

**ON  $\text{CON}(\mathfrak{d}_\lambda > \text{COV}_\lambda(\text{MEAGRE}))$**   
**SH945**

SAHARON SHELAH

ABSTRACT. We prove the consistency of: for suitable strongly inaccessible cardinal  $\lambda$  the dominating number, i.e., the cofinality of  ${}^\lambda\lambda$ , is strictly bigger than  $\text{cov}_\lambda(\text{meagre})$ , i.e. the minimal number of nowhere dense subsets of  ${}^\lambda 2$  needed to cover it. This answers a question of Matet.

§ 0. INTRODUCTION

Cardinal invariants on the continuum have a long tradition of research. For a topologist, it can be viewed as investigating the space  $\beta(\omega)$ , the Stone Čech compactification of  $\omega$ . This point of view is taken, for example, in the celebrated paper of van Douwen [vD84]. From the set theoretic perspective see the recent excellent surveys, Blass [Bla], Bartoszyński [Bar].

For set theorists, it is interesting to check the relationship between the relevant cardinal invariants. In this context, it is natural to generalize the problems to higher cardinals, above  $\aleph_0$ . One finds out, very soon, that for the class of (strongly) inaccessible cardinals, the generalizations are more reasonable and have more affinity to the  $\aleph_0$  case. See Landver [Lan92], Cummings-Shelah [CuSh:541]

We shall define three cardinal invariants (but the paper deals, actually, just with two of them):

**Definition 0.1.** The bounding and dominating numbers. {z1}

Let  $\lambda$  be an inaccessible cardinal.

Let  $f, g \in {}^\lambda\lambda$

- (a)  $f \leq^* g$  if  $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$
- (b)  $A \subseteq {}^\lambda\lambda$  is unbounded if there is no  $h \in {}^\lambda\lambda$  so that  $f \in A \Rightarrow f \leq^* h$
- (c)  $A \subseteq {}^\lambda\lambda$  is dominating when for every  $f \in {}^\lambda\lambda$  there exists  $g \in A$  so that  $f \leq^* g$
- (d) the bounding number for  $\lambda$ , denoted by  $\mathfrak{b}_\lambda$ , is  $\min\{|A| : A \text{ is unbounded in } {}^\lambda\lambda\}$
- (e) the dominating number for  $\lambda$ , denoted by  $\mathfrak{d}_\lambda$ , is  $\min\{|A| : A \text{ is dominating in } {}^\lambda\lambda\}$ .

Notice that the usual definitions of  $\mathfrak{b}$  and  $\mathfrak{d}$  are  $\mathfrak{b}_{\aleph_0}$  and  $\mathfrak{d}_{\aleph_0}$  according to Definition 0.1. The definition of  $\text{cov}_\lambda(\text{meagre})$  involves some topology.

---

*Date:* February 5, 2011.

The author thanks Alice Leonhardt for the beautiful typing. I thank Shimoni Garti for some corrections. This research was supported by the United States-Israel Binational Science Foundation. First version (in TeX Sept/16/2005 as F741). First version in LaTeX January 21, 2009.

**Definition 0.2.** The meagre covering number.

Let  $\lambda$  be a regular cardinal

- (a)  ${}^\lambda 2$  is the space of functions from  $\lambda$  into 2
- (b)  $({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$ , for  $\nu \in \bigcup_{\alpha < \lambda} {}^\alpha 2$
- (c)  $\mathcal{U} \subseteq {}^\lambda 2$  is open in the topology  $({}^\lambda 2)_{< \lambda}$ , iff for every  $\eta \in \mathcal{U}$  there exists  $i < \lambda$  so that  $({}^\lambda 2)^{[\eta \upharpoonright i]} \subseteq \mathcal{U}$
- (d)  $\text{cov}_\lambda(\text{meagre})$  is the minimal cardinality of a family of meagre subsets of  $({}^\lambda 2)_{< \lambda}$ , which covers this space.

The paper deals with the relationship between  $\mathfrak{d}_\lambda$  and  $\text{cov}_\lambda(\text{meagre})$ . Matet asked (a personal communication) whether  $\mathfrak{d}_\lambda \leq \text{cov}_\lambda(\text{meagre})$  is provable in ZFC. We give here a negative answer.

For  $\lambda$  a supercompact cardinal and  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , we force large  $\mathfrak{d}_\lambda$  i.e.,  $\mathfrak{d}_\lambda = \mu$  and small covering number (i.e.,  $\text{cov}_\lambda(\text{meagre}) = \kappa$ ). A similar result should hold also for a wider class of cardinals and we intend to return to this subject.

A point which in a previous version was just a step along the way the referee asked to justify fully, becomes a major point to which §2 is dedicated. A posteriori the point is that the parallel case for  $\lambda = \aleph_0$ , such claim is true. In fact, by Judah-Shelah [JdSh:292], if  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$  is FS iteration of Suslin-c.c.c. forcing notion,  $\mathbb{Q}_\beta$  with the generic  $\eta_\beta \in {}^\omega \omega$  and for notational transparency, its definition is with no parameter and  $\zeta : \beta(*) \rightarrow \alpha(*)$  is increasing and  $\mathbb{P} = \langle \mathbb{P}'_\alpha, \mathbb{Q}'_\beta : \alpha \leq \beta(*), \beta < \beta(*) \rangle$  is FS iteration,  $\mathbb{Q}'_\beta$  defined exactly as  $\mathbb{Q}_{\zeta(\beta)}$  but now in  $\mathbf{V}^{\mathbb{P}'_\beta}$  then  $\Vdash_{\mathbb{P}'_{\alpha(*)}} \langle \eta_{\zeta(\beta)} : \beta < \beta(*) \rangle$  is generic for  $\mathbb{P}'_{\beta(*)}$ .

Now this is not clear to us for  $(< \lambda)$ -support iteration of  $(< \lambda)$ -strategically complete forcing notions. The solution is essentially to change the iteration: we use a “quite generic”  $(< \lambda)$ -support iteration which “include” the one we like and use the complete subforcing it generates. Here we do only what is needed. On the general case we intend to continue in [Sh:F979].

We try to use standard notation. We use  $\theta, \kappa, \lambda, \mu, \chi$  for cardinals and  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$  for ordinals. We use also  $i$  and  $j$  as ordinals. We adopt the Cohen convention that  $p \leq q$  means that  $q$  gives more information, in forcing notions. The symbol  $\triangleleft$  is preserved for “being an initial segment”. Also recall  ${}^B A = \{f : f \text{ a function from } B \text{ to } A\}$  and let  $\alpha \triangleright A = \cup \{{}^\beta A : \beta < \alpha\}$ , some prefer  $<^\alpha A$ , but  $\alpha \triangleright A$  is used systematically in the author’s papers. Lastly,  $J_\lambda^{\text{bd}}$  denotes the ideal of the bounded subsets of  $\lambda$ .

The picture of cardinal invariants related to uncountable  $\lambda$  is related but usually quite different than the one for  $\aleph_0$ , they are more similar if  $\kappa$  is “large” enough, mainly strongly inaccessible.

Let us sketch some known results. These results are related to the inequality number and the covering number for category. Recall:

{z14}

**Definition 0.3.** The inequality number.

Let  $\kappa$  be an infinite cardinal. The inequality number of  $\kappa, \mathfrak{e}_\kappa$ , is the minimal cardinal  $\lambda$  such that there is a set  $\mathcal{F} \subseteq {}^\kappa \kappa$  of cardinality  $\lambda$  such that there is no  $g \in {}^\lambda \lambda$  satisfying  $(\forall f \in \mathcal{F})(\exists^\kappa \alpha < \lambda)(f(\alpha) = g(\alpha))$ .

For  $\kappa = \aleph_0$ ,  $\mathfrak{e}_\kappa = \text{cov}_{\aleph_0}(\text{meagre})$ ; see Bartoszyński (in [Bar87]) and Miller (in [Mil82]).

Now

- (a) the statement  $\mathfrak{e}_\kappa = \text{cov}_\kappa(\text{meagre})$  is valid for  $\kappa > \aleph_0$ , in the case that  $\kappa$  is strongly inaccessible, by [Lan92]. But if  $\kappa$  is a successor cardinal, it may fail
- (b) if  $\mathfrak{d}_\kappa$  is only finitely many cardinals away from  $\kappa$ , then  $\mathfrak{e}_\kappa = \mathfrak{d}_\kappa$ . This can be found in Matet-Shelah [MtSh:804]
- (c) if  $\kappa < \kappa^{<\kappa}$ , then  $\text{cov}_\kappa(\mathcal{M}) = \kappa^+$ . This is due to Landver (in [Lan92])
- (d) it is consistent to get (a) and (b) together, so that  $\text{cov}_\kappa(\text{meagre}) < \mathfrak{e}_\kappa$ . This follows from Cummings-Shelah (in [CuSh:541]).

We intend also to address there:

{z15}

**Problem 0.4.** Can we replace “super-compact” by “strongly inaccessible”?

{z16}

**Problem 0.5.** 1) Can we prove the consistency of  $\text{cov}_\lambda(\text{meagre}) < \mathfrak{b}_\lambda$ ?

2) For  $\lambda$  strongly inaccessible (or just Laver indestructible super-compact) is there a non-trivial  $\lambda^+$ -c.c. ( $< \lambda$ )-strategically complete forcing notion  $\mathbb{Q}$  which is  ${}^\lambda\lambda$ -bounding?

{z19}

**Definition 0.6.** Let  $\mathbb{P}$  be a forcing notion and  $Y \subseteq \mathbb{P}$  and  $\chi$  a regular cardinality.

1) Let  $\mathbb{L}_\chi(Y)$  be the set of sentences formed from  $\{p : p \in \mathbb{P}\}$  closing under the operations  $\neg p$  and  $\bigwedge_{i < \alpha} p_i$ , for  $\alpha < \chi$ ; so propositional logic.

2) For  $\mathbf{G} \subseteq \mathbb{P}$  and  $\psi \in \mathbb{L}_\chi(Y)$  we define the truth value  $\psi[\mathbf{G}]$  naturally (by induction on  $\psi$ ).

3) Let  $\mathbb{L}_\chi^+(Y, \mathbb{P})$  be the  $\mathbb{L}_\chi$ -closure of  $\mathbb{P}$ , ( $Y \subseteq \mathbb{P}$ ; if  $Y = \mathbb{P}$  we may omit  $Y$ ) be the following partial order:

- set of elements  $\{\psi \in \mathbb{L}_\chi(Y) : \not\vdash \psi[\mathbf{G}] = \text{false}\}$
- the order  $\psi_1 \leq \psi_2$  iff  $\Vdash$  “if  $\psi_2[\mathbf{G}] = \text{true}$  then  $\psi_1[\mathbf{G}] = \text{true}$ ”.

4) The completion of  $\mathbb{P}$  is the  $\chi$ -completion,  $\mathbb{L}_\chi(\mathbb{P})$  where  $\chi$  is minimal such that  $\mathbb{P}$  satisfies the  $\chi$ -c.c.

{z21}

**Claim 0.7.** For a forcing notion  $\mathbb{P}$  and  $Y \subseteq \mathbb{P}$  we have:

- (a)  $\mathbb{L}_\chi^+(Y, \mathbb{P})$  is a forcing notion
- (b)  $\mathbb{P} \leq \mathbb{L}_\chi^+(\mathbb{P})$  under the natural identification
- (c)  $\mathbb{L}_\chi^+(Y, \mathbb{P}) \leq \mathbb{L}_\chi^+(\mathbb{P})$
- (d)  $\mathbb{L}_{\chi_1}^+(Y, \mathbb{P}) \leq \mathbb{L}_{\chi_2}^+(Y, \mathbb{P})$  when  $\chi_1 \leq \chi_2$  are regular
- (e) if  $\mathbb{P}$  satisfies the  $\chi_1$ -c.c. and  $\chi_1 < \chi_2$  are regular then  $\mathbb{L}_{\chi_1}^+(Y, \mathbb{P})$  is essentially equal to  $\mathbb{L}_{\chi_2}^+(Y, \mathbb{P})$ , i.e. up to equivalence of elements.

{z23}

**Definition 0.8.** Let  $\lambda$  be inaccessible,  $\langle \theta_\varepsilon : \varepsilon < \lambda \rangle$  be a sequence of regular cardinals  $< \lambda$  satisfying  $\theta_\varepsilon > \varepsilon$ .

1) We define the forcing notion  $\mathbb{Q} = \mathbb{Q}_{\vec{\theta}}$  by

- (a)  $p \in \mathbb{Q}$  iff
- (a)  $p = (\eta, f) = (\eta^p, f^p)$

- (b)  $\eta \in {}^\varepsilon \lambda$  for some  $\varepsilon < \lambda$ , ( $\eta$  is called the trunk of  $p$ )
- (c)  $f \in {}^\lambda \lambda$
- (d)  $\eta \triangleleft f$
- ( $\beta$ )  $p \leq_{\mathbb{Q}} q$  iff
  - (a)  $\eta^p \trianglelefteq \eta^q$
  - (b)  $f^p \leq f^q$ , i.e.  $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$
  - (c) if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$ ; this follows
- 2) The generic is  $\eta = \cup \{\eta^p : p \in \mathbf{G}_{\mathbb{Q}_\delta}\}$ .

## § 1. THE FORCING

{a2}

**Theorem 1.1.** *Assume*

- (a)  $\lambda$  is supercompact
- (b)  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \text{cf}(\mu) = \mu^\lambda$ .

Then for some forcing notion  $\mathbb{P}$  not collapsing cardinals  $\geq \lambda$ ,  $\lambda$  is still supercompact in  $\mathbf{V}^{\mathbb{P}}$  and  $\text{cov}_\lambda(\text{meagre}) = \kappa$ ,  $\mathfrak{d}_\lambda = \mu$ .

*Proof.* By 1.3 below. □<sub>1.1</sub>

Recall {a5}

**Definition 1.2.** 1) We say that a forcing notion  $\mathbb{P}$  is  $\alpha$ -strategically complete when for each  $p \in \mathbb{P}$  in the following game  $\mathfrak{D}_\alpha(p, \mathbb{P})$  between the players COM and INC, the player COM has a winning strategy.

A play lasts  $\alpha$  moves; in the  $\beta$ -th move, first the player COM chooses  $p_\beta \in \mathbb{P}$  such that  $p \leq_{\mathbb{P}} p_\beta$  and  $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$  and second the player INC chooses  $q_\beta \in \mathbb{P}$  such that  $p_\beta \leq_{\mathbb{P}} q_\beta$ .

The player COM wins a play if he has a legal move for every  $\beta < \alpha$ .

2) We say that a forcing notion  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete when it is  $\alpha$ -strategically complete for every  $\alpha < \lambda$ . {a7}

**Lemma 1.3.** 1) If  $\lambda$  is supercompact then after some preliminary forcing of cardinality  $\lambda$ ,  $\lambda$  is still supercompact and  $\square_\lambda$  below holds.

2) If  $\lambda$  is strongly inaccessible and  $\square_\lambda$  below holds and  $\lambda^+ < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , then for some  $\lambda^+$ -c.c.,  $(< \lambda)$ -strategically complete forcing notion  $\mathbb{P}$  we have  $\Vdash_{\mathbb{P}}$  “ $\mathfrak{d}_\lambda = \mu$ ,  $\text{cov}_\lambda(\text{meagre}) = \kappa$ ”

where

$\square_\lambda$  for any regular cardinal  $\chi > \lambda$  and forcing notion  $\mathbb{P} \in \mathcal{H}(\chi)$  which is  $(< \lambda)$ -strategically complete (see Definition 1.2(2)) the following set  $\mathcal{S} = \mathcal{S}_{\mathbb{P}}$  is a stationary subset of  $[\mathcal{H}(\chi)]^{< \lambda}$ :

$\mathcal{S} = \mathcal{S}_{\mathbb{P}} = \mathcal{S}_{\chi, \mathbb{P}}$  is the set of  $N$ 's such that for some  $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N, N' = N'_N, M = M_N, \mathbf{G} = \mathbf{G}_N, P_N$  we have (and say  $(\lambda_s, \chi_s, \dots)$  is a witness for  $N \in \mathcal{S}_{\chi, \mathbb{P}}$  or for  $(N, \mathbb{P}, \chi)$ ):

- (a)  $N \prec (\mathcal{H}(\chi)^{\mathbf{V}}, \in)$  and  $\mathbb{P} \in N$
- (b) the Mostowski collapse  $N'$  of  $N$  is  $\subseteq \mathcal{H}(\chi_N)$ , and let  $\mathbf{j}_N : N \rightarrow N'$  be the unique isomorphism
- (c)  $N \cap \lambda = \lambda_N$  and  $(\lambda_N)^{> N} \subseteq N$  and  $\lambda_N$  is strongly inaccessible
- (d)  $N' \subseteq M := (\mathcal{H}(\chi_N), \in)$  so both  $N'$  and  $M$  are transitive
- (e)  $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$  is generic over  $N'$  for the forcing notion  $\mathbf{j}_N(\mathbb{P})$
- (f)  $M = N'[\mathbf{G}]$ . {a7.3}

*Remark 1.4.* 1) Recall that:

- (a)  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \delta, \beta < \delta \rangle$  be a  $(< \lambda)$ -support iteration of  $(< \lambda)$ -strategically complete forcing notions, then  $\mathbb{P}_\delta$  is also  $\lambda$ -strategically complete; (see e.g. [Sh:546]).

(b) If  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete forcing notion then  $(^{\lambda} \text{Ord})^{\mathbf{V}} = (^{\lambda} \text{Ord})^{\mathbf{V}^{\mathbb{P}}}$ , and consequently  $\lambda$  is strongly inaccessible in  $\mathbf{V}^{\mathbb{P}}$ .

2) In 1.1, clause (c), i.e. “ $\lambda^+ < \kappa$ ” rather than just “ $\lambda < \kappa$ ” which is demanded in (b) is not essential, see in the proof.

{a8}

**Definition 1.5.** We may say  $(N, \lambda_N, \chi_N, \mathbf{j}_N, N'_N, M_N, \mathbf{G}_N)$  is a witness for  $(N, \mathbb{P})$  when clauses (a)-(f) from 1.3(2) hold.

*Proof. Proof of Lemma 1.3* 1) This is essentially by Laver [Lav78] using Laver’s diamond.

2) Stage A: Without loss of generality  $\mathbf{V} \models “\mathfrak{b}_\lambda = \mu = \mathfrak{d}_\lambda”$  as witnessed by  $\langle f_\alpha^* : \alpha < \mu \rangle$ .

No new point, still we elaborate recalling that composition  $(\mathbb{P} * \mathbb{Q}^*)$  preserve “ $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c.”

We use a  $(< \lambda)$ -support iteration  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu, \beta < \mu \rangle$  such that:

(A) if  $\alpha < \mu$  then  $\mathbb{Q}_\alpha$  is the  $(\mathbb{P}_\alpha$ -name of the) dominating forcing,  $\mathbb{Q}_\lambda^{\text{dom}}$ , i.e.  $(\mathbb{Q}_\lambda^{\text{dom}})^{\mathbf{V}[\mathbb{P}_\alpha]}$  where in the universe  $\mathbf{V}'$  the forcing  $\mathbb{Q} = \mathbb{Q}_\lambda^{\text{dom}}$  is

( $\alpha$ )  $p \in \mathbb{Q}$  iff

(a)  $p = (\eta, f) = (\eta^p, f^p)$

(b)  $\eta \in {}^\varepsilon \lambda$  for some  $\varepsilon < \lambda$ , ( $\eta$  is called the trunk of  $p$ )

(c)  $f \in {}^\lambda \lambda$

(d)  $\eta \triangleleft f$

(B)  $p \leq_{\mathbb{Q}} q$  iff

(a)  $\eta^p \leq \eta^q$

(b)  $f^p \leq f^q$ , i.e.  $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$

(c) if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$ ; this follows.

Let  $f_\alpha$  be the generic object for  $\mathbb{Q}_\alpha$  for  $\alpha < \mu$ .

Now:

(\*)<sub>1</sub> for  $\alpha \leq \mu$  the forcing notion  $\mathbb{P}_\alpha$  is  $(< \lambda)$ -complete and, when  $\alpha < \mu$ ,  $\mathbb{Q}_\alpha$  is  $(< \lambda)$ -complete<sup>1</sup>, i.e.  $\Vdash_{\mathbb{P}_\alpha} “\mathbb{Q}_\alpha \text{ is } (< \lambda)\text{-complete}”$ , in fact.

[Why? We prove this by induction on  $\alpha$ ; for  $\mathbb{P}_\alpha$ , for 1.4(1)(a).]

(\*)<sub>2</sub> for each  $\alpha \leq \mu$ ,  $\mathbb{P}_\alpha$  and for  $\alpha < \mu$ , the forcing notions  $\mathbb{Q}_\alpha$  satisfy a strong form of the  $\lambda^+$ -c.c., (see [Sh:80] for definition, preservation and history; or fully [Sh:546, §1])

hence

(\*)<sub>3</sub> (a) forcing with  $\mathbb{P}_\mu$  collapses no cardinal, changes no cofinality, and adds no sequence to  ${}^{\lambda} \mathbf{V}$ ;

(b)  $({}^\lambda \lambda)^{\mathbf{V}[\mathbb{P}_\mu]} = \cup \{({}^\lambda \lambda)^{\mathbf{V}[\mathbb{P}_\alpha]} : \alpha < \mu\}$ .

<sup>1</sup>for this,  $\theta_\alpha > \alpha$  is enough

[Why? By  $(*)_2 + (*)_1$  clause (a) holds, for clause (b) use also the support in the iteration being  $< \lambda$  recalling that  $\mu$  is regular  $> \lambda$ . E.g. if  $\Vdash_{\mathbb{P}_\mu} \check{f} \in {}^\lambda \lambda$  then we can, for  $\alpha < \lambda$  find maximal antichain  $\langle p_{\alpha,i} : i < i_\alpha \leq \lambda \rangle$  of  $\mathbb{P}_\mu$  such that  $p_{\alpha,i}$  forces a value to  $\check{f}(\alpha)$ ; let  $\alpha(*) = \sup(\cup\{\text{dom}(p_{\alpha,i}) : \alpha < \lambda, i < i_\alpha\})$ , so  $\alpha(*) < \mu$  and  $\check{f}$  is a  $\mathbb{P}_{\alpha(*)}$ -name.]

$(*)_4$  in  $\mathbf{V}^{\mathbb{P}_\mu}$ ,  $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$  as witnessed by  $\check{f} = \langle \check{f}_\alpha : \alpha < \mu \rangle$ , in fact  $\Vdash_{\mathbb{P}_{\alpha+1}} \check{f}_\alpha \in {}^\lambda \lambda$  dominates  $({}^\lambda \lambda)^{\mathbf{V}^{\mathbb{P}_\alpha}}$  modulo  $J_\lambda^{\text{bd}}$ .

[Why? Easy using  $(*)_3(b)$ .]

$(*)_5$  in  $\mathbf{V}^{\mathbb{P}_\mu}$  still  $\square_\lambda$  holds.

[Why? Easy, see details in 1.7(2).]

Stage B: In  $\mathbf{V}$  (see Stage A) there are  $\beta(*), \mathbf{q}, \bar{u}, \mathcal{U}_*, \dots$  such that

- $(*)_6$  (a)  $\mathbf{q} = \langle \mathbb{P}_{0,\alpha}, \mathbb{Q}_{0,\beta} : \alpha \leq \beta(*), \beta < \beta(*) \rangle$  is a  $(< \lambda)$ -support iteration
- (b)  $\bar{u} = \langle u_\beta : \beta < \beta(*) \rangle$
- (c)  $u_\beta \subseteq \beta$
- (d)  $\mathbb{Q}_\beta$  has generic  $\eta_\beta \in \prod_{< \lambda_\varepsilon} \theta_\varepsilon$
- (e)  $\mathbb{Q}_{0,\beta}$  is  $\mathbb{Q}_{\check{\beta}}^{\mathbf{V}^{\langle \eta_\alpha : \alpha \in u_\beta \rangle}}$
- (f)  $\mathcal{U}_* \subseteq \gamma(*)$  has order type  $\gamma(*) = \kappa$
- (g) if  $\beta \in \mathcal{U}_*$  then  $\mathcal{U}_* \cap \beta \subseteq u_\beta$
- (h)  $\mathbb{P}_{0,\alpha}$  and  $\mathbb{P}'_i$  are  $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c.

We shall mention more properties later.

[Why? Let  $M$  be isomorphic to  $(\kappa, <)$ , applying 2.28 there is  $\mathbf{m}$  as there. Let  $h$  be a one-to-one order preserving function from  $L_{\mathbf{m}}$  onto some ordinal,  $\beta(*)$ .

So without loss of generality  $h$  is the identity, so let  $L_{\mathbf{m}}$  be  $(\beta(*), <_*)$ ,  $\mathcal{U}_* = M_{\mathbf{m}}, \mathbb{P}_{0,\alpha} = \mathbb{P}_{\mathbf{m}}(\{\beta : \beta < \alpha\})$ ,  $u_\alpha = u_{\mathbf{m},\alpha}$  let  $\langle \beta_i^* : i < \gamma(*) \rangle$  list  $\mathcal{U}_*$  in increasing order so  $\gamma(*) = \kappa$ . For  $i \leq \kappa$ , let  $\mathbb{P}'_i$  is  $\mathbb{P}_{\mathbf{m}}(\{\beta_j^* : j < i\})$  and let  $\check{g}'_i$  be  $\eta_{\beta_i^*}$  and let  $\check{g}' = \langle \check{g}'_i : i < \kappa \rangle$ . Lastly, let  $g_\alpha = \eta_\alpha$ .

$(*)_7$   $\Vdash_{\mathbb{P}_{\alpha+1}} \check{g}_\alpha \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{[< \check{g}_\beta : \beta \in u_\alpha >]}$ ,  
the order being modulo  $J_\lambda^{\text{bd}}$ .

[Why? As in  $\mathbf{V}^{\mathbb{P}_\alpha}$  for each  $g \in (\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}^{\langle \eta_\beta : \beta \in u_\alpha \rangle}}$  the set  $\{(\eta, f) \in \mathbb{Q}_{0,\beta} : \text{for every } \varepsilon \in [\ell g(\eta), \lambda) \text{ we have } g(\varepsilon) \leq f(\varepsilon)\}$  is a dense open subset of  $\mathbb{Q}_\alpha$ .]

$(*)_8$   $\Vdash_{\mathbb{P}'_\kappa} \check{g}' = \langle \check{g}'_i : i < \kappa \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing and cofinal in  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_\lambda^{\text{bd}}})$ .

[Why? By  $(*)_7$  noting that  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}^{\mathbb{P}'_\kappa}} = \cup\{(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}^{\mathbb{P}'_i}} : i < \kappa\}$  which holds by

$(*)_6(f), (g)$ .]

Now

$(*)_9$   $\Vdash_{\mathbb{P}'_\kappa} \text{cov}_\lambda(\text{meagre}) \leq \kappa$ .

[Why? As we can look at  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  instead<sup>2</sup> of  ${}^\lambda 2$  and for each  $\varepsilon < \lambda, i < \kappa$  the set  $B_{\varepsilon,i} = \{\eta \in \prod_{\zeta < \lambda} \theta_\zeta : \text{for every } \zeta \in [\varepsilon, \lambda) \text{ we have } \eta(\zeta) \leq \eta_i(\zeta) < \theta_\zeta\}$  is closed nowhere dense, and by  $(*)_8$  we have  $\mathbf{V}^{\mathbb{P}_\kappa} \Vdash \text{“}\prod_{\zeta < \lambda} \theta_\zeta = \cup\{B_{\varepsilon,i} : \varepsilon < \lambda, i < \kappa\}$ ”. In fact,  $\langle B_{0,i} : i < \kappa \rangle$  suffice.]

$(*)_{10} \Vdash_{\mathbb{P}'_\kappa} \text{“cov(meagre)} \geq \kappa\text{”}$ .

[Why? Let us define the  $\mathbb{P}_{\alpha+1}$ -name  $\eta'_\alpha$  of a member of  ${}^\lambda 2$  by  $\eta'_\alpha(\varepsilon) = 0$  iff  $\eta_\alpha(\varepsilon)$  is even. Now recalling 8.3 clearly  $\Vdash_{\mathbb{P}_{\alpha+1}} \text{“}\eta_\alpha \text{ is a } \lambda\text{-Cohen sequence over } \mathbf{V}^{\mathbb{P}_\alpha}\text{”}$ .

Also every closed nowhere dense subset of  ${}^\lambda 2$  from  $\mathbf{V}^{\mathbb{P}_{\gamma(*)}}$  is from  $\mathbf{V}^{\mathbb{P}_\alpha}$  for some  $\alpha < \gamma(*)$ , so we are done.]  $\square_{1.3}$

**Discussion 1.6.** 1) The reader may justly wonder why we use  $\mathbf{V}' = \mathbf{V}[\bar{g};] = \mathbf{V}[\bar{g} \upharpoonright \mathcal{U}_*]$  rather than simply  $\mathbf{V}[\bar{g}]$ . Of course, nothing is lost by it, but why the extra complication?

2) During the proof we shall use: if  $\zeta(i) \in \mathcal{U}_*$  is increasing with  $i < \gamma(*)$  then also  $\langle g_{\zeta(i)} : i < \kappa \rangle$  is generic over  $\mathbf{V}$  for the subforcing of  $\mathbb{P}_{\beta(*)}$  generated by  $\bar{g} \upharpoonright \mathcal{U}_*$ ; see  $\otimes'_7$  inside the proof of  $\otimes_6$ . But using  $\mathcal{U}_* = \beta(*)$ , we do not know this.

3) Now in the parallel case for  $\lambda = \aleph_0$ , such claim is true, see  $\S 0$ .

4) But we do not know the parallel of 3) for  $\lambda$ , so we use a substitute using  $\mathcal{U}_*$ , i.e.  $\mathbb{P}'_\kappa$ .

*Proof.* Continuation of the proof:

Now we come to the main and last point recalling  $\langle f_\alpha^* : \alpha < \mu \rangle$  from Stage A

$(*)_{11}$  it is forced, i.e.  $\Vdash_{\mathbb{P}'_\kappa}$  that no  $f \in ({}^\lambda \lambda)$  dominate  $\{f_\alpha^* : \alpha < \mu\}$ .

We shall show that it suffices to prove  $(*)_{11}$  for proving 1.3, and that  $(*)_{11}$  holds, thus finishing.

Why it suffices? As  $\langle f_\alpha^* : \alpha < \mu \rangle$  is  $\leq_{J_\lambda^{\text{bd}}}$ -increasing and  $\text{cf}(\mu) = \mu > \lambda$ , this implies  $\Vdash_{\mathbb{P}'_\kappa} \text{“}\mathfrak{d}_\lambda \geq \mu\text{”}$ . Also in  $\mathbf{V}, 2^\lambda = \mu \geq \kappa > \lambda$  and  $|\mathbb{P}'_\kappa| = \kappa^\lambda$  by (g) of 2.28 which is  $= \mu$  and  $\mathbb{P}'_\kappa$  satisfies the  $\lambda^+$ -c.c. hence  $\Vdash_{\mathbb{P}'_\kappa} \text{“}2^\lambda = \mu\text{, hence } \Vdash_{\mathbb{P}'_\kappa} \text{“}\mathfrak{d} = \mu\text{”}$ . Also by  $(*)_6(h)$  “ $\mathbb{P}'_\kappa$  is “ $(< \lambda)$ -strategically complete +  $\lambda^+$ -c.c.” and  $(*)_9 + (*)_{10}$  for “ $\text{cov}_\lambda(\text{meagre}) = \kappa$ ” so we are done; so  $(*)_{11}$  is really the last piece missing. The rest of the proof is dedicated to proving that  $(*)_{11}$  holds.

We shall use further nice properties of  $\mathbb{P}'_j, g'_i(j \leq \gamma(*), i < \gamma(*))$  which holds by  $(*)_6$  and its proof, i.e. 2.28.

$\boxplus_1$  (a)  $\langle g'_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$ , i.e., if  $\mathbf{G}$  is a subset of  $\mathbb{P}'_{\gamma(*)}$  generic over  $\mathbf{V}$  and  $g'_i = \mathbb{P}'_\gamma[\mathbf{G}]$  then  $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_\gamma : \gamma < \gamma(*) \rangle]$

<sup>2</sup>E.g. let  $F : {}^\lambda 2 \rightarrow \prod_{\varepsilon < \lambda} \theta_\varepsilon$  be  $F(\eta) = \rho$  iff  $\eta \in {}^\lambda 2$  and for every  $\varepsilon < \lambda, \rho(\varepsilon) = 0$  iff  $(\forall i < \theta_\varepsilon)(\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + i) = 0)$  and  $\rho(\varepsilon) = 1 + i$  iff  $\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + i) = 1 \wedge (\forall j < i)(\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + j) = 0)$ . Now if  $\prod_{\varepsilon} \theta_\varepsilon = \cup\{\mathcal{U}_i : i < \kappa\}$ , each  $\mathcal{U}_i$  closed nowhere dense then  $\langle F^{-1}(\mathcal{U}_i) : i < \kappa \rangle$  witnesses  $\text{cov}_\lambda(\text{meagre}) \leq \kappa$ .

- (b) if in  $\mathbf{V}[\mathbf{G}]$ ,  $g''_\gamma \in \prod_{\zeta < \lambda} \theta_\zeta$  for  $\gamma < \gamma(*)$  and the set  $\{(\gamma, \zeta) : \gamma < \gamma(*) \text{ and } \zeta < \lambda \text{ and } g''_\gamma(\zeta) \neq g'_\gamma(\zeta)\}$  has cardinality  $< \lambda$  then  $\bar{g}'' = \langle g''_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$  and  $\mathbf{V}[\bar{g}''] = \mathbf{V}[\bar{g}']$
- (c)  $\Vdash_{\mathbb{P}'_\gamma}$  “ $\bar{g}'_\gamma$  dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\gamma]}$ ”
- (d) if  $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$  is an increasing sequence of ordinals  $< \gamma(*)$ , then  $\langle g'_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$
- (e)  $\mathbb{P}'_\gamma$  is  $(< \lambda)$ -strategically complete and satisfies the  $\lambda^+$ -c.c.

We shall use  $\boxplus_1$  freely.]

To prove  $(*)_{11}$  assume toward contradiction that this fails, so  $\mathbb{P}'_{\gamma(*)}$  satisfies the  $\lambda^+$ -c.c. and for some  $\mathbb{P}'_{\gamma(*)}$ -name  $\bar{f}$  and  $\lambda$ -Borel function  $\mathbf{B}$  and  $\rho \in {}^\lambda \gamma(*)$ , moreover  $\rho \in {}^\lambda \mathcal{U}_*$  we have (noting: the “moreover” holds as  $f \in ({}^\lambda \lambda)^{\mathbf{V}[\bar{g}_0 \upharpoonright \mathcal{U}_*]}$ )

$$\boxplus_0 p^* \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\bar{f} \in {}^\lambda \lambda \text{ and dominates } ({}^\lambda \lambda)^{\mathbf{V}} \text{” and } \bar{f} = \mathbf{B}(\langle g_{\rho(i)} : i < \lambda \rangle).$$

Now let  $\chi$  be regular large enough and we choose  $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda \rangle$  such that

- $\boxplus_1$  (a)  $N_\varepsilon$  is as in  $\square_\lambda$  for the forcing notion  $\mathbb{P}'_{\gamma(*)}$ ,  
 $N_\varepsilon \in \mathcal{S}_{\chi, \mathbb{P}'_{\gamma(*)}}$ , see  $\square$  of 1.3
- (b)  $\bar{N} \upharpoonright \varepsilon \in N_\varepsilon$  hence  $\bigcup_{\zeta < \varepsilon} N_\zeta \subseteq N_\varepsilon$  and  $\lambda_\varepsilon := N_\varepsilon \cap \lambda > \lambda_\varepsilon^- := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\}$
- (c)  $\bar{\theta}, \mathbf{q}, p^*, \bar{f}, \mathbf{B}, \rho$  belong to  $N_\varepsilon$
- (d) let  $\delta(\varepsilon) = \text{otp}(\gamma(*) \cap N_\varepsilon)$
- (e)  $\kappa_\varepsilon = \kappa_\varepsilon^{< \kappa_\varepsilon}$  where  $\kappa_\varepsilon = \text{otp}(\kappa \cap N_\varepsilon)$ .

We can find  $f^* \in {}^\lambda \lambda$ , i.e.  $\in ({}^\lambda \lambda)^{\mathbf{V}}$ , such that

$$\boxplus_2 \text{ for arbitrarily large } \varepsilon < \lambda \text{ for some } \zeta \in [\lambda_\varepsilon^-, \lambda_\varepsilon) \text{ we have } f^*(\zeta) > \lambda_\varepsilon.$$

For  $\varepsilon < \lambda$  let  $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, N'_\varepsilon, \mathbf{G}_\varepsilon)$  be a witness for  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)}, \chi)$  recalling  $\square_\lambda$  of Definition 1.5 so  $\lambda_\varepsilon \in (\varepsilon, \lambda)$  is strongly inaccessible and  $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda_\zeta^- < \lambda_\zeta$ , recalling  $\boxplus_1$  and so  $\delta(\varepsilon) = \mathbf{j}_\varepsilon(\delta(*))$ , etc.

Let

- $\boxplus_3$  (a)  $u_\varepsilon = N_\varepsilon \cap \gamma(*)$
- (b)  $i(\varepsilon) = \text{otp}(u_\varepsilon)$
- (c)  $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < i(\varepsilon) \rangle$  list  $u_\varepsilon$  in increasing order
- (d) for  $i < \text{otp}(u_\varepsilon)$ , equivalently  $i < \mathbf{j}_\varepsilon(\gamma(*))$  let  $\eta_i^\varepsilon = (\mathbf{j}_\varepsilon(g'_i))^{N'_\varepsilon[\mathbf{G}_\varepsilon]} \in \prod_{\zeta < \lambda_\varepsilon} \theta_\zeta$  and let  $\bar{\eta}^\varepsilon = \langle \eta_i^\varepsilon : i < \text{otp}(u_\varepsilon) \rangle$ .

Note that clearly

- $\boxplus_4$  (a)  $\bar{\eta}^\varepsilon$  is generic for  $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$ , moreover
- (b) for each  $\varepsilon < \lambda$ , if we change  $\eta_i^\varepsilon(\zeta)$  (legally, i.e.  $< \theta_\zeta$ ) for  $< \lambda_\varepsilon$  pairs  $(i, \zeta) \in \text{otp}(u_\varepsilon) \times \lambda_\varepsilon$  and get  $\bar{\eta}'$ , then also  $\bar{\eta}'$  is generic for  $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$  and  $N'_\varepsilon[\bar{\eta}'] = M_\varepsilon$

(c) like  $\boxplus_1$  with  $\mathbf{V}, \mathbb{P}'_{\gamma(*)}, \lambda$  there standing for  $N_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$  here.

Hence that clearly

$\otimes'_4$  for  $\varepsilon < \lambda$ , if  $\bar{\eta}' = \langle \nu_i : i < i(\varepsilon) \rangle$  recalling  $i(\varepsilon) = \text{otp}(u_\varepsilon)$  is as in  $\otimes_4(b)$ , and  $q \in \mathbb{P}'_{\gamma(*)}$  satisfies  $i < i(\varepsilon) \Rightarrow q \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\underline{g}'_{\gamma_i(\varepsilon)} \upharpoonright \lambda_\varepsilon = \nu_i\text{”}$  then  $q$  is  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$ -generic naturally and  $q \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\mathbf{j}_\varepsilon \text{ can be extended naturally to an isomorphism from } N_\varepsilon[G_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle \underline{g}_\gamma : \gamma \in u_\varepsilon \rangle] \text{ onto } N'_\varepsilon[\bar{\eta}']\text{”}$ .

[Why? Should be clear, see  $\boxplus_1 + \otimes_4(c)$ .]

By the assumption toward contradiction,  $\otimes_0$ , and  $\mathbb{P}'_{\gamma(*)}$  being  $(< \lambda)$ -strategically complete recalling  $\boxplus_1$ , there are  $\zeta(*), p^{**}$  and  $p^+$  such that (recall  $p^* \in \mathbb{P}'_{\gamma(*)} = \ll \mathbb{P}_{0,\beta(*)}$ ):

- $\otimes_5$  (a)  $p^* \leq p^{**} \in \mathbb{P}'_{\gamma(*)}$  and  $p^{**} \leq p^+ \in \mathbb{P}_{0,\gamma(*)}$
- (b)  $\zeta(*) < \lambda$
- (c)  $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}f^*(\zeta) < \underline{f}(\zeta) \text{ whenever } \zeta(*) \leq \zeta < \lambda\text{”}$  where  $f^*$  is from  $\otimes_2$
- (d) if  $\gamma \in \text{Dom}(p^+)$  then  $\eta^{p^+(\gamma)}$  is an object (not just a  $\mathbb{P}'_\gamma$ -name) of length  $\geq \zeta(*)$  (recall that  $\eta^{p^+(\gamma)}$  is the trunk of the condition, see clause  $(\alpha)(b)$  above).

Note that possibly  $\text{Dom}(p^+) \not\subseteq \cup \{u_\varepsilon : \varepsilon < \lambda\}$ . Choose  $\varepsilon(*) < \lambda$  such that  $\lambda_{\varepsilon(*)} > \zeta(*) + |\text{Dom}(p^+)|$  and  $\gamma \in \text{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$  recalling clause (d) of  $\otimes_5$  and  $|\text{Dom}(p^+)| < \lambda$  as  $p^+ \in \mathbb{P}_{0,\beta(*)}$  and  $\mathbb{P}_{0,\beta(*)}$  is the limit of a  $(< \lambda)$ -support iteration.

By  $\otimes_2$  we can add  $(\exists \zeta)[\lambda_{\varepsilon(*)}^- \leq \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$ . Our intention is to find  $q \in \mathbb{P}_{0,\beta(*)}$  above  $p^+$  which is above some  $q' \in \mathbb{P}'_{\gamma(*)}$  which is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ -generic and forces it to include a generic subset of  $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$  which is induced by some  $\bar{\eta}'$  as in  $\otimes_4(b)$ . Toward this in  $\otimes_6$  below the intention is that  $p_{i(*)}^+$  will serve as  $q$ .

Let  $i(*) = i(\varepsilon(*))$  and  $\gamma_i$  for  $i < i(*)$  be such that  $\langle \gamma_i : i < i(*) \rangle$  list  $u_{\varepsilon(*)} \cap \mathcal{U}_*$  in increasing order and let  $\mathcal{U}_{**} = \{\gamma_i : i < i(*)\}$  and  $\gamma_{i(*)} = \gamma(*)$  so  $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in u_{\varepsilon(*)}\} = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$  and  $N_{\varepsilon(*)} \models \text{“}i(*) \text{ is a regular cardinal } > \lambda_\varepsilon\text{”}$  hence  $i(*)$  is really a regular cardinal so call it  $\sigma$ . Now we define a game  $\boxplus$  as follows<sup>3</sup>:

- $\boxplus_2$  (A) each play lasts  $i(*) + 1 = \sigma + 1$  moves and in the  $i$ -th move,
  - (a) if  $i = j + 1$  the antagonist player chooses  $\xi_j = \xi(j) < \sigma$  such that  $j_1 < j \Rightarrow \xi(j_1) < \xi(j)$
  - (b) then, if  $i = j + 1$  the protagonist chooses  $\zeta_j = \zeta(j) \in (\xi(j), \sigma)$ , but there are more restrictions implicit in  $\boxplus_3$
  - (c) in any case the protagonist chooses  $p_i^+, \bar{\nu}^i$  such that  $\boxplus_3$  below holds;
- (B) in the end of the play the protagonist wins the play iff he always has a legal move and in the end  $\{\zeta(i) : i < i(*)\} \in N'_{\varepsilon(*)}$ ;

<sup>3</sup>The idea is to scatter the  $\eta_{\gamma_i}^{\varepsilon(*)}$ 's. Why not use the original places? as then we have a problem in  $\otimes_6$ .

where

- $\boxplus_3$  (a)  $p_i^+ \in \mathbb{P}_{0,\gamma_i}$   
 (b) if  $j < i$  then  $\mathbb{P}_{0,\gamma_i} \models "p_j^+ \leq p_i^+"$   
 (c) if  $\gamma \in \cup\{\text{Dom}(p_j^+) : j < i\}$  then  
 $p_i^+ \upharpoonright \gamma \Vdash_{\mathbb{P}_{0,\gamma_i}} " \eta^{p_i^+(\gamma)} \text{ has length } \geq i(*) \text{ and } \geq \lambda_{\varepsilon(*)}"$   
 moreover  $\eta^{p_i^+(\gamma)}$  is an object,  $\eta^{p_i^+(\gamma)}$  for  $j < i$   
 (d)  $\mathbb{P}_{0,\gamma_i} \models "p^{**} \upharpoonright \gamma_i \leq p_i^+"$   
 (e)  $\bar{v}^i = \langle \nu_{\gamma_j} : j < i \rangle$  and  $\nu_{\gamma_j} \in \prod_{\iota < \lambda_{\varepsilon(*)}} \theta_\iota$   
 (f) for  $j < i$  we have  $\nu_{\gamma_j} \leq \eta^{p_i^+(\gamma_j)}$  so  $p_i^+ \upharpoonright \gamma_j \Vdash " \nu_{\gamma_j} < g'_{\gamma_j} "$  recalling  $\boxplus_1$   
 (g) for  $j < i$  we have (recall  $\bar{\eta}^\varepsilon$  from  $\otimes_3$ )  
 ( $\alpha$ )  $\nu_{\gamma_j} = \eta_{\gamma_{\zeta(j)}}^{\varepsilon(*)}$  recalling  $\eta_{\gamma_j}^{\varepsilon(*)}$  is from  $\otimes_3$  or  
 ( $\beta$ )  $\gamma_j \in \text{Dom}(p^{**})$  and  $\{\iota < \lambda_{\varepsilon(*)} : \eta_{\zeta(j)}^{\varepsilon(*)}(\iota) \neq \nu_{\gamma_j}(\iota)\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ .

We shall prove

- $\otimes_6$  in the game  $\mathcal{D}$   
 (a) the antagonist has no winning strategy  
 (b) in any move the protagonist has a legal move, moreover for any  $\zeta(i) \in (\xi(i), \sigma)$  large enough the protagonist can choose it.

Why  $\otimes_6$  suffice?

By clause (a) of  $\otimes_6$  we can choose a play  $\langle (\xi(i), \zeta(i), p_i^+, \bar{v}^i) : i \leq i(*) \rangle$  in which the protagonist wins. Recalling  $\mathbb{P}'_{\gamma(*)} < \mathbb{P}_{1,\beta(*)}$  and  $\mathbb{P}_{0,\beta(*)}$  is a dense subforcing of  $\mathbb{P}_{1,\beta(*)}$ , clearly

- $\otimes_7$  there is  $p$  such that  
 (a)  $p \in \mathbb{P}'_{\gamma(*)}$   
 (b) if  $\mathbb{P}'_{\gamma(*)} \models "p \leq p'"$  then  $p', p_i^+$  are compatible in  $\mathbb{P}_{0,\beta(*)}$   
 (c)  $p$  is above  $p^{**}$  and it forces  $g'_i \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < \gamma(*)$ .

Then on the one hand

- $\otimes'_7$   $p \in \mathbb{P}'_{\gamma(*)}$  being above  $p^{**}$  forces  $f^* \upharpoonright [\zeta(*), \lambda] < \underline{f} \upharpoonright [\zeta(*), \lambda]$  hence  $f^* \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)}] < \underline{f} \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)}]$  recalling that  $\zeta(*) < \lambda_{\varepsilon(*)}$ .

On the other hand,

- $\otimes''_7$   $p$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ -generic.

[Why? As it forces  $\eta_{\gamma_{1,i}} \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < i(*)$  and  $\langle \nu_{\gamma_i} : i < i(*) \rangle$  is (see  $\otimes_4$ ) "almost equal" to  $\langle \eta_{\zeta(i)}^{\varepsilon(*)} : i < i(*) \rangle$  which is a subsequence of the sequence from  $\otimes_3$  and recalling clause (g) of  $\boxplus_3$ . That is  $\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} \subseteq \cup\{\{(i, \iota) : \iota < \lambda_{\varepsilon(*)} \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} : \gamma \in u_{\varepsilon(*)} \cap \text{Dom}(p^{**})\}$  so is the union of  $\leq |\text{Dom}(p^+)| < \lambda_{\varepsilon(*)}$  sets each of cardinality  $< \lambda_{\varepsilon(*)}$  hence is

of cardinality  $< \lambda_{\varepsilon(*)}$ . Hence by  $\otimes_4(c) + \boxplus_1(d)$  the sequence  $\bar{\nu}^{i(*)}$  is generic for  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ .

As  $f \in N_{\varepsilon(*)}$  it follows from  $\otimes_7''$  that

$$\otimes_7''' \ p \Vdash \text{“} f \upharpoonright \lambda_{\varepsilon(*)} \text{ is a function from } \lambda_{\varepsilon(*)} \text{ to } \lambda_{\varepsilon(*)}\text{”}.$$

Together  $\otimes_7' + \otimes_7'''$  gives a contradiction by the choice of  $f^*$  in  $\otimes_2$  and of  $\varepsilon(*)$  above, hence  $\otimes_6$  is enough.

Why  $\otimes_6$  is true?

Let us prove  $\otimes_6$ ; first for clause (a) choose any strategy  $\mathbf{st}$  for the antagonist and fix a partial strategy  $\mathbf{st}'$  for the protagonist choosing  $(p_i^+, \bar{\nu}^i)$  from the previous choices and  $\zeta(i)$  if relevant and possible. So the only freedom left is for the protagonist to choose the  $\zeta(i)$ . So we have in  $\mathbf{V}$  a function  $F : \sigma^{>}(i(*)) \rightarrow \sigma$  such that:

(\*)<sub>F</sub> playing the game such that the antagonist uses  $\mathbf{st}$  and the protagonist uses  $\mathbf{st}'$ , arriving to the  $i$ -th move,  $\bar{\zeta} = \langle \zeta(j) : j < i \rangle$  is well defined and for the protagonist any choice  $\zeta_i \in (F(\bar{\zeta}), \sigma) \cap \mathcal{U}_{**}$  is legal.

Now we have to find an increasing sequence  $\bar{\zeta} = \langle \zeta(i) : i < i(*) \rangle$  such that  $F(\bar{\zeta} \upharpoonright i) < \zeta(i) < \sigma$  and  $\bar{\zeta} \in N'_{\varepsilon(*)}$ . As  $F \in \mathcal{H}(\chi_\varepsilon)$  and  $\mathcal{H}(\chi_\varepsilon) = N'_\varepsilon[\mathbf{G}_\varepsilon]$  where  $\mathbf{G}_\varepsilon$  is a subset of  $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}) \in N'_\varepsilon$  and  $\mathbf{j}_\varepsilon(\mathbb{P}_{0,\beta(*)})$  satisfies the  $\lambda_\varepsilon^+$ -c.c. and  $\sigma = \text{cf}(\sigma) > \lambda_\varepsilon$  this is possible. We are left with proving  $\otimes_6(b)$ .

Case 1:  $i = 0$ .

Let  $p_0^+ = p^{**} \upharpoonright \gamma_0$ .

Case 2:  $i$  limit.

By clauses (a) and (b), there is  $p_i^+ \in \mathbb{P}_{0,\gamma_i}$  which is an upper bound (even l.u.b.) of  $\{p_j^+ : j < i\}$  and it is easily as required. Also  $\bar{\nu}^i$  is well defined and as required.

Case 3:  $i = j + 1$  and  $\gamma_j \notin \text{Dom}(p^{**})$ .

Clearly  $\gamma_i = \gamma_j + \delta(*)$  and  $\gamma_j \in u_{\varepsilon(*)}$ . As in case 4 below but easier by the properties of the iteration.

Case 4:  $i = j + 1$  and  $\gamma_j \in \text{Dom}(p^{**})$

Again  $\gamma_i = \gamma_j + \delta(*)$  and  $\gamma_j \in u_{\varepsilon(*)}$ . First we find  $p'_j$  such that:

- $\otimes_8$  (a)  $p_j^+ \leq p'_j \in \mathbb{P}_{0,\gamma_j}$
- (b) if  $\gamma \in \text{Dom}(p_j^+)$  then  $p'_j \upharpoonright \gamma \Vdash \text{“} \ell g(\eta^{p'_j(\gamma)}) > i \text{”}$
- (c)  $p'_j$  forces <sup>4</sup> a value to the pair  $(\eta^{p^{**}(\gamma_i)}, f^{p^{**}(\gamma_j)} \upharpoonright \lambda_{\varepsilon(*)})$ ; we call this pair  $q_j$ .

This should be clear.

Second

$\otimes_9$   $p_j^+$  hence  $p'_j$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic and  $\langle \nu_{\gamma_j(1)} : j(1) < j \rangle$  induces the generic.

---

<sup>4</sup>recall that  $\eta^{p^{**}(\gamma)}$  is an object, not a name and  $p_j^+$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic

[Why? As in the proof of  $\textcircled{*}_7''$  above when we assume that we have carried the induction, by  $\textcircled{\boxplus}_2$ , clause (g) and  $\textcircled{*}_4$ .]

Now

- $\textcircled{*}_{10}$  (a)  $f^{q_j} \in (\prod_{\zeta < \lambda_{\varepsilon(*)}} \theta_\zeta)^{N'_{\varepsilon(*)}[\bar{v}^j]}$   
 (b) for some  $\zeta \in (\xi(i), \sigma)$  we have
- $f^{q_j} \leq \eta_\zeta^{\varepsilon(*)} \bmod \mathcal{J}_{\lambda_\varepsilon}^{\text{bd}}$
  - $f^{q_j} \in N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$
  - $\langle \zeta(j_1) : j_1 < j \rangle \in N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$ .
- (c)  $\eta^{q_j} \triangleleft f^{q_j}$ .

[Why? Clause (a) follows from clause (b) and clause (b) should be clear by  $\textcircled{*}_9$  as we can choose  $\zeta(i)$  large enough recalling  $\textcircled{*}_6$ . Also clause (c) follows from (b).]

Now we choose  $\zeta(j)$  as in clause (b) of  $\textcircled{*}_{10}$  and  $\nu_j \in \prod_{\varepsilon < \lambda_{\varepsilon(*)}} \theta_\varepsilon$  such that  $\eta^{p^+(j)} \triangleleft$

$\nu_j$ ,  $f^{q_j} \leq \nu_j$  and  $\{\iota < \lambda_{\varepsilon(*)} : \nu_j(\iota) \neq \eta_{\zeta(j)}^{\varepsilon(*)}\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ . Next choose  $p_i^+ \in \mathbb{P}'_{\gamma(*)}$  such that  $p_i^+ \upharpoonright \gamma_j = p_j'$ ,  $\eta^{p_i^+(\gamma_i)} = \nu_j$  and  $f^{p_i^+(\gamma_i)} \upharpoonright [\lambda_\varepsilon, \lambda] = f^{p^+(\gamma)} \upharpoonright [\lambda_\varepsilon, \lambda]$ .

So we have carried the induction hence proved  $\textcircled{*}_6$  so we are done.  $\square_{1.3}$

{a19}

*Concluding Remark 1.7.* 1) In Theorem 1.1 and Lemma 1.3(2) we can omit “ $\kappa > \lambda^+$ ”; still  $\kappa > \lambda$  is demanded. For this in the proof of 1.3, the assumption  $\kappa > \lambda^+$  is technical, to allow  $\kappa = \lambda^+$  we should just use  $\delta(*)\kappa$  instead of  $\kappa$ .

2) Is the use of  $\bar{g} \upharpoonright \mathcal{U}_*$  rather than  $\bar{g}$  in the proof necessary? See on this [Sh:F979].

3) Why is  $\square_\lambda$  still true in  $\mathbf{V}^\mathbb{P}$  when  $\mathbb{P}$  is  $(< \lambda)$ -strategically complete when in  $\mathbb{P}$  any directed increasing system of cardinality  $< \lambda$  of conditions in  $\mathbb{P}$  has an upper bound?

Let  $\mathbb{Q}$  be a forcing notion in  $\mathbf{V}^\mathbb{P}$  which is  $(< \lambda)$ -strategically complete  $\chi_1$  large enough and we should prove that in  $\mathbf{V}^\mathbb{P}$ , the set  $\mathcal{S}_\mathbb{Q}$  is stationary. So let  $\mathbb{Q}, \bar{E}$  be  $\mathbb{P}$ -names such that for some  $p \in \mathbb{P}$  we have  $p \Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{H}(\chi_1) \text{ is } (< \lambda)\text{-strategically complete forcing notion, } \bar{E} \text{ a club of } [\mathcal{H}(\chi_1)]^{< \lambda} \text{ disjoint to } \mathcal{S}_\mathbb{P}\text{”}$  (no need to use a name for  $\chi_1$  as we can increase it).

Let  $\chi \gg \chi_1$ ; now  $\mathbb{P} * \mathbb{Q} \in \mathcal{H}(\chi)$  is a  $(< \lambda)$ -strategically complete forcing notion and without loss of generality code  $(\chi_1, \bar{E})$ . As  $\square_\lambda$  holds in  $\mathbf{V}$  we can find  $N, \lambda_N, \chi_N, \mathbf{j}_N, N'_N, M_N, \mathbf{G}_N$  such that  $\mathbb{P} * \mathbb{Q} \in N$  so  $\chi_1, \bar{E} \in N$ . Let  $P_N = (p_1, p_2)$  so  $p_2 \in \mathbb{P}$  and  $\Vdash_{\mathbb{P}} \text{“} p_2 \in \mathbb{Q}\text{”}$ . Let  $\mathbf{G}_p$  be a subset of  $\mathbb{P}$  generic over  $\mathbf{V}$  and  $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_p], N_1 = N[\mathbf{G}_p], E_1 = \bar{E}[\mathbf{G}_p], N'_1 = N'[\mathbf{j}''(\mathbf{G}_p \cap N)] = N'[\{p' : (p', q') \in \mathbf{G}\}], \mathbf{G}_1 = \{q[\mathbf{j}''(\mathbf{G}_p \cap N)] : (p, q) \in \mathbf{G}\}$ .

Let  $N_2 = N_2 \upharpoonright \mathcal{H}(\chi_1)^{\mathbf{V}[\mathbf{G}_1]}, \mathcal{S} = \mathcal{S}_\mathbb{Q}, \mathbf{j}_1 =$  the lifting of  $(\mathbf{j} \upharpoonright (N \cap \mathcal{H}(\chi)))$ , to mapping  $N_1$  onto  $N'_1$ .

Now clearly

$$(*) N_2 \in E_1$$

hence

$$(*) N_1 \notin \mathcal{S}_1.$$

But easily  $\lambda_N, \chi_N, \mathbf{j}_1, N'_1, M_1 = M, \mathbf{G}_1 = \mathbf{g}[\mathbf{G}_p]$  witness  $N_1 \in \mathcal{S}_1 \cap E_1$ , a contradiction to the choice of  $\bar{E}$ .

§ 2. A CORRECTED ( $< \lambda$ )-SUPPORT ITERATION

**Explanation 2.1.** For  $\mathbf{m} \in \mathbf{M}$  below

- (a) we use  $L_{\mathbf{m}}$  is the index set for the iteration; really it can be a well ordering (see 2.11), a partial order seem more transparent
- (b)  $M_{\mathbf{m}}$  is the part of the index set we are really in, it may be  $(\kappa, <)$  in §1
- (c) the other part is “generic enough” so that the iteration restricted to  $M$  will be homogenous.

{c0}

**Hypothesis 2.2.** 1)  $\lambda = \lambda^{<\lambda}$  strongly inaccessible.

2)  $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ .

3)  $\theta_\varepsilon$  is an infinite regular cardinal  $> \varepsilon$  and  $< \lambda$ .

4) Assume  $\lambda_0 \geq \lambda$ .

{c1}

*Remark 2.3.* 1) Here no harm in adding

(a)  $\theta_\varepsilon > \prod_{\zeta < \varepsilon} 2^{\theta_\zeta} + 2^{\aleph_0}$  for  $\varepsilon < \lambda$   
or just

(b)  $\bar{\theta}$  is increasing fast enough

(c)  $M$  a partial order, well founded (and even  $M \cong (\kappa, <)$ ,  $\kappa$  regular  $> \lambda$ ).

{c2}

**Definition 2.4.** 1) For a partial order  $L$  let

( $\alpha$ )  $\text{dp}(L) = \cup\{\text{dp}_L(t) + 1 : t \in L\}$ , see below

( $\beta$ )  $\text{dp}_L(t) = \text{dp}(t, L) \in \text{Ord} \cup \{\infty\}$  be defined by  $\text{dp}_L(t) = \cup\{\text{dp}_L(s) + 1 : s <_L t\}$ .

( $\gamma$ )  $L_{<t} = L \upharpoonright \{s \in L : s <_L t\}$

( $\zeta$ )  $L_{\leq t} = L \upharpoonright \{s \in L : s \leq_L t\}$ .

2) Let  $L^+ = L(+)$  be  $L \cup \{\infty\}$  with the natural order (but we may write  $t <_L \infty$  instead of  $t <_{L(+)} \infty$ ).

3) We say the set  $L$  is an initial segment of the partial order  $L_*$  when  $s \in L \Rightarrow s \in L_*$  and  $s <_{L_*} t \wedge t \in L \Rightarrow s \in L$ .

{c4}

**Definition 2.5.** Let  $\mathbf{M}$  be the class of objects  $\mathbf{m}$ , called iteration parameters, of the following form:

(a)  $L$ , a partial order

(b)  $M \subseteq L$

(c) ( $\alpha$ )  $\bar{u} = \langle u_t : t \in L \rangle$

( $\beta$ )  $u_t \subseteq \{s \in L : s <_L t\}$

(d)  $\text{dp}(L) < \infty$

(e) ( $\alpha$ )  $E'$  is a two-place relation

( $\beta$ )  $E'' = E' \upharpoonright (L \setminus M)$  is an equivalence relation on  $L \setminus M$

( $\gamma$ ) if  $s, t \in L \setminus M$  are not  $E''$ -equivalent then

$(s <_L t) \Leftrightarrow (\exists r \in M)(s < r < t)$

( $\delta$ ) if  $sE't$  then  $s \notin M \vee t \notin M$

( $\varepsilon$ ) if  $t \in L \setminus M$  then  $\{s \in L : sE't\} = \{s \in L : tE's\}$ ;

we call it  $t/E'$

( $\zeta$ ) if  $s, i \in L \setminus M$  are  $E''$ -equivalent then  $s/E' = t/E'$ . {c5}

*Notation 2.6.* For  $\mathbf{m} \in \mathbf{M}$ .

- 0) In 2.5 we have  $\mathbf{m} = (L_{\mathbf{m}}, M_{\mathbf{m}}, \bar{u}_{\mathbf{m}}, E'_{\mathbf{m}})$  and  $\bar{u}_{\mathbf{m}} = \langle u_{\mathbf{m},t} : t \in L_{\mathbf{m}} \rangle$ .
- 1) In 2.5, let  $\text{dp}_{\mathbf{m}}(t) = \text{dp}_{L_{\mathbf{m}}}(t)$ ,  $\text{dp}_{\mathbf{m}} = \text{dp}(L_{\mathbf{m}})$  and  $\leq_{\mathbf{m}} = \leq_{L_{\mathbf{m}}}$ .
- 2) For  $L \subseteq L_{\mathbf{m}}$  let  $\mathbf{n} = \mathbf{m} \upharpoonright L$  means  $\mathbf{n} \in \mathbf{M}$ ,  $L_{\mathbf{n}} = L$ ,  $\leq_{\mathbf{n}} = \leq_{\mathbf{m}} \upharpoonright L_{\mathbf{n}}$ ,  $u_{\mathbf{n},t} = u_{\mathbf{m},t} \cap L_{\mathbf{m}}$  and  $M_{\mathbf{n}} = M_{\mathbf{m}} \cap L$ ; let  $\text{dp}_{\mathbf{m}}(L) = \text{dp}(L_{\mathbf{m}} \upharpoonright L)$ .
- 3) For  $t \in L_{\mathbf{m}}$ , let  $\mathbf{m}_{<t} = \mathbf{m}(\langle t \rangle) = \mathbf{m} \upharpoonright L_{<t}$  where  $L_{<t} = L_{\mathbf{m},<t} = \{s : s <_{\mathbf{m}} t\}$ .
- 3A) Also  $\mathbf{m}_{\leq t} = \mathbf{m}(\leq t) = \mathbf{m} \upharpoonright L_{\leq t}$  where  $L_{\leq t} = L_{\mathbf{m},\leq t} = L_{<t} \cup \{t\}$ ; let  $L_{<\infty} = L$ , etc.
- 4)  $\mathbf{M}_{<\mu}$  is the class of  $\mathbf{m} \in \mathbf{M}$  such that  $L_{\mathbf{m}}$  has cardinality  $< \mu$ . Similarly  $\mathbf{M}_{\leq \mu}$ ,  $\mathbf{M}_{=\mu}$ ,  $\mathbf{M}_{>\mu}$ ,  $\mathbf{M}_{\geq \mu}$ ; let  $\mathbf{M}_{\mu} = \mathbf{M}_{=\mu}$ .
- 5) For  $\mathbf{m}, \mathbf{n} \in \mathbf{M}$  let  $\mathbf{m} \approx \mathbf{n}$ , we may say  $\mathbf{m}, \mathbf{n}$  are equivalent mean  $L_{\mathbf{m}} = L_{\mathbf{n}}$  and  $t \in L_{\mathbf{n}} \Rightarrow u_{\mathbf{m},t} = u_{\mathbf{n},t}$ .

**Definition 2.7.** For  $\mu \geq \lambda^+$  and  $\mathbf{m} \in \mathbf{M}_{<\mu}$  let  $L = L_{\mathbf{m}}$  and we define the iteration  $\mathbf{q}_{\mathbf{m}}$  to consist of: {c6}

- (a) a forcing notion  $\mathbb{P}_t$  for  $t \in L^+$ ; we let  $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\infty}$
  - (b)  $\mathbb{Q}_t$  is the  $\mathbb{P}_t$ -name of  $\mathbb{Q}_{\bar{\theta}}$  in some sub-universe of  $\mathbf{V}^{\mathbb{P}_t}$  extending  $\mathbf{V}$  for  $t \in L$ , see below
  - (c)  $p \in \mathbb{P}_t$  iff
    - ( $\alpha$ )  $p$  is a function
    - ( $\beta$ )  $\text{Dom}(p) \subseteq L_{<t}$  has cardinality  $< \lambda$
    - ( $\gamma$ ) if  $\xi \in \text{Dom}(p)$  then for some  $\xi = \xi_{p,t} \leq \lambda$  and  $\bar{r} = \bar{r}_{p,t} = \langle r(\zeta) : \zeta < t \rangle = \langle r_{p,t}(\zeta) : \zeta < \xi_{p,t} \rangle \in {}^\xi(u_t)$  we have  $p(t)$  is a  $\lambda$ -Borel function<sup>5</sup>,  $\mathbf{B} = \mathbf{B}_{p,t} : {}^\xi(\prod_{\varepsilon < \lambda} \theta_\varepsilon) \rightarrow \mathbb{Q}_{\bar{\theta}}$ ; less pedantically  $p(t) = \mathbf{B}(\dots, \eta_{r_{p,t}(\zeta)}, \dots)_{\zeta < \xi}$
    - ( $\delta$ ) moreover  $\text{tr}(p(t))$ , the trunk of  $p(t)$  is well defined, i.e. the trunk of  $\mathbf{B}(\dots, \eta_\zeta, \dots)_{\zeta < \xi}$  does not depend on  $\langle \eta_\zeta : \zeta < \xi \rangle$ , or the universe
  - (d) ( $\alpha$ )  $\eta_s$  is the  $\mathbb{P}_t$ -name when  $t \in L_{\mathbf{m}}^+$ ,  $s \in L_{<t}$  defined by  $\cup \{\text{tr}(p(s)) : p \in \mathbf{G}_{\mathbb{P}_t}\}$ .
  - ( $\beta$ ) For  $p \in \mathbb{P}_t$  and  $s \in \text{Dom}(p)$  we interpret  $p(s)$  as a  $\mathbb{P}_s$ -name  $\mathbf{B}_{p,s}(\dots, \eta_{r_{p,s}(\zeta)}, \dots)_{\zeta < \xi_{p,s}}$
- (e)  $\mathbb{P}_t \models "p \leq q"$  iff
    - ( $\alpha$ )  $p, q \in \mathbb{P}_t$
    - ( $\beta$ )  $\text{Dom}(p) \subseteq \text{Dom}(q)$
    - ( $\gamma$ ) if  $t \in \text{Dom}(p)$  then  $(q \upharpoonright L_{<t}) \Vdash_{\mathbb{P}_{\mathbf{m}(\langle t \rangle)}} "p(t) \leq_{\mathbb{Q}_{\bar{\theta}}} q(t)"$ .{c7}

**Definition 2.8.** 1) For  $p \in \mathbb{P}_{\mathbf{m}}$  let  $\text{fsupp}(p)$ , the full support of  $p$  be  $\cup \{r_{p,s}(\zeta) : \zeta < \xi_{p,s}\} \cup \{s\} : s \in \text{Dom}(p)\}$ .

2) For  $\mathbf{m} \in \mathbf{M}$  let  $\mathbb{P}_{\mathbf{m}}^{\mathbf{m}} = \mathbb{P}_{\mathbf{m},t}$ , etc., in Definition 2.7. {c8}

**Claim 2.9.** For  $\mathbf{m} \in \mathbf{M}$  (so  $\mathbb{P}_t = \mathbb{P}_{\mathbf{m},t}$ , etc.)

- (a) the iteration  $\mathbf{q}_{\mathbf{m}}$  is well defined
- (b) if  $t \in L_{\mathbf{m}}^+$  then  $\mathbb{P}_t$  is indeed a forcing notion and is equal to  $\mathbb{P}_{\mathbf{m}(\langle t \rangle)}$
- (c) if  $s <_L t$  are from  $L_{\mathbf{m}}^+$  then

<sup>5</sup>that is a definition of one

- ( $\alpha$ )  $p \in \mathbb{P}_s \Rightarrow p \in \mathbb{P}_t \wedge p \upharpoonright L_{<s} = p$
- ( $\beta$ ) if  $p, q \in \mathbb{P}_s$  then  $\mathbb{P}_t \models "p \leq q" \Leftrightarrow \mathbb{P}_s \models "p \leq q"$
- ( $\gamma$ )  $p \in \mathbb{P}_t \Rightarrow p \upharpoonright L_{<s} \in \mathbb{P}_s$
- ( $\delta$ )  $\mathbb{P}_t \models "p \leq q" \Rightarrow \mathbb{P}_s \models "p \upharpoonright L_{<s} \leq q \upharpoonright L_{<s}"$
- ( $\varepsilon$ )  $\mathbb{P}_s \triangleleft \mathbb{P}_t$  moreover
- ( $\zeta$ )  $p \in \mathbb{P}_t \wedge (p \upharpoonright L_{<s}) \leq q \in \mathbb{P}_s \Rightarrow q \cup (p \upharpoonright (L_{<t} \setminus L_{<s})) \in \mathbb{P}_\beta$  is a  $\leq$ -lub of  $p, q$ .

*Proof.* Straight. For  $t \in L^+$ , by induction on  $\text{dp}_{\mathbf{m}}(t)$ , define  $\mathbb{P}_t$  and prove the relevant parts of (a),(b),(c).  $\square_{2.9}$

{c11}

**Claim 2.10.** Let  $\mathbf{m} \in \mathbf{M}$ .

0) If  $L_{\mathbf{m}}^+ \models "s < t"$  and  $u = u_{\mathbf{m},s}$  then

- ( $\alpha$ )  $\Vdash_{\mathbb{P}_{\mathbf{m},t}} " \eta_s \in \prod_{i < \lambda} \theta_i "$
- ( $\beta$ ) if  $\mathbf{G} \subseteq \mathbb{P}_t$  is generic over  $\mathbf{V}$  and  $\eta_i = \eta_r[\mathbf{G}]$  for  $r \in L_{\mathbf{m},<t}$  and  $\nu \in \mathbf{V}[\langle \eta_r : r \in u_s \rangle] \subseteq \mathbf{V}[\mathbf{G}]$  then  $\nu <_{J_\lambda^{\text{pd}}} \eta_s$ .

- 1)  $\mathbb{P}_{\mathbf{m}}$  satisfies the  $\lambda^+$ -c.c.
- 2)  $\mathbb{P}_{\mathbf{m}}$  is  $(< \lambda)$ -strategically complete (even  $(\lambda)$ -strategically complete but not used)
- 3)  $\mathbb{Q}_t = \mathbb{Q}_{\bar{\theta}}[\mathbf{V}[\langle \eta_s : s \in u_\gamma \rangle]]$ .
- 4) If  $\bar{p} = \langle p_i : i < \delta \rangle$  is  $\leq_{\mathbb{P}_{\mathbf{m}}}$ -increasing,  $\delta < \lambda$  and  $i < j < \delta \wedge t \in \text{Dom}(p_i) \Rightarrow \text{tr}(p_i(t)) \triangleleft \text{tr}(p_j(t))$  then  $\bar{p}$  has a  $\leq_{\mathbb{P}_{\mathbf{m}}}$ -upper bound.
- 5) If  $p \in \mathbb{P}_{\mathbf{m}}$  then for some  $q \in \mathbb{P}_{\mathbf{m}}$  we have  $p \leq q$  and  $t \in \text{Dom}(p) \Rightarrow \text{tr}(p(t)) \triangleleft \text{tr}(q(t))$ .
- 6) If  $\bar{q}$  is a  $\mathbb{P}_{\mathbf{m}}$ -name of a member of  $\mathbb{Q}_{\bar{\theta}}$  then for some  $\xi \leq \lambda$  and  $\lambda$ -Borel function  $\mathbf{B} : {}^\xi(\mathbb{Q}_{\bar{\theta}}) \rightarrow \mathbb{Q}_{\bar{\theta}}$  and sequence  $\langle r_\zeta : \zeta < \xi \rangle$  of members of  $L_{\mathbf{m}}$  we have  $\Vdash_{\mathbb{P}_{\mathbf{m}}} "q = \mathbf{B}(\dots, \eta_{r_\zeta}, \dots)_{\zeta < \xi}"$ .
- 7) If  $\mathbf{m}, \mathbf{n}$  are equivalent then  $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{n}}$  and  $\mathbb{P}_{\mathbf{m},t} = \mathbb{P}_{\mathbf{n},t}$  for  $t \in L_{\mathbf{m}} = L_{\mathbf{n}}$ .

*Proof.* We prove all parts by induction on  $\text{dp}_{\mathbf{m}}$ .

0) For each  $\mathbf{m}$ , using the induction hypothesis, the problem is only when  $\text{dp}_{\mathbf{m}}(t) = \text{dp}_{\mathbf{m}} - 1$  and use part (6) for  $\mathbb{P}_{\mathbf{m},t}$ .

1) If  $p_\varepsilon \in \mathbb{P}_{\mathbf{m}}$  for  $\varepsilon < \lambda^+$  then we can find  $u$  and unbounded  $S \subseteq \lambda^+$  such that  $\varepsilon \neq \zeta \in S \Rightarrow \text{Dom}(p_\varepsilon) \cap \text{Dom}(p_\zeta) = u$  and  $\langle \text{tr}(p_\varepsilon(\beta)) : \beta \in u \rangle$  is the same for every  $\varepsilon \in S$ . Now  $p_\varepsilon, p_\zeta$  has a common upper bound for every  $\varepsilon, \zeta \in u$ , i.e. we define  $r$  by

- $\text{Dom}(r) = \text{Dom}(p_\varepsilon) \cup \text{Dom}(p_\zeta)$
- $r(\alpha) = p_\varepsilon(\alpha)$  if  $\alpha \in \text{Dom}(p_\varepsilon) \setminus \text{Dom}(p_\zeta)$
- $r(\alpha) = p_\zeta(\alpha)$  if  $\alpha \in \text{Dom}(p_\zeta) \setminus \text{Dom}(p_\varepsilon)$
- if  $\alpha \in u = \text{Dom}(p_\varepsilon) \cap \text{Dom}(p_\zeta)$  and  $p_\varepsilon(\alpha) = (\eta, \underline{f}_\varepsilon), p_\zeta(\alpha) = (\eta, \underline{f}_\zeta)$  for some  $\eta$  then  $r(\alpha) = (\eta, \max\{\underline{f}_\varepsilon, \underline{f}_\zeta\})$ .

2) By (4)+(5) below.

3) Should be clear.

4) We define  $p$  by:

- $\text{Dom}(p) = \cup \{ \text{Dom}(p_i) : i < 0 \}$
- $\text{tr}(p(\alpha)) = \cup \{ \text{tr}(p_i(\alpha)) : i < \delta \text{ satisfies } \alpha \in \text{Dom}(p_i) \}$

- $f_{p(\alpha)} = \sup\{f_{p_i(\alpha)} : i < \delta \text{ satisfies } \alpha \in \text{Dom}(p_i)\}$ .

5) We prove by induction on  $\iota < \lambda$  that:

- (\*) if  $p \in \mathbb{P}_m$  then for some  $q \in \mathbb{P}_m$  we have  $p \leq q$  and  $t \in \text{Dom}(p) \wedge \ell g(\text{tr}(p(t))) < \iota \Rightarrow \ell g(\text{tr}(q(t))) \geq \iota$ .

First note

- ⊞ if  $\langle p_i : i < \delta \rangle$  is  $\leq_{\mathbb{P}_m}$ -increasing, it has a  $\leq_{\mathbb{P}_m}$ -upper bound when
- if  $i < \delta \wedge t \in \text{Dom}(p_i) \wedge \ell g(\text{tr}(p_i(t))) < i$  then  $(\exists j)(i < j < \delta \wedge \ell g(\text{tr}(p_i(t))) < \ell g(\text{tr}(p_j(t)))$ ; can add  $\ell g(\text{tr}(p_i(t))) < \text{cf}(\delta)$ .

Why does ⊞ holds? As in the proof of part (4).

If  $\iota = 0$  this is trivial, if  $\iota$  is a successor ordinal let  $\langle t_i : i < i(*) \rangle$  list  $\text{Dom}(p)$  such that  $t_1 \leq_L t_j \Rightarrow i < j$  and use 2.9(c)( $\zeta$ ), the induction hypothesis and ⊞ is used  $i(*) + 1$  times.

Lastly, if  $\iota$  is a limit ordinal use the induction hypothesis and ⊞ above.

6) By parts (0) + (1) for each  $\zeta < \lambda$  the following subset of  $\mathbb{P}_{m,t}$  is open and dense:  $\mathcal{S}_\zeta = \{p \in \mathbb{P}_{m,t} : \text{for some } \nu \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ (from } \mathbf{V}!) \text{ we have } p \Vdash_{\mathbb{P}_{m,t}} \text{“}\eta_s[\varepsilon = \nu]\text{”}\}$ . Clearly

there is a maximal antichain  $\langle p_{\zeta,\varepsilon} : \varepsilon < \xi_\zeta \rangle$  of  $\mathbb{P}_{m,t}$  included in  $\mathcal{S}_\tau$  and by part (1) without loss of generality  $\varepsilon < \lambda$ , the rest should be clear.

7) Look at the definitions. □<sub>2.10</sub>

**Conclusion 2.11.** Let  $m \in \mathbf{M}$  and for notational transparency for some ordinal  $\beta(*), t \in L_m \Leftrightarrow t \in \beta(*)$  and  $s <_m t \Rightarrow s < t$ . Then  $\mathbf{q}$  is essentially a  $(< \lambda)$ -support iteration with  $\mathbb{Q}_\alpha = \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \eta_\beta : \beta \in U_\alpha \rangle]}$ . {c13}

*Proof.* Should be clear. □

**Claim 2.12.** Let  $m \in \mathbf{M}$  and  $L_m^+ \models \text{“}s < t\text{”}$ . {c15}

1) The following is a dense open subset of  $\mathbb{P}_t$ :

$$\mathcal{S}_{s,t} = \{p \in \mathbb{P}_t : s \in \text{Dom}(p)\}.$$

2) If  $\zeta < \lambda$  then the following is a dense open subset of  $\mathbb{P}_t$ :

$$\mathcal{S}_{s,t,\zeta} = \{p \in \mathbb{P}_t : s \in \text{Dom}(p) \text{ and } \text{tr}(p(s)) \text{ has length } \geq \zeta\}.$$

3) If  $p \in \mathcal{S}_{s,t}$  and  $\zeta_0 = \ell g(\text{tr}(p(s)))$  and  $\zeta_1 < \zeta_0 < \zeta$  then for every  $i < \theta_\varepsilon$  large enough there is  $q \in \mathbb{P}_t$  such that

- $\mathbb{P}_t \models \text{“}p \leq q\text{”}$
- if  $r \in \text{Dom}(p) \setminus L_{m,\leq s}$  then  $q(r) = p(r)$
- if  $r \in \text{Dom}(p) \cap L_{m,< s}$  then  $\text{tr}(p(r)) \triangleleft \text{tr}(q(r))$
- $\text{tr}(q(s))$  has length  $\zeta + 1$
- $q(s)(\zeta) = i$ .

4) If  $p \in \mathcal{S}_{s,t}$  and  $\zeta_0 < \zeta_1 < \lambda, \zeta_0 = \ell g(\text{tr}(p(s)))$  and  $\bar{u} = \langle u_\varepsilon : \zeta_0 \leq \varepsilon < \zeta_1 \rangle$  where  $u_\varepsilon$  is an unbounded subset of  $\theta_\varepsilon$  for  $\varepsilon \in [\zeta_0, \zeta_1)$  then there is  $q \in \mathbb{P}_t$  above  $p$  such that  $p \Vdash_{\mathbb{P}_t} \text{“} \text{name}_{\eta_s}(\varepsilon) \in u_\varepsilon \text{ for } \varepsilon \in [\zeta_0, \zeta_1)\text{”}$ .

*Proof.* Should be clear. □<sub>2.11</sub>

{c19}

**Definition 2.13.** Let  $\mathbf{m} \in \mathbf{M}$ .

- 1) For  $t \in L_{\mathbf{m}}$  and  $\eta \in \prod_{i < \varepsilon} \theta_i$  let  $p = p_{t,\eta}^* \in \mathbb{P}_{\mathbf{m}}$  be the function with domain  $\{t\}$  such that  $p(t) = (\eta, \eta \hat{\ } 0_\lambda)$ , i.e.  $f_{p(t)} \in \prod_{i < \lambda} \theta_i$  is defined by  $f_{p(t)}(i)$  is  $\eta(i)$  if  $i < \ell g(\eta)$  and is zero otherwise.
- 2) For  $L \subseteq L_{\mathbf{m}}$  let  $Y_L = \{p_{t,\eta}^* : \gamma \in L \text{ and } \eta \in \prod_{i < \varepsilon} \theta_i \text{ for some } \varepsilon < \lambda\}$ .
- 3) For  $L \subseteq L_{\mathbf{m}}$  let  $\mathbb{P}_{\mathbf{m}}[L]$  be  $\mathbb{L}_{\lambda^+}[Y_L, \mathbb{P}_{\mathbf{u}}]$ , see Definition 0.6, 0.7.
- 4) For  $L \subseteq L_{\mathbf{m}}$  let  $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} \upharpoonright \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$ , see Definition 2.8(1).

{c23}

**Claim 2.14.** Let  $\mathbf{m} \in \mathbf{M}$  and  $L \subseteq L_{\mathbf{m}}$ .

- 1)  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  is equivalent to  $\mathbb{P}_{\mathbf{m}}$ , in fact  $\mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  and is a dense subset of it under the natural identification (see 0.6, 0.7).
- 2)  $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  is  $(< \lambda)$ -strategically complete and is  $\lambda^+$ -c.c.; also  $\mathbb{P}_{\mathbf{m}}(L)$  is.
- 3)  $\mathbb{P}_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}[L] \triangleleft \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ .
- 4)  $\mathbb{P}_{\mathbf{m}}[L]$  is  $(< \lambda)$ -strategically complete,  $\lambda^+$ -c.c.
- 5) If  $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}$  is generic over  $\mathbf{V}$  and  $\eta_t = \eta_t[\mathbf{G}]$  for  $t \in L$  then  $\mathbf{V}[\mathbf{G} \cap \mathbb{P}_{\mathbf{m}}[L]] = \mathbf{V}[\langle \eta_t : t \in L \rangle]$ .
- 6)  $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_2)$  and  $\mathbb{P}_{\mathbf{m}}[L_1] \triangleleft \mathbb{P}_{\mathbf{m}}[L_2]$  when  $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$ .
- 7) If  $\mathbf{m}, \mathbf{n} \in \mathbf{M}$  are equivalent and  $L \subseteq L_{\mathbf{m}}$  then  $\mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$  and  $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$ .
- 8) Assume  $I_*$  be a  $\lambda_0^+$ -directed partial order and  $\bar{L} = \langle L_r : r \in I_* \rangle$  be such that  $r \in I_* \Rightarrow L_t \subseteq L_m$  as  $r <_{I_*} s \Rightarrow L_r \subseteq L_s$  and  $L = \cup \{L_r : r \in I_*\}$ . Then  $\mathbb{P}_{\mathbf{m}}[L] = \cup \{\mathbb{P}_{\mathbf{m}}[L_r] : r \in I_*\}$  and  $\mathbb{P}_{\mathbf{m}}(L) = \cup \{\mathbb{P}_{\mathbf{m}}(L_r) : r \in I_*\}$ .

*Proof.* 1) Easy.

2) Follows by part (1) and 2.10.

3) " $\mathbb{P}_{\mathbf{m}}[L]$  satisfies the  $\lambda^+$ -c.c. because  $\mathbb{P}_{\mathbf{m}}[L] \triangleleft \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$  and part (2), and similarly  $\mathbb{P}_{\mathbf{m}}[L]$  is  $(< \lambda)$ -strategically complete.

4) By part (2) and (3).

5), 6) Should be clear.

7) Easy, recalling 2.10(7).

8) Easy. □<sub>2.14</sub>Till now  $(E_{\mathbf{m}}, M_{\mathbf{m}})$  have played no role and we could have omitted them.

{c26}

**Definition 2.15.** We define the two-place relation  $\leq = \leq_{\mathbf{M}}$  on  $\mathbf{M} : \mathbf{m} \leq \mathbf{n}$  iff

- (a)  $L_{\mathbf{m}} \subseteq L_{\mathbf{n}}$  as partial orders
- (b)  $M_{\mathbf{m}} = M_{\mathbf{n}}$
- (c)  $u_{\mathbf{m},t} = u_{\mathbf{n},t} \cap L_{\mathbf{m}}$  for  $t \in M$
- (d)  $u_{\mathbf{m},t} = u_{\mathbf{n},t}$  for  $t \in L_{\mathbf{m}} \setminus M$
- (e) if  $t \in L_{\mathbf{m}} \setminus M$  then  $t/E'_{\mathbf{m}} = t/E'_{\mathbf{n}}$  hence  $E'_{\mathbf{m}} = E'_{\mathbf{n}} \upharpoonright L_{\mathbf{m}}$ .

{c29}

**Definition 2.16.** 1) For  $\mathbf{m} \in \mathbf{M}$  let  $\text{cmp}(\mathbf{m})$  be the set of  $L$  such that:

- (a)  $L \subseteq L_{\mathbf{m}}$
- (b) if  $L \subseteq M$  then  $|L| \leq 1$
- (c) if  $t \in L \setminus M$  then  $L = t/E$  hence
  - (α)  $L \setminus M \neq \emptyset$  and  $L \setminus M$  is an  $E_{\mathbf{m}}$ -equivalence class

( $\beta$ ) if  $t \in L \setminus M$  then  $u_{\mathbf{m},t} \subseteq L$ .

2) For  $\mathbf{m} \in \mathbf{M}$

- (a) let  $\text{sub}(\mathbf{m})$  be the set of  $L_1$  such that for some  $L_2 \in \text{cmp}(\mathbf{m})$ ,  $L_1$  is an initial segment of  $L_2$ , i.e.  $L_2 \subseteq L_1 \wedge (\forall s, t)[s \in L_2 \wedge t \in L_1 \wedge s <_{\mathbf{m}} t \rightarrow s \in L_1]$
- (b) let  $\text{sub}_*(\mathbf{m}) = \{L \in \text{sub}(\mathbf{m}) : \text{there is } L_1 \subseteq L \text{ of cardinality } \leq \lambda_0 \text{ such that } L = \{s \in L : (\exists t)(s \leq_{\mathbf{m}} t \in L_1)\}$ .

3) Let

- (a)  $\text{Cmp}(\mathbf{m}) = \{ \bigcup_{\varepsilon < \zeta} L_\varepsilon : L_\varepsilon \in \text{cmp}(\mathbf{m}) \text{ for } \varepsilon < \zeta \}$
- (b)  $\text{Cmp}_*(\mathbf{m}) = \{ \bigcup_{\varepsilon < \zeta} L_\varepsilon : L_\varepsilon \in \text{cmp}(\mathbf{m}) \text{ for } \varepsilon < \zeta \text{ and } \zeta \leq \lambda_0 \}$
- (c)  $\text{Sub}(\mathbf{m}) = \{L_1 : \text{for some } L_2 \in \text{Cmp}(\mathbf{m}), L_1 \text{ is an initial segment of } L_2\}$
- (d)  $\text{Sub}_*(\mathbf{m}) = \{ \bigcup_{\varepsilon < \zeta} L_\varepsilon : \zeta \leq \lambda_0 \text{ and } L_\varepsilon \in \text{sub}_*(\mathbf{m}) \text{ for } \varepsilon < \zeta \}$
- (e) in clauses (b),(c) if  $\iota = 0$  we may omit it. {c30}

**Definition 2.17.** 0) For  $L \subseteq L_{\mathbf{m}}, \mathbf{m} \in \mathbf{M}$  let  $\text{dp}_{\mathbf{m}}^*(L) = \cup\{\text{dp}_{M_{\mathbf{m}}}(t) + 1 : t \in L \cap M_{\mathbf{m}}\}$ .

1) For an ordinal  $\gamma$  let  $\mathbf{M}_{\gamma}^{\text{ec}}$  be the class of  $\mathbf{m} \in \mathbf{M}$  such that:

(\*) if  $L \in \text{Sub}(\mathbf{m})$  and  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$  then  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$ .

2) Let  $\mathbf{M}^{\text{ec}} = M_{\infty}^{\text{ec}}$  be the class of  $\mathbf{m}$  which  $\in \mathbf{M}_{\gamma}^{\text{ec}}$  for every  $\gamma$ .

3) Let  $\mathbf{M}_{\chi, \gamma}^{\text{ec}} = \{\mathbf{m} \in \mathbf{M}_{\gamma}^{\text{ec}} : |L_{\mathbf{m}}| \leq \chi\}$ , similarly  $\mathbf{M}_{\chi, \infty}^{\text{ec}}$ .

4) Let  $\mathbf{M}_{\text{fc}}$  be the class of  $\mathbf{m} \in \mathbf{M}$  satisfying  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \Rightarrow \mathbb{P}_{\mathbf{m}} < \mathbb{P}_{\mathbf{n}}$ . {c32}

**Observation 2.18.** 1)  $\leq_{\mathbf{M}}$  is a partial order on  $\mathbf{M}$ .

2)  $\mathbf{M}_0^{\text{ec}} = \mathbf{M}$  and  $\mathbf{M}_{\chi, 0}^{\text{ec}} = \mathbf{M}_{\chi}$ .

3) If  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $L \in \text{sub}(\mathbf{m})$  then  $\mathbb{P}_{\mathbf{m}}(L), \mathbb{P}_{\mathbf{n}}(L)$  have the same set of elements.

4) If  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  then  $\text{cmp}(\mathbf{m}) \subseteq \text{cmp}(\mathbf{n})$  moreover  $L \in \text{cmp}(\mathbf{m})$  iff  $L \in \text{cmp}(\mathbf{n}) \wedge L \subseteq L_{\mathbf{m}}$ .

5) Like (4) for  $\text{sub}(-)$ .

6) If  $\chi = \chi^{\lambda_0}, \mathbf{m} \in \mathbf{M}_{\chi}, L \in \text{Sub}(\mathbf{m}), \mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $\mathbb{P}_{\mathbf{m}}[L] \neq \mathbb{P}_{\mathbf{n}}[L]$  then we can find such  $\mathbf{n}$  in  $\mathbf{M}_{\chi}$ .

7) If  $\mathbf{m} \in \mathbf{M}$  and  $t \in L_{\mathbf{m}}$  then  $t \in L$  for some  $L \in \text{cmp}(\mathbf{m})$ .

8) If  $\mathbf{m} \in \mathbf{M}$  and  $L_1 \subseteq L_{\mathbf{m}}$  has cardinality  $\leq \lambda_0$  then  $L_1 \subseteq L_2$  for some  $L_2 \in \text{Cmp}_*(\mathbf{m})$ .

*Proof.* Should be clear.  $\square_{2.18}$

**Observation 2.19.** 1) If  $\langle \mathbf{m}_{\alpha} : \alpha < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing then  $\mathbf{m}_{\delta} := \cup\{\mathbf{m}_{\alpha} : \alpha < \delta\}$  is a  $\leq_{\mathbf{M}}$ -upper bound of  $\langle \mathbf{m}_{\alpha} : \alpha < \delta \rangle$ . {c35}

2) Similarly for  $\leq_{\mathbf{M}}$ .

3) Moreover in part (1) if  $\alpha < \delta \Rightarrow \mathbf{m}_{\alpha} \in \mathbf{M}_{\gamma}^{\text{ec}}$  then:

- ( $\alpha$ ) •  $\text{cmp}(\mathbf{m}_{\delta}) = \cup\{\text{cmp}(\mathbf{m}_{\alpha}) : \alpha < \delta\}$   
 •  $\text{sub}(\mathbf{m}_{\delta}) = \cup\{\text{sub}(\mathbf{m}_{\alpha}) : \alpha < \delta\}$
- ( $\beta$ ) if  $\text{cf}(\lambda) \geq \lambda_0^+$  then  
 •  $\text{Cmp}_*(\mathbf{m}_{\delta}) = \cup\{\text{Cmp}(\mathbf{m}_{\alpha}) : \alpha < \delta\}$

- $\text{Sub}_*^t(\mathbf{m}_\delta) = \cup\{\text{Sub}_*(\mathbf{m}_\alpha) : \alpha < \delta\}$ .

4) If  $\gamma_1 < \gamma_2$  then  $\mathbf{M}_{\gamma_1}^{\text{ec}} \subseteq \mathbf{M}_{\gamma_2}^{\text{ec}}$ .

*Proof.* 1) Easy, remembering that  $t \in \mathbf{m}_\alpha \wedge \alpha \leq \beta < \delta \Rightarrow t/E'_{\mathbf{m}_\beta} = t/E'_{\mathbf{m}_\alpha}$ .

2) Easy.

3),4) Easy, too. □<sub>2.19</sub>

{c38}

**Claim 2.20.** 1) In clause (b) of Definition 2.17(1) we can consider only  $L \in \text{Sub}_*(\mathbf{m})$  such that  $\text{dp}_\mathbf{m}^*(L) < \gamma$ .

2) So if  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}$  and  $\mathbf{m} \leq_M \mathbf{n}$  then  $\mathbb{P}_\mathbf{m} \triangleleft \mathbb{P}_\mathbf{n}$ , that is  $\mathbb{P}_\mathbf{m}[L_\mathbf{m}] \triangleleft \mathbb{P}_\mathbf{n}[L_\mathbf{n}]$  and  $L \in \text{Sub}(\mathbf{m}) \Rightarrow \mathbb{P}_\mathbf{m}[L] \triangleleft \mathbb{P}_\mathbf{n}[L]$ .

3) For  $\mathbf{m} \in \mathbf{M}$  we have:  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}$  iff  $\mathbf{m} \in \mathbf{M}_{\text{fc}}$ , i.e. for every  $\mathbf{n}, \mathbf{m} \leq_M \mathbf{n} \Rightarrow \mathbb{P}_\mathbf{m} \triangleleft \mathbb{P}_\mathbf{n}$ .

4)  $\text{Cmp}_*(\mathbf{m})$  has cardinality  $\leq |L_\mathbf{m}|^{\leq \lambda_0}$ .

5)  $\text{Sub}_*(\mathbf{m})$  has cardinality  $\leq |L_\mathbf{m}|^{\lambda_0}$ .

6)  $|\text{cmp}(\mathbf{m})| \leq |L_\mathbf{m}| + 1$  and  $|\text{sub}_*(\mathbf{m})| \leq |L_\mathbf{m}|^{\lambda_0}$ .

*Proof.* 1) Let  $\mathbf{m} \in \mathbf{M}$  satisfies the weaker version, and let  $L \in \text{Sub}(\mathbf{m})$  satisfies  $\text{dp}_\mathbf{m}^*(L) < \gamma$ . Let  $L$  be  $\cup\{L_\varepsilon : \varepsilon < \zeta\}$  where  $L_\varepsilon \subseteq L_\varepsilon^* \in \text{Cmp}(\mathbf{m})$  and  $L_\varepsilon \in \text{sub}_*(\mathbf{m})$ .

For every  $u \in [\zeta]^{\leq \lambda_0}$  let  $L_u = \cup\{L_\varepsilon : \varepsilon \in u\}$  so clearly  $\text{dp}_\mathbf{m}^*(L_u) \leq \text{dp}_\mathbf{m}^*(L) < \gamma$ . Let  $\mathbf{m} \leq_M \mathbf{n}$ . For  $u \in [\zeta]^{\leq \lambda_0}$ , on  $L_u$  the demand holds, i.e.  $\mathbb{P}_\mathbf{m}[L_u] = \mathbb{P}_\mathbf{n}[L_u]$ . Also if  $u \subseteq v \in [\zeta]^{\leq \lambda_0}$  then  $\mathbb{P}_\mathbf{m}[L_u] \subseteq \mathbb{P}_\mathbf{m}[L_v]$ .

Lastly,  $\langle u : u \in [\zeta]^{\leq \lambda_0} \rangle$  is  $\lambda_0^+$ -directed so by 2.14(8) we are done.

2) As  $L_\mathbf{m} \in \text{Sub}(\mathbf{m})$  and  $\text{dp}_\mathbf{m}^*(L_\mathbf{m}) < \infty$ .

3) The implication  $\Rightarrow$  holds by part (2). For the other implication  $\mathbf{m} \notin \mathbf{M}_\infty^{\text{ec}}$  so clause (\*) there fails; this means that there are  $L \in \text{Sub}(\mathbf{m})$  and  $\mathbf{n}$  such that  $\mathbf{m} \leq_M \mathbf{n}$  but  $\mathbb{P}_\mathbf{m}[L] \neq \mathbb{P}_\mathbf{n}[L]$ , by the definitions this implies  $\neg(\mathbb{P}_\mathbf{m} \triangleleft \mathbb{P}_\mathbf{n})$  so we are done.

4) Note that  $\{L \in \text{cmp}(\mathbf{m}) : L \not\subseteq M_\mathbf{m}\} = \{t/E'_\mathbf{m} : t \in L_\mathbf{m} \setminus M_\mathbf{m}\}$  has cardinality  $\leq |L_\mathbf{m} \setminus M_\mathbf{m}| \leq |L_\mathbf{m}|$  and  $\{L \in \text{cmp}(\mathbf{m}) : L \subseteq M_\mathbf{m}\} = \{L \subseteq M_\mathbf{m} : |L| \leq \lambda_0\}$  has cardinality  $\leq |M_\mathbf{m}|^{\lambda_0} \subseteq |L_\mathbf{m}|^{\lambda_0}$ .

5), 6) Similarly. □<sub>2.20</sub>

{c39}

**Claim 2.21.** If  $\gamma \leq \infty$ ,  $\langle \mathbf{m}_\alpha^* : \alpha < \delta \rangle$  is  $\leq_M$ -increasing,  $\text{cf}(\delta) > \lambda_0$  and  $\alpha < \delta \Rightarrow \mathbf{m}_\alpha^* \in \mathbf{M}_\gamma^{\text{ec}}$  then  $\mathbf{m}_* := \cup\{\mathbf{m}_\alpha^* : \alpha < \delta\} \in \mathbf{M}_\gamma^{\text{ec}}$ .

*Proof.* By 2.20(1), toward contradiction assume that  $(\mathbf{n}, L)$  satisfies:

- (\*) (a)  $\mathbf{m}_* \leq_M \mathbf{n}$
- (b)  $L \in \text{Sub}_*(\mathbf{m}_*)$
- (c)  $\text{dp}_{\mathbf{m}_*}^*(L) < \gamma$
- (d)  $\mathbb{P}_{\mathbf{m}_*}[L] \neq \mathbb{P}_\mathbf{n}[L]$ .

As  $L \in \text{Sub}_*(\mathbf{m}_*)$ , we have  $L \subseteq \cup\{L_\varepsilon : \varepsilon < \varepsilon(*)\}$  where  $\varepsilon(*) < \lambda_0^*$  and  $L_\varepsilon \in \text{sub}(\mathbf{m})$  for  $\varepsilon < \varepsilon(*)$ . But then  $L_\varepsilon \in \text{sub}(\mathbf{m}_{\alpha(\varepsilon)})$  for some  $\alpha(\varepsilon) < \delta$  by 2.19(3)(a) hence  $\alpha(*) = \sup\{\alpha(\varepsilon) : \varepsilon < \varepsilon(*)\} < \delta$  so  $L \in \text{Sub}_*(\mathbf{m}_{\alpha(*)})$ .

Now we have assumed  $\text{dp}_{\mathbf{m}_*}^*(L) < \gamma$  but  $M_{\mathbf{m}_*} = M_{\mathbf{m}_{\alpha(*)}}$ , so equivalently  $\text{dp}_{\mathbf{m}_{\alpha(*)}}^*(L) < \gamma$ . By the claim assumption,  $\mathbf{m}_\beta \in \mathbf{M}_\gamma^{\text{ec}}$  and by 2.19(1) we have  $\mathbf{m}_\beta \leq_M \mathbf{m}_*$ .

As  $\leq_M$  is a partial order on  $\mathbf{M}$  and  $\mathbf{m}_* \leq_M \mathbf{n}$  we have  $\mathbf{m}_\beta \leq_M \mathbf{n}$ , so together we deduce  $\mathbb{P}_{\mathbf{m}_*}[L] = \mathbb{P}_{\mathbf{m}_\beta}[L] = \mathbb{P}_\mathbf{n}[L]$ , so we are done. □<sub>2.21</sub>

{c41}

**Crucial Claim 2.22.** *Let  $\chi = \chi^{\lambda_0}$ .*1) *If  $\mathbf{m} \in \mathbf{M}$  of cardinality  $\leq \chi$  there is  $\mathbf{n} \in \mathbf{M}_\infty^{\text{ec}}$  of cardinality  $\leq \chi$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ .*2) *If  $\mathbf{m} \in \mathbf{M}_\chi$  and  $\gamma_1 < \chi^+$  then there is  $\mathbf{n} \in \mathbf{M}_\gamma^{\text{ec}} \cap \mathbf{M}_\chi$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ .**Remark 2.23.* 1) Compare with 2.16(3).2) Use 2.16(3) or  $2^{\mathbf{m}} = 2^{|\mathbf{m}|} \leq \chi$ .*Proof.* 1) By (2) using  $\gamma = \text{dp}_{\mathbf{m}}^*(M_{\mathbf{m}}) + 1 = \text{dp}_{\mathbf{m}}^*(L_{\mathbf{m}}) + 1$  which necessarily is  $< \|M_{\mathbf{m}}\|^+ < \chi^+$ .2) Without loss of generality  $\mathbf{m} \in \mathbf{M}_\chi$ . We prove by induction on  $\gamma$ . In each case consider  $L \in \text{Sub}(\mathbf{m})$  which satisfies  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$ .Case 1:  $\gamma = 0$ .There is no such  $\gamma$  relevant  $L$ .Case 2:  $\gamma = 1$ .So  $\text{dp}_{\mathbf{m}}^*(L) < \gamma$  means  $\text{dp}_{\mathbf{m}}^*(L) = 0$  which means  $L \cap M_{\mathbf{m}} = \emptyset$ . But then  $\mathbf{n} = \mathbf{m}$  is as required because if  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}_1$  then  $\{s \in L_{\mathbf{n}_1} : (\exists t \in L)(s \leq_{\mathbf{n}_1} t)\} = L = \{s \in L_{\mathbf{m}} : (\exists t \in L)(s \leq_{\mathbf{m}} t)\}$ , i.e.  $L$  is an initial segment of both so  $\mathbb{P}_{\mathbf{m},L} = \mathbb{P}_{\mathbf{n}_1,L}$  and the conclusion is obvious.Case 3:  $\gamma = \beta + 2$ .We try to choose  $\mathbf{m}_\varepsilon$  by induction on  $\varepsilon \leq \chi^+$  such that:

- (\*) (a)  $\mathbf{m}_\varepsilon \in \mathbf{M}_\chi$
- (b)  $\langle \mathbf{m}_\zeta : \zeta \leq \varepsilon \rangle$  is  $\leq_{\mathbf{M}}$ -increasing continuous
- (c)  $\mathbf{m}_0 = \mathbf{m}$
- (d) if  $\varepsilon = 2\zeta + 1$  then for some  $L_{2\zeta} \in \text{Sub}_*(\mathbf{m}_{2\zeta})$  we have  $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \neq \mathbb{P}_{\mathbf{m}_\varepsilon}[L]$  and  $\text{dp}_{\mathbf{m}_{2\zeta}}^*(L) < \gamma$
- (e) if  $\varepsilon = 2\zeta + 2$  then  $\mathbf{m}_\varepsilon \in \mathbf{M}_{\beta+1}^{\text{ec}}$ .

Subcase 3A: We succeed to carry the induction.For each  $\delta \in S_{\lambda_0^+}^{\chi^+}$  there is  $L_\delta$  as in (\*) (d). As  $\text{cf}(\delta) = \lambda_0^+ > \lambda_0 \geq |\{t/E'_{\mathbf{m}_\delta} : t \in L \setminus M_{\mathbf{m}_\delta}\}|$  there is  $\alpha(\delta) < \delta$  such that  $L_\delta \subseteq L_{\mathbf{m}_{\alpha(\delta)}}$  hence by 2.19(3) + 2.20(5) for some stationary  $S \subseteq S_{\lambda_0^+}^{\chi^+}$  and  $L_*$  we have  $\delta \in S \Rightarrow L_\delta = L_* \wedge \alpha(\delta) = \alpha(*)$ . Without loss of generality  $\alpha(*)$  is even and a successor. Note that  $\langle \mathbf{m}_{2\zeta+2} : \zeta < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing sequence of members of  $\mathbf{M}_{\beta+1}^{\text{ec}}$  with union  $\mathbf{m}_\delta$  (by clauses (e) and (b) of (\*)) hence  $\mathbf{m}_\delta \in \mathbf{M}_{\beta+1}^{\text{ec}}$  by 2.21; recall  $M_{\mathbf{m}_\alpha} = M_{\mathbf{m}}$  for every  $\alpha$ .Now  $\text{dp}_{\mathbf{m}_\delta}^*(L_*) < \gamma$  and by the previous sentences necessarily  $\text{dp}_{\mathbf{m}_\delta}^*(L_*) = \beta + 1$  hence the set  $\mathcal{Y} := \{t \in L_* \cap M_{\mathbf{m}} : \text{dp}(t, M_{\mathbf{m}}) = \beta\}$  is not empty.Let  $L_0^* = \{s \in L_* : \neg(\exists t \in \mathcal{Y})(t < s)\}$ ,  $L_1^* = L_0^* \cup \mathcal{Y}$ ,  $L_2^* = L_*$ .

Now

- (\*)<sub>1</sub>  $\mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L_0^*]$  is equal to  $\mathbb{P}_{\mathbf{m}_\delta}[L_0^*], \mathbb{P}_{\mathbf{m}_{\delta+1}}[L_0^*]$  for  $\delta \in S$ .

[Why? As  $\mathbf{m}_{\alpha(*)} \in \mathbf{M}_{\beta+1}^{\text{ec}}$  by (\*) (e).]

- (\*)<sub>2</sub> for  $\alpha \in \chi^+ \setminus \alpha(*)$  and  $t \in \mathcal{Y}$  let  $\mathbb{Q}_{\delta,t}^\alpha$  be  $\mathbb{Q}_\delta^{\mathbf{V}[\langle \eta_\gamma : \gamma \in u_t \cap L_* \rangle]}$

- (\*)<sub>3</sub> (a)  $\mathbb{Q}_{\bar{\theta},t}^\alpha$  does not depend on  $\alpha$ , also the order on it does not depend on  $\alpha$   
 (b)  $\Vdash_{\mathbb{P}_{\mathbf{m}_\alpha}} \text{“}\mathbb{Q}_{\bar{\theta},t}^\alpha \subseteq_{\text{ic}} \mathbb{Q}_{\bar{\theta}}\text{”}$   
 (c)  $\mathbb{P}_{\mathbf{m}_\alpha}[L_1^*]/\mathbb{P}_{\mathbf{m}_\alpha}[L_0^*]$  is the same for  $\alpha \in [\alpha(*), \chi^+)$   
 (\*)<sub>4</sub> define  $\mathbb{I}_\alpha = \{\mathcal{I} : \mathcal{I} \text{ is a } \mathbb{P}_{\mathbf{m}_{\alpha(*)}}[L_0^*]\text{-name of a maximal antichain of } \mathbb{P}_{\mathbf{m}_\alpha}[L_1^*]/\mathbb{P}_{\mathbf{m}_\alpha}[L_0^*]\}$ .

Let  $L_{0,\alpha}^* = \{s \in L_{\mathbf{m}_\alpha} : (\exists t \in L_0^*)(s \leq t) \text{ or } (\exists t \in \mathcal{I})(s < t)\}$

- (\*)<sub>5</sub> (a)  $\text{dp}_{\mathbf{m}_\alpha}^*(L_{0,\alpha}^*) < \beta + 1$   
 (b)  $\langle \mathbb{P}_{\mathbf{m}_\alpha}[L_{0,\alpha}^*] : \alpha \in [\alpha(*), \chi^+ \rangle$  is  $\leq$  increasing.

[Why? For clause (a), clearly  $L_\alpha^* \cap M_{\mathbf{m}_\alpha} = L_\alpha^* \cap M_{\mathbf{m}} \setminus \{s \in M_{\mathbf{m}} : (\exists t \in \mathcal{I})(s < t)\}$  hence  $\text{dp}_{\mathbf{m}_\alpha}^*(L_{0,\alpha}^*) = \{\text{dp}(s, M_{\mathbf{m}}) + 1, s \in M_{\mathbf{m}} \text{ and } (\exists t \in \mathcal{I})(s < t)\} = \{\text{dp}(s, M_{\mathbf{m}}) + 1 : s \in L_0^*\} < \gamma$ , so we are done. For clause (b), by the induction hypothesis for  $\alpha < \beta$ ,  $\mathbb{P}_{\mathbf{m}_\alpha}[L_{0,\alpha}^*] = \mathbb{P}_{\mathbf{m}_\beta}[L_{0,\alpha}^*] \leq \mathbb{P}_{\mathbf{m}_\beta}[L_{0,\beta}^*]$ .

So necessarily

- (\*)<sub>6</sub>  $\mathbb{I}_\alpha$  is constant for  $\alpha < \chi^+$  large enough.

Hence for some  $\beta(*) \in (\alpha(*), \chi^+)$  we have

- (\*)<sub>7</sub>  $\langle \mathbb{P}_{\mathbf{m}_\beta}[L_1^*] : \alpha < \chi^+ \rangle$  is constant for  $\beta \in [\beta(*), \chi^+)$ .

But this implies the result for  $L_2^*$  as the  $t \in L_2^* \setminus L_1^*$  does not “have memory outside  $L_1^*$ ”, i.e.  $u_{\mathbf{m}_{\beta(*)},t} \subseteq L_1^*$  and is constant, so we are done.

**Subcase 3B:** We are stuck in  $\varepsilon$ , i.e. cannot define  $\mathbf{m}_\varepsilon$ .

Now  $\varepsilon = 0$  is impossible. If  $\varepsilon$  is a limit, let  $\mathbf{m}_\varepsilon = \cup\{\mathbf{m}_\zeta : \zeta < \varepsilon\}$  so  $\zeta < \varepsilon \Rightarrow \mathbf{m}_\zeta \leq_{\mathbf{N}} \mathbf{m}_\varepsilon \in \mathbf{M}_\chi$ , contradiction. Also  $\varepsilon = \zeta + 1, \zeta = 0$  is impossible.

If  $\varepsilon = 2\zeta + 1$  so then clause (d) of (\*) is relevant so there is no  $\mathbf{n} \in \mathbf{M}_\chi$  such that  $\mathbf{m}_{2\zeta} \leq_{\mathbf{N}} \mathbf{n}$  and  $L \in \text{Sub}_*(\mathbf{m}_{2\zeta})$  such that  $\mathbb{P}_{\mathbf{m}_{2\zeta}}[L] \neq \mathbb{P}_{\mathbf{m}_{2\zeta}}[L]$  and  $\text{dp}_{\mathbf{m}_{2\zeta}}^*(L) < \gamma$ . By 2.20(1) this applies also to  $L \in \text{Sub}(\mathbf{m}_{2\zeta})$ . By claim 2.18(7) this applies also to  $\mathbf{n} \in \mathbf{M}$ . Together so by 2.20(4) we can deduce that  $\mathbf{m}_{2\zeta} \in \mathbf{M}_\gamma^{\text{ec}}$  so we are done.

Lastly, if  $\varepsilon = 2\zeta + 2$  so clause (e) of (\*) applies so use the induction hypothesis to choose  $\mathbf{m}_\varepsilon$ .

**Case 4:**  $\gamma$  is limit.

We choose  $\mathbf{m}_\alpha$  for  $\alpha \leq \lambda_0^+$  such that  $\mathbf{m}_\alpha = \mathbf{m}, \mathbf{m}_\alpha$  is  $\leq_{\mathbf{M}}$ -increasing continuous with  $\alpha$ , and if  $\varepsilon < \gamma$  and  $\alpha = \varepsilon + 1 \pmod{\gamma}$  then  $\mathbf{m}_\alpha \in M_\varepsilon^{\text{ec}}$ . So if  $\mathbf{m}_{\lambda_0^+} \leq_{\mathbf{M}} \mathbf{n}$  and  $L \in \text{Sub}_*(\mathbf{m}_{\lambda_0^+})$  satisfies  $\varepsilon = \text{dp}_{\mathbf{m}_\alpha}(L_{\mathbf{m}})$  and without loss of generality  $\alpha = \varepsilon + 1 \pmod{\gamma}$ , so we use it choice.

So  $\mathbf{m}_{\lambda_0^+}$  is as required by 2.21.

**Case 5:** None of the above.

So  $\gamma = \beta + 1, \beta$  a limit ordinal.

For  $\alpha < \beta$  let  $L_\alpha = \{s \in L : \text{there is no } t \leq s, t \in M_{\mathbf{m}}, \text{dp}(t, M_{\mathbf{m}}) \geq \alpha\}$ . So  $\langle L_\alpha : \alpha < \beta \rangle$  is  $\subseteq$ -increasing, and letting  $L_\beta^* = \cup\{L_\gamma^* : \gamma < \beta\}$  we have  $L \setminus L_\beta^* \subseteq L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ . As in Case 3 we first deal with  $L_\beta^*$ , and it is straight, and finish as there.  $\square_{2.22}$

{c46}

**Claim 2.24.** Assume  $\mathbf{m} \in \mathbf{M}_\infty^{\text{ec}}$ ,  $M \subseteq M_{\mathbf{m}}$  and  $\mathbf{n}$  is defined just like  $\mathbf{m}$  except that  $M_{\mathbf{n}} = M$  and  $E_{\mathbf{n}} = \{(s, t) : s, t \in L_{\mathbf{m}} \text{ but } \{s, t\} \notin M_{\mathbf{m}}\}$ ; so by 2.6(5) clearly  $\mathbf{n}$  is equivalent to  $\mathbf{m}$  and  $M_{\mathbf{n}} \subseteq M_{\mathbf{m}}$ .

Then  $\mathbf{n} \in \mathbf{M}^{\text{fc}}$  in Definition 2.17(1).

*Proof.* Assume that

$$(*)_1 \quad \mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1.$$

We now define  $\mathbf{m}_1 \in \mathbf{M}$  by

$$\begin{aligned} (*)_2 \quad (a) \quad & L_{\mathbf{m}_1} = L_{\mathbf{n}_1} \\ (b) \quad & \bar{u}_{\mathbf{m}_1} = \bar{u}_{\mathbf{n}_1} \\ (c) \quad & M_{\mathbf{m}_1} = M_{\mathbf{m}} \\ (d) \quad & E_{\mathbf{m}_1} = E_{\mathbf{m}} \cup (E_{\mathbf{n}_1} \upharpoonright (L_{\mathbf{n}_1} \setminus M)). \end{aligned}$$

Clearly

$$\begin{aligned} (*)_3 \quad (a) \quad & \mathbf{m}_1 \in \mathbf{M} \\ (b) \quad & \mathbf{m} \leq_{\mathbf{N}} \mathbf{m}_1. \end{aligned}$$

But  $\mathbf{m} \in \mathbf{M}_{\text{fc}}$  hence

$$(*)_4 \quad \mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{m}_1}.$$

But  $\mathbb{P}_{\mathbf{n}} = \mathbb{P}_{\mathbf{m}}$  and  $\mathbb{P}_{\mathbf{n}_1} = \mathbb{P}_{\mathbf{n}}$  by 2.10(7) as  $\mathbf{n}, \mathbf{m}$  are equivalent and  $\mathbf{n}_1, \mathbf{m}_1$  are equivalent, so

$$(*)_5 \quad \mathbb{P}_{\mathbf{n}} \triangleleft \mathbb{P}_{\mathbf{n}_1}.$$

□<sub>2.24</sub>

**The Uniqueness Claim 2.25.** There is an isomorphism from  $\mathbb{P}_{\mathbf{m}_1}[M_1]$  onto  $\mathbb{P}_{\mathbf{m}_2}[M_2]$  mapping  $p_{t,\eta}^*$  to  $p_{h(t),\eta}^*$  for  $\eta \in M_1, \eta \in \cup\{\prod_{\varepsilon < \zeta} \theta_\varepsilon : \zeta < \lambda\}$  when:

$$\begin{aligned} \boxplus (*)_2(a) \quad & \mathbf{m}_\ell \in \mathbf{M}_\infty^{\text{ec}} \text{ for } \ell = 1, 2 \\ (b) \quad & M_\ell \subseteq M_{\mathbf{m}_\ell} \text{ for } \ell = 1, 2 \\ (c) \quad & h \text{ is an isomorphism from } L_{\mathbf{m}_1} \upharpoonright M_1 \text{ onto } L_{\mathbf{m}_2} \upharpoonright M_2. \end{aligned}$$

*Proof.* Without loss of generality  $M_1 = M_2$  call it  $M$  and  $h$  is the identity and  $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$ . Let  $\mathbf{m}'_\ell$  be like  $\mathbf{n}$  in 2.24 with  $\mathbf{m}_\ell, M_\ell$  here standing for  $\mathbf{m}, M$  there so

$$(*)_1 \quad \mathbf{m}'_\ell \in \mathbf{M}_{\text{fc}}.$$

We define  $\mathbf{m}$  by:

$$\begin{aligned} (*)_2 \quad (a) \quad & s \in L_{\mathbf{m}} \text{ iff } s \in L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2} \\ (b) \quad & r \leq_{\mathbf{m}} t \text{ iff } r \leq_{\mathbf{m}_1} t \text{ or } r \leq_{\mathbf{m}_2} t \text{ or } (\exists s \in M)(r \leq_{\mathbf{m}_1} s \leq_{\mathbf{m}_2} t) \text{ or} \\ & (\exists s \in M)(r \leq_{\mathbf{m}_2} s \leq_{\mathbf{m}_1} t) \\ (c) \quad & M_{\mathbf{m}} = M_{\mathbf{m}_1} \cup M_{\mathbf{m}_2} \end{aligned}$$

{c49}

- (d)  $u_{\mathbf{m},t}$  is  $u_{\mathbf{m}_1,t}$  if  $t \in L_{\mathbf{m}_1} \setminus M$ ,  
is  $u_{\mathbf{m}_2,t}$  if  $t \in L_{\mathbf{m}_2} \setminus M$  and  
is  $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$  if  $t \in M$
- (e)  $M_{\mathbf{m}} = M$
- (f)  $E_{\mathbf{m}} = E'_{\mathbf{m}_1} \cup E'_{\mathbf{m}_2}$ .

For  $\ell = 1, 2$ ,  $\mathbb{P}_{\mathbf{m}_\ell}[M] = \mathbb{P}_{\mathbf{m}'_\ell}[M]$  by 2.24 and clearly  $\mathbf{m}'_\ell \leq_{\mathbf{M}} \mathbf{m}$ , but  $\mathbf{m}'_\ell \in \mathbf{M}_{\text{fc}}$ , by 2.24, hence  $\mathbb{P}_{\mathbf{m}'_\ell}[M] = \mathbb{P}_{\mathbf{m}}[M]$ . Together  $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}'_1}[M] = \mathbb{P}_{\mathbf{m}}[M] = \mathbb{P}_{\mathbf{m}'_2}[M] = \mathbb{P}_{\mathbf{m}_2}[M]$  as required.  $\square_{2.24}$

{c51}

**Conclusion 2.26.** For every ordinal  $\delta_*$  there is  $\mathbf{q} = \langle \mathbb{P}_\alpha : \alpha \leq \delta_* \rangle$  such that

- (A) (a)  $\langle \mathbb{P}_\alpha : \alpha \leq \delta_* \rangle$  is  $\triangleleft$ -increasing
- (b)  $\eta_\alpha$  is a  $\mathbb{P}_{\alpha+1}$ -name of a member of  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  which dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}^{\mathbb{P}_\alpha}}$
- (c)  $\eta_\alpha$  is a generic for  $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$ , moreover  $\langle \eta_\beta : \beta < \alpha \rangle$  is a generic for  $\mathbb{P}_\alpha$
- (d) every  $p \in \mathbb{P}_\alpha$  is from  $\mathbb{L}_{\lambda^+}(Y_{<\alpha}, \mathbb{P}_\alpha)$  when  $Y_{<\alpha}$  is as in Definition 2.13
- (e)  $\mathbb{P}_\alpha$  is  $\lambda$ -strategically complete and  $\lambda^+$ -c.c.
- (f) if  $\delta \leq \delta_*$  has cofinality  $> \lambda$  then  $\mathbb{P}_\delta = \cup \{ \mathbb{P}_\alpha : \alpha < \delta \}$
- (g)  $\mathbb{P}_{\delta_*}$  has cardinality  $|\delta_*|^\lambda$ .
- (B) if  $\mathcal{U} \subseteq \delta_*$  then the complete subforcing generated by  $\langle \eta_\alpha : \alpha \in \mathcal{U} \rangle$  is isomorphic to  $\mathbb{P}_{\text{otp}(\mathcal{U})}$
- (C) if  $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$  is generic over  $\mathbf{V}$  and  $\eta_\alpha = \eta_\alpha[\mathbf{G}]$  for  $\alpha < \delta_*$  and  $\eta'_\alpha \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  for  $\alpha < \delta_*$  and  $\{(\alpha, \varepsilon) : \alpha < \delta_*, \varepsilon < \lambda \text{ and } \eta'_\alpha(\varepsilon) \neq \eta_\alpha(\varepsilon)\}$  has cardinality  $< \lambda$  then also  $\langle \eta'_\alpha : \alpha < \delta_* \rangle$  is a generic for  $\mathbb{P}_{\delta_*}$ , determining a different  $\mathbf{G}'$  but  $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\mathbf{G}]$ .

*Remark 2.27.* Check with §1 if more is needed.

*Proof.* We define  $\mathbf{m} \in \mathbf{M}$  by:

- (\*) (a)  $L_{\mathbf{m}} = \delta_*$
- (b)  $M_{\mathbf{m}} = \delta_*$
- (c)  $u_{\mathbf{m},\alpha} = \alpha$  for  $\alpha < \delta_*$
- (d)  $E_{\mathbf{m}} = \emptyset$ .

It is easy to check that indeed  $\mathbf{m} \in \mathbf{M}$ . So by 2.22(1) there is  $\mathbf{n} \in \mathbf{M}_\infty^{\text{ec}}$  such that  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ . Let  $\mathbb{P}_\alpha := \mathbb{P}_{\mathbf{n}}[\{\beta : \beta < \alpha\}]$ , so by 2.25 we are done.  $\square_{2.28}$

{c52}

**Conclusion 2.28.** Assume  $M$  is a partial order and  $\bar{u}' = \langle u'_t : t \in M \rangle$ ,  $u_t \subseteq M_{<t}$ . Then we can find  $\beta(*), h, \mathbb{P}_\beta = \mathbb{P}_{0,\beta}, R, \mathbb{Q}_\alpha, \mathbb{P}_{1,\beta}$  and  $\mathbb{P}'_t$  (for  $\beta \leq \beta(*), \alpha < \beta(*)$  and  $t \in M^+$ ) and  $\bar{u}$

- (A) (a)  $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \beta \leq \beta(*), \alpha < \beta(*) \rangle$  is  $(< \lambda)$ -support iteration
- (b)  $\bar{u} = \langle u_\beta : \beta < \beta(*) \rangle$  such that  $u_\beta \subseteq \beta$
- (c)  $\eta_\alpha$  is a  $\mathbb{P}_{\alpha+1}$ -name of a member of  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$

- (d)  $\langle \eta_\alpha : \alpha < \beta \rangle$  is generic for  $\mathbb{P}_\beta$
- (e)  $\mathbb{Q}_\alpha$  is  $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \eta_i : i \in u_\alpha \rangle]}$
- (f)  $\Vdash_{\mathbb{P}_{\beta(*)}} \text{“} \eta_\beta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon \text{ dominate every } \nu \in \pi\theta_\varepsilon \text{ from } \mathbf{V}[\langle \eta_\alpha : \alpha \in u'_\beta \rangle]\text{”}$
- (B) (a)  $h$  is a one-to-one function from  $M$  into  $\beta(*)$  stipulate  $h(\infty) = \beta(\infty)$
- (b)  $s <_M t \Rightarrow h(s) < h(t)$
- (c)  $u'_{h(t)} \cap \text{Rang}(h) = \{h(s) : s \in u'_t\}$
- (C) (a)  $\mathbb{P}_{1,\beta}$  is  $L_{\lambda^+}(\mathbb{P}_{0,\beta})$  see 0.6
- (b)  $\mathbb{P}_{0,\beta} \triangleleft \mathbb{P}_{1,\beta}$  even  $\mathbb{P}_{0,\beta}$  is a dense in  $\mathbb{P}_{1,\beta}$
- (c) for  $t \in M^+$  let  $\mathbb{P}'_t$  be  $\mathbb{P}_{1,\beta(*)} \upharpoonright L_{\lambda^+}(\{p_{h(s),\nu} : \nu \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ for some } \zeta < \lambda, s <_M t\})$
- (d)  $\langle \eta_{h(s),\nu} : M^+ \models s < t \rangle$  is generic for  $\mathbb{P}'_t$  when  $t \in M^+$
- (D) (a)  $\mathbb{P}'_t \triangleleft \mathbb{P}_{1,h(t)}$
- (b)  $\mathbb{P}_{0,\beta}, \mathbb{P}_{1,\beta}, \mathbb{P}'_t$  are  $(< \lambda)$ -strategically complete and  $\lambda^+$ -c.c.
- (c) if  $M_1, M_2 \subseteq M$  and  $f$  is an isomorphism from  $M_1$  onto  $M_2$  then the mapping  $h(s_i) \mapsto h(f(s_i))$  induce an isomorphism from  $\mathbb{P}_{\beta(*)} \upharpoonright L_{\lambda^+}(\{\eta_{h(s),\nu} : s \in M_1, \nu \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ for some } \zeta < \lambda\})$  onto  $\mathbb{P}_{\beta(*)} \upharpoonright L_{\lambda^+}(\{\eta_{h(s),\nu} : s \in M_2, \nu \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ for some } \zeta < \lambda\})$ .

## § 3. PRIVATE APPENDIX

Old proof of 1.5, pgs.5-13:

2) We use a  $(< \lambda)$ -support iteration  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu + \delta(*), \beta < \mu + \delta(*) \rangle$  such that:

- (A) if  $\alpha < \mu$  then  $\mathbb{Q}_\alpha$  is the  $(\mathbb{P}_\alpha$ -name of the) dominating forcing,  $\mathbb{Q}_\lambda^{\text{dom}}$ , i.e.  $(\mathbb{Q}_\lambda^{\text{dom}})^{\mathbf{V}^{\mathbb{P}_\alpha}}$  where in the universe  $\mathbf{V}^{\mathbb{P}_\alpha}$  the forcing  $\mathbb{Q} = \mathbb{Q}_\lambda^{\text{dom}}$  is
- ( $\alpha$ )  $p \in \mathbb{Q}$  iff
- (a)  $p = (\eta, f) = (\eta^p, f^p)$
  - (b)  $\eta \in {}^\varepsilon \lambda$  for some  $\varepsilon < \lambda$ , ( $\eta$  is called the trunk of  $p$ )
  - (c)  $f \in {}^\lambda \lambda$
  - (d)  $\eta \triangleleft f$
- ( $\beta$ )  $p \leq_{\mathbb{Q}} q$  iff
- (a)  $\eta^p \leq \eta^q$
  - (b)  $f^p \leq f^q$ , i.e.  $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$
  - (c) if  $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$  then  $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$ ; this follows
- (B) fix  $\bar{\theta} = \langle \theta_\alpha : \alpha < \lambda \rangle$  with  $\theta_\alpha = (2^{|\alpha| + \aleph_0})^+$ , or any sequence of cardinals  $\in \text{Reg} \cap \lambda$ , increasing fast enough
- (C) if  $\alpha \in [\mu, \mu + \kappa]$  then  $\mathbb{Q}_\alpha$  is the  $\bar{\theta}$ -dominating forcing, i.e.,  $(\mathbb{Q}_{\bar{\theta}})^{\mathbf{V}^{\mathbb{P}_\alpha}}$  where in the universe  $\mathbf{V}^{\mathbb{P}_\alpha}$  the forcing notion  $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}$  is defined as follows:
- ( $\alpha$ )  $p \in \mathbb{Q}$  iff
- (a)  $p = (\eta, f) = (\eta^p, f^p)$
  - (b)  $\eta \in \prod_{\zeta < \ell g(\eta)} \theta_\zeta$  and  $\ell g(\eta)$  is an ordinal  $< \lambda$
  - (c)  $f \in \prod_{\zeta < \lambda} \theta_\zeta$
  - (d)  $\eta \triangleleft f$
- ( $\beta$ ) order: as in (A)( $\beta$ ).

Let  $f_\alpha$  be the generic object for  $\mathbb{Q}_\alpha$  for  $\alpha < \mu$  and  $g_i$  be the generic object for  $\mathbb{Q}_{\mu+i}$  for  $i < \kappa$ .

Now:

- (\*)<sub>1</sub> for  $\alpha \leq \mu + \kappa$  the forcing notion  $\mathbb{P}_\alpha$  is  $(< \lambda)$ -strategically complete and, when  $\alpha < \mu + \kappa$ ,  $\mathbb{Q}_\alpha$  is  $(< \lambda)$ -strategically complete<sup>6</sup>, i.e.  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q}_\alpha \text{ is } (< \lambda)\text{-strategically complete”}$ , in fact
- ( $\alpha$ ) for  $\alpha \in [\mu, \mu + \kappa)$ ,  $\mathbb{Q}_\alpha$  is not  $(< \lambda)$ -complete but it is  $(< \lambda)$ -strategically complete, and even  $\lambda$ -strategically complete; simply, in a play, COM can keep having the trunk being of length  $\geq$  length of the play so far
- ( $\alpha$ )<sup>+</sup> moreover, COM can guarantee that in limit stage  $\beta$  of the game,  $\langle p_\alpha : \alpha < \beta \rangle$  has a lub
- ( $\beta$ ) for  $\alpha \in [0, \mu)$ ,  $\mathbb{Q}_\alpha$  is  $(< \lambda)$ -complete even for directed systems (hence  $\mathbb{P}_\beta$  for  $\beta \leq \mu$  is)
- ( $\beta$ )<sup>+</sup> moreover, for such systems there is a lub
- ( $\gamma$ ) for  $\alpha \leq \mu$ ,  $\mathbb{P}_\alpha$  is  $(< \lambda)$ -complete

<sup>6</sup>for this,  $\theta_\alpha > \alpha$  is enough

$(\gamma)^+$  for  $\alpha \leq \mu + \kappa$ ,  $\mathbb{P}_\alpha$  is  $(< \lambda)$ -strategically complete.

[Why? We prove this by induction on  $\alpha$ ; for  $\mathbb{P}_\alpha$ , for 1.4(1)(a).]

(\*)<sub>2</sub> for each  $\alpha \leq \mu + \kappa$ ,  $\mathbb{P}_\alpha$  and for  $\alpha < \mu + \kappa$ , the forcing notions  $\mathbb{Q}_\alpha$  satisfy a strong form of the  $\lambda^+$ -c.c., (see [Sh:80] for definition, preservation and history; or fully [Sh:546, §1])

hence

(\*)<sub>3</sub> (a) forcing with  $\mathbb{P}_{\mu+\kappa}$  collapses no cardinal, changes no cofinality, and adds no sequence to  ${}^{\lambda>} \mathbf{V}$ ;  
 (b)  $(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_{\mu+\kappa}]} = \cup\{(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_{\mu+i}]} : i < \delta(*)\}$ ,  
 (c)  $(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\mu]} = \cup\{(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]} : \alpha < \mu\}$ .

[Why? By (\*)<sub>2</sub> + (\*)<sub>1</sub> clause (a) holds, for clauses (b),(c) use also the support in the iteration being  $< \lambda$  recalling that  $\mu, \kappa$  are regular  $> \lambda$ . E.g. if  $\Vdash_{\mathbb{P}_{\mu+1}} \check{f} \in {}^\lambda \lambda$  then we can, for  $\alpha < \lambda$  find maximal antichain  $\langle p_{\alpha,i} : i < i_\alpha \leq \lambda \rangle$  of  $\mathbb{P}_{\mu+\kappa}$  such that  $p_{\alpha,i}$  forces a value to  $\check{f}(\alpha)$ ; let  $\alpha(*) = \sup(\cup\{\text{dom}(p_{\alpha,i}) : \alpha < \lambda, i < i_\alpha\})$ , so  $\alpha(*) < \mu + \kappa$  and  $\check{f}$  is a  $\mathbb{P}_{\alpha(*)}$ -name.]

(\*)<sub>4</sub> in  $\mathbf{V}^{\mathbb{P}_\mu}$ ,  $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$  as witnessed by  $\check{f} = \langle \check{f}_\alpha : \alpha < \mu \rangle$ , in fact  $\Vdash_{\mathbb{P}_{\alpha+1}} \check{f}_\alpha \in {}^\lambda \lambda$  dominates  $(\lambda\lambda)^{\mathbf{V}[\mathbb{P}_\alpha]}$  modulo  $J_\lambda^{\text{bd}}$ .

[Why? Easy using (\*)<sub>3</sub>(c).]

(\*)<sub>5</sub>  $\Vdash_{\mathbb{P}_{\mu+i+1}} \check{g}_i \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  dominates  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_{\mu+i}]}$ , the order being modulo  $J_\lambda^{\text{bd}}$ .

[Why? As in  $\mathbf{V}^{\mathbb{P}_{\mu+i}}$  for each  $g \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  the set  $\{(\eta, f) \in \mathbb{Q}_{\bar{g}} : \text{for every } \varepsilon \in [\ell g(\eta), \lambda) \text{ we have } g(\varepsilon) \leq f(\varepsilon)\}$  is a dense open subset of  $\mathbb{Q}_{\mu+i}$ .]

(\*)<sub>6</sub>  $\Vdash_{\mathbb{P}_{\mu+\kappa}} \check{g} = \langle g_i : i < \kappa \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing and cofinal in  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_\lambda^{\text{bd}}})$ .

[Why? By (\*)<sub>5</sub> noting that  $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_{\mu+\kappa}]} = \cup\{(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_{\mu+i}]} : i < \kappa\}$  which holds by (\*)<sub>3</sub>(b).]

Now

(\*)<sub>7</sub>  $\Vdash_{\mathbb{P}_{\mu+\kappa}} \text{cov}_\lambda(\text{meagre}) \leq \kappa$ .

[Why? As we can look at  $\prod_{\varepsilon < \lambda} \theta_\varepsilon$  instead<sup>7</sup> of  ${}^\lambda 2$  and for each  $\varepsilon < \lambda, i < \kappa$  the set  $B_{\varepsilon,i} = \{\eta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon : \text{for every } \zeta \in [\varepsilon, \lambda) \text{ we have } \eta(\zeta) \leq g_i(\zeta) < \theta_\zeta\}$  is closed nowhere dense, and by (\*)<sub>6</sub> we have  $\mathbf{V}^{\mathbb{P}_{\mu+\kappa}} \models \text{“}\prod_{\zeta < \lambda} \theta_\zeta = \cup\{B_{\varepsilon,i} : \varepsilon < \lambda, i < \kappa\}$ ”.

In fact,  $\langle B_{0,i} : i < \kappa \rangle$  suffice.]

(\*)<sub>8</sub> letting  $\delta(*) = (\lambda^+)^{\lambda^+} < \lambda^{++}$ , ordinal power and  $\mathcal{U}_* = \{\delta(*)^{(\gamma+1)} : \gamma < \kappa\}$ , it is forced, i.e.  $\Vdash_{\mathbb{P}_{\mu+\kappa}}$ , that  $\mathbf{V}' := \mathbf{V}[\check{f}, \check{g} \upharpoonright \mathcal{U}_*]$  satisfies:

<sup>7</sup>E.g. let  $F : {}^\lambda 2 \rightarrow \prod_{\varepsilon < \lambda} \theta_\varepsilon$  be  $F(\eta) = \rho$  iff  $\eta \in {}^\lambda 2$  and for every  $\varepsilon < \lambda, \rho(\varepsilon) = 0$  iff  $(\forall i < \theta_\varepsilon)(\eta \sum_{\zeta < \varepsilon} \theta_\zeta + i) = 0$  and  $\rho(\varepsilon) = 1 + i$  iff  $\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + i) = 1 \wedge (\forall j < i)(\eta(\sum_{\zeta < \varepsilon} \theta_\zeta + j) = 0)$ . Now if  $\prod_{\varepsilon} \theta_\varepsilon = \cup\{\mathcal{U}_i : i < \kappa\}$ , each  $\mathcal{U}_i$  closed nowhere dense then  $\langle F^{-1}(\mathcal{U}_i) : i < \kappa \rangle$  witnesses  $\text{cov}_\lambda(\text{meagre}) \leq \kappa$ .

- (a)  $\mathbf{V}'$  has the same cardinals as  $\mathbf{V}$
- (b) the cofinality of a cardinal is the same in  $\mathbf{V}'$  and  $\mathbf{V}$
- (c)  $(\lambda > \text{Ord})^{\mathbf{V}'} = (\lambda > \text{Ord})^{\mathbf{V}}$
- (d) if  $\theta \geq \mu$  then  $(2^\theta)^{\mathbf{V}'} = (2^\theta)^{\mathbf{V}}$
- (e)  $(2^\lambda)^{\mathbf{V}'} = \mu$
- (f)  $\bar{g} \upharpoonright \mathcal{U}_*$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing cofinal in  $(\prod_{i < \lambda} \theta_i)^{\mathbf{V}'}$ .

[Why? Straightforward as  $\mathbf{V} \subseteq \mathbf{V}[\bar{f}, \bar{g} \upharpoonright \mathcal{U}_*] \subseteq \mathbf{V}[\bar{f}, \bar{g}]$ , and it is well known that  $\mathbf{V}[\bar{f}, \bar{g} \upharpoonright \mathcal{U}_*]$  is a generic extension of  $\mathbf{V}$  by some forcing notion of  $P' \triangleleft \text{com}(\mathbb{P})$ , see 0.6, but  $\text{com}(\mathbb{P})$  has  $\text{card} \leq |\mathbb{P}^\lambda| = \mu^\lambda = \mu$  so  $P'$  has cardinality  $\mu$  (relevant to clause (d)), which is anyhow marginal here.]

**Discussion 3.1.** 1) The reader may justly wonder why we use  $\mathbf{V}' = \mathbf{V}[\bar{f}, \bar{g} \upharpoonright \mathcal{U}_*]$  rather than simply  $\mathbf{V}[\bar{f}, \bar{g}]$ , i.e.  $\mathbf{V}^{\mathbb{P}^{\mu+\kappa}}$ . Of course, nothing is lost by it, but why the extra complication?

2) During the proof we shall use: if  $\zeta(i) \in \mathcal{U}$  is increasing with  $i < \kappa$  then also  $\langle g_{\zeta(i)} : i < \kappa \rangle$  is generic over  $\mathbf{V}[\bar{f}]$  for the subforcing  $\mathbb{P}_{\mu+\kappa}/\mathbb{P}_\mu$  generated by  $\bar{g} \upharpoonright \mathcal{U}_*$ ; see  $\otimes_7''$  inside the proof of  $\otimes_6$ . But using  $\mathcal{U}_* = \kappa$ , we do not know this.

3) Now in the parallel case for  $\lambda = \aleph_0$ , such claim is true. In fact, by Judah-Shelah [JdSh:292], if  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$  is FS iteration of Suslin-c.c.c. forcing notion  $\mathbb{Q}_\beta$  with the generic  $\eta_\beta \in {}^\omega \omega$  and for notational transparency, its definition is with no parameter and  $\zeta : \beta(*) \rightarrow \alpha(*)$  is increasing and  $\mathbb{P} = \langle \mathbb{P}'_\alpha, \mathbb{Q}'_\beta : \alpha \leq \beta(*), \beta < \beta(*) \rangle$  is FS iteration  $\mathbb{Q}'_\beta$  defined exactly as  $\mathbb{Q}_{\zeta(\beta)}$  then  $\Vdash_{\mathbb{P}_{\alpha(*)}} \langle \eta_{\zeta(\beta)} : \beta < \beta(*) \rangle$  is generic for  $\mathbb{P}'_{\beta(*)}$ .

4) But we do not know the parallel of 3) for  $\lambda$ , so we use a substitute using  $\mathcal{U}_*$ .

#### Continuation of the proof:

Now we come to the main and last point

(\*)<sub>9</sub> it is forced, i.e.  $\Vdash_{\mathbb{P}_{\mu+\kappa}}$  that no  $f \in (\lambda^\lambda)^{\mathbf{V}'}$  dominate  $\{f_\alpha : \alpha < \mu\}$ .

We shall note that it suffices to prove (\*)<sub>9</sub> for proving 1.3, and that (\*)<sub>9</sub> holds, thus finishing.

Why it suffices? As  $\langle f_\alpha : \alpha < \mu \rangle$  is  $<_{J_\lambda^{\text{bd}}}$ -increasing and  $\text{cf}(\mu) = \mu > \lambda$ , this implies  $\mathfrak{d}_\lambda \geq \mu$ , and by (\*)<sub>8</sub> this is the last piece missing. The rest of the proof is dedicated to proving that (\*)<sub>9</sub> holds.

Let  $\mathbf{G}_\mu \subseteq \mathbb{P}_\mu$  be generic over  $\mathbf{V}$  and so  $\langle f_\alpha : \alpha < \mu \rangle = \langle f_\alpha[\mathbf{G}_\mu] : \alpha < \mu \rangle$  is well defined. Now  $\mathbb{P}_{\mu+\kappa}/\mathbf{G}_\mu$  is just the limit of the  $(< \lambda)$ -support iteration of  $\langle \mathbb{P}_{\mu+i}/\mathbf{G}_\mu, \mathbb{Q}_{\mu+j} : i \leq \kappa, j < \kappa \rangle$ . Let  $p \in \mathbb{P}_{\mu+\kappa}/\mathbf{G}_\mu$ . For  $i \leq \kappa$  let  $\mathbb{P}_{0,i} = \mathbb{P}_{\mu+i}/\mathbf{G}, \mathbb{Q}_{0,i}$  be the  $\mathbb{P}_{0,i}$ -name of  $\mathbb{Q}_{\mu+i}$ , i.e., of  $\mathbb{Q}_{\bar{\theta}}$  in the universe  $\mathbf{V}[\mathbf{G}_\mu]^{\mathbb{P}_{0,i}}$ .

Without loss of generality (this will help in ...)

(\*)<sub>10</sub>  $p \in \mathbb{P}_{0,\gamma}$  iff:

- (a)  $p$  is a function with domain  $\in [\gamma]^{< \lambda}$
- (b) for every  $\beta \in \text{Dom}(p)$ , for some  $\xi \leq \lambda$  and  $\bar{\beta} = \bar{\beta}_{p,\varepsilon,\beta} \in {}^\xi \gamma$  and  $\lambda$ -Borel function  $\mathbf{B} = \mathbf{B}_{p,\beta} : {}^\xi (\prod_{i < \lambda} \theta_i) \rightarrow \mathbb{Q}_{\bar{\theta}}$  we have  $p(\beta) = \mathbf{B}(\dots, \eta_{\beta(\varepsilon)}, \dots)_{\varepsilon < \xi}$

(\*)<sub>11</sub> for  $p \in \mathbb{P}_{0,\gamma}$  let  $\text{fsup}(p)$ , the full support of  $p$  be  $\cup\{\text{Rang}(\bar{\beta}_{p,\varepsilon}) : \varepsilon \in \text{Dom}(p)\} \cup \text{Dom}(p) \in [\gamma]^{\leq \lambda}$ .

We shall apply  $\square_\lambda$ . Let  $\gamma(*) = \kappa$  (but we shall use  $\gamma(*)$  since the proof applies to any  $\gamma(*)$  of cofinality  $> \lambda$ ). The condition  $\square_\lambda$  is preserved by forcing by  $\mathbb{P}_\mu$  recalling (\*)<sub>1</sub>( $\beta$ ) so  $\mathbf{V}[\mathbf{G}_\mu] = \mathbf{V}^{\mathbb{P}_\mu}$  satisfies  $\square_\lambda$ . So it suffices to prove:

(\*)'\_9 if  $\mathbf{V}$  satisfies  $\square_\lambda$  and  $\mathbf{q} = \langle \mathbb{P}_{0,i}, \mathbb{Q}_{0,j} : i \leq \gamma(*), j < \gamma(*) \rangle$  is a  $(< \lambda)$ -support iteration, such that for every  $j < \gamma(*)$  the forcing notion  $\mathbb{Q}_{0,j}$  is  $(\mathbb{Q}_{\bar{\theta}})^{\mathbf{V}^{\mathbb{P}_{0,j}}}$  and  $\gamma(*)$  is a regular cardinal  $\delta(*) \cdot \gamma(*) = \gamma(*) \wedge \text{cf}(\gamma(*)) \geq \lambda^+$  and  $\mathcal{U}_* = \{(\gamma + 1) : \gamma < \gamma(*)\}$  where  $\delta(*) = (\lambda^+)^{\lambda^+}$ , this means ordinal exponentiation then it is forced, i.e.  $\Vdash_{\mathbb{P}_{0,\gamma(*)}}$ , that no  $f \in {}^{(\lambda\lambda)}\mathbf{V}^{\mathbb{P}_{0,\gamma(*)}}$  dominate  $(\lambda\lambda)^{\mathbf{V}}$  letting  $\bar{g}_0 = \langle \bar{g}_{0,i} : i < \gamma(*) \rangle$  where  $\bar{g}_{0,i} \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$  is the name of the generic for  $\mathbb{Q}_{0,i}$  so  $\bar{g}_{0,i} = g_{\mu+i}$ .

[Why does (\*)'\_9 suffice? We apply it with  $\mathbf{V}, \gamma(*)$  in (\*)'\_9 standing for  $\mathbf{V}^{\mathbb{P}_\mu} = \mathbf{V}[\mathbf{G}_\mu], \kappa$  here; if you wonder by  $\square_\lambda$  holds in  $\mathbf{V}[\mathbf{G}_\lambda]$ , see 6.3(3). So in  $\mathbf{V}[\mathbf{G}_\mu]^{\mathbb{P}_{0,\gamma(*)}} = \mathbf{V}^{\mathbb{P}_{\mu+\kappa}}$ , letting  $f_\alpha = \bar{f}_\alpha[\mathbf{G}_\mu]$  for  $\alpha < \mu$  we have:

- (a)  $f_\alpha \in {}^\lambda \lambda$ , for every  $\alpha < \mu$
- (b)  $\bar{f} = \langle f_\alpha : \alpha < \mu \rangle$  is  $< J_\lambda^{\text{bd}}$ -increasing cofinal in  $\mathbf{V}$
- (c)  $\Vdash_{\mathbb{P}_{0,\gamma(*)}}$  “ $\{f_\alpha : \alpha < \mu\}$  has no common  $\leq J_\lambda^{\text{bd}}$ -upper bound in  $\mathbf{V}[\bar{f}, \bar{g}_0 \upharpoonright \mathcal{U}_*] = \mathbf{V}[\mathbf{G}_\mu][\bar{g}_0 \upharpoonright \mathcal{U}_*]$ ”.

This implies that  $\Vdash_{\mathbb{P}_{\mu+\kappa}}$  “ $\mathbf{V}[\bar{f}, \bar{g}_0 \upharpoonright \mathcal{U}_*]$  satisfies  $\mathfrak{d}_\lambda \geq \mu$ ” as required.]

For  $i \leq \gamma(*)$  let  $\mathbb{P}_{1,i}$  be the completion of  $\mathbb{P}_{0,i}$ , see 0.6 and let  $\mathbb{P}'_i = \mathbb{P}_{2,i}$  be the complete subforcing of  $\mathbb{P}_{1,\delta(*)^{(i+1)}}$  generated by  $g'_j = \langle g'_j : j < i \rangle = \langle g_{0,\delta(*)^{(j+1)}} : j < i \rangle$ .

We shall use the nice properties of  $\mathbb{P}'_i, \bar{g}'_i$ .

Note that

- ⊞<sub>1</sub> (a)  $\langle \bar{g}'_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$ , i.e., if  $\mathbf{G}$  is a subset of  $\mathbb{P}'_{\gamma(*)}$  generic over  $\mathbf{V}$  and  $g'_\gamma = \bar{g}'_\gamma[\mathbf{G}]$  then  $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_\gamma : \gamma < \gamma(*) \rangle]$
- (b) if in  $\mathbf{V}[\mathbf{G}], g''_\gamma \in \prod_{\zeta < \lambda} \theta_\zeta$  for  $\gamma < \gamma(*)$  and the set  $\{(\gamma, \zeta) : \gamma < \gamma(*) \text{ and } \zeta < \lambda \text{ and } g''_\gamma(\zeta) \neq g'_\gamma(\zeta)\}$  has cardinality  $< \lambda$  then  $\bar{g}'' = \langle g''_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$  and  $\mathbf{V}[\bar{g}''] = \mathbf{V}[\bar{g}']$
- (c)  $\mathbb{P}'_{\gamma+1}/\mathbb{P}'_\gamma$  is equivalent to  $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}^{\mathbb{P}'_\gamma}}$
- (d) if  $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$  is an increasing sequence of ordinals  $< \gamma(*)$ , then  $\langle g'_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$
- (e)  $\mathbb{P}'_\gamma$  is  $(< \lambda)$ -strategically complete and satisfies the  $\lambda^+$ -c.c.

[Why? The serious point is clause (d) and (e) which are done similarly. For this it suffices to show that: if  $\langle g_\gamma : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}_{0,\gamma(*)}$  and  $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$  is as there then not only  $\langle g_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$  is generic for  $\mathbb{P}'_{\gamma(*)}$  but also  $\langle g_{\delta(*)^{(\zeta(\gamma)+1)}} : \gamma < \gamma(*) \rangle$  is. This holds and it straightforward translates to saying that the sequences  $\langle \delta(*)^{(\gamma+1)} : \gamma < \gamma(*) \rangle$  and  $\langle \delta(*)^{(\zeta(\gamma)+1)} : \gamma < \gamma(*) \rangle$  realize

the same  $\mathbb{L}_{\lambda^+, \lambda}$ -type in the structure  $(\gamma(*), <)$ , which holds by: (Kino [Kin66]). See details in 6.3(4); more in [Sh:F979].

We shall use  $\boxplus_1$  freely.]

To prove  $(*)'_9$  assume toward contradiction that this fails, so  $\mathbb{P}'_{\gamma(*)}$  satisfies the  $\lambda^+$ -c.c. and for some  $\mathbb{P}'_{\gamma(*)}$ -name  $\underline{f}$  and  $\lambda$ -Borel function  $\mathbf{B}$  and  $\rho \in {}^\lambda \gamma(*)$ , moreover  $\rho \in {}^\lambda (\mathcal{U}_*)$  we have (noting: the “moreover” holds as  $f \in ({}^\lambda \lambda)^{\mathbf{V}[\bar{g}_0 \upharpoonright \mathcal{U}_*]}$ )

$$\textcircled{*}_0 \ p^* \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\underline{f} \in {}^\lambda \lambda \text{ and dominates } ({}^\lambda \lambda)^{\mathbf{V}} \text{” and } \underline{f} = \mathbf{B}(\langle g_{\rho(i)} : i < \lambda \rangle).$$

Now let  $\chi$  be regular large enough and we choose  $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda \rangle$  such that

- $\textcircled{*}_1$  (a)  $N_\varepsilon$  is as in  $\square_\lambda$  for the forcing notion  $\mathbb{P}'_{\gamma(*)}$ ,  
 $N_\varepsilon \in \mathcal{S}_{\chi, \mathbb{P}'_{\gamma(*)}}$ , see  $\square$  of 1.3
- (b)  $\bar{N} \upharpoonright \varepsilon \in N_\varepsilon$  hence  $\bigcup_{\zeta < \varepsilon} N_\zeta \subseteq N_\varepsilon$  and  $\lambda_\varepsilon := N_\varepsilon \cap \lambda > \lambda_\varepsilon^- := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\}$
- (c)  $\bar{\theta}, \mathbf{q}, p^*, \underline{f}, \mathbf{B}, \rho$  belong to  $N_\varepsilon$
- (d) let  $\delta(\varepsilon) = \text{otp}(\delta(*) \cap N_\varepsilon)$
- (e)  $\kappa_\varepsilon = \kappa_\varepsilon^{< \kappa_\varepsilon}$  where  $\kappa_\varepsilon = \text{otp}(\kappa \cap N_\varepsilon)$ .

We can find  $f^* \in {}^\lambda \lambda$ , i.e.  $\in ({}^\lambda \lambda)^{\mathbf{V}}$ , such that

$$\textcircled{*}_2 \text{ for arbitrarily large } \varepsilon < \lambda \text{ for some } \zeta \in [\lambda_\varepsilon^-, \lambda_\varepsilon) \text{ we have } f^*(\zeta) > \lambda_\varepsilon.$$

For  $\varepsilon < \lambda$  let  $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, N'_\varepsilon, \mathbf{G}_\varepsilon)$  be a witness for  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)}, \chi)$  recalling  $\square_\lambda$  of Definition 1.5 so  $\lambda_\varepsilon \in (\varepsilon, \lambda)$  is strongly inaccessible and  $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda_\zeta^- < \lambda_\zeta$ , recalling  $\textcircled{*}_1$  and so  $\delta(\varepsilon) = \mathbf{j}_\varepsilon(\delta(*)$ ), etc.

Let

- $\textcircled{*}_3$  (a)  $u_\varepsilon = N_\varepsilon \cap \gamma(*)$
- (b)  $i(\varepsilon) = \text{otp}(u_\varepsilon)$
- (c)  $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < i(\varepsilon) \rangle$  list  $u_\varepsilon$  in increasing order
- (d) for  $i < \text{otp}(u_\varepsilon)$ , equivalently  $i < \mathbf{j}_\varepsilon(\gamma(*)$ ) let  $\eta_i^\varepsilon = (\mathbf{j}_\varepsilon(\underline{g}'_i))^{N'_\varepsilon}[\mathbf{G}_\varepsilon] \in \prod_{\zeta < \lambda_\varepsilon} \theta_\zeta$  and let  $\bar{\eta}^\varepsilon = \langle \eta_i^\varepsilon : i < \text{otp}(u_\varepsilon) \rangle$ .

Note

- $\textcircled{*}_4$  (a)  $\bar{\eta}^\varepsilon$  is generic for  $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$ , moreover
- (b) for each  $\varepsilon < \lambda$ , if we change  $\eta_i^\varepsilon(\zeta)$  (legally, i.e.  $< \theta_\zeta$ ) for  $< \lambda_\varepsilon$  pairs  $(i, \zeta) \in \text{otp}(u_\varepsilon) \times \lambda_\varepsilon$  and get  $\bar{\eta}'$ , then also  $\bar{\eta}'$  is generic for  $(N'_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$  and  $N'_\varepsilon[\bar{\eta}'] = M_\varepsilon$
- (c) like  $\boxplus_1$  with  $\mathbf{V}, \mathbb{P}'_{\gamma(*)}, \lambda$  there standing for  $N_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$  here.

Hence

- $\textcircled{*}'_4$  for  $\varepsilon < \lambda$ , if  $\bar{\eta}' = \langle \nu_i : i < i(\varepsilon) \rangle$  recalling  $i(\varepsilon) = \text{otp}(u_\varepsilon)$  is as in  $\textcircled{*}_4(b)$ , and  $q \in \mathbb{P}'_{\gamma(*)}$  satisfies  $i < i(\varepsilon) \Rightarrow q \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\underline{g}'_{\gamma_i(\varepsilon)} \upharpoonright \lambda_\varepsilon = \nu_i \text{”}$  then  $q$  is  $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$ -generic naturally and  $q \Vdash_{\mathbb{P}'_{\gamma(*)}} \text{“}\mathbf{j}_\varepsilon \text{ can be extended naturally to an isomorphism from } N_\varepsilon[G_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle \underline{g}_\gamma : \gamma \in u_\varepsilon \rangle] \text{ onto } N'_\varepsilon[\bar{\eta}'] \text{”}$ .

[Why? Should be clear, see  $\boxplus_1 + \boxtimes_4(c)$ .]

By the assumption toward contradiction,  $\boxtimes_0$ , and  $\mathbb{P}'_{\gamma(*)}$  being  $(< \lambda)$ -strategically complete recalling  $\boxplus_1$ , there are  $\zeta(*), p^{**}$  and  $p^+$  such that (recall  $p^* \in \mathbb{P}'_{\gamma(*)} = < \mathbb{P}_{0, \gamma(*)}$ ):

- $\boxtimes_5$  (a)  $p^* \leq p^{**} \in \mathbb{P}'_{\gamma(*)}$  and  $p^{**} \leq p^+ \in \mathbb{P}_{0, \gamma(*)}$
- (b)  $\zeta(*) < \lambda$
- (c)  $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}}$  “ $f^*(\zeta) < \check{f}(\zeta)$  whenever  $\zeta(*) \leq \zeta < \lambda$ ” where  $f^*$  is from  $\boxtimes_2$
- (d) if  $\gamma \in \text{Dom}(p^+)$  then  $\eta^{p^+(\gamma)}$  is an object (not just a  $\mathbb{P}'_\gamma$ -name) of length  $\geq \zeta(*)$  (recall that  $\eta^{p^+(\gamma)}$  is the trunk of the condition, see clause  $(\alpha)(b)$  above).

Note that possibly  $\text{Dom}(p^+) \not\subseteq \cup \{u_\varepsilon : \varepsilon < \lambda\}$ . Choose  $\varepsilon(*) < \lambda$  such that  $\lambda_{\varepsilon(*)} > \zeta(*) + |\text{Dom}(p^+)|$  and  $\gamma \in \text{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$  recalling clause (d) of  $\boxtimes_5$  and  $|\text{Dom}(p^+)| < \lambda$  as  $p^+ \in \mathbb{P}_{0, \gamma(*)}$  and  $\mathbb{P}_{0, \gamma(*)}$  is the limit of a  $(< \lambda)$ -support iteration.

By  $\boxtimes_2$  we can add  $(\exists \zeta)[\lambda_{\varepsilon(*)}^- \leq \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$ . Our intention is to find  $q \in \mathbb{P}_{0, \gamma(*)}$  above  $p^+$  which is above some  $q' \in \mathbb{P}'_{\gamma(*)}$  which is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ -generic and forces it to include a generic subset of  $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$  which is induced by some  $\bar{\eta}'$  as in  $\boxtimes_4(b)$ . Toward this in  $\boxtimes_6$  below the intention is that  $p_{i(*)}^+$  will serve as  $q$ .

Let  $i(*) = i(\varepsilon(*))$  and  $\gamma_i = \gamma_{2,i} = \gamma_{\delta(*) (i+1)}(\varepsilon(*))$  for  $i < i(*)$  so  $\langle \gamma_i : i < i(*) \rangle$  list  $u_{\varepsilon(*)} \cap \mathcal{U}_*$  in increasing order and let  $\mathcal{U}_{**} = \{\gamma_i : i < i(*)\}$  and  $\gamma_{i(*)} = \gamma(*)$  so  $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in u_{\varepsilon(*)}\} = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$  and  $N_{\varepsilon(*)} \models$  “ $i(*)$  is a regular cardinal  $> \lambda_\varepsilon$ ” hence  $i(*)$  is really a regular cardinal so call it  $\sigma$ . Now we define a game  $\boxtimes$  as follows<sup>8</sup>:

- $\boxplus_2$  (A) each play lasts  $i(*) + 1 = \sigma + 1$  moves and in the  $i$ -th move,
  - (a) if  $i = j + 1$  the antagonist player chooses  $\xi_j = \xi(j) < \sigma$  such that  $j_1 < j \Rightarrow \gamma(j_1) < \xi(j)$
  - (b) then, if  $i = j + 1$  the protagonist chooses  $\zeta_j = \zeta(j) \in (\xi(j), \sigma)$ , but there are more restrictions implicit in  $\boxplus_3$
  - (c) in any case the protagonist chooses  $p_i^+, \bar{v}^i$  such that  $\boxplus_3$  below holds;
- (B) in the end of the play the protagonist wins the play iff he always has a legal move and in the end  $\{\zeta(i) : i < i(*)\} \in N'_{\varepsilon(*)}$ ; where
- $\boxplus_3$  (a)  $p_i^+ \in \mathbb{P}_{0, \gamma_i}$
- (b) if  $j < i$  then  $\mathbb{P}_{0, \gamma_i} \models$  “ $p_j^+ \leq p_i^+$ ”
- (c) if  $\gamma \in \cup \{\text{Dom}(p_j^+) : j < i\}$  then
  - $p_i^+ \upharpoonright \gamma \Vdash_{\mathbb{P}_{0, \gamma_i}}$  “ $\eta^{p_i^+(\gamma)}$  has length  $\geq i(*)$  and  $\geq \lambda_{\varepsilon(*)}$ ”
  - moreover  $\eta^{p_i^+(\gamma)}$  is an object,  $\eta^{p_i^+(\gamma)}$  for  $j < i$
- (d)  $\mathbb{P}_{0, \gamma_i} \models$  “ $p^{**} \upharpoonright \gamma_i \leq p_i^+$ ”

<sup>8</sup>The idea is to scatter the  $\eta_{\gamma_i}^{\varepsilon(*)}$ 's. Why not use the original places? as then we have a problem in  $\boxtimes_6$ .

- (e)  $\bar{\nu}^i = \langle \nu_{\gamma_j} : j < i \rangle$  and  $\nu_{\gamma_j} \in \prod_{\iota < \lambda_{\varepsilon(*)}} \theta_\iota$
- (f) for  $j < i$  we have  $\nu_{\gamma_j} \leq \eta_{\zeta(i)}^{p_i^+}$  so  $p_i^+ \upharpoonright \gamma_j \Vdash \text{“}\nu_{\gamma_j} \triangleleft g'_{\gamma_j}\text{”}$  recalling  $\boxplus_1$
- (g) for  $j < i$  we have (recall  $\bar{\eta}^\varepsilon$  from  $\boxtimes_3$ )
- ( $\alpha$ )  $\nu_{\gamma_j} = \eta_{\zeta(j)}^{\varepsilon(*)}$  recalling  $\eta_{\zeta(j)}^{\varepsilon(*)}$  is from  $\boxtimes_3$  or
- ( $\beta$ )  $\gamma_j \in \text{Dom}(p^{**})$  and  $\{\iota < \lambda_{\varepsilon(*)} : \eta_{\zeta(j)}^{\varepsilon(*)}(\iota) \neq \nu_{\gamma_j}(\iota)\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ .

We shall prove

- $\boxtimes_6$  in the game  $\mathcal{D}$
- (a) the antagonist has no winning strategy
- (b) in any move the protagonist has a legal move, moreover for any  $\zeta(i) \in (\xi(i), \sigma)$  large enough the protagonist can choose it.

Why  $\boxtimes_6$  suffice?

By clause (a) of  $\boxtimes_6$  we can choose a play  $\langle (\xi(i), \zeta(i), p_i^+, \bar{\nu}^i) : i \leq i(*) \rangle$  in which the protagonist wins. Recalling  $\mathbb{P}'_{\gamma(*)} \leq \mathbb{P}_{1, \gamma(*)}$  and  $\mathbb{P}_{0, \gamma(*)}$  is a dense subforcing of  $\mathbb{P}_{1, \gamma(*)}$ , clearly

- $\boxtimes_7$  there is  $p$  such that
- (a)  $p \in \mathbb{P}'_{\gamma(*)}$
- (b) if  $\mathbb{P}'_{\gamma(*)} \Vdash \text{“}p \leq p'\text{”}$  then  $p', p_i^+$  are compatible in  $\mathbb{P}_{0, \gamma(*)}$
- (c)  $p$  is above  $p^{**}$  and it forces  $g'_i \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < \gamma(*)$ .

Then on the one hand

- $\boxtimes'_7$   $p \in \mathbb{P}'_{\gamma(*)}$  being above  $p^{**}$  forces  $f^* \upharpoonright [\zeta(*), \lambda) < \underline{f} \upharpoonright [\zeta(*), \lambda)$  hence  $f^* \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)}) < \underline{f} \upharpoonright [\zeta(*), \lambda_{\varepsilon(*)})$  recalling that  $\zeta(*) < \lambda_{\varepsilon(*)}$ .

On the other hand,

- $\boxtimes''_7$   $p$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ -generic.

[Why? As it forces  $\eta_{\gamma_{1,i}} \upharpoonright \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$  for  $i < i(*)$  and  $\langle \nu_{\gamma_i} : i < i(*) \rangle$  is (see  $\boxtimes_4$ ) “almost equal” to  $\langle \eta_{\zeta(i)}^{\varepsilon(*)} : i < i(*) \rangle$  which is a subsequence of the sequence from  $\boxtimes_3$  and recalling clause (g) of  $\boxplus_3$ . That is  $\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} \subseteq \cup \{\{(i, \iota) : \iota < \lambda_{\varepsilon(*)} \text{ and } \nu_{\gamma_i}(\iota) \neq \eta_{\zeta(i)}^{\varepsilon(*)}(\iota)\} : \gamma \in u_{\varepsilon(*)} \cap \text{Dom}(p^{**})\}$  so is the union of  $\leq |\text{Dom}(p^+)| < \lambda_{\varepsilon(*)}$  sets each of cardinality  $< \lambda_{\varepsilon(*)}$  hence is of cardinality  $< \lambda_{\varepsilon(*)}$ . Hence by  $\boxtimes_4(c) + \boxplus_1(d)$  the sequence  $\bar{\nu}^{i(*)}$  is generic for  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$ .]

As  $\underline{f} \in N_{\varepsilon(*)}$  it follows from  $\boxtimes''_7$  that

- $\boxtimes'''_7$   $p \Vdash \text{“}\underline{f} \upharpoonright \lambda_{\varepsilon(*)}$  is a function from  $\lambda_{\varepsilon(*)}$  to  $\lambda_{\varepsilon(*)}\text{”}$ .

Together  $\boxtimes'_7 + \boxtimes'''_7$  gives a contradiction by the choice of  $f^*$  in  $\boxtimes_2$  and of  $\varepsilon(*)$  above, hence  $\boxtimes_6$  is enough.

Why  $\boxtimes_6$  is true?

Let us prove  $\otimes_6$ ; first for clause (a) choose any strategy  $\mathbf{st}$  for the antagonist and fix a partial strategy  $\mathbf{st}'$  for the protagonist choosing  $(p_i^+, \bar{v}^i)$  from the previous choices and  $\zeta(i)$  if relevant and possible. So the only freedom left is for the protagonist to choose the  $\zeta(i)$ . So we have in  $\mathbf{V}$  a function  $F : \sigma^>(i^*) \rightarrow \sigma$  such that:

(\*)<sub>F</sub> playing the game such that the antagonist uses  $\mathbf{st}$  and the protagonist uses  $\mathbf{st}'$ , arriving to the  $i$ -th move,  $\bar{\zeta} = \langle \zeta(j) : j < i \rangle$  is well defined and for the protagonist any choice  $\zeta_i \in (F(\bar{\zeta}), \sigma) \cap \mathcal{U}_{**}$  is legal.

Now we have to find an increasing sequence  $\bar{\zeta} = \langle \zeta(i) : i < i^* \rangle$  such that  $F(\bar{\zeta} \upharpoonright i) < \zeta(i) < \sigma$  and  $\bar{\zeta} \in N'_{\varepsilon(*)}$ . As  $F \in \mathcal{H}(\chi_\varepsilon)$  and  $\mathcal{H}(\chi_\varepsilon) = N'_\varepsilon[\mathbf{G}_\varepsilon]$  where  $\mathbf{G}_\varepsilon$  is a subset of  $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}) \in N'_\varepsilon$  and  $\mathbf{j}_\varepsilon(\mathbb{P}_{0, \gamma(*)})$  satisfies the  $\lambda_\varepsilon^+$ -c.c. and  $\sigma = \text{cf}(\sigma) > \lambda_\varepsilon$  this is possible. We are left with proving  $\otimes_6(b)$ .

Case 1:  $i = 0$ .

Let  $p_0^+ = p^{**} \upharpoonright \gamma_0$ .

Case 2:  $i$  limit.

By clauses (a) and (b), there is  $p_i^+ \in \mathbb{P}_{0, \gamma_i}$  which is an upper bound (even l.u.b.) of  $\{p_j^+ : j < i\}$  and it is easily as required. Also  $\bar{v}^i$  is well defined and as required.

Case 3:  $i = j + 1$  and  $\gamma_j \notin \text{Dom}(p^{**})$ .

Clearly  $\gamma_i = \gamma_j + \delta(*)$  and  $\gamma_j \in u_{\varepsilon(*)}$ . As in case 4 below but easier by the properties of the iteration.

Case 4:  $i = j + 1$  and  $\gamma_j \in \text{Dom}(p^{**})$

Again  $\gamma_i = \gamma_j + \delta(*)$  and  $\gamma_j \in u_{\varepsilon(*)}$ . First we find  $p'_j$  such that:

- $\otimes_8$  (a)  $p_j^+ \leq p'_j \in \mathbb{P}_{0, \gamma_j}$
- (b) if  $\gamma \in \text{Dom}(p_j^+)$  then  $p'_j \upharpoonright \gamma \Vdash \text{“} \ell g(\eta^{p'_j(\gamma)}) > i \text{”}$
- (c)  $p'_j$  forces<sup>9</sup> a value to the pair  $(\eta^{p^+(\gamma_i)}, \underline{f}^{p^+(\gamma_j)} \upharpoonright \lambda_{\varepsilon(*)})$ ; we call this pair  $q_j$ .

This should be clear.

Second

$\otimes_9$   $p_j^+$  hence  $p'_j$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic and  $\langle \nu_{\gamma_{j(1)}} : j(1) < j \rangle$  induces the generic.

[Why? As in the proof of  $\otimes_7''$  above when we assume that we have carried the induction, by  $\boxplus_2$ , clause (g) and  $\otimes_4$ .]

Now

- $\otimes_{10}$  (a)  $f^{q_j} \in (\prod_{\zeta < \lambda_{\varepsilon(*)}} \theta_\zeta)^{N'_{\varepsilon(*)}[\bar{v}^j]}$
- (b) for some  $\zeta \in (\xi(i), \sigma)$  we have
  - $f^{q_j} \leq \eta_\zeta^{\varepsilon(*)} \text{ mod } J_{\lambda_\varepsilon}^{\text{bd}}$
  - $f^{q_j} \in N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$
  - $\langle \zeta(j_1) : j_1 < j \rangle \in N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)} \upharpoonright \zeta]$ .
- (c)  $\eta^{q_j} \triangleleft f^{q_j}$ .

<sup>9</sup>recall that  $\eta^{p^+(\gamma)}$  is an object, not a name and  $p_j^+$  is  $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$ -generic

[Why? Clause (a) follows from clause (b) and clause (b) should be clear by  $\otimes_9$  as we can choose  $\zeta(i)$  large enough recalling  $\otimes_6$ . Also clause (c) follows from (b).]

Now we choose  $\zeta(j)$  as in clause (b) of  $\otimes_{10}$  and  $\nu_j \in \prod_{\varepsilon < \lambda_{\varepsilon(*)}} \theta_\varepsilon$  such that  $\eta^{p^+(j)} \triangleleft \nu_j$ ,  $f^{q_j} \leq \nu_j$  and  $\{\iota < \lambda_{\varepsilon(*)} : \nu_j(\iota) \neq \eta_{\zeta(j)}^{\varepsilon(*)}\}$  is a bounded subset of  $\lambda_{\varepsilon(*)}$ . Next choose  $p_i^+ \in \mathbb{P}'_{\gamma(*)}$  such that  $p_i^+ \upharpoonright \gamma_j = p'_j$ ,  $\eta^{p_i^+(\gamma_i)} = \nu_j$  and  $f^{p_i^+(\gamma_i)} \upharpoonright [\lambda_\varepsilon, \lambda] = f^{p^+(\gamma)} \upharpoonright [\lambda_\varepsilon, \lambda]$ .

So we have carried the induction hence proved  $\otimes_6$  so we are done.

Moved from pgs.15,16:

#### § 4. FIRST TRY

{b2}

#### Hypothesis 4.1.

- (a)  $\lambda = \lambda^{<\lambda}$  strongly inaccessible with
  - ( $\alpha$ )  $\theta_\varepsilon = (2^{|\varepsilon| + \aleph_0})^+$ , or just
  - ( $\beta$ )  $\theta_\varepsilon = \text{cf}(\theta_\varepsilon) < \lambda$  is large enough
- (b)  $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$  as above
- (c)  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \gamma(*), \beta < \gamma(*) \rangle$  is a  $(< \lambda)$ -support iteration,  $\mathbb{Q}_\beta$  is  $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\mathbb{P}^\beta]}$ , see ??
- (d)  $\eta_\alpha$  is the  $\mathbb{Q}_\beta$ -name of the generic.

{b4}

**Definition 4.2.** For  $\gamma \leq \gamma(*)$ ,  $\mathbb{P}'_\gamma$  is the set of  $p$  such that

- (a)  $p$  is a function with domain  $\in [\gamma]^{<\lambda}$
- (b) for every  $\beta \in \text{Dom}(p)$ , for some  $\xi \leq \lambda$  and  $\bar{\beta} = \bar{\beta}_{p,\varepsilon,\beta} \in {}^\xi \gamma$  and  $\lambda$ -Borel function  $\mathbf{B} = \mathbf{B}_{p,\beta} : {}^\xi (\prod_{i < \lambda} \theta_i) \rightarrow \mathbb{Q}_{\bar{\theta}}$  we have  $p(\beta) = \mathbf{B}(\dots, \eta_{\beta(\varepsilon)}, \dots)_{\varepsilon < \xi}$

{b6}

**Claim 4.3.**  $\mathbb{P}'_\gamma$  is a dense subset of  $\mathbb{P}_\gamma$  for  $\gamma \leq \gamma(*)$ .

*Proof.* We prove (B) by induction on  $\gamma$ . □<sub>4.3</sub>

{b9}

**Definition 4.4.** 1) For  $\gamma < \gamma(*)$ ,  $\varepsilon < \lambda$  and  $\eta \in \prod_{i < \varepsilon} \theta_i$  let  $p = p_{\gamma,\eta}^* \in \mathbb{P}_{\gamma(*)}$  be the function with domain  $\{\gamma\}$  such that  $p(\gamma) = (\eta, \eta \hat{\ } 0_\lambda)$ , i.e.  $f_{p(\gamma)} \in \prod_{i < \alpha} \theta_i$  is defined

by  $f_{p(\gamma)}(i)$  is  $\eta(i)$  if  $i < \ell g(\eta)$  and is zero otherwise.

2) For  $\mathcal{U} \subseteq \gamma(*)$  let  $Y_{\mathcal{U}} = \{p_{\gamma,\eta}^* : \gamma \in \mathcal{U} \text{ and } \eta \in \prod_{i < \varepsilon} \theta_i \text{ for some } \varepsilon < \lambda\}$ .

3) For  $\mathcal{U} \subseteq \gamma(*)$  let  $\mathbb{P}[\mathcal{U}]$  be  $L_{\lambda^+}^+(Y_{\mathcal{U}}, \mathbb{P}_{\gamma(*)})$ , see Definition 0.6.

{b11}

**Claim 4.5.** 1)  $\mathbb{P}_{1,\gamma(*)} := \mathbb{L}_{\lambda^+}^+[\mathbb{P}, \mathbb{P}_{\gamma(*)}]$  is equivalent to  $\mathbb{P}$ .

2) If  $\mathcal{U} \subseteq \gamma \leq \gamma(*)$  then  $\mathbb{P}[\mathcal{U}] \triangleleft \mathbb{P}_{1,\gamma}$ .

3) If  $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \gamma(*)$  then  $\mathbb{P}[\mathcal{U}_1] \triangleleft \mathbb{P}[\mathcal{U}_2]$ .

4) If  $\gamma \in \mathcal{U}_1 \subseteq \gamma(*)$  then  $\eta_\gamma$  belongs to  $\mathbf{V}^{\mathbb{P}[\mathcal{U}]}$ , i.e.  $\Vdash_{\mathbb{P}_{\gamma(*)}} \text{“}\eta_\gamma \in \mathbf{V}[G_{\mathbb{P}_{1,\gamma(*)}} \cap \mathbb{P}[\mathcal{U}]]\text{”}$ .

5)  $\mathbf{V}[G_{\mathbb{P}_{1,\gamma(*)}} \cap \mathbb{P}[\mathcal{U}]] = \mathbf{V}[\langle \eta_i : i \in \mathcal{U} \rangle]$ .

*Proof.* Why? Part (1) is the content of  $(*)_{10} + (*)_{11}$  in the proof of 1.3(2).

Part (2)(b) holds as  $\mathbb{P}_{1,\gamma}$  satisfies the  $\lambda^+$ -c.c. and the definition.

Part (3)(c) is obvious by part (2).

Part (4)(d) holds by the choice of  $Y_{\mathcal{U}}$  and the  $p_{\gamma,\eta}^*$ 's.

Part (5) should be clear reflecting the definition. □<sub>4.5</sub>

{b13}

**Definition 4.6.** 1) Let  $\mathcal{F}$  be the set of partial order preserving functions  $f$  from  $\gamma(*)$  to  $\gamma(*)$  of cardinality  $\leq \lambda$ .

2) For  $f \in \mathcal{F}$  let  $f^{cl}$  be the unique function  $g$  such that:

( $\alpha$ )  $\alpha \in \text{dom}(g)$  iff  $\alpha \in \text{dom}(f)$  or  $\alpha = \sup(\alpha \cap \text{dom}(f)) < \gamma(*)$  or  $\alpha = 0$  or  $\alpha = \gamma(*)$

( $\beta$ )  $g(\alpha)$  is  $f(\alpha)$  iff  $\alpha \in \text{dom}(f) \cup \{f(\beta) : \beta \in \alpha \cap \text{dom}(f)\}$  iff  $\alpha = \sup(\alpha \cap \text{dom}(f)) \in \gamma(*) \setminus \text{dom}(f)$  iff  $\alpha = 0 \notin \text{dom}(f), \gamma(*)$  iff  $\alpha = \gamma(*)$ .

3) We define  $\mathcal{F}_\varepsilon$  by induction on  $\varepsilon$  by:  $f \in \mathcal{F}_\varepsilon$  iff  $f \in \mathcal{F}$  and for every  $u_1, u_2 \subseteq \gamma(*)$  of cardinality  $\leq \lambda$  and  $\zeta < \varepsilon$  there is  $f_1 \in \mathcal{F}_\zeta$  such that  $f \subseteq f_1$  and  $u_1 \subseteq \text{Dom}(f), u_2 \subseteq \text{Rang}(f)$

( $d$ ) for  $f \in \mathcal{F}$  let  $\hat{f}$  be the minimal partial function  $\hat{f}$  from  $\mathbb{P}[\text{Dom}(f)]$  into  $\mathbb{P}[\text{Rang}(f)]$  such that:

( $\alpha$ )  $\hat{f}(p_{\gamma, \eta}^*) = f_{f(\gamma), \eta}^*$  for  $\gamma \in \text{Dom}(f)$

( $\beta$ )  $\hat{f}(\neg\psi) = \neg\hat{f}(\psi)$  when  $\hat{f}(\psi)$  is well defined

( $\gamma$ )  $\hat{f}(\bigwedge_{i < \alpha} \psi_i) = \bigwedge_{i < \alpha} \hat{f}(\psi_i)$  when  $\hat{f}(\psi_i)$  is well defined for every  $i < \alpha$ .

## § 5

{d2}

**Definition 5.1.** We say  $\mathbf{m} \in \mathbf{M}$  is  $\mu$ -full (omitting  $\mu$  means  $\mu = \lambda_2$ ) when: if  $\mathbf{n} \in \mathbf{M}_{\lambda_1}$  and  $g$  embed  $\mathbf{n} \upharpoonright M_{\mathbf{n}}$  into  $\mathbf{m} \upharpoonright M_{\mathbf{m}}$  (so  $g_1$  preserves  $s < t, s \in u_t$  and their negations) then there is a sequence  $\langle g_\alpha, L_\alpha : \alpha < \mu \rangle$  such that:

- (a)  $L_\alpha \subseteq L_{\mathbf{m}}$  moreover  $L_\alpha \in \text{sub}(\mathbf{m})$
- (b)  $L_\alpha \cap L_\beta = \text{Rang}(g)$  for  $\alpha < \beta < \mu$
- (c)  $L_\alpha$  is  $E_{\mathbf{m}}$ -closed
- (d)  $L_\alpha \setminus \text{Rang}(g) \subseteq L_{\mathbf{m}} \setminus M_{\mathbf{m}}$
- (e)  $G_\alpha$  is an isomorphism from  $\mathbf{n}$  onto  $\mathbf{m} \upharpoonright L_\alpha$  extending  $g$ .

{d4}

**Claim 5.2.** 1) If  $\mathbf{m}_1 \in M_{\leq \chi}$  and  $\chi = \chi^{\lambda_1}$  then there is  $\mathbf{m}_2 \in \mathbf{M}_\chi$  such that  $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$  and  $\mathbf{M}_2$  is full.

2) If  $\mathbf{m} \in \mathbf{M}$  is full,  $|M_{\mathbf{m}}| = \lambda_1, |L_{\mathbf{m}}| = 2$  and  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  then  $\mathbb{P}_{\mathbf{m}} < \mathbb{P}_{\mathbf{n}}$ .

3) If  $\mathbf{m}_1 \in \mathbf{M}$  is  $\chi$ -full,  $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2 \in \mathbf{M}_\chi$  then  $\mathbf{m}_2, \mathbf{m}_2$  are isomorphic over  $M_{\mathbf{m}_1}$ .

*Proof.* Easy. □

{d6}

**Discussion 5.3.** 1) By using 5.2 it is easy to get “elimination of quantifiers” to some extent, e.g. an automorphism of  $\mathbf{m} \upharpoonright M_{\mathbf{m}}$ , but we need more:  $f$  is a partial automorphism of  $\mathbf{m} \upharpoonright M_{\mathbf{m}}$ .

2) Recall that for our purpose  $L_{\mathbf{m}} \upharpoonright M_{\mathbf{m}}$  is a linear order,  $u_t \cap M_{\mathbf{m}} = \{s \in M_{\mathbf{m}} : s <_{\mathbf{m}} t\}$ .

{d9}

**Definition 5.4.** For  $\chi = \chi^\lambda$ . We define  $\mathbf{M}_{\chi, \gamma}$  by induction on  $\gamma < \chi^+$  as follows:

- (a) for  $\gamma = 0, \mathbf{M}_{\chi, \gamma} = \mathbf{M}_\chi$
- (b) for  $\gamma = \beta + 1, \mathbf{M}_{\chi, \gamma}$  is the class of  $\mathbf{m} \in \mathbf{M}_{\chi, \beta}$  such that: if  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_{\chi, \beta}$  and  $L \in \text{Sub}(\mathbf{m})$  then  $dp_{\mathbf{m}}(L \cap M_{\mathbf{m}}) < \gamma$  then  $\mathbb{P}_{\mathbf{m}}(L) < \mathbb{P}_{\mathbf{n}}$
- (c) for  $\gamma$  a limit ordinal  $\mathbf{m} \in \mathbf{M}_{\chi, \gamma}$  iff  $\mathbf{m} \in \mathbf{M}_{\chi, \beta}$  for every  $\beta < \gamma$ .

{d11}

**Claim 5.5.** Let  $\chi = \chi^\lambda$  or better or  $\chi = \lambda_1$  and  $\gamma < \chi^+$ .

1) If  $\mathbf{m} \in \mathbf{M}_\chi$  then for some  $\mathbf{n} \in \mathbf{M}_{\chi, \gamma}$  we have  $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ .

2) If  $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$  is  $\leq_{\mathbf{M}}$ -increasing,  $\mathbf{m}_\alpha \in \mathbf{M}_{\chi, \gamma}$  for  $\alpha < \delta$  and  $\delta < \chi^+$  has cofinality  $> \lambda$  then  $\mathbf{m} := \cup \{ \mathbf{m}_\alpha : \alpha < \delta \}$  belongs to  $\mathbf{M}_{\chi, \gamma}$ .

3) If  $\mathbf{m} \in \mathbf{M}_{\chi, \gamma}$  and  $\mathbf{n} \in \mathbf{M}$  is  $2^\chi$ -full and  $L \subseteq L_{\mathbf{m}}$  and  $\mathbf{m} \cong \mathbf{n} \upharpoonright L$  then  $\mathbb{P}_{\mathbf{n} \upharpoonright L} < \mathbb{P}_{\mathbf{n}}$ .

## § 6. PRIVATE APPENDIX

We have used proving  $\textcircled{*}_4(c)$  inside the proof of 1.3

**Claim/Definition 6.1.** Assume  $\lambda$  is strongly inaccessible,  $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$  is an increasing sequence of regular cardinals such that  $\theta_\varepsilon > 2^{\sup\{\theta_\zeta : \zeta < \varepsilon\}}$  for  $\varepsilon < \lambda$ . Assume further  $\mathbb{Q} = \langle \mathbb{P}_\beta, \mathbb{Q}_\gamma : \gamma < \gamma(*), \beta \leq \gamma(*) \rangle$  is a  $(< \lambda)$ -support iteration and  $\mathbb{Q}_\gamma$  is  $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\mathbb{P}_\gamma]}$  the  $\bar{\theta}$ -dominating forcing (see in the proof above).

1) For  $u \subseteq \gamma \leq \gamma(*)$  we define  $\mathbb{P}_u$  and prove  $\mathbb{P}_u \subseteq \mathbb{P}_\gamma$  and moreover  $\mathbb{P}_u \triangleleft \mathbb{P}_\gamma$  and  $\mathbb{P}_u$  does not depend on  $\gamma$  by induction on  $\gamma : \mathbb{P}_u = \{p : p \in \mathbb{P}_\gamma \text{ with support } \subseteq u \text{ and for each } \beta \in \text{Dom}(p), p(\beta) \text{ is a } \mathbb{P}_{u \cap \beta}\text{-name of a condition in } \mathbb{Q}_\beta\}$ .

2)  $\mathbb{P}_u$  is isomorphic to  $\mathbb{P}_{\text{otp}(u)}$  naturally.

*Remark 6.2.* Saharon recheck. This is a special case of [Sh:F979]. Alternate.

*Proof.* The main point is:

(\*) assume  $\langle p_i : i < i(*) \rangle$  is a maximal antichain of  $\mathbb{P}_u$  then it is a maximal antichain of  $\mathbb{P}_\gamma$ .

Why? Without loss of generality  $\eta^{p_i(\alpha)}$ ,  $\alpha \in \text{Dom}(p_i)$ , is a real object (not a name).

Case 1:  $\gamma = 0$ .

Trivial.

Case 2:  $\beta := \sup(u) < \gamma$ .

Use  $\mathbb{P}_\beta \triangleleft \mathbb{P}_\gamma$ .

Case 3:  $\gamma$  is a limit ordinal, neither Case 1 nor Case 2.

So necessary  $\text{cf}(\gamma) < \lambda$ ; let  $\partial = \text{cf}(\gamma)$  and let  $\langle \gamma_\varepsilon : \varepsilon < \partial \rangle$  be increasing continuous with limit  $\gamma$  and let  $\gamma_\partial = \gamma$  and let  $\beta_\varepsilon = \min(u \cup \{\gamma\} \setminus \gamma_\varepsilon)$  for  $\varepsilon \leq \partial$ , so  $\beta = \gamma = \gamma_\partial$  and  $\varepsilon < \partial \Rightarrow \beta_\varepsilon < \gamma$ .

For any  $q \in \mathbb{P}_\gamma$  we choose  $p_\varepsilon$  by induction on  $\varepsilon \leq \partial$  such that

- $\textcircled{*}$  (a)  $p_\varepsilon \in N \cap \mathbb{P}_{u \cap \beta_\varepsilon}$
- (b)  $p_\varepsilon$  witness  $\mathbb{P}_v \triangleleft \mathbb{P}_\beta$  for  $q \upharpoonright \gamma_\varepsilon$  which means
  - (\*) if  $\alpha \in \text{Dom}(p_\varepsilon) \upharpoonright \alpha \in \mathbb{P}_{\alpha \cap u}, p_\varepsilon \upharpoonright \alpha \leq r \in \mathbb{P}_\gamma, q \upharpoonright \alpha \leq_{\mathbb{P}_\alpha} r$  and  $p_\varepsilon \upharpoonright (\alpha + 1) \leq_{\mathbb{P}_{(\alpha+1) \cap u}} p \in \mathbb{P}_{(\alpha+1) \cap u}$  and  $p \upharpoonright \alpha \leq_{\mathbb{P}_\alpha} r$  then  $r, p$  are compatible in  $\mathbb{P}_{\alpha+1}$
- (c) if  $\zeta < \varepsilon$  then  $p_\zeta \leq_{\mathbb{P}_{u \cap \beta_\varepsilon}} p_\varepsilon$ .

Case 4:  $\gamma = \beta + 1, \beta \in u$ .

Let  $v = u \cap \beta$  and let  $\mathbf{G}_\beta$  be a subset of  $\mathbb{P}_\beta$  generic over  $\mathbf{V}$  so  $\mathbf{G}_v = \mathbf{G}_\beta \cap \mathbb{P}_v$  is a subset of  $\mathbb{P}_v$  generic over  $\mathbf{V}$ . So we are interested in the forcing notion  $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\mathbf{G}_v]}$ . But in  $\mathbf{V}[G]$  the cardinal  $\lambda$  is weakly compact (and even supercompact). But easily this implies that " $\langle p_i : i < i_* \leq \lambda \rangle$  is predense in  $\mathbb{Q}_{\bar{\theta}}$ " is  $\lambda$ -Borel, so is absolute from  $\mathbf{V}[G_v]$  to  $\mathbf{V}[G_\beta]$ , i.e.,  $\mathbf{V}^{\mathbb{P}_v}$  to  $\mathbf{V}^{\mathbb{P}_\beta}$ .  $\square$

*Remark 6.3. Concluding Remarks:* 0) So see  $(*)_7$  in the proof of 1.3, if  $\aleph_0 \leq \theta_\varepsilon \leq \lambda$  for  $\varepsilon < \lambda$ , then  $\text{cov}_\lambda(\text{meagre}) \leq \text{cf}(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J^{\text{bd}}})$ .

1) In  $\textcircled{*}_0$  we can require  $\gamma(*) < \lambda^+$ . Why? If  $\kappa = \lambda^+$  so  $\gamma(*) = \kappa^+$  then by " $\mathbb{P}'_{\gamma(*)}$  satisfies the  $\lambda^+$ -c.c." so if  $p^* \Vdash_{\mathbb{P}'_\kappa} \text{"} f \text{ dominates } (\lambda \lambda)^{\mathbf{V}} \text{"}$  then for some  $\gamma(*) < \kappa, f$

{a9}

{a10}

is a  $\mathbb{P}'_{\gamma(*)}$ -name. Generally we can use a parallel of nep see [Sh:630]. We may treat this more generally.

2) We may control the various  $\text{cf}(\prod_{\varepsilon < \lambda} \theta_\varepsilon, < J_\lambda^{\text{bd}})$ , see [Sh:F923].

3) We may consider

{a11}

**Definition 6.4.** Assume  $\lambda$  is strongly inaccessible ( $> \aleph_0$ ),  $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$  where  $\theta_\varepsilon = \text{cf}(\theta_\varepsilon) < \lambda$  and  $\langle \kappa_\varepsilon : \varepsilon < \lambda \rangle$  are increasing continuous sequences of cardinals of length  $\lambda$  with limit  $\lambda$ . We say that the forcing notion  $\mathbb{Q}$  is  $(\bar{\theta}, \bar{\kappa})$ -centered when: there is a function  $f$  witnessing it which means

- (\*)  $f$  is a function from  $\mathbb{Q}$  to  $\lambda$  such that: if  $\kappa_\varepsilon \leq \alpha < \kappa_{\varepsilon+1}$  then  $\{p \in \mathbb{Q} : f(p) = \alpha\}$  is  $\theta_\varepsilon$ -directed.

## § 7. PRIVATE APPENDIX

Moved 2009/3/26 from end of §1 (right paper ? of 935?) (ref 4d.3)

**Claim 7.1.** *If  $\kappa = \mathfrak{s}$  and  $\text{cf}(\kappa) < \mathfrak{a}$  then the conclusions of ?? holds.*

{4d.6}

*Proof.* Similar to the proof of ??. Let  $\langle \gamma_i^x : i < \text{cf}(\kappa) \rangle$  be increasing with limit  $\kappa$ . In the proof we add

$\boxplus_1(h)$  if  $\ell g(\eta) \notin \{\gamma_i^* : i < \text{cf}(\kappa)\}$  then  $A_\eta = \emptyset$ .

□

§2 Weakly compact

**Theorem 7.2.** *If  $\lambda$  is weakly compact,  $\lambda < \kappa = \text{cf}(\kappa) < \mu = \mu^\lambda$ , then for some  $\lambda^+$ -c.c. ( $< \lambda$ )-strategically complete forcing notion  $\mathbb{P}$ , we have  $\Vdash_{\mathbb{P}} \mathfrak{d}_\lambda = \mu$ ,  $\text{cov}_\lambda(\text{meagre}) = \kappa$ .*

{b7}

*Proof. Stage A:*

We define  $\mathbb{Q}_{\mu+\kappa} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu + \kappa, \beta < \kappa \rangle$  as in the proof of 7.2 and let  $\underline{f}_\alpha, \underline{g}_i$  be as there and the proof up to  $(*)_7$  including  $(*)_7$  works.

Stage B: For every set  $u \subseteq \mu + \kappa$  we define  $\bar{\mathbb{Q}}_u = \langle P_\alpha^u, \mathbb{Q}_\beta^u : \alpha \in u \cup \{\mu + \kappa\}$  and  $\beta \in u \rangle$  by induction on  $\text{otp}(u)$  such that  $p \in \mathbb{P}_\alpha$  iff  $\text{Dom}(p) \subseteq u \cap \alpha$  and for every  $\beta \in \text{Dom}(\beta)$  the object  $p(\beta)$  is a  $\mathbb{P}_\alpha^u$ -name of a member of  $\mathbb{Q}_\beta$ , noting  $\mathbb{P}_\alpha^u = \mathbb{P}_{\mu+\kappa}^{u \cap \alpha}$ .

Also the order is natural

$\boxplus_1$  if  $u \subseteq \mu + \kappa$  then  $\mathbb{P}_{\mu+\kappa}^u \leq \mathbb{P}_{\mu+\kappa}$ .

[Why? As each  $\mathbb{Q}_\beta$  is absolutely enough defined and is  $2 - \kappa$ -linked.]

Add details? Use conditions  $p$  in  $\mathbb{P}_\alpha^\circledast$  ( $\alpha \leq \mu + \kappa$ )

$\boxplus_2$  (a)  $\mathbb{P}_\alpha^*$  :=  $\{f \in \mathbb{P}_\alpha : \text{the trunk } p(\beta_1) \text{ is an object for every } \beta \in \text{Dom}(p)\}$

(b)  $\mathbb{P}_\alpha^*$  is a dense subset of  $\mathbb{P}_\alpha$

(c)  $\mathbb{P}_\alpha^{u,*}$  is defined similarly

(d)  $\mathbb{P}_\alpha^{u,*}$  is a dense subset of

$\boxplus_3$   $\mathbb{P}_{\mu+\kappa}^* = \cup \{\mathbb{P}_{\mu+\kappa}^{u,*} : u \subseteq \mu + \kappa, |u| \leq \lambda\}$ .

Stage C:

$(*)_8$   $\Vdash_{\mathbb{P}_{\mu+\kappa}}$  “no  $\underline{f} \in {}^\lambda \lambda$  dominate  $\{f_\alpha : \alpha < \mu\}$ ”.

Toward contradiction assume

$\boxminus_1$   $p^* \Vdash_{\mathbb{P}_{\mu+\kappa}}$  “ $\underline{f} \in {}^\lambda \lambda$  is dominate  $\{f_\alpha : \alpha < \lambda\}$  equivalently  $({}^\lambda \lambda)^\mathbf{V}[\mathbb{P}_\lambda]$ ”.

Let  $u \in [\lambda + \kappa]^{< \lambda}$  be such that  $p^* \in \mathbb{P}_{\mu+\kappa}^u$  and  $\underline{f}$  is a  $\mathbb{P}_{\lambda+\kappa}^u$ -name, i.e. for every  $\alpha < \lambda$  there is a maximal antichain  $\mathcal{S}$  of  $\mathbb{P}_{\mu+\kappa}^u$  of conditions forcing a value to  $\underline{f}(\alpha)$  and is  $\subseteq \mathbb{P}_{\mu+\kappa}^{u,*}$ . As  $|u| = \lambda < \text{cf}(\mu)$  there is  $\beta_* \in (\text{sup}(u \cap \mu), \mu)$  and let  $v = u \cup \{\beta_*\}$  and let  $f_* = \underline{f}_{\beta_*}[\mathbf{G}_{\mathbb{P}_\mu}]$ .

Let  $\mathbf{G} \subseteq \mathbb{P}_{\mu+\kappa}^{u \cap \mu}$  be generic over  $\mathbf{V}$  such that  $p_* \upharpoonright \mu = p^* \upharpoonright (u \cap \mu) \in \mathbf{G}$ . We continue as there. □<sub>7.2</sub> □

§ 8. ON  $\lambda^+$ -C.C. 2-LINKED  $\lambda$ -NEP ITERATION

{c11.golen}

**Definition 8.1.** 1) Let  $K_{\lambda,2}$  be the class of  $\mathbf{p}$  consisting of

- (a)  $\bar{Q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha(*), \beta < \beta(*) \rangle$  be  $(< \lambda)$ -support iteration
- (b)  $\vdash_{\mathbb{P}_\beta}$  “ $\mathbb{Q}_\beta \rightarrow \lambda$  and  $\mathbb{Q}_\alpha$  is strategically  $(< \lambda)$ -complete
- (c)  $\xi$  is a limit ordinal  $< \lambda$
- (d)  $\vdash_{\mathbb{P}_\beta}$  “if  $\langle p_\zeta^\ell : \zeta < \xi \rangle$  is  $<_{\mathbb{Q}_\beta}$ -increasing for  $\ell = 1, 2$  and  $\mathfrak{c}_\beta(p_\zeta^1) = \mathfrak{c}_\beta(p_\zeta^2)$  for  $\zeta < \xi$  then  $\{p_\zeta^\ell : \ell = 1, 2 \text{ and } \zeta < \xi\}$  has a common upper bound
- (e) the set element of  $\mathbb{Q}_\beta$  are  $\subseteq \mathcal{H}_{< \lambda}(\chi_\beta)$  and it has a generic  $f_\beta \in {}^\lambda \lambda$  for simplicity
- (f)  $\mathfrak{c}_\beta$  is definable using  $\langle \eta_\gamma; \gamma \in u_\beta \rangle$  as parameters  $u_\beta \in [\beta]^{\leq \lambda}$ .

**Definition 8.2.** 1)  $K'_{\lambda,2}$  is defined similarly but now  $\mathfrak{c}_\beta(\langle p_\varepsilon^1 : \varepsilon < \zeta \rangle) = \mathfrak{c}_\beta(\langle p_\varepsilon^2 : \varepsilon < \zeta \rangle)$  for every  $\zeta < \xi$ . The amount of absoluteness, see below.2) In 4.5 and 4.6(1) let  $\mathbb{P}_\alpha = \mathbb{P}_\alpha^* = \mathbb{P}_{\mathbf{p},\alpha}$ ,  $\alpha(*) = \alpha_{\mathbf{p}}(*)$ , etc.

{c15}

**Definition 8.3.** For  $\mathbf{p} \in K_{\lambda,2}$ .

- 1) We say  $u$  is  $\mathbf{p}$ -closed when  $u \subseteq \alpha_{\mathbf{p}}(*)$  and  $\beta \in u \Rightarrow u_{\mathbf{p},\beta} \subseteq u$ .
- 2) For  $\mathbf{p}$ -closed we define  $bbQ_u = \mathbb{Q}_{\mathbf{p},u} = \langle \mathbb{P}_{\mathbf{p},u,\alpha}^*, \mathbb{Q}_{\mathbf{p},u,\beta} : \alpha \in u \cup \{\alpha_{\mathbf{p}}(*)\} \text{ and } \beta \in u \rangle$ . The definition is by induction on  $\text{otp}(u)$ :

$$p \in \mathbb{P}_{\mathbf{p},u,\alpha}^* = \{f : f \text{ is a function with domain } \in [\alpha \cap u]^{< \lambda} \\ \text{and for every } \beta \in \text{Dom}(f), f \upharpoonright \beta \in \mathbb{P}_{\mathbf{p},u,\beta}^* \text{ and there is} \\ \langle p_{\beta,\zeta}, \Upsilon_\zeta : \zeta < \varepsilon \rangle \text{ such that } f \upharpoonright \beta \vdash_{\mathbb{P}_{\mathbf{p},u,\beta}^*} \\ \text{“}\langle p_{\beta,\zeta} : \zeta < \xi \rangle \text{ is } \leq_{\mathbb{Q}_\beta[v[p_{\mathbf{p},u,\beta}^*]} \text{-increasing} \\ \text{and } \mathfrak{c}_\beta(p_{\beta,\zeta}) = \Upsilon_\zeta \text{”}\}.$$

**Claim 8.4.** Assume  $\mathbf{p} \in K_{\lambda,2}$ . If  $u \subseteq v$  are  $\mathbf{p}$ -closed then  $\alpha \in u \Rightarrow \mathbb{P}_{\mathbf{p},u,\alpha} < \mathbb{P}_{\mathbf{p},v,\alpha}$ .*Proof.* FILL. □

## § 9. PRIVATE APPENDIX 2

Moved 2010.12.22, was old 9.1 and proof, pg.20:

**Claim 9.1.** *If  $\mathbf{m}_1 \in \mathbf{M}_\gamma, L \subseteq L_{\mathbf{m}_1}$  and  $\mathbf{m} = \mathbf{m}_2 \upharpoonright L, \mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$  and  $L \in \text{Sub}(\mathbf{m}_1)$ ,  $\text{dp}_{\mathbf{m}_1}(L) < \gamma$  then  $\mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{n}}$ .*

{c37}

*Remark 9.2.* If in  $\mathbf{m}_1$  “everything possible of cardinality  $\leq \lambda_1$  occurs”, this is straight.

*Proof.* We prove this by induction on  $\gamma$ . Hence  $\mathbb{P}_{\mathbf{m}} \subseteq_{\text{ic}} \mathbb{P}_{\mathbf{n}}$ . Without loss of generality  $L_{\mathbf{m}} \cap L_{\mathbf{m}_1} = L$ .

Toward contradiction, assume  $\neg(\mathbb{P}_{\mathbf{m}} \triangleleft \mathbb{P}_{\mathbf{n}})$  so let  $\mathcal{S}$  be a maximal antichain of  $\mathbb{P}_{\mathbf{m}}$  and  $q \in \mathbb{P}_{\mathbf{n}}$  incompatible with every  $p \in \mathcal{S}$ . We can find  $L_1, L_2$  such that

- (\*) (a)  $L_1 \subseteq L_{\mathbf{m}}, |L_1| \leq \lambda_1$  and  $\mathcal{S} \subseteq \mathbb{P}_{\mathbf{m} \upharpoonright L_1}$
- (b)  $L_2 \subseteq L_{\mathbf{n}}, |L_2| \leq \lambda_1$  and  $q \in \mathbb{P}_{\mathbf{n} \upharpoonright L_2}$
- (c)  $L_1 = L_2 \cap L_{\mathbf{m}}$
- (d)  $\mathbb{P}_{\mathbf{m} \upharpoonright L_1} \triangleleft \mathbb{P}_{\mathbf{m}}$
- (e)  $\mathbb{P}_{\mathbf{n} \upharpoonright L_2} \triangleleft \mathbb{P}_{\mathbf{n}}$
- (f) if  $t \in L_1 \setminus L_{\mathbf{m}}$  then there is  $L'_1 \in \text{sub}(\mathbf{m}), t \in L'_1 \subseteq L_1$ , hence  $t/E_{\mathbf{m}} \subseteq L_1$
- (g) similarly for  $L_2, \mathbf{n}$ .

Now we define  $\mathbf{n}_1$  by:

- (\*) (a)  $L_{\mathbf{n}_1} = L_{\mathbf{m}} \cup L_2$
- (b)  $\leq_{\mathbf{n}_1}$  is the transitive closure of  $\leq_{\mathbf{m}} \cup \leq_{L_2}$
- (c)  $E_{\mathbf{n}_1} = E_{\mathbf{m}} \cup \{(ab) : a, b \in L_2 \setminus L_{\mathbf{m}}\}$ .

See 9.3.

□<sub>9.1</sub>

Moved 2010.12.22, was proof of 2.21, pgs.20,21:

*Proof.* We prove this by induction on  $\gamma$ .

So let  $\mathbf{m} \in \mathbf{M}, L \in \text{Sub}(\mathbf{m})$  and  $g$  is an isomorphism from  $\mathbf{m}_* \upharpoonright L_\gamma(\mathbf{m}_*)$  onto  $\mathbf{m} \upharpoonright L$  and let  $L_\alpha := \{g(t) : t \in L_\gamma(m_\alpha^*)\}$  for  $\alpha < \delta$  and  $L_\delta = L$ .

By xxx we have  $L_\gamma(m_*) \cap L_{\mathbf{m}_\alpha^*} = L_\gamma(\mathbf{m}_\alpha^*)$ . We should prove  $\mathbb{P}_{\mathbf{m}}(L) \triangleleft \mathbb{P}_{\mathbf{m}}$ . By the assumption for every  $\alpha < \delta$  we have  $\mathbb{P}_{\mathbf{m}}(L_\alpha) \triangleleft \mathbb{P}_{\mathbf{m}}$  and let  $\mathbb{P}' = \mathbb{P}_{\mathbf{m}} \upharpoonright \{p : p \in \mathbb{P}_{\mathbf{m}}(L)\}$ , so  $\mathbb{P}' \subseteq \mathbb{P}_{\mathbf{m}}(L)$  and  $\mathbb{P}_{\mathbf{m}}(L_\alpha) \subseteq \mathbb{P}'$ ; we know that  $\langle \mathbb{P}_{\mathbf{m}}(L_\alpha) : \alpha < \delta \rangle$  is a  $\triangleleft$ -increasing sequence and its union is  $\mathbb{P}'$  hence  $\alpha < \delta \Rightarrow \mathbb{P}_{\mathbf{m}}(L_\alpha) \subseteq_{\text{ic}}, \mathbb{P}' \subseteq_{\text{ic}} \mathbb{P}_{\mathbf{m}}$ .

Next let  $p_* \in \mathbb{P}_{\mathbf{m}}$  be given

- (\*) there is  $p \in \mathbb{P}'$  such that  $\mathbb{P}' \Vdash p \leq q \Rightarrow q, p_*$  are compatible.

[Why? There is  $p_\alpha \in \mathbb{P}_{\mathbf{m}}(L_\alpha)$  such that  $\mathbb{P}_{\mathbf{m}}(L_\alpha) \Vdash p_\alpha \leq q \Rightarrow q, p$  are compatible.

Let  $\mathcal{S} \subseteq \mathbb{P}'$  be a maximal subset of  $\{q \in \mathbb{P}' : p_*, q \text{ are incompatible in } \mathbb{P}_{\mathbf{m}}\}$  such that  $q_1 \neq q_2 \in \mathcal{S} \Rightarrow q_1, q_2$  are incompatible in  $\mathbb{P}'$ . As  $\mathbb{P}_{\mathbf{m}}$  satisfies the  $\lambda^+$ -c.c., clearly  $|\mathcal{S}| \leq \lambda$  and as  $p \in \mathbb{P}' \Leftrightarrow (\exists \alpha < \delta)(p \in \mathbb{P}_{\mathbf{m}}(L_\alpha) \text{ and } \text{cf}(\delta) > \lambda$  necessarily there is  $\alpha(*) < \delta$  such that  $\mathcal{S} \subseteq \mathbb{P}_{\mathbf{m}}(L_{\alpha(*)})$ . But  $\mathbb{P}_{\mathbf{m}}(L_{\alpha(*)}) \triangleleft \mathbb{P}_{\mathbf{m}}$  so  $p_*$  witness that

there is  $q \in \mathbb{P}'_{\mathbf{m}}(L_{\alpha(*)})$  incompatible with every (in  $\mathbb{P}'$ , equivalently in  $\mathbb{P}_{\mathbf{m}}(L_{\alpha(*)})$ ) member of  $\mathcal{S}$ .

By the choice of  $\mathcal{S}$ ,  $\mathbb{P}' \models "q \leq q_1" \Rightarrow q_1, p_*$  are compatible as promised in (\*).]

Hence

$$(*) \mathbb{P}' \triangleleft \mathbb{P}_{\mathbf{m}}.$$

Next

$$(*) \text{ if } t \in L \text{ then } \mathbb{P}_{\mathbf{m}}(L \cap L_{\mathbf{m}, < t}) \triangleleft \mathbb{P}_{\mathbf{m}}.$$

[Why? We use the induction hypothesis on  $\gamma$ .]

$$(*) \mathbb{P}_{\mathbf{m} \upharpoonright L} \subseteq \mathbb{P}_{\mathbf{m}}.$$

[Why?  $p \in \mathbb{P}_{\mathbf{m} \upharpoonright L} \Rightarrow p \in \mathbb{P}_{\mathbf{m}}$  by their definition.]

By the (\*) above and the definition of the orders in  $\mathbb{P}_{\mathbf{m} \upharpoonright L}$  and  $\mathbb{P}_{\mathbf{m}}$  we have  $\mathbb{P}_{\mathbf{m} \upharpoonright L} \subseteq \mathbb{P}_{\mathbf{m}}$  as partial order.

Case 1:  $\gamma = 0$  trivial.

Case 2:  $\gamma = \beta + 1$ .

$$(*) \mathbb{P}_{\mathbf{m} \upharpoonright L} \subseteq_{\text{ic}} \mathbb{P}_{\mathbf{m}}$$

$$(*) \mathbb{P}_{\mathbf{m} \upharpoonright L} \triangleleft \mathbb{P}_{\mathbf{m}}.$$

Case 3:  $\gamma$  is a limit ordinal.

FILL. □

Moved 2010.12.22, was after 9.3, pg.21:

{c36}

**Definition 9.3.** 1) Let  $\mathbf{M}^{\text{ec}}$  be the class of  $\mathbf{m}_* \in \mathbf{M}$  such that: if  $\mathbf{m} \in \mathbf{M}, L \in \text{Sub}(\mathbf{m})$  and  $\mathbf{m} \upharpoonright L$  is isomorphic to  $\mathbf{m}_*$  then  $\mathbb{P}_{\mathbf{m}}((L) \triangleleft \mathbb{P}_{\mathbf{m}} \text{ or } \mathbb{P}_{\mathbf{m} \upharpoonright L} \triangleleft \mathbb{P}_{\mathbf{m}}$  (notation!) (note that always  $\mathbb{P}_{\mathbf{n}_*}$  is isomorphic to  $\mathbb{P}_{\mathbf{m}}(L)$ ).

2) Let  $M_{\gamma}^{\text{ec}}$  be the class of  $\mathbf{m}_* \in \mathbf{M}$  such that: if  $\mathbf{m} \in \mathbf{M}, L \in \text{Sub}(\mathbf{m})$  and  $\mathbf{m} \upharpoonright L \cong \mathbf{m}_* \upharpoonright L_{\gamma}(\mathbf{m}_*)$  see below then  $\mathbb{P}_{\mathbf{m}}(L) \triangleleft \mathbb{P}_{\mathbf{m}}$ .

$$3) \mathbf{M}_{< \gamma}^{\text{ec}} = \bigcap_{\beta < \gamma} \mathbf{M}_{\beta}^{\text{ec}}.$$

4) Let  $L_{\mathbf{n}_*}(\gamma) = \{s \in L_{\mathbf{n}_*} : \text{dp}_{\mathbf{n}_*}(\{t \in M_{\mathbf{m}_*} : t \leq s\}) < \gamma\}$ .

## § 10. PRIVATE APPENDIX

*Question 10.1.* : (here? complete) Assume  $\langle \mathbb{P}_1^*, \mathbb{Q}_i^* : i < i(*) \rangle$  is a  $(< \lambda)$ -support iteration with limit  $\mathbb{P}_{i(*)}$ .

{a5x}

- 1) If  $\mathbb{Q}_i^*$  is the  $\theta$ -dominating forcing in  $\mathbf{V}^{\mathbb{P}_i^*}$ , as in the proof of ?? and  $u \leq i(*)$  and  $\langle \mathbb{P}_i^u, \mathbb{Q}_i^u : i \in u \rangle$  is defined similarly with limit  $\mathbb{P}'_u$  then  $\mathbb{P}_i^u \triangleleft \mathbb{P}_{i(*)}$ .
- 2) Like Suslin forcing/nep.

*Proof.* Let  $P_{u \cap j} = P_{\min(u \setminus j)}^u$ . We prove this by induction on  $i(*)$ . For  $i(*) = 0$  this is obvious. Generally clearly we interpret the iteration such that every  $p$  satisfies:  $i \in \text{Dom}(p) = \text{trunk}(p(i))$  is an object (not just a  $\mathbb{P}_i$ -name) and the names in the  $f^p(\alpha)$ ,  $(\alpha < \lambda)$  of  $p$  use maximal antichains of cardinality  $\leq \lambda$ .

Now

- ⊕<sub>1</sub> if  $p \in \mathbb{P}_u$  then  $p \in \mathbb{P}_{i(*)}$ ,  $j < i(*) \Rightarrow \mathbb{P}_{u \cap j}^u \triangleleft \mathbb{P}_j$
- ⊕<sub>2</sub>  $\mathbb{P}_u^u \subseteq \mathbb{P}_{i(*)}$  as partial order (similarly)
- ⊕<sub>3</sub>  $\mathbb{P}_u^u \triangleleft \mathbb{P}_{i(*)}$ .

[Why? Let  $\langle p_\varepsilon : \varepsilon < \varepsilon(*) \leq \lambda \rangle$  be a maximal antichain of  $\mathbb{P}_u^u$  and  $q \in \mathbb{P}_{i(*)}$ .] □

## § 11

Of course we need “no dominated  $f \in {}^\lambda\lambda$  is added”. On  $\mathbb{Q}_\theta$  O.K., but iterations?

Our true problem is as follows: for characteristic  $\lambda_\varepsilon$  having chosen the  $\mathbb{Q}$  we iterate (e.g.  $\mathbb{Q}_{\bar{\theta}}$  or see below, or can use a family  $\{\mathbb{Q}_*^i : i < i^*\}$  of definitions of  $\lambda^+$ -c.c. ( $< \lambda$ )-strategically).

Now repeating here the argument in the end of §1, the problem is that when we like to say that the value  $q_i \in Q_{\gamma_i}$  forced for  $(\eta^{p^+(\gamma_i)}, f^{p^+(\gamma_i)} \upharpoonright \lambda_\varepsilon)$  to be satisfied by a “variant of  $\eta_{\gamma_i}^\varepsilon = \text{changing } < \lambda_\varepsilon \text{ places}$ ”. But  $f^{p^+(\gamma_i)} \upharpoonright \lambda_\varepsilon$  depends on  $\eta_{\gamma_j}^\varepsilon, j \in (i, i(\varepsilon))$ .

So let us use in each  $\lambda_\varepsilon$  an iteration of length say  $\chi_\varepsilon$  of  $\mathbb{Q}^i$  (or of  $\mathbb{Q}^i$ 's (those are definitions) each occurring  $\chi_\varepsilon$  times and  $(\forall \alpha < \chi_\varepsilon)(|\alpha|^{\lambda_\varepsilon} < \chi_\varepsilon)$ ).

Now we will not use  $\langle \eta_{\gamma_i}^\varepsilon : i < \text{otp}(u_\varepsilon) \rangle$ , rather we choose  $\xi(i) < \chi_\varepsilon$  such that  $Q_{\xi(i)}^\varepsilon$  (is of the right kind and)  $\langle \xi(i) : \text{otp}(u_\varepsilon) \rangle$  has to increase fast enough such that  $\eta_{\xi(i)}^\varepsilon$  is generic also for the condition  $q_i$ . We can use a game and it suffices that the nice guy does not lose. Hence it suffices to consider

- (\*)  $\langle \bar{\xi}^\alpha : \alpha < \chi_\varepsilon \rangle, \bar{\xi}^\alpha$  increases of length  $< \lambda_\varepsilon^+$  such that for every  $F : \lambda_\varepsilon^+ \rightarrow \chi_\varepsilon$  and  $\beta < \lambda_\varepsilon^+$  there is  $\alpha$  such that  $\ell g(\bar{\xi}^\alpha) \geq \beta$  and  $\bar{\zeta} \triangleleft \bar{\xi}^\alpha \Rightarrow \xi_{\ell g(\bar{\zeta})}^\alpha$ .

So the question is

- assume  $\langle P_\alpha, \mathbb{Q}_\alpha : \alpha < \chi_\varepsilon \rangle$  is  $(< \lambda)$ -support iteration  $u \subseteq \chi_\varepsilon$  (with much space (a) is  $\mathbb{P}_\mu \triangleleft \mathbb{P}_{\chi_\varepsilon}$ ?
- a weaker version: in the iteration, on  $\mathbb{Q}_\alpha$  has a strange memory
  - part (1):  $v_\alpha \subseteq \alpha, |v_\alpha| \leq \lambda_\varepsilon$  such that  $\beta \in v_\alpha \Rightarrow v_\beta \triangleleft v_\alpha$ .
  - part (2): one condition  $p_\alpha$  from  $\mathbb{P}_\alpha$  (e.g. chosen).

We may use

**Definition 11.1.** For  $\bar{\theta}$  as before (or constantly  $\lambda$ ) let  $\mathbb{Q}_{\bar{\theta}}^{\text{ed}} = \{(\eta, \mathcal{F}) : \mathcal{F} \subseteq \prod_{\varepsilon < \lambda} \theta_\varepsilon, |\mathcal{F}| < \lambda, \eta \in \prod_{\varepsilon < \zeta} \theta_\varepsilon \text{ for some } \zeta < \lambda \text{ and } \varepsilon < \ell g(\eta) \wedge f \in \mathcal{F} \Rightarrow \eta(\varepsilon) \neq f(\varepsilon)\}$ .

order:  $(\eta_1, \mathcal{F}_1) \subseteq (\eta_2, \mathcal{F}_2)$  iff  $\eta_1 \triangleleft \eta_2 \wedge \mathcal{F}_1 \subseteq \mathcal{F}_2$ .

Problem: For FS iteration of definable nep c.c.c. forcing (e.g. Suslin) i.e. absolute maximal anti-chain we have  $\mathbb{P}_u \triangleleft \mathbb{P}_\alpha$ . How does it generalize to  $(< \lambda)$ -support iteration?

Question: Other uses of the machinery of §1?

## § 12. PRIVATE APPENDIX

Moved from pgs.4,5:

First Version:

Let  $\langle q_{\varepsilon, \zeta} : \zeta < \lambda \rangle$  be a maximal antichain of  $\mathbb{P}'_\kappa$  such that  $q_{\varepsilon, \zeta}$  forces a value  $\check{f}(\varepsilon)$ , say  $q_{\varepsilon, \zeta} \Vdash_{\mathbb{P}'_\kappa} \check{\xi}_{\varepsilon, \zeta}$ . Let  $u = \cup \{\text{Dom}(q_{\varepsilon, \zeta}) : \varepsilon, \zeta < \lambda\}$  so without loss of generality  $u \in [\kappa]^\lambda$ . Let  $\langle u_\varepsilon : \varepsilon < \lambda \rangle$  be an increasing continuous sequence of members of  $[u]^{<\lambda}$  with union  $u$ ,  $|u_\varepsilon| \leq |\varepsilon|$ . For each  $\varepsilon$  let  $\langle \bar{\eta}_\gamma^\varepsilon : \gamma < \gamma_\varepsilon \rangle$  list the  $\{\bar{\eta} : \eta = \langle \eta_j : j \in u \rangle, \eta_j \in \prod_{\zeta < \varepsilon} \theta_\zeta\}$ . So  $\gamma_\varepsilon < \theta_\varepsilon$  and  $\bar{\eta}_\gamma^\varepsilon = \langle \eta_{\gamma, j}^\varepsilon : j \in u_\varepsilon \rangle$ .

For each  $\varepsilon < \lambda$  and  $\gamma < \gamma_\varepsilon$  let

$$\mathcal{I}_{\varepsilon, \gamma} = \{q : q \in \mathbb{P}'_\kappa \text{ and } u_\varepsilon \subseteq \text{Dom}(q) \text{ and } j \in u_\varepsilon \Rightarrow q \upharpoonright j \Vdash_{\mathbb{P}'_\kappa} \text{“trunk}(q(j)) = \eta_{\gamma, j}^\varepsilon \text{”}\}.$$

Note

□  $\mathcal{I}_{\varepsilon, \gamma}$  is  $(< \theta_\varepsilon)$ -directed.

Let

⊠  $\mathcal{I}_{\varepsilon, \gamma}$  for  $\gamma < \lambda$  is  $\{q : q \in \mathbb{P}'_\kappa, u_\varepsilon \subseteq \text{Dom}(q), j \in u_\varepsilon = q \upharpoonright j \Vdash \text{trunk}(q) = \eta_{\gamma, j}^\varepsilon \text{ and } q \text{ forces a value of } \check{f}(\varepsilon)\}$ .

Now we use “in  $\mathbf{V}^{\mathbb{P}^\mu}$ ,  $\lambda$  is still weakly compact” to find  $\gamma_\varepsilon < \lambda$  such that

⊙ for every  $q \in \mathcal{I}_{\varepsilon, \gamma}$  there are  $\gamma < \gamma_\varepsilon$  and  $q' \in \mathcal{I}_{\varepsilon, \gamma}$  such that  $q \leq q'$ .

Now define  $g$

$$g(\varepsilon) = \text{Min}\{\alpha < \lambda : \text{for every } \gamma < \gamma_\varepsilon \text{ and } q \in \mathcal{I}_{\varepsilon, \gamma} \text{ for some } \alpha' < \alpha \text{ we have } q \not\Vdash \check{f}(\varepsilon) \neq \alpha'\}.$$

This is O.K. as  $\text{UI}_{\varepsilon, \gamma_\varepsilon}$  is the union of  $\leq \Sigma\{2^{|u_\varepsilon|+|\gamma'|} : \gamma' < \gamma\}$  directed sets.

Now  $g \in \mathbf{V}[\mathbf{G}_{\mathbb{P}^\mu}]$  so for some  $\alpha < \mu$ ,  $g <_{J_\lambda^{\text{bd}}} f_\alpha$ . But this means  $\Vdash_{\mathbb{P}'_\kappa}$  “ $\check{f}$  is not dominated by  $f_\alpha$  (and more: every stationary set modulo the clubs).”

So we are done.

So it is enough to prove  $(*)_9$  below. But before this...

## § 13. PRIVATE APPENDIX

**Discussion 13.1.** : 1)  $D$ -completeness for  $D$  a filter on  $\lambda$ . In preservation under iteration  $D$  is reasonable with regular ultrafilter, in every move only finitely many cardinals are active (832)?

2) We can define  $\mathbb{P}_u'', u \subseteq \chi$ . When is  $\mathbb{P}_u'' \subseteq \mathbb{P}_\kappa''$ ? That is, is being maximal antichains preserved.

## REFERENCES

- [Bar] Tomek Bartoszyński, *Invariants of Measure and Category*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), vol. 1, Springer, pp. 491–556.
- [Bar87] ———, *Combinatorial aspects of measure and category*, Fundamenta Mathematicae **127** (1987), 225–239.
- [Bla] Andreas Blass, *Combinatorial Cardinal Characteristics of the Continuum*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), vol. 1, Springer, pp. 395–490.
- [Kin66] Akiko Kino, *On definability of ordinals in logic with infinitely long expressions*, Journal of Symbolic Logic **31** (1966), 365–375.
- [Lan92] A. Landver, *Baire numbers, uncountable cohen sets and perfect-set forcing*, Journal of Symbolic Logic **57** (1992), 1086–1107.
- [Lav78] Richard Laver, *Making the supercompactness of  $\kappa$  indestructible under  $\kappa$ -directed closed forcing*, Israel J. of Math. **29** (1978), 385–388.
- [Mil82] Arnold W. Miller, *A characterization of the least cardinal for which the baire category theorem fails*, Proceedings of the American Mathematical Society **86** (1982), 498–502.
- [vD84] Eric K. van Douwen, *The integers and topology*, Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), Elsevier Science Publishers, 1984, pp. 111–167.
- [Sh:80] Saharon Shelah, *A weak generalization of MA to higher cardinals*, Israel Journal of Mathematics **30** (1978), 297–306.
- [JdSh:292] Jaime Ihoda (Haim Judah) and Saharon Shelah, *Souslin forcing*, The Journal of Symbolic Logic **53** (1988), 1188–1207.
- [CuSh:541] James Cummings and Saharon Shelah, *Cardinal invariants above the continuum*, Annals of Pure and Applied Logic **75** (1995), 251–268, math.LO/9509228.
- [Sh:546] Saharon Shelah, *Was Sierpiński right? IV*, Journal of Symbolic Logic **65** (2000), 1031–1054, math.LO/9712282.
- [Sh:630] ———, *Properness Without Elementarity*, Journal of Applied Analysis **10** (2004), 168–289, math.LO/9712283.
- [MtSh:804] Pierre Matet and Saharon Shelah, *Positive partition relations for  $P_\kappa(\lambda)$* , Preprint, math.LO/0407440.
- [Sh:F923] Saharon Shelah, *Iterated creatures for inaccessibles*.
- [Sh:F979] ———, *Iterating reasonable  $\lambda$ -complete definable forcing*.

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL, AND, DEPARTMENT OF MATHEMATICS, HILL CENTER - BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA

*E-mail address:* shelah@math.huji.ac.il

*URL:* <http://shelah.logic.at>