \mathbb{P}_{\mathbf{m}}^{\text{cr}}$ . So the conclusion holds.

2) Easy, too.  $\square_{4.26}$

§ 5. ON A RELATIVE TO §2 HAVING  $\lambda_0$  AND  $\lambda_1$ 

**Hypothesis 5.1.** As in 2.2, but now we have parameter  $\bar{\lambda} = (\lambda_1, \lambda_0)$ ,  $\lambda_1 \geq \lambda_0 \geq \lambda$  and use  $\mathbf{q}$  the usual  $< \lambda$ -support iteration. {c0}

**Discussion 5.2.** (2012.2.02) 1)  $\lambda_1$  has the role of  $\lambda_0$  in clause (e)( $\theta$ ) of Definition 2.7. {c4}

2)  $\lambda_0$  has the role of  $\lambda_0$  in Definition 5.4 and in 2.7. {c29}

3) At the moment, 5.11 seems unclear. {c46}

4) Even with this separation to  $\lambda_0, \lambda_1$  we can prove 2.26 but it requires more care ( $\mathcal{X}$  more information). {c41}

5) We may change Definition 5.5(1) to (\*) below, getting weaker results for  $\mathbf{M}$  but ones which fit more general contents {c30}

(\*) by induction on the ordinal  $\gamma$  we define the class  $\mathbf{M}_\gamma^{\text{ec}}$  as the class of  $\mathbf{m} \in \mathbf{M}$  such that:

- if  $L \in \text{Sub}(\mathbf{m})$  and  $\beta = \text{dp}_\mathbf{m}^*(L) < \gamma$  then  $m \leq_\mathbf{M} n \in M_\beta^{\text{ec}} \Rightarrow P_\mathbf{m}[L] = \mathbb{P}_\mathbf{n}[L]$ .

6) We may assume  $\chi = \chi^{\lambda_0}$  instead of  $\chi = \chi^{\lambda_1}$  but using  $E$  in the proof more information  $\mathcal{X}$  and  $\mathcal{E}$ .

**Claim 5.3.** *Continuing claim:*

10) If  $L_\mathbf{m}^+ \models "s < t"$  and  $p \in \mathcal{I}_{s,t}$  and  $\zeta = \text{lg}(\text{tr}(p(s)))$  then for every  $i < \theta_\varepsilon$  large enough there is  $q \in \mathbb{P}_t$  such that:

- $\mathbb{P}_t \models "p \leq q"$
- if  $r \in \text{Dom}(p) \setminus L_{\mathbf{m}, \leq s}$  then  $q(r) = p(r)$
- if  $r \in \text{Dom}(p) \cap L_{\mathbf{m}, < s}$  then  $\text{tr}(p(r)) \triangleleft \text{tr}(q(r))$
- $\text{tr}(q(s))$  has length  $\zeta + 1$
- $q(s)(\zeta) = i$ .

11) If  $L_\mathbf{m}^+ \models "s < t"$  and  $p \in \mathcal{I}_{s,t}$  and  $\zeta_0 < \zeta_1 < \lambda$ ,  $\zeta_0 = \text{lg}(\text{tr}(p(s)))$  and  $\bar{u} = \langle u_\varepsilon : \zeta_0 \leq \varepsilon < \zeta_1 \rangle$  where  $u_\varepsilon$  is an unbounded subset of  $\theta_\varepsilon$  for  $\varepsilon \in [\zeta_0, \zeta_1)$  then there is  $q \in \mathbb{P}_t$  above  $p$  such that  $p \Vdash_{\mathbb{P}_t} " \eta_s(\varepsilon) \in u_\varepsilon \text{ for } \varepsilon \in [\zeta_0, \zeta_1) "$ .

12) If  $\xi \leq \lambda$ ,  $r_\zeta \in u_t$  for  $\zeta < \xi$  and  $\mathbf{B}$  is a  $\lambda$ -Borel function from  ${}^\xi(\Pi\bar{\theta})$  to  $\mathbb{Q}_{\bar{\theta}}$  and  $p \in \mathbb{P}_\mathbf{m}$  then for some  $q$  we have:

- $\mathbb{P}_\mathbf{m} \models "p \leq q"$
- $q \upharpoonright L_{< t} \Vdash_{\mathbb{P}_t} " \mathbf{B}(\dots, \eta_{r_\zeta}, \dots)_{\zeta < \xi} \leq_{\mathbb{Q}_{\bar{\theta}}} q(t) \text{ or } \mathbf{B}(\dots, \eta_{r_\zeta}, \dots)_{\zeta < \varepsilon}, q(t) \text{ are incompatible in } \mathbb{Q}_{\bar{\theta}}, p \in \mathbb{P}_\mathbf{m} "$ .

13) If  $t \in L_\mathbf{m}$ ,  $j \leq \lambda$ ,  $\xi_i \leq \lambda$  for  $i < j$ ,  $r_{i,\zeta} \in u_t$  for  $i < j$ ,  $\zeta < \xi_i$  and  $p \upharpoonright L_{< t} \Vdash_{\mathbb{P}_t} " \langle \mathbf{B}_i(\dots, \eta_{r_{i,\zeta}}, \dots)_{\zeta < \xi_i} : i < j \rangle \text{ is a maximal antichain of } \mathbb{Q}_t^{\mathbf{V}[(\eta_r : r \in u_t)]} "$  then for some  $q$  and  $i < j$  we have:

- $\mathbb{P}_\mathbf{m} \models "p \leq q"$
- $q \upharpoonright L_{< t} \Vdash_{\mathbb{P}_t} "q(t) \text{ is above } \mathbf{B}_i(\dots, \eta_{r_{i,\zeta}}, \dots)_{\zeta < \varepsilon_i} \text{ in } \mathbb{Q}_{\bar{\theta}} "$ .

**Definition 5.4.** 1) For  $\mathbf{m} \in \mathbf{M}$  let  $\text{cmp}(\mathbf{m})$  be the set of  $L$  such that: {c29}

- (a)  $L \subseteq L_\mathbf{m}$

- (b) if  $L \subseteq M_{\mathbf{m}}$  then  $|L| \leq 1$
- (c) if  $t \in L \setminus M_{\mathbf{m}}$  then  $L = t/E'_{\mathbf{m}}$  hence:
  - ( $\alpha$ )  $L \setminus M_{\mathbf{m}} \neq \emptyset$  and  $L \setminus M$  is an  $E_{\mathbf{m}}$ -equivalence class
  - ( $\beta$ ) if  $t \in L \setminus M_{\mathbf{m}}$  then  $u_{\mathbf{m},t} \subseteq L$
  - ( $\gamma$ )  $|L| \leq \lambda_0$ .

2) For  $\mathbf{m} \in \mathbf{M}$

- (a) let  $\text{sub}(\mathbf{m})$  be the set of  $L_1$  such that for some  $L_2 \in \text{cmp}(\mathbf{m})$ ,  $L_1$  is an initial segment of  $L_2$ , i.e.  $L_2 \subseteq L_1 \wedge (\forall s, t)[s \in L_2 \wedge t \in L_1 \wedge s <_{\mathbf{m}} t \rightarrow s \in L_1]$
- (b) let  $\text{sub}_*(\mathbf{m}) = \{L \in \text{sub}(\mathbf{m}) : \text{there are}^{17} L' \in \text{cmp}(\mathbf{m}) \text{ and } Y_1, Y_2 \subseteq L' \text{ of cardinality } \leq \lambda_0 \text{ such that } L = \{s \in L' : (\exists t)(s \leq_{\mathbf{m}} t \in \mathcal{Y}_1 \vee s <_{\mathbf{m}} \in \mathcal{Y}_2)\}\}$ .

3) Let

- (a)  $\text{Cmp}(\mathbf{m}) = \{\bigcup_{\varepsilon < \zeta} L_\varepsilon : L_\varepsilon \in \text{cmp}(\mathbf{m}) \text{ for } \varepsilon < \zeta\}$
- (b)  $\text{Cmp}_*(\mathbf{m}) = \{\bigcup_{\varepsilon < \zeta} L_\varepsilon : L_\varepsilon \in \text{cmp}(\mathbf{m}) \text{ for } \varepsilon < \zeta \text{ and } \zeta \leq \lambda_0\}$
- (c)  $\text{Sub}(\mathbf{m}) = \{L_1 : \text{for some } L_2 \in \text{Cmp}(\mathbf{m}), L_1 \text{ is an initial segment of } L_2\}$
- (d)  $\text{Sub}_*(\mathbf{m}) = \{\bigcup_{\varepsilon < \zeta} L_\varepsilon : \zeta \leq \lambda_0 \text{ and } L_\varepsilon \in \text{sub}_*(\mathbf{m}) \text{ for } \varepsilon < \zeta\}$ .

{c30}

**Definition 5.5.** 0) For  $L \subseteq L_{\mathbf{m}}$ ,  $\mathbf{m} \in \mathbf{M}$  let  $\text{dp}_{\mathbf{m}}^*(L) = \cup\{\text{dp}_{M_{\mathbf{m}}}(t)+1 : t \in L \cap M_{\mathbf{m}}\}$ .

1) For an ordinal  $\gamma$  let  $\mathbf{M}_{\gamma}^{\text{ec}}$  be the class of  $\mathbf{m} \in \mathbf{M}$  such that:

- (\*) if  $L \in \text{Sub}(\mathbf{m})$  and  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$  then  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$ .

2) Let  $\mathbf{M}^{\text{ec}} = M_{\infty}^{\text{ec}}$  be the class of  $\mathbf{m}$  which  $\in \mathbf{M}_{\gamma}^{\text{ec}}$  for every  $\gamma$ .

3) Let  $\mathbf{M}_{\chi, \gamma}^{\text{ec}} = \{\mathbf{m} \in \mathbf{M}_{\gamma}^{\text{ec}} : |L_{\mathbf{m}}| \leq \chi\}$ , similarly  $\mathbf{M}_{\chi, \infty}^{\text{ec}}$ .

4) Let  $\mathbf{M}_{\text{fc}}^{\text{ec}}$  be the class of  $\mathbf{m} \in \mathbf{M}$  satisfying  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} < \mathbb{P}_{\mathbf{n}}$ .

{c32}

**Observation 5.6.** 1)  $\leq_{\mathbf{M}}$  is a partial order on  $\mathbf{M}$ .

2)  $\mathbf{M}_0^{\text{ec}} = \mathbf{M}$  and  $\mathbf{M}_{\chi, 0}^{\text{ec}} = \mathbf{M}_{\chi}$ .

3) If  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $L \in \text{sub}(\mathbf{m})$  then  $\mathbb{P}_{\mathbf{m}}(L), \mathbb{P}_{\mathbf{n}}(L)$  have the same set of elements.

4) If  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  then  $\text{cmp}(\mathbf{m}) \subseteq \text{cmp}(\mathbf{n})$  moreover  $L \in \text{cmp}(\mathbf{m})$  iff  $L \in \text{cmp}(\mathbf{n}) \wedge L \subseteq L_{\mathbf{m}}$ .

5) Like (4) for  $\text{sub}(-), \text{Cmp}(-), \text{Sub}(-), \text{sub}_*(-), \text{Cmp}_*(-), \text{Sub}_*(-)$ .

6) If  $\chi = \chi^{\lambda_0}$ ,  $\mathbf{m} \in \mathbf{M}_{\chi}$ ,  $L \in \text{Sub}(\mathbf{m})$ ,  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $\mathbb{P}_{\mathbf{m}}[L] \neq \mathbb{P}_{\mathbf{n}}[L]$  then we can find such  $\mathbf{n}$  in  $\mathbf{M}_{\chi}$ .

7) If  $\mathbf{m} \in \mathbf{M}$  and  $t \in L_{\mathbf{m}}$  then  $t \in L$  for some  $L \in \text{cmp}(\mathbf{m})$ .

8) If  $\mathbf{m} \in \mathbf{M}$  and  $L_1 \subseteq L_{\mathbf{m}}$  has cardinality  $\leq \lambda_0$  then  $L_1 \subseteq L_2$  for some  $L_2 \in \text{Cmp}_*(\mathbf{m})$ .

9) If  $L_1 \in \text{sub}(\mathbf{m})$  then  $L_1 = \cup\{L_1 \cap L_{\mathbf{m}, \leq t} : t \in L_1\}$  and  $t \in L_1 \Rightarrow L_1 \cap L_{\mathbf{m}, \leq t} \in \text{sub}_*(\mathbf{m})$ .

*Proof.* Should be clear. □<sub>5.6</sub>

<sup>17</sup>version A:  $s_\varepsilon$  of  $L_{\mathbf{m}}$  for  $\varepsilon < \varepsilon(*) < \lambda$  and  $\lambda$ -Borel function  $\mathbf{B}$  from  $\varepsilon(*)_4$  to 2 and  $L = \{t \in L' : \mathbf{B}(\dots, \mathfrak{A}_I(s, t), \dots) = 1 \text{ where } \mathfrak{A}_I(s, t) \text{ is: } 0 \text{ if } s <_I t, 1 \text{ if } s = t, 2 \text{ if } t <_I s \text{ and } 3 \text{ otherwise.}\}$



{c35}

**Observation 5.7.** 1) If  $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing then  $\mathbf{m}_\delta := \cup\{\mathbf{m}_\alpha : \alpha < \delta\}$  is a  $\leq_{\mathbf{M}}$ -upper bound of  $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$ .

2) Moreover in part (1):

- ( $\alpha$ ) •  $\text{cmp}(\mathbf{m}_\delta) = \cup\{\text{cmp}(\mathbf{m}_\alpha) : \alpha < \delta\}$
- $\text{sub}(\mathbf{m}_\delta) = \cup\{\text{sub}(\mathbf{m}_\alpha) : \alpha < \delta\}$
- $\text{sub}_*(\mathbf{m}_\delta) = \cup\{\text{sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$
- ( $\beta$ ) if  $\text{cf}(\lambda) \geq \lambda_0^+$  then
  - $\text{Cmp}_*(\mathbf{m}_\delta) = \cup\{\text{Cmp}(\mathbf{m}_\alpha) : \alpha < \delta\}$
  - $\text{Sub}_*(\mathbf{m}_\delta) = \cup\{\text{Sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$ .

3) If  $\gamma_1 < \gamma_2$  then  $\mathbf{M}_{\gamma_1}^{\text{ec}} \subseteq \mathbf{M}_{\gamma_2}^{\text{ec}}$ .

*Proof.* 1) Easy, remembering that  $t \in \mathbf{m}_\alpha \wedge \alpha \leq \beta < \delta \Rightarrow t/E'_{\mathbf{m}_\beta} = t/E'_{\mathbf{m}_\alpha}$ .

2) Easy.

3) Easy, too. □<sub>5.7</sub>

**Claim 5.8.** 1) In Definition 5.5(1) we can consider only  $L \in \text{Sub}_*(\mathbf{m})$  such that  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$ . {c38}

2) So if  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}$  and  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  then  $\mathbb{P}_{\mathbf{m}} < \mathbb{P}_{\mathbf{n}}$ , that is  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] < \mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$  and  $L \in \text{Sub}(\mathbf{m}) \Rightarrow \mathbb{P}_{\mathbf{m}}[L] < \mathbb{P}_{\mathbf{n}}[L]$ . {c30}

3) For  $\mathbf{m} \in \mathbf{M}$  we have:  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}$  iff  $\mathbf{m} \in \mathbf{M}_{\text{fc}}$ , i.e. for every  $\mathbf{n}, \mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} < \mathbb{P}_{\mathbf{n}}$ .

4)  $\text{Cmp}_*(\mathbf{m})$  has cardinality  $\leq |L_{\mathbf{m}}|^{\leq \lambda_0}$ .

5)  $\text{Sub}_*(\mathbf{m})$  has cardinality  $\leq |L_{\mathbf{m}}|^{\lambda_0}$ .

6)  $|\text{cmp}(\mathbf{m})| \leq |L_{\mathbf{m}}| + 1$  and  $|\text{sub}_*(\mathbf{m})| \leq |L_{\mathbf{m}}|^{\lambda_0}$ .

*Proof.* 1) Let  $\mathbf{m} \in \mathbf{M}$  satisfies the weaker version, and let  $L \in \text{Sub}(\mathbf{m})$  satisfies  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$ . By Definition 5.4(3)(c) and Observation 5.6(9) we can find  $\langle L_\varepsilon, L_\varepsilon^* : \varepsilon < \zeta \rangle$  such that  $L \in \cup\{L_\varepsilon : \varepsilon < \zeta\}$  where  $L_\varepsilon \subseteq L_\varepsilon^* \in \text{Cmp}(\mathbf{m})$  and  $L_\varepsilon \in \text{sub}_*(\mathbf{m})$ . {c29}

For every  $u \in [\zeta]^{\leq \lambda_0}$  let  $L_u = \cup\{L_\varepsilon : \varepsilon \in u\}$  so clearly  $\text{dp}_{\mathbf{m}}^*(L_u) \leq \text{dp}_{\mathbf{m}}^*(L) < \gamma$ . Let  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ . For  $u \in [\zeta]^{\leq \lambda_0}$ , for  $L_u$  the demand holds, i.e.  $\mathbb{P}_{\mathbf{m}}[L_u] = \mathbb{P}_{\mathbf{n}}[L_u]$ . Also if  $u \subseteq v \in [\zeta]^{\leq \lambda_0}$  then  $\mathbb{P}_{\mathbf{m}}[L_u] \subseteq \mathbb{P}_{\mathbf{m}}[L_v]$ .

Lastly,  $\langle u : u \in [\zeta]^{\leq \lambda_0} \rangle$  is  $\lambda_0^+$ -directed so by ??(8) and the Definition 5.5(1) we are done. {c20}

2) As  $L_{\mathbf{m}} \in \text{Sub}(\mathbf{m})$  and  $\text{dp}_{\mathbf{m}}^*(L_{\mathbf{m}}) < \infty$  by clause (d) Definition 2.7. {c4}

3) The implication  $\Rightarrow$  holds by part (2). For the other implication assume  $\mathbf{m} \notin \mathbf{M}_\infty^{\text{ec}}$  so clause (\*) there fails for some  $\gamma$ ; this means that there are  $L \in \text{Sub}(\mathbf{m})$  and  $\mathbf{n}$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  but  $\mathbb{P}_{\mathbf{m}}[L] \neq \mathbb{P}_{\mathbf{n}}[L]$ , by the definitions this implies  $\neg(\mathbb{P}_{\mathbf{m}} < \mathbb{P}_{\mathbf{n}})$  so we are done.

4) Note that  $\{L \in \text{cmp}(\mathbf{m}) : L \not\subseteq M_{\mathbf{m}}\} = \{t/E'_{\mathbf{m}} : t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}\}$  has cardinality  $\leq |L_{\mathbf{m}} \setminus M_{\mathbf{m}}| \leq |L_{\mathbf{m}}|$  and  $\{L \in \text{cmp}(\mathbf{m}) : L \subseteq M_{\mathbf{m}}\} = L \subseteq M_{\mathbf{m}} : |L| \leq \lambda_0$  has cardinality  $\leq |M_{\mathbf{m}}|^{\lambda_0} \subseteq |L_{\mathbf{m}}|^{\lambda_0}$ .

5), 6) Similarly. □<sub>5.8</sub>

**Claim 5.9.** 1) If  $\gamma \leq \infty$ ,  $\langle \mathbf{m}_\alpha^* : \alpha < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing,  $\text{cf}(\delta) > \lambda_0$  and  $\alpha < \delta \Rightarrow \mathbf{m}_\alpha^* \in \mathbf{M}_\gamma^{\text{ec}}$  then  $\mathbf{m}_* := \cup\{\mathbf{m}_\alpha^* : \alpha < \delta\} \in \mathbf{M}_\gamma^{\text{ec}}$ . {c39}

2) Similarly for  $\lambda_0^+$ -directed union.

*Proof.* 1) By 5.8(1), toward contradiction assume that  $(\mathbf{n}, L)$  satisfies: {c38}

- (\*) (a)  $\mathbf{m}_* \leq_{\mathbf{M}} \mathbf{n}$
- (b)  $L \in \text{Sub}_*(\mathbf{m}_*)$
- (c)  $\text{dp}_{\mathbf{m}_*}^*(L) < \gamma$
- (d)  $\mathbb{P}_{\mathbf{m}_*}[L] \neq \mathbb{P}_{\mathbf{n}}[L]$ .

{c35} As  $L \in \text{Sub}_*(\mathbf{m}_*)$ , we have  $L \subseteq \cup\{L_\varepsilon : \varepsilon < \varepsilon(*)\}$  where  $\varepsilon(*) < \lambda_0^*$  and  $L_\varepsilon \in \text{sub}_*(\mathbf{m})$  for  $\varepsilon < \varepsilon(*)$ . But then  $L_\varepsilon \in \text{sub}_*(\mathbf{m}_{\alpha(\varepsilon)})$  for some  $\alpha(\varepsilon) < \delta$  by 5.7(2)( $\alpha$ ) but  $\text{cf}(\delta) > \lambda_0$  hence  $\alpha(*) = \sup\{\alpha(\varepsilon) : \varepsilon < \varepsilon(*)\} < \delta$  so  $L \in \text{Sub}_*(\mathbf{m}_{\alpha(*)})$ .

{c35} Now we have assumed  $\text{dp}_{\mathbf{m}_*}^*(L) < \gamma$  but  $M_{\mathbf{m}_*} = M_{\mathbf{m}_{\alpha(*)}}$ , so equivalently  $\text{dp}_{\mathbf{m}_{\alpha(*)}}^*(L) < \gamma$ . By the claim assumption,  $\mathbf{m}_\beta \in \mathbf{M}_\gamma^{\text{ec}}$  and by 5.7(1) we have  $\mathbf{m}_\beta \leq_{\mathbf{M}} \mathbf{m}_*$ .

As  $\leq_{\mathbf{M}}$  is a partial order on  $\mathbf{M}$  and  $\mathbf{m}_* \leq_{\mathbf{M}} \mathbf{n}$  by (\*) (a) clearly  $\mathbf{m}_{\alpha(*)} \leq_{\mathbf{M}} \mathbf{n}$ , but  $\mathbf{m}_{\alpha(*)} \in M_\delta^{\text{ec}}$  so together we deduce  $\mathbb{P}_{\mathbf{m}_*}[L] = \mathbb{P}_{\mathbf{m}_\beta}[L] = \mathbb{P}_{\mathbf{n}}[L]$ , so we are done.  $\square_{5.9}$

{c41y}

**Crucial Claim 5.10.** Let  $\chi = \chi^{\lambda_0}$ .

- 1) If  $\mathbf{m} \in \mathbf{M}$  of cardinality  $\leq \chi$  there is  $\mathbf{n} \in \mathbf{M}_\infty^{\text{ec}}$  of cardinality  $\leq \chi$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ .
- 2) If  $\mathbf{m} \in \mathbf{M}_\chi$  and  $\gamma < \chi^+$  then there is  $\mathbf{n} \in \mathbf{M}_\gamma^{\text{ec}} \cap \mathbf{M}_\chi$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ .

*Proof.* 1) By (2) using  $\gamma = \text{dp}_{\mathbf{m}}^*(M_{\mathbf{m}}) + 1 = \text{dp}_{\mathbf{m}}^*(L_{\mathbf{m}}) + 1$  which necessarily is  $< \|M_{\mathbf{m}}\|^+ \leq \chi^+$ .

{c30} 2) We prove the statement by induction on  $\gamma$ . Recalling Definition 5.5(1), in each  
{c38} case consider  $L \in \text{Sub}(\mathbf{m})$  which satisfies  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$  and by 5.8(1) without loss of generality  $L \in \text{Sub}_*(\mathbf{m})$ .

Case 1:  $\gamma = 0$ .

There is no such relevant  $L$ .

Case 2:  $\gamma = 1$ .

{c13} So  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$  means  $\text{dp}_{\mathbf{m}}^*(L) = 0$  which means  $L \cap M_{\mathbf{m}} = \emptyset$ . But then  $\mathbf{n} = \mathbf{m}$  is as required because if  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}_1$  then  $\{s \in L_{\mathbf{n}_1} : (\exists t \in L)(s \leq_{\mathbf{n}_1} t)\} = L = \{s \in L_{\mathbf{m}} : (\exists t \in L)(s \leq_{\mathbf{m}} t)\}$ , i.e.  $L$  is an initial segment of both so  $\mathbb{P}_{\mathbf{m},L} = \mathbb{P}_{\mathbf{n}_1,L}$  and the conclusion is obvious, or see 2.15.

Case 3:  $\gamma = \beta + 2$ .

We try to choose  $\mathbf{m}_\varepsilon$  by induction on  $\varepsilon \leq \chi^+$  such that:

- (\*) (a)  $\mathbf{m}_\varepsilon \in \mathbf{M}_\chi$
- (b)  $\langle \mathbf{m}_\zeta : \zeta \leq \varepsilon \rangle$  is  $\leq_{\mathbf{M}}$ -increasing continuous
- (c)  $\mathbf{m}_0 = \mathbf{m}$
- (d) if  $\varepsilon = 2\zeta + 1$  then for some  $L \in \text{Sub}_*(\mathbf{m}_{2\zeta})$  we have  $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \neq \mathbb{P}_{\mathbf{m}_\varepsilon}[L]$  and  $\text{dp}_{\mathbf{m}_{2\zeta}}^*(L) < \gamma$
- (e) if  $\varepsilon = 2\zeta + 2$  then  $\mathbf{m}_\varepsilon \in \mathbf{M}_{\beta+1}^{\text{ec}}$ .

Subcase 3A: We succeed to carry the induction.

For each  $\delta \in S_{\lambda_0^+}^{\chi^+} := \{\delta < \chi^+ : \text{cf}(\delta) = \lambda_0^+\}$  (i.e.  $\varepsilon = 2\delta + 1$  and note  $2\delta = \delta$ ) there is  $L_\delta$  as in  $(*)$ (d). Recalling Definition 5.4(3)(d), as  $\text{cf}(\delta) = \lambda_0^+ > \lambda_0 \geq |\{t/E'_{\mathbf{m}_\delta} : t \in L_\delta \setminus M_{\mathbf{m}_\delta}\}|$  there is  $\alpha(\delta) < \delta$  such that  $L_\delta \subseteq L_{\mathbf{m}_{\alpha(\delta)}}$  hence by 5.7(2) + 5.8(5) for some stationary  $S \subseteq S_{\lambda_0^+}^{\chi^+}$  and  $L_*$  we have  $\delta \in S \Rightarrow L_\delta = L_* \wedge \alpha(\delta) = \alpha(*)$ . Without loss of generality  $\alpha(*)$  is even and a successor ordinal. For  $\delta \in S$ , note that  $\langle \mathbf{m}_{2\zeta+2} : \zeta < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing sequence of members of  $\mathbf{M}_{\beta+1}^{\text{ec}}$  with union  $\mathbf{m}_\delta$  (by clauses (e) and (b) of  $(*)$ ) hence  $\mathbf{m}_\delta \in \mathbf{M}_{\beta+1}^{\text{ec}}$  by 5.9; recall  $\text{cf}(\delta) > \lambda_0$  and  $M_{\mathbf{m}_\alpha} = M_{\mathbf{m}}$  for every  $\alpha$ .

Now  $\text{dp}_{\mathbf{m}_\delta}^*(L_*) < \gamma$  and by the previous sentences necessarily  $\text{dp}_{\mathbf{m}_\delta}^*(L_*) = \beta + 1$  hence the set  $\mathcal{Y} := \{t \in L_* \cap M_{\mathbf{m}} : \text{dp}(t, M_{\mathbf{m}}) = \beta\}$  is not empty.

Let

- ⊞<sub>1</sub> (a)  $L_0^* = \{s \in L_* : \neg(\exists t \in \mathcal{Y})(t \leq s)\}$
- (b)  $L_1^* = L_0^* \cup \mathcal{Y}$
- (c)  $L_2^* = L_*$ .

Now

- ⊞<sub>2</sub>  $\mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L_0^*]$  is equal to  $\mathbb{P}_{\mathbf{m}_\alpha}[L_0^*]$  and to  $\mathbb{P}_{\mathbf{m}_{\alpha+1}}[L_0^*]$  for  $\alpha \in S \setminus \alpha(*)$  moreover for  $\alpha \geq \alpha(*)$ .

[Why?  $L_0^*$  is an initial segment of  $L_*$  but  $\delta \in S \Rightarrow L_* = L_\delta \in \text{Sub}_*(\mathbf{m}_\delta)$ , see above hence, see Definition 5.4(2)(b),(3)(c) we have  $L_0^* \in \text{Sub}_*(\mathbf{m}_\alpha)$  for  $\alpha \geq \alpha(*)$ . We are done as  $\mathbf{m}_{\alpha(*)} \in \mathbf{M}_{\beta+1}^{\text{ec}}$  by  $(*)$ (e) and 5.5(1).] {c29}  
{c30}

- ⊞<sub>3</sub> (a)  $L_{0,\alpha}^* = \{s \in L_{\mathbf{m}_\alpha} : (\exists t \in L_0^*)(s \leq t)\}$  or  $(\exists t \in \mathcal{Y})(s \leq t)$
- (b)  $L_{1,\alpha}^* = L_{0,\alpha}^* \cup Y$
- (c)  $L_{2,\alpha}^* = L_\alpha^0 \cup L_*$
- ⊞<sub>4</sub> (a)  $\text{dp}_{\mathbf{m}_\alpha}^*(L_{0,\alpha}^*) < \beta + 1$
- (b)  $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L_{0,\alpha}^*] : \alpha \in [\alpha(*), \chi^+] \rangle$  is  $\triangleleft$ -increasing.
- ⊞<sub>5</sub> (a)  $L_{1,\alpha}^* \subseteq L_{\mathbf{m}_\alpha}$  is downward closed in  $L_{\mathbf{m}_\alpha}$
- (b)  $\langle \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}^*) : \alpha \in [\alpha(*), \chi^+] \rangle$  is  $\subseteq$ -increasing
- (c) moreover  $\subseteq_{\text{ic}}$ -increasing.

[Why? Should be clear.]

- ⊞<sub>6</sub> there is a club  $E$  of  $\chi^+$  such that  $\delta \in E \cap S_{\lambda_0^+}^{\chi^+} \setminus \alpha(*) \wedge \delta \leq \alpha < \chi^+ \Rightarrow P_{\mathbf{m}_\gamma}(L_{1,\delta}^*) \triangleleft \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}^*)$ .

[Why? By ⊞<sub>5</sub> as every  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}^*)$  satisfies the  $\lambda$ -c.c.,  $\lambda_0 \geq \lambda$  and  $\chi^{\lambda_0} = \chi$ .]

Now

- ⊞<sub>7</sub> (a)  $L_{2,\alpha}^*$  is a downward closed subset of  $L_{\mathbf{m}_\alpha}$  for  $\alpha \in [\alpha(*), \chi^+]$
- (b) if  $\alpha(*) \leq \alpha \leq \beta < \chi^+$  and  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}^*) \triangleleft \mathbb{P}_{\mathbf{m}_\beta}(L_{1,\beta}^*)$  then
  - ( $\alpha$ )  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}^*) = \mathbb{P}_{\mathbf{m}_\beta}(L_{2,\alpha}^*) \triangleleft \mathbb{P}_{\mathbf{m}_\beta}$
  - ( $\beta$ )  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}^*) \triangleleft \mathbb{P}_{\mathbf{m}}(L_{2,\beta}^*)$
  - ( $\gamma$ )  $\mathbb{P}_{\mathbf{m}_\alpha}[L_{2,\alpha}^*] = \mathbb{P}_{\mathbf{m}_\beta}[L_{2,\beta}^*]$
  - ( $\delta$ )  $\mathbb{P}_{\mathbf{m}_\alpha}[L_1^*] = \mathbb{P}_{\mathbf{m}_\beta}[L_1^*]$ .

[Why? Clause (a) by inspection. Clause (b)( $\alpha$ ), the equality is obvious, the  $\leq \mathbb{P}_{\mathbf{m}_\beta}$  holds by clause (a). Now we prove clause (b)( $\beta$ ), toward this

- <sub>1</sub>  $t \in L_{2,\beta} \setminus L_{1,\alpha} \Leftrightarrow t \in Y \Leftrightarrow t \in L_{2,\alpha} \setminus L_{1,\alpha}$  (why? check)
- <sub>2</sub>  $t \in L_{2,\alpha} \setminus L_{1,\alpha} \Rightarrow u_{\mathbf{m}_\beta, t} \subseteq L_{1,\alpha}$  (why? check)
- <sub>3</sub>  $\mathbb{P}_{\mathbf{m}_\alpha}(L_{2,\alpha}) \subseteq \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\beta})$  (by assumption of (b)).

Next assume  $p_* \in \mathbb{P}_{2,\beta}(L_{2,\beta})$  then  $q_* = p_* \upharpoonright L_{2,\alpha} \in \mathbb{P}_{2,\alpha}(L_{1,\alpha})$  and also  $q' = p_\beta \upharpoonright L_{1,\alpha} \in \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha})$ ,  $p' = p \upharpoonright L_{1,\beta} \in \mathbb{P}_{\mathbf{m}_\beta}$ . Also  $q' \leq q'' \in \mathbb{P}_{\mathbf{m}_\alpha}(L_{1,\alpha}) \Rightarrow q'' \cup (p \upharpoonright (L_{1,\beta} \setminus L_{1,\alpha}))$  is a common upper bound of  $q'', p'$  in  $\mathbb{P}_{\mathbf{m}_\beta}(L_{1,\beta})$ . Hence  $q_* = q' \cup (p \upharpoonright Y)$  satisfies  $q_* \in P_{\mathbf{m}_\alpha}(L_{2,\alpha})$  and  $q_* \leq q'_* \in P_{\mathbf{m}_\alpha}(L_{2,\alpha}) \Rightarrow q'_* \cup (p \upharpoonright (L_{1,\beta} \setminus L_{1,\alpha}))$  is a common upper bound of  $q'_*, p$  in  $\mathbb{P}_{\mathbf{m}_\beta}(L_{2,\beta})$ .

Together with •<sub>3</sub> this paragraph shows

- <sub>4</sub>  $\mathbb{P}_{\mathbf{m}_\alpha}[L_{2,\alpha}] \leq \mathbb{P}_{\mathbf{m}_\beta}[L_{2,\beta}]$  that is clause (b)( $\beta$ ).

Clause (b)( $\gamma$ ) holds by (b)( $\beta$ ) and (a) and clause (b)( $\beta$ ) follows as  $L_2^* \subseteq L_{2,\alpha}^* \subseteq L_{2,\alpha}^*$  so  $\boxplus_7$  holds indeed.

Now choose  $\delta \in E \cap S \setminus \alpha(*) \subseteq S_{\lambda_0^+}^{\chi^+}$  and we get a contradiction to the choice of  $L_\delta = L_*$  so we are done with subcase 3A.

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(\*)<sub>2</sub> for  $\alpha \in \chi^+ \setminus \alpha(*)$  and  $t \in \mathcal{Y}$  let  $\mathbb{Q}_{\bar{\theta}, t}^\alpha$  be the  $\mathbb{P}^{\mathbf{m}_\alpha}$ -name of  $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \eta_\gamma : \gamma \in u(t) \cap L_* \rangle]}$  where  $u(t) = u_{\mathbf{m}_{\alpha(*)}, t}$

(\*)<sub>3</sub> (a)  $\mathbb{Q}_{\bar{\theta}, t}^\alpha$  does not depend on  $\alpha$ , also the order on it does not depend on  $\alpha$

(b)  $\Vdash_{\mathbb{P}_{\mathbf{m}_\alpha}} \text{“} \mathbb{Q}_{\bar{\theta}, t}^\alpha \subseteq_{\text{ic}} \mathbb{Q}_{\bar{\theta}}, \text{ i.e. } \mathbb{Q}_{\bar{\theta}, t}^\alpha \subseteq \mathbb{Q}_{\bar{\theta}} \text{ as quasi orders and any } p_1, p_2 \in \mathbb{Q}_{\bar{\theta}, t}^\alpha \text{ compatible in } \mathbb{Q}_{\bar{\theta}} \text{ are compatible in } \mathbb{Q}_{\bar{\theta}, t}^\alpha \text{”}$

(c)  $\mathbb{P}_{\mathbf{m}_\alpha}[L_1^*] / \mathbb{P}_{\mathbf{m}_\alpha}[L_0^*]$  is the same for  $\alpha \in [\alpha(*), \chi^+]$

(\*)<sub>4</sub> define  $\mathbb{I}_\alpha = \{ \mathcal{J} : \mathcal{J} \text{ is a } \mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L_0^*]\text{-name of a maximal antichain of } \mathbb{P}_{\mathbf{m}_\alpha}[L_1^*] / \mathbb{P}_{\mathbf{m}_\alpha}[L_0^*] \}$ .

Let  $L_{0,\alpha}^* = \{ s \in L_{\mathbf{m}_\alpha} : (\exists t \in L_0^*)(s \leq t) \text{ or } (\exists t \in \mathcal{Y})(s < t) \}$  and  $S_* = \{ \alpha < \chi^+ : (\exists \varepsilon)(\alpha = 2\varepsilon + 2) \text{ or } \text{cf}(\alpha) = \lambda_0^+ \}$

(\*)<sub>5</sub> (a)  $\text{dp}_{\mathbf{m}_\alpha}^*(L_{0,\alpha}^*) < \beta + 1$

(b)  $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L_{0,\alpha}^*] : \alpha \in S_* \setminus \alpha(*) \subseteq [\alpha(*), \chi^+] \text{ and } \alpha = (\exists \varepsilon)(\alpha = 2\varepsilon + 2) \text{ or } \alpha \in S \rangle$  is  $\leq$  increasing.

[Why? For clause (a):

{c2} if  $s \in L_{0,\alpha} \cap M_{\mathbf{m}_\alpha}$  then for some  $t \in Y$  we have  $s <_{L_\alpha} t$  hence  $\text{dp}(s, M_{\mathbf{m}}) < \text{dp}(t, M_{\mathbf{m}}) = \beta$  hence  $\text{dp}(s, M_{\mathbf{m}}) \leq \beta$ , recalling Definition 2.5(1)( $\beta$ ) hence we have  $\text{dp}_{\mathbf{m}_\alpha}^*(L_{0,\alpha}) = \sup\{ \text{dp}(s, M_{\mathbf{m}}) + 1, s \in L \cap M_{\mathbf{m}} \} \leq \beta$  as promised.

{c39} For clause (b), by the induction hypothesis for  $\alpha_1 < \alpha_2$  from  $\{ 2\varepsilon + 2 : \varepsilon < \chi^+ \}$  we have  $\mathbb{P}_{\mathbf{m}_{\alpha_1}}[L_{0,\alpha_1}^*] = \mathbb{P}_{\mathbf{m}_{\alpha_2}}[L_{0,\alpha_1}^*] \leq \mathbb{P}_{\mathbf{m}_{\alpha_2}}[L_{0,\alpha_2}^*]$ . This implies the result also when we allow  $\text{cf}(\alpha_\ell) = \lambda$  by 5.9.]

So necessarily

(\*)<sub>6</sub>  $\mathbb{I}_\alpha$  is constant for  $\alpha < \chi^+$  large enough.

Hence recalling  $(*)_3$  for some  $\beta(*) \in (\alpha(*), \chi^+)$  we have

$$(*)_7 \langle \mathbb{P}_{\mathbf{m}_\beta}[L_1^*] : \alpha < S_* \rangle \text{ is constant for } \beta \in [\beta(*), \chi^+).$$

But this implies the result for  $L_2^*$  as the  $t \in L_2^* \setminus L_1^*$  does not “have memory outside  $L_1^*$ ”, i.e.  $u_{\mathbf{m}_{\beta(*)}, t} \subseteq L_1^*$  and is constant, so we are done.

Subcase 3B: We are stuck in  $\varepsilon$ , i.e. cannot define  $\mathbf{m}_\varepsilon$ .

Now  $\varepsilon = 0$  is impossible. If  $\varepsilon$  is a limit, let  $\mathbf{m}_\varepsilon = \cup\{\mathbf{m}_\zeta : \zeta < \varepsilon\}$  so  $\zeta < \varepsilon \Rightarrow \mathbf{m}_\zeta \leq_{\mathbf{N}} \mathbf{m}_\varepsilon \in \mathbf{M}_\chi$ , contradiction. Also  $\varepsilon = \zeta + 1, \zeta = 0$  is impossible.

If  $\varepsilon = 2\zeta + 1$  so then clause (d) of  $(*)$  is relevant so there is no  $\mathbf{n} \in \mathbf{M}_\chi$  such that  $\mathbf{m}_{2\zeta} \leq_{\mathbf{N}} \mathbf{n}$  and  $L \in \text{Sub}_*(\mathbf{m}_{2\zeta})$  such that  $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \neq \mathbb{P}_{\mathbf{m}_{2\zeta}}[L]$  and  $\text{dp}_{\mathbf{m}_{2\zeta}}^*(L) < \gamma$ . By 5.8(1) this applies also to  $L \in \text{Sub}(\mathbf{m}_{2\zeta})$ . By claim 5.6(7) this applies also to  $\mathbf{n} \in \mathbf{M}$ . Together so by 5.8(4) we can deduce that  $\mathbf{m}_{2\zeta} \in \mathbf{M}_\gamma^{\text{ec}}$  so we are done. {c32}

Lastly, if  $\varepsilon = 2\zeta + 2$  so clause (e) of  $(*)$  applies, then use the induction hypothesis for  $\beta + 1$  to choose  $\mathbf{m}_\varepsilon$ . {c38}

Case 4:  $\gamma$  is limit.

Let  $\delta < \chi^+$  be divisible by  $\gamma$  and of cofinality  $\lambda_0^+$ , recalling  $\lambda_0^+ \leq \chi$ .

We choose  $\mathbf{m}_\alpha$  for  $\alpha \leq \delta_0^+$  such that:

- $\mathbf{m}_\alpha \in \mathbf{M}_\chi$
- $\alpha = 0 \Rightarrow \mathbf{m}_\alpha = \mathbf{m}$
- $\mathbf{m}_\alpha$  is  $\leq_{\mathbf{M}}$ -increasing continuous with  $\alpha$
- if  $\varepsilon < \gamma$  and  $\alpha = \varepsilon + 1 \pmod{\gamma}$  then  $\mathbf{m}_\alpha \in \mathbf{M}_\varepsilon^{\text{ec}}$ .

Clearly possible: if  $\alpha = 0$  trivial, if  $\alpha$  is a successor ordinal use the induction hypothesis and if  $\alpha$  is a limit recall  $\delta < \chi^+$  and use 5.9. Now  $\mathbf{m}_\delta$  is as required. {c39}

Why? Toward contradiction assume  $\mathbf{m}_\delta \leq_{\mathbf{M}} \mathbf{n}$  and  $L \in \text{Sub}_*(\mathbf{m}_{\lambda_0^+})$  satisfies  $\gamma > \text{dp}_{\mathbf{m}_\delta}(L)$ ; let  $\varepsilon := \text{dp}_{\mathbf{m}_\alpha}(L)$ . By 5.7(2)( $\beta$ ) there is  $\alpha < \delta$  such that  $L \in \text{Sub}_*(\mathbf{m}_\alpha)$  and without loss of generality  $\alpha = \varepsilon + 1 \pmod{\gamma}$ , so we use the choice of  $\mathbf{m}_{\alpha+1}$  to get contradiction. {c35}

So  $\mathbf{m}_\delta$  is as required by 5.9. {c39}

Case 5: None of the above.

So  $\gamma = \beta + 1, \beta$  a limit ordinal.

For  $\alpha \leq \beta$  let  $L_\alpha = \{s \in L : \text{there is no } t \leq s \text{ satisfying } t \in M_{\mathbf{m}} \text{ and } \text{dp}(t, M_{\mathbf{m}}) \geq \alpha\}$ . So  $\langle L_\alpha : \alpha \leq \beta \rangle$  is  $\subseteq$ -increasing continuous and  $L \setminus L_\beta^* \subseteq L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ . As in Case 3 we first deal with  $L_\beta$ , and it is straight, and finish as there. □<sub>2.26</sub> {c46}

**Claim 5.11.** Assume  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}, M \subseteq M_{\mathbf{m}}$  and  $\mathbf{n}$  is defined just like  $\mathbf{m}$  except that  $M_{\mathbf{n}} = M$  and  $E_{\mathbf{n}} = \{(s, t) : s, t \in L_{\mathbf{m}} \text{ but } \{s, t\} \not\subseteq M_{\mathbf{n}}\}$ ; so by 2.8(5) clearly  $\mathbf{n}$  is equivalent to  $\mathbf{m}$  and  $M_{\mathbf{n}} \subseteq M_{\mathbf{m}}$  but we have to change the context by replacing  $\lambda_0$  by any  $\lambda'_0 \geq \lambda_0 + \|L_{\mathbf{m}}\|$ . {c5}

Then  $\mathbf{n} \in \mathbf{M}^{\text{fc}}[\lambda'_0]$ , see Definition 5.5(1). {c30}

*Proof.* Assume that

$$(*)_1 \mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1.$$

We now define  $\mathbf{m}_1 \in \mathbf{M}$  by

$$(*)_2 (a) \quad L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$$

- (b)  $\bar{u}_{\mathbf{m}_1} = \bar{u}_{\mathbf{n}_1}$
- (c)  $M_{\mathbf{m}_1} = M_{\mathbf{m}}$
- (d)  $E_{\mathbf{m}_1} = E_{\mathbf{m}} \cup (E_{\mathbf{n}_1} \upharpoonright (L_{\mathbf{n}_1} \setminus (L_{\mathbf{n}} \setminus M)))$ .

Clearly

- (\*)<sub>3</sub> (a) indeed  $\mathbf{m}_1 \in \mathbf{M}$
- (b)  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1$
- (c)  $\mathbf{m}_1, \mathbf{n}_1$  are equivalent.

But  $\mathbf{m} \in \mathbf{M}_{\text{fc}}$  hence

- (\*)<sub>4</sub>  $\mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{m}_1}$ .

{c11} But  $\mathbb{P}_{\mathbf{n}} = \mathbb{P}_{\mathbf{m}}$  and  $\mathbb{P}_{\mathbf{n}_1} = \mathbb{P}_{\mathbf{n}}$  by 2.13(7) as  $\mathbf{n}, \mathbf{m}$  are equivalent and  $\mathbf{n}_1, \mathbf{m}_1$  are equivalent, so

- (\*)<sub>5</sub>  $\mathbb{P}_{\mathbf{n}} \triangleleft \mathbb{P}_{\mathbf{n}_1}$ .

So we are done. □<sub>5.11</sub>

{c49y}

**The Uniqueness Claim 5.12.** *There is an isomorphism from  $\mathbb{P}_{\mathbf{m}_1}[M_1]$  onto  $\mathbb{P}_{\mathbf{m}_2}[M_2]$  which (recalling Definition ??) maps  $p_{t,\eta}^*$  to  $p_{h(t),\eta}^*$  for  $t \in M_1, \eta \in \cup\{\prod_{\varepsilon < \zeta} \theta_\varepsilon : \zeta < \lambda\}$  when:*

{c19}

- ⊕ (a)  $\mathbf{m}_\ell \in \mathbf{M}_\infty^{\text{ec}}$  for  $\ell = 1, 2$
- (b)  $M_\ell \subseteq M_{\mathbf{m}_\ell}$  for  $\ell = 1, 2$
- (c)  $h$  is an isomorphism from  $\mathbf{m}_1 \upharpoonright M_1$  onto  $\mathbf{m}_2 \upharpoonright M_2$ .

{c46} *Proof.* By renaming without loss of generality  $M_1 = M_2$  call it  $M$  and  $h$  is the identity and  $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$ . For  $\ell = 1, 2$  let  $\mathbf{m}'_\ell$  be like  $\mathbf{n}$  in 5.11 with  $\mathbf{m}_\ell, M_\ell$  here standing for  $\mathbf{m}, M$  there so

- (\*)<sub>1</sub>  $\mathbf{m}'_\ell \in \mathbf{M}_{\text{fc}}[\lambda'_0]$ .

We define  $\mathbf{m}$  by:

- (\*)<sub>2</sub> (a)  $s \in L_{\mathbf{m}}$  iff  $s \in L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$
- (b)  $r \leq_{\mathbf{m}} t$  iff  $r \leq_{\mathbf{m}_1} t$  or  $r \leq_{\mathbf{m}_2} t$  or  $(\exists s \in M)(r \leq_{\mathbf{m}_1} s \leq_{\mathbf{m}_2} t)$  or  $(\exists s \in M)(r \leq_{\mathbf{m}_2} s \leq_{\mathbf{m}_1} t)$
- (c)  $u_{\mathbf{m},t}$  is  $u_{\mathbf{m}_1,t}$  if  $t \in L_{\mathbf{m}_1} \setminus M$ ,  
is  $u_{\mathbf{m}_2,t}$  if  $t \in L_{\mathbf{m}_2} \setminus M$  and  
is  $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$  if  $t \in M$
- (d)  $M_{\mathbf{m}} = M$
- (e)  $E_{\mathbf{m}} = E'_{\mathbf{m}_1} \cup E'_{\mathbf{m}_2}$ .

{c46} For  $\ell = 1, 2$ ,  $\mathbb{P}_{\mathbf{m}_\ell}[M] = \mathbb{P}_{\mathbf{m}'_\ell}[M]$  by 5.11 and clearly  $\mathbf{m}'_\ell \leq_{\mathbf{M}} \mathbf{m}$ , but  $\mathbf{m}'_\ell \in \mathbf{M}_{\text{fc}}[\lambda'_0]$ ,  
{c46} by 5.11, hence  $\mathbb{P}_{\mathbf{m}'_\ell}[M] = \mathbb{P}_{\mathbf{m}}[M]$ . Together  $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}'_1}[M] = \mathbb{P}_{\mathbf{m}}[M] = \mathbb{P}_{\mathbf{m}'_2}[M] = \mathbb{P}_{\mathbf{m}_2}[M]$  as required. □<sub>5.11</sub>

{c51y}

**Conclusion 5.13.** *For every ordinal  $\delta_*$  there is  $\mathbf{q} = \langle \mathbb{P}_\alpha, \eta_\alpha : \alpha \leq \delta_* \rangle$  such that*

- (A) (a)  $\langle \mathbb{P}_\alpha : \alpha \leq \delta_* \rangle$  is  $\triangleleft$ -increasing

- (b)  $\eta_\alpha$  is a  $\mathbb{P}_{\alpha+1}$ -name of a member of  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  which dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_\alpha]}$
- (c)  $\eta_\alpha$  is a generic for  $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$ , moreover  $\langle \eta_\beta : \beta < \alpha \rangle$  is a generic for  $\mathbb{P}_\alpha$
- (d) every  $p \in \mathbb{P}_\alpha$  is from  $\mathbb{L}_{\lambda^+}(Y_{<\alpha}, \mathbb{P}_\alpha)$  when  $Y_{<\alpha}$  is as in Definition ?? {c19}
- (e)  $\mathbb{P}_\alpha$  is  $\lambda$ -strategically complete and  $\lambda^+$ -c.c.
- (f) if  $\delta \leq \delta_*$  has cofinality  $> \lambda$  then  $\mathbb{P}_\delta = \cup \{\mathbb{P}_\alpha : \alpha < \delta\}$
- (g)  $\mathbb{P}_{\delta_*}$  has cardinality  $|\delta_*|^\lambda$ .
- (B) if  $\mathcal{U} \subseteq \delta_*$  then the complete subforcing generated by  $\langle \eta_\alpha : \alpha \in \mathcal{U} \rangle$  is isomorphic to  $\mathbb{P}_{\text{otp}(\mathcal{U})}$
- (C) if  $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$  is generic over  $\mathbf{V}$  and  $\eta_\alpha = \eta_\alpha[\mathbf{G}]$  for  $\alpha < \delta_*$  and  $\eta'_\alpha \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  for  $\alpha < \delta_*$  and  $\{(\alpha, \varepsilon) : \alpha < \delta_*, \varepsilon < \lambda \text{ and } \eta'_\alpha(\varepsilon) \neq \eta_\alpha(\varepsilon)\}$  has cardinality  $< \lambda$  then also  $\langle \eta'_\alpha : \alpha < \delta_* \rangle$  is a generic for  $\mathbb{P}_{\delta_*}$ , determining a different  $\mathbf{G}'$  but  $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\mathbf{G}]$ .

*Proof.* We define  $\mathbf{m} \in \mathbf{M}$  by:

- (\*) (a)  $L_{\mathbf{m}} = \delta_*$   
 (b)  $M_{\mathbf{m}} = \delta_*$   
 (c)  $u_{\mathbf{m}, \alpha} = \alpha$  for  $\alpha < \delta_*$   
 (d)  $E_{\mathbf{m}} = \emptyset$ .

It is easy to check that indeed  $\mathbf{m} \in \mathbf{M}$ . So by 2.26(1) there is  $\mathbf{n} \in \mathbf{M}_\infty^{\text{ec}}$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ . Let  $\mathbb{P}_\alpha := \mathbb{P}_{\mathbf{n}}[\{\beta : \beta < \alpha\}]$ , so by ?? we are done.  $\square_{3.12}$  {c41} {c33}

Similarly we can prove

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EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL, AND, DEPARTMENT OF MATHEMATICS, HILL CENTER - BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA

*E-mail address:* [shelah@math.huji.ac.il](mailto:shelah@math.huji.ac.il)

*URL:* <http://shelah.logic.at>