PCF WITH LITTLE CHOICE
SH955

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Anotated Content

§0 Introduction

§1 On pseudo true cofinality, pg.2

[We continue [Sh:938, §5] to try to generalize the pcf theory for $\aleph_1$-complete filters D on Y assuming DC + $\mathrm{AC}_{\mathcal{P}(Y)}$. So is similar to [Sh:b, ChXII]. We suggest to replace cofinality by pseudo cofinality.]

§2 Depth of reduced power of ordinals, pg.10

[Using the independence property for a sequence of filters we can bound the relevant depth.]

0. Introduction

In the first section we deal with generalizing the pcf theory in the direction as started in [Sh:938, §5]. The point is that we assume $\mathrm{AC}_{\mathcal{U}}$ for any set $\mathcal{U}$ of power $\le |\mathcal{P}(\mathcal{P}(Y))|$ or actually working harder, just $\le |\mathcal{P}(Y)|$ analyzing $\prod\limits_{t \in Y} \alpha_t$, but we <u>do not</u> assume $\mathrm{AC}_{\sup\{\alpha_t : t \in Y\}}$. The price is that we replace (true) cofinality by pseudo (true) cofinality.

In the second section we prove a relative of [Sh:513, §3]; again dealing with depth (instead of rank as in [Sh:938]) adding some information even under ZFC. Assuming that the sequence $\langle D_n : n < \omega \rangle$ of filters has the independence property (IND), see Definition 2.3, with D_n a filter on Y_n we can bound the depth of ${}^{(Y_n)}\zeta$ by ζ for many m's, see 2.4. Of course, we can generalize this to $\langle D_s : s \in S \rangle$. This is incomparable with the results of [Sh:938, §4], also we add some cases to [Sh:938, §4].

Note that the assumptions like $\mathrm{IND}(\bar{D})$ are complimentary to ones used in [Sh:835] to get considerable information. Our original hope was to arrive to a dichotomy. The first possibility will say that one of the versions of an axiom suggested in [Sh:835] holds, which means "for some suitable algebra", there is no independent ω-sequence; in this case [Sh:835] tells us much. The second possibility will be a

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case of IND, and then we try to show that there is a rank system in the sense of [Sh:938]. But presently for this we need too much choice. The dichotomy we have has o-Depth in one side, the results of [Sh:835] on the other side. It would be better to have ps-o-Depth in the first side.

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Question 0.1. $[\mathrm{AC}_{\mathcal{P}(Y)}]$
Assume

- (a) $\bar\alpha \in {}^Y\mathrm{Ord}$
- (b) $\mathrm{cf}(\alpha_t) \ge \theta(\mathcal{P}(Y)$ for every $t \in Y$
- (c) $\lambda_t \in \ \mathrm{pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$ for $t \in Z$, in fact, $\lambda_t = \ \text{ps-tcf}(\Pi\bar\alpha, <_{D_t}), D_t$ is a $\aleph_1$-complete filter on Y
- (d) $\lambda = \ \text{ps-tcf}_{\aleph_1\text{-comp}}(\langle\lambda_t : t \in Z\rangle$
- (e) (a possible help) $X_t \in D_t, \langle X_t : t \in Y\rangle$ are pairwise disjoint.

(A) Now does $\lambda \in \ \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$? Can we say something on D_λ from 8.9? in [Sh:938, 5.9]

(B) At least when trying to generalize the RGCH, see [Sh:460] and [Sh:829].

1. On pseudo true cofinality

We continue [Sh:938, §5]. A central definition here is

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Definition 1.1. For $\bar\alpha \in {}^Y\mathrm{Ord}$ let $J^{\aleph_1\text{-comp}}_{<\lambda}(\bar\alpha) = \{X \subseteq Y\colon \text{ps-pcf}_{\aleph_1\text{-com}}(\bar\alpha \restriction X) \subseteq \lambda\}$ and $J^{\aleph_1\text{-comp}}_{\le\lambda}$ is $J^{\aleph_2\text{-comp}}_{<\lambda^+}$.

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Claim 1.2. *<u>The Generator Existence Claim</u>*
Let $\bar\alpha \in {}^Y(\mathrm{Ord}\backslash\{0\})$.
1) $[\mathrm{AC}_{\aleph_0}]$ $J^{<\aleph_1\text{-comp}}_{<\lambda}(\bar\alpha)$ *is an* $\aleph_1$*-complete ideal on* Y *for any cardinal* λ *but may be* $\mathcal{P}(Y)$.
2) $[\mathrm{AC}_{\mathcal{P}(Y)}]$ *Assume* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \theta(\mathcal{P}(Y))$. *If* $\lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$ *<u>then</u> for some* $X \subseteq Y$

- (A) $J^{\aleph_1\text{-comp}}_{<\lambda^+}[\bar\alpha] = J^{\aleph_1\text{-comp}}_{<\lambda}[\bar\alpha] + X$
- (B) $\lambda = \text{ps-tcf}(\Pi\bar\alpha, <_{J^{\aleph_1\text{-comp}}_{=\lambda}[\bar\alpha]})$ *where* $J^{\aleph_1\text{-comp}}_{=\lambda}[\bar\alpha] := J^{\aleph_1\text{-comp}}_{<\lambda}[\bar\alpha] + (Y\backslash X)$
- (C) $\lambda \notin \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha \restriction (Y\backslash X))$.

Proof. 1) Easy by the definition.
2) Recall λ is a regular cardinal by [Sh:938, 5.8(0)] and $\lambda \ge \theta(\mathcal{P}(Y))$ by [Sh:938, 5.16].

Let $D = D^{\bar\alpha}_\lambda$ be from [Sh:938, 5.19], i.e. $\Pi\bar\alpha/D$ has pseudo true cofinality λ and contains any other such $\aleph_1$-complete filter on Y. Now if $X \in D^+$ then $\lambda = \ \text{ps-tcf}_{\aleph_1\text{-comp}}(\bar\alpha \restriction X, <_{(D+X)\cap\mathcal{P}(X)})$ hence $X \notin J^{\aleph_1\text{-comp}}_{<\lambda}[\bar\alpha]$, so

$$(*)_1 \ \ X \in J^{\aleph_1\text{-comp}}_{<\lambda}[\bar\alpha] \Rightarrow X = \emptyset \bmod D.$$

A major point is

$$(*)_2 \ \ \text{some } X \in D \text{ belongs to } J^{\aleph_1\text{-comp}}_{<\lambda^+}[\bar\alpha].$$

Why $(*)_2$? The proof will take awhile; assume that not, we have $\mathrm{AC}_{\mathcal{P}(Y)}$ hence AC_D, so we can find $\langle(\bar{\mathcal{F}}^X, D_X, \lambda_X) : X \in D\rangle$ such that

- (a) λ_X is a regular cardinal $\geq \lambda^+$, i.e. $> \lambda$
- (b) D_X is an $\aleph_1$-complete filter on Y such that $X \in D_X$ and $\lambda_X = \text{ ps-tcf}(\Pi\bar{\alpha}, <_{D_X})$
- (c) $\bar{\mathcal{F}}^X = \langle \mathcal{F}^X_\alpha : \alpha < \lambda_X\rangle$ exemplifies that $\lambda_X = \text{ ps-tcf}(\Pi\bar{\alpha}, <_{D_X})$
- (d) moreover $\bar{\mathcal{F}}^X$ is as in [Sh:938, 5.17(2)].

Let

- (e) $D^*_1 = \{A \subseteq Y\colon$ for some $X_1 \in D$ we have $X \in D \wedge X \subseteq X_1 \Rightarrow A \in D_X\}$.

Clearly

- (f) D^*_1 is an $\aleph_1$-complete filter on Y extending D.

[Why? First, if $A \in D$ then $X \in D \wedge X \subseteq A \Rightarrow A \in D_X$ hence choosing $X_1 = A$ the demand for "$A \in D^*_1$" holds so indeed $D \subseteq D^*_1$. Second, assume $\bar{A} = \langle A_n : n < \omega\rangle$ and "$A_n \in D^*_1$" for $n < \omega$, <u>then</u> for each A_n there is a witness $X_n \in D$ so by $\mathrm{AC}_{\aleph_0}$ there is an ω-sequence $\langle X_n : n < \omega\rangle$ with X_n witnessing $A_n \in D^*_1$. Then $X = \cap\{X_n : n < \omega\}$ belongs to D and witness that $A := \cap\{A_n : n < \omega\} \in D^*_1$ because D is $\aleph_1$-complete.]

- (g) we can choose an $\aleph_1$-complete filter $D^*_2 \supseteq D^*_1$ on Y such that $(\Pi\bar{\alpha}, <_{D^*_2})$ has a true pseudo cofinality.

[Why? By [Sh:938, 5.9].]

- (h) assume $\langle \mathcal{F}_\alpha : \alpha < \lambda\rangle$ is $<_D$-increasing, i.e. $\alpha < \lambda \Rightarrow \mathcal{F}_\alpha \subseteq \Pi\bar{\alpha}$ and $\alpha_1 < \alpha_2 \wedge f_1 \in \mathcal{F}_{\alpha_1} \wedge f_2 \in \mathcal{F}_{\alpha_1} \Rightarrow f_1 <_D f_2$ and $\mathcal{F}_\alpha \neq \emptyset$ for every or at least unboundedly many $\alpha < \lambda$ <u>then</u> $\bigcup\limits_{\alpha<\lambda} \mathcal{F}_\alpha$ has a common $<_{D^*_2}$-upper bound.

[Why? For each $X \in D$ recall $(\Pi\bar{\alpha}, <_{D_X})$ has true cofinality λ_X which is $> \lambda$ hence by [Sh:938, 5.7(1)] is pseudo λ^+-directed hence there is a common $<_{D_X}$-upper bounded h_X of $\cup\{\mathcal{F}^*_\alpha : \alpha < \lambda\}$. As we have $\mathrm{AC}_{\mathcal{P}(Y)}$ we can find a sequence $\langle h_X : X \in D\rangle$ with each h_X as above. Define $h \in \Pi\bar{\alpha}$ by $h(t) = \sup\{h_X(t) : X \in D\}$, it belongs to $\pi\bar{\alpha}$ as we are assuming $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \geq \theta(\mathcal{P}(Y))$. So $h \in \Pi\bar{\alpha}$ is a $<_{D_X}$-upper bound of $\cup\{\mathcal{F}_\alpha : \alpha < \lambda\}$ for every $X \in D$, hence is a $<_{D^*_1}$-upper bound of $\cup\{\mathcal{F}_\alpha : \alpha < \lambda\}$ but $D^*_1 \subseteq D^*_2$ hence it is also a $<_{D^*_2}$-upper bound of $\cup\{\mathcal{F}_\alpha : \alpha < \lambda\}$.]

But by the choice of D in the beginning of the proof we have $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ so there is a sequence $\langle \hat{\mathcal{F}}_\alpha : \alpha < \lambda\rangle$ witnessing it. Clearly $\langle \hat{\mathcal{F}}_\alpha : \alpha < \lambda\rangle$ is $<_{D^*_2}$-increasing hence we can apply clause (h) to the sequence $\langle \hat{\mathcal{F}}_\alpha : \alpha < \lambda\rangle$ and got a $<_{D^*_2}$-upper bound $f \in \Pi\bar{\alpha}$, contradiction to the choice of $\langle \hat{\mathcal{F}}_\alpha : \alpha < \lambda\rangle$ because $D \subseteq D^*_1 \subseteq D^*_2$. So $(*)_2$ really holds.

Choose X as in $(*)_2$, now

- $(*)_3$ $D = \text{ dual}(J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}] + (Y \backslash X))$.

[Why? The inclusion $\supseteq$ holds by $(*)_1$ and $(*)_2$, i.e. the choice of X as a member of D. Now for every $Z \subseteq X$ which does not belong to $J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}]$, by the definition of $J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}]$ there is an $\aleph_1$-complete filter D_Z on Y to which Z belongs such that $\theta :=$ ps-cf$(\Pi\bar{\alpha}, <_D)$ is well defined and $\geq \lambda$. But $\theta \geq \lambda^+$ is impossible as we know that $Z \subseteq X \in J^{\aleph_1\text{-comp}}_{<\lambda^+}[\bar{\alpha}]$, so necessarily $\theta = \lambda$, hence by the choice of $D, D \subseteq D_Z$, hence $Z \neq \emptyset$ mod D. Together we are done.]

$(*)_4$ $\lambda =$ ps-tcf$(\Pi\bar{\alpha}, <_{J^{\aleph_1\text{-comp}}_{=\lambda}})$, see clause (B) of the conclusion of 1.2(2).

[Why? By $(*)_3$, the choice of $J^{\aleph_1\text{-comp}}_{=\lambda}[\bar{\alpha}]$ and as $\lambda =$ ps-tcf$(\Pi\bar{\alpha}, <_D)$.]

$(*)_5$ $\lambda \notin$ ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha} \restriction (Y \backslash X))$.

[Why? Otherwise there is an $\aleph_1$-complete filter D on Y such that $Y \backslash X \in D$ and $\lambda =$ tcf$(\Pi\bar{\alpha}, <_D)$. But this contradicts the choice of D.]

So X is as required in the desired conclusion of 1.2(2): clause (B) by $(*)_4$, clause (C) by $(*)_5$ and clause (A) follows. Note that the notation $J^{\aleph_1\text{-comp}}_{=\lambda}[\bar{\alpha}]$ is justified, as if X' satisfies the requirements on X then $X' = X$ mod $J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}]$. $\square_{1.2}$

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Conclusion 1.3. [AC$_{\aleph_0}$ + AC$_{\mathcal{P}(Y)}$] Assume $\bar{\alpha} \in {}^Y\text{Ord}$ and each α_t a limit ordinal of cofinality $\geq \theta(\mathcal{P}(Y))$.
1) There is a function h such that

- the domain of h is $\mathcal{P}(Y)$
- the range of h is ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha}) \cup \{\mu : \mu = \sup(\mu \cap \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}))$ has cofinality $\aleph_0\}$
- $A \subseteq B \subseteq Y \Rightarrow h(A) \leq h(B)$
- $h(A) =$ min$\{\lambda : A \in J^{\aleph_1\text{-comp}}_{\leq\lambda}[\bar{\alpha}]\}$
- if $h(A) = \lambda$, cf$(\lambda) > \aleph_0$ then for some $\aleph_1$-complete filter D on Y we have $A \in D$ and ps-tcf$(\Pi\bar{\alpha}, <_D) = \lambda$
- the set ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha})$ has cardinality $< \theta(\mathcal{P}(Y))$
- if $h(A) = \lambda$ and cf$(\lambda) = \aleph_0$ then we can find a sequence $\langle A_n : n < \omega\rangle$ such that $A = \cup\{A_n : n < \omega\}$ and $h(A_n) < \lambda$ for $n < \omega$
- $J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}] = \{A \subseteq Y : h(A) < \lambda\}$ when cf$(\lambda) > \aleph_0$
- if cf(otp(ps-pcf$_{\aleph_1\text{-com}}(\bar{\alpha})$)) $> \aleph_0$ then ps-pcf$_{\aleph_1\text{-com}}(\bar{\alpha})$ has a last member.

2) Assume also $\alpha < \theta(\mathcal{P}(Y)) \Rightarrow$ AC$_\alpha$. We can find $\langle X_\lambda : \lambda \in$ ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha})\rangle$ such that for each $\lambda \in$ ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha})$, the ideal $J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}]$ is the $\aleph_1$-complete ideal generated by $\{X_\mu : \mu < \lambda\}$ and the filter on Y generated by $\{Y \backslash X_\mu : \mu \in \lambda \cap$ ps-pcf$_{\aleph_1\text{-com}}(\bar{\alpha})\} \cup \{X_\lambda\}$ is the filter from [Sh:938, 5.19].

Proof. 1) Let $\Theta =$ ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha})$. We define function h from $\mathcal{P}(Y)$ into Θ^+ which is defined as the closure of Θ, i.e. $\Theta \cup \{\mu : \mu = \sup(\mu \cap \Theta)\}$, by $h(X) =$ Min$\{\lambda \in \Theta^+ : X \in J^{\aleph_1\text{-comp}}_{\leq\lambda}[\bar{\alpha}]\}$. It is well defined as ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha})$ is a set (see [Sh:938, 5.8(2),(8)]), and $J^{\aleph_1\text{-comp}}_{\leq\lambda}[\bar{\alpha}] = \mathcal{P}(Y)$ when $\lambda \geq$ sup(ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha})$). This function, its range is included in Θ^+, but otp$(\Theta^+) \leq$ otp$(\Theta) + 1$; also clearly the first clause of the conclusion holds.

Now first if $\lambda \in \Theta^+$ and cf$(\lambda) = \aleph_0$ then clearly $\lambda \in \Theta^+ \backslash \Theta$ and we can find an increasing sequence $\langle \lambda_n : n < \omega\rangle$ of members of ps-pcf$_{\aleph_1\text{-comp}}(\bar{\alpha})$ with limit λ. For each n there is $X_n \in J^{\aleph_1\text{-comp}}_{\leq\lambda_n}[\bar{\alpha}] \backslash J^{\aleph_1\text{-comp}}_{<\lambda_n}[\bar{\alpha}]$ by 1.2(2), but AC$_{\aleph_0}$ holds hence such a sequence $\langle X_n : n < \omega\rangle$ exists. Easily $A := \cup\{X_n : n < \omega\} \in \mathcal{P}(Y)$ satisfies $h(A) = \lambda$ hence $\lambda \in \text{Rang}(h)$. Second, $\lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha) \Rightarrow (\exists A)h(A) = \lambda$ by 1.2(2). Third, if $\lambda = \sup(\text{ps-pcf}_{\aleph_1\text{-com}}(\bar\alpha))$ and $\text{cf}(\lambda) > \aleph_0$, then $J_{<\lambda}[\bar\alpha] \neq \mathcal{P}(Y)$ while $J_{<\lambda^+}(\bar\alpha) = \mathcal{P}(Y)$ so $h(Y) = \lambda$.

Fourth, assume $\lambda = h(A), \lambda \notin \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$ and $\text{cf}(\lambda) > \aleph_0$, we can find $\langle \lambda_i : i < \text{cf}(\lambda)\rangle$, an increasing sequence with limit λ, so without loss of generality all are members of $\text{ps-pcf}_{\aleph_1\text{-comp}}(\Pi\bar\alpha)$. Now $\langle J_i := J^{\aleph_1\text{-comp}}_{\lambda_i}[\bar\alpha] : i < \text{cf}(\lambda)\rangle$ is a $\subseteq$-increasing sequence of $\aleph_1$-complete ideals on Y, no choice is needed, and by our present assumption $\aleph_0 < \text{cf}(\lambda)$ hence the union $J = \cup\{J_i : i < \text{cf}(\lambda)\rangle$ is an $\aleph_1$-complete ideal on Y and obviously $A \notin J$. So also $D_1 = \text{dual}(J)$ is an $\aleph_1$-complete filter hence for some $\aleph_1$-complete ideal D_2 extending D_1 we have $\mu' = \text{ps-tcf}(\Pi\alpha, <_{D_2})$ is well defined, but $(\Pi\bar\alpha, <_{J_i})$ is pseudo λ_i-directed hence $\mu' \geq \lambda_i$ for every $i < \text{cf}(\lambda)$ but λ is singular so $\mu' > \lambda$ and $\mu' \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$. Hence $\mu = \min(\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)\backslash\lambda)$ is well defined and $J^{\aleph_1\text{-comp}}_{<\mu}[\bar\alpha] = J$. But as $h(A) < \mu$ we get that $A \in J^{\aleph_1\text{-comp}}_{<\mu}[\bar\alpha]$, contradiction.

So we have proved $\lambda \in \Theta^+ \wedge \text{cf}(\lambda) > \aleph_0 \Rightarrow \lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$, this proves the fifth clause of the conclusion and also proves $\text{Rang}(h) \subseteq \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha) \cup \{\mu : \text{cf}(\mu) = \aleph_0$ and $\mu = \sup(\mu \cap \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)\}$. The other inclusion was proved earlier, so the second clause of the conclusion holds. The other clauses follow from the properties of h.

2) Obvious: by part (1) and 1.2, as $\text{pcf}_{\aleph_1\text{-comp}}(\bar\alpha) < \theta(\mathcal{P}(Y))$. $\square_{1.3}$

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Definition 1.4. Assume $\text{cf}(\mu) < \theta(Y)$ and μ is a singular limit of regulars. We let

(a) $\text{pp}^*_Y(\mu) = \sup\{\lambda :$ for some $\bar\alpha, D$ we have
- (a) $\lambda = \text{ps-tcf}(\Pi\bar\alpha, <_D)$,
- (b) D is an $\aleph_1$-complete filter on Y
- (c) $\bar\alpha = \langle \alpha_t : t \in Y\rangle$, each α_t regular
- (d) $\mu = \lim_D \bar\alpha\}$

(b) $\text{pp}^+_Y(\mu) = \sup\{\lambda^+ : \lambda$ as above$\}$.

(c) similarly $\text{pp}^*_{\kappa\text{-comp},Y}(\mu), \text{pp}^+_{\kappa\text{-comp},Y}(\mu)$ restricting ourselves to κ-complete filters D; similarly for other properties

(d) we can replace Y by an $\aleph_1$-complete filter D on Y, this means we fix D but not $\bar\alpha$ above.

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Conclusion 1.5. [DC + $\text{AC}_{\mathcal{P}(Y)}$] Assume $\theta_1 = \theta(\mathcal{P}(Y)) < \mu, \mu$ is as in Definition 1.4, $\mu_0 < \mu, \sigma = |\text{Reg} \cap \mu\backslash\mu_0| < \mu$ and $\kappa = |\text{Reg} \cap \text{pp}^+_Y(\mu)\backslash\mu_0|$ <u>then</u> $\kappa < \theta(\theta_1 \times {}^Y\sigma)$.

Proof. Obvious by Definition [Sh:938, 5.6] noting Conclusion 1.3 above and 1.6 below. That is, letting $\Lambda_0 = \text{Reg} \cap \mu\backslash\mu_0$, for every $\bar\alpha \in {}^Y\Lambda$ by 1.3 the set $\text{ps-cf}_{\aleph_1\text{-comp}}(\bar\alpha)$ is a subset of $\Xi = \text{Reg} \cap \text{pp}^+_Y(\mu)\backslash\mu_0$, of cardinality $< \theta(\mathcal{P}(Y))$. By 1.6 below we have $\Xi = \cup\{\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha) : \bar\alpha \in {}^Y\Lambda\}$. So there is a function h with domain $\theta(\mathcal{P}(Y)) \times {}^Y\sigma$ such that $\varepsilon < \theta(\mathcal{P}(Y)) \cap \bar\alpha \in {}^Y\sigma \Rightarrow (h(\varepsilon, \bar\alpha)$ is the ε-th member of $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$ if there is one, $\min(\Lambda)$ otherwise). So h is a function from $\theta(\mathcal{P}(Y)) \times {}^Y\sigma$ onto a set of cardinality κ, so we are done. $\square_{1.5}$

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Claim 1.6. *<u>The no hole claim</u>[DC] 1) If $\bar\alpha \in {}^Y\text{Ord}$ and $\lambda_2 \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$ and $\theta(\mathcal{P}(Y)) \leq \lambda_1 = \text{cf}(\lambda_1) < \lambda_2$, <u>then</u> for some $\bar\alpha' \in \Pi\bar\alpha$ we have $\lambda_1 = \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha')$.*
2) If in addition $\text{AC}_{\mathcal{P}(Y)} + \text{AC}_{<\kappa}$ <u>then</u> even witnessed by the same filter (on Y).

Proof. 1) Let D be an $\aleph_1$-complete filter on Y such that $\lambda_2 =$ ps-tcf$(\Pi\bar\alpha, <_D)$, let $\langle \mathcal{F}_\alpha : \alpha < \lambda_2\rangle$ exemplify this.

First assume $\theta(\mathrm{Fil}^1_{\aleph_1}(Y)) \le \lambda_1$, clearly $f \in \mathcal{F}_{\lambda_1} \Rightarrow \mathrm{rk}_D(f) \ge \lambda_1$ hence there is $f \in {}^Y\mathrm{Ord}$ such that $\mathrm{rk}_D(f) = \lambda_1$ and now use [Sh:938, 5.9] but we change the filter D, i.e. [Sh:938, 5.9]. In general, i.e. without this extra assumption, use 1.7(1),(2) below.

2) Similarly using 1.7. $\square_{1.6}$

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Claim 1.7. *Assume* $D \in \mathrm{Fil}^1_\kappa(Y), \kappa > \aleph_0, \mathcal{F}_\alpha \subseteq {}^Y\mathrm{Ord}$ *non-empty for* $\alpha < \delta$ *and* $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha < \delta\rangle$ *is* $<_D$*-increasing,* δ *a limit ordinal.*

1) [DC] *There is* $f^* \in \Pi\bar\alpha$ *which satisfies* $f \in \cup\{\mathcal{F}_\alpha : \alpha < \lambda_1\} \Rightarrow f <_D f^*$ *but there is no such* $f^{**} \in \Pi\bar\alpha$ *satisfying* $f^{**} <_D f$.

2) [DC + $AC_{<\kappa}$*] For* f^* *as above, let* $D_1 = D_{f^*,\bar{\mathcal{F}}} = \{Y\backslash A : A = \emptyset \mod D$ *or* $A \in D^+$ *and there is* $f^{**} \in {}^Y\mathrm{Ord}$ *such that* $f^{**} <_{D+(Y\backslash A)} f^*$ *and* $f \in \cup\{\mathcal{F}_\alpha : \alpha < \lambda_1\} \Rightarrow f <_{D+A} f^{**}\}$. *Now* D_1 *is a* κ*-complete filter and* $\emptyset \notin D_1$*, and if* $\mathrm{cf}(\delta) \ge \theta(\mathcal{P}(Y))$ *then* $\langle \mathcal{F}_\alpha : \alpha < \delta\rangle$ *witness that* f^* *is a* $<_{D_1}$*-exact upper bound of* $\bar{\mathcal{F}}$ *hence* $(\prod\limits_{y\in Y} f^*(y), <_{D_1})$ *has pseudo-true-cofinality* $\mathrm{cf}(\delta)$.

3) [DC + $AC_{\mathcal{P}(Y)}$*]*

There is $f' \in {}^Y\mathrm{Ord}$ *which is an* $<_D$*-exact upper bound of* $\bar{\mathcal{F}}$*, i.e.* $f <_D f' \Rightarrow (\exists\alpha < \delta)(\exists g \in \mathcal{F}_\alpha)[f < g \mod D]$ *and* $f \in \bigcup\limits_{\alpha<\delta} \mathcal{F}_\alpha \Rightarrow f <_{D_1}$.

Proof. 1) If not then by DC we can find $\bar f = \langle f_n : n < \omega\rangle$ such that:

(*a*) $f_n \in {}^Y\mathrm{Ord}$

(*b*) $f_{n+1} < f_n \mod D$

(*c*) if $f \in \bigcup\limits_{\alpha<\delta} \mathcal{F}_\alpha$ and $n < \omega$ then $f < f_n \mod D$.

So $A_n = \{t \in Y : f_{n+1}(t) < f_n(t)\} \in D$ hence $\cap\{A_n : n < \omega\} \in D$, contradiction.

2) First, clearly $D_1 \subseteq \mathcal{P}(Y)$ and by the assumption $\emptyset \notin D_1$. Second, if f^{**} witness $A \in D_1$ and $A \subseteq B \subseteq Y$ then f^{**} witness $B \in D_1$.

Third, we prove D_1 is closed under intersection of $< \kappa$ members, so assume $\zeta < \kappa$ and $\bar A = \langle A_\varepsilon : \varepsilon < \zeta\rangle$ is a sequence of members of D_1. Let $A := \cap\{A_\varepsilon : \varepsilon < \zeta\}, B_\varepsilon = Y\backslash A_\varepsilon$ for $\varepsilon < \zeta$ and $B'_\varepsilon = B_\varepsilon\backslash \cup\{B_\xi : \xi < \varepsilon\}$ and $B = \cup\{B_\varepsilon : \varepsilon < \zeta\}$. Clearly $B = Y\backslash A, A \subseteq Y$ and $\langle B'_\varepsilon : \varepsilon < \zeta\rangle$ is a sequence of pairwise disjoint subsets of Y with union B. But AC_ζ holds hence we can find $\langle f^{**}_\varepsilon : \varepsilon < \zeta\rangle$ such that f^{**}_ε witness $A_\varepsilon \in D_1$. Let $f^{**} \in {}^Y\mathrm{Ord}$ be defined by $f^{**}(t) = f^{**}_\varepsilon(t)$ if $t \in B'_\varepsilon$ or $\varepsilon = 0 \wedge t \in Y\backslash B$; easily $f \in \bigcup\limits_{\alpha<\delta} \mathcal{F}_\alpha \Rightarrow f < f^{**} \mod (D + A_\varepsilon)$. So clearly f^{**} witness $A = \bigcap\limits_{\varepsilon<\zeta} A_\varepsilon \in D_1$ so D_1 is indeed κ-complete.

Lastly, assume $\mathrm{cf}(\delta) \ge \theta(\mathcal{P}(Y))$ and we shall show that f^* is an exact upper bound of $\bar{\mathcal{F}}$ modulo D_1. So assume $f^{**} \in {}^Y\mathrm{Ord}$ and $f^{**} < f^* \mod D_1$.

Let $\mathcal{A} = \{A \in D^+_1$: there is $f \in \bigcup\limits_{\alpha<\delta} \mathcal{F}_\alpha$ such that $f^{**} \le f \mod (D + A)\}$, yes, not D_1!

<u>Case 1</u>: For every $B \in D^+_1$ there is $A \in \mathcal{A}, A \subseteq B$.

For every $A \in \mathcal{A}$ let $\alpha_A = \min\{\beta$: there is $f \in \mathcal{F}_\beta$ such that $f^{**} \le f \mod D + A\}$.

So the sequence $\langle \alpha_A : A \in \mathcal{A} \rangle$ is well defined.

Let $\alpha(*) = \sup\{\alpha_A + 1 : A \in \mathcal{A}\}$, it is $< \delta$ as $\mathrm{cf}(\delta) \geq \theta(\mathcal{P}(Y)) \geq \theta(\mathcal{A})$.

Now any $f \in \mathcal{F}_{\alpha(*)}$ is as required as $\{t \in Y : f^{**}(t) \geq f(t)\}$ contains no $A \in \mathcal{A}$ hence is $= \emptyset$ mod D_1 and the case assumption.

<u>Case 2</u>: $B \in D_1^+$ and there is no $A \in \mathcal{A}$ such that $A \subseteq B$.

So $f \in \bigcup\limits_{\alpha<\delta} \mathcal{F}_\alpha$ implies $\{t \in B : f^{**}(t) \leq f(t)\} = \emptyset$ mod D_1, i.e. $f < f^{**}$ $\mathrm{mod}(D_1 + B)$ so by the definition of D_1 we have $Y \backslash B \in D_1$, contradiction to the case assumption.

3) By [Sh:938, §5] without loss of generality $\bar{\mathcal{F}}$ is $\aleph_0$-continuous. For every $A \in D^+$ the assumptions hold even if we replace D by $D + A$ and so there are D_1, f^* as in part (2), we are allowed to use part (2) as by 1.8 below we have $\mathrm{AC}_{<\kappa}$. As we are assuming $\mathrm{AC}_{\mathcal{P}(Y)}$ there is a sequence $\langle (D_A, f_A) : A \in D^+ \rangle$ such that:

$(*)_1$ (a) D_A is a κ-complete filter extending $D + A$

(b) $f_A \in {}^Y\mathrm{Ord}$ is a $<_{D_A}$-exact upper bound of $\bar{\mathcal{F}}$.

Recall $|A| \leq_* |B|$ is defined as: A is empty or there is a function from B onto A. Of course, this implies $\theta(A) \leq \theta(B)$.

Let $\bar{\mathcal{U}} = \langle \mathcal{U}_t : t \in Y \rangle$ be defined by $\mathcal{U}_t = \{f_A(t) : A \in D^+\} \cup \{\sup\{f(t) : f \in \bigcup\limits_{\alpha<\delta} \mathcal{F}_\alpha\}\}$ hence $t \in Y \Rightarrow 0 < |\mathcal{U}_t| \leq_* \mathcal{P}(Y)$ even uniformally but we have AC_Y so there is a sequence $\langle h_t : t \in Y \rangle$ such that h_t is a function from $\mathcal{P}(Y)$ onto $\mathcal{U}_t$ hence $|\prod\limits_{t \in Y} \mathcal{U}_t| \leq_* \mathcal{P}(Y) \times Y \leq_* \mathcal{P}(Y \times Y)$ but $\mathrm{AC}_{\mathcal{P}(Y)}$ holds hence Y can be well ordered but without loss of generality Y is infinite hence $|Y \times Y| = Y$, so $|\prod\limits_{t \in Y} \mathcal{U}_t| \leq_* |\mathcal{P}(Y)|$.

Let $\mathcal{G} = \{g : g \in \prod\limits_{t \in Y} \mathcal{U}_t$ and not for every $f \in \bigcup\limits_{\alpha<\delta} \mathcal{F}_\alpha$ do we have $f < g$ mod $D\}$, so $|\mathcal{G}| \leq |\prod\limits_{t \in Y} \mathcal{U}_t| \leq_* |\mathcal{P}(Y \times Y)| = |\mathcal{P}(Y)|$ hence $\theta(\mathcal{G}) \leq \mathrm{cf}(\delta)$.

Now for every $g \in \mathcal{G}$ the sequence $\langle \{\{t \in Y : g(t) \leq f(t)\} : f \in \bigcup\limits_{\beta<\alpha} \mathcal{F}_\beta\} : \alpha < \delta \rangle$ is a $\subseteq$-increasing sequence of subsets of $\mathcal{P}(Y)$, but $\theta(\mathcal{P}(Y)) \leq \mathrm{cf}(\delta)$ hence the sequence is eventually constant and let $\alpha(g) < \delta$ be minimal such that

$$(*)_g \quad (\forall \beta)[\alpha(g) \leq \beta < \delta \Rightarrow \{\{t \in Y : g(t) \leq f(t)\} : f \in \bigcup_{\gamma<\beta} \mathcal{F}_\gamma\} = \{\{t \in Y : g(t) \leq f(t)\} : f \in \bigcup_{\gamma<\alpha} \mathcal{F}_\gamma\}].$$

But recalling $\theta(\mathcal{G}) \leq \mathrm{cf}(\delta)$, the ordinal $\alpha(*) := \sup\{\alpha(g) : g \in \mathcal{G}\}$ is $< \delta$. Now choose $f^* \in \mathcal{F}_{\alpha(*)+1}$ and define $g^* \in \prod\limits_{t \in Y} \mathcal{U}_t$ by $g^*(t) = \min(\mathcal{U}_t \backslash f^*(t))$, well defined as $\sup\{f(t) : t \in \bigcup\limits_{\alpha<\delta} \mathcal{F}_\alpha\} \in \mathcal{U}_t$. It is easy to check that g^* is as required. $\square_{1.7}$

{r25}

Observation 1.8. *Let D be a filter on Y.*
If D is κ-complete for every κ <u>then</u> for every $f \in {}^Y\mathrm{Ord}$ and $A \in D^+$ there is $B \subseteq A$ from D^+ such that $f \restriction B$ is constant.

Proof. Straight. $\square_{1.8}$

{r26}

Definition 1.9. Let $\bar{\alpha} = \langle \alpha_y : y \in Y \rangle \in {}^Y\text{Ord}$ be such that $t \in Y \Rightarrow \alpha_t > 0$.
1) Let ps-$\mathbf{T}_D(\bar{\alpha}) = \sup\{\theta(\mathbf{F}) : \mathbf{F}$ is a family of non-empty subsets of $\Pi\bar{\alpha}$ such that for every $\mathcal{F}_1 \neq \mathcal{F}_2$ from $\mathbf{F}$ we have $f_1 \in \mathcal{F}_1 \wedge f_2 \in \mathcal{F}_2 \Rightarrow f_1 \neq_D f_2\}$.
2) Let ps-$\mathbf{T}_{\kappa\text{-comp}}(\bar{\alpha}) = \sup\{\theta(\mathbf{F})$: for some κ-complete filter D on Y, $\mathbf{F}$ is as above for $D\}$.

{r29}

Theorem 1.10. *[DC + $AC_{\mathcal{P}(Y)}$] Assume that D is a κ-complete filter on Y. For any μ the following cardinals are equal <u>or</u> at least $\lambda_1, \lambda_2, \lambda_3$ are $(\text{Fil}^1_\kappa(D)$-almost equal which means: for $\ell_1, \ell_2 \in \{1,2,3\}$, if $\alpha < \lambda_{\ell_1}$ then α is included in the union of $|Fil^1_\kappa(D)|$ sets each of order type $< \lambda_{\ell_2}$:*

- (a) $\lambda_1 = \sup\{|\text{rk}_{D_1}(\mu)|^+ : D_1 \in \text{Fil}^1_\kappa(D)\}$
- (b) $\lambda_2 = \sup\{\lambda^+$: *there are* $D_1 \in \text{Fil}^1_\kappa(D)$ *and a* $<_{D_1}$*-increasing sequence* $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ *such that* $\mathcal{F}_\alpha \subseteq {}^Y\mu$ *is non-empty*$\}$
- (c) $\lambda_3 = \sup\{\text{ps} - \mathbf{T}_{D_1}(\mu)^+ : D_1 \in \text{Fil}^1_\kappa(D)\}$.

Remark 1.11. 1) The conclusion gives slightly less than equality of $\lambda_1, \lambda_1, \lambda_3$, see 2.1 later.
2) We may replace κ-complete by $(\leq Z)$-complete) if $\aleph_0 \leq |Z|$.
3) See Definition 2.1.

Proof. Note

(∗) $\mu < \lambda_\ell$ for $\ell = 1, 2, 3$.

[Why? For clause (a), for any D_1, $\text{rk}_{D_1}(\mu) \geq \mu$ and for clauses (b),(c) use $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha < \mu \rangle$ such that $\mathcal{F}_\alpha = Y_{\{\alpha\}} := \{h : h$ is the unique function from Y to $\{\alpha\}\}$.]

<u>Stage A</u>: if $\chi < \lambda_1$ <u>then</u> $\chi < \lambda_2, \lambda_3$.

Why? As $\chi < \lambda_1$ there is $D_1 \in \text{Fil}^1_\kappa(D)$ such that $\text{rk}_{D_1}(\mu) \geq \chi$. By [Sh:938, 1.11(5)] letting $X_{D_2} = \{\alpha < \kappa$: some $f \in {}^Y\mu$ satisfies $D_2 = \text{dual}(J[f, D_1])$ and $\alpha = \text{rk}_{D_1}(f)\}$, for any $D_2 \in \text{Fil}^1_\kappa(D_1)$ we have $\chi = \bigcup\{X_{D_2} : D_2 \in \text{Fil}^1_\kappa(D_1)\}$ and clearly $D_2 \in \text{Fil}^1_\kappa(D_1) \Rightarrow |\text{otp}(X_{D_2})| < \lambda_2, \lambda_3$; this is enough.

Why? Letting $\mathcal{F}_{\alpha,i} = \{f \in {}^Y\mu : \text{rk}_{D_1}(f) = \alpha$ and $J[f, D_1] = \text{dual}(D_2)\}$, we have: $i < j \wedge i \in X_{D_2} \wedge j \in X_{D_2} \wedge f \in \mathcal{F}_{\alpha,i} \wedge g \in \mathcal{F}_{\alpha,j} \Rightarrow f < g \mod D_2$ so $\text{otp}(X_{D_2}) < \lambda_2, \lambda_3$.

<u>Stage B</u>: If $\chi < \lambda_2$ then $\chi < \lambda_1, \lambda_3$.

Why? Let D_1 and $\langle \mathcal{F}_\alpha : \alpha < \chi \rangle$ exemplify $\chi < \lambda_2$. Let $\gamma_\alpha = \min\{\text{rk}_{D_1}(f) : f \in \mathcal{F}_\alpha\}$ so easily $\alpha < \beta < \chi \Rightarrow \gamma_\alpha < \gamma_\beta$ hence $\text{rk}_D(\mu) \geq \chi$. So $\chi^+ < \lambda_1$ and as for $\mu < \lambda_3$ this holds by Definition 1.9(2) as $\alpha < \beta \wedge f \in \mathcal{F}_\alpha \wedge g \in \mathcal{F}_\beta \Rightarrow f < g \mod D_1 \Rightarrow f \neq g \mod D_1$.

<u>Stage C</u>: If $\mu < \lambda_3$ then $\mu < \lambda_1, \lambda_2$.

Why? Let $\langle \mathcal{F}_\alpha : \alpha < \chi \rangle$ exemplify $\chi < \lambda_3$. For each $\alpha < \chi$ let $\mathbf{J}_\alpha = \{\text{dual}(J[f, D]) : f \in \mathcal{F}_\alpha\}$ so a non-empty subset of $\text{Fil}^1_\kappa(Y)$. Now for every $J \in \cup\{\mathbf{J}_\alpha : \alpha < \lambda\}$ let $X_J = \{\alpha < \chi : J \in \mathbf{J}_\alpha\}$ and for $\alpha \in X_J$ let $\zeta_{J,\alpha} = \min\{\text{rk}_D(f) : f \in \mathcal{F}_\alpha$ and $J = \text{dual}(J[f, D])\}$. Now easily $\alpha \mapsto \zeta_{J,\alpha}$ is a one-to-one function with domain X_J and if $\alpha < \beta$ are from X_J then $\zeta_{J,\alpha} < \zeta_{J,\beta}$, etc. $\square_{1.10}$

{r32}

Discussion 1.12. 1) We like to measure $({}^Y\mu)/D$ in some ways and show their equivalence, as was done in ZFC. Natural candidates are:

- (A) $\text{pp}_D(\mu)$: say of length of increasing sequence $\bar{P}$ (not $\bar{p}$! i.e. sets) ordered by $<_D$
- (B) $\text{pp}^+_Y(\mu) = \sup\{\text{pp}^+_D(\mu) : D \text{ an } \aleph_1\text{-complete filter on } Y\}$
- (C) As in 1.9.

2) We may measure ${}^Y\mu$ by considering all ∂-complete filters.
3) We may be more lenient in defining "same cardinality". E.g.

- (A) we define when sets have similar powers say by divisions to $\mathcal{P}(\mathcal{P}(Y))$ sets we measure $({}^Y\mu)/\approx_{\mathcal{P}(\mathcal{P}(Y))}$ where $\approx_B$ is the following equivalence relation on sets:
  $X \approx_B Y$ when we can find sequences $\langle X_b : b \in B\rangle, \langle Y_b : b \in B\rangle$ such that:
  - (a) $X = \cup\{X_b : b \in B\}$
  - (b) $Y = \cup\{Y_b : b \in B\}$
  - (c) $|X_b| = |Y_b|$
- (B) we may demand more: the $\langle X_b : b \in B\rangle$ are pairwise disjoint and the $\langle Y_b : b \in B\rangle$ are pairwise disjoint
- (C) we may demand less: e.g.
  - $(c)'$ $|X_b| \le_* |Y_b| \le_* |X_b|$
    and/or
  - $(c)_*$ $(\forall b \in B)(\exists c \in B)(|X_b| \le |Y_c|)$ and
    $(\forall b \in B)(\exists c \in B)(|Y_b| \le |X_c|)$.

Note that some of the main results of [Sh:835] can be expressed this way.

- (D) $\text{rk-sup}_{Y,\partial}(\mu) = \text{ rk-sup}\{\text{rk}_D(\mu) : D \text{ is } \partial\text{-complete filters on } Y\}$
- (E) for each non-empty $X \subseteq {}^Y\mu$ let

$$\text{sp}^1_\alpha(X) = \{(D,J) : D \text{ an } \aleph_1\text{-complete filter on } Y, J = J[f,D], \alpha = \text{ rk}_D(f) \text{ and } f \in X\}$$

$$\text{sp}_1(X) = \cup\{\text{sp}^1_\alpha(X) : \alpha\}$$

- (F) <u>question</u>: If $\{\text{sp}(X_s) : s \in S\}$ is constant, can we bound J?
- (G) X, Y are called connected <u>when</u> $\text{sp}(X_1), \text{sp}(X_2))$ are non-disjoint or equal.

4) We hope to prove, at least sometimes $\gamma := \Upsilon({}^Y\mu) \le \text{pp}_\kappa(\mu)$ that is we like to immitate [Sh:835] without the choice axioms on ${}^\omega\mu$. So there is $\bar{f} = \langle f_\alpha : \alpha < \delta\rangle$ witnessing $\gamma < \Upsilon({}^Y\mu)$. We define $u = u_{\bar{f}} = \{\alpha$: there is no $\bar{\beta} \in {}^\omega\alpha$ such that $(\forall it \in Y)(f_\alpha(t) \in \{f_{\beta_n}(t) : n < \omega\})$. You may say that $u_{\bar{f}}$ is the set of $\alpha < \delta$ such that f_α is "really novel".

By DC this is O.K., i.e.

$\boxplus_1$ for every $\alpha < \delta$ there is $\bar{\beta} \in {}^\omega(u_{\bar{f}} \cap \alpha)$ such that $(\forall t \in Y)(f_\alpha(t)) = \{f_{\beta_n}(t) : n < \omega\}$.

Next for $\alpha \in u_{\bar{f}}$ we can define $D_{\bar{f},\alpha}$, the $\aleph_1$-complete filter on Y generated by $\Big\{\{t \in Y : f_\beta(t) = f_\alpha(t)\} : \beta < \alpha\Big\}$. So clearly $\alpha \neq \beta \in u_{\bar{f}} \wedge D_{\bar{f},\alpha} = D_{\bar{f},\beta} \Rightarrow f_\alpha \neq_D f_\beta$. Now for each pair $\bar{D} = (D_1, D_2) \in \mathrm{Fil}^4_Y$ (i.e. for the $\aleph_1$-complete case) let $\Lambda_{\bar{f},\bar{D}} = \{\alpha \in u_{\bar{f}} : D_{\bar{f},\alpha} = D_1 \text{ and } J[f_\alpha, D_1]\} = \mathrm{dual}(D_2)$. So γ is the union of $\leq \mathcal{P}(\mathcal{P}(Y))$-sets (as $|Y| = |Y| \times |Y|$, well ordered.

So

$(*)_1$ $\gamma \leq \theta(({}^Y\omega \times {}^\omega(\mu))$

$(*)_2$ u is the union of $\mathcal{P}(\mathcal{P}(\kappa))$-sets each of cardinality $< \mathrm{pp}^+_{Y,\aleph_1}(\mu)$

(I) what about $\theta({}^\kappa\mu) < \text{ps-pp}_{Y,\aleph_1}(\mu)$?

We are given $\langle \mathcal{F}_\alpha : \alpha < \kappa \rangle \neq F_\alpha \neq \emptyset, \mathcal{F}_\alpha \subseteq \mu, \alpha \neq \beta \Rightarrow \mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$.

Easier: looking modulo a fix filter D.

$(*)_2$ for $D \in \mathrm{Fil}_{Y,\aleph_1}$, let $\mathcal{F}_{\alpha,D} = \{f \in \mathcal{F}_\alpha : \neg(\exists g \in \mathcal{F}_\alpha)(g <_D f)\}$.

Maybe we have somewhere a bound on the size of $\mathcal{F}_{\alpha,D}$.

2. Depth of reduced power of ordinals

Our intention has been to generalize a relative of [Sh:460], but actually we are closed to [Sh:513, §3] using IND but unlike [Sh:938] we deal with depth.

{r28}

Definition 2.1. 1) Let $\mathrm{suc}_X(\alpha)$ be the first ordinal β such that we cannot find a sequence $\langle \mathcal{U}_x : x \in X \rangle$ of subsets of β, each of order type $< \alpha$ such that $\beta = \cup\{\mathcal{U}_x : x \in X\}$.
2) We define $\mathrm{suc}^{[\varepsilon]}_X(\alpha)$ by induction on ε naturally: if $\varepsilon = 0$ it is α, if $\varepsilon = \zeta + 1$ it is $\mathrm{suc}_X(\mathrm{suc}^{[\zeta]}_X(\alpha))$ and if ε is a limit ordinal then it is $\cup\{\mathrm{suc}^{[\zeta]}_X(\alpha) : \zeta < \varepsilon\}$.
3) For a quasi-order P let the pseudo ordinal depth of P, denoted by ps-o-Depth(P) be $\sup\{\gamma$: there is a $<_P$-increasing sequence $\langle X_\alpha : \alpha < \gamma \rangle$ of non-empty subsets of $P\}$.
4) o-Depth(P) is defined similarly demanding $|X_\alpha| = 1$ for $\alpha < \gamma$.
5) Omitting the "ordinal" and the o, means γ is replaced by $|\gamma|$.
6) Let ps-o-Depth$^+(P) = \sup\{\gamma + 1$: there is an increasing sequence $\langle X_\alpha : \alpha < \gamma \rangle$ of non-empty subsets of $P\}$. Similarly for the other variants; without $\mathcal{O}$ we use $|\gamma|^+$.

Remark 2.2. Note that 1.7 can be phrased using this definition.

{k4}

Definition 2.3. 0) We say $\mathbf{x}$ is a filter ω-sequence when $\mathbf{x} = \langle (Y_n, D_n) : n < \omega \rangle = \langle Y_{\mathbf{x},n}, D_{\mathbf{x},n} : n < \omega \rangle$ is such that D_n is a filter on Y_n for each $n < \omega$; we may omit Y_n when it is $\cup\{Y : Y \in D\}$ and may write D if $\bigwedge\limits_n D_n = D$.
1) Let IND$(\mathbf{x}), \mathbf{x}$ has the independence property, means that there for every sequence $\bar{F} = \langle F_{m,n} : m < n < \omega \rangle$ from alg$(\mathbf{x})$, see below, there is $\bar{t} \in \prod\limits_{n<\omega} Y_n$ such that $m < n < \omega \Rightarrow t_m \notin F_{m,n}(\bar{t} \restriction (m,n])$. Let NIND$(\mathbf{x})$ be the negation.
2) Let alg$(\mathbf{x})$ be the set of sequence $\langle F_{n,m} : n < m < \omega \rangle$ such that $F_{m,n} : \prod\limits_{\ell=m+1}^{n} Y_\ell \to \mathrm{dual}(D_n)$.
3) We say $\mathbf{x}$ is κ-complete when each $D_{\mathbf{x},n}$ is a κ-complete filter.

{k6}

Theorem 2.4. *Assume* IND$(\mathbf{x})$ *where* $\mathbf{x} = \langle (Y_n, D_n) : n < \omega \rangle$ *is as in Definition 2.3,* D_n *is* κ_n*-complete,* $\kappa_n \geq \aleph_1$.
*1) [*DC + AC$_{Y_n}$ *for* $n < \omega$*] For every ordinal* ζ*, for infinitely many* n*'s* ps-o-Depth$({}^{(Y_n)}\zeta, <_{D_n}) \leq \zeta$.
*2) [*DC*] For every ordinal* ζ *for infinitely many* n*,* o-Depth$({}^{(Y_n)}\zeta, <_D) \leq \zeta$*, equivalently there is no* $<_{D_n}$*-increasing sequence of length* $\zeta + 1$.

{k7}

Remark 2.5. 0) Note that the present results are incomparable with [Sh:938, §4] - the loss is using depth instead of rank and possibly using "pseudo".
1) If in 2.4, for every n we have $\mathrm{rk}_{D_n}(\zeta) > \mathrm{suc}_{\mathrm{Fil}^1_\kappa(D_n)}(\zeta)$ then for some $D^1_n \in \mathrm{Fil}^1_u(Y_n)$ for $n < \omega$ we have NIND$(\langle Y_n, D^1_n) : n < \omega \rangle$. (Why? By [Sh:938, 5.9]). But we do not know much of the D^1_n's.
2) This theorem applies to e.g. $\zeta = \aleph_\omega, Y_n = \aleph_n, D_n = \mathrm{dual}(J^{\mathrm{bd}}_{\aleph_n})$. So even in ZFC, it tells us things not covered by [Sh:513, §3]. Note that Depth and pcf are closely connected but only for sequences of length $\geq \theta(\mathcal{P}(Y))$.
3) If we assume IND$(\langle Y_{\eta(n)}, D_{\eta(n)} : n < \omega \rangle$ for every increasing $\eta \in {}^\omega\omega$, which is quite reasonable then in Theorem 2.4 we can strengthen the conclusion "for every $n < \omega$ large enough".

4) Note that 2.4(2) is complimentary to [Sh:835].

{k8}

Observation 2.6. *If* $\mathbf{x}$ *is a filter* ω*-sequence and* $n_* < \omega$ *and* $IND(\mathbf{x}\restriction[n_*,\omega)$ *<u>then</u>* $IND(\mathbf{x})$.

Proof. Let $\bar F = \langle F_{n,m} : n < m < \omega\rangle \in$ alg$(\mathbf{x})$, so $\langle F_{n,m} : n \in [n_*,\omega)$ and $m \in (n,\omega)\rangle$ belongs to alg$(\mathbf{x}\restriction[n_*,\omega)$ hence by the assumption "IND$(\mathbf{x}\restriction[n_*,\omega))$ there is $\bar t = \langle t_n : n \in [n_*,\omega)\rangle \in \prod\limits_{n\ge n_*} Y_n$ such that $t_n \notin F_{n,m}(\bar t\restriction(n,m))$ when $n_* \le n < \omega$. Now by downward induction on $n < n_*$ we choose $t_n \in Y_n$ such that $t_n \notin \{F_{n,m}(\langle t_k : k \in [n+1,m])$ for $m \in [n+1,\omega)$. This is possible as the countable union of members of dual(D_n) is not equal to Y_n. We can carry the induction and $\langle t_n : n < \omega\rangle$ is as required to verify IND$(\mathbf{x})$. $\square_{2.6}$

Proof. We concentrate on proving part (1), part (2) is easier.

Assume this fails. So for some $n_* < \omega$ for every $n \in [n_*,\omega)$ there is a counterexample. As AC$_{\aleph_0}$ holds we can find a sequence $\langle \bar{\mathcal{F}}_n : n \in [n_*,\omega)\rangle$ such that

- $\odot$ for $n \in [n_*,\omega)$
  - (a) $\bar{\mathcal{F}}_n = \langle \mathcal{F}_{n,\varepsilon} : \varepsilon \le \zeta\rangle$
  - (b) $\mathcal{F}_{n,\varepsilon} \subseteq {}^{Y_n}\zeta$ is non-empty
  - (c) $\bar{\mathcal{F}}_n$ is a $<_{D_n}$-increasing sequence of sets, i.e. $\varepsilon_1 < \varepsilon_2 \le \zeta \wedge f_1 \in \mathcal{F}_{n,\varepsilon_1} \wedge f_2 \in \mathcal{F}_{n,\varepsilon_2} \Rightarrow f_1 <_{D_n} f_2$.

Now by AC$_{\aleph_0}$ we can choose $\langle f_n : n < \omega\rangle$ such that $f_n \in \mathcal{F}_{n,\zeta}$ for $n \in [n_*,\omega)$.

$(*)$ without loss of generality $n_* = 0$.

[Why? As $\mathbf{x}\restriction[n_*,\omega)$ satisfies the assumptions on $\mathbf{x}$ by 2.6.]

Let

- $\boxplus_1$ for $m < n$ let $Y^0_{m,n} = \prod\limits_{\ell=m}^{n-1} Y_\ell$ and $Y^1_{m,n} := \cup\{Y^0_{k,n} : k \in [m,n]\}$ so $Y^0_{m,n} = \emptyset = Y^1_{m,n}$ if $m > n$ and $Y^0_{m,n} = \{<>\} = Y^1_{m,n}$ if $m = n$
- $\boxplus_2$ for $m \le n$ let $\mathcal{G}^1_{m,n}$ be the set of functions g such that:
  - (a) g is a function from $Y^1_{m,n}$ into $\zeta + 1$
  - (b) $\langle\rangle \ne \eta \in Y^1_{m,n} \Rightarrow g(\eta) < \zeta$
  - (c) if $k \in [m,n)$ and $\eta \in Y^0_{k+1,n}$ then the sequence $\langle g(\eta\hat{\ }\langle y\rangle) : y \in Y_k\rangle$ belongs to $\mathcal{F}_{k,g(\eta)}$
- $\boxplus_3$ $\mathcal{G}^1_{m,n,\varepsilon} := \{g \in \mathcal{G}^1_{m,n} : g(\langle\rangle) = \varepsilon\}$ for $\varepsilon \le \zeta$ and $m \le n < \omega$.

Now the sets $\mathcal{G}^1_{m,n}$ are non-trivial, i.e.

- $\boxplus_4$ if $m \le n$ and $\varepsilon \le \zeta$ then $\mathcal{G}^1_{m,n,\varepsilon} \ne \emptyset$.

[Why? We prove it by induction on n; first if $n = m$ this is trivial. The unique function g with domain $\{<>\}$ and value ε. Next, if $m < n$ we choose $f \in \mathcal{F}_{n-1,\varepsilon}$ hence the sequence $\langle \mathcal{G}^1_{m,n-1,f(s)} : s \in Y_{n-1}\rangle$ is well defined and by the induction hypothesis each set in the sequence is non-empty. As AC$_{Y_{n-1}}$ holds there is a sequence $\langle g_s : s \in Y_{n-1}\rangle$ such that $s \in Y_{n-1} \Rightarrow g_s \in \mathcal{G}^1_{m,n-1,f(s)}$. Now define g as the function with domain $Y^1_{m,n}$:

$$g(\langle\rangle) = \varepsilon$$

$$g(\langle s\rangle\hat{\ }\nu) = g_s(\nu) \text{ for } \nu \in Y^1_{m,n-1}.$$

It is easy to check that $g \in \mathcal{G}^1_{m,n,\varepsilon}$ indeed so $\boxplus_4$ holds.]

$\boxplus_5$ if $g, h \in \mathcal{G}^1_{m,n}$ and $g(\langle\rangle) < h(\langle\rangle)$ <u>then</u> there is an (m,n)-witness Z for (h,g) which means (just being a witness means we omit (d)):

- (a) $Z \subseteq Y^1_{m,n}$ is closed under initial segments, i.e. if $\eta \in Y^0_{k,n} \cap Z$ and $m \le k < \ell \le n$ then $\eta{\restriction}[\ell,n) \in Y^0_{\ell,n} \cap Z$
- (b) $\langle\rangle \in Z$
- (c) if $\eta \in Z \cap Y^0_{k+1,n}, m \le k < n$ then $\{y \in Y_k : \eta\hat{\ }\langle y\rangle \in Z\} \in D_k$
- (d) if $\eta \in Z$ then $g(\eta) < h(\eta)$.

[Why? By induction on n, similarly to the proof of $\boxplus_4$.]

$\boxplus_6$ (a) we can find $\langle g_n : n < \omega\rangle$ such that $g_n \in \mathcal{G}^1_{0,n,\zeta}$ for $n < \omega$

(b) for $n < \omega, s \in Y_n$ let $g_{n+1,s} \in \mathcal{G}^1_{m,n}$ be defined by $g_{n+1,s}(\nu) = g_{n+1}(\langle s\rangle\hat{\ }\nu)$.

[Why? Clause (a) by $\boxplus_4$ as $\mathrm{AC}_{\aleph_0}$ holds, clause (b) is obvious by the definitions in $\boxplus_2 + \boxplus_3$.]

$\boxplus_7$ there is $\langle\langle Z_{n,s} : s \in Y_n\rangle : n < \omega\rangle$ such that $Z_{n,s}$ witness $(g_n, g_{n+1,s})$ for $n < \omega, s \in Y_n$.

[Why? For a given $n < \omega, s \in Y_n$ we know that $g_{n+1}(\langle s\rangle) < \zeta = g_n(\langle\rangle)$ hence $Z_{n,s}$ as required exists by $\boxplus_5$. By AC_{Y_n} for each n a sequence $\langle Z_{n,s} : s \in Y_n\rangle$ as required exists, and by $\mathrm{AC}_{\aleph_0}$ we are done.]

$\boxplus_8$ $Z_n := \{\langle s\rangle\hat{\ }\nu : s \in Y_{n-1}, \nu \in Z_{n-1,s}\}$ is a $(0,n)$-witness

$\boxplus_9$ there is $\bar{F}$ such that:

- (a) $\bar{F} = \langle F_{m,n} : m < n < \omega\rangle$
- (b) $F_{m,n} : Y^1_{m+1,n} \rightarrow$ dual(D_m)
- (c) $F_{m,n}(\nu)$ is $\{s \in Y_m : \nu\hat{\ }\langle s\rangle \notin Z_{n-1}\}$ when $\nu \in Z_n$ and is $\emptyset$ otherwise.

[Why? Check.]

$\boxplus_{10}$ $\bar{F}$ witness $\mathrm{IND}(\langle(Y_n, D_n) : n < \omega\rangle)$ fail.

[Why? Clearly $\bar{F} = \langle F_{m,n} : m < n < \omega\rangle$ has the right form.

So toward contradiction assume $\bar{t} = \langle t_n : n < \omega\rangle \in \prod\limits_{n<\omega} Y_n$ is such that

$(*)_1$ $m < n < \omega \Rightarrow t_m \notin F_{n,n}(\bar{t}{\restriction}(m,n])$.

Now

$(*)_2$ $\bar{t}{\restriction}[m,n) \in Z_n$ for $m \le n < \omega$.

[Why? By $(*)_1$ recalling $\boxtimes_9(c)$.]

$(*)_3$ $g_{n+1}(\bar{t}{\restriction}[m,n]) < g_n(\bar{t}{\restriction}[m,n))$.

[Why? Note that Z_{n,t_n} is a witness for (g_n, g_{n+1,t_n}) by $(*)_2 + \boxplus_7$. So by $\boxplus_5$ we have $\eta \in Z_{n,t_n} \Rightarrow g_{n+1,t_n}(\eta) < g_n(\eta)$. But $m < n \Rightarrow \bar{t}\restriction[m,n] \in Z_n \Rightarrow \bar{t}\restriction[m,n) \in Z_{n,t_n}$ hence $g_{n+1}(\bar{t}\restriction[m,n]) = g_{n+1,t_n}(\bar{t}\restriction[m,n)) < g_n(\bar{t}\restriction[m,n))$ as required.]

So for each $m < \omega$ the sequence $\langle g_n(\bar{t}\restriction[m,n) : n < \omega\rangle$ is a decreasing sequence of ordinals, contradiction. Hence there is no $\bar{t}$ as above, so indeed $\boxplus_{10}$ holds. But $\boxplus_{10}$ contradicts an assumption, so we are done. $\square_{2.4}$

{k10}

Remark 2.7. 1) Note that in 2.4 there were no use of completeness demands except of $\aleph_1$-complete when we get rid of n^*, still natural to assume $\aleph_1$-completeness because: if D'_n is the $\aleph_1$-completion of D_n then $\mathrm{IND}(\langle D'_n : n < \omega\rangle)$ is equivalent to $\mathrm{IND}(D_n : n < \omega)$.
2) Recall that by [Sh:513], iff $\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_1}$ then for every $\lambda > \aleph_\omega$ for infinitely many $n < \omega$ we have $(\forall \mu_1 < \lambda)(\mathrm{cf}(\mu) = \aleph_n \Rightarrow \mathrm{pp}(\mu) \le \lambda)$.

{k13}

Claim 2.8. [DC] *For* $\mathbf{x} = \langle Y_n, D_n : n < \omega\rangle$ *with each* D_n *being an* $\aleph_1$*-complete filter on* Y_n*, each of the following is a sufficient condition for* $\mathrm{IND}(\mathbf{x})$*, letting* $Y(<n) = \prod\limits_{m<n} Y_m$

(*a*)
- D_n *is a* $(\le (\prod\limits_{m<n} (D_m)^{Y(<n)})$*-complete ultrafilter*
- *for each* n *in the following game* $\Game_{\mathbf{x},n}$ *the non-empty player has a winning strategy. A play last* ω*-moves. In the* k*-th move the empty player chooses* $A_k \in D_n$ *and* $\langle X_t : t \in (\sum\limits_{m<n} D_m)^{Y(<n)}\rangle$ *a partition of* A_k *and the non-empty player chooses* t_k*. In the end the non-empty player wins the play if* $\bigcap\limits_{k<\omega} X_{t_k}$ *is non-empty*

(*b*) *like (a) but in the second part the non-empty player instead* t_k *chooses* $S_k \subseteq (\sum\limits_{m<n} D_n)^{Y(<n)}$ *satisfying* $|S_k| \le_X |S|$ *and every* $D_{\mathbf{x},n}$ *is* $(\le S)$*-complete,* S *is infinite*

(*c*) *if* $m < n < \omega$ *then* D_m *is* $(\le \prod\limits_{k=m+1}^{n} Y_k)$*-complete*[1]

Proof. Straight. $\square$

---

[1] so the Y_k's are not well ordered! If $\alpha < \theta(Y_n) \Rightarrow D$ is $|\alpha|^+$-complete then $\alpha^{Y_m}/D_n \cong \alpha$. If α is counterexample D project onto a uniform $\aleph_1$-complete filter on some $\mu \le \alpha$.

3

§(3A) $Y \in$ Card is enough

{c3.2}

Definition 3.1. Assume D is a filter on Y.
1) Let $\mathrm{oq}(Y) = \mathrm{oq}(Y,D) = \{f : f$ a function from Y <u>onto</u> some ordinal$\}$.
2) For $f \in \mathrm{oq}(Y)$ let $e_f = \{(y_1, y_2) : y_1 \in Y, y_2 \in Y$ and $f(y_1) = f(y_2)\}$.
3) Let $\mathrm{oeq}(Y) = \{e_f : f \in \mathrm{oq}(Y,d)\}$.
4) For $h \in \mathrm{oq}(Y,D)$ let D/h be $\{x \subseteq \mathrm{Rang}(h) : h^{-1}(X) \in D\}$, a filter on $\mathrm{Rang}(f)$ which necessarily is an ordinal $< \theta(Y)$.
5) For $f \in {}^Y\mathrm{Ord}$ let g_f be the following function:

- (a) $\mathrm{Dom}(g_f) = \mathrm{otp}(\mathrm{Rang}(f))$
- (b) $g_f(i) = \alpha$ iff $(\exists y)(y \in Y \wedge f(y) = \alpha \wedge i = \mathrm{otp}(f(y) \cap \mathrm{Rang}(f))$.

6) For $f \in {}^Y\mathrm{Ord}$ let h_f be the following function:

- (a) $\mathrm{Dom}(h_f) = Y$
- (b) $h_f(y) = \mathrm{otp}(f(y) \cap \mathrm{Rang}(f)) \in \mathrm{oq}(Y,d)$.

7) Assume $D \in \mathrm{Fil}^1_\kappa(Y)$ and $\bar f = \langle f_\alpha : \alpha < \alpha(*)\rangle$ is a $<_D$-increasing sequence of members of ${}^Y\mathrm{Ord}$

- (a) we let $\bar u = \langle u_{\bar f,h} : h \in \mathrm{oq}(Y,D)$ where $u_{\bar f,g} = \{\alpha < \alpha(*) : h_{f_\alpha} = h\}$
- (b) $\bar f_0^{[h]} = \langle g_{f_\alpha} : \alpha \in u_{\bar f,h}\rangle$ is $<_D$-increasing.

{c3.5}

Claim 3.2. *Assume $D \in \mathrm{Fil}^1_\kappa$.*
1) Assume $\bar f = \langle f_\alpha : \alpha < \delta\rangle$ is a $<_D$-increasing sequence of members of ${}^Y\mathrm{Ord}$

- (a) *$\langle u_{\bar f,h} : h \in \mathrm{oq}(Y,D)\rangle$ is a partition of Y*
- (b) *$\mathrm{cf}(\delta) \geq \theta(\mathrm{oq}(Y,D))$ <u>then</u> for some $h \in \mathrm{oq}(Y,D)$ the set $u_{\bar f,h}$ is an unbounded subset of δ*
- (c) *for $h \in \mathrm{oq}(Y)$ the sequence $\langle g_{f_\alpha} : \alpha \in u_{\bar f,h}\rangle$ is a $<_{D/h}$-increasing sequence of members of ${}^{\mathrm{Dom}(h)}\mathrm{Ord}$*
- (d) *in (b); if $\delta = |\delta|$ then for some $h \in \mathrm{oq}(Y)$ the set $u_{\bar f,h}$ has order-type δ.*

2) For $\bar\alpha \in {}^Y$Ord for every regular $\lambda \geq \theta(Y)$ we have

- (a) $\lambda \in \text{ps-tcf}_{\kappa\text{-com}}(\bar\alpha)$ <u>*iff*</u> $\lambda \in \text{ps-tcf}_{\kappa\text{-com}}(g_{\bar\alpha})$
- (b) $\lambda \in \text{dp-tcf}_{\kappa\text{-com}}(\bar\alpha)$ <u>*iff*</u> $\lambda \in \text{dp-tcf}_{\kappa\text{-com}}(g_{\bar\alpha})$ *recalling* $\text{dp-tcf}_{\kappa\text{-com}}(\bar\alpha) = \{\lambda$*: for some* $D \in \mathrm{Fil}^1_\kappa(Y), \lambda = \mathrm{tcf}(\Pi\bar\alpha, D)$, *equivalently there is a cofinal sequence of members of* $\Pi\bar\alpha\}$.

{c3.7}

Observation 3.3. *If $AC_{\theta(Y)}$ <u>then</u> $\theta(Y)$ is a successor cardinal.*

Proof. Toward contradiction assume $\theta(Y)$ is a limit cardinal say $\aleph_{\delta(*)}$.

For $\alpha < \theta(Y)$ let $\mathcal{F}^1_\alpha = \{g : g$ a function from Y onto $\alpha\}$, by the definition of $\theta(Y)$ it is non-empty, hence by AC_α the set $\mathcal{F}^2_\alpha = \{f : f$ a one-to-one function from α into $Y\}$ is non-empty. As $\langle \mathcal{F}^2_\alpha : \alpha < \theta(Y)\rangle$ exists and $\mathrm{AC}_{\theta(Y)}$ holds, there is a sequence $\langle f_\alpha : \alpha < \theta(Y)\rangle$ with $f_\alpha \in \mathcal{F}^2_\alpha$. Define the function pr with domain $\{(\alpha, \zeta) : \alpha < \aleph_\zeta < \theta(Y)\}$ by $\mathrm{pr}(\alpha,\zeta) = \sum\limits_{\varepsilon<\zeta} \aleph_\varepsilon + \alpha$, now $\mathrm{pr}(\alpha,\zeta) < \aleph_{\zeta+1} \leq \theta(Y)$ so pr is one-to-one into $\theta(Y)$, also the range of pr is an initial segment of Ord, and

$|\mathrm{Rang}(\mathrm{pr}| =$ Dom(pr) as it is one-to-one and obviously $|\mathrm{Dom}(\mathrm{pr})| \geq \theta$; together pr is onto $\theta(Y)$. We define $\langle y_\gamma : \gamma < \theta(Y)\rangle$ by $y_{\mathrm{pr}(\alpha,\zeta)} = f_\zeta(\alpha)$ for $\alpha < \aleph_\zeta < \theta(Y)$; let $u = \{\gamma < \theta(Y) : (\forall \beta < \gamma)(y_\gamma \neq y_\beta)\}$, so easily $\zeta < \delta(*) \Rightarrow \aleph_{\zeta+1} = |u \cap [\aleph_\zeta, \aleph_{\zeta+1})|$, hence $|u| = \theta(Y)$, hence $\langle y_\gamma : \gamma \in \theta(Y)\rangle$ exemplify $\Upsilon(Y) > \theta(Y)$, contradiction. $\square$

{c3.11}

Claim 3.4. *Assume [?]*

- (a) $\langle \bar{\mathrm{cf}}_D : D \in \text{ps-tcf-fil}(\bar\alpha)\rangle$ *is as in ?*
- (b) *for any* $\bar{\bar{=}}\langle D_i : i < i(*) < \kappa\rangle \in {}^{\kappa>}\text{ps-tcf-fil}_\kappa(\bar\alpha)$
- (c) *for* $\bar D$ *as above and* $\bar\beta \in \prod\limits_i \mathrm{tcf}(\pi\alpha, <_{D_i})$ *let* $\mathcal{F}_{D,\bar\beta} = \{\sup\{f_{\beta_i} : i < \ell g(\bar\beta)\} : \bar f \in \prod\limits_{i<\ell g(\bar\beta)} F_{D_i,\beta_i}\}$ *where* $f = \sup\{f_{\beta_i} : i < \ell g(\bar\beta)\}$ *which means* $s \in Y \Rightarrow f(s) = \sup\{f_{\beta_i}(i) : i < \ell g(\bar\beta)\}$
- (d) $\{\mathcal{F}_{\bar D,\bar\beta} : \bar D \in {}^{\kappa>}(\text{ps-tcf-fil}_\kappa(D))$ *and* $\bar\beta \in \prod\limits_{i<\ell g(\beta)} \mathrm{tcf}(\pi\bar\alpha, <_{D_i})\}$ *is cofinal*
- (e) $\text{ps-cf}^\kappa(\pi\bar\alpha) = \sup(\text{ps-pcf}_\kappa(\pi\bar\alpha))$ *where we define* $\text{ps-cf}^\kappa(\pi\bar\alpha) \leq S$ *<u>when</u> ... ?*

{c13}

Claim 3.5. *Assume*

- (a) $D \in \mathrm{Fil}^1_\kappa(Y), \kappa \geq \aleph_1$ *and* $\alpha_y > 1$ *for* $y \in Y$
- (c) $\mathrm{rk}_D(\bar\alpha) = \zeta = |\zeta|$
- (d) $\mathrm{cf}(\zeta) > \theta(Fil^1_\kappa(Y))$.

1) For some $J \in \{J[f,D] : f \in {}^Y\mathrm{Ord}\}$ *we have* $\zeta = \mathrm{otp}(\{\gamma$*: there is* $\bar\beta \in \Pi\bar\alpha$ *such that* $\mathrm{rk}_D(\bar\beta) = \gamma$ *and* $J[\bar\beta, D] = J\})$.
2) ? In (1) if $\mathrm{dual}(I) \subseteq D_1 \in \mathrm{Fil}^1_\kappa(Y)$ *<u>then</u>* $\mathrm{rk}_{D_1}(\bar\alpha) = \zeta$ *and ?*
3) ? Moreover in (1) if $\bar\beta \in \Pi\bar\alpha$, $\mathrm{rk}_D(\bar\beta) = \gamma, J[\bar\beta, D] = J$ *then* $\mathrm{rk}_{D_1}(\bar\beta) \subseteq$ *??*

Proof. 1) For $\varepsilon < \zeta$ let $\mathcal{F}_\varepsilon = \{\bar\beta \in \Pi\bar\alpha$: $\mathrm{rk}_D(\bar\beta) = \varepsilon\}$ so $\bar{\mathcal{F}} = \langle \mathcal{F}_\varepsilon : \varepsilon < \zeta\rangle$ exists and $\varepsilon < \zeta \Rightarrow \mathcal{F}_\varepsilon \neq \emptyset$ by xxxx and $\cup\{\mathcal{F}_\varepsilon : \varepsilon < \zeta\} = \Pi\bar\alpha$.

Let $\mathcal{F}_{\varepsilon,E} = \{\bar\beta \in \mathcal{F}_\varepsilon : J[\bar\beta, D] = \mathrm{dual}(E)\}$ for $E \in \mathrm{Fil}^1_\kappa(f))$ extending D and let $u_E = \{\varepsilon < \zeta : \mathcal{F}_{\varepsilon,E} \neq \emptyset\}$, so $\mathcal{F}_\varepsilon = \cup \bigcup\{\mathcal{F}_{\varepsilon,E} : E \in \mathrm{Fil}^1_\kappa(D)\}$ and $\zeta = \cup\{u_E : D \subseteq E \in \mathrm{Fil}^1_\kappa(Y)\}$. As $\mathrm{cf}(\zeta) > \theta(\mathrm{Fil}^1_\kappa(Y))$ necessarily for some $E, |u_E| = \zeta$ but $u_E \subseteq \zeta = |\zeta|$ hence $\mathrm{otp}(u_E) = \zeta$, so $\mathrm{dual}(E)$ is as required.
2) By (3). ?
3) ? So J is from (1) and toward contradiction assume $\mathrm{dual}(J) \subseteq D_1 \in \mathrm{Fil}^1_\kappa(Y)$ and $\bar\alpha_1 \in \Pi\bar\alpha$, but $\mathrm{rk}_{D_1}(\bar\alpha_1) \geq \zeta$; without loss of generality $y \in Y \Rightarrow \alpha_{1,y} > 0$ and $\mathrm{rk}_{D_1}(\bar\alpha_1) = \zeta_1$. Now we choose $\mathcal{F}^1_\varepsilon, \mathcal{F}^1_{\varepsilon,E}, E_2$ as in the proof of part (1) starting with $\bar\alpha_1, \zeta_1$. $\square_{3.4}$

4. Private Appendix

We can add to [Sh:938, 2.6,2.7]

{k17}

Claim 4.1. *The filter D_2 4-commutes with the filter D_1 (see* [Sh:938, 3.1]*)* <u>*when*</u>*:*

(*a*) $D_\ell \in \mathrm{Fil}_{\mathrm{cc}}(Y_\ell)$ *for* $\ell = 1, 2$

(*b*) D_1 *is* σ*-complete*

(*c*) *if* $J_1 \in \{J[f, D_1] : f \in {}^{Y_1}Ord\}$ *or just* J_1 *is a* σ*-complete ideal extending* $\mathrm{dual}(D_1)$ *then* $A \subseteq Y_1$ *but* $\mathrm{dual}(J_1) \in \{D_1 + A : A \in D_1^+\}$*; this follows from clause (b) + DC*$_\sigma$ $\mathrm{VAC}_{\mathcal{P}(Y_1)}$ *when* D_1 *is* σ*-c.c., i.e. there is no sequence* $\langle A_i : i < \sigma\rangle$ *of a pairwise disjoint sets from* D_1^+

(*d*) DC_σ *and* AC_{Y_1}, AC_{Y_2}

(*e*) (α) D_1 *is* $\mathcal{P}(Y_2)$*-complete or just*

(β) *if* $\langle B_s : s \in A_1\rangle \in {}^{A}(J_2^+)$ *and* $A \in J_1^+, J_\ell \in \{J[f, D_\ell] : f\}$ *for* $\ell = 1, 2$ <u>*then*</u> *for some* $B_* \in J_2^+$ *and we have* $A_* \subseteq A, A_* \in J_1$ *we have* $s \in A_* \Rightarrow B_s \supseteq B_s$.

Proof. <u>Stage A</u>:

Let $A \in D_2$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^{A}(D_2)$ and $\bar{J}^1 = \langle J_t^1 : t \in Y_2\rangle$ where $J_t^1 \in \{J[f, D_1) : f \in {}^{Y_2}\mathrm{Ord}\}$ and $J_2 \in \{J[f, D_2) : f \in {}^{I_2}\mathrm{Ord}\}$, i.e. as in the assumption of $\boxplus_4$ of Definition [Sh:938, 2.1]. We should find A_*, B_* as there.

<u>Stage B</u>:

For each $t \in I_2$ there is $A_t \in D_1^+$ such that $J_t^1 = \mathrm{dual}(D_1 + A_t)$, hence as AC_{Y_2} holds such that $\langle A_t : t \in Y_2\rangle$ exist. Why? By clauses (b),(c) of the assumption.

<u>Stage C</u>:

Choice of B_*, A_*. Apply clause (d) of the assumption applied to $(J_2, \langle A_t : t \in I_2\rangle)$. $\square_{5.11}$

{k19}

Remark 4.2. 1) We can weaken "D_1 is σ-complete, σ-c.c." to "D_2 is σ-complete, σ^+-c.c." when we have some normality conditions.
2) We can replace this by "any $J[f, D_1]$ is of the form $D_1 + A$ for some $A \in D_1^+$".

We can add in [Sh:938, §4]

{k23}

Conclusion 4.3. [$\mathrm{AC}_{<\mu}$ and μ a limit singular cardinality]

Assume $\mu = \sup\{\kappa < \mu$: for some $\lambda \in [\kappa, \mu)$ on λ there is a κ-complete κ-c.c. filter D on $\lambda\}$. <u>Then</u> for every ordinal ζ for some $\kappa_* < \mu$, for every $\lambda \in [\kappa, \mu)$ and κ-complete κ-c.c. filter D on λ we have $\mathrm{rk}_D(\zeta) = \zeta$.

Proof. By 5.11 and [Sh:938, 4.1]. $\square_{4.3}$

We define $f : Y_1 \to \mathcal{P}(Y_2)$ by $f(s) = \{t \in Y_2 : s \in A_t\}$; as D_1 is $(P(Y_2))$-complete filters on Y_1 necessarily also J_2 is a $(\mathcal{P}(Y_2))$-complete ideal on Y_1 hence there is

- $Y_2^* \subseteq Y_2$ such that $A^* := \{y \in A_1 : f(y) = Y_2^*\}$ belongs to J_2^+.

Choose $s_* \in A^*$ so $Y_2^* = f(s) = \{t \in Y_2 : s \in A_t\}$.

5. Private Appendix

Remark 5.1. pcf inventory (August 2009)
1) See [Sh:F663] lecture - [Sh:430, §6] is locality proved for $\mathrm{pcf}_{\theta\text{-com}}(-), \theta > |\mathfrak{a}|$.
2) See Rinot question [Sh:F893].
3) See the notes for Larson [Sh:F814] - on HOD.
4) Continue [**?**], see [Sh:F878].
5) Failed try to continue [Sh:460, §5B], [Sh:F563].
6) [Sh:F355] - on consistency - answer Gitik?
7) [Sh:F354] $\lambda = \sup(\lambda \cap \ \mathrm{pcf}(\mathfrak{a}))$ is weakly inaccessible.
8) Densities of basic product [Sh:F132], covered by paper with Moti?
9) [Sh:F50] to Shimoni.
10) Hopes rank for precipiousness?
11) Sort out? Y_n is well ordered, need $\mathrm{IND}_{\aleph_1}(\bar{D})$?
12) (09.10.19) A related question: let $\mathbf{x} = \langle (Y_n, D_n, h_n) : n < \omega \rangle$ is here $h_n : Y_n \rightarrow Y$ and D a filter on Y and we try to prove

- $(*)$ for every $f \in {}^Y\mathrm{Ord}$, for every large enough n we have $\mathrm{rk}_{D_n}(f \circ h_n) \subseteq \ \mathrm{rk}_D(f)$ or similarly for Depth.

13) (09.10.26, old thought) As we pass from cofinality to pseudo-cofinality, iterate this notion and then have strong dichotomies.
14) (09.11.15) Think of a problem where:

- (a) $\mathrm{Depth}({}^\omega(\aleph_n), \mathcal{D}_{\aleph_n})$ large given an answer.

15) Tasks (2010.1.08)

- (a) if $Y = \chi$, then we can replace $\mathrm{AC}_{\mathcal{P}(Y)}$ by DC_{χ^+}
- (b) replace Y by all $\mu < \theta(Y)$, just split to some ?
- (c) Definition $\text{dp-pcf}_\kappa(Y) = \{x : \lambda$ regular and there is a filter D such that $\lambda = \text{dp-tcf}(\pi\bar{\alpha}, <_D)\}$ where: $\text{dp-tcf}(\pi\bar{\alpha}, <_D)$ means there is an increasing cofinal of this length
- (d) nice results but no existence
- (e) given $\bar{\alpha}$, how much choice needed to find D with $\mathrm{dual}(D) = ([Z]^{<\kappa} + (Y \backslash Z)$ for some Z?
- (f) for a λ-sequence of length λ, $<_{D_1}$-increasing in ${}^Y\mathrm{Ord}$, is there $<_{D_2}$-lub for some $D_2 \supseteq D_1$?
- (g) smooth closed generating sequence: by $\mathrm{DC}_{|Y|}$?
- (h) generalize [Sh:460]
- (i) get bound or Depth $\aleph_{\omega_1}$
- (j) try for a dichotomy: with IND

Moved 2010.1.08 from 8.8, p.7:
2) $[\mathrm{AC}_{\mathcal{P}(Y)}]$ If D is κ-complete but not $(< \infty)$-complete then AC_κ.
2) So without loss of generality D is κ-complete not κ^+-complete hence there is a sequence $\bar{A} = \langle A_\alpha : \alpha < \kappa \rangle$ of members of D with $\cap \{A_\alpha : \alpha < \kappa\} \notin D$ and without loss of generality $\bar{A}$ is with no repetition. This implies $\kappa < \theta(\mathcal{P}(Y))$, but we have $\mathrm{AC}_{\mathcal{P}(Y)}$ hence we have AC_κ as promised.

* * *

Moved from pg.8:

For $\aleph_1$-complete ultrafilter we get more

{r31} **Claim 5.2.** *[true??] Let D be an $\aleph_1$-complete ultrafilter on Y. Then for any $f \in {}^Y(\mathrm{Ord}\backslash\{0\})$ we have $\mathrm{rk}_D(f) = \text{ps-o-Depth}(\prod\limits_{t\in Y} f(t), <_D)$ and the supremum on the left is obtained.*

Proof. Obvious. $\square_{5.2}$

Question 5.3. 1) Can we prove parallel of the ZFC results?
2) (09.7.19) Is this not $\theta(\Pi\bar{\alpha}/D)$?

Moved from Anotated Content:

§(2A) Getting quasi-rank systems with $\mathrm{AC}_{<\mu}$, pg.7 (090909)?

[We start with pre-rank-system $\mathbf{p}$ and define rank trying to get a strict rank system using IND we get that the ranks are $<\infty$. Has to be read together with [Sh:938]. While this has to be checked we still use $\mathrm{AC}_{<\mu}, \mu = \sum\limits_n \kappa_n$. A new suggestion in f6.2, f6.3d, f6.9(5) has not been elaborated on.]

§3 Connection to IND, pg.13

§4 Appendix, pg.19

[We repeat [Sh:938, §5].]

NOTE: pg.9I - can't read the top of this page

{k10} **Discussion 5.4.** Whereas our orignal intention was to use $\mathrm{IND}(\mathbf{x})$, we actualy use only $\mathrm{IND}'(\mathbf{x})$, which is much better.

{k12} **Definition 5.5.** 1) $\mathrm{IND}'(\langle (Y_n, D_n) : n < \omega\rangle)$ means that if no $\bar{F} = \langle F_n : n < \omega\rangle$ is a witness against it which means:

- (a) F_n is a two-place function from $I_{n+1} \cup \{\mathbf{x}\}$ into $\mathrm{dual}(D_n)$
- (b) there are no $\bar{t}_n = \langle t_{n,\ell} : \ell < n\rangle \in I^1_{0,n}$ for $n < \omega$, stipulating $t_{n,n} = x$ we have $m < n \Rightarrow t_{n,m}t_{n-1,m} \notin F_m(t_{n,m+1}, t_{n+1,m})$.

2) Let $\mathrm{IND}''(\langle (Y_n, D_n) : n < \omega\rangle)$ means that there is no $\langle F_{m,n} : m, n < \omega\rangle$ a witness against it which means:

- (a) $F_{m,n}$ is a two-place function from $I_n \cup \{\}$ into $\mathrm{dual}(D_n)$
- (b) $u_{n,\varepsilon,\xi,t} \subseteq \{(\varepsilon_1, \xi_1 : \varepsilon_1 < xi_1 \le \zeta\}$ coming from $(\mathcal{F}_{n,\varepsilon}, \mathcal{F}_{n,\xi})$.

{k14} **Question 5.6.** 1) If we try to prove 2.4 with choosing $\ll \sum\limits_n (I/n)$?

2) Try $\zeta_n =$ oDepth$({}^{\mathcal{I}_n}\zeta, <_n)$ is $\gg \zeta$. Really for every $\bar{\zeta} \in \prod\limits_{n<\omega} \xi_n$ we have $\bar{F}^{\bar{\zeta}}$ for the Y's witnessing failure of $\mathrm{IND}(\mathbf{x})$ can we combine to get a contradiction? We have the Z's colouring by large subsets of $Y_{0,n}$ with sub-additivity.

{k16}

Claim 5.7. [ZFC] *1) If* $Y_n = \lambda, D_n = \{u \subseteq Y : Y \backslash u \leq \kappa\}$ *and* $Y \rightarrow (\omega)^3_\kappa$ *<u>then</u>* *IND*$'(\langle (Y_n, D_n) : n < \omega)$.
2) If $Y_n = \kappa_2, Y - D_n$*-co-countable.*

Discussion 5.8. We may wonder on relatives on 2.4. First, if instead ps-Depth we use Depth it seems that $\bigwedge\limits_n \mathrm{AC}_{I_n}$ is not necessary. Second, we may try to use ranks instead of depth.

* * *

{k18} Does looking at the proof of 2.4 give more?

Definition 5.9. 1) We say $\bar{f}$ is an $(\mathbf{x}, \zeta)$-system or $(\bar{A}, \mathbf{x}, \zeta)$ is a system <u>when</u>

- (a) $\mathbf{x} = \langle (Y_n, D_n : n < \omega \rangle, D_n$ a filter on Y_n
- (b) ζ an ordinal
- (c) $\bar{f} = \langle f_{n,\varepsilon} : n < \omega, \varepsilon \leq \zeta)$
- (d) $f_{n,\varepsilon} \in {}^{I_n}\zeta$ (with full choice without a more complicated
- (e) $\varepsilon < \xi \leq \zeta$ and $n < \omega$ then $f_{n,\varepsilon} <_{D_n} f_{n,\xi}$.

2) we say the pair $(\bar{t}, \bar{\varepsilon})$ solve the system $(\bar{A}, \mathbf{x}, \zeta)$ when

- (a) $\bar{t} \in \prod\limits_{n<\omega} Y_n$
- (b) $\bar{\varepsilon} = \langle \bar{\varepsilon}_n : n < \omega \rangle$ where $\bar{\varepsilon}_n = \langle \varepsilon_{n,\ell} : \ell \leq n \rangle, \varepsilon_{n,\ell} \leq \zeta$.

Remark 5.10. With little choice for $n < \omega, \varepsilon < \xi \leq \varepsilon$ we have $\langle u_{n,\varepsilon,\xi,t} : t \in I_n \rangle$.
If D_{n+1} is λ^+_n-complete <u>then</u> ?

{k17}

Theorem 5.11. *[*AC_{Y_n} *for* $n < \omega$*.]*
Assume D_n *is an* $\aleph_1$*-complete on* Y_n *for* $n < \omega$ *and* $\mathrm{IND}(\langle D_n : n < \omega \rangle$ *then for every* ζ*, for some* n *we have* $\mathrm{rk}_{D_n}(\zeta) = \zeta$.

Definition 5.12. $\mathrm{AC}_{Y,2}$ where for every $\langle A_y : y \in Y \rangle$ there is $\langle B_y : y \in Y \rangle$ such that $Ay \neq \emptyset \Rightarrow B_y \neq \emptyset, |B_y| \leq_* |Z|$.

Question 5.13. Interesting? Natural for a sequence $(\leq Z)$-complete filter, as in we can use $\langle \bigcap\limits_{a/\in By} : y \in Y \rangle$.

Proof. We choose g_n, Z_n as in the proof of 2.4 using the definition. □

Remark 5.14. 1) In (5B), **??**(2) silly? We can find disjoint $Y_1 Y_2$ with $\mathrm{id}(Y_1) = \mathrm{id}(Y_2)$.
2) Definition **??**(2) line 2: $I \mapsto J$.

Discussion 5.15. Seemingly [Sh:835] connect well to [Sh:F955].
So ssume $\langle \lambda_i : i < \kappa \rangle$ is increasing with limit μ and that is we should deal with a game, where..?

* * *

6. Private Appendix
Using pure Σ: July 2009

{m6}

Definition 6.1. We say $\mathbf{s}$ is a frame <u>when</u> $\mathbf{s}$ consists of the following objects satisfying the following conditions:

- (a) $\langle \kappa_i : i < \text{ cf}(\mu)\rangle$ is increasing with limit μ
- (b) set $\mathbb{D}$
- (c) $D_{\mathbf{d}}$ a filter on $I_{\mathbf{d}} = I[\mathbf{d}]$ for $\mathbf{d} \in \mathbb{D}$
- (d) for $\mathbf{d} \in \mathbb{D}$
  - (α) $\Sigma(\mathbf{d}) \subseteq \{(\mathbf{e}, h) : \mathbf{e} \in \mathbb{D}$ and h a function from $I_{\mathbf{e}}$ onto $I_{\mathbf{e}}$ such taht $D_{\mathbf{d}} = \{h''(A) : A \in D_{\mathbf{e}}\}$
  - (β) $\Sigma_{\text{pr}}(\mathbf{d}) \subseteq \Sigma(\mathbf{d})$, a set of so called pure extensions
  - (γ) $\Sigma_{\text{ap}}(\mathbf{d}) \subseteq \Sigma(\mathbf{d})$, a set of so called a-pure extensions such that $(\mathbf{e}, h) \in \Sigma_{\text{ap}}(\mathbf{d}) \Rightarrow I_{\mathbf{e}} = I_{\mathbf{d}} \wedge h = \text{ id}_{I_{\mathbf{d}}}$
  - (δ) $\mathbf{d} \in \Sigma_{\text{pr}}(\mathbf{d}) \cap \Sigma_{\text{ap}}(\mathbf{d})$
  - (ε) transitivity of Σ? Σ_{pr}? Σ_{ap}?
- (e) $\mathbf{j}$ is a function from $\mathbb{D}$ to $\text{cf}(\mu)$ and $D_{\mathbf{d}}$ is $\kappa_{\mathbf{j}(\mathbf{d})}$-complete and $c \in \ell \text{ par}(\mathbf{d}) \Rightarrow |S_\ell| < \kappa_{\mathbf{j}(\mathbf{d})}(?)$
- (k) $\text{par}(\mathbf{d})$ and for $p \in \text{ part}(\mathbf{d}), \bar{X}_p = \langle X_{p,s} : s \in S_p\rangle$ is a sequence of pairwise disjoint subsets of $I_{\mathbf{d}}$ with union $\in D_{\mathbf{d}}$ and $\langle \mathbf{e}_{p,s} : s \in S\rangle$ is such that $\mathbf{e}_{p,s} \in \mathbb{D}, I_{\mathbf{e}_{p,s}} = I_{\mathbf{d}}, D_{\mathbf{e}_{p,s}} = D_{\mathbf{d}} + X_{p,s}$ so $\mathbf{e}_{p,s} = \mathbf{d} + X_{p,s}$
- (l) (α) if $\mathbf{d}_1 \in \Sigma_{\text{pr}}(\mathbf{d}_0)$ and $\mathbf{d}_2 \in \Sigma_{\text{ap}}(\mathbf{d}_0)$ then $\mathbf{d}_1 +_{\mathbf{d}_0} \mathbf{d}_2 = \mathbf{d}_1 +^{\mathbf{s}}_{\mathbf{d}_0} \mathbf{d}_2$ is a well defined member of $\mathbb{D}_{\mathbf{s}}$ and $\mathbf{d}_3 \in \Sigma_{\text{pr}}(\mathbf{d}_2) \cap \Sigma_{\text{ap}}(\mathbf{d}_1)$
  - (β) above
  - (γ) above if $\mathbf{e} \in \Sigma(\mathbf{d}_1) \cap \Sigma_1(\mathbf{d}_2)$ then $\mathbf{e} \in \Sigma(\mathbf{d})$.

Question 6.2. Maybe $\text{cf}(\kappa)$ replaced by a linear order (which can have a pseudo cofinality)?

We now give examples

{m8}

Definition/Claim 6.3. 1) Assume $\bar{\kappa} = \langle \kappa_n : n < \omega\rangle, \bar{J} = \langle J_n : n < \omega\rangle$, when J_n is a κ_n-complete ideal on I_n, and $\kappa_n < \kappa_{n+1}$ (or just $\kappa_n \leq \kappa_{n+1}$)? We define $\mathbf{s} = \mathbf{s}_{\bar{\kappa},\bar{J}}$ and prove that $\mathbf{s}$ is a pre-system as follows (so $\mu = \mu_{\mathbf{s}}$, etc.)

- (a) $\mu = \Sigma\kappa_n$ and κ is given
- (b) $\mathbb{D}$ is the st of $\mathbf{d} : \mathbf{d} = (\eta, A) = (\eta_{\mathbf{d}}, A_{\mathbf{d}})$ and for some $m = m_{\mathbf{d}} \leq n = n_{\mathbf{d}} < \omega$ we have
  - (α) $\mathbf{F}_{\mathbf{d}} = \{\bar{F} : \bar{F} = \langle F_{m_1,n_1} : m_{\mathbf{d}} < m_1 \leq n_1 \leq n_{\mathbf{d}}\rangle = \langle F^{\mathbf{d}}_{m_1,n_1} : m_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{d}}\rangle$ and $F_{m_1,1} : \prod\limits_{\ell=m_1+1}^{n_1} I_{\eta(\ell)} \rightarrow J_{\eta(m_1)}$
  - (β) $\eta = \langle n, n-1, \ldots, m\rangle$
  - (γ) $I_{\mathbf{d}} = \prod\limits_{\ell=m}^{n} I_\ell$
  - (δ) $D_{\mathbf{d}} = \{X \subseteq I_{\mathbf{d}}$: there are $X_\ell \in J_\ell$ for some $\bar{F} \in \mathbf{F}_{\bar{d}}$ for $\ell \in [m,n]$ such that $A \cap X \supseteq \{\rho \in I_{\mathbf{d}} : \rho(m_1) \notin F_{m_1,n_1}(\rho{\restriction}[m_1,n_1])$ whenever $m_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{d}}\}\}$

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  - (ζ) $\emptyset \notin D_{\mathbf{d}}$ and nec $D_{\mathbf{d}}$ is κ_m-complete
- (c) for $\mathbf{d} \in \mathbb{D}$
  - (α) let $\Sigma(\mathfrak{d})$ be the set of pairs $(\mathbf{e}, h)$ such that $\mathbf{e} \in \mathbb{D}, m_{\mathbf{e}} = m_{\mathbf{d}} \leq n_{\mathbf{d}} \leq n_{\mathbf{e}}, F^{\mathbf{e}}_{m_1,n_1} = F^{\mathbf{d}}_{m_1,n_1}$ when $n_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{d}}$ and h is $h(\rho) = \rho{\restriction}[m_{\mathbf{d}}, n_{\mathbf{d}}]$
  - (β) $\Sigma_{\text{pr}}(\mathbf{d}) = \{(\mathbf{d}, h) \in \Sigma(\mathbf{d}) : F^{\mathbf{e}}_{m_1,n_1}$ is constantly $\emptyset$ when $n_{\mathbf{d}} < n_1$ (and $m_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{e}}$)
  - (γ) $\Sigma_{\text{ap}}(\mathbf{d}) = \{(\mathbf{e}, h) \in \Sigma(\mathbf{d}) : h = \text{ id}_{I_{\mathbf{d}}}$ so $n_{\mathbf{e}} = n_{\mathbf{d}}$
- (d) for $\mathfrak{d} \in \mathbb{D}_{\mathbf{s}}$ and $A \in D^+_{\mathbf{d}}$ let $\mathbf{e} = \mathbf{d} + A \in \mathbb{D}$ be defined naturally, it is $(\eta_{\mathbf{d}}, A_{\mathbf{d}} \cap A, \bar{F})$
- (e) part $(\mathfrak{d})$ is the set of $p = ((X_{p,s}, \mathbf{e}_{p,s}) : s \in S\rangle$ such that: for some so called witness $\bar{G} = \ langleG_{m_1,n_1} : m_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{d}}\rangle, G_{m_1,n_1} : I_{[m_1+1,n_1]} \rightarrow \kappa_{m_1}$ with bounded range letting $S' = \{\langle \alpha_{m_1,n_1} : m_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{d}}\rangle : \alpha_{m_1,n_1} < \kappa_{m_1}\}$ and $A_{\bar{a}} = \{\rho \in I_{\mathbf{d}} : G_{m_1,n_1}(\rho{\restriction}[m_1+1, n_1]) = \alpha_{m_1,n_1}$ for $m_1 < n_1$ from $[m_{\mathbf{d}}, n_{\mathbf{d}}]$ <u>we have</u> $S_p = \{\bar{a} \in A' : \emptyset \in D_{\mathbf{d}} + A_{\bar{a}}\}$ and $\bar{e}_{\bar{d},p,s} = \mathbf{d} + A_{\bar{a}}$

  (question): should we allow $|\text{Rang}(G_{m_1,n_1})|$ be large, etc.?
- (f) $\text{part}(\mathbf{d}) = \{p \in \text{ par}(\mathbf{d}) : |S_p| < \kappa_{m_{\mathbf{d}}}\}$

  (question: should we have $\text{par}(\mathbf{d}) \subseteq \{(\mathbf{e}, d, p) : (\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and as above$\}$?

Discussion 6.4. (09.8.17) 1) Discuss (here?) to achieve our hope (dichotomy using [Sh:835]). We would like for every $\eta \in \mathbb{D}_{\mathbf{x}} = \text{ dec}_{<\omega}(\mathbf{O})$ to define what are η-objects which are a replacement for $(I_\eta)_{\text{Ord}}$. Maybe we should repalce $\text{dec}_{<\omega}(\theta)$ by closing $\mathbf{O}$ by ordered pairing,but first ignore this.

A natural try define when $x \in \text{ obj}(\eta)$ by induction on $\ell g(\eta)$.

If $\ell g(\eta) = 0$ then x is just an ordinal.

If $\ell g(\eta) = n + 1$ <u>then</u> x consists of a non-empty set $\mathcal{F} \in {}^{(I_{\eta(\theta)})}\text{Ord}$, a set $A \in D^+_{\eta(\theta)}, A_B = \{t \in A : f(t) > 0\}$ (or $\langle A_f : f \in \mathcal{F}\rangle, A_g \in D^+_{\eta(\theta)}$?) and a fucntion which gives for every $f \in \mathcal{F}$ and $t \in A_f$ and object $x_{f,t} \in \text{ obj}(\langle \eta(1 + \ell) : \ell < n\rangle)$. We have to: (A) define rank, (B) using DC criterion for the rank being an ordinal, (C) reprove [Sh:938] main Theorem.

2) (09.8.26) The example in [Sh:938, §0] can be pushed up: use $\lambda + \aleph_\omega$ ordinal addition, $(\lambda, \text{rk}_J(\lambda) = \lambda$ for all relevant J's. Hence it seems there is no hope for $\mu = \aleph_\omega$ but there may be for $\mu = \beth_\omega$. At least combine $\mu = \beth_\omega, \theta(\mathcal{P}(\lambda_n)) < \mu_{n+1}, \mu = \sum\limits_{n<\omega} \lambda_n$ and $\text{IND}(\langle \lambda_n : n < \omega\rangle)$ or try the proof of [Sh:460, §1].

{m10}

Claim/Definition 6.5. Like 6.3 but $\bar{J} = \langle J_n : n \in \mathbf{O}\rangle$, FILL. Now $\eta_{\mathbf{d}}$ is a decreasing sequence of length $n_{\mathbf{d}} + 1$, so $D_{\mathbf{d}}$ is $\kappa_{\eta_{\mathbf{d}}(n_{\mathbf{d}})}$-complete and $\mathbf{e} \in \Sigma(\mathbf{d})$ implies $\eta_{\mathbf{e}}(n_{\mathbf{e}}) = \eta_{\mathbf{d}}(n_{\mathbf{d}})$, $\text{Rang}(\eta_{\mathbf{d}}) \subseteq \text{rang}(\eta_{\mathbf{e}})$.

{m7}

Convention 6.6. We naturally let $\mathbf{s} = \langle \bar{\kappa}_{\mathbf{s}}, \mu_{\mathbf{s}}, \mathbb{D}_{\mathbf{s}}, \text{ par}(-,-), \ell\text{par}(-,-)\rangle$ and $I_{\mathbf{s},\mathbf{d}}, D_{\mathbf{s},\mathbf{d}}, S_{\mathbf{s},p}, X_{\mathbf{s},p,s}, D_{\mathbf{s},p,2}$.

{m12}

Definition 6.7. Given a frame $\mathbf{s}$ let $\text{tru}(\mathbf{s})$ be the set of objects $\mathbf{t}$ consisting of:

- (a) $\mathcal{T}_{\mathbf{t}}$ a set of finite sequences closed under initial segments
- (b) $\mathbf{d}_{\mathbf{t},\rho} \in \mathbb{D}$ for $t \in \mathbf{T}$
- (c) $\bar{h}_{\mathbf{t}} = \langle h^{\mathbf{t}}_{\rho,\varrho} : \rho \trianglelefteq \varrho \in \mathcal{T}_{\mathbf{t}}\rangle$

(*d*) for non-Δ-maxiam $\rho \in \mathcal{T}_{\mathbf{t}}, (\mathbf{d}^+_{\mathbf{t},\rho}, h^+_{\mathbf{t},\rho}) \in \Sigma_{\rm pr}(\mathbf{d}_{\mathbf{t},\rho})$ and $p_\rho \in$ par$(\mathbf{d}^+_{\mathbf{t},\rho})$ satisfied $\text{suc}_{\mathcal{T}_{\mathbf{t}}}(\rho) = \{\rho\hat{\ }\langle s\rangle : S \in S_{p_\rho}\}$ and $\mathbf{d}_{\mathbf{t},\rho\hat{\ }<s>} = \mathbf{e}_{\mathbf{d}_{\mathbf{t},\rho}}$ and $h_{\rho,rho\hat{\ }<s>} = h^+_{\mathbf{t}}$ for $\rho \triangleleft \varrho \in \mathcal{T}_{\mathbf{t}}, \ell g(\rho) = m, \ell g(\varrho) = n$ then $h_{\rho,\varrho} : I_{\mathbf{d}_{\mathbf{t},\varrho}} \to I_{\mathbf{d}_{\mathbf{t},\rho}}$ is $h_{\mathbf{t}}, h_{\rho_{m+1}} \circ \ldots \circ h_{\mathbf{t},\rho_n}$ where $h_{\mathbf{t},\rho_{\ell+1}} := h^+_{\mathbf{t},\rho_\ell}$ and $\rho_\ell = \varrho \restriction \ell$ for $\ell = m, \ldots, n-1$

(*e*) if $\rho \triangleleft \varrho \in \mathcal{T}_{\mathbf{t}}$ we have: $h_{\rho,\varrho}$ maps $A_{\mathbf{d}_\varrho}$ into $A_{\mathbf{d}_\rho}$
question: put this in Definition 6.1?

{m16}

Definition 6.8. Given a candidate $\mathbf{s}$ we try to define a rank; (we may omit the subscript $\mathbf{s}$ as its value is fixed).

If $\mathbf{d} \in \mathbb{D}_{\mathbf{s}}$ and $f \in {}^{I[\mathbf{d}]}\text{Ord}$ we define $\text{rk}^{\rm tr}_{\mathbf{d}}(f) = \text{rk}^{\rm tr}_{\mathbf{d}}(f, \mathbf{s}) \in \text{Ord} \cup \{\infty\}$; or we may replace "tr" by 1 or omit it; by defining by induction on the ordinal ζ when $\text{rk}^{\rm tr}_{\mathbf{d}}(f) \supseteq \zeta$: it holds iff for every $\zeta_1 < \lambda$ there is a pair $(\mathbf{t}, \bar{g})$ such that

(*a*) $\mathbf{t} \in$ tree$(\mathbf{s})$ where $\mathcal{T}_{\mathbf{t}}$ is well founded, i.e. with no ω-branch
(*b*) $\mathbf{d}_{\mathbf{t}=<>} = \mathbf{d}$
(*c*) $\bar{g} = \langle g_\rho : \rho \in \max(\mathcal{T}_{\mathbf{t}})\rangle$
(*d*) $g_\rho : I_d[\mathbf{d}_{\mathbf{t},\rho}] \to \text{Ord}$
(*e*) $g_\rho < f \circ h_{\mathbf{t},<>,\rho} \mod D_{\mathbf{d}_{\mathbf{t},\rho}}$
(*f*) $\text{rk}^{\rm tr}_{\mathbf{d}_{\mathbf{t},\rho}}(g_\rho) \geq \zeta_1$.

The choice in **??** though more transparent than the following relative, need more use of choice.

{m18}

Definition 6.9. Like 7.9 - FILL - $\text{rk}^2_{\mathbf{d}}(f)$, <u>but</u> maybe rk^1 is enough.
Check.

{m21}

Claim 6.10. *Let* $\mathbf{s}$ *be a candidate and* $k = 0, 1$.
1) The rank $rk^k_{\mathbf{d}}(f)$ *for* $f \in {}^{I[\mathbf{d}]}Ord$ *is well defined* $(\in Ord \cup \{\infty\})$.
2) If $(\mathbf{d}_2, h) \in \Sigma_{pr}(\mathbf{d}_1)$ *and* $f_1 \in {}^{I[\mathbf{d}_1]}Ord$ <u>*then*</u> $rk^k_{\mathbf{d}_1}(f) = rk^k_{\mathbf{d}_2}(f \circ h)$.
3) If $\mathbf{d} \in \mathbb{D}_{\mathbf{s}}$ *and* $f \in {}^{I[\mathbf{d}]}Ord$ *and* $p \in \ell par(\mathbf{d})$ <u>*then*</u> $rk_{\mathbf{d}}(f) = min\{rk_{\mathbf{e}_{\mathbf{d},s}}(f) : s \in S_p\}$.

Proof. 1) Easy.
2) Use + on $\mathbb{D}$ - FILL.
3) By induction - FILL. ☐

{m25}

Claim 6.11. *For a free?* $\mathbf{s}$ *the following condition (a),(b) are equivalent: and if* $\mathbf{s} = \mathbf{s}_{\bar{\kappa},\bar{J}}$ *from 6.3 we can add (c), and if* $\mathbf{s} = \mathbf{s}_{\bar{\kappa},\bar{J}}$ *is from 6.5 we can add clause* $(c)^+$*:*

(*a*) $\text{rk}_{\mathbf{d}}(f) = \infty$ *for some* $\mathbf{d} \in \mathbb{D}_{\mathbf{s}}$ *and* $f \in {}^{I[\mathbf{d}]}Ord$
(*b*) *there* $\mathbf{t} \in \text{tree}(\mathbf{s})$ *and* $\mathcal{Y} \subseteq \mathcal{T}_{\mathbf{t}}$ *such that* $(\forall \rho \in \lim_\omega(\mathcal{T}_{\mathbf{t}}))(\exists^\infty n)[\rho \restriction n \in \mathcal{Y})$ *and* $f_\rho \in {}^{I[\mathbf{d}_{\mathbf{t},\rho}]}\text{Ord}$ *for* $\eta \in \mathcal{Y}$ *such that for any* $\rho < \varrho$ *from* $\mathcal{Y}$ *we have* $f_\varrho < f_\rho \circ h^{\mathbf{t}}_{\rho,\varrho} \mod D_{\mathbf{d}_{\mathbf{t},\varrho}}$
(*c*) $\neg\text{IND}(\bar{\kappa}, J)$ *when...?*

{m26}

Definition 6.12. For $(\bar{\kappa}, \bar{J})$ as in 6.3 or 6.5 let $\text{IND}(\bar{\kappa}, J)$ mean that:

<u>Case 1</u>: Definition 6.3 for every $F_{m,n} : I_{[m+1,n]} \to J_m$ for $m < n < \omega$ there is $\eta \in \prod\limits_{\ell<n} I_\ell$ such that $m < n < \omega \Rightarrow \eta(\ell) \notin F_{m,n}(\eta \restriction [m+1, \eta])$.

Case 2: Definition 6.5

[copied] 1) Above $\bar{p}^1_{\bar{J}}$ is not well 0-founded iff: there are $\bar{\varepsilon}, \bar{f}$ such that

$\circledast_{\bar{\varepsilon},\bar{f}}$ (a) $\bar{\varepsilon} = \langle \varepsilon_i : i < \omega \rangle$ is increasing

(b) $\bar{f} = \langle f_{i,j} : i < j < \omega \rangle$

(c) $f_{i,j}$ is a function from $I_{\langle \varepsilon_j, \varepsilon_{j-1}, \dots, \varepsilon_{i+1} \rangle}$ into J_{ε_i}

(d) for every $\bar{\alpha} \in \prod\limits_{i<\omega} \kappa_{\varepsilon_i}$ for some $i < j$ we have $\alpha_i \in f(\alpha_{n_j}, \alpha_{n_{j-1}}, \dots, \alpha_{n_{i+1}})$.

Proof. FILL $\square$

We quote [Sh:938]

Definition 6.13. Main Definition: We say that $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_\mathbf{p}, \mathrm{rk}_\mathbf{p}, \Sigma_\mathbf{p}, \mathbf{j}_\mathbf{p}, \mu_\mathbf{p})$ is a weak (rank) 1-system when:

(a) μ is singular

(b) each $\mathbf{d} \in \mathbb{D}$ is (or just we can compute from it) a pair $(I, D) = (I_\mathbf{d}, D_\mathbf{d}) = (I[\mathbf{d}], D_\mathbf{d}) = (I_{\mathbf{p},\mathbf{d}}, D_{\mathbf{p},\mathbf{d}})$ such that:
  - (α) $\theta(I_\mathbf{d}) < \mu$, on $\theta(-)$ see **??**
  - (β) $D_\mathbf{d}$ is a filter on $I_\mathbf{d}$

(c) for each $\mathbf{d} \in \mathbb{D}$, a definition of a function $\mathrm{rk}_\mathbf{d}(-)$ with domain ${}^{I[\mathbf{d}]}\mathrm{Ord}$ and range $\subseteq$ Ord, that is $\mathrm{rk}_{\mathbf{p},\mathbf{d}}(-)$ or $\mathrm{rk}^\mathbf{p}_\mathbf{d}(-)$

(d) (α) Σ is a function with domain $\mathbb{D}$ such that $\Sigma(\mathbf{d}) \subseteq \mathbb{D}$
  - (β) if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{e} \in \Sigma(\mathbf{d})$ then $I_\mathbf{e} = I_\mathbf{d}$ [natural to add $D_\mathbf{d} \subseteq D_\mathbf{e}$,

    this is not demanded but see **??**(2)]

(e) (α) $\mathbf{j}$ is a function from $\mathbb{D}$ onto $\mathrm{cf}(\mu)$
  - (β) let $\mathbb{D}_{\geq i} = \{\mathbf{d} \in \mathbb{D} : \mathbf{j}(\mathbf{d}) \geq i\}$ and $\mathbb{D}_i = \mathbb{D}_{\geq i} \backslash \mathbb{D}_{i+1}$
  - (γ) $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow \mathbf{j}(\mathbf{e}) \geq \mathbf{j}(\mathbf{d})$

(f) for every $\sigma < \mu$ for some $i <$ $\mathrm{cf}(\mu)$, if $\mathbf{d} \in \mathbb{D}_{\geq i}$, then $\mathbf{d}$ is $(\mathbf{p}, \leq \sigma)$-complete where:
  - $(*)$ we say that $\mathbf{d}$ is $(\mathbf{p}, \leq X)$-complete (or $(\leq X)$-complete for $\mathbf{p}$) when: if $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ and $\zeta =$ $\mathrm{rk}_\mathbf{d}(f)$ and $\langle A_j : j \in X \rangle$ a partition[2] of $I_\mathbf{d}$, then for some $\mathbf{e} \in \Sigma(\mathbf{d})$ and $j < \sigma$ we have $A_j \in D_\mathbf{e}$ and $\zeta =$ $\mathrm{rk}_\mathbf{e}(f)$; so this is not the same as "$D_\mathbf{d}$ is $(\leq X)$-complete"; we define $(\mathbf{p}, |X|^+)$-complete, i.e. $(\mathbf{p}, < |X|^+)$-complete similarly

(g) no hole[3]: if $\mathrm{rk}_\mathbf{d}(f) > \zeta$ then for some pair $(\mathbf{e}, g)$ we have: $\mathbf{e} \in \Sigma(\mathbf{d})$ and $g <_{D[\mathbf{e}]} f$ and $\mathrm{rk}_\mathbf{e}(g) = \zeta$

(h) if $f = g + 1 \mod D_\mathbf{d}$ then $\mathrm{rk}_\mathbf{d}(f) =$ $\mathrm{rk}_\mathbf{d}(g) + 1$

(i) if $f \leq g \mod D_\mathbf{d}$ then $\mathrm{rk}_\mathbf{d}(f) \leq$ $\mathrm{rk}_\mathbf{d}(g)$.

{m30}

Definition 6.14. We say $\mathbf{p}$ is a quasi rank ι-system when $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_\mathbf{p}, \mathrm{rk}_\mathbf{p}, \Sigma_\mathbf{p}, \mathbf{j}_\mathbf{p}, \mu_\mathbf{p})$ satisfies Definition m4.3 of §3 of [Sh:938] if $\iota = 1$, Definition m4.4 of §3 of [Sh:938] if $\iota = 2$ except that the rank may be ∞; we write $\mathrm{rk}_\mathbf{d}(f, \mathbf{d})$ for $\mathbf{d} \in \mathbb{D}_\mathbf{P}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$.

---

[2] as long as σ is a well ordered set it does not matter whether we use a partition or just a covering, i.e. $\cup\{A_j : j \in \sigma\} = I_\mathbf{d}$

[3] we may use another function Σ here, as in natural examples here we use $\Sigma(\mathbf{d}) = \{\mathbf{d}\}$ and not so in clause (f)

{m32}
Definition/Claim 6.15. For a frame $\mathbf{s}$ let $\mathbf{p}$ be the following quasi rank system:

- $\mu, \mathbb{D}, \Sigma, \mathbf{j}$ are as in Definition 6.1
- $\mathrm{rk}_{\mathbf{d}}(f)$ is as in Definition ?

{m34}
Claim 6.16. *1) If $(\bar\kappa, \bar J)$ is as in Definition 6.3 or 6.5 and $\mathrm{IND}(\bar\kappa, J)$ holds, see Definition 6.12 then $p_{\mathbf{s}(\bar\kappa, J)}$ is a rak system.*
2) Moreover it is a strict one.

Saharon copied. 1) As in the proof of e5.g of §4 of [Sh:938, §4,e5.g] or better see the proof of 7.17 except that we use 7.9 instead of 7.8 which simplify clause (f), but is cumbersome in other places.
2) We check Definition m4.3 of §3 of [Sh:938, §3,m4.3].

Clause (a): μ is singular.
As $\mu = \sum_n \kappa_n$ and $\kappa_n < \kappa_{n+1}$ this is obvious.

Clause (b): Let $\mathbf{d} \in \mathbb{D}, \eta = \eta_{\mathbf{d}}, J = J_n$ now clause (α) says $\theta(I_\eta) = \theta(|I_\eta|) = \kappa_{\eta(0)}, \kappa_{\eta(0)+1} < \mu$ so as for clause (β), "$D_{\mathbf{p}}$ is a filter on I_η", it holds by the choice of $\mathbf{p}$.

Clause (c): $\mathrm{rk}^{\mathbf{p}}_{\mathbf{d}}(f) = \mathrm{rk}_{\mathbf{d}}(f, \mathbf{p})$ is an ordinal as defined in 7.9.

Clause (d):
Clearly $\Sigma(\mathbf{d})$ is of the right form.

Clause (e):
On $\mathbf{j}$ - see 7.13(2)(c).

Clause (f):
We prove by induction on the ordinal ζ that:

$(*)$ if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{j}(\mathbf{d}) > \varepsilon$ and $A = \cup\{A_\alpha : \alpha < \kappa_\varepsilon\} \in D_{\mathbf{d}}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ and $A_\alpha \in D^+_{\mathbf{d}} \Rightarrow \mathrm{rk}_{\mathbf{d}+A_\alpha}(f) \geq \alpha$ then $\mathrm{rk}_{\mathbf{d}}(f) \geq \alpha$.

Now Definition 7.9 is tailored made for this.

Older version using 7.8 recheck:
For $\alpha = 0$ and α a limit ordinal this is obvious. For $\alpha = \beta + 1$ let $\mathcal{Y} = \{\alpha < \kappa_\varepsilon : A_\alpha \in D^+_{\mathbf{d}}\}$ and for $\alpha \in \mathcal{Y}$ let $\mathbf{n}_\alpha = \min\{n$: there is $(\mathbf{e}, h) \in \Sigma(\mathbf{d} + A_\alpha)$ such that $\mathrm{rk}_{\mathbf{e}}(f \circ h) \geq \beta$ and $\eta_{\mathbf{e}}(0) = n\}$. Clearly $\mathbf{n}_\alpha$ is well defined for $\alpha \in Y$, and let $w := \{n : \cup\{A_\alpha : \alpha \in \mathcal{Y}$ and $\mathbf{n}_\alpha = n\} \in D^+_{\mathbf{d}}\}$ and also the rest should be clear.

Clause (g): (no-hole)
By the Definition 7.8 or 7.8 of rk. Saharon 09.5.31 recheck.

Clause (h): $\mathrm{rk}_{\mathbf{d}}(f + 1) = \mathrm{rk}_{\mathbf{d}}(f) + 1$.
We prove by induction on the ordinal α that:

$(*)$ for every $\mathbf{d} \in \mathbb{D}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ we have $\mathrm{rk}_{\mathbf{d}}(f) \geq \alpha \Leftrightarrow \mathrm{rk}_{\mathbf{d}}(f+1) \geq \alpha + 1$.

Clause (i): Obvious. $\square$

{m37}

Question 6.17. (09.7.19) Assume little choice and $\mu_* = \min\{\mu : \text{IND}(\mu)\}$. So up to μ we can apply [Sh:835]. Now above it seemed that if $\alpha < \mu \Rightarrow \text{AC}_\alpha$ and μ is a limit cardinal, we can find bound above to $\text{rk}_{\mathbf d}$ hence to $\text{rk}_J(-)$ for J quite complete ideal.
1) Assume $\text{cf}(\mu_*) = \aleph_0$, we try to apply the above replacing $\text{AC}_{<\mu_*}$ by DC $+(\forall \alpha < \mu_*)(\neg\text{IND}(\mu))$. So the problem is, on the one hand, about [Sh:938, §3] with weaker form of choice (as in [Sh:835]) and on the other hand the right use of $\text{IND}(\mu_*)$ here.
2) What above μ is a successor?
3) Even with choice, the bound on rank does not give a bound on pp or $\text{tcf}(\mu^{\kappa_n}, <_D)$ well above $\theta(\mathcal{P}(\kappa_n))$ it gives with choice/without much choice - as can be done in §1.

Claim 6.18. *1) If $\langle f_\alpha : \alpha < \delta\rangle$ is $<_D$-increasing in $(\Pi, \bar\alpha, <_D)$ then $\text{rk}_D(\alpha) \ge \delta$.*
2) If $\langle f_\alpha : \alpha < \mu\rangle$ are $\ne_D$-distinct in $(\Pi\bar\alpha, <_D)$ and $\mu > \theta(\mathcal{P}(\ell g(\bar\alpha))$ then we can use [Sh:E38] *which continues* [Sh:497].
3) As in (1) devise μ to $\le \mathcal{P}(\kappa_n)$ on each for some $D_2 \supseteq D$ the sequence is increasing.

{m37}

Theorem 6.19. *1) If* $\text{IND}(\langle\kappa_n : n < \omega\rangle$ *<u>then</u>* [**?**] *- FILL.*
2) For $\aleph_\omega$ - [FILL].

The following information is not presently

{m40}

Claim 6.20. *1) Assume $(\bar\kappa, \bar J)$ is as in 6.3 and $n < \omega \Rightarrow |\mathcal{P}(I_n)| < \kappa_{n+1}$. <u>Then</u> for $\mathbf{s} = \mathbf{s}_{\bar\kappa, \bar J}$, for every $\mathbf{d} \in \mathbb{D}_{\mathbf s}$ we can find $A_\ell \in J^+_\ell$ for $\ell \in$ Rang$(\eta_{\mathbf d})$ such that $\prod\limits_{\ell \in m_{\mathbf d}}^{n(\mathbf d)} A_\ell \in D^+_{\mathbf d}$ and $D_{\mathbf d} + \Pi A_\ell = D_{\mathbf e}$ for some $\mathbf e$ such that $I_{\mathbf e} = I_{\mathbf d}, A_{\mathbf e} = \prod\limits_\ell A_\ell$.*
2) Moreover, for every $p \in$ par$(\mathbf d)$ there is a refinement q such that each $\mathbf{e}_{q,s}(s \in S_q)$ is of the form in (1).
3) In part (1) if $J_n = J^{bd}_{\lambda_n}$ where $\lambda_n =$ cf(λ_n) in$[\kappa_n, \kappa_{n+1})$ <u>then</u> in fact $D_{\mathbf d} + \Pi A_\ell$ is isomorphic to $D_{\mathbf e}$ where $\eta_{\mathbf e} = \eta_{\mathbf d}, A_\ell = I_{\mathbf d} = I_{\mathbf e}$.

Proof. FILL. □

7. Connection to IND

§(2A) Getting quasi-rank system with $\text{AC}_{<\mu}$

{f6.2}

Remark 7.1. 1) Below we can concentrate on the case $\ell g(\bar J) = \omega, \langle\kappa_n : n < \omega\rangle$ increasing, even $2^{\kappa_n} < \kappa_{n+1}$ and $\kappa_n = \text{cf}(\kappa_n)$.
2) We like to use less choice say only DC not $\text{AC}_{<\mu}, \mu = \sum\limits_n \kappa_n$. This is not achieved for $\mathbf{q}^1_{\bar J}, \mathbf{q}^3_{\bar J}$, so it seems. So we may like to change [Sh:938, §3]. Consider $k = 2, 4$ in 7.13(2) to use.
3) (09.7.18) We may hope that if $J_n = [\kappa_n]^{\le\sigma}$ we need only, e.g. DC + $\text{AC}_{\mathcal{P}(\sigma)}$. But then we do not look at $J_{n+1} + A, |A| = \kappa_{n+1}$. So maybe have $\langle J^1_n, J^2_n : n < \omega\rangle$, see 7.14 or maybe have $J_{m,n}$ an ideal on $\kappa_n, J_{m,n} = [\kappa_n]^{\le\kappa_m}$, see 7.19.
4) (09.7.18) Try $\text{IND}_\kappa(\mu)$ or so $(\tau(\mathfrak{A})| = \kappa, |\mathfrak{A}| = \mu$, no ω-end-independent sequence or $\text{IND}\langle\mu_i, I_i : i < \kappa\rangle$ looking for $i_n < i_{n+1} < ldots \alpha_m \in \mu_m, \alpha_m \notin F(\alpha_{n+1}, \dots, \alpha_m) \in I_{\alpha_m}$. Can we connect by Fodor?

5) (09.7.18) To define the ranks for $\mathbf{p}$ we better revise the pre-rank-system as follows. For every $\mathbf{d}$ we have $\Sigma_{\rm pr}(\mathbf{d}) = \Sigma^{\rm pr}_{\mathbf{p}}(\mathbf{d})$, the pure successors and $\Sigma_{\rm ap}(\mathbf{d}) = \Sigma^{\rm ap}_{\mathbf{p}}(\mathbf{d})$ the apure ones and we have interpolation. In the conclusion we try.

In clause (f), $\mathbf{p}$-completeness, we shall try to get $\mathbf{e} \in \Sigma_{\rm pr}(\mathbf{d})$.

In clause (i), also if $(\mathbf{e}, h) \in \Sigma_{\rm pr}(\mathbf{d}), f \in {}^{I_{\mathbf{d}}}{\rm Ord}, g = f \circ h \in {}^{I_{\mathbf{e}}}{\rm Ord}$ then ${\rm rk}_{\mathbf{d}}(f) = {\rm rk}_{\mathbf{e}}(g)$.

In the definition of ${\rm rk}_{\mathbf{p}}$, **??**, $(\mathbf{e}, h) \in \Sigma_{\rm pr}(\mathbf{d})$, we may instead of ${\rm rk}_{\mathbf{p},\zeta}(-,-)$ ask for a tree of pure extensions, but well founded tree.
5A) The natural case is $\bar{J} = \langle J_n : n < \omega \rangle, \mathbb{D} = \{\eta : \eta \text{ is } \langle n, n-1, \dots, m \rangle\}, \Sigma_{\rm pr}(\mathbf{d})$ is as there but $\eta_{\mathbf{e}} = \varrho\hat{\ }\eta_{\mathbf{d}}$ but on $I_{\varrho(f)}$ we use the original J. This fine to see that it fits. If $\mathbf{O}$ or κ larger, we allow "side extension of η" but min ${\rm Rang}(\eta)$ remains.
6) (09.7.18) But later we have preservation of ranks when we use isomorphic $\mathbf{p}$ or $\mathbf{p}$ restricted to "$\mathbf{d}$ and above". So if $J_n = J^{\rm bd}_{\kappa_n}, \kappa_n$ regular, $J^{\rm bd}_{\kappa_n}, J^{\rm bd}_{\kappa_n} + A$ are the same.
6A) Maybe legal partitions of $\prod\limits_{\ell} I_{\eta,\ell}$ is when $I_{\eta(\ell)}$ is divided to $< \kappa_{\eta(\ell)}$.

{f6.3}

Definition 7.2. 1) Let $\bar{J}$ be called a candidate or δ-candidate when:

(a) $\bar{J} = \langle J_\varepsilon : \varepsilon < \delta \rangle$, δ a limit ordinal
(b) J_ε is an ideal on κ_ε
(c) $\delta < \kappa_0$ and κ_ε is non-decreasing.

2) We say that $\bar{J}$ is a generalized candidate when for some $\mathbf{O}$:

(a) $\mathbf{O}$ is a linear order with no last element
(b) $J = \langle J_\varepsilon : \varepsilon \in \mathbf{O} \rangle$
(c) J_ε is a $\aleph_1$-complete ideal on $I_\varepsilon := {\rm Dom}(J_\varepsilon) = \cup\{u : u \in J_\varepsilon\}$.

In some sense the simplest example is

{f6.3d}

Example 7.3. Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of ordinals, $J_n := [\kappa_n]^{\leq \aleph_0}$.

{f6.4}

Discussion 7.4. (08.6.27) 1) We shall try to define a rank (from a p.r.s. or p.r.s.*) such that clause (j) of m4.6 of §3 of [Sh:938] follows. It seems that a necessary condition for the rank to be $< \infty$ we need ${\rm IND}(\mathbf{p})$.
2) Naturally we can define $\mathbf{p}$ from $\bar{J}$ and a reasonable condition is ${\rm IND}(\bar{J})$ at least when $\ell g(\bar{J}) = \omega$.
3) We can below use generalized candidates.

{f6.5}

Definition 7.5. 1) We say $\mathbf{p} = (\mathbb{D}, \Sigma, \mathbf{j})$ be a ι-p.r.s. (pre-rank-ι-system with $\iota = 1, 2$; if $\iota = 2$ we may omit it) when in Definition m4.3 or m4.4 of §3 of [Sh:938, §3,m4.4] it satisfies clauses (a),(b),(d),(e) and we add in (d):

$(*)$ Σ is transitive: if $(h_1, \mathbf{d}_1) \in \Sigma(\mathbf{d}_0)$ and $(h_2, \mathbf{d}_2) \in \Sigma(d_1)$ then $(h_2 \circ h_1, \mathbf{d}_2) \in \Sigma(\mathbf{d}_0)$

[check where used].

2) We say $\mathbf{p}$ is a quasi rank ι-system when $\mathbf{p} = (\mathbb{D}, {\rm rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_{\mathbf{p}}, {\rm rk}_{\mathbf{p}}, \Sigma_{\mathbf{p}}, \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{p}})$ satisfies Definition m4.3 of §3 of [Sh:938] if $\iota = 1$, Definition m4.4 of §3 of [Sh:938] if $\iota = 2$ except that the rank may be ∞; we write ${\rm rk}_{\mathbf{d}}(f, \mathbf{d})$ for $\mathbf{d} \in \mathbb{D}_{\mathbf{P}}$ and $f \in {}^{I[\mathbf{d}]}{\rm Ord}$.

2A) Alternatively: $\mathrm{rk}_{\mathbf{p}}$ is defined as in 7.8 below [or 7.9].

{f6.6} **Convention 7.6.** 1) $\mathbf{p}$ is a 2-p.r.s.
2) We usually omit the $\mathbf{p}$ when clear from the context, similarly for $\mathrm{rk}_{\mathbf{d}}(f, \mathbf{p})$ defined below.

{f6.6.3} *Remark* 7.7. 1) We shall try to define rk. We shall try to prove mainly (f) [the version with $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$].

{f6.7} **Definition 7.8.** For $\mathbf{p}$ a p.r.s., $\mathbf{d} \in \mathbb{D}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ we define $\mathrm{rk}_{\mathbf{d}}(f, \mathbf{p}) = \mathrm{rk}^0_{\mathbf{d}}(f, \mathbf{p})$ by defining when $\mathrm{rk}_{\mathbf{d}}(f, \mathbf{p}) \geq \alpha$ for an ordinal α by induction on α for all pairs $(\mathbf{d}, f)$; so $\mathrm{rk}^0_{\mathbf{d}}(f, \mathbf{p}) = \alpha$ when it is $\geq \alpha$ but not $\geq \alpha + 1$, and is ∞ otherwise; by monotonicity well defined.

$\underline{\alpha = 0}$: always.

$\underline{\alpha \text{ limit}}$: when $\mathrm{rk}^0_{\mathbf{d}}(f, \mathbf{p}) \geq \beta$ for every $\beta < \alpha$.

$\underline{\alpha = \beta + 1}$: when for some $(h, \mathbf{e}) \in \Sigma_{\mathbf{p}}(\mathbf{d})$ and $g \in {}^{I[\mathbf{e}]}\mathrm{Ord}$ we have $g <_{D_{\mathbf{e}}} f \circ h$ and $\mathrm{rk}^0_{\mathbf{e}}(g, \mathbf{p}) \geq \beta$.

{f6.8} **Definition 7.9.** [Saharon 09.06.01: check that this definition satisfies additivity and $\mathrm{rk}(f + 1) = \mathrm{rk}(f) + 1$.

We define $\mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p})$ and $\mathrm{dp}^1_{\mathbf{d},\zeta}(f, \mathbf{p})$ from $\mathrm{Ord} \cup \{\infty\}$ for $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}, f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ by defining by induction on the ordinal ζ:

(a) when $\mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p}) \geq \zeta$ and
(b) when $\mathrm{dp}^1_{\mathbf{d},\zeta}(f, \mathbf{p}) \geq \xi$ for any ordinal ξ.

Arriving to ζ we let:

- $\mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p}) \geq \zeta$ $\underline{\text{iff}}$ for every $\zeta_1 < \zeta$ and $\xi < \infty$ there is $(h, \mathbf{e}) \in \Sigma(\mathbf{d})$ such that $\mathrm{rk}^1_{\mathbf{e}}(f \circ h, \mathbf{p}) \geq \zeta_1$ and $\mathrm{dp}^1_{\mathbf{e},\zeta_1}(f \circ h, \mathbf{p}) \geq \xi$
- we define by induction on $\xi < \infty$ when $\mathrm{dp}^1_{\mathbf{d},\zeta}(f, \mathbf{p}) \geq \xi$; it holds if $\mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p}) \geq \zeta$ and for every $\xi_1 < \xi$ and partition $\langle A_\varepsilon : \varepsilon < \varepsilon_* \rangle$ of $I_{\mathbf{d}}$ with $\varepsilon_* < \kappa_{\mathbf{j}(\mathbf{d})}$ parts, there is $(h, \mathbf{e}) \in \Sigma(\mathbf{d})$ such that $\mathrm{rk}^1_{\mathbf{e}}(f \circ h, \mathbf{p}) \geq \zeta$ and $\mathrm{dp}^1_{\mathbf{e},\zeta}(f \circ h, \mathbf{p}) \geq \xi_1$ and $I_{\mathbf{e}} = I_{\mathbf{d}}$ (Saharon 09.06.01: or use Σ_1.)

Remark 7.10. 1) In a $\underline{\text{variant}}$ we demand: and $I_{\mathbf{e}} = I_{\mathbf{d}} \wedge h = \mathrm{id}_{I[\mathbf{d}]}$.
2) By 7.9 we may derive a quasi rank system from a p.r.s., but we deal with the special case which seems most interesting.

{f6.8d} **Claim 7.11.** *1) The rank in Definition 7.8, 7.9 are well defined.*
2) $\mathrm{rk}^0_{\mathbf{d}}(f, \mathbf{p}) \leq \mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p})$.

{f6.8g} **Discussion 7.12.** (09.06.01) 1) We would like to use $\mathrm{AC}_{\mathcal{U}}$ for constant $\mathcal{U}$ or at most $\mathcal{U}$ depend on $\mathbf{0}$. By the amount of completeness we need (approaching μ), if we use $\mathrm{rk}^1_{\mathbf{d}}(-, \mathbf{f}^1_{\bar{J}})$ is it O.K.? Does it?

{f6.9} **Definition 7.13.** 1) For $\ell = 1, 2$ and $\mathbf{p}$ a p.r.s. we say $\mathbf{p}$ is well ℓ-founded $\underline{\text{when}}$ $\mathrm{rk}^\ell_{\mathbf{d}}(f, \mathbf{p}) < \infty$ for every $\mathbf{d} \in \mathbb{D}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$.
2) Similarly for $\mathbf{p}$ a quasi rank system (so now $\mathrm{rk}_{\mathbf{d}}(f, \mathbf{p})$ is not as defined in Definition 7.9, but is from Definition 7.5(2)).

{f6.10} **Definition 7.14.** For a candidate $\bar{J} = \langle J_\varepsilon : \varepsilon \in \delta \rangle, J_\varepsilon$ an ideal on κ_ε we define $\mathbf{p} = \mathbf{p}_{\bar{J}}$ as follows:

(a) $\mathbb{D}_{\mathbf{p}}$ is the set of $\mathbf{d} = (I, D) = (I_{\mathbf{d}}, D_{\mathbf{d}})$ such that for some $\eta = \eta_{\mathbf{d}}$ we have:

- (α) η a non-empty decreasing sequence of ordinals $< \delta$
- (β) $I = \prod\limits_{\ell < \ell g(\eta)} \kappa_{\eta(\ell)}$
- (γ) $D = D_\eta + A_{\mathbf{d}}$ for some $(\bar\kappa, \eta)$-large subset of I_η which means
- (δ) $A \subseteq I_\eta$ is $(\bar\kappa, \eta)$-large when $A = \prod\limits_{\ell < n} Y_\ell$ for some $Y_\ell \in [\kappa_{\eta(\ell)}]^{\kappa_{\eta(\ell)}}$ for $\ell < n$ and
- (ε) let $u_{\mathbf{d}} = \text{Rang}(\eta_{\mathbf{d}}), D_{u_{\mathbf{d}}} = D_\eta$
- (ζ) $D = \{Y \subseteq \prod\limits_{\ell < n} \kappa_{\eta(\ell)} :$ there is a sequence $\langle Y_\ell : \ell \le \ell g(\eta)\rangle$
  such that $Y_n = Y, Y_0 = \{<>\}$ and $\ell \le \ell g(\eta) \Rightarrow Y_\ell \subseteq \prod\limits_{m < \ell} \kappa_{\eta(m)}$
  and $\ell < \ell g(\eta) \wedge \rho \in Y_\ell \Rightarrow \{\alpha < \kappa_{\eta(\ell)} : \rho\hat{\ }\langle\alpha\rangle \notin Y_{\ell+1}\} \in J_{\eta(\ell)}\}$

(b) $\Sigma(\mathbf{d}) = \{(h, \mathbf{e})$: for some ϱ we have $\eta_{\mathbf{e}} = \varrho\hat{\ }\eta_{\mathbf{d}} \in \mathbb{D}$ and $h : I_\nu \to I_\eta$ is defined by $h(\rho) = \langle \rho(\ell g(\varrho) + \ell) : \ell < \ell g(\eta)\rangle$ and h induces a mapping from $D_{\mathbf{e}}$ into $D_{\mathbf{d}}\}$

(c) $\mathbf{j}(\eta) = \eta(\ell g(\eta) - 1)$

(d) $\mu = \cup\{\kappa_\varepsilon : \varepsilon < \delta\}$.

{f6.11}

Definition 7.15. 1) Similarly to 7.14 for a generalized candidate $\bar J = \langle J_\varepsilon : \varepsilon \in \mathbf{O}\rangle$.
2) For a candidate $\bar J = \langle J_n : n < \omega\rangle$ we define $\mathbf{p}^2_{\bar J} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu)$ as in 7.14 but:

$(a)'$ $\mathbb{D} = \{\mathbf{d} : \mathbf{d}$ as in clause (a) of Definition 7.14 but $\eta_{\mathbf{d}} = \langle n, n-1, \dots, m\rangle$ where $m \le n\}$

$(e)'$ rk is as defined in Definition 7.8.

3) We define $\mathbf{p}^{\ell+2}_{\bar J}$ as in part (1) or by **??** but replace clause $(a)(\delta)$ of **??** or part (1) by:

$(\delta)'$ $D_\eta = \{Y \subseteq \prod\limits_{\ell < n} \kappa_{\eta(\ell)}$: for some $Y_\ell \in J_\ell$ for $\ell < n$ we have $\prod\limits_{\ell < n} \kappa_{\eta(\ell)} \backslash \{\rho \in \prod\limits_{\ell < n} \kappa_{\eta(\ell)} : (\exists \ell < n)[\rho(\ell) \in Y_\ell]\}$.

4) For $\ell = 0, 1$ let $\mathbf{q}^{k,\ell}_{\bar J}$ be the $\mathbf{q}^k_J$ expanded by $\text{rk}^\ell_{\mathbf{d}}(f, \mathbf{p}^k_{\bar J})$. If $\ell = 1$ we may omit it.

{f6.12}

Claim 7.16. *1) Above $\bar p^1_{\bar J}$ is not well 0-founded iff: there are $\bar\varepsilon, \bar f$ such that*

$\circledast_{\bar\varepsilon, \bar f}$
- (a) $\bar\varepsilon = \langle \varepsilon_i : i < \omega\rangle$ *is increasing*
- (b) $\bar f = \langle f_{i,j} : i < j < \omega\rangle$
- (c) $f_{i,j}$ *is a function from* $I_{\langle \varepsilon_j, \varepsilon_{j-1}, \dots, \varepsilon_{i+1}\rangle}$ *into* J_{ε_i}
- (d) *for every* $\bar\alpha \in \prod\limits_{i < \omega} \kappa_{\varepsilon_i}$ *for some* $i < j$ *we have* $\alpha_i \in f(\alpha_{n_j}, \alpha_{n_{j-1}}, \dots, \alpha_{n_{i+1}})$.

2) Similarly for $\mathbf{p}^2_J$ (i.e. $\delta = \omega$ we can above demand $\varepsilon_i = i$, so it is equivalent to $\neg\text{IND}\langle J_n : n < \omega\rangle$.

Proof. 1) As in [Sh:513].
2) Easy as we can add to a function dummy variables. $\square_{7.16}$

Task: 1) Prove $\mathbf{p}^2_{\bar J}$ satisfies clause (f) for $\text{rk} = \text{rk}^1_{\mathbf{p}}$ defined as in 7.8.
2) Check the $\text{rk}(f + 1) = \text{rk}(f) + 1$, but see below.

{f6.17}

Claim 7.17. *1) If* $\bar{J} = \langle J_\varepsilon : \varepsilon \in \mathbf{O} \rangle$ *is a generalized candidate and* $k = 1, 3$ *then* $\mathbf{p}^k_{\bar{J}}$ *is a p.r.s. provided that "*J_ε *is* $\theta(\mathbf{O})$*-complete"(?)*
2) If $\bar{J} = \langle J_n : n < \omega \rangle$ *is a candidate and* $k = 2, 4$ *then* $\mathbf{p}^k_{\bar{J}}$ *is a p.r.s.*
3) In part (1), $\mathbf{q}^k_{\bar{J}}$ *is a quasi rank system.*
4) Assume $\bar{J} = \langle J_n : n < \omega \rangle, J_n$ *an ideal on* $\kappa_n, \kappa^+_n < \kappa_{n+1}, \mu = \Sigma \kappa_n$*. Then* $\mathbf{q}^k_{\bar{J}}$ *is a quasi rank system.*

Proof. 1) As in the proof of e5.g of §4 of [Sh:938, §4,e5.g] or better see the proof of 7.17(?) except that we use 7.9 instead of 7.8 which simplify clause (f), but is cumbersome in other places.
2) We check Definition m4.3 of §3 of [Sh:938, §3,m4.3].

Clause (a): μ is singular.
As $\mu = \sum_n \kappa_n$ and $\kappa_n < \kappa_{n+1}$ this is obvious.

Clause (b): Let $\mathbf{d} \in \mathbb{D}, \eta = \eta_{\mathbf{d}}, J = J_n$ now clause (α) says $\theta(I_\eta) = \theta(|I_\eta|) = \kappa_{\eta(0)}, \kappa_{\eta(0)+1} < \mu$ so as for clause (β), "$D_{\mathbf{p}}$ is a filter on I_η", it holds by the choice of $\mathbf{p}$.

Clause (c): $\text{rk}^{\mathbf{p}}_{\mathbf{d}}(f) = \text{rk}_{\mathbf{d}}(f, \mathbf{p})$ is an ordinal as defined in 7.9.

Clause (d):
Clearly $\Sigma(\mathbf{d})$ is of the right form.

Clause (e):
On $\mathbf{j}$ - see 7.13(2)(c).

Clause (f):
We prove by induction on the ordinal ζ that:

$(*)$ if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{j}(\mathbf{d}) > \varepsilon$ and $A = \cup\{A_\alpha : \alpha < \kappa_\varepsilon\} \in D_{\mathbf{d}}$ and $f \in {}^{I[\mathbf{d}]}\text{Ord}$ and $A_\alpha \in D^+_{\mathbf{d}} \Rightarrow \text{rk}_{\mathbf{d}+A_\alpha}(f) \geq \alpha$ then $\text{rk}_{\mathbf{d}}(f) \geq \alpha$.

Now Definition 7.9 is tailored made for this.

Older version using 7.8 recheck:
For $\alpha = 0$ and α a limit ordinal this is obvious. For $\alpha = \beta + 1$ let $\mathcal{Y} = \{\alpha < \kappa_\varepsilon : A_\alpha \in D^+_{\mathbf{d}}\}$ and for $\alpha \in \mathcal{Y}$ let $\mathbf{n}_\alpha = \min\{n$: there is $(\mathbf{e}, h) \in \Sigma(\mathbf{d} + A_\alpha)$ such that $\text{rk}_{\mathbf{e}}(f \circ h) \geq \beta$ and $\eta_{\mathbf{e}}(0) = n\}$. Clearly $\mathbf{n}_\alpha$ is well defined for $\alpha \in Y$, and let $w := \{n : \cup\{A_\alpha : \alpha \in \mathcal{Y}$ and $\mathbf{n}_\alpha = n\} \in D^+_{\mathbf{d}}\}$ and also the rest should be clear.

Clause (g): (no-hole)
By the Definition 7.8 or 7.8 of rk. Saharon 09.5.31 recheck.

Clause (h): $\text{rk}_{\mathbf{d}}(f + 1) = \text{rk}_{\mathbf{d}}(f) + 1$.
We prove by induction on the ordinal α that:

$(*)$ for every $\mathbf{d} \in \mathbb{D}$ and $f \in {}^{I[\mathbf{d}]}\text{Ord}$ we have $\text{rk}_{\mathbf{d}}(f) \geq \alpha \Leftrightarrow \text{rk}_{\mathbf{d}}(f+1) \geq \alpha + 1$.

Clause (i): Obvious. $\square_{7.17}$

{f6.19}

Claim 7.18. *Assume* $\bar J = \langle J_n : n < \omega\rangle$ *is a candidate and* $\mathrm{IND}(\bar J)$.

Then $\mathbf{p}^2_{\bar J}$ *is a strict rank system.*

Proof. By 7.17 and the definition, it is a weak rank system. So we should prove the "strict", i.e. clause (j) of Definition m4.6 of §3 of [Sh:938] which we do by m4.16 of §3 of [Sh:938]. We use $\Sigma_1(\mathbf{d}) = \Sigma(\mathbf{d})$.

On $(*)_2$:

Given $\mathbf{d}$ we choose $j < \omega$ such that $j > \eta_{\mathbf{d}}(0)$ and assume $\mathbf{e} \in \mathbb{D}_{\geq j}$. ☐

§(2B) Revisiting

The simplest case below is: $\mathbf{x}$ consist $I_n = \kappa_n, \kappa_n < \kappa_{n+1}, J_{1,n} = [\kappa_n]^{<\theta}, J_{2,n} = [\kappa_n]^{<\kappa_n}$, $\mathrm{ind}(\mu,\theta), \mu = \Sigma\kappa_n, \mu$ minimal (or $\mu = \infty$) $\mathrm{ind}_{\mathbf{x}} :\in \mathrm{Ord}_* \cup \{\infty\}$.

For μ there are algebras on γ with no independent ω-sequence hence [Sh:835] and see §5 apply. But if using $\mathbf{x}$ we have a rank 2-system for which Theorem m4.13 of §3 of [**?**] apply (check!)

We may consider the pseudo version (using $\mathrm{comp}_\gamma(J)$. We have to sort out the amount of choice needed -seemingly.

{k2}

Definition 7.19. We say that $\mathbf{x}$ is a ω-candidate when it consists of

- (a) set I_n for $n < \omega$ (κ a cardinal and $\theta(< \kappa) = \kappa$
- (b) ideal $J_{n,k}$ on I_n for $k < \omega, n < \omega$
- (c) $J_{n,k} \subseteq J_{n,k+1}$
- (d) κ_n.

{k5}

Definition 7.20. For a 2-candidate $\mathbf{x}$ we define by induction on $i < \omega$ what is an $\mathbf{x}$-object $\mathfrak{c} = \mathbf{d}$ of depth i, such that

- $(*)_\iota$ for some $n_{\mathbf{d}} < m_{\mathbf{d}} < \omega, \iota$ is an $\subseteq$-increasing sequence $\langle J_{\iota,k} : k < \omega\rangle$ of ideals on $I_{m_{\mathbf{d}},n_{\mathbf{d}}} = \Pi\{I_k : k \in [m,n)\}$.

The case $i = 0$:

$n_{\mathbf{d}} = m_{\mathbf{d}} + 1$ and let $h_{\mathbf{d}}$ be the one-to-one function from $I_{m_{\mathbf{d}}}$ onto $I_{\mathbf{m}_\iota,n_\iota}$ and $J_{\iota,k} \in \hat h_\iota(I_{m_\iota,k} + A_k)$ where $A_k \in J^+_{m_{\mathbf{d}}}$ and $A_k \supseteq A_{k+1}$ for $k < \omega$.

The case $i + 1$:

For some $k, \iota(1), \iota(2)$ we have

- (a) $k \in (m_{\mathbf{d}}, n_{\mathbf{d}})$
- (b) $\iota(\ell)$ is an i_ℓ-pair for some $i_\ell \leq i$ for $\ell = 1, 2$
- (c) $m_{\mathbf{d}(1)} = m_\iota, n_{\mathbf{d}(1)} = k$
- (d) $m_{\mathbf{d}(2)} = k, n_{\mathbf{d}(1)} = k$
- (e) $m_{\mathbf{d}(2)} = k, n_{\mathbf{d}(2)} = n_\iota$
- (f) there are $\langle A_{1,k}, A_{2,k} : k < \omega\rangle$ such that
  - (α) $A_{\ell,k} \in J_{\iota(\ell),k+1}$
  - (β) $B \in J_{\mathbf{d},k}$ iff $B \subseteq I_{m_{\mathbf{d}},n_{\mathbf{d}}}$ and for some $B_1 \in I_{\iota(1),k}$ we have $\eta \in A_{2,k} \subseteq I_{n_{\mathbf{d}(2)},n_{\mathbf{d}(2)}} \Rightarrow \{\nu \in I_{m_{\mathbf{d}(2)},n_{\mathbf{d}(2)}} : \eta \cup \nu \in B\} \in\ ?$

Remark 7.21. 1) Definition 7.20? seemingly does not behave transitively.
2) We may allow $n_{\mathbf{d}} = m_{\mathbf{d}}$.

{k7} **Definition 7.22.** For $\mathbf{x}$ an ω-candidate, we define a p.c.s. $\mathbf{p} = \mathbf{p}^0_{\mathbf{x}}$ as follows:

- (a) $\mathbb{D}_{\mathbf{p}} = \{\mathbf{d} : \mathbf{d}$ is an $\mathbf{x}$-object$\}$
- (b) $\Sigma(\mathbf{d}_\iota) = \{\mathbf{d}$: for some $\mathbf{d}_2$ the triple $(\mathbf{d}, \mathbf{d}_{i_1}, \mathbf{d}_{i_2})$ is as in Definition 7.20
- (c) $\mathbf{j}(\mathbf{d})$ is $m_{\mathbf{d}}$
- (d) $\mu = \cup\{\kappa_n : n < \omega\}$.

{k9} **Claim 7.23.** *If* $\mathbf{x}$ *is an* ω*-candidate* *<u>then</u>* $p^0_{\mathbf{x}}$ *is a quasi rank sytem.*

Proof. FILL. $\square$

{k11} **Definition 7.24.** 1) For an ω-candidate $\mathbf{x}$ we say it is well founded when the p.r.s. $\mathbf{p}^0_{\mathbf{x}}$ is well founded, e.g. $\mathbf{p}_{\mathbf{x}}$ is a weak rank system.
2) For a well founded.

{k13} **Claim 7.25.** *If* $\mathbf{x}$ *is a well founded* ω*-candidate* *<u>then</u>* $\mathbf{p}_{\mathbf{x}}$ *is a strict rank system.*

Proof. <u>Stage A</u>: We have to check clause (1) from Definition m4.6 of §3 of [Sh:938].

So assume $\mathbf{d}, \zeta, \xi, f$ are as in $\boxplus$ there. Choose $j < \omega$ such that $j > n_{\mathbf{d}}$ and toward contradiction assume $\mathbf{e}, g$ are as in $\oplus$ there.

<u>Stage B</u>: We find $(\mathbf{e}_1, g_1)$ satisfying $\oplus$ of clause (j) of m4.6 of §3 of [Sh:938] and $m_{\mathbf{e}_1} = n_{\mathbf{d}}$; note if we define as in [**?**](2) rather than as in 7.13(3), we would not need this step, but then we may have to reconsider the proof of (f) of Definition m4.3 of §3 of [Sh:938].

<u>Stage C</u>: We use $\mathrm{AC}_{I[\mathbf{e}]}$ we continue as in 7.18 and in §4. But see footnote to $\bullet_3$ in $\oplus$ in clause (j) of m4.6 of §3 of [**?**]. $\square_{7.25}$

8. Appendix: psuedo true cofinality

We repeat here [Sh:938, §5].

Pseudo PCF

We try to develop pcf theory with little choice. We deal only with $\aleph_1$-complete filters, and replace cofinality and other basic notions by pseudo ones, see below. This is quite reasonable as with choice there is no difference.

This section main result are 8.9, existence of filters with pseudo-true-cofinality; 8.19, giving a parallel of $J_{<\lambda}[\alpha]$; and 1.2, on generators of $J^{[\bar{\alpha}]}_{<\lambda^+}$.

In the main case we may (in addition to ZF) assume DC + $\mathrm{AC}_{\mathcal{P}(\mathcal{P}(Y))}$; this will be continued in [Sh:938].

{r1} **Hypothesis 8.1.** ZF

{r2} **Definition 8.2.** 1) We say that a partial order P is $(< \kappa)$-directed when every subset A of P of power $< \kappa$ has a common upper bound.
1A) Similarly P is $(\leq S)$-directed.
2) We say that a partial order P is pseudo $(< \kappa)$-directed <u>when</u> it is $(< \kappa)$-directed and moreover every subset $\cup\{P_\alpha : \alpha < \delta\}$ has a common upper bound <u>when</u>:

- (a) if $\delta < \kappa$ is a limit ordinal
- (b) $\bar{P} = \langle P_\alpha : \alpha < \delta \rangle$ is a sequence of non-empty subsets of P

(c) if $\alpha_1 < \alpha_2, p_1 \in P_{\alpha_1}$ and $p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$.

2A) For a partial order S we say that the partial order P is pseudo ($\leq S$)-directed when $\cup\{P_s : s \in S\}$ has a common upper bound whenever

(a) $\langle P_s : s \in S\rangle$ is a sequence

(b) $P_s \subseteq P$

(c) if $s <_S t$ and $f \in P_s, g \in P_t$ then $f <_p g$

(d) if $s \in S$ then P_s has a common upper bound (so if S has no minimal member this is redundant).

{r3}

Definition 8.3. We say that a partial (or quasi) order P has pseudo true cofinality δ when: δ is a limit ordinal and there is a sequence $\langle P_\alpha : \alpha < \delta\rangle$ such that

(a) $P_\alpha \subseteq P$ and $\delta = \sup\{\alpha < \delta : P_\alpha$ non-empty$\}$

(b) if $\alpha_1 < \alpha_2 < \delta, p_1 \in P_{\alpha_1}, p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$

(c) if $p \in P$ then for some $\alpha < \delta$ and $q \in P_\alpha$ we have $p \leq_P q$.

{r4}

Remark 8.4. 0) See 8.2(2) and 8.8(1).
1) We could replace δ by a partial order Q.
2) The most interesting case is in Definition 8.6.
3) We may in Definition 8.3 demand δ is a regular cardinal.
4) Usually in clause (a) without loss of generality $\bigwedge\limits_{\alpha} P_\alpha \neq \emptyset$, as without loss of generality $\delta = \mathrm{cf}(\delta)$ using $P'_\alpha = P_{f(\alpha)}$ where $f(\alpha) =$ the α-th member of $\{\beta < \delta : P_\beta \neq \emptyset\}$. Why do we allow $P_\alpha = \emptyset$? as it is more natural in 8.17(1), but can usually ignore it.

{r5}

Example 8.5. Suppose we have a limit ordinal δ and a sequence $\langle A_\alpha : \alpha < \delta\rangle$ of sets with $\prod\limits_{\alpha<\delta} A_\alpha = \emptyset$; moreover $u \subseteq \delta = \sup(u) \Rightarrow \prod\limits_{\alpha\in u} A_\alpha = \emptyset$. Define a partial order P by:

(a) its set of elements is $\{(\alpha, a) : a \in A_\alpha$ and $\alpha < \delta\}$

(b) the order is $(\alpha_1, a_1) <_P (\alpha_2, a_2)$ iff $\alpha_1 < \alpha_2$ (and $a_\ell \in A_{\alpha_\ell}$ for $\ell = 1, 2$).

It seems very reasonable to say that P has true cofinality but there is no increasing cofinal sequence.

{r6}

Definition 8.6. 1) For a set Y and sequence $\bar\alpha = \langle \alpha_t : t \in Y\rangle$ of ordinals and cardinal κ we define

$$\text{ps-tcf-fil}_\kappa(\bar\alpha) = \{D : \; D \text{ a } \kappa\text{-complete filter on } Y \text{ such that } (\Pi\bar\alpha/D) \text{ has a pseudo true cofinality}\};$$

see below.
2) We say that $\Pi\bar\alpha/D$ or $(\Pi\bar\alpha, D)$ or $(\Pi\bar\alpha, <_D)$ has pseudo true cofinality γ when D is a filter on $Y = \mathrm{Dom}(\bar\alpha)$ and γ is a limit ordinal and the partial order $(\Pi\bar\alpha, <_D)$ essentially does[4], i.e., there is a sequence $\bar{\mathcal{F}} = \langle \mathcal{F}_\beta : \beta < \gamma\rangle$ satisfying:

---

[4] so necessarily $\{s \in Y : \alpha_s > 0\}$ belongs to D but is not necessarily empty; if it is non-empty then $\Pi\bar\alpha = \emptyset$, so pedantically this is wrong, but we shall ignore this or assume $\bigwedge\limits_t \alpha_t \neq 0$ when not said otherwise.

$\circledast_{\bar{\mathcal{F}}}$ (a) $\mathcal{F}_\beta \subseteq \{f \in {}^Y\text{Ord} : f <_D \bar{\alpha}\}$

(b) $\mathcal{F}_\beta \neq 0$

(c) if $\beta_1 < \beta_2, f_1 \in \mathcal{F}_{\beta_1}$ and $f_2 \in \mathcal{F}_{\beta_2}$ then $f_1 < f_2$ mod D

(d) if $f \in {}^Y\text{Ord}$ and $f < \bar{\alpha}$ mod D then for some $\beta < \gamma$ we have $g \in \mathcal{F}_\beta \Rightarrow f < g$ mod D (by clause (c) this is equivalent to: for some $\beta < \gamma$ and some $g \in \mathcal{F}_\beta$ we have $f \le g$ mod D).

3) $\text{ps-pcf}_\kappa(\bar{\alpha}) = \text{ps-pcf}_{\kappa\text{-comp}}(\bar{\alpha}) := \{\gamma$: there is a κ-complete filter D on Y such that $\Pi\bar{\alpha}/D$ has pseudo true cofinality γ and γ is minimal for $D\}$.

4) $\text{pcf-fil}_{\kappa,\gamma}(\bar{\alpha}) = \{D : D$ a κ-complete filter on Y such that $\Pi\bar{\alpha}/D$ has true cofinality $\gamma\}$.

5) In part (2) if γ is minimal we call it ps-tcf$(\Pi\bar{\alpha}, D)$ or simply ps-tcf$(\Pi\bar{\alpha}, <_D)$; note that it is a well defined (regular cardinal).

{r7}

Claim 8.7. *1) If* $\lambda =$ ps-tcf$(\Pi\bar{\alpha}, <_D)$, *then* $(\Pi\bar{\alpha}, <_D)$ *is pseudo* $(< \lambda)$*-directed.*

1A) If $\theta(S) < \lambda =$ *ps-tcf*$(\Pi\bar{\alpha}, <_D)$ *then* $(\Pi\bar{\alpha}, <_D)$ *is pseudo* $(\le S)$*-directed.*

2) Similarly for any quasi order.

3) Assume AC_α *for* $\alpha < \lambda$. *If* $\text{cf}(\alpha_t) \ge \lambda = \text{cf}(\lambda)$ *for* $t \in Y$ *then* $(\Pi\bar{\alpha}, <_D)$ *is* λ*-directed.*

4) Assume $\text{AC}_{Y \times \lambda}$. *If* $\text{cf}(\alpha_s) > \lambda$ *for* $s \in Y$ *then* $(\Pi\bar{\alpha}, <_D)$ *is pseudo* λ^+*-directed.*

Proof. As in 8.8(1) below. $\square_{8.7}$

{r8}

Claim 8.8. *Let* $\bar{\alpha} = \langle \alpha_s : s \in Y\rangle$ *and* D *is a filter on* Y.

0) If $\Pi\bar{\alpha}/D$ *has pseudo true cofinality then* ps-tcf$(\Pi\bar{\alpha}, <_D)$ *is a regular cardinal; similarly for any partial order.*

1) If $\Pi\bar{\alpha}/D$ *has pseudo true cofinality* γ_1 *and true cofinality* γ_2 *then* $\text{cf}(\gamma_1) = \text{cf}(\gamma_2) =$ ps-tcf$(\Pi\bar{\alpha}, <_D)$, *similarly for any partial order.*

2) $\text{ps-pcf}_\kappa(\bar{\alpha})$ *is a set of regular cardinals so if* $\Pi\bar{\alpha}/D$ *has pseudo true cofinality then* ps-tcf$(\Pi\bar{\alpha}, <_D)$ *is* γ *where* $\gamma = \text{cf}(\gamma)$ *and* $\Pi\bar{\alpha}/D$ *has pseudo cofinality* γ.

3) Always $\text{ps-pcf}_\kappa(\bar{\alpha})$ *has cardinality* $< \theta(\{D : D$ *a* κ*-complete filter on* $Y\})$.

4) If $\bar{\beta} = \langle \beta_s : s \in Y\rangle \in {}^Y\text{Ord}$ *and* $\{s : \beta_s = \alpha_s\} \in D$ *then* ps-tcf$(\Pi\bar{\alpha}/D) =$ ps-tcf$(\Pi\bar{\beta}/D)$ *so one is well defined iff the other is.*

Proof. 0) By the definitions.

1) Let $\langle \mathcal{F}^\ell_\beta : \beta < \gamma_\ell\rangle$ exemplify "$\Pi\bar{\alpha}/D$ has pseudo true cofinality γ_ℓ" for $\ell = 1, 2$. Now

$(*)$ if $\ell \in \{1, 2\}$ and $\beta_\ell < \gamma_\ell$ then for some $\beta_{3-\ell} < \gamma_{3-\ell}$ we have $g_1 \in \mathcal{F}^\ell_{\beta_\ell} \wedge g_2 \in \mathcal{F}^{3-\ell}_{\beta_{\ell-1}} \Rightarrow g_1 <_D g_2$.

[Why? Choose $g^\ell \in \mathcal{F}^\ell_{\beta_\ell+1}$, choose $\beta_{3-\ell} < \gamma_{3-1}$ and $g_{3-\ell} \in \mathcal{F}^{3-\ell}_{\beta_{3-\ell}}$ such that $g^\ell < g^{3-\ell}$ mod D.]

Hence

$(*)$ $h_1 : \gamma_1 \to \gamma_2$ is well defined when

$$h_1(\beta_1) = \text{Min}\{\beta_2 < \gamma_2 : (\forall g_1 \in \mathcal{F}^1_{\beta_1})(\forall g_2 \in \mathcal{F}^2_{\beta_2})(g_1 < g_2 \text{ mod } D)\}.$$

Clearly h is non-decreasing and it is not eventually constant (as $\cup\{\mathcal{F}^1_\beta : \beta < \gamma_1\}$ is cofinal in $\Pi\bar{\alpha}/D$) and has range unbounded in γ_2 (similarly).

The rest should be clear.
2) Follows.
3),4) Easy. $\square_{8.8}$

Concerning [Sh:835]

{r9}

Claim 8.9. *<u>The Existence of true cofinality filter</u> [$\kappa > \aleph_0 + \mathrm{DC} + \mathrm{AC}_{<\kappa}$] If*

- (a) *D is a κ-complete filter on Y*
- (b) $\bar\alpha \in {}^Y\mathrm{Ord}$
- (c) $\delta := \mathrm{rk}_D(\bar\alpha)$ *satisfies* $\mathrm{cf}(\delta) \geq \theta(\mathrm{Fil}^1_\kappa(Y))$*, see below.*

<u>Then</u> for some D' we have

- (α) *D' is a κ-complete filter on Y*
- (β) $D' \supseteq D$
- (γ) *$\Pi\bar\alpha/D'$ has pseudo true cofinality, in fact,* $\text{ps-tcf}(\Pi\bar\alpha, <_D) = \mathrm{cf}(\mathrm{rk}_D(\bar\alpha))$.

Recall from [Sh:835]

{r9a}

Definition 8.10. 0) $\mathrm{Fil}^1_\kappa(Y) = \{D : D$ a κ-complete filter on $Y\}$ and if $D \in \mathrm{Fil}^1_\kappa(Y)$ then $\mathrm{Fil}^1_\kappa(D) = \{D' \in \mathrm{Fil}^1_\kappa(Y) : D \subseteq D'\}$.
1) $\mathrm{Fil}^4_\kappa(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are κ-complete filters on $Y\}$.
2) $J[f, D]$ where D is a filter on Y and $f \in {}^Y\mathrm{Ord}$ is $\{A \subseteq Y : A = \emptyset \bmod D$ or $\mathrm{rk}_{D+A}(f) > \mathrm{rk}_D(f)\}$.

{r9b}

Remark 8.11. 1) On the Definition of pseudo $(< \kappa, 1+\gamma)$-complete D see [Sh:938, 1.13=0z.51]; we may consider changing the definition of $\mathrm{Fil}^1_\kappa(Y)$ to D is $\aleph_1$-complete and pseudo$(< \kappa, 1+\gamma))$-complete filter on Y.
2) Related to [Sh:835].

Proof. <u>Proof of the Claim of 8.9</u>

Recall $\{y \in Y : \alpha_y = 0\} = \emptyset \bmod D$ as $\mathrm{rk}_D(\langle \alpha_y : y \in Y\rangle) = \delta > 0$ but $f_1, f_2 \in {}^Y\mathrm{Ord} \wedge (f_1 = f_2 \bmod D) \Rightarrow \mathrm{rk}_D(f_1) = \mathrm{rk}_D(f_2)$ hence without loss of generality $y \in Y \Rightarrow \alpha_y > 0$.

Let $\mathbb{D} = \{D' : D'$ is a filter on Y extending D which is κ-complete$\}$. So $\theta(\mathbb{D}) \leq \theta(\mathrm{Fil}^1_{\aleph_1}(Y)) \leq \mathrm{cf}(\delta)$. For any $\gamma < \mathrm{rk}_D(\bar\alpha)$ and $D' \in \mathbb{D}$ let

$(*)_2$ (a) $\mathcal{F}_{\gamma,D'} = \{f \in \Pi\bar\alpha : \mathrm{rk}_D(f) = \gamma$ and D' is $\mathrm{dual}(J[f, D])\}$
(b) $\mathcal{F}_{D'} = \cup\{\mathcal{F}_{\gamma,D'} : \gamma < \mathrm{rk}_D(\bar\alpha)\}$
(c) $\Xi_{\bar\alpha,D'} = \{\gamma < \mathrm{rk}_D(\bar\alpha) : \mathcal{F}_{\gamma,D'} \neq \emptyset\}$
(d) $\mathcal{F}_\gamma = \cup\{\mathcal{F}_{\gamma,D''} : D'' \in \mathbb{D}\}$.

Now

$(*)_3$ if $\gamma < \mathrm{rk}_D(\bar\alpha)$ then $\mathcal{F}_\gamma \neq \emptyset$.

[Why? By [Sh:938, 1.8(2)=z0.23(2)] there is $g \in {}^Y\mathrm{Ord}$ such that $g < f \bmod D$ and $\mathrm{rk}_D(g) = \gamma$ and without loss of generality $g \in \Pi\bar\alpha$. Now let $D' = \mathrm{dual}(J[g, D])$, so $(D, D') \in \mathrm{Fil}^4_\kappa(Y), D' \in \mathbb{D}$ and $g \in \mathcal{F}_{\gamma,D'}$, see [Sh:938, 1.7(2)=z0.23(2)], Claim [Sh:835, 0.10(2)], here we use $\mathrm{AC}_{<\kappa}$.]

$(*)_4$ $\{\sup(\Xi_{\bar\alpha,D'}) : D' \in \mathbb{D}$ and $\Xi_{\bar\alpha,D'}$ is bounded in $\mathrm{rk}_D(\bar\alpha)\}$ is a subset of $\mathrm{rk}_{D'}(\bar\gamma)$ which has cardinality $< \theta(\mathbb{D}) \leq \theta(\mathrm{Fil}^1_\kappa(Y)) \leq \mathrm{cf}(\delta)$.

[Why? The function $D' \mapsto \sup(\Xi_{\bar{\alpha},D'})$ witness this.]

$(*)_5$ the set in $(*)_4$ is bounded below $\mathrm{rk}_D(\bar{\alpha})$ so let $\gamma(*) < \mathrm{rk}_D(\bar{\alpha})$ be its supremum.

[Why? By $(*)_4$.]

$(*)_6$ there is $D' \in \mathbb{D}$ such that $\Xi_{\bar{\alpha},D'}$ is unbounded in $(\Pi\bar{\alpha}, <_{D'})$.

[Why? Choose $\gamma < \mathrm{rk}_D(\bar{\alpha})$ such that: $\gamma > \gamma(*)$. By $(*)_3$ there for some $f \in \mathcal{F}_{\gamma(*)}$ and $D' \in \mathbb{D}$ we have $f \in \mathcal{F}_{\gamma(*),D'}$ so by the choice of $\gamma(*)$ the set $\Xi_{\bar{\alpha},D'}$ cannot be bounded in $\mathrm{rk}_D(\bar{\alpha})$.]

$(*)_7$ if $\gamma_1 < \gamma_2$ are from $\Xi_{\bar{\alpha},D'}$ and $f_1 \in \mathcal{F}_{\gamma_1,D'}, f_2 \in \mathcal{F}_{\gamma_2,D'}$ <u>then</u> $f_1 <_{D'} f_2$.

[Why? By [Sh:938, 1.7=z0.23], [Sh:835, 0.10(2)].]

Together we are done: by $(*)_6$ there is $D' \in \mathbb{D}$ such that $\Xi_{\bar{\alpha},D'}$ is unbounded in $\mathrm{rk}_D(\bar{\alpha})$. Let $\bar{\mathcal{F}} = \langle \mathcal{F}_{\gamma,D'} : \gamma \in \Xi_{\bar{\alpha},D'} \rangle$ witness that $(\Pi\bar{\alpha}, <_{D'})$ has pseudo true cofinality, and so $\text{ps-tcf}(\Pi\bar{\alpha}, <_D) = \mathrm{cf}(\mathrm{otp}(\Xi_{\bar{\alpha},D'})) = \mathrm{cf}(\mathrm{rk}_D(\bar{\alpha}))$, so we are done. $\square_{8.9}$

So we have

{r10}

Definition/Claim 8.12. 1) We say that $\delta = \text{ps-tcf}_{\bar{D}}(\bar{\alpha})$, where δ is a limit ordinal <u>when</u>, for some set Y:

(a) $\bar{\alpha} \in {}^Y\mathrm{Ord}$
(b) $\bar{D} = (D_1, D_2)$
(c) $D_1 \subseteq D_2$ are $\aleph_1$-complete filters on Y
(d) $\mathrm{rk}_{D_1}(\bar{\alpha}) = \delta = \sup(\Xi_{\bar{D},\bar{\alpha}})$ where $\Xi_{\bar{D},\bar{\alpha}} = \{\gamma < \mathrm{rk}_{D_1}(\bar{\alpha})$: for some $f < \bar{\alpha}$ mod D_1, we have $\mathrm{rk}_{D_1}(f) = \gamma$ and $D_2 = \mathrm{dual}(J[f, D_1]\}$.

2) If D_1 is $\aleph_1$-complete filter on $Y, \bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ and $\mathrm{cf}(\alpha_t) \geq \theta(\mathrm{Fil}^1_{\aleph_1}(Y))$ for $t \in Y$ <u>then</u> for some $\aleph_1$-complete filter D_2 on Y extending D_1 we have $\text{ps-tcf}_{(D_1,D_2)}(\bar{\alpha})$ is well defined.
3) Moreover in part (2) there is a definition giving for any $(Y, D_1, D_2, \bar{\alpha})$ as there, a sequence $\langle \mathcal{F}_\gamma : \gamma < \delta \rangle$ exemplifying the value of $\text{ps-tcf}_{\bar{D}}(\bar{\alpha})$.

Proof. Let $\delta := \mathrm{rk}_{D_1}(f)$, so by Claim 8.16 below $\mathrm{cf}(\delta) \geq \theta(\mathrm{Fil}^1_{\aleph_1}(Y))$ hence has Claim 8.9 above and its proof the conclusion holds: the proof is needed for "$\delta = \sup(\Xi_{\bar{D},\alpha})$", noting observation 8.13 below. $\square_{8.12}$

{r10d}

Observation 8.13. *1) [DC] or just [$AC_{\aleph_0}$].*

Assume D is an $\aleph_1$-complete filter on Y and $f, f_n \in {}^Y\mathrm{Ord}$ for $n < \omega$ and $f(t) = \sup\{f_n(t) : n < \omega\}$. Then $\mathrm{rk}_D(f) = \sup\{\mathrm{rk}_D(f_n) : n < \omega\}$.

Remark 8.14. Similarly for other amounts of completeness, see 8.18.

Proof. As $\mathrm{rk}_D(f) = \min\{\mathrm{rk}_{D+A_n}(f) : n < \omega\}$ if $\cup\{A_n : n < \omega\} \in D, A_n \in D^+$ by [Sh:71] or see [Sh:835, 1.9=z0.25]. $\square_{8.13}$

Remark 8.15. Also in [Sh:835, 1.9(2)=z0.25(2)] can use AC_I only, i.e. omit the assumption DC, a marginal point here.

{r11}

Claim 8.16. *[$\mathrm{AC}_{<\theta}$] The ordinal δ has cofinality $\geq \theta$ <u>when</u>:*

$\circledast$ (a) $\delta = \mathrm{rk}_D(\bar{\alpha})$

(*b*) $\bar\alpha = \langle \alpha_y : y \in Y\rangle \in {}^Y\mathrm{Ord}$
(*c*) *D is an* $\aleph_1$*-complete filter on* Y
(*d*) $y \in Y \Rightarrow \mathrm{cf}(\alpha_y) \geq \theta$.

Proof. Note that $y \in Y \Rightarrow \alpha_y > 0$. Toward contradiction assume $\mathrm{cf}(\delta) < \theta$ so δ has a cofinal subset C of cardinality $< \theta$. For each $\beta < \delta$ for some $f \in {}^Y\mathrm{Ord}$ we have $\mathrm{rk}_D(f) = \beta$ and $f <_D \bar\alpha$ and without loss of generality $f \in \prod\limits_{y\in Y} \alpha_y$. By $\mathrm{AC}_{<\theta}$ there is a sequence $\langle f_\beta : \beta \in C\rangle$ such that $f_\beta \in \prod\limits_{y\in Y} \alpha_y, f <_D \bar\alpha$ and $\mathrm{rk}_D(f_\beta) = \beta$. Define $g \in \prod\limits_{y\in Y} \alpha_y$ by $g(y) = \cup\{f_\beta(y) : \beta \in C \text{ and } f_\beta(y) < \alpha_t\}$. By clause (d) we have $[y \in Y \Rightarrow g(y) < \alpha_y]$, so $g <_D \bar\alpha$, hence $\mathrm{rk}_D(\bar g) < \mathrm{rk}_D(\alpha)$ but by the choice of g we have $\beta \in C \Rightarrow f_\beta \leq_D g$ hence $\beta \in C \Rightarrow \beta = \mathrm{rk}_D(f_\beta) \leq \mathrm{rk}_D(g)$ hence $\delta = \sup(C) \leq \mathrm{rk}_D(g)$, contradiction. $\square_{8.16}$

{r12}

Observation 8.17. *1) Assume* $(\bar\alpha, D)$ *satisfies*

(*a*) D *a filter on* Y *and* $\bar\alpha = \langle \alpha_t : t \in Y\rangle$ *and each* α_t *is a limit ordinal*
(*b*) $\bar{\mathcal F} = \langle \mathcal F_\beta : \beta < \partial\rangle$ *exemplify* $\partial = \text{ps-tcf}(\Pi\bar\alpha, <_D)$ *so we demand just* $\partial = \sup\{\beta < \partial : \mathcal F_\beta \neq \emptyset\}$
(*c*) $\mathcal F'_\beta = \{f \in \prod\limits_{t\in Y} \alpha_t$*: for some* $g \in \mathcal F_\beta$ *we have* $f = g \bmod D\}$.

Then: $\langle \mathcal F'_\beta : \beta < \partial\rangle$ *exemplify* $\partial = \text{ps-tcf}(\Pi\bar\alpha, <_D)$ *that is*

(α) $\bigcup\limits_{\beta<\gamma} \mathcal F'_\beta$ *is cofinal in* $(\Pi\bar\alpha, <_D)$
(β) *for every* $\beta_1 < \beta_2 < \partial$ *and* $f_1 \in \mathcal F'_{\beta_1}$ *and* $f_2 \in \mathcal F'_{\beta_2}$ *we have* $f_1 \leq f_2$.

2) Similarly, if $D, \bar{\mathcal F}$ *satisfies clauses (a),(b) above and* D *is* $\aleph_1$*-complete and* $\partial = \mathrm{cf}(\partial) > \aleph_0$ *then we can "correct"* $\bar{\mathcal F}$ *to make it* $\aleph_0$*-continuous that is* $\langle \mathcal F''_\beta : \beta < \partial\rangle$ *defined in* $(c)_1 + (c)_2$ *below satisfies* $(\alpha) + (\beta)$ *above and* (γ) *below and so is* $\aleph_0$*-continuous, (see below) where*

$(c)_1$ *if* $\beta < \partial$ *and* $\mathrm{cf}(\beta) \neq \aleph_0$ *then* $\mathcal F''_\beta = \mathcal F'_\beta$
$(c)_2$ *if* $\beta < \partial$ *and* $\mathrm{cf}(\beta) = \aleph_0$ *then* $\mathcal F''_\beta = \{\sup\langle f_n : n < \omega\rangle$*: for some increasing sequence* $\langle \beta_n : n < \omega\rangle$ *with limit* β *we have* $n < \omega \Rightarrow f_n \in \mathcal F'_{\beta_n}\}$*, see below*
(γ) *if* $\beta < \partial$ *and* $\mathrm{cf}(\beta) = \aleph_0$ *and* $f_1, f_2 \in \mathcal F''_\beta$ *then* $f_1 = f_2$ *mod D.*

3) This applies to an increasing sequence $\langle \mathcal F_\beta : \beta < \delta\rangle, \mathcal F_\beta \subseteq {}^Y\mathrm{Ord}, \delta$ *a limit ordinal.*

Proof. Straightforward. $\square_{8.17}$

{r13}

Definition 8.18. 0) If $f_n \in {}^Y\mathrm{Ord}$ for $n < \omega$, then $\sup\langle f_n : n < \omega\rangle$ is defined as the function f with domain Y such that $f(t) = \cup\{f_n(t) : n < \omega\}$.
1) We say $\bar{\mathcal F} = \langle \mathcal F_\beta : \beta < \lambda\rangle$ exemplifying $\lambda = \text{ps-tcf}(\Pi\bar\alpha, <_D)$ is weakly $\aleph_0$-continuous when:

if $\beta < \partial$, $\mathrm{cf}(\beta) = \aleph_0$ and $f \in \mathcal F_\beta$ then for some sequence $\langle (\beta_n, f_n) : n < \omega\rangle$ we have $\beta = \cup\{\beta_n : n < \omega\}, \beta_n < \beta_{n+1} < \beta, f_n \in \mathcal F_{\beta_\alpha}$ and $f = \sup\langle f_n : n < \omega\rangle$; so if D is $\aleph_1$-complete then $\{f/D : f \in \mathcal F_\beta\}$ is a singleton.
2) We say it is $\aleph_0$-continuous if we can replace the last "then" by "iff".

{r14}

Theorem 8.19. *__The Canonical Filter Theorem__ Assume* DC *and* $\mathrm{AC}_{\mathcal{P}(Y)}$.

Assume $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle \in {}^Y\mathrm{Ord}$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \geq \theta(\mathcal{P}(Y))$ *and* $\partial \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ *hence is a regular cardinal. __Then__ there is* $D = D^{\bar{\alpha}}_{\partial}$*, an* $\aleph_1$*-complete filter on* Y *such that* $\partial = \text{ps-tcf}(\Pi\bar{\alpha}/D)$ *and* $D \subseteq D'$ *for any other such* $D' \in \mathrm{Fil}^1_{\aleph_1}(D)$.

{r14b}

Remark 8.20. 1) By 8.9 there are some such ∂.
2) We work to use just $\mathrm{AC}_{\mathcal{P}(Y)}$ and not more.

Proof. Let

$\boxplus_1$ (*a*) $\mathbb{D} = \{D : D$ is an $\aleph_1$-complete filters on Y such that $(\Pi\bar{\alpha}/D)$ has pseudo true cofinality $\partial\}$,

(*b*) $D_* = \cap\{D : D \in \mathbb{D}\}$.

Now obviously

(*c*) D_* is an $\aleph_1$-complete filter on Y.

For $A \subseteq Y$ let $\mathbb{D}_A = \{D \in \mathbb{D} : A \notin D\}$ and let $\mathcal{P}_* = \{A \subseteq Y : \mathbb{D}_A \neq \emptyset\}$. As $\mathrm{AC}_{\mathcal{P}(Y)}$ we can find $\langle D_A : A \in \mathcal{P}_* \rangle$ such that $D_A \in \mathbb{D}_A$ for $A \in \mathcal{P}_*$. Let $\mathbb{D}_* = \{D_A : A \in \mathcal{P}_*\}$, clearly

$\boxplus_2$ $D_* = \cap\{D : D \in \mathbb{D}_*\}$ and $\mathbb{D}_* \subseteq \mathbb{D}$ is non-empty.

As $\mathrm{AC}_{\mathcal{P}_*}$ holds clearly

$(*)_0$ we can choose $\langle \bar{\mathcal{F}}^A : A \in \mathcal{P}_* \rangle$ such that $\bar{\mathcal{F}}_A$ exemplifies $D_A \in \mathbb{D}$ as in 8.17(1),(2), so in particular is $\aleph_0$-continuous.

For each $\beta < \partial$ let $\mathcal{F}^*_\beta = \cap\{\mathcal{F}^A_\beta : A \in \mathcal{P}_*\}$, now

$(*)_1$ $\mathcal{F}^*_\beta \subseteq \Pi\bar{\alpha}$.

[Why? As by 8.17(1)(c) we have $\mathcal{F}^A_\beta \subseteq \Pi\bar{\alpha}$ for each $A \in \mathcal{P}_*$.]

$(*)_2$ if $\beta_1 < \beta_2 < \partial, f_1 \in \mathcal{F}^*_{\beta_1}$ and $f_2 \in \mathcal{F}^*_{\beta_2}$ then $f_1 < f_2 \bmod D_*$.

[Why? As $A \in \mathcal{P}_* \Rightarrow f_1 <_{D_A} f_2$ by the choice of $\langle \mathcal{F}^*_\beta : \beta < \partial \rangle$, hence the set $\{t \in Y : f_1(t) < f_2(t)\}$ belongs to D_A for every $A \in \mathcal{P}_*$ hence by $\boxplus_2$ it belongs to D_* which means that $f_1 <_{D_*} f_2$ as required.]

$(*)_3$ if $f \in \Pi\bar{\alpha}$ then for some $\beta_f < \partial$ we have $f' \in \cup\{\mathcal{F}^*_\beta : \beta \in [\beta_f, \partial)\} \Rightarrow f < f' \bmod D_*$.

[Why? For each $A \in \mathcal{P}_*$ there are β, g such that $\beta < \partial, g \in \mathcal{F}^A_\beta$ and $f < g \bmod D$ hence $\beta' \in [\beta + 1, \partial) \wedge f' \in \mathcal{F}^A_{\beta'} \Rightarrow f < g < f' \bmod D_A$. Let β_A be the minimal such ordinal $\beta_A < \delta$. As $\mathrm{cf}(\delta) \geq \theta(\mathcal{P}(Y)) \geq \theta(\mathcal{P}_*)$, clearly $\beta_* = \sup\{\beta_A + 1 : A \in \mathcal{P}_*\}$ is $< \delta$. So $A \in \mathcal{P}_* \wedge g \in \cup\{\mathcal{F}^*_\beta : \beta \in [\beta_*, \delta)\} \Rightarrow f <_D g$. By $\boxplus_2$ the ordinal α_* is as required on α_ℓ.]

Moreover

$(*)_4$ there is a function $f \mapsto \beta_f$ in $(*)_3$.

[Why? As we can (and will) choose β_f as minimal β such that ...]

$(*)_5$ for every $\beta_* < \partial$ there is $\beta \in (\beta_*, \partial)$ such that $\mathcal{F}^*_\beta \neq \emptyset$.

[Why? We choose by induction on n, a sequence $\bar{\beta}_n = \langle \beta_{n,A} : A \in \mathcal{P}_* \rangle$ and a sequence $\bar{f}_n = \langle f_{n,A} : A \in \mathcal{P}_* \rangle$ and a function f_n such that

(α) $\beta_n < \partial$ and $m < n \Rightarrow \beta_m < \beta_n$

(β) $\beta_0 = \beta_*$ and for $n > 0$ we let $\beta_n = \sup\{\beta_{m,A} : m < n, A \in \mathcal{P}_*\}$

(γ) $\beta_{n,A} \in (\beta_n, \partial)$ is minimal such that there is $f_{n,A} \in \mathcal{F}^A_{\beta_{n,A}}$ satisfying $n = m+1 \Rightarrow f_m < f_{\beta_n,A}$ mod D_A

(δ) $\langle f_{n,A} : A \in \mathcal{P}_* \rangle$ is a sequence such that each $f_{n,A}$ are as in clause (γ)

(ε) $f_n \in \Pi\bar{\alpha}$ is defined by $f_n(t) = \sup\{f_{m,A}(t) + 1 : A \in \mathcal{P}_*$ and $m < n\}$.

[Why can we carry the induction? Arriving to n first, f_n is well defined $\in \Pi\bar{\alpha}$ by clause (ε) as $\mathrm{cf}(\alpha_t) \geq \theta(\mathcal{P}_*)$ for $t \in Y$. Second by clause (γ), $\langle \beta_{n,A} : A \in \mathcal{P}_* \rangle$ is well defined. Third by clause (δ) we can choose $\langle f_{m,A} : A \in \mathcal{P}_* \rangle$ as $\mathrm{AC}_{\mathcal{P}_*}$.

Lastly, the inductive construction is possibly by DC.]

Let $\beta^* = \cup\{\beta_n : n < \omega\}$ and $f = \sup\langle f_n : n < \omega \rangle$. Easily $f \in \cap\{\mathcal{F}^A_{\beta^*} : A \in \mathcal{P}_*\}$ as each $\langle \mathcal{F}^A_\beta : \beta < \partial \rangle$ is $\aleph_0$-continuous.]

$(*)_6$ if $f \in \Pi\bar{\alpha}$ then for some $\beta < \gamma$ and $f' \in \mathcal{F}^*_\beta$ we have $f < f'$ mod D^*.

[Why? By $(*)_3 + (*)_4$.]

So we are done. $\square_{8.19}$

Definition 8.21. For $\bar{\alpha} \in {}^Y\mathrm{Ord}$ let $J^{\aleph_1\text{-comp}}_{<\lambda}(\bar{\alpha}) = \{X \subseteq Y\colon \text{ps-pcf}_{\aleph_1\text{-com}}(\bar{\alpha} \restriction X) \subseteq \lambda\}$ and $J^{\aleph_1\text{-comp}}_{\leq\lambda}$ is $J^{\aleph_2\text{-comp}}_{<\lambda^+}$. {r16}

Remark 8.22. In 8.21, see Definition 8.6(3). {r17}

On this and more see [Sh:F955].

9. Appendix: Definition of Rank-System

Moved from pg.3:
We define a function H from $\Pi\bar{\alpha}$ into $\Pi\{\lambda_X : X \in D\}$ by:

(α) $(H(f))(X) = \text{ Min}\{\beta < \lambda_X$: if $f' \in \mathcal{F}^X_\beta$ then $f \le f'$ mod $D_X\}$.

We let

(β) $\check{D}$ be the following filter on the set $\check{Y} := D$:
$Z \in \check{D}$ iff $Z \subseteq D$ and $(\exists X \in D)[Z \supseteq \{X' \in D : X' \subseteq X\}]$.

Now

(γ) $\check{D}$ is an $\aleph_1$-complete filter on $\check{Y}$
(δ) if $f_1, f_2 \in \Pi\bar{\alpha}$ and $f_1 \le f_2$ mod D^*_1 then $H(f_1) \le H(f_2)$ mod $\check{D}$
(ε) $(\prod\limits_{t \in \check{Y}} \lambda_t, <_{\check{D}})$ is pseudo $(< \lambda^+)$-directed.

[Why? By claim 8.7, i.e. 8.7 of §5 of [Sh:938].]
Because by an assumption

(ζ) if $f_1, f_2 \in \mathcal{F}_\alpha$ and $\alpha < \delta$ then $H(f_1) = H(f_2)$ mod $\check{D}$.

Why? $f_1 = f_2$ mod D hence by ? we have $f_1 = f_2$ mod D^*_1 hence by (yyy), $H(f_1) = H(f_2)$ mod $\check{D}$. FILL
Now by $(\varepsilon) + (zzz)$ we are done proving (h).]

(i) $D \subseteq D^*_1$.

[Why? Because if $A \in D$ then $X_1 := A$ witness $A \in D$, as $X \in D \wedge X \subseteq X_1 \Rightarrow X \in D \wedge X \subseteq A \Rightarrow X \in D_X \wedge X \subseteq A \subseteq Y \Rightarrow A \in D_X$.]

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