PSEUDO PCF
SH955

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Abstract. We continue our investigation on pcf with weak forms of the axiom of choice. Characteristically, we assume DC + $\mathscr{P}(Y)$ when looking at $\prod\limits_{s \in Y} \delta_s$. We get more parallels of pcf theorems.

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Anotated Content

§ 0. Introduction

{intro}

In the first section we deal with generalizing the pcf theory in the direction started in [Sh:938, §5] trying to understand the pseudo true cofinality of small products of regular cardinals. The difference with earlier works is that here we assume $\mathrm{AC}_{\mathscr{U}}$ for any set $\mathscr{U}$ of power $\le |\mathscr{P}(\mathscr{P}(Y))|$ or, actually working harder, just $\le |\mathscr{P}(Y)|$ when analyzing $\prod\limits_{t\in Y} \alpha_t$, whereas in [Sh:497] we assumed $\mathrm{AC}_{\sup\{\alpha_t:t\in Y\}}$ and in [Sh:835] we have (in addition to $\mathrm{AC}_{\mathscr{P}(\mathscr{P}(Y))}$) assumptions like "$[\sup\{\alpha_t : t \in Y\}]^{\aleph_0}$ is well ordered". In [Sh:938, §1-§4] we assume only $\mathrm{AC}_{<\mu} + DC$ and consider $\aleph_1$-complete filters on μ but in the characteristic case μ is a limit of measurable cardinals.

Note that generally in this work, though we try occasionally not to use DC, it will not be a real loss to assume it all the time. More specifically, we prove the existence of a minimal $\aleph_1$-complete filter D on Y such that $\lambda = \text{ ps-tcf}(\Pi\bar{\alpha}, <_D)$ assuming $\mathrm{AC}_{\mathscr{P}(Y)}$ and (of course) DC and α_t of large enough cofinality. We then prove the existence of one generator, that is, of $X \subseteq Y$ such that $J^{\aleph_1\text{-comp}}_{\le\lambda}[\bar{\alpha}] = J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}] + X$, see 1.6 and even (in 1.8) the parallel of the existence of a $<_{D_1}$-lub for an $<_D$-increasing sequence $\langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$, generalize the no-hole claim in 1.13, and give a bound on pp for non-fix points (in 1.11).

In §2 we further investigate true cofinality. In Claim 2.2, assuming AC_λ and D an $\aleph_1$-complete filter on Y, we start from ps-tcf$(\Pi\bar{\alpha}, <_D)$, dividing by $\mathrm{eq}(\bar{\alpha}) = \{(s,t) : \alpha_s = \alpha_t\}$. We also prove the composition Theorem 2.6: it tells us when ps-tcf$(\prod\limits_i$ ps-tcf$(\prod\limits_j \lambda_{i,j}, <_{D_i}), <_E)$ is equal to ps-tcf$(\prod\limits_{(i,j)} \lambda_{i,j}, <_D)$.

We then prove the pcf closure conclusion: giving a sufficient condition for the operation $\text{ps-pcf}_{\aleph_1\text{-comp}}$ to be idempotent. Lastly, we revisit the generating sequence.

In §(3A) we measure $\prod\limits_{t\in Y} g(t)$ modulo a filter D on Y for $g \in {}^Y(\mathrm{Ord}\backslash\{0\})$ in three ways and show they are almost equal in 3.2. The price is that we replace (true) cofinality by pseudo (true) cofinality, which is inevitable. We try to sort out the "almost equal" in 3.5 - 3.7.

In §(3B) we prove a relative of [Sh:513, §3]; again dealing with depth (instead of rank as in [Sh:938]) adding some information even under ZFC. Assuming that the sequence $\langle D_n : n < \omega\rangle$ of filters has the independence property (IND), see Definition 3.12, with D_n a filter on Y_n we can bound the depth of $({}^{(Y_n)}\zeta, <_{D_n})$ by ζ, for every ζ for many n's, see 3.13. Of course, we can generalize this to $\langle D_s : s \in S\rangle$. This is incomparable with the results of [Sh:938, §4]. See a continuation of [Sh:835] in [Sh:1005].

Note that the assumptions like $\mathrm{IND}(\bar{D})$ are complementary to ones used in [Sh:835] to get considerable information. Our original hope was to arrive to a dichotomy. The first possibility will say that one of the versions of an axiom suggested in [Sh:835] holds, which means "for some suitable algebra", there is no independent ω-sequence; in this case [Sh:835] tells us much. The second possibility will be a case of IND, and then we try to show that there is a rank system in the sense of [Sh:938]. But presently for this we need too much choice. The dichotomy we succeed to prove is with small o-Depth in one side, the results of [Sh:835] on the other side. It would be better to have ps-o-Depth in the first side.

{r15}

Question 0.1. [DC + $\mathrm{AC}_{\mathscr{P}(Y)}$]

Assume

(a) $\bar{\alpha} \in {}^Y \mathrm{Ord}$
(b) $\mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(\mathscr{P}(Y)$ for every $t \in Y$
(c) $\lambda_t \in \mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha})$ for $t \in Z$, in fact, $\lambda_t = \text{ps-tcf}(\Pi\bar{\alpha}, <_{D_t}), D_t$ is an $\aleph_1$-complete filter on Y
(d) $\lambda = \text{ps-tcf}_{\aleph_1-\mathrm{comp}}(\langle \lambda_t : t \in Z \rangle$
(e) (a possible help) $X_t \in D_t, \langle X_t : t \in Y \rangle$ are pairwise disjoint.

(A) Now does $\lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$? (See 2.6.)

(B) Can we say something on D_λ from [Sh:938, 5.9] improved in 1.3?

{r15f}

Question 0.2. How well can we generalize the RGCH, see [Sh:460] and [Sh:829]; the above may be relevant; see [Sh:938] and here in §(3C).

{z12}

Recall

Notation 0.3. 1) For any set X let $\mathrm{hrtg}(X) = \min\{\alpha : \alpha$ an ordinal such that there is no function from X onto $\alpha\}$.
2) $A \leq_{\mathrm{qu}} B$ means that either $A = \emptyset$ or there is a function from A onto B.

{z15}

Central in this work is

Definition 0.4. For a quasi order P we say P has pseudo-true-cofinality λ or "λ is the pseudo true cofinality of P" when λ is a regular cardinal and λ is a pseudo true cofinality of P which means that there is a sequence $\bar{\mathscr{F}}$ such that:

(a) $\mathscr{F} = \langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$
(b) $\mathscr{F}_\alpha \subseteq P$
(c) if $\alpha_1 < \alpha_2, p_1 \in \mathscr{F}_{\alpha_1}$ and $p_2 \in \mathscr{F}_{\alpha_2}$ then $p_1 \leq_P p_2$
(d) if $q \in \mathscr{F}$ then for some $\alpha < \lambda$ and $p \in \mathscr{F}_\alpha$ we have $q <_P p_1$
(e) $\lambda = \sup\{\alpha < \lambda : \mathscr{F}_\alpha \neq \emptyset\}$.

We may consider replacing AC_A by more refined version, $\mathrm{AC}_{A,B}$ defined below (e.g. in 1.1, 2.6) but we have not dealt with it systematically.

{z17}

Definition 0.5. 1) $\mathrm{AC}_{A,B}$ means: if $\langle X_a : a \in A \rangle$ is a sequence of non-empty sets then there is a sequence $\langle Y_a : a \in A \rangle$ such that $Y_a \subseteq X_a$ is not empty and $Y_a \leq_{\mathrm{qu}} B$.
2) $\mathrm{AC}_{A,<\kappa}, \mathrm{AC}_{A,\leq B}$ are defined similarly but $|Y_a| < \kappa, |Y_a| \leq |B|$ respectively in the end.

{z20}

Observation 0.6. 1) We have AC_A iff $\mathrm{AC}_{A,1}$.
2) $\mathrm{AC}_{A,B}$ fails if $B = \emptyset$.
3) If $\mathrm{AC}_{A,B}$ and $|A_1| \leq |A|$ and $B \leq_{\mathrm{qu}} B_1$ then AC_{A_1,B_1}.

§ 1. On pseudo true cofinality

{onpseudo}

We continue [Sh:938, §5].

Below we improve [Sh:938, 5.19] by omitting DC from the assumptions but first we observe

{r15}

Claim 1.1. *Assume* AC_Z.
1) We have $\theta \ge \mathrm{hrtg}(Z)$ *when* $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ *and* $\theta \in \text{ps-pcf}(\Pi\bar{\alpha})$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(Z)$.
2) We have $\mathrm{cf}(\mathrm{rk}_D(\bar{\alpha})) \ge \mathrm{hrtg}(Z)$ *when* $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(Z)$.

{r17f}

Remark 1.2. We can weaken the assumption $\mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(Z)$ by using the ideal of small cofinality, $\mathrm{cf} - \mathrm{id}_\theta(\bar{\alpha})$, see [Sh:1005, 1.1=Lc2]. This can be done systematically in this work.

Proof. 1) If we have AC_α for every $\alpha < \mathrm{hrtg}(Z)$ then we can use [Sh:938, 5.7(4)] but we do not assume this. In general let D be a filter on Y such that $\theta = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, exists as we are assuming $\theta \in \text{ps-pcf}(\Pi\bar{\alpha})$. Let $\bar{\mathscr{F}} = \langle \mathscr{F}_\alpha : \alpha < \theta \rangle$ witness $\theta = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, i.e. as in [Sh:938, 5.6(2)] or see 0.4 here; note $t \in Y \Rightarrow \alpha_t > 0$, as we are assuming $\mathscr{F}_\alpha \subseteq \Pi\bar{\alpha}$ for some $\alpha < \theta$; also if $\Pi\bar{\alpha}$ is non-empty then we can assume $\mathscr{F}_\alpha \ne \emptyset$ for every $\alpha < \theta$.

Toward contradiction assume $\theta < \mathrm{hrtg}(Z)$. As $\theta < \mathrm{hrtg}(Z)$, there is a function h from Z onto θ, so the sequence $\langle \mathscr{F}_{h(z)} : z \in Z \rangle$ is well defined. As we are assuming AC_Z, there is a sequence $\langle f_z : z \in Z \rangle$ such that $f_z \in \mathscr{F}_{h(z)}$ for $z \in Z$. Now define $g \in {}^Y(\mathrm{Ord})$ by $g(s) = \cup\{f_z(s) : z \in Z\}$; clearly g exists and $g \le \bar{\alpha}$. But for each $s \in Y$, the set $\{f_z(s) : z \in Z\}$ is a subset of α_s of cardinality $\le \theta < \mathrm{hrtg}(Z)$ hence $< \mathrm{cf}(\alpha_s)$ hence $g(s) < \alpha_s$. Together $g \in \Pi\bar{\alpha}$ is a $<_D$-upper bound of $\cup\{\mathscr{F}_\varepsilon : \varepsilon < \theta\}$, contradiction to the choice of $\bar{\mathscr{F}}$.
2) Otherwise let $\theta = \mathrm{cf}(\mathrm{rk}_D(\bar{\alpha}))$ so $\theta < \mathrm{hrtg}(Z), \langle \alpha_\varepsilon : \varepsilon < \theta \rangle$ be increasing with limit $\mathrm{rk}_D(\bar{\alpha})$ and again let g be a function from Z onto θ. As AC_Z holds, we can find $\langle f_z : z \in Z \rangle$ such that for every $z \in Z$ we have $\mathrm{rk}_D(f_z) \ge \alpha_{h(z)}$ and $f_z <_D \bar{\alpha}$ and without loss of generality$f_z \in \Pi\bar{\alpha}$. Let $f \in \Pi\bar{\alpha}$ be defined by $f(t) = \sup\{f_{h(z)}(t) : z \in Z\}$ so $\mathrm{rk}_D(f) \ge \sup\{\alpha_z : z \in Z\} = \mathrm{rk}_D(\bar{\alpha}) > \mathrm{rk}_D(f)$, contradiction. $\square_{1.1}$

{r16}

Theorem 1.3. The Canonical Filter Theorem *Assume* $\mathrm{AC}_{\mathscr{P}(Y)}$.

Assume $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle \in {}^Y\mathrm{Ord}$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(\mathscr{P}(Y))$ *and* $\partial \in \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha})$ *hence is a regular cardinal. Then there is* $D = D^{\bar{\alpha}}_{\partial}$*, an* $\aleph_1$*-complete filter on* Y *such that* $\partial = \text{ps-tcf}(\Pi\bar{\alpha}/D)$ *and* $D \subseteq D'$ *for any other such* $D' \in \mathrm{Fil}^1_{\aleph_1}(D)$.

{c17d}

Remark 1.4. 1) By [Sh:938, 5.9] there are some such ∂ if DC holds.
2) We work more to use just $\mathrm{AC}_{\mathscr{P}(Y)}$ and not more.
3) If $\kappa > \aleph_0$ we can replace "$\aleph_1$-complete" by "κ-complete".
4) If we waive "∂ regular" so just ∂, an ordinal, is a pseudo true cofinality of $(\Pi\bar{\alpha}, <_D)$ for $D \in \mathbb{D} \subseteq \mathrm{Fil}^1_{\aleph_1}(Y)$, exemplified by $\bar{\mathscr{F}}^D, \mathbb{D} \ne \emptyset$ the proof gives some $\partial', \mathrm{cf}(\partial') = \mathrm{cf}(\partial)$ and $\bar{\mathscr{F}}$ witnessing $(\Pi\bar{\alpha}, <_{D_*})$ has pseudo true cofinality ∂' where $D_* = \cap\{D : D \in \mathbb{D}\}$ for $\mathbb{D}$ as below.

Proof. Note that by 1.1

$\boxplus_1$ $\partial \geq \text{hrtg}(\mathscr{P}(Y))$.

Let

$\boxplus_2$ (a) $\mathbb{D} = \{D : D$ is an $\aleph_1$-complete filter on Y such that $(\Pi\bar{\alpha}/D)$ has pseudo true cofinality $\partial\}$,

(b) $D_* = \cap\{D : D \in \mathbb{D}\}$.

Now obviously

$\boxplus_3$ (a) $\mathbb{D}$ is non-empty

(b) D_* is an $\aleph_1$-complete filter on Y.

For $A \subseteq Y$ let $\mathbb{D}_A = \{D \in \mathbb{D} : (Y \backslash A) \notin D\}$ and let $\mathscr{P}_* = \{A \subseteq Y : \mathbb{D}_A \neq \emptyset\}$, equivalently $\mathscr{P}_* = \{A \subseteq Y : A \neq \emptyset \mod D$ for some $D \in \mathbb{D}\}$. As $\text{AC}_{\mathscr{P}(Y)}$ holds also $\text{AC}_{\mathscr{P}_*}$ holds hence we can find $\langle D_A : A \in \mathscr{P}_*\rangle$ such that $D_A \in \mathbb{D}_A$ for $A \in \mathscr{P}_*$. Let $\mathbb{D}_* = \{D_A : A \in \mathscr{P}_*\}$, clearly

$\boxplus_4$ (a) $D_* = \cap\{D : D \in \mathbb{D}_*\}$

(b) $\mathbb{D}_* \subseteq \mathbb{D}$ is non-empty.

As $\text{AC}_{\mathscr{P}_*}$ holds clearly

$(*)_1$ we can choose $\langle \bar{\mathscr{F}}^A : A \in \mathscr{P}_*\rangle$ such that $\bar{\mathscr{F}}^A$ exemplifies $D_A \in \mathbb{D}$ as in [Sh:938, 5.17,(1),(2)], so in particular $\bar{\mathscr{F}}^A$ is $\aleph_0$-continuous and without loss of generality $\mathscr{F}^A_\alpha \neq \emptyset, \mathscr{F}^A_\alpha \subseteq \Pi\bar{\alpha}$ for every $\alpha < \partial$.

For each $\beta < \partial$ let

$(*)_2$ $\mathbf{F}^1_\beta = \{\bar{f} = \langle f_A : A \in \mathscr{P}_*\rangle : \bar{f}$ satisfies $A \in \mathscr{P}_* \Rightarrow f_A \in \mathscr{F}^A_\beta\}$

$(*)_3$ for $\bar{f} \in \mathbf{F}^1_\beta$ let $\sup\{f_A : A \in \mathscr{P}_*\}$ be the function $f \in {}^Y\text{Ord}$ defined by $f(y) = \sup\{f_A(y) : A \in \mathscr{P}_*\}$

$(*)_4$ $\mathscr{F}^1_\beta = \{\sup\{f_A : A \in \mathscr{P}_*\} : \bar{f} = \langle f_A : A \in \mathscr{P}_*\rangle$ belongs to $\mathbf{F}^1_\beta\}$.

Now

$(*)_5$ (a) $\langle \mathscr{F}^1_\beta : \beta < \partial\rangle$ is well defined, i.e. exist

(b) $\mathscr{F}^1_\beta \subseteq \Pi\bar{\alpha}$.

[Why? Clause (a) holds by the definitions, clause (b) holds as $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathscr{P}(Y))$.]

$(*)_6$ $\mathscr{F}^1_\beta \neq \emptyset$ for $\beta < \partial$.

[Why? As for $\beta < \lambda$, the sequence $\langle \bar{\mathscr{F}}^A_\beta : A \in \mathscr{P}_*\rangle$ is well defined (as $\langle \bar{\mathscr{F}}^A : A \in \mathscr{P}_*\rangle$ is) and $A \in \mathscr{P}_* \Rightarrow \mathscr{F}^A_\beta \neq \emptyset$, so we can use $\text{AC}_{\mathscr{P}(Y)}$ to deduce $\mathscr{F}^1_\beta \neq \emptyset$.]

Define

$(*)_7$ (a) for $f \in \Pi\bar{\alpha}$ and $A \in \mathscr{P}_*$ let
$\beta_A(f) = \min\{\beta < \partial : f < g \mod D_A$ for every $g \in \mathscr{F}^A_\beta\}$

(b) for $f \in \Pi\bar{\alpha}$ let $\beta(f) = \sup\{\beta_A(f) : A \in \mathscr{P}_*\}$.

Now

$(*)_8$ (a) for $A \in \mathscr{P}_*$ and $f \in \Pi\bar{\alpha}$, the ordinal $\beta_A(f) < \partial$ is well defined

(b) for $f \in \Pi\bar{\alpha}$ the sequence $\langle \beta_A(f) : A \in \mathscr{P}_* \rangle$ is well defined.

[Why? Clause (a) holds because $\langle \mathscr{F}^A_\gamma : \gamma < \partial \rangle$ is cofinal in $(\Pi, \bar{\alpha}, <_{D_A})$, clause (b) holds by $(*)_7(a)$.]

$(*)_9$ (a) for $f \in \Pi\bar{\alpha}$ the ordinal $\beta(f)$ is well defined and $< \partial$

(b) if $f \le g$ are from $\Pi\bar{\alpha}$ then $\beta(f) \le \beta(g)$.

[Why? For clause (a), first, $\beta(f)$ is well defined and $\le \partial$ by $(*)_8$ and the definition of $\beta(f)$ in $(*)_7(b)$. Second, recalling that ∂ is regular $\ge \mathrm{hrtg}(\mathscr{P}(Y)) \ge \mathrm{hrtg}(\mathscr{P}_*)$ clearly $\beta(f) < \partial$. Clause (b) is obvious.]

Now

$(*)_{10}$ (a) if $A \in \mathscr{P}_*, \gamma < \partial$ and $f \in \mathscr{F}^A_\gamma$ then $\beta_A(f) > \gamma$

(b) if $\gamma < \partial$ and $f \in \mathscr{F}^1_\gamma$ then $\beta(f) > \gamma$.

[Why? Clause (a) holds because $\beta < \gamma \wedge g \in \mathscr{F}^A_\beta \Rightarrow g < f \bmod D_A$ and $\beta = \gamma \Rightarrow f \in \mathscr{F}^A_\gamma \wedge f \nleq f \bmod D_A$. Clause (b) holds because for some $\langle f_B : B \in \mathscr{P}_* \rangle \in \Pi\{\mathscr{F}^B_\gamma : B \in \mathscr{P}_*\}$ we have $f = \sup\{f_B : B \in \mathscr{P}_*\}$ hence $B \in \mathscr{P}_* \Rightarrow f_B \le f$ hence in particular $f_A \le f$; now recalling $\beta(f_A) > \gamma$ by clause (a) it follows that $\beta(f) > \gamma$.]

$(*)_{11}$ (a) for $\xi < \partial$ let $\gamma_\xi = \min\{\beta(f) : f \in \mathscr{F}^1_\xi\}$

(b) for $\xi < \partial$ let $\mathscr{F}^2_\xi = \{f \in \mathscr{F}^1_\xi : \beta(f) = \gamma_\xi\}$

$(*)_{12}$ (a) $\langle(\gamma_\xi, \mathscr{F}^2_\xi) : \xi < \partial\rangle$ is well defined, i.e. exists

(b) if $\xi < \partial$ then $\xi < \gamma_\xi < \partial$.

[Why? γ_ξ is the minimum of a set of ordinals which is non-empty by $(*)_6$ and $\subseteq \partial$, by $(*)_9(a)$, and all members are $> \gamma$ by $(*)_{10}(b)$.]

$(*)_{13}$ for $\xi < \partial$ we have $\mathscr{F}^2_\xi \subseteq \Pi\bar{\alpha}$ and $\mathscr{F}^2_\xi \ne \emptyset$.

[Why? By $(*)_{11}$ as $\mathscr{F}^1_\xi \ne \emptyset$ and $\mathscr{F}^1_\xi \subseteq \Pi\bar{\alpha}$.]

$(*)_{14}$ we try to define $\beta_\varepsilon < \partial$ by induction on the ordinal $\varepsilon < \partial$

<u>$\varepsilon = 0$</u>: $\beta_\varepsilon = 0$

<u>ε limit</u>: $\beta_\varepsilon = \cup\{\beta_\zeta : \zeta < \varepsilon\}$

<u>$\varepsilon = \zeta + 1$</u>: $\beta_\varepsilon = \gamma_{\beta_\zeta}$

$(*)_{15}$ (a) if $\varepsilon < \partial$ then $\beta_\varepsilon < \partial$ is well defined $\ge \varepsilon$

(b) if $\zeta < \varepsilon$ is well defined then $\beta_\zeta < \beta_\varepsilon$.

[Why? Clause (a) holds as ∂ is a regular cardinal so the case ε limit is O.K., the case $\varepsilon = \zeta + 1$ holds by $(*)_{12}(b)$. As for clause (b) we prove this by induction on ε; for $\varepsilon = 0$ this is empty, for ε a limit ordinal use the induction hypothesis and the choice of β_ε in $(*)_{14}$ and for $\varepsilon = \xi + 1$, clearly by $(*)_{12}(b)$ and the choice of γ_ε in $(*)_{14}$ we have $\beta_\xi < \beta_\varepsilon$ and use the induction hypothesis.]

$(*)_{16}$ if $f \in \Pi\bar{\alpha}$, then for some $g \in \cup\{\mathscr{F}^2_{\beta_\varepsilon} : \varepsilon < \partial\}$ we have $f < g \bmod D_*$.

[Why? Recall that $\beta_A(f)$ for $A \in \mathscr{P}_*$ and $\beta(f)$ are well defined ordinals $< \partial$ and $A \in \mathscr{P}_* \Rightarrow \beta_A(f) \le \beta(f)$. Now let $\zeta < \partial$ be such that $\beta(f) < \beta_\zeta$, exists as we can prove by induction on ε (using $(*)_{15}(b)$) that $\beta_\varepsilon \ge \varepsilon$. As $\bar{\mathscr{F}}^A$ is $<_{D_A}$-increasing for $A \in \mathscr{P}_*$ clearly $A \in \mathscr{P}_* \wedge g \in \mathscr{F}^A_{\beta_\zeta} \Rightarrow f < g \bmod D_A$. So by the definition of $\mathscr{F}^1_{\beta_\zeta}$ we have $A \in \mathscr{P}_* \wedge g \in \mathscr{F}^1_{\beta_\zeta} \Rightarrow f < g \bmod D_A$ hence $g \in \mathscr{F}^1_{\beta_\zeta} \Rightarrow f < g \bmod D_*$. As $\mathscr{F}^2_{\beta_\zeta} \subseteq \mathscr{F}^1_{\beta_\zeta}$ we are done.]

$(*)_{17}$ if $\zeta < \xi < \partial$ and $f \in \mathscr{F}^2_\zeta$ and $g \in \mathscr{F}_\xi$ then $f < g \bmod D_*$.

[Why? As in the proof of $(*)_{16}$ but now $\beta(f) = \gamma_\zeta$.]

Together by $(*)_{13} + (*)_{16} + (*)_{17}$ the sequence $\langle \mathscr{F}^2_{\beta_\varepsilon} : \varepsilon < \partial \rangle$ is as required. $\square_{1.3}$

{r17g} A central definition here is

Definition 1.5. 1) For $\bar{\alpha} \in {}^Y\mathrm{Ord}$ let $J^{\aleph_1\text{-comp}}_{<\lambda}[\bar{\alpha}] = \{X \subseteq Y\colon \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha} \restriction X) \subseteq \lambda\}$. So for $X \subseteq Y, X \notin J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}]$ iff there is an $\aleph_1$-complete filter D on Y such that $X \neq \emptyset \bmod D$ and $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is well defined $\ge \lambda$ <u>iff</u> there is an $\aleph_1$-complete filter D on Y such that $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is well defined $\ge \lambda$ and $X \in D$.
2) $J^{\aleph_1-\mathrm{comp}}_{\le\lambda}$ is $J^{\aleph_1-\mathrm{comp}}_{<\lambda^+}$ and we can use a set $\mathfrak{a}$ of ordinals instead of $\bar{\alpha}$.

{r18} **Claim 1.6.** <u>*The Generator Existence Claim*</u>
Let $\bar{\alpha} \in {}^Y(\mathrm{Ord}\backslash\{0\})$.
1) $J^{<\aleph_1-\mathrm{comp}}_{<\lambda}(\bar{\alpha})$ *is an* $\aleph_1$*-complete ideal on* Y *for any cardinal* λ *except that it may be* $\mathscr{P}(Y)$.
2) [$\mathrm{AC}_{\mathscr{P}(Y)}$] *Assume* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \ge \mathrm{hrtg}(\mathscr{P}(Y))$. *If* $\lambda \in \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha})$ <u>*then*</u> *for some* $X \subseteq Y$ *we have*

(*A*) $J^{\aleph_1-\mathrm{comp}}_{<\lambda^+}[\bar{\alpha}] = J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}] + X$

(*B*) $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_{J^{\aleph_1-\mathrm{comp}}_{=\lambda}[\bar{\alpha}]})$ *where* $J^{\aleph_1-\mathrm{comp}}_{=\lambda}[\bar{\alpha}] := J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}] + (Y\backslash X)$

(*C*) $\lambda \notin \text{ps-pcf}_{\aleph_1-\mathrm{comp}}(\bar{\alpha} \restriction (Y\backslash X))$.

{r18d} *Remark* 1.7. 1) Recall that if $\mathrm{AC}_{\mathscr{P}(Y)}$ <u>then</u> without loss of generality $\mathrm{AC}_{\aleph_0}$ holds. Why? Otherwise by $\mathrm{AC}_{\mathscr{P}(Y)}$ we have Y is well ordered and AC_Y hence $|Y| = n$ for some $n < \omega$ and in this case our claims are obvious, e.g. 1.6(2), 1.8.
2) Note that $J^{\aleph_1-\mathrm{comp}}_{=\lambda}[\bar{\alpha}]$ is a well defined ideal in 1.6(2)(B) though X is not uniquely determined.
3) Note that if $\theta = \mathrm{ps}-\mathrm{tcf}(\Pi\bar{\alpha}, <_D)$ and $X \in D^+$ then $\theta = \mathrm{ps}-\mathrm{tcf}(\Pi(\bar{\alpha}\restriction X), <_{(D+X)\cap\mathscr{P}(X)})$.

Proof. 1) Clearly $J^{<\aleph_1\text{-comp}}_{<\lambda}(\bar{\alpha})$ is a $\subseteq$-downward closed subset of $\mathscr{P}(Y)$. If the desired conclusion fails, then we can find a sequence $\langle A_n : n < \omega \rangle$ of members of $J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}]$ such that their union $A := \cup\{A_n : n < \omega\}$ does not belong to it. As $A \notin J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}]$, by the definition there is an $\aleph_1$-complete filter D on Y such that $A \neq \emptyset \bmod D$ and $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is well defined, so let it be $\mu = \mathrm{cf}(\mu) \ge \lambda$ and let $\langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$ exemplify it.

As D is $\aleph_1$-complete and $A = \cup\{A_n : n < \omega\} \neq \emptyset \bmod D$ necessarily for some $n, A_n \neq \emptyset \bmod D$ but then D witness $A_n \notin J^{\aleph_1-\mathrm{comp}}_{<\lambda}[\bar{\alpha}]$, contradiction.
2) Recall λ is a regular cardinal by [Sh:938, 5.8(0)] and $\lambda \ge \mathrm{hrtg}(\mathscr{P}(Y))$ by 1.1.

Let $D = D^{\bar{\alpha}}_\lambda$ be as in [Sh:938, 5.19] when DC holds, and as in 1.3 in general, i.e. $\Pi\bar{\alpha}/D$ has pseudo true cofinality λ and D contains any other such $\aleph_1$-complete filter on Y. Now if $X \in D^+$ then $\lambda = \text{ps-tcf}_{\aleph_1-\text{comp}}(\bar\alpha \restriction X, <_{(D+X)\cap\mathscr{P}(X)})$ hence $X \notin J^{\aleph_1-\text{comp}}_{<\lambda}[\bar\alpha]$, so

$(*)_1$ $X \in J^{\aleph_1-\text{comp}}_{<\lambda}[\bar\alpha] \Rightarrow X = \emptyset \bmod D$.

A major point is

$(*)_2$ some $X \in D$ belongs to $J^{\aleph_1-\text{comp}}_{<\lambda^+}[\bar\alpha]$.

Why $(*)_2$? The proof will take awhile; assume that not, we have $\text{AC}_{\mathscr{P}(Y)}$ hence AC_D, so we can find $\langle(\bar{\mathscr{F}}^X, D_X, \lambda_X) : X \in D\rangle$ such that:

- (a) λ_X is a regular cardinal $\geq \lambda^+$, i.e. $> \lambda$
- (b) D_X is an $\aleph_1$-complete filter on Y such that $X \in D_X$ and $\lambda_X = \text{ps-tcf}(\Pi\bar\alpha, <_{D_X})$
- (c) $\bar{\mathscr{F}}^X = \langle \mathscr{F}^X_\alpha : \alpha < \lambda_X\rangle$ exemplifies that $\lambda_X = \text{ps-tcf}\ (\Pi\bar\alpha, <_{D_X})$
- (d) moreover $\bar{\mathscr{F}}^X$ is as in [Sh:938, 5.17(2)], that is, it is $\aleph_0$-continuous and $\alpha < \lambda_X \Rightarrow \mathscr{F}^X_\alpha \neq \emptyset$.

Let

- (e) $D^*_1 = \{A \subseteq Y$: for some $X_1 \in D$ we have $X \in D \wedge X \subseteq X_1 \Rightarrow A \in D_X\}$.

Clearly

- (f) D^*_1 is an $\aleph_1$-complete filter on Y extending D.

[Why? First, clearly $D^*_1 \subseteq \mathscr{P}(Y)$ and $\emptyset \notin D^*_1$ as $X \in D \Rightarrow \emptyset \notin D_X$. Second, if $A \in D$ then $X \in D \wedge X \subseteq A \Rightarrow A \in D_X$ by clause (b) hence choosing $X_1 = A$ the demand for "$A \in D^*_1$" holds so indeed $D \subseteq D^*_1$. Third, assume $\bar A = \langle A_n : n < \omega\rangle$ and "$A_n \in D^*_1$" for $n < \omega$, then for each A_n there is a witness $X_n \in D$, so by $\text{AC}_{\aleph_0}$, recalling 1.7, there is an ω-sequence $\langle X_n : n < \omega\rangle$ with X_n witnessing $A_n \in D^*_1$. Then $X = \cap\{X_n : n < \omega\}$ belongs to D and witness that $A := \cap\{A_n : n < \omega\} \in D^*_1$ because every D_X is $\aleph_1$-complete. Fourth, if $A \subseteq B \subseteq Y$ and $A \in D^*_1$, then some X_1 witness $A \in D^*_1$, i.e. $X \in D \wedge X \subseteq X_1 \Rightarrow A \in D_X$; but then X_1 witness also $B \in D^*_1$.]

- (g) assume $\langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$ is $<_D$-increasing in $\Pi\bar\alpha$, i.e. $\alpha < \lambda \Rightarrow \mathscr{F}_\alpha \subseteq \Pi\bar\alpha$ and $\alpha_1 < \alpha_2 \wedge f_1 \in \mathscr{F}_{\alpha_1} \wedge f_2 \in \mathscr{F}_{\alpha_2} \Rightarrow f_1 <_D f_2$ and $\mathscr{F}_\alpha \neq \emptyset$ for every or at least unboundedly many $\alpha < \lambda$ then $\bigcup\limits_{\alpha<\lambda} \mathscr{F}_\alpha$ has a common $<_{D^*_1}$-upper bound.

[Why? For each $X \in D$ recall $(\Pi\bar\alpha, <_{D_X})$ has true cofinality λ_X which is regular $> \lambda$ hence by [Sh:938, 5.7(1A)] is pseudo λ^+-directed hence there is a common $<_{D_X}$-upper bounded h_X of $\cup\{\mathscr{F}_\alpha : \alpha < \lambda\}$. As we have $\text{AC}_{\mathscr{P}(Y)}$ we can find a sequence $\langle h_X : X \in D\rangle$ with each h_X as above. Define $h \in \Pi\bar\alpha$ by $h(t) = \sup\{h_X(t) : X \in D\}$, it belongs to $\Pi\bar\alpha$ as we are assuming $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathscr{P}(Y)) \geq \text{hrtg}(D)$. So $h \in \Pi\bar\alpha$ is a $<_{D_X}$-upper bound of $\cup\{\mathscr{F}_\alpha : \alpha < \lambda\}$ for every $X \in D$, hence by the choice of D^*_1 it is a $<_{D^*_1}$-upper bound of $\cup\{\mathscr{F}_\alpha : \alpha < \lambda\}$.]

But by the choice of D in the beginning of the proof we have $\lambda = \text{ps-tcf}(\Pi\bar\alpha, <_D)$ so there is a sequence $\langle \hat{\mathscr{F}}_\alpha : \alpha < \lambda\rangle$ witnessing it. By clause (f) we have $D \subseteq D^*_1$ so clearly $\langle \hat{\mathscr{F}}_\alpha : \alpha < \lambda\rangle$ is also $<_{D^*_1}$-increasing hence we can apply clause (g) to the sequence $\langle \hat{\mathscr{F}}_\alpha : \alpha < \lambda \rangle$ and got a $<_{D_1^*}$-upper bound $f \in \Pi\bar{\alpha}$, contradiction to the choice of $\langle \hat{\mathscr{F}}_\alpha : \alpha < \lambda \rangle$ recalling 0.4(d) because $D \subseteq D_1^*$, contradiction. So $(*)_2$ really holds.

Choose X as in $(*)_2$, now

$(*)_3$ $D = \text{dual}(J^{\aleph_1-\text{comp}}_{<\lambda}[\bar{\alpha}] + (Y \backslash X))$.

[Why? The inclusion $\supseteq$ holds by $(*)_1$ and $(*)_2$, i.e. the choice of X as a member of D. Now for every $Z \subseteq X$ which does not belong to $J^{\aleph_1-\text{comp}}_{<\lambda}[\bar{\alpha}]$, by the definition of $J^{\aleph_1-\text{comp}}_{<\lambda}[\bar{\alpha}]$ there is an $\aleph_1$-complete filter D_Z on Y to which Z belongs such that $\theta := \text{ps-cf}(\Pi\bar{\alpha}, <_D)$ is well defined and $\geq \lambda$. But $\theta \geq \lambda^+$ is impossible as we know that $Z \subseteq X \in J^{\aleph_1\text{-comp}}_{<\lambda^+}[\bar{\alpha}]$, so necessarily $\theta = \lambda$, hence by the choice of D by using 1.3 we have $D \subseteq D_Z$, hence $Z \neq \emptyset \mod D$. Together we are done.]

$(*)_4$ $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_{J^{\aleph_1-\text{comp}}_{=\lambda}})$, see clause (B) of the conclusion of 1.6(2).

[Why? By $(*)_3$, the choice of $J^{\aleph_1-\text{comp}}_{=\lambda}[\bar{\alpha}]$ and as $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ by the choice of D.]

$(*)_5$ $\lambda \notin \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha} \restriction (Y \backslash X))$.

[Why? Otherwise there is an $\aleph_1$-complete filter D' on Y such that $Y \backslash X \in D'$ and $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_{D'})$. But this contradicts the choice of D by using 1.3.]

So X is as required in the desired conclusion of 1.6(2): clause (B) by $(*)_4$, clause (C) by $(*)_5$ and clause (A) follows. Note that the notation $J^{\aleph_1-\text{comp}}_{=\lambda}[\bar{\alpha}]$ is justified, as if X' satisfies the requirements on X then $X' = X \mod J^{\aleph_1-\text{comp}}_{<\lambda}[\bar{\alpha}]$. $\square_{1.6}$

{r19} *Conclusion* 1.8. [$\text{AC}_{\mathscr{P}(Y)}$] Assume $\bar{\alpha} \in {}^Y\text{Ord}$ and each α_t a limit ordinal of cofinality $\geq \text{hrtg}(\mathscr{P}(Y))$ and $\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$ is not empty.
1) If $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\text{Fil}^1_{\aleph_1}(Y))$ <u>then</u> there is a function h such that:

- $\bullet_1$ the domain of h is $\mathscr{P}(Y)$
- $\bullet_2$ Rang(h) includes $\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$ and is included in $\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\bar{\alpha}) \cup \{0\} \cup \{\mu : \mu = \sup(\mu \cap \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha}))\}$, also Rang$(h)$ includes $\{\text{cf}(\alpha_t) : t \in Y\}$, but see $\bullet_5$
- $\bullet_3$ $A \subseteq B \subseteq Y \Rightarrow h(A) \leq h(B)$ and $h(A) = 0 \Leftrightarrow A = \emptyset$
- $\bullet_4$ $h(A) = \min\{\lambda : A \in J^{\aleph_1-\text{comp}}_{\leq\lambda}[\bar{\alpha}]\}$
- $\bullet_5$ if $h(A) = \lambda$ and $\text{cf}(\lambda) > \aleph_0$ then λ is regular and $\lambda \in \text{ps-tcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$, i.e. for some $\aleph_1$-complete filter D on Y we have $A \in D$ and $\text{ps-tcf}(\Pi\bar{\alpha}, <_D) = \lambda$
- $\bullet_6$ the set $\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$ has cardinality $< \text{hrtg}(\mathscr{P}(Y))$
- $\bullet_7$ if $h(A) = \lambda$ and $\text{cf}(\lambda) = \aleph_0$ then we can find a sequence $\langle A_n : n < \omega \rangle$ such that $A = \cup\{A_n : n < \omega\}$ and $h(A_n) < \lambda$ for $n < \omega$
- $\bullet_8$ $J^{\aleph_1-\text{comp}}_{<\lambda}[\bar{\alpha}] = \{A \subseteq Y : h(A) < \lambda\}$ when $\text{cf}(\lambda) > \aleph_0$
- $\bullet_9$ if $\text{cf}(\text{otp}(\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha}))) > \aleph_0$ then $\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar{\alpha})$ has a last member.

2) Without the extra assumption of part (1), still there is h such that:

- $\bullet_1$ h is a function with domain $\mathscr{P}(Y)$

- $\bullet_2$ the range of h is ps-pcf$_{\aleph_1-\text{comp}}(\bar\alpha)\cup\{0\}\cup\{\mu:\mu=\sup(\mu\cap$ ps-pcf$_{\aleph_1-\text{comp}}(\bar\alpha))$ and $\text{cf}(\mu)=\aleph_0$ or just $\text{cf}(\mu)<\text{hrtg}(\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha))$ and $J^{\aleph_1-\text{comp}}_{<\mu}[\bar\alpha]\neq\cup\{J^{\aleph_1-\text{comp}}_{<\chi}[\bar\alpha]:\chi<\mu\}\}$
- $\bullet_3$ $A\subseteq B\subseteq Y\Rightarrow h(A)\le h(B)$ and $h(A)=0\Leftrightarrow A=\emptyset$
- $\bullet_4$ $h(A)=\ \min\{\lambda:A\in J^{\aleph_1-\text{comp}}_{\le\lambda}[\bar\alpha]\}$
- $\bullet_5$ if $h(A)=\lambda$ and $\text{cf}(\lambda)\ge\text{hrtg}(\text{ ps-pcf}_{\aleph_1-\text{comp}}[\bar\alpha])$ then $\lambda\in\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$, i.e. there is an $\aleph_1$-complete filter D on Y such that $(\Pi\bar\alpha,<_D)$ has true cofinality λ
- $\bullet_6$ as above
- $\bullet_7$ as above
- $\bullet_8$ as above.

3) The set $\mathfrak{c}:=\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$ satisfies $\mathfrak{c}\le_{\text{qu}}\mathscr{P}(Y)$. If also AC_α holds for $\alpha<\text{hrtg}(\mathscr{P}(Y))$ or just $\text{AC}_{\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)}$ then we can find a sequence $\langle X_\lambda:\lambda\in\mathfrak{c}\rangle$ of subsets of Y such that for every cardinality μ, $J^{\aleph_1-\text{comp}}_{<\mu}[\bar\alpha]$ is the $\aleph_1$-complete ideal on Y generated by $\{X_\lambda:\lambda<\mu$ and $\lambda\in\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)\}$.

Proof. 1) Let $\Theta=\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$. We define the function h from $\mathscr{P}(Y)$ into Θ^+ which is defined as the closure of $\Theta\cup\{0\}$, i.e. $\Theta\cup\{\mu:\mu=\sup(\mu\cap\Theta)\}$, by $h(X)=\ \text{Min}\{\lambda\in\Theta^+:X\in J^{\aleph_1-\text{comp}}_{\le\lambda}[\bar\alpha]\}$. It is well defined as ps-pcf$_{\aleph_1-\text{comp}}(\bar\alpha)$ is a set, that is as $\mu_*=\text{hrtg}(\Pi\bar\alpha)$ is well defined and so $J^{\aleph_1-\text{comp}}[\bar\alpha]=\mathscr{P}(Y)$ (see [Sh:938, 5.8(2)]), non-empty by an assumption and $J^{\aleph_1-\text{comp}}_{\le\lambda}[\bar\alpha]=\mathscr{P}(Y)$ when $\lambda\ge\ \sup(\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha))$. This function h, its range is included in Θ^+, but $\text{otp}(\Theta^+)\le\text{otp}(\Theta)+1$; also clearly $\bullet_1$ of the conclusion holds. Also if $\lambda\in\Theta$ and X is as in 1.6(2) then $h(X)=\lambda$; so h is a function from $\mathscr{P}(Y)$ into Θ^+ and its range include Θ hence $|\Theta|<\text{hrtg}(\mathscr{P}(Y))$ so $\bullet_2$ first clause holds; the second clause of $\bullet_2$ holds as trivially $h(\emptyset)=0$ and the definition of Θ^+ and the third clause by $t\in Y\Rightarrow h(\{t\})=\text{cf}(\alpha_t)$ holds. Now first by 1.1 we have $\theta\in\Theta\Rightarrow\theta\ge\text{hrtg}(\mathscr{P}(Y))$, hence $\theta\in\Theta\Rightarrow\theta>\sup(\Theta\cap\theta)$ so the range of h is as required in $\bullet_2$.

Second, if $\lambda\in\Theta^+$ and $\text{cf}(\lambda)=\aleph_0$ then clearly $\lambda\in\Theta^+\backslash\Theta$ and we can find an increasing sequence $\langle\lambda_n:n<\omega\rangle$ of members of ps-pcf$_{\aleph_1\text{-comp}}(\bar\alpha)$ with limit λ. For each n there is $X_n\in J^{\aleph_1-\text{comp}}_{\le\lambda_n}[\bar\alpha]\backslash J^{\aleph_1\text{-comp}}_{<\lambda_n}[\bar\alpha]$ by 1.6(2), but $\text{AC}_{\aleph_0}$ holds, see 1.7 hence such a sequence $\langle X_n:n<\omega\rangle$ exists. Easily $A:=\cup\{X_n:n<\omega\}\in\mathscr{P}(Y)$ satisfies $h(A)=\lambda$ hence $\lambda\in\ \text{Rang}(h)$. Third, if $\lambda=\sup(\text{ ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha))$ and $\text{cf}(\lambda)>\aleph_0$, then $\bigcup\limits_{\mu<\lambda}J_{<\mu}[\bar\alpha]\neq\mathscr{P}(Y)$ because Y does not belong to the union while $J_{<\lambda^+}(\bar\alpha)=\mathscr{P}(Y)$ so $h(Y)=\lambda$.

Fourth, assume $\lambda=h(A),\lambda\notin\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$ and $\text{cf}(\lambda)>\aleph_0$, we can find $\langle\lambda_i:i<\text{cf}(\lambda)\rangle$, an increasing sequence with limit λ, but by the definition of h necessarily $\lambda\cap$ ps-pcf$_{\aleph_1-\text{comp}}(\bar\alpha)$ is an unbounded subset of λ so without loss of generality all are members of ps-pcf$_{\aleph_1-\text{comp}}(\Pi\bar\alpha)$. Now $\langle J_i:=J^{\aleph_1-\text{comp}}_{<\lambda_i}[\bar\alpha]:i<\text{cf}(\lambda)\rangle$ is a $\subseteq$-increasing sequence of $\aleph_1$-complete ideals on Y, no choice is needed, and by our present assumption $\aleph_0<\text{cf}(\lambda)$ hence the union $J=\cup\{J_i:i<\ \text{cf}(\lambda)\}$ is an $\aleph_1$-complete ideal on Y and obviously $A\notin J$. So also $D_1=\ \text{dual}(J)+A$ is an $\aleph_1$-complete filter hence by [Sh:938, 5.9] (recalling the extra assumption $t\in Y\Rightarrow\text{cf}(\alpha_t)\ge\text{hrtg}(\text{Fil}^1_{\aleph_1}(Y)))$

for some $\aleph_1$-complete filter D_2 extending D_1 we have $\mu = \text{ps-tcf}(\Pi\alpha, <_{D_2})$ is well defined, so by 1.6(2) we have some $D_2 \cap J^{\aleph_1-\text{comp}}_{\le\mu}[\bar\alpha] \ne \emptyset$ but $\emptyset = D_2 \cap J_i = D_2 \cap J^{\aleph_1-\text{comp}}_{<\lambda_i}[\bar\alpha]$ hence $\mu \ge \lambda_i$. Hence $\mu \ge \lambda_i$ for every $i < \text{cf}(\lambda)$ but λ is singular so $\mu > \lambda$ and $\mu \in \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$. Hence $\chi := \min(\text{ ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)\backslash\lambda)$ is well defined and $J^{\aleph_1-\text{comp}}_{<\chi}[\bar\alpha] = J$ trivially $\chi \ge \lambda$, but as χ is regular while λ is singular clearly $\chi > \lambda$. But as $h(A) = \lambda < \chi$ we get that $A \in J^{\aleph_1-\text{comp}}_{<\chi}[\bar\alpha]$, contradiction to the definition of h.

So we have proved $\bullet_5$, the fifth clause of the conclusion. The other clauses follow from the properties of h.

2) Similar proof.

3) We define a function g with domain $\mathscr{P}(Y)$ by $g(A) = \min\{\lambda : A \in J_{<\lambda^+}[\bar\alpha]\}$. This function is well defined as if $\lambda = \text{hrtg}(\Pi\bar\alpha)$ then $A \subseteq Y \Rightarrow A \in J_{\le\lambda}[\bar\alpha]$; and the cardinals are well ordered. Also $\mathfrak{c} \subseteq \text{Rang}(h)$ because if $\lambda \in \mathfrak{c}$, then by 1.6(2) we are done recalling that we are assuming $\text{AC}_{\mathscr{P}(Y)}$.

So clearly $\mathfrak{c} \le_{\text{qu}} \mathscr{P}(Y)$ so as $\mathfrak{c}$ is a set of cardinals, clearly $\text{otp}(\mathfrak{c}) < \text{hrtg}(\mathscr{P}(Y))$ hence $|\mathfrak{c}| < \text{hrtg}(\mathscr{P}(Y))$.

For the second sentence in 1.8(3) by the last sentence it suffices to assume $\text{AC}_{\mathfrak{c}}$. For $\lambda \in \mathfrak{c}$ let $\mathscr{P}_\lambda = \{X \subseteq Y : X \text{ as in 1.6(2)}\}$, so $\mathscr{P}_\lambda \ne \emptyset$. By $\text{AC}_{\mathfrak{c}}$ there is a sequence $\langle X_\lambda : \lambda \in \mathfrak{c}\rangle \in \prod\limits_{\lambda\in\mathfrak{c}} \mathscr{P}_\lambda$. For $\lambda \in \mathfrak{c}$, let J^*_λ be the $\aleph_1$-complete ideal on Y generated by $\{X_\mu : \mu \in \mathfrak{c} \cap \lambda\}$, so by the definitions of $\mathscr{P}_\lambda$ we have $\mu < \lambda \wedge \mu \in \mathfrak{c} \Rightarrow X_\mu \in J_{\le\mu}[\bar\alpha] \subseteq J_{<\lambda}[\bar\alpha]$, also $J_{<\lambda}[\bar\alpha]$ is $\aleph_1$-complete hence $\lambda \in \mathfrak{c} \Rightarrow J^*_\lambda \subseteq J_{<\lambda}[\bar\alpha]$.

If for every λ equality holds we are done, otherwise there is a minimal counterexample and use 1.6(2). $\square_{1.8}$

{r20}

Definition 1.9. Assume $\text{cf}(\mu) < \text{hrtg}(Y)$ and μ is singular of uncountable cofinality limit of regulars. We let

(a) $\text{pp}^*_Y(\mu) = \sup\{\lambda :$ for some $\bar\alpha, D$ we have
   (a) $\lambda = \text{ps-tcf}(\Pi\bar\alpha, <_D)$,
   (b) D is an $\aleph_1 -$ complete filter on Y
   (c) $\bar\alpha = \langle \alpha_t : t \in Y\rangle$, each α_t regular
   (d) $\mu = \lim_D \bar\alpha\}$

(b) $\text{pp}^+_Y(\mu) = \sup\{\lambda^+ : \lambda \text{ as above}\}$.

(c) similarly $\text{pp}^*_{\kappa-\text{comp},Y}(\mu), \text{pp}^+_{\kappa-\text{comp},Y}(\mu)$ restricting ourselves to κ-complete filters D; similarly for other properties

(d) we can replace Y by an $\aleph_1$-complete filter D on Y, this means we fix D but not $\bar\alpha$ above.

{r20d}

Remark 1.10. 1) of course, if we consider sets Y such that AC_Y may fail, it is natural to omit the regularity demands, so $\bar\alpha$ is just a sequence of ordinals.
2) We may use $\bar\alpha$ a sequence of cardinals, not necessarily regular; see §3.

{r21}

Conclusion 1.11. [DC + $\text{AC}_{\mathscr{P}(Y)}$] Assume $\theta = \text{hrtg}(\mathscr{P}(Y)) < \mu, \mu$ is as in Definition 1.9, $\mu_0 < \mu$ and $\bar\alpha \in {}^Y(\text{Reg} \cap \mu_0^+) \wedge \text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\bar\alpha) \ne \emptyset \Rightarrow \text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\bar\alpha) \subseteq \mu$. If $\sigma = |\text{Reg } \cap \mu\backslash\mu_0| < \mu$ and $\kappa = |\text{Reg } \cap \text{ pp}^+_Y(\mu)\backslash\mu_0|$ <u>then</u> $\kappa < \text{hrtg}(\theta \times {}^Y\sigma)$.

Remark 1.12. In the ZFC parallel the assumption on $\mu_0 < \mu$ is not necessary.

Proof. Obvious by Definition [Sh:938, 5.6] noting Conclusion 1.8 above and 1.13 below. That is, letting $\Xi := \text{Reg} \cap \text{pp}^+_Y(\mu)\backslash\mu_0$ so $|\Xi| = \kappa$ and $\Lambda = \text{Reg} \cap \mu\backslash\mu_0$, for every $\bar\alpha \in {}^Y\Lambda$ by Definition 1.9 the set $\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$ is a subset of $\text{Reg} \cap \text{pp}^+_Y(\mu)\backslash\mu_0$, and by claim 1.8 it is a set of cardinality $< \text{hrtg}(\mathscr{P}(Y))$. By Definition 1.9 and Claim 1.13 below we have $\Xi = \cup\{\text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\bar\alpha) : \bar\alpha \in {}^Y\Lambda\}$. Clearly there is a function h with domain $\text{hrtg}(\mathscr{P}(Y)) \times {}^Y\sigma$ such that $\varepsilon < \text{hrtg}(\mathscr{P}(Y)) \wedge \bar\alpha \in {}^Y\sigma \Rightarrow (h(\varepsilon,\bar\alpha)$ is the ε-th member of $\text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$ if there is one, $\min(\Lambda)$ otherwise). So h is a function from $\text{hrtg}(\mathscr{P}(Y)) \times {}^Y\sigma$ onto a set including Ξ which has cardinality κ, so we are done. $\square_{1.11}$

{r22}

Claim 1.13. *<u>The No Hole Claim</u>[DC]*
1) If $\bar\alpha \in {}^Y\text{Ord}$ and $\lambda_2 \in \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha)$, for transparency $t \in Y \Rightarrow \alpha_t > 0$ and $\text{hrtg}(\mathscr{P}(Y)) \le \lambda_1 = \text{cf}(\lambda_1) < \lambda_2$, <u>then</u> for some $\bar\alpha' \in \Pi\bar\alpha$ we have $\lambda_1 = \text{ps-pcf}_{\aleph_1-\text{comp}}(\bar\alpha')$.
2) In part (1), if in addition AC_Y <u>then</u> without loss of generality $\bar\alpha' \in {}^Y\text{Reg}$.
3) If in addition $\text{AC}_{\mathscr{P}(Y)} + \text{AC}_{<\kappa}$ <u>then</u> even witnessed by the same filter (on Y).

Proof. 1) Let D be an $\aleph_1$-complete filter on Y such that $\lambda_2 = \text{ps-tcf}(\Pi\bar\alpha, <_D)$, let $\langle \mathscr{F}_\alpha : \alpha < \lambda_2\rangle$ exemplify this.

First assume $\text{hrtg}(\text{Fil}^1_{\aleph_1}(Y)) \le \lambda_1$, clearly $f \in \mathscr{F}_\alpha \Rightarrow \text{rk}_D(f) \ge \alpha$ for every $\alpha < \lambda_2$, hence in particular for $\alpha = \lambda_1$ hence there is $f \in {}^Y\text{Ord}$ such that $\text{rk}_D(f) = \lambda_1$ and now use [Sh:938, 5.9] but there we change the filter D, (extend it), so is O.K. for part (1). In general, i.e. without the extra assumption $\text{hrtg}(\text{Fil}^1_{\aleph_2}(Y)) \le \lambda_1$, use 1.14(1),(2) below.
2) Easy, too.
3) Similarly using 1.14(3) below. $\square_{1.13}$

{r24}

Claim 1.14. *Assume $D \in \text{Fil}^1_\kappa(Y), \kappa > \aleph_0, \mathscr{F}_\alpha \subseteq {}^Y\text{Ord}$ non-empty for $\alpha < \delta$ and $\bar{\mathscr{F}} = \langle \mathscr{F}_\alpha : \alpha < \delta\rangle$ is $<_D$-increasing, δ a limit ordinal.*
1) [DC] There is $f^* \in \Pi\bar\alpha$ which satisfies $f \in \cup\{\mathscr{F}_\alpha : \alpha < \lambda_1\} \Rightarrow f <_D f^*$ but there is no such $f^{**} \in \Pi\bar\alpha$ satisfying $f^{**} <_D f$.
2) [$\text{AC}_{<\kappa}$] For f^* as above, let $D_1 = D_{f^*,\bar{\mathscr{F}}} := \{Y\backslash A : A = \emptyset \mod D$ or $A \in D^+$ and there is $f^{**} \in {}^Y\text{Ord}$ such that $f^{**} <_{D+A} f^*$ and $f \in \cup\{\mathscr{F}_\alpha : \alpha < \lambda_1\} \Rightarrow f <_{D+A} f^{**}\}$. Now D_1 is a κ-complete filter and $\emptyset \notin D_1, D_1$ extends D and if $\text{cf}(\delta) \ge \text{hrtg}(\mathscr{P}(Y))$ then $\langle \mathscr{F}_\alpha : \alpha < \delta\rangle$ witness that f^* is a $<_{D_1}$-exact upper bound of $\bar{\mathscr{F}}$ hence $(\prod_{y\in Y} f^*(y), <_{D_1})$ has pseudo-true-cofinality $\text{cf}(\delta)$.
3) [DC + $\text{AC}_{<\kappa}$+ $\text{AC}_{\mathscr{P}(Y)}$]
If $\text{cf}(\delta) \ge \text{hrtg}(\mathscr{P}(Y))$ <u>then</u> there is $f' \in {}^Y\text{Ord}$ which is an $<_D$-exact upper bound of $\bar{\mathscr{F}}$, i.e. $f <_D f' \Rightarrow (\exists\alpha < \delta)(\exists g \in \mathscr{F}_\alpha)[f < g \mod D]$ and $f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha \Rightarrow f <_D f'$.

Proof. 1) If not then by DC we can find $\bar f = \langle f_n : n < \omega\rangle$ such that:

- (a) $f_n \in {}^Y\text{Ord}$
- (b) $f_{n+1} < f_n \mod D$
- (c) if $f \in \bigcup_{\alpha<\delta} \mathscr{F}_\alpha$ and $n < \omega$ then $f < f_n \mod D$.

So $A_n = \{t \in Y : f_{n+1}(t) < f_n(t)\} \in D$ hence $\cap\{A_n : n < \omega\} \in D$, contradiction.

2) First, clearly $D_1 \subseteq \mathscr{P}(Y)$ and by the assumption $\emptyset \notin D_1$. Second, if f^{**} witness $A \in D_1$ and $A \subseteq B \subseteq Y$ then f^{**} witness $B \in D_1$.

Third, we prove D_1 is closed under intersection of $< \kappa$ members, so assume $\zeta < \kappa$ and $\bar{A} = \langle A_\varepsilon : \varepsilon < \zeta \rangle$ is a sequence of members of D_1. Let $A := \cap\{A_\varepsilon : \varepsilon < \zeta\}, B_\varepsilon = Y \backslash A_\varepsilon$ for $\varepsilon < \zeta$ and $B'_\varepsilon = B_\varepsilon \backslash \cup \{B_\xi : \xi < \varepsilon\}$ and $B = \cup\{B_\varepsilon : \varepsilon < \zeta\}$. Clearly $B = Y \backslash A, A \subseteq Y$ and $\langle B'_\varepsilon : \varepsilon < \zeta \rangle$ is a sequence of pairwise disjoint subsets of Y with union B. But AC_ζ holds and $\varepsilon < \zeta \Rightarrow A_\varepsilon \in D_1$ hence we can find $\langle f^{**}_\varepsilon : \varepsilon < \zeta \rangle$ such that $f^{**}_\varepsilon \in {}^Y\mathrm{Ord}$ and if $A_\varepsilon \notin D$ then f^{**}_ε witness $A_\varepsilon \in D_1$. Let $f^{**} \in {}^Y\mathrm{Ord}$ be defined by $f^{**}(t) = f^{**}_\varepsilon(t)$ if $t \in B'_\varepsilon$ or $\varepsilon = 0 \wedge t \in Y \backslash B$; easily $B'_\varepsilon \in D^+ \wedge f \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha \Rightarrow f < f^{**}_\varepsilon = f^{**} \mod (D + B'_\varepsilon)$ but $B = \cup\{B'_\varepsilon : \varepsilon < \zeta\}$ and D is κ-complete hence $f \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha \Rightarrow f < f^{**} \mod (D+B)$. So as $A = Y \backslash B$ clearly f^{**} witness $A = \bigcap\limits_{\varepsilon<\zeta} A_\varepsilon \in D_1$ so D_1 is indeed κ-complete.

Lastly, assume $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ and we shall show that f^* is an exact upper bound of $\bar{\mathscr{F}}$ modulo D_1. So assume $f^{**} \in {}^Y\mathrm{Ord}$ and $f^{**} < f^* \mod D_1$ and we shall prove that there are $\alpha < \delta$ and $f \in \mathscr{F}_\alpha$ such that $f^{**} \leq f \mod D_1$.

Let $\mathscr{A} = \{A \in D_1^+$: there is $f \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha$ such that $f^{**} \leq f \mod (D+A)\}$, yes, not D_1!

<u>Case 1</u>: For every $B \in D_1^+$ there is $A \in \mathscr{A}, A \subseteq B$.

For every $A \in \mathscr{A}$ let $\alpha_A = \min\{\beta$: there is $f \in \mathscr{F}_\beta$ such that $f^{**} \leq f \mod (D + A)\}$.

So the sequence $\langle \alpha_A : A \in \mathscr{A} \rangle$ is well defined.

Let $\alpha(*) = \sup\{\alpha_A + 1 : A \in \mathscr{A}\}$, it is $< \delta$ as $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y)) \geq \mathrm{hrtg}(\mathscr{A})$.

Choose $f \in \mathscr{F}_{\alpha(*)}$ and let $B_f := \{t \in Y : f^{**}(t) > f(t)\}$. Now if $A \in \mathscr{A}$ (so $A \in D_2^+$) and $f' \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha$ witness this (i.e. $f^{**} \leq f' \mod (D + A)$); without loss of generality $f' \in \mathscr{F}_{\alpha_A}$ hence $f' < f \mod D$ recalling $\alpha_A < \alpha(*)$, then $A \nsubseteq B_f$ as otherwise $f^{**} \leq f' < f < f^{**} \mod (D + A)$. So B_f contains no $A \in \mathscr{A}$ hence necessarily B_f is $= \emptyset \mod D_1$ by the case assumption; this means that $f^{**} \leq f \mod D_1$. So recalling $f \in \mathscr{F}_{\alpha(*)} \subseteq \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha$, we have "$f$ is as required" thus finishing the proof of "f^* is an exact upper bound of $\bar{\mathscr{F}} \mod D$".

<u>Case 2</u>: $B \in D_1^+$ and there is no $A \in \mathscr{A}$ such that $A \subseteq B$.

For $f \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha$ let $B_f = \{t \in B : f(t) < f_{**}(t)\}$ and for $\alpha < \delta$ we define $\mathscr{B}_\alpha = \{B_f : f \in \mathscr{F}_\alpha\}$ and we define a partial function h from $\mathscr{P}(Y)$ into δ by $h(A) = \sup\{\alpha < \delta : A \in \mathscr{B}_\alpha\}$. As $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ necessarily $\alpha(*) = \sup(\delta \cap \mathrm{Rang}(h))$ is $< \delta$. Choose $g \in \mathscr{F}_{\alpha(*)+1}$, hence $u := \{\alpha : \alpha \in [\alpha(*), \delta]$ and $B_g \in \mathscr{B}_\alpha\}$ is an unbounded subset of δ.

Let $A = B \cap B_g$, now if $A \in D^+$ then $\alpha \in u \Rightarrow \bigvee\limits_{f \in \mathscr{F}_\alpha} f < f_{**} \mod (D + A)$ but $\bar{\mathscr{F}}$ is $<_D$-increasing and $\delta = \sup(u)$ hence $f \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha \Rightarrow f < f_{**} \mod (D + A)$ hence by the definition of D_1, f_{**} witness that $Y \backslash A \in D_1$, hence $A = \emptyset \mod D_1$. As $B \in D_1^+$ and $A = B \cap B_g$ it follows that $B \backslash B_g \in D_1^+$ and by the choice of $\mathscr{A}$ the set $B \backslash B_g$ belongs to $\mathscr{A}$. But $B \backslash B_g \subseteq B$ by its definition so we get a contradiction to the case assumption.

3) By [Sh:938, 5.12] without loss of generality $\bar{\mathscr{F}}$ is $\aleph_0$-continuous. For every $A \in D^+$ the assumptions hold even if we replace D by $D+A$ and so there are D_1, f^* as in part (2), we are allowed to use part (1) as we have DC and part (2) as we have $\mathrm{AC}_{<\kappa}$. As we are assuming $\mathrm{AC}_{\mathscr{P}(Y)}$ there is a sequence $\langle (D_A, f_A) : A \in D^+\rangle$ such that:

$(*)_1$ (a) D_A is a κ-complete filter extending $D + A$
(b) $f_A \in {}^Y\mathrm{Ord}$ is a $<_{D_A}$-exact upper bound of $\bar{\mathscr{F}}$.

Recall $|A| \le_{\mathrm{qu}} |B|$ is defined as: A is empty or there is a function from B onto A. Of course, this implies $\mathrm{hrtg}(A) \le \mathrm{hrtg}(B)$.

Let $\bar{\mathscr{U}} = \langle \mathscr{U}_t : t \in Y\rangle$ be defined by $\mathscr{U}_t = \{f_A(t) : A \in D^+\} \cup \{\sup\{f(t) : f \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha\}\}$ hence $t \in Y \Rightarrow 0 < |\mathscr{U}_t| \le_{\mathrm{qu}} \mathscr{P}(Y)$ even uniformly so there is a sequence $\langle h_t : t \in Y\rangle$ such that h_t is a function from $\mathscr{P}(Y)$ onto $\mathscr{U}_t$ hence $|\prod\limits_{t\in Y} \mathscr{U}_t| \le_{\mathrm{qu}} \mathscr{P}(Y) \times Y \le_{\mathrm{qu}} \mathscr{P}(Y \times Y)$ but $\mathrm{AC}_{\mathscr{P}(Y)}$ holds hence Y can be well ordered however without loss of generality Y is infinite hence $|Y \times Y| = Y$, so $|\prod\limits_{t\in Y} \mathscr{U}_t| \le_{\mathrm{qu}} |\mathscr{P}(Y)|$.

Let $\mathscr{G} = \{g : g \in \prod\limits_{t\in Y} \mathscr{U}_t$ and not for every $f \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha$ do we have $f < g \bmod D\}$, so $|\mathscr{G}| \le |\prod\limits_{t\in Y} \mathscr{U}_t| \le_{\mathrm{qu}} |\mathscr{P}(Y \times Y)| = |\mathscr{P}(Y)|$ hence $\mathrm{hrtg}(\mathscr{G}) \le \mathrm{hrtg}(\mathscr{P}(Y)) \le \mathrm{cf}(\delta)$.

Now for every $g \in \mathscr{G}$ the sequence $\langle\{\{t \in Y : g(t) \le f(t)\} : f \in \bigcup\limits_{\beta<\alpha} \mathscr{F}_\beta\} : \alpha < \delta\rangle$ is a $\subseteq$-increasing sequence of subsets of $\mathscr{P}(Y)$, but $\mathrm{hrtg}(\mathscr{P}(Y)) \le \mathrm{cf}(\delta)$ hence the sequence is eventually constant and let $\alpha(g) < \delta$ be the minimal α such that

$(*)_g$ $(\forall\beta)[\alpha \le \beta < \delta \Rightarrow \{\{t \in Y : g(t) \le f(t)\} : f \in \bigcup\limits_{\gamma<\beta} \mathscr{F}_\gamma\} = \{\{t \in Y : g(t) \le f(t)\} : f \in \bigcup\limits_{\gamma<\alpha} \mathscr{F}_\gamma\}]$.

But recalling $\mathrm{hrtg}(\mathscr{G}) \le \mathrm{cf}(\delta)$, the ordinal $\alpha(*) := \sup\{\alpha(g) : g \in \mathscr{G}\}$ is $< \delta$. Now choose $f^* \in \mathscr{F}_{\alpha(*)+1}$ and define $g^* \in \prod\limits_{t\in Y} \mathscr{U}_t$ by $g^*(t) = \min(\mathscr{U}_t \backslash f^*(t))$, well defined as $\sup\{f(t) : t \in \bigcup\limits_{\alpha<\delta} \mathscr{F}_\alpha\} \in \mathscr{U}_t$. It is easy to check that g^* is as required. $\square_{1.14}$

Observation 1.15. 1) Let D be a filter on Y. {r25}
If D is κ-complete for every κ <u>then</u> for every $f \in {}^Y\mathrm{Ord}$ and $A \in D^+$ there is $B \subseteq A$ from D^+ such that $f{\restriction}B$ is constant.
2) If $\bar{\alpha} = \langle \alpha_s : s \in Y\rangle$ and $X_\varepsilon \subseteq Y$ for $\varepsilon < \alpha < \kappa$ and $X = \bigcup\limits_{\varepsilon} X_\varepsilon$ <u>then</u> $\mathrm{ps}-\mathrm{pcf}_{\kappa-\mathrm{comp}}(\bar{\alpha}{\restriction}X) = \bigcup\limits_{\varepsilon} \mathrm{ps}-\mathrm{pcf}_{\kappa-\mathrm{comp}}(\bar{\alpha}{\restriction}X_\varepsilon)$.

Remark 1.16. 1) Note that 1.15(1) is not empty; its assumptions hold when Y is an infinite set such that: for every $X \subseteq Y, |X| < \kappa \vee |Y \backslash X| < \kappa$ and $D = \{X \subseteq Y : |Y \backslash X| \nleq \kappa\}$.

Proof. Straightforward. $\square_{1.15}$

{comp}

§ 2. Composition and generating sequence for pseudo pcf

How much choice suffice to show $\lambda = \text{ps} - \text{tcf}(\prod\limits_{(i,j)\in Y} \lambda_{i,j}/D)$ when λ_i is the pseudo true equality of $(\prod\limits_{j\in Y_i} \lambda_{i,j}, <_{D_i})$ for $i \in Z$ where $Z = \{i : (i,j) \in Y\}$ and $Y_i = \{(i,j) : i \in Z, j \in Y_i\}$ and $\lambda =$ ps-tcf$(\prod\limits_{i\in Z} \lambda_i, <_E)$? This is 2.6, the parallel of [Sh:g, Ch.II,1.10,pg.12].

{e1}

Claim 2.1. *If* $\boxplus$ *below holds* *then* *for some partition* (Y_1, Y_2) *of* Y *and club* E *of* λ *we have*

$\oplus$ (a) *if* $Y_1 \in D^+$ *and* $f, g \in \cup\{\mathscr{F}_\alpha : \alpha \geq \min(E)\}$ *then* $f = g \bmod (D + Y_1)$

(b) *if* $Y_2 \in D^+$ *then* $\langle \mathscr{F}_\alpha : \alpha \in E\rangle$ *is* $<_{D+Y_2}$*-increasing*

where

$\boxplus$ (a) λ *is regular* $\geq \text{hrtg}(\mathscr{P}(Y)))$

(b) $\mathscr{F}_\alpha \subseteq {}^Y\text{Ord}$ *for* $\alpha < \lambda$ *is non-empty*

(c) D *is an* $\aleph_1$*-complete filter on* Y

(d) *if* $\alpha_1 < \alpha_2 < \lambda$ *and* $f_\ell \in \mathscr{F}_{\alpha_\ell}$ *for* $\ell = 1, 2$ *then* $f_1 \leq f_2 \bmod D$.

Proof. For $Z \in D^+$ let

$(*)_1$ (a) $S_Z = \{(\alpha, \beta) : \alpha \leq \beta < \lambda$ and for some $f \in \mathscr{F}_\alpha$ and $g \in \mathscr{F}_\beta$ we have $f < g \bmod (D + Z)\}$

(b) $S^+_Z = \{(\alpha, \beta) : \alpha \leq \beta < \lambda$ and for every $f \in \mathscr{F}_\alpha$ and $g \in \mathscr{F}_\beta$ we have $f < g \bmod (D + Z)\}$.

Note

$(*)_2$ (a) if $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ and $(\alpha_2, \alpha_3) \in S_Z$ then $(\alpha_1, \alpha_4) \in S_Z$

(b) similarly for S^+_Z

(c) if $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ and $(\alpha_1 \neq \alpha_2) \wedge (\alpha_3 \neq \alpha_4)$ and $(\alpha_2, \alpha_3) \in S_Z$ then $(\alpha_1, \alpha_4) \in S^+_Z$

(d) $S_Z \subseteq S^+_Z$.

[Why? By the definitions.]

Let

$(*)_3$ $J := \{Z \subseteq Y : Z \in \text{dual}(D)$ or $Z \in D^+$ and $(\forall \alpha < \lambda)(\exists \beta)((\alpha, \beta) \in S^+_Z)$.

Next

$(*)_4$ (a) J is an $\aleph_1$-complete ideal on Y

(b) if D is κ-complete then J is κ-complete[1]

(c) $J = \{Z \subseteq Y : Z \in \text{dual}(D)$ or $Z \in D^+$ and $(\forall \alpha < \lambda)(\exists \beta)$ $((\alpha, \beta) \in S_Z)\}$.

[Why? For clauses (a),(b) check and for clause (c) recall $(*)_2(c)$.]

Let

[1] not used; note that AC_κ holds in the non-trivial case as $\text{AC}_{\mathscr{P}(Y)}$ holds, see 1.15

$(*)_5$ (a) for $Z \in J^+$ let $\alpha(Z) = \min\{\alpha < \lambda$: for no $\beta \in (\alpha, \lambda)$ do we have $(\alpha, \beta) \in S_Z\}$

(b) $\alpha(*) = \sup\{\alpha_Z : Z \in J^+\}$

$(*)_6$ (a) for $Z \in J^+$ we have $\alpha(Z) < \lambda$

(b) $\alpha(*) < \lambda$.

[Why? Clause (a) by the definition of the ideal J, and clause (b) as $\lambda = \mathrm{cf}(\lambda) \geq \mathrm{hrtg}(\mathscr{P}(Y))$.]

Let

$(*)_7$ (a) for $Z \in D^+$ let $f_Z : \lambda \to \lambda + 1$ be defined by $f_Z(\alpha) = \mathrm{Min}\{\beta : (\alpha, \beta) \in S_Z^+$ or $\beta = \lambda\}$

(b) $f_* : \lambda \to \lambda$ be defined by: $f_*(\alpha) = \sup\{f_Z(\alpha) : Z \in D^+ \cap J\}$

(c) $E = \{\delta : \delta$ a limit ordinal $< \lambda$ such that $\alpha < \delta \Rightarrow f_*(\alpha) < \delta\} \backslash \alpha(*)$.

Hence

$(*)_8$ (a) if $Z \in D^+ \cap J$ then f_Z is indeed a function from λ to λ

(b) f_* is indeed a function from λ to λ

(c) f_* is non-decreasing

(d) E is a club of λ.

[Why? Clause (a) by the definition of J and of f_* and clause (b) as $\lambda = \mathrm{cf}(\lambda) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ and clause (c) by $(*)_2$ and clause (d) follows from (b)+(c).]

$(*)_9$ Let $\alpha_0 = \min(E), \alpha_1 = \min(E \backslash (\alpha_0 + 1))$ choose $f_0 \in \mathscr{F}_{\alpha_0}, f_1 \in \mathscr{F}_{\alpha_1}$ and let $Y_1 = \{y \in Y : f_0(y) = f_1(y)\}$ and $Y_2 = Y \backslash Y_1$

$(*)_{10}$ (Y_1, Y_2, E) are as required.

[Why? Think.] $\square_{2.1}$

Claim 2.2. *We have* $\lambda = \mathrm{ps} - \mathrm{tcf}(\Pi \bar{\alpha}_1, <_{D_1}) = \mathrm{ps} - \mathrm{tcf}(\Pi \bar{\alpha}, <_D)$, *this means also that one of them is well defined iff the other is, <u>when</u>* {e2}

(a) $\bar{\alpha} \in {}^Y\mathrm{Ord}$ *and* $t \in Y \Rightarrow \mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(Y)$

(b) E *is the equivalence relation on* Y *such that* $sEt \Leftrightarrow \alpha_s = \alpha_t$

(c) D *is a filter on* X

(d) $Y_1 = Y/E$

(e) $D_1 = \{Z \subseteq Y/E : \cup\{X : X \in Z\} \in D\}$, *so a filter on* Y_1

(f) $\bar{\alpha}_1 = \langle \alpha_{1,y_1} : y_1 \in Y_1 \rangle$ *where* $y_1 = y/E \Rightarrow \alpha_{1,y_1} = \alpha_y$.

Remark 2.3. We can for the "only if" direction in 2.2 weaken the demand on $\mathrm{cf}(\alpha_t)$ to $\mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(t/E)$.

Proof. The claim means

$(*)$ $\lambda = \mathrm{ps} - \mathrm{tcf}(\Pi \bar{\alpha}_1, <_{D_1})$ if and only if $\lambda = \mathrm{ps} - \mathrm{tcf}(\Pi \bar{\alpha}_2, <_{D_2})$.

First "only if" direction holds by 2.4.

Second, for the "if direction", assume that $\text{ps} - \text{pcf}(\Pi\bar{\alpha}_1, <_{D_1})$ is well defined and call it λ_1. Let $\langle \mathscr{F}_{1,\alpha} : \alpha < \lambda \rangle$ witness this, for $f \in \mathscr{F}_{1,\alpha}$ let $f^{[0]} \in {}^Y\text{Ord}$ be defined by $f^{[0]}(s) = f(s/E)$ and let $\mathscr{F}_\alpha = \{f^{[0]} : f \in \mathscr{F}_{1,\alpha}\}$. It is easy to check that $\langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$ witness $\lambda_1 = \text{ps} - \text{tcf}(\Pi\bar{\alpha}, <_D)$ recalling $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Y)$ by clause (d), so we have proved also the "if" implication. $\square_{2.2}$

By the following claims we do not really lose by using $\mathfrak{a} \subseteq \text{Reg}$ instead $\bar{\alpha} \in {}^Y\text{Ord}$ as by 2.5 below, without loss of generality $\alpha_t = \text{cf}(\alpha_t)$ (when AC_Y) and by 2.2.

{e23} **Claim 2.4.** *Assume* $\bar{\alpha} \in {}^Y\text{Ord}$, $D \in \text{Fil}(Y)$ *and* $\lambda = \text{ps-pcf}(\Pi\bar{\alpha}, <_D)$ *so* λ *is regular, and* $y \in Y \Rightarrow \alpha_y < \lambda$.

If $\langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$ *witness* $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ *and* $y \in Y \Rightarrow \text{cf}(\alpha_y) \geq \text{hrtg}(Y)$ *and* $\lambda \geq \text{hrtg}(Y)$ *<u>then</u> for some e:*

(a) $e \in \text{eq}(Y) = \{e : e$ *an equivalence relation on* $Y\}$

(b) *the sequence* $\mathscr{F}_e = \langle \mathscr{F}_{e,\alpha} : \alpha < \lambda \rangle$ *witness* $\text{ps-tcf}(\langle \alpha_{y/e} : y \in Y/e \rangle, D/e\rangle$ *where*

(c) $\alpha_{y/e} = \alpha_y, D/e = \{A/e : A \in D\}$ *where* $A/e = \{y/e : y \in A\}$ *and* $\mathscr{F}_{e,\alpha} = \{f^{[*]} : f \in \mathscr{F}_\alpha\}, f^{[*]} : Y/e \to \text{Ord}$ *is defined by* $f^{[*]}(t/e) = \sup\{f(s) : s \in t/e\}$*; noting* $\text{hrtg}(Y/e) \leq \text{hrtg}(Y)$

(d) $e = \{(s_1, s_2) : \alpha_{s_1} = \alpha_{s_2}\}$.

Proof. Let $e = \text{eq}(\bar{\alpha}) = \{(y_1, y_2) : y_1 \in Y, y_2 \in Y$ and $\alpha_{y_1} = \alpha_{y_2}\}$. For each $f \in \Pi\bar{\alpha}$ let the function $f^{[*]} \in \Pi\bar{\alpha}$ be defined by $f^{[*]}(y) = \sup\{f(z) : z \in y/e\}$. Clearly $f^{[*]}$ is a function from $\prod_{y \in Y}(\alpha_y + 1)$ and it belongs to $\Pi\bar{\alpha}$ as $y \in Y \Rightarrow \text{cf}(\alpha_y) \geq \text{hrtg}(Y) \geq \text{hrtg}(y/E)$. Let $H : \lambda \to \lambda$ be: $H(\alpha) = \min\{\beta < \lambda : \beta > \alpha$ and there are $f_1 \in \mathscr{F}_\alpha$ and $f_2 \in \mathscr{F}_\beta$ such that $f_1^{[*]} < f_2 \mod D\}$, well defined as $\mathscr{F}$ is cofinal in $(\Pi\bar{\alpha}, <_D)$. We choose $\alpha_i < \lambda$ by induction on i by: $\alpha_i = \cup\{H(\alpha_j) + 1 : j < i\}$. So $\alpha_0 = 0$ and $\langle \alpha_i : i < \lambda \rangle$ is increasing continuous. Let $\mathscr{F}'_i = \{f^{[*]} : f \in \mathscr{F}_{\alpha_i}$ and there is $g \in \mathscr{F}_{H(\alpha_i)} = \mathscr{F}_{\alpha_{i+1}-1}$ such that $f^{[*]} < g \mod D\}$.

So

$(*)_1$ $\mathscr{F}'_i \subseteq \{f \in \Pi\bar{\alpha} : \text{eq}(\bar{\alpha})$ refine $\text{eq}(f)\}$.

[By the choice of $\mathscr{F}'_i$ and of e]

$(*)_2$ $\mathscr{F}'_i$ is non-empty.

[Why? By the choice of $H(\alpha_i)$.]

$(*)_3$ if $i(1) < i(2) < \lambda$ and $h_\ell \in \mathscr{F}'_{i_\ell}$ for $\ell = 1, 2$ then $h_1 < h_2 \mod D$.

[Why? For $\ell = 1, 2$ let $g_\ell \in \mathscr{F}_{H(\alpha_{i(\ell)})}$ be such that $h_\ell = f_\ell^{[*]} < g_\ell \mod D$, exists by the definition of $\mathscr{F}'_{i(\ell)}$. But $H(\alpha_{i(1)}) < \alpha_{i(1)+1} \leq \alpha_{i(2)}$ hence $g_1 \leq f_2 \mod D$ so together $h_1 = f_1^{[*]} < g_1 \leq f_2 \leq f_2^{[*]} = h_2 \mod D$ hence we are done.]

$(*)_4$ $\bigcup_{i<\lambda} \mathscr{F}'_i$ is cofinal in $(\Pi\bar{\alpha}, <_D)$.

[Easy, too.]

Lastly, let $\mathscr{F}_i^+ = \{f/e : e \in \mathscr{F}_i'\}$ where $f/e \in {}^{Y/e}\text{Ord}$, is defined by $(f/e)(y/e) = f(y)$, clearly well defined. $\square_{2.4}$

{e29}

Claim 2.5. *Assume* AC_Y *and* $\bar\alpha_\ell = \langle \alpha_y^\ell : y \in Y\rangle \in {}^Y\text{Ord}$ *for* $\ell = 1, 2$. *If* $y \in Y \Rightarrow \text{cf}(\alpha_y^1) = \text{cf}(\alpha_y^2)$ *then* $\lambda = \text{ps-tcf}(\Pi\bar\alpha_1, <_D)$ *iff* $\lambda = \text{ps} - \text{tcf}(\Pi\bar\alpha_2, <_D)$.

Proof. Straightforward. $\square_{2.5}$

Now we come to the heart of the matter

{e3}

Theorem 2.6. *The Composition Theorem [Assume* AC_Z *and* $\kappa \geq \aleph_0$*]*

We have $\lambda = \text{ps-tcf}(\prod\limits_{(i,j)\in Y} \lambda_{i,j}, <_D)$ *and* D *is a* κ*-complete filter on* Y *when:*

(a) *E is a κ-complete filter on Z*

(b) *$\langle \lambda_i : i \in Z\rangle$ is a sequence of regular cardinals*

(c) $\lambda = \text{ps-tcf}(\prod\limits_{i\in Z} \lambda_i, <_E)$

(d) $\bar Y = \langle Y_i : i \in Z\rangle$

(e) $\bar D = \langle D_i : i \in Z\rangle$

(f) *D_i is a κ-complete filter on Y_i*

(g) *$\bar\lambda = \langle \lambda_{i,j} : i \in Z, j \in Y_i\rangle$ is a sequence of regular cardinals (or just limit ordinals)*

(h) $\lambda_i = \text{ps-tcf}(\prod\limits_{j\in Y_i} \lambda_{i,j}, <_{D_i})$

(i) $Y = \{(i,j) : j \in Y_i \text{ and } i \in Z\}$

(j) $D = \{A \subseteq Y$*: for some* $B \in E$ *we have* $i \in B \Rightarrow \{j : (i,j) \in A\} \in D_i\}$.

Proof.

$(*)_0$ D is a κ-complete filter on Y.

[Why? Straightforward (and do not need any choice).]

Let $\langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i, i \in Z\rangle$ be such that

$(*)_1$ (a) $\bar{\mathscr{F}}_i = \langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i\rangle$ witness $\lambda_i = \text{ps} - \text{tcf}(\prod\limits_{j\in Y_i} \lambda_{i,j}, <_{D_i})$

(b) $\mathscr{F}_{i,\alpha} \neq \emptyset$.

[Why? Exists by clause (h) of the assumption and AC_Z, for clause (b) recall [Sh:938, 5.6].]

By clause (c) of the assumption let $\bar{\mathscr{G}}$ be such that

$(*)_2$ (a) $\bar{\mathscr{G}} = \langle \mathscr{G}_\beta : \beta < \lambda\rangle$ witness $\lambda = \text{ps-tcf}(\prod\limits_{i\in Z} \lambda_i, <_E)$

(b) $\mathscr{G}_\beta \neq \emptyset$ for $\beta < \lambda$.

Now for $\beta < \lambda$ let

$(*)_3$ $\mathscr{F}_\beta := \{f : f \in \prod\limits_{(i,j)\in Y} \lambda_{i,j}$ and for some $g \in \mathscr{G}_\beta$ and $\bar h = \langle h_i : i \in Z\rangle \in \prod\limits_{i\in Z} \mathscr{F}_{i,g(i)}$ we have $(i,j) \in Y \Rightarrow f((i,j)) = h_i(j)\}$

$(*)_4$ the sequence $\langle \mathscr{F}_\beta : \beta < \lambda\rangle$ is well defined (so exists).

[Why? Obviously.]

$(*)_5$ if $\beta_1 < \beta_2, f_1 \in \mathscr{F}_{\beta_1}$ and $f_2 \in \mathscr{F}_{\beta_2}$ <u>then</u> $f_1 <_D f_2$.

[Why? Let $g_\ell \in \mathscr{G}_{\beta_\ell}$ and $\bar{h}_\ell = \langle h^\ell_i : i \in Z\rangle \in \prod\limits_{i \in Z} \mathscr{F}_{i,g_\ell(i)}$, witness $f_\ell \in \mathscr{F}_{\beta_\ell}$ for $\ell = 1,2$. As $\beta_1 < \beta_2$ by $(*)_2$ we have $B := \{i \in Z : g_1(i) < g_2(i)\} \in E$. For each $i \in B$ we know that $g_1(i) < g_2(i) < \lambda_i$ and so $h^1_i \in \mathscr{F}_{i,g_i(i)}, h^2_i \in \mathscr{F}_{i,g_2(i)}$; hence recalling the choice of $\langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i\rangle$, see $(*)_1$, we have $A_i \in D_i$ where for every $i \in Z$ we let $A_i := \{j \in Y_i : h^1_i(j) < h^2_i(j)\}$. As $\bar{h}_1, \bar{h}_2$ exists clearly $\langle A_i : i \in Z\rangle$ exist hence $A = \{(i,j) : i \in B$ and $j \in A_i\}$ is a well defined subset of Y and it belongs to D by the definition of D.

Lastly, $(i,j) \in A \Rightarrow f_1((i,j)) < f_2((i,j))$, shown above; so by the definition of D we are done.]

$(*)_6$ for every $\beta < \lambda$ the set $\mathscr{F}_\beta$ is non-empty.

[Why? Recall $\mathscr{G}_\beta \neq \emptyset$ by $(*)_2(b)$ and let $g \in \mathscr{G}_\beta$. As $\langle \mathscr{F}_{i,g(i)} : i \in Z\rangle$ is a sequence of non-empty sets (recalling $(*)_2(b)$), and we are assuming AC_Z there is a sequence $\langle h_i : i \in Z\rangle \in \prod\limits_{i \in Z} \mathscr{F}_{i,g(i)}$. Let f be the function with domain Y defined by $f((i,j)) = h_i(j)$; so $g, \bar{h}$ witness $f \in \mathscr{F}_\beta$, so $\mathscr{F}_\beta \neq \emptyset$ as required.]

$(*)_7$ if $f_* \in \prod\limits_{(i,j)\in Y} \lambda_{i,j}$ <u>then</u> for some $\beta < \lambda$ and $f \in \mathscr{F}_\beta$ we have $f_* < f \bmod D$.

[Why? We define $\bar{f} = \langle f^*_i : i \in Z\rangle$ as follows: f^*_i is the function with domain Y_i such that

$$j \in Y_i \Rightarrow f^*_i(j) = f((i,j)).$$

Clearly $\bar{f}$ is well defined and for each $i, f^*_i \in \prod_{j \in Y_i} \lambda_{i,j}$ hence by $(*)_1(a)$ for some $\alpha < \lambda_i$ and $h \in \mathscr{F}_{i,\alpha}$ we have $f^*_i < h \bmod D_i$ and let α_i be the first such α so $\langle \alpha_i : i \in Z\rangle$ exists.

By the choice of $\langle \mathscr{G}_\beta : \beta < \lambda\rangle$ there are $\beta < \lambda$ and $g \in \mathscr{G}_\beta$ such that $\langle \alpha_i : i \in Z\rangle < g \bmod E$ hence $A := \{i \in Z : \alpha_i < g(i)\}$ belongs to E. So $\langle \mathscr{F}_{i,g(i)} : i \in Z\rangle$ is a (well defined) sequence of non-empty sets hence recalling AC_Z there is a sequence $\bar{h} = \langle h_i : i \in Z\rangle \in \prod\limits_{i \in Z} \mathscr{F}_{i,g(i)}$. By the property of $\langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i\rangle$ and the choice of h_i recalling the definition of A, we have $i \in A \Rightarrow f^*_i < h_i \bmod D_i$, exists as $\langle h_i : i \in Z\rangle$ exist.

Lastly, let $f \in \prod\limits_{(i,j)\in Y} \lambda_{i,j}$ be defined by $f((i,j)) = h_i(j)$. Easily $g, \bar{h}$ witness that $f \in \mathscr{F}_\beta$, and by the definition of D, recalling $A \in E$ and the choice of $\bar{h}$ we have $f_* < f \bmod D$, so we are done.]

Together we are done proving the theorem. $\square_{2.6}$

{e5} *Conclusion* 2.7. <u>The pcf closure conclusion</u> Assume $\mathrm{AC}_{\mathscr{P}(\mathfrak{a})}$. We have $\mathfrak{c} = \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{c})$ <u>when</u>:

- (a) $\mathfrak{a}$ a set of regular cardinals, non-empty
- (b) $\text{hrtg}(\mathscr{P}(\mathfrak{a})) \leq \min(\mathfrak{a})$
- (c) $\mathfrak{c} = \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$.

Proof. Note that $\mathfrak{c}$ is non-empty because $\mathfrak{a} \subseteq \mathfrak{c}$.

Assume $\lambda \in$ ps-pcf$_{\aleph_1-\text{comp}}(\mathfrak{c})$, hence there is E an $\aleph_1$-complete filter on $\mathfrak{c}$ such that $\lambda =$ ps-tcf$(\Pi\mathfrak{c}, <_E)$. As we have AC$_{\mathscr{P}(\mathfrak{a})}$ by 1.3 (as the D there is unique) there is a sequence $\langle D_\theta : \theta \in \mathfrak{c}\rangle, D_\theta$ an $\aleph_1$-complete filter on $\mathfrak{a}$ such that $\theta =$ ps-tcf$(\Pi\mathfrak{a}, <_{D_\theta})$, also by 1.8 there is a function h from $\mathscr{P}(\mathfrak{a})$ onto $\mathfrak{c}$, let $E_1 = \{S \subseteq \mathscr{P}(\mathfrak{a}) : \{\theta \in \mathfrak{c} : h^{-1}\{\theta\} \subseteq S\} \in E\}$. By claim 2.2, the "if" direction with $\mathscr{P}(Y)$ here standing for Y there, we have $\lambda = \text{ps} - \text{tcf}(\Pi\{h(\mathfrak{b}) : \mathfrak{b} \in \mathscr{P}(\mathfrak{a})\}, <_{E_1})$ and E_1 is an $\aleph_1$-complete filter on $\mathscr{P}(\mathfrak{a})$.

Now we apply Theorem 2.6 with $E_1, \langle D_{h(\mathfrak{b})} : \mathfrak{b} \in \mathscr{P}(\mathfrak{a})\rangle, \lambda, \langle h(\mathfrak{b}) : \mathfrak{b} \in \mathscr{P}(\mathfrak{a})\rangle, \langle\theta : \theta \in \mathfrak{a}\rangle$ here standing for $E, \langle D_i : i \in Z\rangle, \lambda, \langle\lambda_i : i \in Z\rangle, \langle\lambda_{i,j} : j \in Y_i\rangle$ for every $j \in Z$ (constant here). We get a filter D_1 on $Y = \{(\mathfrak{b}, \theta) : \mathfrak{b} \in \mathscr{P}(\mathfrak{a}), \theta \in \mathfrak{a}\rangle$ such that $\lambda = \text{ps} - \text{tcf}(\Pi\{\theta : (\mathfrak{b}, \theta) \in Y\}, <_{D_1})$.

Now $|Y| = |\mathscr{P}(\mathfrak{a})|$ as $\mathfrak{a}$ can be well ordered (hence $\aleph_0 \leq |\mathfrak{a}|$ or $\mathfrak{a}$ finite and all is trivial) so applying 2.2 again we get an $\aleph_1$-complete filter D on $\mathfrak{a}$ such that $\lambda =$ ps-tcf$(\Pi\mathfrak{a}, <_D)$, so we are done. $\square_{2.7}$

Definition 2.8. Let a set $\mathfrak{a}$ of regular cardinals. {e10}
1) We say $\bar{\mathfrak{b}} = \langle\mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ is a generating sequence for $\mathfrak{a}$ <u>when</u>:

(α) $\mathfrak{b}_\lambda \subseteq \mathfrak{a} \subseteq \mathfrak{c} \subseteq$ ps-pcf$_{\aleph_1-\text{comp}}(\mathfrak{a})$

(β) $J_{<\lambda^+}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda$ for every $\lambda \in \mathfrak{c}$, hence for every cardinal λ we have $J_{<\lambda}[\mathfrak{a}]$ is the $\aleph_1$-complete ideal on $\mathfrak{a}$ generated by $\{\mathfrak{b}_\theta : \theta \in \text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$ and $\theta < \lambda\}$.

2) We say $\bar{\mathscr{F}}$ is a <u>witness</u> for $\bar{\mathfrak{b}} = \langle\mathfrak{b}_\lambda : \lambda \in \mathfrak{c} \subseteq \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})\rangle$ when:

(α) $\bar{\mathscr{F}} = \langle\bar{\mathscr{F}}_\lambda : \lambda \in \mathfrak{c}\rangle$

(β) $\bar{\mathscr{F}}_\lambda = \langle\mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ witness $\lambda = \text{ps} - \text{tcf}(\Pi\mathfrak{a}, <_{J_{=\lambda}[\mathfrak{a}]})$.

3) Above $\bar{\mathfrak{b}}$ is <u>closed</u> when $\mathfrak{b}_\lambda = \mathfrak{a} \cap$ ps-pcf$_{\aleph_1-\text{comp}}(\mathfrak{b}_\lambda)$; if $\mathfrak{a}$ is not mentioned it means $\mathfrak{a} = \mathfrak{c}$.
3A) Above $\bar{\mathfrak{b}}$ is <u>smooth</u> when $\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda$.
4) We say above $\bar{\mathfrak{b}}$ is <u>full</u> when $\mathfrak{c} = \text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a})$.

Remark 2.9. 1) Note that 1.8 gives sufficient conditions for the existence of $\bar{\mathfrak{b}}$ as in 2.8(1) which is full. {e11}
2) Of course, Definition 2.8 is interesting particularly when $\mathfrak{a} =$ ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{a})$.

Theorem 2.10. *Assume* AC$_\mathfrak{c}$ *and* AC$_{\mathscr{P}(\mathfrak{a})}$. <u>*Then*</u> $\mathfrak{c} =$ ps-pcf$_{\aleph_1-\text{comp}}(\mathfrak{c})$ *has a full closed generating sequence for $\aleph_1$-complete filters (see below)* <u>*when*</u>: {e7}

(a) $\mathfrak{a}$ *is a set of regular cardinals*

(b) hrtg$(\mathscr{P}(\mathfrak{a})) < \min(\mathfrak{a})$

(c) $\mathfrak{c} =$ ps-pcf$_{\aleph_1-\text{comp}}(\mathfrak{a})$.

Proof. <u>Proof of 2.10</u>

$(*)_1$ $\mathfrak{c} =$ ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{c})$.

[Why? By 2.7 using AC$_{\mathscr{P}(\mathfrak{a})}$.]

$(*)_2$ there is a generating sequence $\langle\mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ for $\mathfrak{a}$.

[Why? By 1.8(3) using also $AC_{\mathfrak{c}}$.]

$(*)_3$ let $\mathfrak{b}^*_\lambda = $ ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{b}_\lambda)$ for $\lambda \in \mathfrak{c}$.

Now

$(*)_4$ (a) $\bar{\mathfrak{b}}^* = \langle \mathfrak{b}^*_\lambda : \lambda \in \mathfrak{c}\rangle$ is well defined
(b) $\mathfrak{b}_\lambda \subseteq \mathfrak{b}^*_\lambda \subseteq \mathfrak{c}$
(c) $\mathfrak{b}^*_\lambda = $ ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{b}^*_\lambda)$
(d) $\lambda = \max(\mathfrak{b}^*_\lambda)$
(e) $\lambda \notin \text{pcf}(\mathfrak{c}\backslash\mathfrak{b}^*_\lambda)$.

[Why? First, $\bar{\mathfrak{b}}^*$ is well defined as $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ is well defined. Second, $\mathfrak{b}_\lambda \subseteq \mathfrak{b}^*_\lambda$ by the choice of $\mathfrak{b}^*_\lambda$ and $\mathfrak{b}^*_\lambda \subseteq \mathfrak{c}$ as $\mathfrak{b}_\lambda \subseteq \mathfrak{a}$ hence $\mathfrak{b}^*_\lambda = $ ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{b}^*_\lambda) \subseteq$ ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{c}) = \mathfrak{c}$, the last equality by 2.7. Third, $\mathfrak{b}^*_\lambda = $ ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{b}^*_\lambda)$ by Conclusion 2.7, it is easy to check that its assumption holds recalling $\mathfrak{b}_\lambda \subseteq \mathfrak{a}$. Fourth, $\lambda \in \mathfrak{b}^*_\lambda$ as $J_{=\lambda}[\mathfrak{a}]$ witness $\lambda \in \text{ps}-\text{pcf}_{\aleph_1-\text{com}}(\mathfrak{b}_\lambda) = \mathfrak{b}^*_\lambda$ and $\max(\mathfrak{b}^*_\lambda) = \lambda$ by $(*)_2$ recalling Definition 2.8.

Lastly, note that $\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a}) = \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{b}_\lambda) \cup \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a}\backslash\mathfrak{b}_\lambda)$ by 1.15(2) hence $\mu \in \mathfrak{c}\backslash\mathfrak{b}^*_\lambda \Rightarrow \mu \in \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{a}\backslash\mathfrak{b}_\lambda)$; so if $\lambda \in \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{c}\backslash\mathfrak{b}^*_\lambda)$ by 2.7 it follows that $\lambda \in \text{pcf}(\mathfrak{a}\backslash\mathfrak{b}^*_\lambda)$ which contradict 1.8(3), 1.6(2) so $\lambda \notin \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{c}\backslash\mathfrak{b}^*_\lambda)$ that is, clause (e) holds.]

We can now choose $\bar{\bar{\mathscr{F}}}$ such that

$(*)_5$ (a) $\bar{\bar{\mathscr{F}}} = \langle \bar{\mathscr{F}}_\lambda : \lambda \in \mathfrak{c}\rangle$
(b) $\bar{\mathscr{F}}_\lambda = \langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$
(c) $\bar{\mathscr{F}}_\lambda$ witness $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <_{J_{=\lambda}[\mathfrak{a}]})$
(d) if $\lambda \in \mathfrak{a}, \alpha < \lambda$ and $f \in \mathscr{F}_{\lambda,\alpha}$ then $f(\lambda) = \alpha$.

[Why? For each λ there is such $\bar{\bar{\mathscr{F}}}$ as $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <_{J_{=\lambda}[\mathfrak{a}]})$. But we are assuming $AC_{\mathfrak{c}}$ and for clause (d) it is easy; in fact it is enough to use $AC_{\mathscr{P}(\mathfrak{a})}$ and h as in 2.7, getting $\langle \bar{\mathscr{F}}_{\mathfrak{b}} : \mathfrak{b} \in \mathscr{P}(\mathfrak{a})\rangle, \bar{\mathscr{F}}_{\mathfrak{b}}$ witness $h(\mathfrak{b}) = $ ps-tcf$(\Pi\mathfrak{a}, <_{J_{=\lambda}}[\mathfrak{a}])$ and putting $\langle \bar{\mathscr{F}}_{\mathfrak{b}} : \mathfrak{b} \in h^{-1}\{\lambda\}\rangle$ together for each $\lambda \in \mathfrak{c}$.]

$(*)_6$ (a) for $\lambda \in \mathfrak{c}$ and $f \in \Pi\mathfrak{b}_\lambda$ let $f^{[\lambda]} \in \Pi\mathfrak{b}^*_\lambda$ be defined by: $f^{[\lambda]}(\theta) =$
$\min\{\alpha < \lambda$: for every $g \in \mathscr{F}_{\theta,\alpha}$ we have
$f{\restriction}\mathfrak{b}_\lambda \le (g{\restriction}\mathfrak{b}_\lambda) \bmod J_{=\theta}[\mathfrak{b}_\lambda]\}$
(b) for $\lambda \in \mathfrak{c}$ and $\alpha < \lambda$ let $\mathscr{F}^*_{\lambda,\alpha} = \{(f{\restriction}\mathfrak{b}_\lambda)^{[\lambda]} : f \in \mathscr{F}_{\lambda,\alpha}\}$.

Now

$(*)_7$ (a) $f^{[\lambda]}{\restriction}\mathfrak{a} \ge f$ for $f \in \Pi\mathfrak{b}_\lambda, \lambda \in \mathfrak{c}$
(b) $\langle \mathscr{F}^*_{\lambda,\alpha} : \lambda \in \mathfrak{c}, \alpha < \lambda\rangle$ is well defined (hence exist)
(c) $\mathscr{F}^*_{\lambda,\alpha} \subseteq \Pi\mathfrak{b}^*_\lambda$.

[Why? Obvious, e.g. for clause (a) note that $\theta \in \mathfrak{a} \Rightarrow \{\theta\} \in (J_{=\theta}[\mathfrak{b}_\lambda])^+$.]

$(*)_8$ let J_λ be the $\aleph_1$-complete ideal on $\mathfrak{b}^*_\lambda$ generated by $\{\mathfrak{b}^*_\theta \cap \mathfrak{b}^*_\lambda : \theta \in \mathfrak{c} \cap \lambda\}$

$(*)_9$ $J_\lambda \subseteq J^{\aleph_1-\text{comp}}_{<\lambda}[\mathfrak{b}^*_\lambda]$.

[Why? As for $\theta_0, \dots, \theta_n \dots \in \mathfrak{c} \cap \lambda$ by 1.15(2) we have $\text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\cup\{\mathfrak{b}^*_{\theta_n} : n < \omega\}) = \cup\{\text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{b}^*_{\theta_n}) : n < \omega\} = \cup\{\mathfrak{b}^*_{\theta_n} : n < \omega\} \in J^{\aleph_1-\text{comp}}_{<\lambda}[\mathfrak{c}]$.]

$\odot_1$ if $\lambda \in \mathfrak{c}$ and $\alpha_1 < \alpha_2 < \lambda$ and $f_\ell \in \mathscr{F}_{\lambda,\alpha_\ell}$ for $\ell = 1,2$ <u>then</u> $f_1^{[\lambda]} \le f_2^{[\lambda]}$ mod J_λ.

[Why? Let $\mathfrak{a}_* = \{\theta \in \mathfrak{b}_\lambda : f_1(\theta) \ge f_2(\theta)\}$, hence by the assumption on $\langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ we have $\mathfrak{a}_* \in J^{\aleph_1-\text{comp}}_{<\lambda}[\mathfrak{a}]$, hence we can find a sequence $\langle \theta_n : n < \mathbf{n} \le \omega\rangle$ such that $\theta_n \in \mathfrak{c} \cap \lambda$ and $\mathfrak{a}_* \subseteq \mathfrak{b}_* := \cup\{\mathfrak{b}_{\theta_n} : n < \mathbf{n}\}$ hence $\mathfrak{c}_* := \text{ps} - \text{pcf}_{\aleph_1-\text{com}}(\mathfrak{a}_*) \subseteq \cup\{\mathfrak{b}^*_{\theta_n} : n < \mathbf{n}\} \in J_\lambda$. So it suffices to prove $f_1^{[\lambda]} {\restriction} (\mathfrak{b}^*_\lambda \backslash \mathfrak{c}_*) \le f_2^{[\lambda]} {\restriction} (\mathfrak{b}^*_\lambda \backslash \mathfrak{c}_*)$, so let $\theta \in \mathfrak{b}^*_\lambda \backslash \bigcup\limits_n \mathfrak{b}^*_{\theta_n}$, by $(*)_4(d)$ we have $\theta \le \lambda$, let $\alpha := f_2^{[\lambda]}(\theta)$, so by the definition of $f_2^{[\lambda]}(\theta)$ we have $(\forall g \in \mathscr{F}_{\theta,\alpha})((f_2 {\restriction} \mathfrak{b}_\lambda) \le (g {\restriction} \mathfrak{b}_\lambda) \bmod J_{=\theta}[\mathfrak{b}_\lambda])$. But $\mathfrak{a}_* \subseteq \bigcup\limits_n \mathfrak{b}_{\theta_n}$ and $n < \omega \Rightarrow \theta \notin \mathfrak{b}^*_{\theta_n} = \text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{b}_{\theta_n})$ hence by 1.15(2) we have $\theta \notin \text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\bigcup\limits_n \mathfrak{b}_{\theta_n})$ hence $\bigcup\limits_n \mathfrak{b}_{\theta_n} \in J^{\aleph_1-\text{comp}}_{<\theta}[\mathfrak{b}_\lambda]$ hence $\mathfrak{a}_* \in J^{\aleph_1-\text{comp}}_{=\theta}[\mathfrak{b}_\lambda]$. So (first inequality by the previous sentence and the choice of $\mathfrak{a}_*$, second by the earlier sentence)

$$(f_1 {\restriction} \mathfrak{b}_\lambda) \le (f_2 {\restriction} \mathfrak{b}_\lambda) \le (g {\restriction} \mathfrak{b}_\lambda) \mod J^{\aleph_1-\text{comp}}_{=\theta}[\mathfrak{b}_\lambda]$$

hence by the definition of $f_1^{[\lambda]}, f_1^{[\lambda]}(\theta) \le \alpha = f_2^{[\lambda]}(\theta)$. So we are done.]

$\odot_2$ if $\lambda \in \mathfrak{c}$ and $g \in \Pi\mathfrak{b}^*_\lambda$ <u>then</u> for some $\alpha < \lambda$ and $f \in \mathscr{F}_{\lambda,\alpha}$ we have $g < f$ mod J_λ.

[Why? We choose $\langle h_\theta : \theta \in \mathfrak{b}^*_\lambda\rangle$ such that $h_\theta \in \mathscr{F}_{\theta,g(\theta)}$ for each $\theta \in \mathfrak{b}^*_\lambda$; this is possible as we are assuming $\text{AC}_{\mathfrak{c}}$ and $\mathfrak{b}^*_\lambda \subseteq \mathfrak{c}$. Let $h_1 \in \Pi\mathfrak{b}^*_\lambda$ be defined by $h_1(\kappa) = \sup\{h_\theta^{[\lambda]}(\kappa) : \kappa \in \mathfrak{b}_\theta \text{ and } \theta \in \mathfrak{b}^*_\lambda\}$ for $\kappa \in \mathfrak{b}^*_\lambda$, the result is $< \kappa$ because the supremum is on $\le |\mathfrak{b}_\theta|$ ordinals and $\kappa \ge \min(\mathfrak{b}^*_\lambda) \ge \min(\mathfrak{c}) = \min(\mathfrak{a}) \ge \text{hrtg}(\mathscr{P}(\mathfrak{a}))$. Hence there are $\alpha < \lambda$ and $h_2 \in \mathscr{F}_{\lambda,\alpha}$ such that $h_1 \le h_2$ mod $J_{=\lambda}[\mathfrak{a}]$. Now $f := h_2^{[\lambda]} \in \Pi\mathfrak{b}^*_\lambda$ recalling $(*)_7(a)$ is as required, in particular $f \in \mathscr{F}^*_{\lambda,\alpha}$.]

$\odot_3$ the sequence $\langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ witness $\lambda = \text{ps} - \text{tcf}(\Pi\mathfrak{b}^*_\lambda, <_{J_\lambda})$.

[Why? In $(*)_7(b),(c) + \odot_1 + \odot_2$.]

$\odot_4$ if $\lambda \in \mathfrak{c}$ then $J_{<\lambda} = J^{\aleph_1-\text{comp}}_{<\lambda}[\mathfrak{b}^*_\lambda]$.

[Why? By $(*)_4, (*)_8, (*)_9$ and $\odot_3$.]

So

$\odot_5$ $\bar{\mathfrak{b}}^* = \langle \mathfrak{b}^*_\lambda : \lambda \in \mathfrak{c}\rangle$ is a generating sequence for $\mathfrak{c}$.

[Why? By $\odot_4, (*)_8$ recalling that $\lambda \notin \text{ps} - \text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{c} \backslash \mathfrak{b}^*_\lambda)$ by $(*)_4(e)$.] $\square_{2.10}$

{e30}

Remark 2.11. Clearly $\bar{\mathfrak{b}}^*$ is full and closed, but what about smoooth? Is this necessary for generalizing [Sh:460]?

{e31}

Discussion 2.12. Naturally the definition now of $\bar{\mathscr{F}}$ as in 2.8(2) for $\Pi\mathfrak{a}$ is more involved where $\bar{\mathscr{F}} = \langle \bar{\mathscr{F}}_\lambda : \lambda \in \text{ps-pcf}_{\kappa-\text{com}}(\mathfrak{a})\rangle, \bar{\mathscr{F}}_\lambda = \langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ exemplifies $\text{ps-tcf}(\Pi\mathfrak{a}, J_{=\lambda}(\mathfrak{a}))$.

{e33}

Claim 2.13. *[DC + AC$_{<\kappa}$] Assume*

(*a*) $\mathfrak{a}$ *a set of regular cardinals*

(*b*) κ *is regular* $> \aleph_0$

(*c*) $\mathfrak{c} = \text{ps-pcf}_{\kappa-\text{comp}}(\mathfrak{a})$

(*d*) $\min(\mathfrak{a})$ *is* $\geq \text{hrtg}(\mathscr{P}(\mathfrak{c}))$ *or at least* $\geq \text{hrtg}(\mathfrak{c})$

(*e*) $\bar{\mathscr{F}} = \langle \bar{\mathscr{F}}_\lambda : \lambda \in \mathfrak{c}\rangle, \bar{\mathscr{F}}_\lambda = \langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ *witness*[2] $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <^{J^{\kappa-\text{comp}}}_{=\lambda}[\mathfrak{a}])$.

Then

⊞ *for every* $f \in \Pi\mathfrak{a}$ *for some* $g \in \Pi\mathfrak{c}$*, if* $g \leq g_1 \in \Pi\mathfrak{c}$ *and* $\bar{h} \in \Pi\{\mathscr{F}_{\lambda,g_1(\lambda)} : \lambda \in \mathfrak{c}\}$ *then* $(\exists \mathfrak{d} \in [\mathfrak{c}]^{<\kappa})(f < \sup\{h_\lambda : \lambda \in \mathfrak{d}\})$.

Proof. Let $f \in \Pi\mathfrak{a}$. For each $\lambda \in \text{ps-pcf}_{\kappa-\text{com}}(\mathfrak{a})$ let $\alpha_{f,\lambda} = \min\{\alpha < \lambda : f < g \mod J_{=\lambda}[\mathfrak{a}]$ for every $g \in \mathscr{F}_{\lambda,\alpha}\}$, so clearly each α_f is well defined hence $\bar{\alpha} = \langle \alpha_{f,\lambda} : \lambda \in \text{ps-pcf}_{\kappa-\text{com}}(\mathfrak{a})\rangle$ exists. So $g = \langle \alpha_{f,\lambda} : \lambda \in \mathfrak{c}\rangle \in \Pi\mathfrak{c}$ is well defined. Assume $g_1 \in \Pi\mathfrak{c}$ and $g \leq g_1$. Let $\langle h_\lambda : \lambda \in \mathfrak{c}\rangle$ be any sequence from $\prod\limits_{\lambda\in\mathfrak{c}} \mathscr{F}_{\lambda,g_1(\lambda)}$, at least one exists when AC$_\mathfrak{c}$ holds but this is not needed here. Let $\mathfrak{a}_{f,\lambda} = \{\theta \in \mathfrak{a} : f(\theta) < h_\lambda(\theta)\}$ so $\langle \mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{c}\rangle$ exists and we claim that for some $\mathfrak{d} \in [\mathfrak{c}]^{<\kappa}$ we have $\mathfrak{a} = \cup\{\mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{d}\}$. Otherwise let J be the κ-complete ideal on $\mathfrak{a}$ generated by $\{\mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{c}\}$, it is a κ-complete ideal. So by [Sh:938, 5.9=r9], applicable by our assumptions, there is a κ-complete ideal J_1 on $\mathfrak{a}$ extending J such that $\lambda_* = \text{ps-tcf}(\Pi\mathfrak{a}, <_{J_1})$ is well defined. So $\lambda_* \in \mathfrak{c}$ and $\mathfrak{a}_{f,\lambda_*} \in J_1$, easy contradiction. $\square_{2.13}$

{e40}

Claim 2.14. *[AC$_{\aleph_0}$] We can uniformly define*[3] *a* $\aleph_0$*-continuous witness for* $\lambda = \text{ps}-\text{pcf}_{\kappa-\text{comp}}(\bar{\Pi}\bar{\alpha}, <_D)$ *where:*

(*a*) $\bar{\alpha} \in {}^Y\text{Ord}$

(*b*) *each* α_t *is a limit ordinal with* $\text{cf}(\alpha_t) \geq \text{hrtg}(S)$

(*c*) λ *is regular* $\geq \text{hrtg}(S)$

(*d*) $\bar{\mathscr{F}} = \langle \bar{\mathscr{F}}_a : a \in S\rangle$ *satisfies: each* $\bar{\mathscr{F}}_a$ *is a witness for* $\lambda = \text{pcf}_{\kappa-\text{comp}}(\Pi\bar{\alpha}, <_D)$

(*e*) *if* $a \in S$ *then* $\bar{\mathscr{F}}_a$ *is* $\aleph_0$*-continuous and* $f_1, f_2 \in \mathscr{F}_{a,\alpha} \Rightarrow f_1 = f_2 \mod D$.

Proof.

$(*)_0$ $\text{hrtg}(S \times S)$ is $\leq \lambda$ and $\leq \text{cf}(\alpha_t)$ for $t \in Y$.

[Why? As $\lambda, \text{cf}(\alpha_t)$ are regular cardinals.]

For $a, b \in S$ let

$(*)_1$ (*a*) $E_{a,b} = \{\delta < \lambda$: if $\alpha < \delta$ then for some $\beta \in (\alpha,\delta)$ and $f_1 \in \mathscr{F}_{a,\alpha}$, $f_2 \in \mathscr{F}_{b,\beta}$ we have $f_1 < f_2 \mod D\}$

(*b*) define $g_{a,b} : \lambda \to \lambda$ by $g_{a,b}(\alpha) = \min\{\beta < \lambda$: there are $f_1 \in \mathscr{F}_{a,\alpha}$ and $f_2 \in \mathscr{F}_{b,\beta}$ such that $f_1 < f_2 \mod D\}$

$(*)_2$ $g_{a,b}$ is well defined.

[2] So we are assuming it is well defined, now if AC$_{\mathscr{P}(Y)}$ such $\bar{\mathscr{F}}$ exists.

[3] Of course, mere existence is already given by the assumptions.

[Why? As $\bar{\mathscr{F}}_b$ is cofinal in $(\Pi\bar{\alpha}, <_D)$.]

$(*)_3$ $g_{a,b}$ is non-decreasing.

[Why? As $\bar{\mathscr{F}}_a$ is $<_D$-increasing.]

Hence

$(*)_4$ $E_{a,b} = \{\delta < \lambda : \delta$ a limit ordinal and $(\forall\alpha < \delta)(g_{a,b}(\alpha) < \delta)\}$.

Also

$(*)_5$ $E_{a,b}$ is a club of λ.

[Why? By its defintion, $E_{a,b}$ is a closed subset of λ and it is unbounded as $\mathrm{cf}(\lambda) = \lambda > \aleph_0$, because for every $\alpha < \lambda$ letting $\alpha_0 = \alpha, \alpha_{n+1} = g_{a,b}(\alpha_n) + 1 < \lambda$ clearly $\beta := \cup\{\alpha_n : n < \omega\}$ is $< \lambda$ and $\gamma < \delta \Rightarrow (\exists n)(\gamma < \alpha_n) \Rightarrow (\exists n)(g_{a,b}(\gamma) < \alpha_{n+1}))$.]

$(*)_6$ let $g : \lambda \to \lambda$ be $g(\alpha) = \sup\{g_{a,b}(\alpha) : a, b \in S\}$

$(*)_7$ g is a (well defined) non-decreasing function from λ to λ.

[Why? "Non-decreasing trivial", and it is "into λ" as $\mathrm{hrtg}(S \times S) \le \lambda$ recalling $(*)_0$.]

$(*)_8$ $E = \cap\{E_{a,b} : a, b \in S\} = \{\delta < \lambda : (\forall\alpha < \delta)(g(\alpha) < \delta)\}$ is a club of λ.

[Why? Like $(*)_7$.]

$(*)_9$ let $E_1 = \{\delta \in E : \mathrm{cf}(\delta) = \aleph_0\}$ so $E_1 \subseteq \lambda = \sup(\lambda), \mathrm{otp}(E_1) = \lambda$

$(*)_{10}$ for $\delta \in E$ of cofinality $\aleph_0$ let $\mathscr{F}_\delta = \{\sup\{f_n : n < \omega\}$: for some $a \in S$ and $\bar{\alpha} = \langle\alpha_n : n < \omega\rangle$ increasing of cofinality $\aleph_0$ we have $\langle f_n : n < \omega\rangle \in \prod\limits_n \mathscr{F}_{a,\alpha_n}\rangle\}$

$(*)_{11}$ $\langle\mathscr{F}_\delta : \delta \in E_1\rangle$ is $<_D$-increasing cofinal in $(\Pi\bar{\alpha}, <_D)$ in particular $\mathscr{F}_\delta \ne \emptyset$.

[Why? $\mathscr{F}_\delta \ne \emptyset$ as $\delta \in E, \mathrm{cf}(\delta) = \aleph_0$ and $\mathrm{AC}_{\aleph_0}$.]

We can correct $\langle\mathscr{F}_\delta : \delta \in E_1\rangle$ to be $\aleph_0$-continuous easily (and as in [Sh:938, §5]). $\square_{2.14}$ {2.14}

Question 2.15. 1) Can we in 2.5 get smoothness?
2) If 2.10 does it suffice to assume $\mathrm{AC}_{\mathscr{P}(\mathfrak{a})}$ (and omit $\mathrm{AC}_{\mathfrak{a}}$) and we can conclude that $\mathfrak{c} = \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{c})$ has a full closed generating sequence.

We may try to repeat the proof of 2.10, only in the proof of $(*)_5$ we use claim 2.16 below. {e46}

Claim 2.16. *In 2.10 we can add "$\bar{\mathfrak{b}}$ is weakly smooth" which means $\theta \in \mathfrak{b}_\lambda \Rightarrow \theta \notin \mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{c}\backslash\mathfrak{b}_*)$.*

Proof. Let $\bar{\mathfrak{b}} = \langle\mathfrak{b}_\lambda : \lambda \in \mathfrak{c}\rangle$ be a full closed generating sequence.

We choose $\mathfrak{b}^1_\lambda$ by induction on $\lambda \in \mathfrak{c}$ such that

$(*)_1$ (a) $J_{\le\lambda}[\mathfrak{a}] = J^{\aleph_1-\mathrm{comp}}_{\le\lambda}[\mathfrak{a}] + \mathfrak{b}^1_\lambda$

(b) $\mathrm{ps}-\mathrm{pcf}_{\aleph_1-\mathrm{comp}}(\mathfrak{b}^1_\lambda) = \mathfrak{b}^1_\lambda$

(c) $\max(\mathfrak{b}^1_\lambda) = \lambda$

(d) if $\theta \in \mathfrak{b}^1_\lambda$ then $\mathfrak{b}^1_\lambda \supseteq \mathfrak{b}_\theta$ mod $J_{=\lambda}[\mathfrak{a}]$, i.e. $\mathfrak{b}^1_\theta\backslash\mathfrak{b}^1_\lambda \in J^{\aleph_1-\mathrm{comp}}_{<\theta}[\mathfrak{a}]$.

Arriving to λ let $\mathfrak{d}_\lambda = \{\theta \in \mathfrak{b}_\lambda : \mathfrak{b}^1_\theta \backslash \mathfrak{b}^1_\lambda \notin J^{\aleph_1-\text{comp}}_{<\theta}[\mathfrak{a}]\}, \mathfrak{d}^1_\lambda = \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{d}_\lambda)$.

Now

$(*)_2$ $\text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{d}_\lambda) \subseteq \mathfrak{b}_\lambda \cap \lambda$.

[Why? $\subseteq \mathfrak{b}_\lambda$ is obvious; recalling $\mathfrak{b}^1_\lambda = \text{ps}-\text{pcf}(\mathfrak{b}^1_\lambda \cap \mathfrak{a})$ because $\bar{\mathfrak{b}}$ is closed. If "$\not\subseteq \lambda$" recall $\mathfrak{d}^1_\lambda = \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{d}_\lambda)$, now $\mathfrak{d}_\lambda \subseteq \mathfrak{b}_\lambda$ hence $\mathfrak{d}^1_\lambda \subseteq \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{b}_\lambda) \subseteq \lambda^+$. So the only problematic case is $\lambda \in \mathfrak{d}^1_\lambda = \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{d}_\lambda)$. But then, $\mathfrak{d}_\lambda \subseteq \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{c}\backslash\mathfrak{b}_\lambda)$ by the definition of $\mathfrak{d}_\lambda$ hence by the composition theorem we have $\lambda \in \text{ps}-\text{pcf}_{\aleph_1-\text{comp}}(\mathfrak{c}\backslash\mathfrak{b}_\lambda)$, contradicting an assumption on $\bar{\mathfrak{b}}$.]

$(*)_3$ there is a countable $\mathfrak{e}_\lambda \subseteq \mathfrak{d}^1_\lambda$ such that $\mathfrak{d}^1_\lambda \subseteq \cup\{\mathfrak{b}^1_\sigma : \sigma \in \mathfrak{e}_\lambda\}$.

[Why? Should be clear.]

Lastly, let $\mathfrak{b}^1_\lambda = \cup\{\mathfrak{b}^1_\theta : \theta \in \mathfrak{e}_\lambda\} \cup \mathfrak{b}_\lambda$ and check. $\square_{2.16}$

§ 3. Measuring reduced products

{meas}

{onps}

§ 3(A). On ps-$\mathbf{T}_D(g)$.

Now we consider some ways to measure the size of ${}^\kappa\mu/D$ and show that they essentially are equal; see Discussion 3.9 below.

{r26}

Definition 3.1. Let $\bar\alpha = \langle \alpha_t : t \in Y\rangle \in {}^Y\text{Ord}$ be such that $t \in Y \Rightarrow \alpha_t > 0$.
1) For D a filter on Y let ps-$\mathbf{T}_D(\bar\alpha) = \sup\{\text{hrtg}(\mathbf{F}) : \mathbf{F}$ is a family of non-empty subsets of $\Pi\bar\alpha$ such that for every $\mathscr{F}_1 \neq \mathscr{F}_2$ from $\mathbf{F}$ we have $f_1 \in \mathscr{F}_1 \wedge f_2 \in \mathscr{F}_2 \Rightarrow f_1 \neq_D f_2\}$, recalling $f_1 \neq_D f_2$ means $\{t \in Y : f_1(t) \neq f_2(t)\} \in D$.
2) Let ps-$\mathbf{T}_{\kappa-\text{comp}}(\bar\alpha) = \sup\{\text{hrtg}(\mathbf{F})$: for some κ-complete filter D on Y, $\mathbf{F}$ is as above for $D\}$.
3) If we allow $\alpha_t = 0$ just replace $\Pi\bar\alpha$ by $\Pi^*\bar\alpha := \{f : f \in \prod\limits_t(\alpha_t+1)$ and $\{t : f(t) = \alpha_t\} = \emptyset \bmod D\}$.

{r29}

Theorem 3.2. *[DC + $\text{AC}_{\mathscr{P}(Y)}$] Assume that D is a κ-complete filter on Y and $\kappa > \aleph_0$ and $g \in {}^Y(\text{Ord}\setminus\{0\})$, if g is constantly α we may write α. The following cardinals are equal <u>or</u> at least $\lambda_1, \lambda_2, \lambda_3$ are $\text{Fil}^1_\kappa(D)$-almost equal which means:*

for $\ell_1, \ell_2 \in \{1,2,3\}$ we have $\lambda_{\ell_1} \leq^{\text{sal}}_{\text{Fil}^1_\kappa(D)} \lambda_{\ell_2}$ which means if $\alpha < \lambda_{\ell_1}$ then α is included in the union of S sets each of order type $< \lambda_{\ell_2}$:

- (a) $\lambda_1 = \sup\{|\text{rk}_{D_1}(g)|^+ : D_1 \in \text{Fil}^1_\kappa(D)\}$
- (b) $\lambda_2 = \sup\{\lambda^+$: *there are $D_1 \in \text{Fil}^1_\kappa(D)$ and a $<_{D_1}$-increasing sequence $\langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$ such that $\mathscr{F}_\alpha \subseteq \prod\limits_{t\in Y} g(t)$ is non-empty*$\}$
- (c) $\lambda_3 = \sup\{\text{ps} - \mathbf{T}_{D_1}(g) : D_1 \in \text{Fil}^1_\kappa(D)\}$.

{r30}

Remark 3.3. 1) Recall that for D a κ-complete filter on Y we let $\text{Fil}^1_\kappa(D) = \{E : E$ is a κ-complete filter on Y extending $D\}$.
2) The conclusion gives slightly less than equality of $\lambda_1, \lambda_1, \lambda_3$.
3) See 3.10(6) below, by it $\lambda_2 = \text{ps-Depth}^+({}^\kappa\mu, <_D)$ recalling 3.10(5).
4) We may replace κ-complete by $(\leq Z)$-complete if $\aleph_0 \leq |Z|$.
5) Compare with Definition 3.10.
6) Note that those cardinals are $\leq \text{hrtg}(\Pi^* g)$, see 3.1(3).

Proof. <u>Stage A</u>: $\lambda_1 \leq^{\text{sal}}_{\text{Fil}^1_\kappa(D)} \lambda_2, \lambda_3$.

Why? Let $\chi < \lambda_1$, so by clause (a) there is $D_1 \in \text{Fil}^1_\kappa(D)$ such that $\text{rk}_{D_1}(g) \geq \chi$. Let $X_{D_2} = \{\alpha < \chi$: some $f \in \prod\limits_{t\in Y} g(t)$ satisfies[4] $D_2 = \text{dual}(J[f, D_1])$ and $\alpha = \text{rk}_{D_1}(f)\}$, for any $D_2 \in \text{Fil}^1_\kappa(D_1)$. By [Sh:938, 1.11(5)] we have $\chi = \bigcup\{X_{D_2} : D_2 \in \text{Fil}^1_\kappa(D_1)\}$.

Now

- $\odot$ $D_2 \in \text{Fil}^1_\kappa(D_1) \Rightarrow |\text{otp}(X_{D_2})| < \lambda_2, \lambda_3$; this is enough.

[4] recall $\text{dual}(J[f, D_1]) = \{X \subseteq Y : X \in D_1$ or $\text{rk}_{D_1+(X\setminus Y)}(f) > \text{rk}_{D_1}(f)\}$.

Why does this hold? Letting $\mathscr{F}_{D_2,i} = \{f \in {}^Y\mu : \text{rk}_{D_1}(f) = i \text{ and } J[f,D_1] = \text{dual}(D_2)\}$, by [Sh:938, 1.11(2)] we have: $i < j \wedge i \in X_{D_2} \wedge j \in X_{D_2} \wedge f \in \mathscr{F}_{D_2,i} \wedge g \in \mathscr{F}_{D_2,j} \Rightarrow f < g \mod D_2$ so by the definitions of λ_2, λ_3 we have $\text{otp}(X_{D_2}) < \lambda_2, \lambda_3$.

Stage B: $\lambda_2 \le^{\text{sal}}_{\text{Fil}^1_\kappa(D)} \lambda_1, \lambda_3$, moreover $\lambda_2 \le \lambda_1, \lambda_3$.

Why? Let $\chi < \lambda_2$ and let D_1 and $\langle \mathscr{F}_\alpha : \alpha < \chi \rangle$ exemplify $\chi < \lambda_2$. Let $\gamma_\alpha = \min\{\text{rk}_{D_1}(f) : f \in \mathscr{F}_\alpha\}$ so easily $\alpha < \beta < \chi \Rightarrow \gamma_\alpha < \gamma_\beta$ hence $\text{rk}_D(g) \ge \chi$. So $\chi < \lambda_1$ by the definition of λ_1 and as for $\chi < \lambda_3$ this holds by Definition 3.1(2) as $\alpha < \beta \wedge f \in \mathscr{F}_\alpha \wedge g \in \mathscr{F}_\beta \Rightarrow f < g \mod D_1 \Rightarrow f \ne g \mod D_1$ as $\chi^+ = \text{hrtg}(\chi) \le \lambda_3$.

Stage C: $\lambda_3 \le^{\text{sal}}_{\text{Fil}^1_\kappa(D)} \lambda_1, \lambda_2$.

Why? Let $\chi < \lambda_3$. Let $\langle \mathscr{F}_\alpha : \alpha < \chi \rangle$ exemplify $\chi < \lambda_3$. For each $\alpha < \chi$ let $\mathbf{D}_\alpha = \{\text{dual}(J[f,D]) : f \in \mathscr{F}_\alpha\}$ so a non-empty subset of $\text{Fil}^1_\kappa(Y)$. Now for every $D_1 \in \mathbf{D}_* := \cup\{\mathbf{D}_\alpha : \alpha < \lambda\}$ let $X_{D_1} = \{\alpha < \chi : D_1 \in \mathbf{D}_\alpha\}$ and for $\alpha \in X_{D_1}$ let $\zeta_{D_1,\alpha} = \min\{\text{rk}_D(f) : f \in \mathscr{F}_\alpha \text{ and } D_1 = \text{dual}(J[f,D])\}$ and let $\mathscr{F}_{D_1,\alpha} = \{f \in \mathscr{F}_\alpha : D_1 = J[f,D] \text{ and } \text{rk}_{D_1}(f) = \zeta_{D_1,\alpha}\}$ so a non-empty subset of $\mathscr{F}_\alpha$ and clearly $\langle (\zeta_{D_1,\alpha}, \mathscr{F}_{D_1,\alpha}) : \alpha \in X_{D_1} \rangle$ exists.

Now

- (a) $\alpha \mapsto \zeta_{D_1,\alpha}$ is a one-to-one function with domain X_{D_1} for $D_1 \in \mathbf{D}_*$
- (b) $\chi = \cup\{X_{D_1} : D_1 \in \mathbf{D}_*\}$ noting $\mathbf{D}_* \subseteq \text{Fil}^1_\kappa(D)$
- (c) for $D \in \mathbf{D}_*$ if $\alpha < \beta$ are from X_{D_1} and $\zeta_{D_1,\alpha} < \zeta_{D_1,\beta}, f \in \mathscr{F}_{D_1,\alpha}, g \in \mathscr{F}_{D_2,\beta}$ then $f < g \mod D_1$.

[Why? For clause (a), if $\alpha \ne \beta \in X_{\zeta_1}, f \in \mathscr{F}_{D_1,\alpha}, g \in \mathscr{F}_{D_1,\beta}$ then $f \ne g \mod D$ hence by [Sh:938, 1.11] we have $\zeta_{D_1,\alpha} \ne \zeta_{D_1,\beta}$. For clause (b), it follows by the choices of $\mathbf{D}_\alpha, X_{D_1}$. Lastly, clause (c) follows by [Sh:938, 1.11(2)].]

Hence (by clause (c))

- (d) $\text{otp}(X_{D_1})$ is $< \lambda_2$ and is $\le \text{rk}_{D_1}(g)$ for $D_1 \in \cup\{\mathbf{D}_\alpha : \alpha < \chi\} \subseteq \text{Fil}^1_\kappa(D)$.

Together clause (d) shows that $D \in \mathbf{D}_* \Rightarrow |X_D| < \lambda_1, \lambda_2$ so by clause (b), $\lambda_3 \le^{\text{sal}}_{\text{Fil}^1_*(D)} \lambda_1, \lambda_2$ hence we are done. $\square_{3.2}$

{r31f}

Observation 3.4. If D is a filter on Y and $\bar{a} \in {}^Y(\text{Ord}\backslash\{0\})$ then

$\text{ps} - \mathbf{T}_D(\bar{a}) = \sup\{\lambda^+$: there is a sequence $\langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$ such that $\mathscr{F}_\alpha \subseteq \Pi\bar{a}, \mathscr{F}_\alpha \ne \emptyset$ and $\alpha \ne \beta \wedge f_1 \in \mathscr{F}_\alpha \wedge f_2 \in \mathscr{F}_\beta \Rightarrow f_1 \ne_D f_2\}$.

Proof. Clearly the new definition gives a cardinal $\le \text{ps} - \mathbf{T}_D(\bar{a})$. For the other inequality assume $\lambda < \text{ps} - \mathbf{T}_D(\bar{a})$ so there is $\mathbf{F}$ as there such that $\lambda < \text{hrtg}(\mathbf{F})$. As $\lambda < \text{hrtg}(\mathbf{F})$ there is a function h from $\mathbf{F}$ onto λ. For $\alpha < \lambda$ define $\mathscr{F}'_\alpha = \cup\{\mathscr{F} : \mathscr{F} \in \mathbf{F} \text{ and } h(\mathscr{F}) = \alpha\}$. So $\langle \mathscr{F}'_\alpha : \alpha < \lambda \rangle$ exists and is as required. $\square_{3.4}$

Concerning Theorem 3.2 we may wonder "when does λ_1, λ_2 being S-almost equal implies they are equal". We consider a variant this time for sets (or powers, not just cardinals).

{r32}

Definition 3.5. 1) We say "the power of $\mathscr{U}_1$ is S-almost smaller than the power of $\mathscr{U}_2$", or write $|\mathscr{U}_1| \le |\mathscr{U}_2| \mod S$ or $|\mathscr{U}_1| \le^{\text{alm}}_S |\mathscr{U}_2|$ when: we can find a sequence $\langle u_{1,s} : s \in S \rangle$ such that $\mathscr{U}_1 = \cup\{u_{1,s} : s \in S\}$ and $s \in S \Rightarrow |\mathscr{U}_{1,s}| \le |\mathscr{U}_2|$.

2) We say the power $|\mathscr{U}_1|, |\mathscr{U}_2|$ are S-almost equal (or $|\mathscr{U}_1| = |\mathscr{U}_2|$ mod S or $|\mathscr{U}_1| =^{\rm alm}_S |\mathscr{U}_2|$) <u>when</u> $|\mathscr{U}_1| \le^{\rm alm}_S |\mathscr{U}_2| \le^{\rm alm}_S |\mathscr{U}_2|$.
3) Let $|\mathscr{U}_1| \le^{\rm alm}_{<S} |\mathscr{U}_2|$ be defined naturally.
4) In particular this applies to cardinals.
5) Let $|\mathscr{U}_1| <^{\rm alm}_S |\mathscr{U}_2|$ means there is a sequence $\langle u_{1,s} : s \in S\rangle$ with union $\mathscr{U}_1$ such that $s \in S \Rightarrow |\mathscr{U}_s| < |\mathscr{U}_2|$.
6) Let $|\mathscr{U}_1| \le^{\rm sal}_S |\mathscr{U}_2|$ means that if $|\mathscr{U}| < |\mathscr{U}_1|$ then $|\mathscr{U}| <^{\rm alm}_S |\mathscr{U}_2|$.

{r34}

Observation 3.6. 1) If $|\mathscr{U}_1| \le |\mathscr{U}_2|$ and $S \ne \emptyset$ <u>then</u> $|\mathscr{U}_1| \le^{\rm alm}_S |\mathscr{U}_2|$.
2) If $\lambda_1 \le \lambda_2$ and $S \ne \emptyset$ then $\lambda_1 \le^{\rm sal}_S \lambda_2$.
3) If $\lambda_2 = \lambda_1^+$ and $\mathrm{cf}(\lambda_2) < \mathrm{hrtg}(S)$ then the power of λ_2 is S - almost smaller than S.

Proof. Immediate. $\square_{3.6}$

{r36}

Observation 3.7. 1) The cardinals λ_1, λ_2 are equal <u>when</u> $\lambda_1 =^{\rm alm}_S \lambda_2$ and $\mathrm{cf}(\lambda_1)$, $\mathrm{cf}(\lambda_2) \ge \mathrm{hrtg}(\mathscr{P}(S))$.
2) The cardinals λ_1, λ_2 are equal <u>when</u> $\lambda_1 =^{\rm alm}_S \lambda_2$ and λ_1, λ_2 are limit cardinals $> \mathrm{hrtg}(\mathscr{P}(S))$.
3) If $\lambda_1 \le^{\rm alm}_S \lambda_2$ and $\partial = \mathrm{hrtg}(\mathscr{P}(S))$ <u>then</u> $\lambda_1 \le^{\rm alm}_{<\partial} \lambda_2$.
4) If $\lambda_1 \le^{\rm alm}_{<\theta} \lambda_2$ and $\mathrm{cf}(\lambda_1) \ge \theta$ <u>then</u> $\lambda_1 \le \lambda_2$.
5) If $\lambda_1 \le^{\rm alm}_{<\theta} \lambda_2$ and $\theta \le \lambda_2^+$ <u>then</u> $\lambda_1 \le \lambda_2^+$.

Proof. 1) Otherwise, let $\partial = \mathrm{hrtg}(\mathscr{P}(S))$, without loss of generality $\lambda_2 < \lambda_1$ and by part (3) we have $\lambda_1 \le^{\rm alm}_{<\partial} \lambda_2$ and by part (4) we have $\lambda_1 \le \lambda_2$ contradiction.
2) Otherwise letting $\partial = \mathrm{hrtg}(\mathscr{P}(S))$ without loss of generality $\lambda_2 < \lambda_1$ and by part (3) we have $\lambda_1 \le^{\rm alm}_{<\partial} \lambda_2$ but $\partial < \lambda_2$ is assume and $\lambda_2^+ < \lambda_1$ as λ_2 is a limit cardinal so together we get contradiction to part (5).
3) If $\langle u_s : S \in S\rangle$ witness $\lambda_1 \le^{\rm alm}_S \lambda_2$, let $w = \{\alpha < \lambda_1$: for no $\beta < \alpha$ do we have $(\forall s \in S)(\alpha \in u_s \equiv \beta \in u_s)\}$ so clearly $|w| < \mathrm{hrtg}(\mathscr{P}(S)) = \theta$ and for $\alpha \in w$ let $v_\alpha = \{\beta < \lambda_1 : (\forall s \in S)(\alpha \in u_s \equiv \beta \in u_s)\}$ so $\langle v_\alpha : \alpha \in w\rangle$ witness $\lambda_1 \le^{\rm alm}_w \lambda_2$ hence $\lambda_1 \le^{\rm alm}_{<\theta} \lambda_2$.
4),5) Let $\sigma < \theta$ be such that $\lambda_1 \le^{\rm alm}_\sigma \lambda_2$ and let $\langle u_\varepsilon : \varepsilon < \sigma\rangle$ witness $\lambda_1 \le^{\rm alm}_\sigma \lambda_2$, that is $|u_\varepsilon| \le \lambda_2$ for $\varepsilon < \sigma$ and $\cup\{u_\varepsilon : \varepsilon < \sigma\} = \lambda_1$.

For part (4), if $\lambda_2 < \lambda_1$, then we have $\varepsilon < \sigma \Rightarrow |u_\varepsilon| < \lambda_1$, but $\mathrm{cf}(\lambda_1) > \sigma$ hence $|\{\cup\{u_\varepsilon : \varepsilon < \sigma\}| < \lambda_1$, contradiction.

For part (5) for $\varepsilon < \sigma$, let $u'_\varepsilon = u_\varepsilon \backslash \cup \{u_\zeta : \zeta < \varepsilon\}$ and so $\mathrm{otp}(u'_\varepsilon) \le \mathrm{otp}(u_\varepsilon) < |u_\varepsilon|^+ \le \lambda_2^+$ so easily $|\lambda_1| = |\cup\{u_\varepsilon : \varepsilon < \sigma\}| = |\cup\{u'_\varepsilon : \varepsilon < \sigma\}| \le \sigma \cdot \lambda_2^+ \le \lambda_2^+ \cdot \lambda_2^+ = \lambda_2^+$. $\square_{3.7}$

Similarly

{r37}

Observation 3.8. 1) If $\lambda_1 <^{\rm alm}_S \lambda_2$ and $\partial = \mathrm{hrtg}(\mathscr{P}(S))$ <u>then</u> $\lambda_1 <^{\rm alm}_{<\partial} \lambda_2$.
2) If $\lambda_1 <^{\rm alm}_{<\theta} \lambda_2$ and $\mathrm{cf}(\lambda_1) \ge \theta$ <u>then</u> $\lambda_1 < \lambda_2$.
3) If $\lambda_1 <^{\rm alm}_{<\theta} \lambda_2$ and $\theta \le \lambda_2^+$ <u>then</u> $\lambda_1 \le \lambda_2$.
4) If $\lambda_1 \le^{\rm sal}_S \lambda_2$ and $\partial = \mathrm{hrtg}(\mathscr{P}(S))$ <u>then</u> $\lambda_1 <^{\rm sal}_{<\partial} \lambda_2$.
5) If $\lambda_1 \le^{\rm sal}_{<\theta} \lambda_2$ and $\partial \le \lambda_2^+, \theta < \lambda_2$ and $\mathrm{cf}(\lambda_2) \ge \theta$ <u>then</u> $\lambda_1 \le \lambda_2$.
6) If $\lambda_1 \le^{\rm sal}_{<\theta} \lambda_2$ and $\theta \le \lambda_2^+$ <u>then</u> $\lambda_1 \le \lambda_2^+$.

Proof. Similar, e.g.
1) Like the proof of 3.7(3). $\square_{3.8}$

{r38}

Discussion 3.9. 1) We like to measure $({}^Y\mu)/D$ in some ways and show their equivalence, as was done in ZFC. Natural candidates are:

- (A) $\mathrm{pp}_D(\mu)$: say of length of increasing sequence $\bar{P}$ (not $\bar{p}$!, i.e. sets) ordered by $<_D$
- (B) $\mathrm{pp}^+_Y(\mu) = \sup\{\mathrm{pp}^+_D(\mu) : D$ an $\aleph_1$-complete filter on $Y\}$
- (C) As in 3.1.

2) We may measure ${}^Y\mu$ by considering all ∂-complete filters.
3) We may be more lenient in defining "same cardinality". E.g.

- (A) we define when sets have similar powers say by divisions to $\mathscr{P}(\mathscr{P}(Y))$ sets we measure $({}^Y\mu)/\approx_{\mathscr{P}(\mathscr{P}(Y))}$ where $\approx_B$ is the following equivalence relation on sets:
  $X \approx_B Y$ when we can find sequences $\langle X_b : b \in B\rangle, \langle Y_b : b \in B\rangle$ such that:
  - (a) $X = \cup\{X_b : b \in B\}$
  - (b) $Y = \cup\{Y_b : b \in B\}$
  - (c) $|X_b| = |Y_b|$
- (B) we may demand more: the $\langle X_b : b \in B\rangle$ are pairwise disjoint and the $\langle Y_b : b \in B\rangle$ are pairwise disjoint
- (C) we may demand less: e.g.
  - $(c)'$ $|X_b| \le_* |Y_b| \le_* |X_b|$
    and/or
  - $(c)_*$ $(\forall b \in B)(\exists c \in B)(|X_b| \le |Y_c|)$ and
    $(\forall b \in B)(\exists c \in B)(|Y_b| \le |X_c|)$.

Note that some of the main results of [Sh:835] can be expressed this way.

- (D) $\text{rk-sup}_{Y,\partial}(\mu) = \text{rk-sup}\ \{\mathrm{rk}_D(\mu) : D$ is ∂-complete filters on $Y\}$
- (E) for each non-empty $X \subseteq {}^Y\mu$ let

$$\mathrm{sp}^1_\alpha(X) = \{(D, J) : D \text{ an } \aleph_1\text{-complete filter on } Y, J = J[f, D], \alpha = \ \mathrm{rk}_D(f) \text{ and } f \in X\}$$

$$\mathrm{sp}_1(X) = \cup\{\mathrm{sp}^1_\alpha(X) : \alpha\}$$

- (F) question: If $\{\mathrm{sp}(X_s) : s \in S\}$ is constant, can we bound J?
- (G) X, Y are called connected when $\mathrm{sp}(X_1), \mathrm{sp}(X_2))$ are non-disjoint or equal.

4) We hope to prove, at least sometimes $\gamma := \Upsilon({}^Y\mu) \le \mathrm{pp}_\kappa(\mu)$ that is we like to immitate [Sh:835] without the choice axioms on ${}^\omega\mu$. So there is $\bar{f} = \langle f_\alpha : \alpha < \delta\rangle$ witnessing $\gamma < \Upsilon({}^Y\mu)$. We define $u = u_{\bar{f}} = \{\alpha$: there is no $\bar{\beta} \in {}^\omega\alpha$ such that $(\forall it \in Y)(f_\alpha(t) \in \{f_{\beta_n}(t) : n < \omega\})$. You may say that $u_{\bar{f}}$ is the set of $\alpha < \delta$ such that f_α is "really novel".

By DC this is O.K., i.e.

- $\boxplus_1$ for every $\alpha < \delta$ there is $\bar{\beta} \in {}^\omega(u_{\bar{f}} \cap \alpha)$ such that $(\forall t \in Y)(f_\alpha(t)) = \{f_{\beta_n}(t) : n < \omega\}$.

Next for $\alpha \in u_{\bar f}$ we can define $D_{\bar f,\alpha}$, the $\aleph_1$-complete filter on Y generated by $\Big\{\{t \in Y : f_\beta(t) = f_\alpha(t)\} : \beta < \alpha\Big\}$. So clearly $\alpha \ne \beta \in u_{\bar f} \wedge D_{\bar f,\alpha} = D_{\bar f,\beta} \Rightarrow f_\alpha \ne_D f_\beta$. Now for each pair $\bar D = (D_1, D_2) \in \text{Fil}^4_Y$ (i.e. for the $\aleph_1$-complete case) let $\Lambda_{\bar f,\bar D} = \{\alpha \in u_{\bar f} : D_{\bar f,\alpha} = D_1$ and $J[f_\alpha, D_1]\} = \text{dual}(D_2)$. So γ is the union of $\le \mathscr{P}(\mathscr{P}(Y))$-sets (as $|Y| = |Y| \times |Y|$, well ordered.

So

$(*)_1$ $\gamma \le \text{hrtg}(({}^Y\omega \times {}^\omega(\mu))$

$(*)_2$ u is the union of $\mathscr{P}(\mathscr{P}(\kappa))$-sets each of cardinality $<$ $\text{pp}^+_{Y,\aleph_1}(\mu)$

(I) what about $\text{hrtg}({}^\kappa\mu) <$ $\text{ps-pp}_{Y,\aleph_1}(\mu)$?

We are given $\langle \mathscr{F}_\alpha : \alpha < \kappa\rangle \ne F_\alpha \ne \emptyset, \mathscr{F}_\alpha \subseteq \mu, \alpha \ne \beta \Rightarrow \mathscr{F}_\alpha \cap \mathscr{F}_\beta = \emptyset$.

Easier: looking modulo a fix filter D.

$(*)_2$ for $D \in$ $\text{Fil}_{Y,\aleph_1}$, let $\mathscr{F}_{\alpha,D} = \{f \in \mathscr{F}_\alpha : \neg(\exists g \in \mathscr{F}_\alpha)(g <_D f)\}$.

Maybe we have somewhere a bound on the size of $\mathscr{F}_{\alpha,D}$.

§ 3(B). **Depth of Reduced Power of Ordinals.** {depth}

Our intention has been to generalize a relative of [Sh:460], but actually we are closed to [Sh:513, §3]. So as there we use IND but unlike [Sh:938] rather than with rank we deal with depth.

Definition 3.10. {k1} 1) Let $\text{suc}_X(\alpha)$ be the first ordinal β such that we cannot find a sequence $\langle \mathscr{U}_x : x \in X\rangle$ of subsets of β, each of order type $< \alpha$ such that $\beta = \cup\{\mathscr{U}_x : x \in X\}$.

2) We define $\text{suc}^{[\varepsilon]}_X(\alpha)$ by induction on ε naturally: if $\varepsilon = 0$ it is α, if $\varepsilon = \zeta + 1$ it is $\text{suc}_X(\text{suc}^{[\zeta]}_X(\alpha))$ and if ε is a limit ordinal then it is $\cup\{\text{suc}^{[\zeta]}_X(\alpha) : \zeta < \varepsilon\}$.

3) For a quasi-order P let the pseudo ordinal depth of P, denoted by ps-o-Depth(P) be $\sup\{\gamma$: there is a $<_P$-increasing sequence $\langle X_\alpha : \alpha < \gamma\rangle$ of non-empty subsets of $P\}$.

4) o-Depth(P) is defined similarly demanding $|X_\alpha| = 1$ for $\alpha < \gamma$.

5) Omitting the "ordinal" means γ is replaced by $|\gamma|$; similarly in the other variants including omitting the letter o in ps-o-Depth.

6) Let $\text{ps-o-Depth}^+(P) = \sup\{\gamma + 1$: there is an increasing sequence $\langle X_\alpha : \alpha < \gamma\rangle$ of non-empty subsets of $P\}$. Similarly for the other variants, e.g. without o we use $|\gamma|^+$ instead of $\gamma + 1$ in the supremum.

7) For D a filter on Y and $\bar\alpha \in {}^Y(\text{Ord} \setminus \{0\})$ let $\text{ps-o-Depth}^+_D(\bar\alpha) = \text{ps-o-Depth}^+(\Pi\bar\alpha, <_D)$. Similarly for the other variants and we may allow $\alpha_t = 0$ as in 3.1(3).

8) Let $\text{ps-o-depth}^+_D(\bar\alpha)$ be the cardinality of $\text{ps-o-Depth}^+_D(\bar\alpha)$.

Remark 3.11. Note that 1.14 can be phrased using this definition.

Definition 3.12. {k4} 0) We say $\mathbf{x}$ is a filter ω-sequence <u>when</u> $\mathbf{x} = \langle (Y_n, D_n) : n < \omega\rangle = \langle Y_{\mathbf{x},n}, D_{\mathbf{x},n} : n < \omega\rangle$ is such that D_n is a filter on Y_n for each $n < \omega$; we may omit Y_n as it is $\cup\{Y : Y \in D\}$ and may write D if $\bigwedge_n D_n = D$.

1) Let IND$(\mathbf{x}), \mathbf{x}$ has the independence property, mean that for every sequence $\bar F = \langle F_{m,n} : m < n < \omega\rangle$ from alg$(\mathbf{x})$, see below, there is $\bar t \in \prod_{n<\omega} Y_n$ such that $m < n < \omega \Rightarrow t_m \notin F_{m,n}(\bar t{\restriction}(m,n])$. Let NIND$(\mathbf{x})$ be the negation.

2) Let alg($\mathbf{x}$) be the set of sequence $\langle F_{n,m} : m < n < \omega\rangle$ such that $F_{m,n} : \prod\limits_{\ell=m+1}^{n} Y_\ell \to$ dual(D_n).

3) We say $\mathbf{x}$ is κ-complete when each $D_{\mathbf{x},n}$ is a κ-complete filter.

{k6} **Theorem 3.13.** *Assume* IND($\mathbf{x}$) *where* $\mathbf{x} = \langle (Y_n, D_n) : n < \omega\rangle$ *is as in Definition 3.12,* D_n *is* κ_n*-complete,* $\kappa_n \geq \aleph_1$.

*1) [*DC + AC$_{Y_n}$ *for* $n < \omega$*] For every ordinal* ζ*, for infinitely many* n*'s* ps-o-Depth$({}^{(Y_n)}\zeta, <_{D_n}) \leq \zeta$.

*2) [*DC*] For every ordinal* ζ *for infinitely many* n*,* o-Depth$({}^{(Y_n)}\zeta, <_{D_n}) \leq \zeta$*, equivalently there is no* $<_{D_n}$*-increasing sequence of length* $\zeta + 1$.

{k7} *Remark* 3.14. 0) Note that the present results are incomparable with [Sh:938, §4] - the loss is using depth instead of rank and possibly using "pseudo".

1) [Assume AC$_{\aleph_0}$] If for every n we have $\mathrm{rk}_{D_n}(\zeta) >$ $\mathrm{suc}_{\mathrm{Fil}^1_\kappa(D_n)}(\zeta)$ then for some $D^1_n \in$ $\mathrm{Fil}^1_{\aleph_1}(Y_n)$ for $n < \omega$ we have NIND($\langle Y_n, D^1_n) : n < \omega\rangle$. (Why? By [Sh:938, 5.9]). But we do not know much on the D^1_n's.

2) This theorem applies to e.g. $\zeta = \aleph_\omega, Y_n = \aleph_n, D_n =$ dual$(J^{\mathrm{bd}}_{\aleph_n})$. So even in ZFC, it tells us things not covered by [Sh:513, §3]. So it also tells us that it is easy by forcing to get, e.g. NIND($\langle(\aleph_{n+1}, \mathrm{dual}(J^{\mathrm{bd}}_{\aleph_{n+1}})) : n < \omega\rangle$), see 3.19. Note that Depth and pcf are closely connected but only for sequences of length $\geq$ hrtg$(\mathscr{P}(Y))$ and see 3.19 below.

3) If we assume IND($\langle Y_{\eta(n)}, D_{\eta(n)} : n < \omega\rangle$) for every increasing $\eta \in {}^\omega\omega$, which is quite reasonable then in Theorem 3.13 we can strengthen the conclusion, replacing "for infinitely many n's" by "for every $n < \omega$ large enough".

4) Note that 3.13(2) is complimentary to [Sh:835].

{k8} **Observation 3.15.** 1) If $\mathbf{x}$ is a filter ω-sequence, $\mathbf{x}$ is $\aleph_1$-complete and $n_* < \omega$ and IND($\mathbf{x}{\restriction}[n_*, \omega)$ then IND($\mathbf{x}$).

2) If $\mathbf{x}$ is a filter ω-sequence and IND($\mathbf{x}$) and $\eta \in {}^\omega\omega$ is increasing and $\mathbf{y} = \langle Y_{\mathbf{x},\eta(n)}, D_{\mathbf{x},\eta(n)} : n < \omega\rangle$ then $\mathbf{y}$ is a filter ω-sequence and IND($\mathbf{y}$).

Proof. 1) Let $\bar F = \langle F_{n,m} : n < m < \omega\rangle \in$ alg($\mathbf{x}$), so $\langle F_{n,m} : n \in [n_*, \omega)$ and $m \in (n, \omega)\rangle$ belongs to alg($\mathbf{x}{\restriction}[n_*, \omega)$ hence by the assumption "IND($\mathbf{x}{\restriction}[n_*, \omega)$)" there is $\bar t = \langle t_n : n \in [n_*, \omega)\rangle \in \prod\limits_{n \geq n_*} Y_n$ such that $t_n \notin F_{n,m}(\bar t{\restriction}(n, m))$ when $n_* \leq n < \omega$. Now by downward induction on $n < n_*$ we choose $t_n \in Y_n$ such that $t_n \notin F_{n,m}(\langle \bar t{\restriction}[n+1, m])$ for $m \in [n+1, \omega)$. This is possible as the countable union of members of dual(D_n) is not equal to Y_n. We can carry the induction and $\langle t_n : n < \omega\rangle$ is as required to verify IND($\mathbf{x}$).

2) Let $\bar F = \langle F_{i,j} : i < j < \omega\rangle \in$ alg($\mathbf{y}$). For $m < n$ we define $F'_{m,n}$ as the following function from $\prod\limits_{k=m-1}^{n} Y_{\mathbf{x},k}$ into dual$(D_{\mathbf{x},m})$ by

- if $i < j, m = \eta(i), n = \eta(j)$ and $\bar s = \langle s_k : k \in (m, n]\rangle \in \prod\limits_{k=m+1}^{n} Y_{\mathbf{x},k}$ then $F'_{m,n}(\bar s) = F_{i,j}(\langle s_{\eta(i+k)} : k \in [1, j - i)]\rangle)$
- if there are no such i, j then $F_{m,n}$ is constantly $\emptyset$.

As IND($\mathbf{x}$) holds there is $\bar t \in \prod\limits_n Y_{\mathbf{x},k}$ such that $m < n \Rightarrow t_m \notin F_{m,n}(\bar t{\restriction}(m, n))$. Now $\bar t' = \langle t_{\eta(k)} : k < \omega\rangle \in \prod\limits_n Y_{\mathbf{x},\eta(n)} = \prod\limits_n Y_{\mathbf{y},n}$ is necessarily as required. $\square_{3.15}$

Proof. Proof of Theorem 3.13

We concentrate on proving part (1), part (2) is easier, (i.e. below each $\mathscr{F}_{n,\varepsilon}$ is a singleton hence so is $\mathscr{G}^1_{m,n,\varepsilon}$ so there is no need to use AC_{Y_n}).

Assume this fails. So for some $n_* < \omega$ for every $n \in [n_*, \omega)$ there is a counterexample. As $\mathrm{AC}_{\aleph_0}$ holds we can find a sequence $\langle \bar{\mathscr{F}}_n : n \in [n_*, \omega) \rangle$ such that:

$\odot$ for $n \in [n_*, \omega)$
- (a) $\bar{\mathscr{F}}_n = \langle \mathscr{F}_{n,\varepsilon} : \varepsilon \le \zeta \rangle$
- (b) $\mathscr{F}_{n,\varepsilon} \subseteq {}^{Y_n}\zeta$ is non-empty
- (c) $\bar{\mathscr{F}}_n$ is a $<_{D_n}$-increasing sequence of sets, i.e. $\varepsilon_1 < \varepsilon_2 \le \zeta \wedge f_1 \in \mathscr{F}_{n,\varepsilon_1} \wedge f_2 \in \mathscr{F}_{n,\varepsilon_2} \Rightarrow f_1 <_{D_n} f_2$.

Now by $\mathrm{AC}_{\aleph_0}$ we can choose $\langle f_n : n \in [n_*, \omega) \rangle$ such that $f_n \in \mathscr{F}_{n,\zeta}$ for $n \in [n_*, \omega)$.

$(*)$ without loss of generality $n_* = 0$.

[Why? As $\mathbf{x}{\restriction}[n_*, \omega)$ satisfies the assumptions on $\mathbf{x}$ by 3.15(2).]

Now

$\boxplus_1$ for $m \le n < \omega$ let $Y^0_{m,n} = \prod\limits_{\ell=m}^{n-1} Y_\ell$ and for $m, n < \omega$ let $Y^1_{m,n} := \cup\{Y^0_{k,n} : k \in [m,n]\}$ so $Y^0_{m,n} = \emptyset = Y^1_{m,n}$ if $m > n$ and $Y^0_{m,n} = \{<>\} = Y^1_{m,n}$ if $m = n$; so if $\eta \in Y^0_{m+1,n}$ and $s \in Y_m, t \in Y_{n+1}$ we define $\langle s \rangle \hat{}\, \eta \in Y^0_{m,n}$ and $\eta \hat{}\, \langle t \rangle \in Y_{m+1,n+1}$ naturally

$\boxplus_2$ for $m \le n$ let $\mathscr{G}^1_{m,n}$ be the set of functions g such that:
- (a) g is a function from $Y^1_{m,n}$ into $\zeta + 1$
- (b) $\langle\rangle \ne \eta \in Y^1_{m,n} \Rightarrow g(\eta) < \zeta$
- (c) if $k \in [m,n)$ and $\eta \in Y^0_{k+1,n}$ then the sequence $\langle g(\langle s \rangle \hat{}\, \eta) : s \in Y_k \rangle$ belongs to $\mathscr{F}_{k,g(\eta)}$

$\boxplus_3$ $\mathscr{G}^1_{m,n,\varepsilon} := \{g \in \mathscr{G}^1_{m,n} : g(\langle\rangle) = \varepsilon\}$ for $\varepsilon \le \zeta$ and $m \le n < \omega$.

Now the sets $\mathscr{G}^1_{m,n}$ are non-trivial, i.e.

$\boxplus_4$ if $m \le n$ and $\varepsilon \le \zeta$ then $\mathscr{G}^1_{m,n,\varepsilon} \ne \emptyset$.

[Why? We prove it by induction on n; first if $n = m$ this is trivial because the unique function g with domain $\{<>\}$ and value ε belongs to $\mathscr{G}^1_{m,n,\varepsilon}$. Next, if $m < n$ we choose $f \in \mathscr{F}_{n-1,\varepsilon}$ hence the sequence $\langle \mathscr{G}^1_{m,n-1,f(s)} : s \in Y_{n-1} \rangle$ is well defined and by the induction hypothesis each set in the sequence is non-empty. As $\mathrm{AC}_{Y_{n-1}}$ holds there is a sequence $\langle g_s : s \in Y_{n-1} \rangle$ such that $s \in Y_{n-1} \Rightarrow g_s \in \mathscr{G}^1_{m,n-1,f(s)}$. Now define g as the function with domain $Y^1_{m,n}$:

$$g(\langle\rangle) = \varepsilon$$

$$g(\nu \hat{}\, \langle s \rangle) = g_s(\nu) \text{ for } \nu \in Y^1_{m,n-1} \text{ and } s \in Y_n.$$

It is easy to check that $g \in \mathscr{G}^1_{m,n,\varepsilon}$ indeed so $\boxplus_4$ holds.]

$\boxplus_5$ if $g, h \in \mathscr{G}^1_{m,n}$ and $g(\langle\rangle) < h(\langle\rangle)$ <u>then</u> there is an (m,n)-witness Z for (h,g) which means (just being an (m,n)-witness means we omit clause (d)):

- (a) $Z \subseteq Y^1_{m,n}$ is closed under initial segments, i.e. if $\eta \in Y^0_{k,n} \cap Z$ and $m \le k < \ell \le n$ then $\eta{\restriction}[\ell,n) \in Y^0_{\ell,n} \cap Z$
- (b) $\langle\rangle \in Z$
- (c) if $\eta \in Z \cap Y^0_{k+1,n}, m \le k < n$ then $\{s \in Y_k : \langle s\rangle\char`^\eta \in Z\} \in D_k$
- (d) if $\eta \in Z$ then $g(\eta) < h(\eta)$.

[Why? By induction on n, similarly to the proof of $\boxplus_4$.]

$\boxplus_6$ (a) we can find $\bar{g} = \langle g_n : n < \omega\rangle$ such that $g_n \in \mathscr{G}^1_{0,n,\zeta}$ for $n < \omega$

(b) for $\bar{g}$ as above for $n < \omega, s \in Y_n$ let $g_{n+1,s} \in \mathscr{G}^1_{0,n}$ be defined by $g_{n+1,s}(\nu) = g_{n+1}(\nu\char`^\langle s\rangle)$ for $\nu \in Y_{0,n}$.

[Why? Clause (a) by $\boxplus_4$ as $\mathrm{AC}_{\aleph_0}$ holds, clause (b) is obvious by the definitions in $\boxplus_2 + \boxplus_3$.]

We fix $\bar{g}$ as in $\boxplus_6(a)$ for the rest of the proof.

$\boxplus_7$ There is $\langle\langle Z_{n,s} : s \in Y_n\rangle : n < \omega\rangle$ such that $Z_{n,s}$ witness $(g_n, g_{n+1,s})$ for $n < \omega, s \in Y_n$.

[Why? For a given $n < \omega, s \in Y_n$ we know that $g_{n+1}(\langle s\rangle) < \zeta = g_n(\langle\rangle)$ hence $Z_{n,s}$ as required exists by $\boxplus_5$. By AC_{Y_n} for each n a sequence $\langle Z_{n,s} : s \in Y_n\rangle$ as required exists, and by $\mathrm{AC}_{\aleph_0}$ we are done.]

$\boxplus_8$ $Z_n := \{\langle\rangle\} \cup \{\nu\char`^\langle s\rangle : s \in Y_{n-1}, \nu \in Z_{n-1,s}\}$ is a $(0,n)$-witness.

[Why? By our definitions.]

$\boxplus_9$ there is $\bar{F}$ such that:

- (a) $\bar{F} = \langle F_{m,n} : m < n < \omega\rangle$
- (b) $F_{m,n} : Y^1_{m+1,n+1} \to \mathrm{dual}(D_m)$
- (c) $F_{m,n}(\nu)$ is $\{s \in Y_m : \nu\char`^\langle s\rangle \notin Z_{n-1}\}$ when $\nu \in Z_n$ and is $\emptyset$ otherwise.

[Why? As clauses (a),(b),(c) define $\bar{F}$.]

$\boxplus_{10}$ $\bar{F}$ witness $\mathrm{IND}(\langle(Y_n, D_n) : n < \omega\rangle)$ fail.

[Why? Clearly $\bar{F} = \langle F_{m,n} : m < n < \omega\rangle$ has the right form.

So toward contradiction assume $\bar{t} = \langle t_n : n < \omega\rangle \in \prod\limits_{n<\omega} Y_n$ is such that

$(*)_1$ $m < n < \omega \Rightarrow t_m \notin F_{m,n}(\bar{t}{\restriction}(m,n])$.

Now

$(*)_2$ $\bar{t}{\restriction}[m,n) \in Z_n$ for $m \le n < \omega$.

[Why? For each n, we prove this by downward induction on m. If $m = n$ then $\bar{t}{\restriction}[m,n) = \langle\rangle$ but $\langle\rangle \in Z_n$ by its definition. If $m < n$ and $\bar{t}{\restriction}[m+1,n) \in Z_n$ then $t_m \notin F_{m,n-1}(\bar{t}{\restriction}(m,n])$ by $(*)_1$ so $\bar{t}{\restriction}[m,n) = \langle t_m\rangle\char`^(\bar{t}{\restriction}[m+1,n)) \in Z_n$ holds by clause $\boxplus_9(c)$.]

$(*)_3$ $g_{n+1}(\bar{t}{\restriction}[m,n]) < g_n(\bar{t}{\restriction}[m,n))$.

[Why? Note that Z_{n,t_n} is a witness for (g_n, g_{n+1,t_n}) by $\boxplus_7$. So by $\boxplus_5$ (see clause (d) there) we have $\eta \in Z_{n,t_n} \Rightarrow g_{n+1,t_n}(\eta) < g_n(\eta)$. But $m < n \Rightarrow \bar{t}{\restriction}[m,n] \in Z_{n+1} \Rightarrow \bar{t}{\restriction}[m,n) \in Z_{n,t_n}$, the first implication by $(*)_2$, the second implication by the definition of Z_{n+1} in $\boxplus_8$. Hence by $\boxplus_6(b)$ and the last sentence, and by the sentence before last $g_{n+1}(\bar{t}{\restriction}[m,n]) = g_{n+1,t_n}(\bar{t}{\restriction}[m,n)) < g_n(\bar{t}{\restriction}[m,n))$ as required. So $(*)_3$ holds indeed.]

So for each $m < \omega$ the sequence $\langle g_n(\bar{t}{\restriction}[m,n) : n \in [m,\omega)\rangle$ is a decreasing sequence of ordinals, contradiction. Hence there is no $\bar{t}$ as above, so indeed $\boxplus_{10}$ holds. But $\boxplus_{10}$ contradicts an assumption, so we are done. $\square_{3.13}$

{k10}

Remark 3.16. 1) Note that in the proof of 3.13 there was no use of completeness demands, still natural to assume $\aleph_1$-completeness because: if D'_n is the $\aleph_1$-completion of D_n then $\text{IND}(\langle D'_n : n < \omega\rangle)$ is equivalent to $\text{IND}(D_n : n < \omega)$.
2) Recall that by [Sh:513, 2.7], iff $\text{pp}(\aleph_\omega) > \aleph_{\omega_1}$ then for every $\lambda > \aleph_\omega$ for infinitely many $n < \omega$ we have $(\forall \mu < \lambda)(\text{cf}(\mu) = \aleph_n \Rightarrow \text{pp}(\mu) \le \lambda)$.
3) Concerning 3.17 below recall that:

- (A) if Y_n is a regular cardinal, D_n witness Y_n is a measurable cardinal, then clause (a) of 3.17 holds, but [Sh:938, §4] says more
- (B) if $\mu = \mu^{<\mu}$ and $\mathbb{P}_\mu$ is the Levy collapse a measurable cardinal $\lambda > \mu$ to be μ^+ with D a normal ultrafilter on λ, then $\Vdash_{\mathbb{P}_\mu}$ "the filter which D generates is as required in (b) with μ in the role of Z_n", by Jech-Magidor-Mitchel-Prikry [JMMP80].

So we can force that $n < \omega \Rightarrow Y_n = \aleph_{2n}$.
4) So

- (a) if $\text{pp}(\aleph_\omega) > \aleph_{\omega_1}$ and $\aleph_\omega$ divides $\delta, \text{cf}(\delta) < \aleph_\omega$ and $\delta < \aleph_\delta$ then $\text{pp}(\aleph_\delta) < \aleph_{|\delta|^+}$
- (b) we can replace $\aleph_\omega$ by any singular $\mu < \aleph_\mu$
- (c) if, e.g. $\delta_n < \lambda_n = \aleph_{\delta_n}, \delta_n < \delta_{n+1}$ and $\text{cf}(\delta_n) < \aleph_{\delta_0}$ for $n < \omega$, then, except for at most one $n, \text{pp}(\aleph_{\lambda_n}) < \aleph_{\lambda_n^+}$.

5) We had thought that maybe: if μ is singular and $\text{pp}(\mu) \ge \aleph_{\mu^+}$ then some case of IND follows. Why? Because by [Sh:513, 2.8] this holds if $\mu < \aleph_{\mu^+}$ provided that $\mu = \aleph_\delta \wedge |\delta|^{\aleph_0} < \mu$, (even getting $\text{IND}(\langle \text{dual}(J^{\text{bd}}_{\lambda_n}) : n < \omega\rangle)$ for some increasing sequence $\langle \lambda_n : n < \omega\rangle$ of regular cardinals $< \mu$ with limit μ if $\text{cf}(\mu) = \aleph_0$ and $\subseteq \{\lambda^+ : \lambda \in E\}$ for any pre-given club E of μ if $\text{cf}(\mu) > \aleph_0$). If only $\mu = \aleph_\delta \wedge |\delta| < \mu$ then in [Sh:513] we get a weaker version of IND.

{k15}

Claim 3.17. [DC] *For* $\mathbf{x} = \langle Y_n, D_n : n < \omega\rangle$ *with each* D_n *being an* $\aleph_1$*-complete filter on* Y_n*, each of the following is a sufficient condition for* $\text{IND}(\mathbf{x})$*, letting* $Y(<n) := \prod_{m<n} Y_m$ *and for* $m < n$*, let* $Z_{m,n} = \{t : t$ *is a function from* $\prod_{\ell=m+1}^{n-1} Y_\ell$ *into* $Y_m\}$ *and let* $Z_n = \prod_{m<n} Z_{m,n}$

- (a) D_n *is a* $(\le Z_n)$*-complete ultrafilter*
- (b) • D_n *is a* $(\le Z_n)$*-complete filter*

- *for each n in the following game $\partial_{\mathbf{x},n}$ the non-empty player has a winning strategy. A play last ω-moves. In the k-th move the empty player chooses $A_k \in D_n$ and $\langle X^k_t : t \in Z_n\rangle$, a partition of A_k and the non-empty player chooses $t_k \in Z_n$. In the end the non-empty player wins the play if $\bigcap\limits_{k<\omega} X^k_{t_k}$ is non-empty*

(c) *like clause (b) but in the second part the non-empty player instead t_k chooses $S_k \subseteq Z_n$ satisfying $|S_k| \le_X |S|$ and every $D_{\mathbf{x},n}$ is $(\le S)$-complete, S is infinite*

(d) *if $m < n < \omega$ then D_m is $(\le \prod\limits_{k=m+1}^{n} Y_k)$-complete*[5]

Proof. Straightforward. E.g.

Clause (b):

Let $\langle \mathbf{st}_n : n < \omega\rangle$ be such that $\mathbf{st}_n$ is a winning strategy of the non-empty player in the game $\partial_{\mathbf{x},n}$.

Let $\bar F = \langle F_{m,n} : m < n < \omega\rangle \in \mathrm{alg}(\mathbf{x})$ and we should find a member of $\prod\limits_n Y_n$ as required in Definition 3.12(2). We now, by induction on $i < \omega$, choose the following objects satisfying the following condition

$(*)_i$ (a) for $k < m$ and $j < i, G_{j,k,m}$ is a function from $\prod\limits_{\ell=k+1}^{m} Y_\ell$ into Y_k

(b)(α) for $m < \omega, \langle(\bar X_{j,m}, \mathbf{t}_{j,m}) : j < i\rangle$ is an initial segment of a play of the game $\partial_{\mathbf{x},m}$ in which the non-empty player uses the strategy $\mathbf{st}_m$;

(β) we have $\bar X_{j,m} = \langle X_{j,m,t} : \mathbf{t} \in Z_m\rangle$ so $X_{j,m,\mathbf{t}} \subseteq Y_m$

(γ) $\mathbf{t}_{j,m} = \langle t_{j,k,m} : k < m\rangle$ and $t_{j,k,m} \in Z_k$

(δ) $X_{j,m,\mathbf{t}} = \bigcap\limits_{k<m} X_{j,k,m,t_k}$, see clause (e) when $\mathbf{t} = \langle t_k : k < m\rangle \in Z_m, \bigwedge\limits_k t_k \in Z_{k,m}$

(c)(α) $Y_{j,m}$ is Y_m if $j = 0$

(β) $Y_{j,m}$ is $\cap\{X_{\iota,m,k,t_{j,k,m}} : \iota < j\} \subseteq Y_m$ if $j \in (0, i)$

(d)(α) <u>if</u> $j = 0 < i$ then $G_{j,k,m}$ is $F_{k,m}$

(β) if $j \in (0, i)$ then $G_{j,k,m}$ is defined by: for $\langle y_{k+1}, \ldots, y_m\rangle \in \prod\limits_{\ell=k+1}^{m} Y_\ell$ we have $G_{j,k,m}(\langle y_{k+1}, \ldots, y_m\rangle) = G_{j-1,k,m+1}(\langle y_{k+1}, \ldots, y_{m+1}\rangle)$ for any $y_{m+1} \in Y_{j,m+1}$ (so the value does not depend on y_{m+1}!)

(e) for $k < m$ and $t \in Z_{k,m}$ let $X_{j,k,m,t}$ be $\{y \in Y_m$: if $\langle y_{k+1}, \ldots, y_{m-1}\rangle \in \prod\limits_{\ell=k+1}^{m-1} Y_\ell$ then $G_{j,k,m}(y_{k+1}, \ldots, y_{m-1}, y) = (y_{k+1}, \ldots, y_{m-1})\}$.

---

[5]So the Y_k's are not well ordered! But, on the one hand, if $\alpha < \mathrm{hrtg}(Y_n) \Rightarrow D_n$ is $|\alpha|^+$-complete then $\alpha^{Y_n}/D_n \cong \alpha$. On the other hand, if D_n is $\aleph_1$-complete and $\alpha^{Y_n}/D \cong \alpha$ then D projects onto a uniform $\aleph_1$-complete filter on some $\mu \le \alpha$ and those projections cover the ultra-power.

Clearly $(*)_0$ holds emptily.

For $i \geq 1$, let $j = i-1$ clearly $\langle Y_{j,m} : m < \omega \rangle$ is well defined by clause (c), hence we can define $\langle X_{j,k,m,t} : t \in Z_{k,m} \rangle$ by clause (e) and let $X_{j,m,\mathbf{t}} = \cap\{X_{j,k,m,t_k} : k < m\}$ when $\mathbf{t} = \langle t_k : k < m \rangle$.

So $\bar{X}_{j,m} = \langle X_{j,m,\mathbf{t}} : \mathbf{t} \in Z_m \rangle$ is a legal j-move of the empty player in the game $\Game_{\mathbf{x},m}$, so we can use $\mathbf{st}_m$ to define $\mathbf{t}_{j,m} = \langle t_{j,k,m} : k < m \rangle$ as the j-th move of the non-empty player.

Lastly, the function $G_{j,k,m}$ is well defined. Having carried the induction, for each m clearly $\langle (\bar{X}_{j,m}, \mathbf{t}_{j,m}) : j < \omega \rangle$ is a play of the game $\Game_{\mathbf{x},m}$ in which the non-empty player uses the strategy $\mathbf{st}_m$ hence win in the play, so $\cap\{X_{j,m,\mathbf{t}_{j,m}} : j < \omega\}$ is non-empty so by $\mathrm{AC}_{\aleph_0}$ we can choose $\bar{y} = \langle y_m : m < \omega \rangle$ such that $y_m \in \cap\{X_{j,m,\mathbf{t}_{j,m}} : j < \omega\}$.

It is easy to see that $\bar{y}$ is as required in Definition 3.12(2). $\square_{3.17}$

{k20}

Conclusion 3.18. [DC] Assume $\langle \kappa_n : n \rangle$ is increasing and κ_n is measurable as witnessed by the ultrafilter D_n or just D_n is a uniform[6] $\Upsilon(\mathscr{P}(\kappa_{n-1}))$-complete ultrafilter on κ_n.

Then for every ordinal ζ, for every large enough n we have $\text{o-Depth}^+_{D_n}(\zeta) \leq \zeta$.

Proof. By 3.17 we know that $\mathrm{IND}(\langle D_n : n < \omega \rangle)$ and by 3.13(2) we get the desired conclusion. $\square_{3.18}$

{k24}

Claim 3.19. *(*ZFC *for simplicity).*

If (A) then (B) where

(A) (a) $\lambda_n = \mathrm{cf}(\lambda_n)$ *and* $(\lambda_n)^{<\lambda_n} < \lambda_{n+1}$ *and* $\mu = \Sigma\{\lambda_n : n < \omega\}$ *and* $\lambda = \mu^+$

(b) $\mathbb{P}_n$ *is the natural* λ_n*-complete* λ_n^+*-c.c. forcing adding* $\langle \underset{\sim}{f}_{n,\alpha} : \alpha < \lambda \rangle$ *of members of* ${}^{\lambda_n}(\lambda_n), <_{J^{\mathrm{bd}}_{\lambda_n}}$*-increasing*

(c) $\mathbb{P}$ *is the product* $\prod\limits_n \mathbb{P}_n$ *with full support*

(B) in $\mathbf{V}^{\mathbb{P}}$ *we have* $\mathrm{NIND}(\langle \mathrm{dual}(J^{\mathrm{bd}}_{\lambda_n}) : n < \omega \rangle)$ *and a cardinal* θ *is not collapsed if* $\theta \notin (\mu^+, \mu^{\aleph_0}]$.

Proof. So $p \in \mathbb{P}_n$ ff p is a function from some $u \in [\lambda^+]^{<\lambda_n}$ into $\cup\{{}^{\zeta}(\lambda_n) : \zeta < \lambda_n\}$, ordered by $\mathbb{P}_n \models$ "$p \leq q$" iff $\alpha \in \mathrm{Dom}(q) \Rightarrow \alpha \in \mathrm{Dom}(q) \wedge p(\alpha) \subseteq q(\alpha)$. Now use 3.13. $\square_{3.19}$

{boun}

§ 3(C). **Bounds on the Depth.** We continue 3.2. We try to get a bound for singulars of uncountable cofinality say for the depth, recalling that depth, rank and ps-T_D are closely related.

{c1}

Hypothesis 3.20. D an $\aleph_1$-complete filter on a set Y.

Remark 3.21. Some results do not need the $\aleph_1$-completeness.

[6]Recall $\Upsilon(A) = \min\{\theta$: there is no one-to-one function from θ into $A\}$.

{c2}

Claim 3.22. *Assume* $\bar{\alpha} \in {}^Y\mathrm{Ord}$.
1) [DC] *(No-hole-Depth) If* $\zeta + 1 \le \text{ps-o-Depth}^+_D(\bar{\alpha})$ *<u>then</u> for some* $\bar{\beta} \in {}^Y\mathrm{Ord}$, *we have* $\bar{\beta} \le \bar{\alpha} \bmod D$ *and* $\zeta + 1 = \text{ps-o-Depth}^+(\bar{\beta})$.
2) In Definition 3.1 we may allow $\mathscr{F}_\varepsilon \subseteq {}^Y\mathrm{Ord}$ *such that* $g \in \mathscr{F}_\varepsilon \Rightarrow g < f \bmod D$.
3) If $\bar{\beta} \in {}^Y\mathrm{Ord}$ *and* $\bar{\alpha} = \bar{\beta} \bmod D$ *<u>then</u>* $\text{ps-o-Depth}^+(\bar{\alpha}) = \text{ps-o-Depth}^+(\bar{\beta})$.
4) If $\{y \in Y : \alpha_y = 0\} \in D^+$ *<u>then</u>* $\text{ps-o-Depth}^+(\bar{\alpha}) = 1$.
5) Similarly for the other versions of depth from Definition 3.10.

Proof. 1) By DC without loss of generality there is no $\bar{\beta} <_D \bar{\alpha}$ such that $\zeta + 1 \le \text{ps-o-Depth}^+(\bar{\beta})$. Without loss of generality $\bar{\alpha}$ itself fails the desired conclusion hence $\zeta + 1 < \text{ps-o-Depth}^+(\beta)$. By parts (3),(4) without loss of generality $s \in Y \Rightarrow \alpha_s > 0$. As $\zeta + 1 < \text{ps-o-Depth}^+(\bar{\alpha})$ there is a $<_D$-increasing sequence $\langle \mathscr{F}_\varepsilon : \varepsilon < \zeta + 1\rangle$ with $\mathscr{F}_\varepsilon$ a non-empty subset of $\Pi\bar{\alpha}$. Now any $\bar{\beta} \in \mathscr{F}_\zeta, \zeta + 1 \le \text{ps-o-Depth}^+(\bar{\beta})$ as witnessed by $\langle \mathscr{F}_\varepsilon : \varepsilon < \zeta\rangle$, recalling part (2); contradicting the extra assumption on $\bar{\alpha}$ (being $<_D$-minimal such that...).
2) Let $\mathscr{F}'_\varepsilon = \{f^{[\bar{\alpha}]} : f \in \mathscr{F}_\varepsilon\}$ where $f^{[\bar{\alpha}]}(s)$ is $f(s)$ if $f(s) < \alpha_s$ and is zero otherwise.
3),4) Obvious.
5) Similarly. $\square_{3.22}$

{c4}

Claim 3.23. *[DC + AC_Y] If* $\bar{\alpha}, \bar{\beta} \in {}^Y\mathrm{Ord}$ *and* D *is a filter on* Y *and* $s \in Y \Rightarrow |\alpha_s| = |\beta_s|$ *<u>then</u>* $\text{ps-}\mathbf{T}_D(\bar{\alpha}) = \text{ps} - \mathbf{T}_D(\bar{\beta})$.

Proof. Straightforward. $\square_{3.23}$

Assuming full choice the following is a relative of Galvin-Hajnal theorem.

{c7}

Theorem 3.24. *[DC + AC_Y] Assume* $\alpha(1) < \alpha(2) < \lambda^+$, $\text{ps-o-Depth}^+(\lambda) \le \lambda^{+\alpha(1)}$ *and* $\xi = \mathrm{hrtg}({}^Y\alpha(2)/D)$. *<u>Then</u>* $\text{ps-o-Depth}^+_D(\lambda^{+\alpha(2)}) < \lambda^{+(\alpha(1)+\xi)}$.

Proof. Let $\Lambda = \{\mu : \lambda^{+\alpha(1)} < \mu \le \lambda^{\alpha(1)+\xi}\}$ and for every $\mu \in \Lambda$ let

- $(*)_1$ $\mathscr{F}_\mu = \mathscr{F}(\mu) = \{f : f \in {}^Y\{\lambda^{+\alpha} : \alpha < \alpha(2)\}$ and $\mu = \text{ps-Depth}^+_D(f)\}$
- $(*)_2$ obviously $\langle \mathscr{F}_\mu : \mu \in \Lambda\rangle$ is a sequence of pairwise disjoint subsets of ${}^Y\alpha(2)$ closed under equality modulo D.

By the no-hole-depth claim 3.22(1) above we have

- $(*)_3$ if $\mu_1 < \mu_2$ are from Λ and $f_2 \in \mathscr{F}_{\mu_2}$ <u>then</u> for some $f_1 \in \mathscr{F}_{\mu_1}$ we have $f_1 < f_2 \bmod D$
- $(*)_4$ $\xi > \sup\{\zeta + 1 : \mathscr{F}(\lambda^{+(\alpha+\zeta)}) \neq \emptyset\}$ implies the conclusion.

Lastly, as $\xi = \mathrm{hrtg}({}^Y\alpha(2)/D)$ we are done. $\square_{3.24}$

Remark 3.25. 0) Compare this with conclusion 1.11.
1) We may like to lower ξ to $\text{ps-Depth}^+_D(\alpha(2))$, toward this let

- $(*)_1$ $\mathscr{F}'_\mu = \{f \in \mathscr{F}_\mu :$ there is no $g \in \mathscr{F}_\mu$ such that $g < f \bmod D\}$.

By DC

- $(*)_2$ if $f \in \mathscr{F}_\mu$ then there is $g \in \mathscr{F}'_\mu$ such that $g \le_D f \bmod D$.

2) Still the sequence of those $\mathscr{F}'_\mu$ is not $<_D$-increasing.

Instead of counting cardinals we can count regular cardinals.

{c13}

Theorem 3.26. [DC+AC$_Y$] *The number of regular cardinals in the interval* $(\lambda^{+\alpha(1)}, \text{ps-depth}^+_D(\lambda^{+\alpha(2)})$ *is at most* $\text{hrtg}({}^Y\alpha(2)/D)$ *when:*

(*a*) $\alpha(1) < \alpha(2) < \lambda^+$

(*b*) $\kappa > \aleph_0$

(*c*) *D is a* κ*-complete filter on* Y

(*d*) $\lambda^{+\alpha(1)} = \text{ps-Depth}_D(\lambda)$.

Proof. Straightforward, using the No-Hole Claim 1.13. $\square_{3.26}$

§ 4. Private Appendix

{pa}

When ready §3D will be moved to the paper or to a new one.

{rgch}

§ 4(A). RGCH Revisited.

Discussion 4.1. (2013.2.12) More try to continue [Sh:386] with games for $D \in \mathrm{Fil}^1_\kappa(Y)$ giving rank to $2^\theta < \kappa$, function from ${}^Y\mathrm{Ord}$.

{h4}

Theorem 4.2. *Assume* $\mathrm{DC} + \mathrm{AC}_{<\mu}$.

If μ is strong limit (i.e. $\chi < \mu \Rightarrow 2^\chi < \mu$ and mu uncountable) then for every $\lambda \geq \mu$ for some $\kappa < \mu$ we have: if $\xi < \mu, \chi < \lambda, D$ is a κ-complete filter on ξ then $\mathrm{Depth}_D(\lambda) \leq \lambda$, *that is,* $\mathrm{depth}_D(\lambda) = \mathrm{Depth}({}^\xi\lambda, <_D) \leq \lambda$.

{h7}

Theorem 4.3. *The second composition theorem. Assume* AC_Z *we have* $\lambda < \mathrm{Depth}^+(\prod\limits_{i \in Z} P_i, <_D)$ *when:*

(a) *E is a filter on Z*

(b) *$\langle P_i : i \in Z\rangle$ is a sequence of partial orders*

(c) $\lambda < \mathrm{Depth}^-(\prod\limits_{i<Z} \lambda_i, M_D)$

(d) $\lambda_u < \mathrm{Depth}^+(P_i)$

(e) *$<_D$ is the following partial orders on $P = \prod\limits_{i \in Z} P_i : f <_D g \Leftrightarrow \{i \in Z : f(i) <_{P_i} g(i)\} \in E$.*

Proof.

(a) E is a κ-complete filter on Z

(b) $\langle Y_i : i \in Z\rangle$ is a sequence of regular cardinals

(c) $\lambda = \text{ps-tcf}(\prod\limits_{i \in Z} \lambda_i, <_E)$

(d) $\bar{Y} = \langle Y_i : i \in Z\rangle$

(e) $\bar{D} = \langle D_i : i \in Z\rangle$

(f) D_i is a κ-complete filter on Y_i

(g) $\bar{P} = \langle P_{i,j} : i \in Z, j \in Y_i\rangle$ is a sequence of regular cardinals (or just limit ordinals)

(h) $\lambda_i = \text{ps-tcf}(\prod\limits_{j \in Y_i} P_{i,j}, <_{D_i})$

(i) $Y = \{(i,j) : j \in Y_i \text{ and } i \in Z\}$

(j) $D = \{A \subseteq Y$: for some $B \in E$ we have $i \in B \Rightarrow \{j : (i,j) \in A\} \in D_i\}$.

□

Proof.

$(*)_0$ D is a κ-complete filter on Y.

[Why? Straightforward (and do not need any choice).]

Let $\langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i, i \in Z\rangle$ be such that

$(*)_1$ (a) $\bar{\mathscr{F}}_i = \langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i\rangle$ witness $\lambda_i = \mathrm{ps} - \mathrm{tcf}(P_i)$

(b) $\mathscr{F}_{i,\alpha} \neq \emptyset$.

[Why? Exists by clause (d) of the assumption, for clause (b) recall [Sh:938, 5.6].]

By clause (c) of the assumption let $\bar{\mathscr{G}}$ be such that

$(*)_2$ (a) $\bar{\mathscr{G}} = \langle \mathscr{G}_\beta : \beta < \lambda \rangle$ witness $\lambda = \text{ps-tcf}(\prod\limits_{i \in Z} \lambda_i, <_E)$

(b) $\mathscr{G}_\beta \neq \emptyset$ for $\beta < \lambda$.

Now for $\beta < \lambda$ let

$(*)_3$ $\mathscr{F}_\beta := \{f : f \in \prod\limits_{i \in Z} P_i$ and for some $g \in \mathscr{G}_\beta$ we have $i \in Z \Rightarrow f(i) = \mathscr{F}_{i,g(i)}\}$

$(*)_4$ the sequence $\langle \mathscr{F}_\beta : \beta < \lambda \rangle$ is well defined (so exists).

[Why? Obviously.]

$(*)_5$ if $\beta_1 < \beta_2, f_1 \in \mathscr{F}_{\beta_1}$ and $f_2 \in \mathscr{F}_{\beta_2}$ <u>then</u> $f_1 <_D f_2$.

[Why? Let $g_\ell \in \mathscr{G}_{\beta_\ell}$, witness $f_\ell \in \mathscr{F}_{\beta_\ell}$ for $\ell = 1, 2$. As $\beta_1 < \beta_2$ by $(*)_2$ we have $B := \{i \in Z : g_1(i) < g_2(i)\} \in E$. For each $i \in B$ we know that $g_1(i) < g_2(i) < \lambda_i$ and as $f_1(i) \in \mathscr{F}_{i,g_i(i)}, f_2(i) \in \mathscr{F}_{i,g_2(i)}$; hence recalling the choice of $\langle \mathscr{F}_{i,\alpha} : \alpha < \lambda_i \rangle$, see $(*)_1$, we have $f_1(i) <_{P_i} f_2(i)$. As $B \in E$ and $f_1, f_2 \in \prod\limits_{i \in Z} P_i$ it follows that $f_1 <_D f_2$.]

$(*)_6$ for every $\beta < \lambda$ the set $\mathscr{F}_\beta$ is non-empty.

[Why? Recall $\mathscr{G}_\beta \neq \emptyset$ by $(*)_2(b)$ and let $g \in \mathscr{G}_\beta$. As $\langle \mathscr{F}_{i,g(i)} : i \in Z \rangle$ is a sequence of non-empty sets (recalling $(*)_2(b)$), and we are assuming AC_Z there is a function $g \in \prod\limits_{i \in Z} \mathscr{F}_{i,g(i)}$ so $g \in \mathscr{F}_\beta$, so $\mathscr{F}_\beta \neq \emptyset$ as required.]

$(*)_7$ if $f_* \in \prod\limits_{i \in Z} P_i$ <u>then</u> for some $\beta < \lambda$ and $f \in \mathscr{F}_\beta$ we have $f_* < f \mod D$.

[Why? For each $i \in Z$ let $\alpha_i = \min\{\alpha < \lambda_i$: there is $g \in \mathscr{F}_\alpha$ such that $f_*(i) <_{P_i} g\}$, clearly well defined so $\bar{\alpha} = \langle \alpha_i : i \in Z \rangle$ exists. By the choice of $\bar{\mathscr{G}}$ there are $\beta < \lambda$ and $g \in \mathscr{G}_\beta$ such that $\bar{\alpha} <_E g$. Recalling $\mathscr{F}_\beta \neq \emptyset$ choose $f \in \mathscr{F}_\gamma$, it is as required.]

Together we are done proving the theorem. $\square$

{h10} *Conclusion* 4.4. The third composition theorem: assume AC_Z and $\kappa \geq \lambda$.

We have $\lambda < \text{Depth}^+(\prod\limits_{(i,j) \in Y} P_{i,j}, <_D)$ and D is a κ-complete filter on Y <u>when</u>?

Proof. Combine the proof of 2.6 and 4.3. $\square_{4.4}$

ADD 39A? NOT SENT

§ 5. Private Appendix

§(3D) Concluding Remarks, pg.**??**

[Comments to [Sh:938].]

Old Proof of 2.2 moved from pgs.18,19:

First for the "only if" direction, assume $\text{ps}-\text{tcf}(\Pi\bar{\alpha},<_D)$ is well defined and call it λ.

Let $\bar{\mathscr{F}} = \langle \mathscr{F}_\alpha : \alpha < \lambda\rangle$ witness $\lambda =$ ps-tcf$(\bar{\alpha},<_D)$. For $f \in \cup\{\mathscr{F}_\alpha : \alpha < \lambda\}$ let $f^{[*]} \in {}^Y\text{Ord}$ be defined by $f^{[*]}(s) = \sup\{f(t) : t \in s/E\}$. Clearly $f^{[*]} \in \Pi\bar{\alpha}$ as $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Y)$ by clause (a) of the assumption. Let $\mathscr{F}^{[*]}_\alpha = \{f^{[*]} : f \in \mathscr{F}_\alpha\}$ for $\alpha < \lambda$ so $\langle \mathscr{F}^{[*]}_\alpha : \alpha < \lambda\rangle$ exists and $\mathscr{F}^{[*]}_\alpha \subseteq \Pi\bar{\alpha}$. Also $f_1 \in \mathscr{F}_{\alpha_1} \wedge f_2 \in \mathscr{F}_{\alpha_2} \wedge \alpha_1 < \alpha_2 < \lambda \Rightarrow f_1 <_D f_2 \Rightarrow f_1 \leq_D f_2 \Rightarrow f^{[*]}_1 \leq_D f^{[*]}_2$ hence $\alpha_1 < \alpha_2 < \lambda \wedge f_2 \in \mathscr{F}^{[*]}_1 \wedge f_2 \in \mathscr{F}^{[*]}_2 \Rightarrow f_1 \leq_D f_2$.

Now apply 2.1, getting (Y'_1, Y'_2) as there, but by the choice of $\bar{\mathscr{F}}$ necessarily $Y'_1 = \emptyset \mod D$. Hence for some club E of λ, $\langle \mathscr{F}^{[*]}_\alpha : \alpha \in E\rangle$ is $<_D$-increasing cofinal in $\Pi\bar{\alpha}$.

Lastly, for $f \in \cup\{\mathscr{F}^{[*]}_\alpha : \alpha \in E\}$ let $f^{[**]} \in {}^{(Y_1)}\text{Ord}$ be defined by $f^{[**]}(t/E) = f(t)$, well defined as $f{\restriction}(t/E)$ is constant. Let $\mathscr{F}^{[**]}_\alpha := \{f^{[**]} : f \in \mathscr{F}^{[*]}_\alpha\}$ for $\alpha \in E$. Easily $\langle \mathscr{F}^{[**]}_\alpha : \alpha \in E\rangle$ witness the desired conclusions, that is, $\text{ps}-\text{tcf}(\Pi\bar{\alpha}_1,<_{D_1})$ is well defined and equal to λ, so we have proved the "only if" implication.

§ 5(A). **Concluding Remarks.**

Those are comments to [Sh:938].

Definition 5.1. We say $(\Pi\bar{\alpha},<_{D_*})$ has weak κ-true cofinality δ, omitting κ means $\kappa = \aleph_0$, if there is some witness or $(\mathbb{D},\bar{f})$-witness $\bar{\mathscr{F}}$ which means:

- (a) $\mathbb{D} \subseteq \{D : D$ an κ-complete filter on Y extending $D\}$ is not empty
- (b) $D_* = \cap\{D : D \in \mathbb{D}\}$
- (c) $\bar{\mathscr{F}} = \langle \mathscr{F}_{D,\alpha} : D \in \mathbb{D}, \alpha < \delta\rangle$
- (d) $\langle \mathscr{F}_{D,\alpha} : \alpha < \delta\rangle$ witness $(\Pi\bar{\alpha},<_D)$ has pseudo-true-cofinality δ.

Definition 5.2. $\delta = \text{wtcf}_\kappa(\Pi\bar{\alpha},<_{D_*})$ means $(\Pi\bar{\alpha},<_{D_*})$ has weak κ-true cofinality δ and δ is minimal (hence a regular cardinal).

Discussion 5.3. 1) Why do not ask δ to be regular always? We may consider a sequence of δ's and as in $\text{id}-\text{cf}_\kappa(\bar{\alpha})$ in [Sh:1005].
2) Can we $(\text{ZF}+\text{DC}+\text{AC}_{\beth_\omega})$ prove [Sh:460], using $\text{ps}-\mathbf{T}_D(\bar{\alpha})$? Use [Sh:460, §1].
3) Can we generalize the proof of [Sh:829, §1] using $\text{ps}-\mathbf{T}_D(f)$? We get λ is $\text{ps}-\mathbf{T}_D(\prod\limits_{i<\kappa}\lambda_i), \kappa < \mu$ as witnessed by $\langle \mathscr{F}^+_\alpha : \alpha < \lambda\rangle$, but toward contradiction we have $D_n \in \text{Fil}^1_{\kappa^+_n}(\kappa_{n+1})$.

Remark 5.4. For $D \in \text{Fil}^1_\kappa(Y), \text{ps}-\mathbf{T}_D(f)$ is closely related to $\sup\{\text{ps}-\mathbf{T}_{D_1}(f)$. D_1 is a filter on some $\theta < \text{hrtg}(Y)$ such that $D_2 \leq_{\text{RK}} D$ so natural to define $\text{ps}-\mathbf{T}_{\mathbb{D}}$.

Definition 5.5. 1) Assume $\mathbb{D}_1$ is a set of filters and let $\mathrm{prj}(\mathbb{D}_1)$ be

$$\{D_2 : \quad \text{for some } D_1 \in \mathbb{D}_2, \mu < \mathrm{hrtg}(\mathrm{Dom}(D_2)) \text{ and } h : \mathrm{Dom}(D_1) \to \mu \text{ we have } D_2 = h(D_1)\}.$$

2) Let $\mathrm{ps} - \mathbf{T}_{\mathbb{D}}(\bar{\alpha}) = \sup\{\mathrm{ps} - \mathbf{T}_D(\bar{\alpha}) : D \in \mathbb{D}\}$.

Claim 5.6. *Let $\mathbb{D}_1$ be a set of $\aleph_1$-complete filters, $\mathbb{D}_2 = \mathrm{pry}(\mathbb{D}_1)$. Then the following cardinals are S-almost equivalent where $S = \mathrm{Fil}^1_{\aleph_1}(\mathbb{D}_1) = \cup\{\mathrm{Fil}^1_{\aleph_1}(D_1) : D_1 \in \mathbb{D}_1\}$:*

- (a) $\mathrm{ps} - \mathbf{T}_{\mathbb{D}_1}(\bar{\alpha})$
- (b) $\mathrm{ps} - \mathbf{T}_{\mathbb{D}_1}(\bar{\alpha})$
- (c) *FILL.*

§ 6. On RGCH with little choice

If we assume (ZF + DC of course and) Ax_4 can we prove a theorem parallel to the RGCH from [Sh:460]? See [Sh:1005]. We like to prove such a result just that assuming DC; so if we have enough cases of IND, we use [Sh:955, §(3B)] if not, assume for every κ we have $\mathbf{p}$ more or less as in [Sh:938, 3.1], i.e. omitting the ranks such that $(\forall\lambda)\mathrm{Ax}^\theta_{\lambda,\mu_0,\aleph_0}$ all $D \in \mathbb{D}_{\mathbf{p}}$ are μ_1-complete. We try to repeat.

So trying to immitate, e.g. [Sh:829] in the main case we have $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}, \bar\alpha \in {}^{Y(\mathbf{d})}\alpha$. Without loss of generality $(\forall t \in Y_{\mathbf{d}})[(\alpha_t, \mathbf{i}_1)$ is as required], using the induction hypothesis.

For $s \in Y_{\mathbf{d}}$, using $c\ell : [\alpha]^{<\mu(\mathbf{P})} \to [\alpha]^{<\mu(\mathbf{P})}$ which exists by $\mathrm{Ax}^0_\lambda, \ldots$ we have $\big\langle\langle f^t_{\mathbf{e},\mathbf{y},\beta} : \beta < \alpha_t\rangle : \mathbf{e} \in \mathbb{D}_{\geq \mathbf{i}_1}, \beta, \mathbf{y} \in \mathrm{Fil}^4_{\kappa(\mathbf{i}_1,\mathbf{p})}(D_{e_x})\big\rangle$ such that every: if $\mathbf{d} \in \mathbb{D}_{\geq\mathbf{i}_1}, s \in Y_{\mathbf{d}}, f \in {}^{Y[\mathbf{e}]}(\alpha_s)$ then for some set $\langle(y_i,\beta_i) : i < \zeta_p < \kappa(\mathbf{i},\mathbf{p})\rangle$, $\bigwedge\limits_{t\in Y_{\mathbf{e}}} \bigvee\limits_i f(s) = f_{\mathbf{e},\mathbf{y}_i,\beta_i}(s)$.

Why? Given $(\mathbf{e}, f)$ if there is no such sequence, we can find a filter $\kappa(\mathbf{i}_1,\mathbf{p})$-complete filter on $Y_{\mathbf{e}}$ such that...

But we need more: given $\bar f = \langle f_s : s \in Y_{\mathbf{s}}\rangle, f_s \in {}^{Y[\mathbf{e}]}\alpha_s$ and we like to consider all f_s simultaneously, say find $\langle(y_{s,i},\beta_{s,i}) : s \in Y_s, i < i_s\rangle$ as above.

If we have $\mathbf{d} \in \mathbb{D}_{\mathbf{p}} \Rightarrow \mathrm{AC}_{Y_{\mathbf{d}}}$ this can be done. So the status of Ax^0_λ change: given $\mathbf{p}$ we say? If $(\forall x A^0_y, \ldots)$ fix. If not, then for some $\lambda(*)$ we have $\mathbf{i} < \mathrm{cf}(\mu) \Rightarrow \neg\mathrm{Ax}^0_{\lambda,\kappa(\mathbf{i},\mathbf{p}),\aleph_1}$ (can determine the other cases).

We get

$(*)_1$ if $\partial < \mu_{\mathbf{p}}$ then $I = [\lambda]^{<\partial}$ and $D_n = \mathrm{dual}(I)$ then $\mathrm{IND}(\langle I, D : n\rangle)$.

{d2}

Question 6.1. Can we use $\langle([\lambda]^{\kappa(i,\mathbf{p})}, I_{\lambda,\kappa(\mathbf{i},p)}) : n < \omega\rangle$?

Can we avoid using $\langle \mathrm{AC}_{\kappa(i,\mathbf{p})} : i < \mathrm{cf}(\mu)\rangle$? Given $\bar f = \langle f_\alpha : s \in Y_{\mathbf{d}}\rangle$ we can consider $Y_* = Y_{\mathbf{d}} \times Y_{\mathbf{e}}$ and for every sequence $\mathbf{x} = \langle(\mathbf{y}_s, f_s) : s \in Y_{\mathbf{d}}\rangle, f_s \in {}^{Y[\mathbf{e}]}(\alpha_s)$ let $A_{\mathbf{x}} = \{(s,t)(Y_{\mathbf{d}} \times Y_{\mathbf{e}}) : f_s(t) = f_s(t)\}$.

Now we may look at (R not too large)

$$D^* = \{Z \subseteq Y_* : \text{ there is } \langle \mathbf{x}_r : r \in P\rangle \text{ such that } Y \backslash Z \subseteq \bigcup_{r\in R} A_{\mathbf{x}_r}\}.$$

So D_R is a $\kappa_p(i_1,\mathbf{p})$-complete filter.

Let $D^*_{R,s}$ be the projection of D^*_R to $\{s\} \in Y_{\mathbf{e}}$. Clearly it is the filter defined by (α_s, f_s).

Recall [Sh:835, 2.2].

{d4}

Definition 6.2. We say $\mathrm{Ax}^0_{\alpha,\kappa,\mu}$ <u>when</u> some $c\ell$ exemplifies it which means:

(a) $c\ell : [\alpha]^{<\kappa} \to [\alpha]^{<\mu}$

(b) $u \subseteq c\ell(u)$

(c) $u_1 \subseteq u_2 \Rightarrow c\ell(u_1) \subseteq c\ell(u_2)$

(d) there is no sequence $\langle\alpha_n : n < \omega\rangle \in {}^\omega\alpha$ such that $\alpha_n \notin c\ell\{\alpha_k : k > n\}$.

{d6}

Definition 6.3. We say $\mathbf{x}$ is a filter system (as in [Sh:938, 3.1], add $\kappa_{\mathbf{p},\mathbf{d}}, \mathrm{Rep}_{\kappa(\mathbf{d},\mathbf{p})}(D_{\mathbf{p}})$ but no rk

(a) μ is singular

(b) each $\mathbf{d} \in \mathbb{D}$ is (or just we can compute from it) a pair $(Y, D) = (Y_{\mathbf{d}}, D_{\mathbf{d}}) = (Y[\mathbf{d}], D_{\mathbf{d}}) = (Y_{\mathbf{p},\mathbf{d}}, D_{\mathbf{p},\mathbf{d}})$ such that:

(α) $\mathrm{hrtg}(Y_{\mathbf{d}}) < \mu$, on hrtg(−) see Definition **??**

(β) $D_{\mathbf{d}}$ is a filter on $Y_{\mathbf{d}}$

(c) (α) $\kappa_\iota = \kappa_{\mathbf{p},\iota} = \kappa(\mathbf{i}, \mathbf{p})$ is a cardinal $< \mu$

(β) $i_1 < i_2 \Rightarrow \kappa_{\mathbf{p},i_1} < \kappa_{\mathbf{p},i_2}$

(γ) $(\forall \sigma < \mu)[\exists i < \mathrm{cf}(\mu))(\sigma < \kappa_{\mathbf{p},i}$

(δ) if $\mathbf{d} \in \mathbb{D}_{\geq i}$ then $D_{\mathbf{d}}$ is $\kappa_{\mathbf{p},i}$-complete

(ε) μ is strong limit

(d) (α) Σ is a function with domain $\mathbb{D}$ such that $\Sigma(\mathbf{d}) \subseteq \mathbb{D}$

(β) if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{e} \in \Sigma(\mathbf{d})$ then $Y_{\mathbf{e}} = Y_{\mathbf{d}}$ [natural to add $D_{\mathbf{d}} \subseteq D_{\mathbf{e}}$, this is not demanded but see **??**(2)]

(e) (α) $\mathbf{j}$ is a function from $\mathbb{D}$ onto $\mathrm{cf}(\mu)$

(β) let $\mathbb{D}_{\geq i} = \{\mathbf{d} \in \mathbb{D} : \mathbf{j}(\mathbf{d}) \geq i\}$ and $\mathbb{D}_i = \mathbb{D}_{\geq i} \backslash \mathbb{D}_{i+1}$

(γ) $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow \mathbf{j}(\mathbf{e}) \geq \mathbf{j}(\mathbf{d})$

(f) for every $\sigma < \mu$ for some $i <$ cf(μ), if $\mathbf{d} \in \mathbb{D}_{\geq i}$, then $\mathbf{d}$ is $(\mathbf{p}, \leq \sigma)$-complete where

(g) $\mathbf{p}$ is complete when $\mathbb{D}_{\geq i} = \{(\kappa, D) : \kappa \in [\kappa_{\mathbf{p},i}, \mu), D$ a $\kappa_{\mathbf{p},i}$-complete filter on $\kappa\}$.

{d8}

Definition 6.4. Let $\mathrm{Ax}^0_{\alpha,\mathbf{p}}$ means that: there is a function $c\ell$ satisfying (a)-(c) of 6.2 and:

(d) if $\mathbf{d} \in \mathbb{D}$ and $u \in [\alpha]^{<\mathrm{hrtg}(Y[\mathbf{d}])}$ then $|c\ell(u)| < \kappa(\mathbf{d}, \mathbf{p})$.

FILL

{d10}

Claim 6.5. *Assume* $\mathrm{Ax}^0_{\alpha,\mu,\leq\kappa}, D$ *a filter on* Y *and* $\bar{\alpha} \in {}^Y(\alpha_* + 1)$.
Then $\mathrm{ps-o-Depth}_D(\bar{\alpha}) \leq_S \mathrm{o-Depth}_D(\bar{\alpha})$.
Why?

Proof. Let $c\ell$ witness $Ax^0_{\alpha_*}$, and assume $u \in [\alpha]^{<\mathrm{hrtg}(Y)} \Rightarrow c\ell(u) \in [\alpha]^{<\mu}$. Let $\kappa = \sup\{|c\ell(u)|^+ : u \in [\alpha_*]^{<\mathrm{hrtg}(Y)}\}$. For transparency as $0 \notin \mathrm{Rang}(\bar{\alpha})$, assume $\beta_* < \mathrm{ps-o-Depth}_D(\bar{\alpha})$, so there is a sequence $\langle \mathscr{F}_\beta : \beta < \beta_* \rangle$ witnessing it so $f \in \mathscr{F}_\beta \Rightarrow f < \bar{\alpha}$.

For each $\beta < \beta_*$ so $\mathscr{F}_\beta \subseteq \Pi\bar{\alpha}, f \in \mathscr{F} := \cup\{\mathscr{F}_\beta : \beta < \beta_*\}$, there is $y \in \mathrm{Rep}_\kappa(D)$ which represents f which means:

$(*)_{f,\mathbf{y}}$ (a) $\mathbf{y} \equiv (Y, D, A, h)$

(b) if $B \in D$ and $B \subseteq A$ then $c\ell\{f(t) : t \in \beta\} = c\ell\{f(t) : t \in A\}$

(c) h is a function with domain $A_{\mathbf{y}}$ such that: $h(t) = \mathrm{otp}(f(t) \cap c\ell\{f(s)\colon s \in A_{\mathbf{y}}\})$ so $< \mu$

$\boxplus$ if $f_1, f_2 \in \mathscr{F}_\beta$ are represented by $\mathbf{y}$ then $f_1{\restriction}A_{\mathbf{y}} = f_2{\restriction}A_{\mathbf{y}}$.

Now

$\boxplus$ $|\mathrm{Rep}_\kappa(D)| = |D \times {}^Y\kappa|$

$\boxplus$ for $\mathbf{y} \in \mathrm{Rep}_\kappa(D)$ let $\mathscr{U}_{\mathbf{y}} = \{\beta < \beta_*\colon$ there is $f \in \mathscr{F}_\beta$ represented by $\mathbf{y}\}$

$\boxplus$ $\langle \mathscr{U}_{\mathbf{y}} : y \in \mathrm{Rep}(D, \kappa) \rangle$ is well defined

⊞ $\beta_* = \cup\{\mathscr{U}_{\mathbf{y}} : \mathbf{y} \in \mathrm{Rep}(D,\kappa)$

⊞ for $\mathbf{y} \in \mathrm{Rep}_\kappa(D)$ and $\alpha \in \mathscr{U}_{\mathbf{y}}$ let $g_{\mathbf{y},\beta}$ is the unique member of $\Pi\bar{\alpha}$ such that: if $f \in \mathscr{F}_\beta$ is represented by $\mathbf{y}$ then $g_{\mathbf{y},\beta} \restriction A_{\mathbf{y}} = f \restriction A_{\mathbf{y}}$ and $g_{\mathbf{y},\beta}(t) = 0$ for $t \in Y \backslash A_{\mathbf{y}}$

⊞ $\langle g_{\mathbf{y},\beta} : \beta \in \mathscr{U}_{\mathbf{y}}\rangle$ is $<_D$-increasing sequence in $\Pi\bar{\alpha}$.

□

{d12}

Claim 6.6. *Assume D_* is a κ-complete filter on $Y, \kappa \geq \aleph_1$ and $\mathrm{Ax}^0_{\lambda,\leq\gamma,\leq\kappa}$ so γ acts as an ordinal and $\mu = \chi$ and $S = \mathrm{Fil}^4_\kappa(D_*,\gamma)$, so γ fixes the order type of $c\ell(\{f(s) : s \in Y\})$ and $\mathbb{D} = \{\mathrm{dual}(J[f,D]) : f \in {}^Y\mathrm{Ord}\}$.*

The following cardinals are S-almost equal for $\bar{\alpha} \in {}^Y\mathrm{Ord}$

(a) $0-\mathrm{Depth}^+_{\mathbb{D}}(\bar{\alpha})$
(b) $\mathrm{ps}-0-\mathrm{Depth}(\bar{\alpha})$
(c) $\mathrm{ps}-\mathbf{T}_{\mathbb{D}}(\bar{\alpha})$
(d) $\cup\sup\{\mathrm{rk}_D(\bar{\alpha})+1 : D \in \mathbb{D}\}$.

Proof. FILL. □

{d14}

Theorem 6.7. *Let $\mathbf{p} = (\mathbb{D},\mu,\ldots)$ be a filter system and $(\forall\alpha)(\forall^\infty, i < \mathrm{cf}(\mu))(\mathrm{Ax}^0_{\alpha,\kappa_{\mathbf{p},i+1},\aleph_1})$.*

Assume further $\mathrm{AC}_{\kappa(i,\mathbf{p})}$ for $i < \mathrm{cf}(\mu)$. For $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ let obey??

For every α (question: or λ?) such that $\mathrm{Ax}^0_{\alpha,\kappa(\mathbf{p}),\aleph_1}$ there is $i < \mathrm{cf}(\mu_{\mathbf{p}})$ such that: if $\mathbf{d} \in \mathbb{D}_{\geq i}$ then the following as $\mathrm{Rep}_{\kappa(\bar{\mathbf{d}},\mathbf{p})}(D_{\mathbf{d}})$-almost equal

(a) α
(b) $o-\mathrm{Depth}_{D_{\mathbf{d}}}(\alpha)$
(c) $\mathrm{ps}-o-\mathrm{Depth}_{D_{\mathbf{d}}}(\bar{\alpha})$
(d) $\mathrm{ps}-\mathbf{T}_{D_{\mathbf{d}}}(\alpha)$
(e) $\mathrm{rk}_{D_{\mathbf{d}}}(\alpha)$.

Remark 6.8. 1) For (b),(c) their being almost equal we already know, see §(3A).
2) Use $\mathrm{rk}_{\mathbf{d}}$ or $\mathrm{rk}_{D_{\mathbf{d}}}$? Presently, rk_d.

Proof. Case 1: $\alpha < \mu$
Obvious.

Case 2: $\alpha < \mu^+$
Easy.

Case 3: $\alpha \geq \mu^+$ and for $\mathbf{d} \in \mathbb{D}$ and $\bar{\alpha} \in {}^{Y[\mathbf{d}]}\alpha$ do we have $\alpha < \mathrm{ps}-0-\mathrm{Depth}(\bar{\alpha})$.
Easy by the definitions.

Case 4: as ab there are $\mathbf{d} \in \mathbb{D}$ and $\bar{\alpha} \in {}^{Y[\mathbf{d}]}\alpha$.

Choose $\langle g^*_\varepsilon : \varepsilon < \alpha\rangle$ witness $\alpha < o-\mathrm{Depth}^+_{D_{\mathbf{d}}}(\alpha)$ or more: such that $J[g^*_\varepsilon, D_{\mathbf{d}}]$ is constant; D_2 the dual.

For $s \in Y_{\mathbf{d}}$ clearly $\mathbf{i}(s) = \min\{i < \mathrm{cf}(\mu)$: for α_t, i is as required in the claim$\}$. Clearly $\mathbf{i}(s) < \mathrm{cf}(\mu)$ is well defined by the induction hypothesis

(∗) without loss of generality for some $\mathbf{i}_0, A = \{s \in Y_{\mathbf{d}} : \mathbf{i}(s) = \mathbf{i}_0\} \in D_{\mathbf{d}}$.

[Why? See Definition **??**, clause (∗)?]

We choose $\mathbf{i}_1 \in (\mathbf{i}_0, \mathrm{cf}(\mu)))$ such that

(∗) FILL.

Now let $\mathbf{e} \in \mathbb{D}_{\geq \mathbf{i}_1}$ and $\beta_* < 0 - \text{Depth}^+(\alpha)$ and let $\langle f_\beta : \beta < \beta_* \rangle$ witness this. Define $\langle f_{\beta,s} : \beta < \beta_*, s \in Y_{\mathbf{d}} \rangle$ with $f_{\beta,s}$ the function from $Y_{\mathbf{e}}$ into α_* defined by $f_{\beta,s}(t) = g_{f_\beta(t)}(s)$.

let $\langle \xi_{\beta,s} : \beta < \beta_*, s \in Y_{\mathbf{d}} \rangle$ be defined by

- $\xi_{\beta,s} = \text{rk}_{D_{\mathbf{e}}}(f_{\beta,s})$.

Now

(∗) $\xi_{\beta,s} < \alpha_s$.

Lastly, let $\langle \xi_\beta : \beta < \beta_* \rangle$ be defined by

- $\xi_\beta = \text{rk}_{\mathbf{d}}(\bar{\xi}_\beta)$ where $\bar{\xi}_\beta = (\xi_{\beta,s} : s \in Y_{\mathbf{d}})$.

As $\text{rk}_{\mathbf{d}}(\bar{\alpha} = \alpha$ and $\langle \xi_{\beta,s} : s \in Y_{\mathbf{d}} \rangle <_{D_{\mathbf{d}}} \bar{\alpha}$ we have

(∗) $\xi_\beta \leq \alpha$ (or $\xi_\beta < \alpha$).

Now for each $\xi \leq \alpha$ let

(∗) $u_\xi = \{\beta < \beta_* : \xi_\beta = \xi\}$.

It suffices (check formulation) to prove

⊞ $|u_\xi| < \text{hrtg}(\text{Fil}^1_{\aleph_1}(D_{\mathbf{d}}) \times \text{Fil}^1_{\aleph_1}(D_{\mathbf{e}}))$.

Why? For every $\beta < \beta_*$ let $\mathbf{x}^1_\beta = (J\langle \bar{\xi}_{\beta,s} : s \in Y_{\mathbf{d}} \rangle, D_{\mathbf{d}}), \mathbf{x}^2_\beta = \langle J[\langle g^*_{f_{\beta,s}(t)}(s) : t \in D_{\mathbf{e}} \rangle, D_{\mathbf{e}}] : s \in Y_{\mathbf{d}} \rangle, \mathbf{x}^3_\beta = J[f_\beta, D_{\mathbf{e}}, \mathbf{x}^4_\beta = \langle J[g^*_{f_{\beta,t}}, D_{\mathbf{d}}] : t \in Y_{\mathbf{e}} \rangle.]$

Now

- if $\beta_1 < \beta_1 < \beta_*$ and $(\xi_{\beta_1}, \mathbf{x}^\ell_{\beta_1}) = (\xi_{\beta_2}, \mathbf{x}^\ell_{\beta_2})$ then $\xi_1 = \xi$.

[The delicate point: how much should $\mathbf{i}_1$ or comp($\mathbf{e}$) be above $\mathbf{d}$? or too similar to [Sh:938, §2].]

∗ ∗ ∗

Let $J = J[\langle \xi_{\beta_\ell,1} : s \in \mathbf{d} \rangle, D_{\mathbf{d}}], J_s = J[\langle g_{f_{\beta_\ell},2(t)}(s) : t \in D_{\mathbf{d}} \rangle]$.

First, note that as $\xi_{\beta_1} = \xi_{\beta_2}$, clearly $A = \{s \in Y_{\mathbf{d}} : \xi_{\beta_{1,s}} = \xi_{\beta_2,s}\} = Y_{\mathbf{d}} \mod J$. Also for every $s \in A$ we have $B_s := \{t \in Y_{\mathbf{e}} : g_{f_{\beta_1},s(t)} := g_{f_{\beta,s}(t)}(s)\} = Y_{\mathbf{e}} \mod J$.

Is $\mathbf{i}_1$ large enough?

∗ ∗ ∗

- $A_{\beta_1,\beta_2} = \{t \in Y_{\mathbf{e}} : f_{\beta_1}(t) < f_{\beta_2}(t)\} = Y_{\mathbf{e}} \mod D_{\mathbf{e}}$
- for $t \in Y_{\mathbf{d}} : A^t_{\beta_1,\beta_2} = \{s \in Y_{\mathbf{d}} : g_{f_{\beta_1}(t)}(s) < g_{f_{\beta_2}(t)}(s)\}$.

So

- $A_{\beta_1,\beta_2} = Y_{\mathbf{e}} \mod D_{\mathbf{e}}$
- $A^t_{\beta_1,\beta_2} = Y_{\mathbf{e}} \mod D_{\mathbf{d}}$ for $t \in A_{\beta_1,\beta_2}$.

As $\text{hrtg}(D_{\mathbf{d}}) < \text{comp}(D_{\mathbf{e}})$ by the choice of $\mathbf{i}_2$ and "$\mathbf{e} \in \mathbb{D}_{\geq \mathbf{i}_1}$", for some $A_* \in D_{\mathbf{d}}$ we have

- $B_* = \{t \in Y_{\mathbf{e}} : A^t_{\beta_1,\beta_2} = A_*\} \neq \emptyset \mod J$ where $J = J[f_{\beta_1}, D_{\mathbf{e}}] = J[f_{\beta_2}, D_{\mathbf{e}}]$.

Hence

- for every $s \in A_*, t \in B_*$ we have $g_{f_{\beta_1}(t)}(s) < g_{f_{\beta_2}(t)}(s)\}$.

$\square$

* * *

§ 7. Private Appendix
Bounds

Saharon: check with [Sh:F1039]

Moved from pg.2:

§4 Bounds

§(4B) Minimality for ps-0-Depth

[We define "f is (Y, D, γ)-ps-0-Depth$^{(+)}$-minimal and variants (clarify which we deal with). Note existence and how it commutes wit $\langle D + A_i : i < \partial\rangle \mapsto \langle D + \bigcup\limits_{i<\partial} A_i\rangle$.

§(4C) Depth is regular and obtained

[A main claim is that: $f \in {}^Y\text{Ord}, (Y, D, \lambda^+)$-minimal then $\{y : f(y)$ is regular$\} \in D^+$ (see 7.8), existence 7.10.]

§(4D) Weakly inaccessible (to [Sh:F1039])

[We like to show that if $\aleph_0 < \text{cf}(\mu) < \mu$ and μ is not the accumulation point of the class of inaccessible cardinals then there is no (weakly) inaccessible cardinals $\in (\mu, \text{pp}^+_{\aleph_1\text{-com}}(\mu))$. This will be the main result of this section. In [Sh:F1039] we shall get a similar theorem with somewhat different assumptions.]

§5 Try to immitate [Sh:460], pg. 28 [to [Sh:F1039]? till the end?], pg.29

[Check carefully.]

§6 Absoluteness for non-well founded ultra-powers, pg.36

§7 More pcf with little choice, a try, pg.39

§(7A) Semi-filter

[Is it helpful to use semi-filters in [Sh:938, §3,§4]?]

§(7B) Games and Rank, pg.40

[This is an alternative to the present [Sh:F1039] using games and forcing.]

§(7C) Various

[In 11.1, 11.2 we show that investigating ps-tcf it is enough to consider Y a cardinal. In 11.3 we note $\text{AC}_{\text{hrtg}(Y)\Rightarrow\text{hrtg}(Y)}$ successor. In **??** we (? check). In 11.6 we show $\aleph_0 < \kappa = \text{cf}(\mu) < \mu \Rightarrow \text{rk}_{J^{\text{bd}}_\kappa}(\mu) > \mu^+$. In 11.5 we use pigeon $\perp$ hull for $J[f, D]$, nec?]

§8 More, pg.42-44

§ 7(A). **Replacing rank_D by Depth_D - [FILL].**

In ZFC we know that, e.g. for μ singular strong limit of uncountably cofinality, if $\lambda \in (\mu, 2^\mu]$ is weakly inaccessible then weakly inaccessible are unbounded below μ. We like to prove such results with little choice, for this we look at the minimal case.

{c23} **Definition 7.1.** 1) We say $f \in {}^Y\mathrm{Ord}$ is (Y, D, γ)-ps-o-Depth$^+$-minimal when (may omit Y, D in this section), $\gamma \leq$ ps-o-depth$^+_D(f)$ but for no $g \in {}^\lambda\mathrm{Ord}$ satisfying $g < f$ mod D do we have $\gamma \leq$ ps-depth$_D(f)$.
2) Similarly for other variants.

{c25} **Claim 7.2.** *1) If $\gamma \leq$ ps-o-Depth$^+_D(f)$ where $f \in {}^Y\mathrm{Ord}$ then ps-o-Depth$^+$-minimal $g \in {}^Y\mathrm{Ord}$ is such that $g \leq f$ mod D.*
2) Similarly for other variants.

{c26} **Claim 7.3.** *If $f = g + 1 \in {}^Y\mathrm{Ord}$ then ps-o-Depth$^+_D(f) = \cup\{\alpha + 2 : \alpha <$ ps-o-Depth$^+_D(g)\}$.*

{c27} **Claim 7.4.** *1) If f is (Y, D, λ)-ps-depth-minimal and λ is a limit ordinal then $\{y \in Y : f(y)$ limit$\} \in D^+$.*
2) If $\gamma = \delta + 1, \delta$ a limit ordinal and f is (Y, D, γ)-ps-minimal, then $\{y \in Y : f(y)$ a limit ordinal$\} \neq \emptyset$ mod D.

Proof. Fill more? ☐

{c29} **Definition 7.5.** Let $f \in {}^Y\mathrm{Ord}$.
1) Let $J_{\text{ps-o-depth}}[f, D] = \{A \subseteq Y : A = \emptyset$ mod D or $A \in D^+$ but ps-o-Depth$_D(f) <$ ps-o-depth$_{D+A}(f)$.
2) Similarly for other variants, but we write ps-o-depth(+).

{c31} **Claim 7.6.** *1) $\gamma \leq$ ps-o-Depth$_{D+A_\ell}(f)$ for $\ell = 1, 2$ then $\gamma \leq$ ps-depth$_{D+A_1 \cup A_2}(f)$.*
2) $[\mathrm{AC}_\partial]$ If D is $(\leq \partial)$-complete and $\gamma \leq$ ps-Depth$_{D+A_i}(f)$ for $i < \partial$ and $A = \cup\{A_i : i < \sigma\}$ then $\gamma \leq$ ps-o-Depth$_{D+A}(f)$.
3) $[\mathrm{AC}_{<\kappa}]$ If D is κ-complete and f is ps-Depth-minimal then $J_{\text{ps-o-depth}}[f, D]$ is κ-complete ideal disjoint to D.

Proof. FILL ☐

§ 7(B). **Depth is regular and obtained.**

Recall

{c35} **Definition 7.7.** We call λ inaccessible when λ is regular uncountable limit cardinal.

{c37} **Claim 7.8.** $[\mathrm{AC}_Y]$ *Assume λ is regular and $f \in {}^Y\mathrm{Ord}$ is (Y, D, λ^+)-ps-o-depth-minimal. Then $\{y \in Y : f(y)$ is regular$\} \neq \emptyset$ mod D.*

Remark 7.9. The assumption is equivalent to $(Y, D, \lambda + 1)$-ps-o-Depth$^+$-minimal.

Proof. Assume that not, so without loss of generality

$(*)_1$ $f(y)$ is not regular for $y \in Y$
$(*)_2$ $f(y)$ is > 0 for $y \in Y$.

Let

$(*)_3$ (a) $Y_1 = \{y : f(y)$ successor ordinal$\}$
(b) $Y_2 = \{y \in Y : \mathrm{cf}(f(y)) < f(y)\}$.

Clearly

$(*)_4$ $\langle Y_1, Y_2\rangle$ is a partition of Y.

By 7.6(1) without loss of generality

$(*)$ $\ell(*) \in \{1,2\}, Y_{\ell(*)} \in D$, so without loss of generality $Y_{\ell(*)} = Y$.

Case 1: $\ell(*) = 1$
We get a contradiction by 7.3 to the minimality.

Case 2: $\ell(*) = 2$
By AC_Y we can find $\langle C_y : y \in Y\rangle$ such that C_y is an unbounded subset of $f(y)$ of order-type $\mathrm{cf}(f(y))$. Let $\bar\beta = \langle \beta_y : y \in Y\rangle$ with $\beta_y =$ otp(C_y), let $\mathbf{F} : \Pi f \to \Pi\bar\beta$ be defined by $(\mathbf{F}(f))(y) =$ otp$(C_y \cap f(y))$ and let $\mathbf{H} : \Pi\bar\beta \to \Pi f$ be $\mathbf{H}(g)(y) =$ the $h(y)$-th member of C_y).
Let $\bar h = \langle h_y : y \in Y\rangle, h_y$ is the function with domain otp(C_y) such that $h_y(\varepsilon) = \alpha \Leftrightarrow \varepsilon <$ otp$(C_y) \wedge \alpha \in C_y \wedge \varepsilon =$ otp$(C_y \cap \alpha)$.
Clearly $\langle \beta_y : y \in Y\rangle < f$ mod D hence $\gamma(*) :=$ dp-o-Depth$_D(\langle \mathrm{otp}(C_y) : y \in Y\rangle) < \lambda$. Define $\mathscr{F}'_\alpha \subseteq \prod\limits_y \beta_y$ as $\{\mathbf{F}(g) : f \in \mathscr{F}_\alpha\}$

$(*)_5$ $\bar{\mathscr{F}}' := \langle \mathscr{F}'_\alpha : \alpha < \lambda\rangle$ is as in 2.1 below.

[Why? As $g_1 \le g_2$ mod $D \Rightarrow \mathbf{F}(g_1) \le \mathbf{F}(g_2)$ mod D.]
So let $\langle Y_1, Y_2, E\rangle$ be as in 2.1, hence

$(*)_6$ if $Y_1 \in D^+$ then for some $g \in \Pi f$ we have $\lambda^+ <$ ps-Depth$^+_{D+Y_1}(g)$.

[Why? Choose $g_1 \in \mathscr{F}_{\min(E)}$ then

(a) $\mathbf{F}(g_1) < \bar\beta$ mod D.

So letting $g_2 = \mathbf{H}(\mathbf{F}(g_1)) \in \Pi f$ we have $g_2 < f$ mod D and even $g_2 + 1 < f$ mod D.
Also $\langle \mathscr{F}''_\alpha : \alpha \in E\rangle$ witness $\lambda <$ ps-Depth$^+_D(g_2+1)$ where $\mathscr{F}''_\alpha = \{g^{[*]} : g \in \mathscr{F}_\alpha\}$ where $g^{[*]}(y)$ is $g(y)$ if $g(y) \le g_2(y)$ and is zero otherwise.]

$(*)$ if $Y_2 \in D^+$ then $\lambda <$ ps-Depth$_{D+Y_2}(\bar\beta)$.

[Why? $\langle \mathscr{F}'_\alpha : \alpha \in E\rangle$ witness it, or pedantically for $\alpha < \lambda$ let $\gamma(\alpha)$ be the α-th member of E and $\mathscr{F}^*_\alpha := \mathscr{F}'_{\gamma(\alpha)}$ and $\langle \mathscr{F}^*_\alpha : \alpha < \lambda\rangle$ witness.]
Together by 7.6 we are done (check ps/ps-o). $\square_{7.8}$

Claim 7.10. [DC $+\mathrm{AC}_Y$] If $\mathrm{cf}(\lambda) = \lambda > \mathrm{hrtg}(\mathscr{P}(Y))$ and $\lambda^+ <$ ps-Depth$^+_D(f)$ and λ is regular $\ge \mathrm{hrtg}(\mathscr{P}(Y))$ then there is f_1 such that {c53}

(a) $f_1 \in {}^Y\mathrm{Reg}$
(b) $f_1 \le f$ mod D
(c) f_1 is $(Y, D, \lambda^+)-$ ps-Depth$^+$-minimal.

{c55}

Remark 7.11. Use just 7.8 and the existence of minimality.

So we can replace "regular" by any property which satisfies a parallel statement.

Proof. We try to choose (f_n, y_n, ι_m) by induction on n such that

$\boxplus$ (a) $f_n \in {}^Y\text{Ord}$

(b) $f_0 = f$

(c) $n = m+1 \Rightarrow f_n \le f_m$

(d) $Y_n = \{y \in Y : f_n(y) \text{ is regular}\} \setminus \cup\{Y_m : m < n\}$ and $\iota_m = 1\}$

(e) $i_n = 1$ iff $\iota_n \ne 2$ iff $Y_n \in D^+$ and f_n is $(Y, D, \lambda^+)-$ps-Depth$^+$-minimal

(f) if $m < n$ and $\iota_m = 1$ then $f_n \restriction Y_m = f_m \restriction Y_m$

(g) if $n = m+1, Y_m \in D^+$ and $\iota_m = 2$ then $f_n < f_m \mod (D + Y_m)$

(h) if $n = m+1, Z_m = Y \setminus Y_m \setminus \cup\{Y_k : k < n$ and $\iota_k = 1\} \in D^+$ then $f_n < f_m \mod (D + Z_m)$.

Each step is O.K. (for (h) by 7.6) and so by DC we can carry the inductive choice. In this case a D is $\aleph_1$-complete, let $Z \in D$ be the set of y's such that all the relevant inequalities mentioned hold. As $\langle f_n(y) : n < \omega\rangle$ is not decreasing, for some $m, y \in Y_m \wedge \iota_m = 1$, so $Z' := \cup\{Y_m : m < \omega, \iota_m = 1\} \in D$ and let $g \in {}^Y\text{Ord}$ be $g \restriction y_m = f_m \restriction y_m$ if $\iota_m = 1, g(y) = \aleph_1$ otherwise.

Easily g is as required by 7.6. Check. $\square_{7.10}$

{c55}

Conclusion 7.12. In 7.8 we can weaken the assumption - FILL.

{c56}

Remark 7.13. The point is that we do not have to change the filter, hence the demand on "λ large enough is weaker".

§(3D) Weakly inaccessible (to [Sh:F1039]?)

{c61}

Claim 7.14. *[DC+AC$_{\mathscr{P}(Y)}$ (?)]*

Assume $f \in {}^Y\text{Ord}$ is (Y, D, λ^+)-ps-Depth$^+$-minimal. If λ is weakly inaccessible then $\{t : f(t)$ is weakly inaccessible$\} \in D^+$.

Proof. Let

$(*)_1$ (a) $Y_0 = \{t : f(t) = 0\}$

(b) $Y_1 = \{t : f(t)$ a successor ordinal$\}$

(c) $Y_2 = \{t : f(t)$ a limit ordinal of cofinality $< f(t)\}$

(d) $Y_3 = \{t : f(t)$ is regular cardinal which is a successor

(e) $Y_4 = \{t : f(t)$ is weakly inaccessible$\}$.

Obviously

$(*)_2$ $\langle Y_\ell : \ell \le 4\rangle$ is a partition of Y.

By 7.6 without loss of generality

$(*)_3$ (a) $\ell(*) \le 4$ and $Y_{\ell(*)} \in D$

(b) moreover $Y_{\ell(*)} = Y$.

The cases $\ell(*) < 3$ are easily discarded.

Also if $\ell(*) = 4$ then the desired conclusion holds, so we can assume $\ell(*) = 3$ and eventually will get a contradiction.

Choose f_* such that

(∗) $f_* \in {}^Y\text{Card}$ such that $(f_*(g))^+ = f(y)$ for $y \in Y$.

By the assumption on f we can find

(∗) $\bar{\mathscr{F}} = \langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$ is $<_D$-increasing, $\mathscr{F}_\alpha \subseteq \Pi f$ is non-empty.

FILL. $\square_{7.14}$

Discussion 7.15. By AC_Y and 3.1(?) we can get $\lambda \le \text{ps-Depth}_D(f_*)$. But does this suffice? Or can we do the regular for ps-Depth-minimal?

§(3E) Higher rank (to [Sh:F1039]?)

1) We like to repeat [Sh:g, V,VI],but there are some different points; fix $\kappa = \text{cf}(\kappa) > \aleph_0$, e.g. $\aleph_1$.

First, suppose that we have $\text{AC}_{\mathscr{P}^k(\kappa)}, k$ large enough and $\mathscr{H}(\chi)$ we have choice and we know that $\text{rk}^\ell_E(f,\mathbb{E}) < \infty$ for $f \in {}^\kappa\chi$, does this imply the same for $f \in {}^\kappa\text{Ord}$? The remedy we take here is DC_{κ^+}. It is enough to use $\text{rk}^{5/4}_E(f,\mathbb{E})$, so the "antagonist" can chose any "legal filter".

2) Fix $\mathbb{E} = \mathbb{E}^k$. Now if λ is regular (or less?) we can find $\text{rk}^4_E(f_0,\mathbb{E}) = \text{rk}^5_E(f_0,\mathbb{E}) = \lambda$ or just $\text{rk}^4_{E_0}(f,\mathbb{E}) \ge \lambda$. So for every $\alpha < \lambda, \mathbb{E}_\alpha := \{E : E \ge E_*$ and for some $g <_{\text{fil}(E_*)} f_0$ we have $\text{rk}^4_E(g,\mathbb{E}) = \text{rk}^5_E(g,E) = \alpha\}$. Hence for some $E_1 \ge E_0$, the set $\mathscr{U} := \{\alpha : E_1 \in \mathbb{E}_\alpha\}$ is unbounded in λ (and has order type λ). For $\alpha \in \mathscr{U}$ let $\mathscr{F}_\alpha = \{f \le f_0 : \text{rk}^4_{E_1}(g,\mathbb{E}) = \text{rk}^5_{E_1}(g,\mathbb{E}) = \alpha\}$. So $\langle \mathscr{F}_\alpha : \alpha \in \mathscr{U}\rangle$ is $<_u$-increasing and let $f_1 \le f_0$ be a $<_{\text{fil}(E_1)}$-lub.

Hence (forgetting f_0) we have $\text{rk}^4_{E_1}(f_1,\mathbb{E}) = \text{rk}^5$. Suppose we force by $\mathbb{P} = \{(E_*) : D \in E\}$ getting $\mathbf{G}, \underset{\sim}{D}[G]$ what is $\theta(\pi f_1/\underset{\sim}{D})$¿ [Maybe better: what is $\text{hrtg}(\Pi f'/D)$ for $f' \in (\Pi f_1)^{\mathbf{V}}$?]

Clearly $> \lambda^*$. Toward contradiction assume $\lambda_1 = \lambda_2 = \text{cf}(\lambda)$ or just $\lambda_2 \ge \text{Suc}_{pl(\kappa)}(\lambda_1), \lambda_1 > \lambda, E \Vdash_{\mathbb{P}}$ "$\text{hrtg}(\Pi f1/\underset{\sim}{D}) > \lambda_1$ say $\underset{\sim}{F}$ witness this. Hence for $\alpha < \lambda_2$ the following set is non-empty

$$\mathbb{D}_{2,\alpha,\beta} = \{D : (E_1)_{D_1} \Vdash_{\mathbb{P}} \text{"}(\exists g \in (\pi f_1)^{\mathbf{V}})(\underset{\sim}{F}(g)) = \alpha \text{ and } \text{rk}^4(g,\mathbb{E}^{k-1}) = \text{rk}^5_D(g,\mathbb{E}) = \beta\}.$$

So for some $E_2 = (E_1)_{[D_2]}$, the set $\mathscr{U}_2 = \{\alpha < \lambda_1 : (\exists\beta)D_2 \in \mathbb{D}_{2,\alpha,\beta}\}$ has order type λ_1.

Let $\beta_\alpha = \min\{\beta : D_2 \in \mathbb{D}_{2,\alpha,\beta}\}$ for $\alpha \in u_2$. Let $\mathscr{F}^2_\alpha = \{g \in (\pi f_1)^{\mathbf{V}} : D_2 \Vdash_{\mathbb{P}} \underset{\sim}{F}(g) = \alpha, \text{rk}^{4/5}_{D_2}(g,\mathbb{E}) = \beta_\alpha\}$.

Again $\langle \mathscr{F}^2_\alpha : \alpha \in \mathscr{U}_2\rangle$ is increasing.

3) Similarly with $\text{rk}^{2/3}_D(f_0) = \lambda$ forcing with $(D^+, \supseteq)$.

4) Now go back to [Sh:460]. The above is just going back to [Sh:386], [Sh:333], an avenue I had tried and failed, but why?

5) Instead of DC_{κ^+} we may consider a definition of a filter on $[\lambda]^\theta$ with $\theta \ge \beth_2(\kappa)^+$ or so; we do not use real sets just definitions of the sets used. Now to prove in the game $\eth_\kappa(\lambda)$ the protagonist wins, we use χ such that $A \subseteq \lambda, |A| \Rightarrow \mathbf{K}[\mathscr{A}] \models \chi \to (\lambda)^{<\omega}_\kappa$.

§ 8. A try on immitating [Sh:460]

Question 8.1. (to [Sh:F1039]?)

{f2} **Theorem 8.2.** *For every* λ *there is* $n \le \omega$ *such that for no set* $\mathfrak{a} \subseteq \lambda \cap \text{Reg} \backslash \mu_n$ *of cardinality* $< \mu$ *and* μ_n*-complete ideal* I *on* $\mathfrak{a}$ *do we have* $\text{ps-tcf}(\Pi\mathfrak{a}, <_I)$ *is a well defined (regular) cardinal* $\ge \lambda$, *when :*

(a) $\langle \mu_n : n < \omega \rangle$ *is increasing with limit* μ
(b) AC_{μ_n}
(c) $\text{AC}_{\mathscr{P}(\mu_n)}$
(d) DC*?*
(e) $\text{hrtg}(\mathscr{P}(\mu_n)) < \mu_{n+1}$ *moreover* $\text{hrtg}(\text{Fil}^4(\mu_n)) < \mu_{n+1}$.

Proof. We prove this by induction on λ; there is such n let $\mathbf{n}(\lambda)$ be the minimal such λ.

Case 1: $\lambda < \mu$

Easy: even for $n = 0$, as if $\kappa = \text{cf}(\kappa) > \mu$ and $\mathfrak{a} \subseteq \text{Reg} \cap \lambda$ so trivially $|\mathfrak{a}| < \mu$ and I is $\aleph_1$-complete ideal and $\langle P_\alpha : \alpha < \lambda_* \rangle$ is witness to $\lambda_* = \text{ps-tcf}(\Pi\mathfrak{a}, <_I)$ then $\lambda_* < \text{hrtg}(\mathscr{P}(\mathscr{P}(\sup(\mathfrak{a}))))$ (can use less)?.

Case 2: $\lambda = \mu$

Let $n = 1$ and use the $\aleph_1$-completeness to get that without loss of generality $\mathfrak{a}$ is bounded in λ and use the proof of Case 1.

Case 3: $\text{cf}(\lambda) > \aleph_0, \lambda > \mu$

We let $\langle \lambda_\varepsilon : \varepsilon < \text{cf}(\lambda) \rangle$ be an increasing sequence of cardinals $< \lambda$ with limit λ so $\varepsilon \mapsto \mathbf{n}(\lambda_\varepsilon)$ is a function from $\text{cf}(\lambda)$ to ω hence for some n_1 we have $\lambda = \sup\{\lambda_\varepsilon : \mathbf{n}(\lambda_\varepsilon) = n_1\}$.

Let n_2 be such that $\text{cf}(\lambda) < \mu \Rightarrow \text{cf}(\lambda) < \mu_n$. Now $\max\{n_1, n_2\}$ can serve.

Case 4: $\lambda_1 = \lambda_*^+$ or $\sup(\lambda \cap \text{Reg}) < \lambda$.

Easy.

Case 5: $\text{cf}(\lambda) = \aleph_0$ and $\lambda > \mu$ and $\lambda = \sup(\lambda \cap \text{Reg})$.

Toward contradiction assume this fails. We first choose $\mathfrak{a}_1, D_1$ such that

$(*)_1$ (a) $\mathfrak{a}_1 \subseteq \text{Reg} \cap \lambda$ of cardinality $< \mu$
(b) D_1 as $\aleph_1$-complete filter on $\mathfrak{a}_1$ such that
(c) $\lambda_1 = \text{ps-tcf}(\Pi\mathfrak{a}_1, <_{D_1})$ is well defined and $\ge \lambda$ hence $> \lambda$.

Without loss of generality

$(*)_2$ (a) $\mathfrak{a}_1 \cap \mu^{++} = \emptyset$ and $\sup(\mathfrak{a}_1) < \lambda$ and $n_3 = n(*) \ge \max\{\mathbf{n}(\theta) : \theta \in \mathfrak{a}_1\} < \omega$
(b) $\sup(ga_1) < \lambda_0 < \lambda$
(c) $(\mathfrak{a}_1) < \mu_{n_3} < \mu$.

[Why? Note that $\mathbf{n}(\theta)$ is well defined for $\theta \in \mathfrak{a}_1$ by the induction hypothesis. As D_1 is $\aleph_1$-complete, for some $n_2 < \omega$ the set $\mathfrak{a}'_1 = \{\theta \in \mathfrak{a}_1 : \mathbf{n}(\theta) \le n\} \in D_1^+$. By xxx we can replace D_1 by $D_1 + \mathfrak{a}'_1$ and even replace $\mathfrak{a}_1, D_1$ by $\mathfrak{a}'_1, D'_1$, also without loss of generality $n_2 \ge n_1$ and without loss of generality $n_2 \ge n_1$ and without loss of generality some $\min(\text{Reg}\backslash\mu) < \min(\mathfrak{a}_1)$.

Also, alternatively, $\mathbf{n}(\sup(\mathfrak{a}_1)) \le n_2$. Let $n_3 \ge n_1, n_2$ be such that $\mu_{n_3} > \text{hrtg}(\text{Fil}^4(\mathfrak{a}_1))$.]

By the assumption toward contradiction, there is a pair $(\mathfrak{a}_2, D_2)$ such that

$\boxdot_1$ (a) $\mathfrak{a}_2 \subseteq \text{Reg} \cap \lambda\backslash\lambda^+_\theta$
(b) $|\mathfrak{a}_2| < \mu$
(c) D_2 is a $\mu^+_{n(*)}$-complete filter on $\mathfrak{a}$
(d) $\text{ps-tcf}(\Pi\mathfrak{a}_2, <_{D_2})$ is well defined $\ge \lambda$ hence $> \lambda$.

As $\text{hrtg}(\text{Fil}^4(\mathfrak{a}_1)) < \mu_{n_3}$ and $\min(\mathfrak{a}_1) > \min(\text{Reg}\backslash\mu)$ by 1.13, the no-hole claim, we know

$\boxdot_2$ for every $\kappa \in \mathfrak{a}_2$ there is a sequence $\bar\lambda_\kappa = \langle \lambda_{\kappa,\theta} : \theta \in \mathfrak{a}_1\rangle$ such that
(a) $\lambda_{\kappa,\theta} \in \text{Reg} \cap \theta\backslash\mu$
(b) $\kappa = \text{ps-tcf}(\Pi\bar\lambda_\kappa, <_{D_1})$.

As we assume $\text{AC}_{\mathfrak{a}_2}$ recalling $|\mathfrak{a}_2| < \mu$

$\boxdot_3$ (a) there is a sequence $\langle \bar\lambda_\kappa : \kappa \in \mathfrak{a}_2\rangle$ as above
(b) $\bar{\mathfrak{a}}_3 = \langle \mathfrak{a}_{3,\theta} : \theta \in \mathfrak{a} - 1\rangle$ where $\mathfrak{a}_{3,\theta} = \{\lambda_{\kappa,\theta} : \kappa \in \mathfrak{a}_2\} \in [\text{Reg} \cap \theta]^{\le \mathfrak{a}_2|}$
(c) let $\bar\lambda^\theta = \langle \lambda_{\kappa,\theta} : \kappa \in \mathfrak{a}_2\rangle$.

By the choice of n_2, etc. and Theorem xxx using clauses (x),(y) of the assumption

$\boxdot_4$ for each $\theta \in \mathfrak{a}_1$ there is a set $\mathscr{S}_\theta \subseteq \{\mathfrak{b} \subseteq \mathfrak{a}_{3,\theta} : \sup(\text{ps-pcf}_{\aleph_1-\text{com}}(\mathfrak{b}) \le \lambda\}$ of cardinality $< \mu_{n_3}$ with union $\mathfrak{a}_2$

$\boxdot_5$ here is $\langle \mathscr{S}_\theta : \theta \in \mathfrak{a}_1\rangle$ as above.

[Why? By $\text{AC}_{\mathfrak{a}_1}$ because $\text{AC}_{\mu_{n_1}}$.]

$\boxdot_6$ there is $A \in D_2^+$ such that $(\forall\theta \in \mathfrak{a}_1)(\exists B \in \mathscr{F}_\theta)(A \subseteq B)$.

[Why? $\mathscr{S} := \cup\{\mathscr{S}_\theta : \theta \in \mathfrak{a}_1\}$ is a set of cardinality $\le \mu_{n_2}$ as we have $|\mathscr{P}_\theta| \le \mu_2$ and $\text{AC}_{\mathfrak{a}_1}$ holds and $|\mathfrak{a}_1| \le \mu_{n_1}$ and $n_1 < n_2$. Define an equivalence relation e on $\mathfrak{a}_3 : \kappa_1 e \kappa_2$ <u>iff</u> $(\forall A \in \mathscr{S})(\kappa_1 \in A \Leftrightarrow \kappa_2 \in A)$. So the function $\kappa \mapsto \{A \in \mathscr{S} : \kappa \in A\}$ witness that $|\mathfrak{a}_2/e| < \text{hrtg}(\mathscr{P}(\mathscr{S})) \le \text{hrtg}(\mathscr{P}(\mu_{n_2}))$. But D_2 is $\mu^+_{n_3}$-complete and $\mu_{n_3} > \theta(\mathscr{P}(\mu_{n_2}))$, so we are done.]

$\boxdot_7$ (a) without loss of generality $A \in \mathscr{S}_\theta \wedge \theta \in \mathfrak{a}_1 \Rightarrow A \in \{\mathfrak{a}_2, \emptyset\}$.

[Why? By xxx.]

$\boxdot_8$ $\theta \ge \sup(\text{ps-pcf}_{\aleph_1-\text{com}}(\mathfrak{a}_{3,\theta}))$ for $\theta \in \mathfrak{a}_1$.

[Why? By $\boxdot_7$ and the assumption on $\mathscr{S}_\theta$.]

$\boxdot_9$ let D_3 be the following filter on $Y = \mathfrak{a}_2 \times \mathfrak{a}_1$

$$D_2 \times D_1 := \{A \subseteq \mathfrak{a}_2 \times \mathfrak{a}_1 : \{\kappa \in \mathfrak{a}_2 : \{\theta \in \mathfrak{a}_1 : (\kappa,\theta) \in A\} \in D_1\} \in D_2\}$$

$\boxdot_{10}$ (a) ps-tcf$(\Pi\bar\lambda, <_{D_3}) = \lambda_* :=$ ps-tcf$(\Pi\mathfrak{a}_2, <_{D_2}) > \lambda$ where $\bar\lambda := \langle \lambda_{\kappa,\theta} : (\kappa,\theta) \in Y\rangle$

(b) let $\langle \theta_\alpha : \alpha < \lambda_*\rangle$ witness it

$\boxdot_{11}$ $D_* = \{A \subseteq \mathfrak{a}_2 \times \mathfrak{a}_1\}$??

We shall try to prove that $(\Pi\bar\lambda, <_{D_*})$ has a small cofinality. Let $\bar{\mathfrak{c}} = \langle \mathfrak{c}_\theta : \theta \in \mathfrak{a}_1\rangle$ be $\mathfrak{c}_\theta =$ pcf$_{\aleph_1-\text{com}}(\mathfrak{a}_{3,\theta})$ so $|\mathfrak{c}_\theta| < \mu_{n(2)}$.

For every $f \in \Pi\{\mathfrak{c}_\theta : \theta \in \mathfrak{a}_2\}$ or $\bar{\mathfrak{d}} = \langle \mathfrak{d}_\theta : \theta \in \mathfrak{a}_1\rangle \in \prod\limits_{\theta\in\mathfrak{a}_1} [\mathfrak{c}_\theta]^{\le\aleph_0}$ let $D_{\bar{\mathfrak{d}}} = \{A \subseteq Y : \{\theta \in \mathfrak{a}_1 : \{\kappa : (\kappa,\theta) \in A\} = \mathfrak{a}_2 \bmod J_{=\partial}[\bar\lambda^\theta]/J_{f(\theta)}[\bar\lambda^\theta]\} \in D_1\}$

$\boxplus_1$ (a) D_f is an $\aleph_1$-complete filter on Y

(b) $\lambda \ge \sup($ps-pcf$_{\aleph_1-\text{com}}(\Pi\bar\lambda, <_{D_f}))$ define!, by the minimality of $\mathfrak{a}_1$ see $(*)_1(d)$

$\odot_2$ (a) let $\mathfrak{c} = \lambda \cap$ ps-pcf$_{\aleph_1-\text{com}}(\Pi\bar\lambda)$

(b) let $\langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda \in \mathfrak{c}\rangle$ be such that $\bar{\mathscr{F}}_\lambda = \langle \mathscr{F}_{\lambda,\alpha} : \alpha < \lambda\rangle$ witness $\lambda =$ ps-tcf$(\pi\bar\lambda, <_{J_{=\lambda}[\bar\lambda]})$.

$\square_{8.2}$

Discussion 8.3. We try to continue below but §5 seems to solve it another way.

Discussion 8.4. We try to analyze the remaining cases. If we add $|\mathscr{P}(\mu_n)| < \mu$ for $n < \omega$ by forcing without loss of generality

- otp$(\mathfrak{a}_\ell) = \partial_\ell =$ cf(∂_ℓ)
- $D_\ell =$ dual$(J_{<\lambda^+}[\mathfrak{a}_\ell])$
- let $E = \{\mathfrak{b}_2 \times \mathfrak{b}_1 : \mathfrak{b}_\ell \subseteq \mathfrak{a}_\ell, |\mathfrak{b}_\partial| < \partial_\ell$ for $\ell = 1,2\}$.

So let

- $\sigma \in \mathfrak{a}_1 \Rightarrow \mathfrak{c}_\sigma :=$ ps-pcf$_{\partial_2-\text{com}}(\{\lambda_{\kappa,\sigma} : \kappa \in \mathfrak{a}_2\}) \subseteq N$
- $\mathscr{F} = \Pi\mathfrak{c}_\sigma$
- $\bar\alpha \in \mathscr{F} \Rightarrow \mathfrak{d}_\alpha =$ ps-pcf$_{\partial_1-\text{com}}(\bar\alpha, <_{J^{\text{bd}}_{\partial_1}})$ define naturally
- $\mathfrak{d} = \cup\{\mathfrak{d}_{\bar\alpha} : \bar\alpha \in \mathscr{F}\}$
- $\langle A_\chi : \chi \in$ ps-pcf$_{\partial_1-\text{com}}(\bar\lambda)\rangle$.

So

- $\chi \in \mathfrak{d} \Rightarrow (\forall^{\partial_2}\kappa \in \mathfrak{a}_2)(\forall^{\partial_1}\mathfrak{a}_1)[(\kappa,\sigma) \notin A_\chi]$.

By forcing without loss of generality

- $|\mathfrak{c}_\sigma| = \partial_2$.

Question 8.5. Assume $\mathfrak{a}$ is the disjoint union of $\langle \mathfrak{a}_\varepsilon : \varepsilon < \partial\rangle, \mathfrak{a} \subseteq$ Reg$\backslash\mu, |\mathfrak{a}| < \mu$.

Do we have ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{a}) = \cup\{$ps-pcf$_{\aleph_1-\text{com}}(\bigcup\limits_{\varepsilon<\partial} d_\varepsilon), \mathfrak{d}_\varepsilon \subseteq ($ps-pcf$_{\aleph_1-\text{com}}(\mathfrak{a}_\varepsilon))$ countable?

This is a consequence of the existence of smooth closed generating sequences; but does it exist here?

Question 8.6. Does it help to collapse $2^{|\mathfrak{a}_1|}$ and so find as an ultrafilter E_* on $\mathfrak{a}_1$ such that $\mathbf{V}^{\mathfrak{a}_1}/E_*$ has standard $\mathbb{N}$, etc.?

§ 9. Absoluteness for non-well founded ultra-powers

Question 9.1. (to [Sh:F1039])

This may be used in §5 to immitate [Sh:460]. Here we try to avoid using the "smooth closed generating sequences".

<u>Check</u>: What does this give directly?

{m0}

Hypothesis 9.2.

(a) $\mathrm{AC}_{\kappa(*)}, \kappa_* = \kappa(*) > \aleph_0$

(b) D_{**} is a uniform $\aleph_1$-complete ultrafilter on $\kappa_* = \kappa(*)$

(c) $\mathbb{P}$ a forcing notion, $\underset{\sim}{D}$ a $\mathbb{P}$-name of an ultrafilter on $\mathscr{P}(\kappa)^{\mathbf{V}}$ extending $D_{**}, \mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$, in $\mathbf{V}[G]; \underset{\sim}{D}[\mathbf{D}] = D_*$

(d) $\mathbf{W} = \mathbf{W}^{\kappa(*)}/D_*$, so in general not well founded, computed in $\mathbf{V}[G]$

(e) $\mathbf{j} = \mathbf{j}_{\mathbf{G}}$ is the canonical embedding of $\mathbf{V}$ into $\mathbf{W}$.

{m1}

Remark 9.3. 1) We may demand $\mathscr{P}(\mathbb{P})$ well ordered and $\mathrm{AC}_{\mathscr{P}(\mathbb{P}_*)}$ holds.
2) Natural to choose $\mathbb{P} = (\{D : D$ and $\aleph_1$-complete filter on κ extending $D_{**}\}, \geq)$.

{m2}

Claim 9.4. *If $\boxplus_1 + \boxplus_2$ then $\oplus$ when*

$\boxplus_1$ (a)
- $\kappa_1 < \kappa_2 < \kappa_3$
- $\kappa_2 = |(\kappa_2)^{(\kappa_1)}/D_2)$ *can we use less?* $\mathrm{AC}_{<\mu}$
- $\mathbf{V}$ *satisfies enough for Theorem gxxx with* (κ_2, κ_3) *here standing for* $(\kappa, |Y|)$ *there*

(b) D_1 *is an* $\aleph_1$*-complete ultrafilter on* κ_1

(c) $\mathbf{W}$ *is* $\mathbf{V}^{\kappa_1}/D_1$*, i.e.* $(\mathbf{V}_j \in)^{\kappa_1}_{D_1}$

(d) $\mathbf{j}$ *is the canonical elementary embedding of* $\mathbf{V}$ *into* $\mathbf{W}$

(e) $\mathbf{W} \models$ "$\mathfrak{a}$ *is a set of regular cardinals* $> \mathbf{j}(\mu)$ *of cardinality* $\leq j(\kappa_3)$"

(f) $\mathbf{W} \models$ "$\langle A_\theta : \theta \in \mathfrak{c}\rangle$ *where* $\mathfrak{c} = \text{ps-pcf}_{\kappa_2-\text{com}}(\mathfrak{a})\rangle$ *is a generating force* $\mathfrak{c}$ *(not just* $(\mathfrak{a})$ *as in gxxx")*

(g) $Y = \{\theta : \mathbf{W} \models$ "$\theta \in \mathfrak{a}$"$\}$

(h) *for* $\theta \in Y$ *let* I_θ *be* $\{\alpha : \mathbf{W} \models \alpha < \theta\}$ *linearly ordered by* $<^{\mathbf{W}}$ *so* $\langle I_\theta : \theta \in Y\rangle$ *exists in* $\mathbf{v}$

(i) $\bar{\lambda} = \langle \lambda_\theta : \theta \in Y\rangle$ *where* $\lambda_\theta = \mathrm{cf}(I_0)$

(j) $J = \{\{\theta \in Y : \mathbf{W} \models \theta \in \mathfrak{b}\} : \mathbf{W} \models$ "$\mathfrak{b} \subseteq \mathfrak{a}$ *have cardinality* $< \mathbf{j}(\kappa_2)\}$

(k) $J^+ = \{Z \subset Y : (\exists u)(Z \subseteq u \in J)\}$

(l) J_θ *is* $\{\{\theta : \mathbf{W} \models \theta \in u\} : \mathbf{W} \models$ "$u \in J_{=\theta}[\mathfrak{c}]$"$\}$

(m) $J^+_\theta := \{\mathbf{W} : (\exists u)(W \subseteq u \in J_\theta)\}$

$\oplus$ (a) Y *is of cardinality* $(\kappa_3)^{(\kappa_1)}/D_2$ *a cardinal*

(b) *if* $Z \in J$ *then* $|Z| \leq (\kappa_2)^{(\kappa_1)}/D_1$ *(is this well ordered?) no real harm assuming yes; similarly* Y

(c) *the following are equivalent*
- $Z \subseteq \mathbf{W}$ *has cardinality* $\leq \kappa_2$
- *for some* $u \in \mathbf{W}, \mathbf{W} \models$ "$|u| \leq \mathbf{j}(\kappa_2)$" *and* $Z \subseteq \{a : W \models$ "$a \in u$"$\}$

(d) J^+ *is an ideal of subsets of* Y*, in fact in* $[Y]^{\leq \kappa_2}$

(e) $\lambda_\theta = \text{ps-tcf}(\Pi\bar{\lambda}, J^+_\theta)$ *for* $\theta \in Y$

(f) $J^+_\theta = \{A : A \subseteq \bigcup_{\sigma \in \mathfrak{b}} A_\sigma : \mathfrak{b} \in [Y]^{\le \kappa_2}$ *and* $(\forall \sigma \in \mathfrak{b})(\sigma <_{\mathbf{W}} \theta)\}$

(g) $\text{ps-pcf}_{\kappa_2^+-\text{com}}(\{\lambda_\theta : \theta \in \mathfrak{c}\}) = \{\lambda_\theta : \theta \in \mathfrak{c}\}$.

{k2d}
Remark 9.5. 1) Applying this in §4 we let $\mathfrak{a}, \kappa_2, \kappa_3$ stand for $\langle\{\lambda_{\kappa,\theta} : \kappa \in \mathfrak{a}_2\} : \theta \in \mathfrak{a}_1\rangle/D, |(\mu_{n_2})^{\mathfrak{a}_1}/D|, \mathbf{j}^{(|\mathfrak{a}_2|)}$ there.
2) Well the problem may come from undefinable Dedekind cuts in $(Y, <_{\mathbf{W}} \restriction Y)$. However $\mathfrak{a}_1 = \langle\theta_i : i < \kappa_1\rangle, \mathfrak{a}_2 = \langle\theta_\varepsilon : \varepsilon < \kappa_2\rangle$ let D_2 be a κ_2^+-complete filter on $\mathfrak{a}_2$ such that $\lambda_* =$ ps-tcf$(\Pi\mathfrak{a}_2, <_{D_2})$ is too large. So we use $\bar{\lambda}^i = \langle\lambda_{\varepsilon,i} : \varepsilon < \kappa_2\rangle \in$ Reg $\cap (\theta_i), \mathbf{W} \models$ "$\mathfrak{a}$, i.e. $Y = \langle\theta_\iota : \iota \in \mathbf{j}(\kappa_1)\rangle, j(D_2)$ is a $\mathbf{j}(\kappa_2^+)$-complete filter on $\mathbf{j}(\kappa_2)$.

We may wonder: what filter does $\mathbf{j}(D_2)$ induce on $\langle\bar{\lambda}_\varepsilon/D_1 = \langle\lambda_{\varepsilon,i} : i < \kappa_1\rangle/D_2 : \varepsilon < \kappa_1\rangle$ (from the outside)?

Exactly D_2 by the completeness.

Proof. Clause (a): Straight

Clause (b): Follows from clause (c).

Clause (c): If $Z \subseteq \mathbf{W}$ and $|Z| \le \kappa_2$ (in $\mathbf{V}$) this member of Z has the form f/D_1 with $f \in {}^{(\kappa_1)}V$, so by AC_{κ_2} we can find a sequence $\langle f_i : i < i(*) \le \kappa_2\rangle$ such that $Z = \{f_i/D_1 : i < i(*)\}$. For $\varepsilon < \kappa_1$ let $Z_\varepsilon = \{f_i(\varepsilon) : i < i(*)\}$ so $\langle Z_\varepsilon : \varepsilon < \kappa_1\rangle \in {}^{\kappa_1}V$ hence $Z^* = \langle Z_\varepsilon : \varepsilon < \kappa_1\rangle/D_2 \in \mathbf{W}$.

As $\mathbf{V} \models$ "$|Z_\varepsilon| \le \kappa_2$" for $\varepsilon < \kappa_2$ by the relevant version of Łos theorem (quote use AC_{κ_1}!) we have $\mathbf{W} \models$ "$|Z^*| \le j(\kappa_2)$" and obviously $i < i(*) \Rightarrow f_i \in \prod_{\varepsilon<\kappa_1} Z_\varepsilon \Rightarrow W \models$ "$f_i/D_1 \in Z^*$". So we have proved one direction. The other is even easier. □

{k7}
Observation 9.6. $[\text{AC}_{\kappa(*)}]$
Łos theorem holds and so $\mathbf{j}$ is an elementary embedding.

{k10}
Claim 9.7. *If* $\theta = \theta^\kappa/D_*$ *(in* $\mathbf{V}$*)* *then for every* $w \subseteq \mathbf{W}$ *the following are equivalent*

- $|\{a : \mathbf{W} \models a \in w\}| \le \theta$
- *for some* $w \in \mathbf{W}$ *we have* $\mathbf{W} \models$ "$|w| \le \mathbf{j}(\theta)$" *and* $u \subseteq \{a : \mathbf{W} \models$ "$a \in w$"$\}$.

Proof. See above. □

{m10}
Claim 9.8. *If* I *is a linear order of cofinality* $\theta > \kappa$ *then* $\{\mathbf{j}(s) : s \in I\}$ *is a cofinal subset of* $I^{\mathbf{W}} = I[\mathbf{W}]$ *the linear order with set of elements* $\{a : \mathbf{W} \models$ "$a \in I$"$\}$ *and* $I^{\mathbf{W}} \models$ "$a < b$" *iff* $\mathbf{W} \models$ "$(I \models a < b)$".

{m13}
Claim 9.9. *(Also 3.10!) 1) If* $\mathbf{W} \models$ "*I is the linear order* $(a, <) \in \text{Reg}\backslash\mathbf{j}(\mu)$" *then in* $\mathbf{V}[\mathbf{G}]$, $\text{tcf}(I^{\mathbf{W}}) \in \text{Reg}^{\mathbf{V}}$.
2) Moreover if $I = f/D_*, f : \kappa_* \to$ *(the class of regular cardinals) then for some* $p \in \mathbf{G}$ *and* $\lambda \in \text{Reg}\backslash\mu$ *we have* $\lambda = \text{ps-tcf}(\Pi f, <_{D_p})$.

{m15}
Claim 9.10. *If* θ *is a regular cardinal* $> |\mathbb{P}| + \kappa$ *and* $\bar{u} = \langle u_\alpha : \alpha < \theta\rangle$ *is a sequence of non-empty subsets of* $\text{Ord}^{\mathbf{W}}$ *and* $a \in u_\alpha \wedge b \in u_\beta \wedge \alpha < \beta \Rightarrow \mathbf{W} \models a < b$ *then* $\bar{u}$ *has an lub, i.e. there is* a_* *such that*

- $a_* \in \text{Ord}^{\mathbf{W}}$

- $\alpha < \theta \wedge a \in u_\alpha \Rightarrow \mathbf{W} \models a < a_2$
- *if* a'_* *satisfies* $\bullet_1 + \bullet_2$ *then* $\mathbf{W} \models$ "$a_* \leq a'_*$".

Proof. By xxx. $\square$

{m17}

Claim 9.11. *(Like 9.5(2).)*

{m19}

Claim 9.12. *A sufficient condition for* $\mathbf{W} \models$ "$\theta \in (f/D) \cap (\text{Reg} \backslash$ "$\mathbf{j}(\mu)$" $\Rightarrow \lambda_\theta < \chi$ *is:* $\mathbb{P} = (D^+_{**}, \supseteq)$ *and* (f, D_{**}) *or niceness (check!).*

§ 10. More pcf with little choice: a try

Question 10.1. (To [Sh:F1039])

§(7A) Introductory Remark

Discussion 10.2. We observe [Sh:938, §3,§4] works if we demand just that $D_{\mathbf{d}}$ a semi-filter. Then we replace measurable by the chosen win in a cut and choose game. Third, ?

Lastly, let the chosen choose few instead?

{z2}
Definition 10.3. We say D is a semi-filter on $\mathscr{Y}$ when:

(a) $D \subseteq \mathscr{P}(Y)$
(b) if $A \subseteq B \subseteq Y$ and $A \in D$ then $B \in D$
(c) $\emptyset \notin D$ and $Y \in D$.

{z4}
Claim 10.4. *If in* [Sh:938, Def.3.1(b)(β)] *we weaken the demand "$D_{\mathbf{d}}$ is a filter on $Y_{\mathbf{d}}$" to "$D_{\mathbf{d}}$ is a semi-filter on $Y_{\mathbf{d}}$" still all the claims (and definitions) in* [Sh:938, §3,§4] *works.*

§(7B) Games and Rank

{g2}
Definition 10.5. We say $\mathbf{x}$ is appropriate when:

(a) $\mathbf{x} = (\kappa, \theta, \sigma, D_1, D_2) = (\kappa_{\mathbf{x}}, \theta_{\mathbf{x}}, \sigma_{\mathbf{x}}, D_{\mathbf{x},1}, D_{\mathbf{x},2})$
(b) $\kappa > \theta > \sigma$ are cardinals
(c) $D_{\mathbf{x},1} \subseteq D_{\mathbf{x},2}$ are filters on κ.

{g4}
Definition 10.6. 1) We say $\mathbf{x}$ is large when the chooser has a winning strategy in the game $\partial_{\mathbf{x}}$ defined below.
2) The game $\partial_{\mathbf{x}}$ between the player cutter and chooser last ω moves in the n-th move a set $A_{n+1} \in D^+_{\mathbf{x},2}$ is chosen, letting $A_0 = \kappa$. In the n-th move the cutter chooses $\zeta_n < \theta$ and $f_n : A_n \to \alpha_n$, and the chooser chooses $w_n \in [\zeta_n]^{<(1+\sigma)}$ and let $A_{n+1} = \{\alpha \in A_n : f_n(\alpha) \in w_n\}$.

In the end the chooser wins iff $\cap\{A_n : n < \omega\} \in D^+_{\mathbf{x},1}$.

For the rest of this section

{g8}
Hypothesis 10.7. We assume $\mathbf{x}$ is large and $\mathbf{st}$ is a winning strategy for the chooser and $\sigma_{\mathbf{x}} = 1$.

{g10}
Definition 10.8. 1) $P =$ pos($\mathbf{x}$,$\mathbf{st}$) is the set of finite initial segments of a play of the game $\partial_{\mathbf{x}}$ during which the chooser uses the strategy G; we denote such initial segments by s and A_s is A_n for the maximal $n < \omega$ such that it is well defined.
2) For $s, t \in P$ let $s \le t$ iff s is an initial segment of t.
3) Let $P_{\ge s} = \{t \in P : s \le t\}$.

{g12}
Definition 10.9. 1) For $s \in P$ let $D_s = D_{\mathbf{x},\mathbf{st},s} = \{A \subseteq \kappa_{\mathbf{x}}$: for no t do we have $s \le t \wedge A \cap A_t = \emptyset \mod D_{\mathbf{x},2}\}$.
2) We define $\text{rk}_s(f) \in$ Ord $\cup\{\infty\}$ by defining when $\text{rk}_s(f) = \alpha$ for $s \in P, f \in {}^{\kappa}\text{Ord}$ and $\alpha \in$ Ord (and let $\text{rk}_s(f) =^* \alpha$ when below $t = s$ is O.K.)

⊞ $\mathrm{rk}_s(f) = B$ iff $\beta < \alpha \neq \neg(\mathrm{rk}_s(f) = \beta)$ for some $t \in P_{\geq s}$ for every $g \in {}^\kappa\mathrm{Ord}$ satisfying $g <_{D_{\mathbf{x},2}+A_t} f$ we can find $\beta < \alpha$ such that $\mathrm{rk}_t(g) = \beta$.

{g14} **Claim 10.10.** *1) For $s \in P$ and $f \in {}^\kappa\mathrm{Ord}$, exactly one $\alpha \in \mathrm{Ord} \cup \{\infty\}$ we have $\mathrm{rk}_s(f) = \alpha$.*
2) Assume $f, g \in {}^\kappa\mathrm{Ord}$ and $s \in P$. If $f = g \bmod(D_{\mathbf{x},\kappa} + A_s)$ then $\mathrm{rk}_s(f) = \mathrm{rk}_t(g)$ and if $f \leq g \bmod(D_{\mathbf{x},2} + A_s)$ then $\mathrm{rk}_s(f) \leq \mathrm{rk}_t(g)$.
3) [DC] For $s \in P$ and $f \in {}^\kappa\mathrm{Ord}$ we have $\mathrm{rk}_s(f) \in \mathrm{Ord}$.

Proof. Easy. □

Claim 10.11. *If $\zeta = \mathrm{rk}_s(f)$ and $h : A_s \to \theta$ then for some $\varepsilon < \theta$ and $t \in P_{\geq s}$ we have $\mathrm{rk}_t(f) = \zeta$ and $h{\restriction}A_t$ is constant.* {g17}

Proof. Without loss of generality $\mathrm{rk}_s(f) =^* \zeta$.
Not sure, try definition by forcing when ... ? □

§ 11. Various

{c3.2} §(7C)

Definition 11.1. Assume D is a filter on Y.
1) Let $\mathrm{oq}(Y) = \mathrm{oq}(Y,D) = \{f : f$ a function from Y onto some ordinal$\}$.
2) For $f \in \mathrm{oq}(Y)$ let $e_f = \{(y_1,y_2) : y_1 \in Y, y_2 \in Y$ and $f(y_1) = f(y_2)\}$.
3) Let $\mathrm{oeq}(Y) = \{e_f : f \in \mathrm{oq}(Y,d)\}$.
4) For $h \in \mathrm{oq}(Y,D)$ let D/h be $\{x \subseteq \mathrm{Rang}(h) : h^{-1}(X) \in D\}$, a filter on $\mathrm{Rang}(f)$ which necessarily is an ordinal $< \mathrm{hrtg}(Y)$.
5) For $f \in {}^Y\mathrm{Ord}$ let g_f be the following function:

- (a) $\mathrm{Dom}(g_f) = \mathrm{otp}(\mathrm{Rang}(f))$
- (b) $g_f(i) = \alpha$ iff $(\exists y)(y \in Y \wedge f(y) = \alpha \wedge i = \mathrm{otp}(f(y) \cap \mathrm{Rang}(f))$.

6) For $f \in {}^Y\mathrm{Ord}$ let h_f be the following function:

- (a) $\mathrm{Dom}(h_f) = Y$
- (b) $h_f(y) = \mathrm{otp}(f(y) \cap \mathrm{Rang}(f)) \in \mathrm{oq}(Y,d)$.

7) Assume $D \in \mathrm{Fil}^1_\kappa(Y)$ and $\bar f = \langle f_\alpha : \alpha < \alpha(*)\rangle$ is a $<_D$-increasing sequence of members of ${}^Y\mathrm{Ord}$

- (a) we let $\bar u = \langle u_{\bar f,h} : h \in \mathrm{oq}(Y,D)$ where $u_{\bar f,g} = \{\alpha < \alpha(*) : h_{f_\alpha} = h\}$
- (b) $\bar f_0^{[h]} = \langle g_{f_\alpha} : \alpha \in u_{\bar f,h}\rangle$ is $<_D$-increasing.

{c3.5} **Claim 11.2.** *Assume $D \in \mathrm{Fil}^1_\kappa$.*
1) Assume $\bar f = \langle f_\alpha : \alpha < \delta\rangle$ is a $<_D$-increasing sequence of members of ${}^Y\mathrm{Ord}$

- (a) *$\langle u_{\bar f,h} : h \in \mathrm{oq}(Y,D)\rangle$ is a partition of Y*
- (b) *$\mathrm{cf}(\delta) \ge \mathrm{hrtg}(\mathrm{oq}(Y,D))$ then for some $h \in \mathrm{oq}(Y,D)$ the set $u_{\bar f,h}$ is an unbounded subset of δ*
- (c) *for $h \in \mathrm{oq}(Y)$ the sequence $\langle g_{f_\alpha} : \alpha \in u_{\bar f,h}\rangle$ is a $<_{D/h}$-increasing sequence of members of ${}^{\mathrm{Dom}(h)}\mathrm{Ord}$*
- (d) *in (b); if $\delta = |\delta|$ then for some $h \in \mathrm{oq}(Y)$ the set $u_{\bar f,h}$ has order-type δ.*

2) For $\bar\alpha \in {}^Y\mathrm{Ord}$ for every regular $\lambda \ge \mathrm{hrtg}(Y)$ we have

- (a) *$\lambda \in \mathrm{ps\text{-}tcf}_{\kappa\text{-com}}(\bar\alpha)$ iff $\lambda \in \mathrm{ps\text{-}tcf}_{\kappa\text{-com}}(g_{\bar\alpha})$*
- (b) *$\lambda \in \mathrm{dp\text{-}tcf}_{\kappa\text{-com}}(\bar\alpha)$ iff $\lambda \in \mathrm{dp\text{-}tcf}_{\kappa\text{-com}}(g_{\bar\alpha})$ recalling $\mathrm{dp\text{-}tcf}_{\kappa\text{-com}}(\bar\alpha) = \{\lambda$: for some $D \in \mathrm{Fil}^1_\kappa(Y), \lambda = \mathrm{tcf}(\Pi\bar\alpha, D)$, equivalently there is a cofinal sequence of members of $\Pi\bar\alpha\}$.*

{c3.7} **Observation 11.3.** If $\mathrm{AC}_{\mathrm{hrtg}(Y)}$ then $\mathrm{hrtg}(Y)$ is a successor cardinal.

Proof. Toward contradiction assume $\mathrm{hrtg}(Y)$ is a limit cardinal say $\aleph_{\delta(*)}$.

For $\alpha < \mathrm{hrtg}(Y)$ let $\mathscr{F}^1_\alpha = \{g : g$ a function from Y onto $\alpha\}$, by the definition of $\mathrm{hrtg}(Y)$ it is non-empty, hence by AC_α the set $\mathscr{F}^2_\alpha = \{f : f$ a one-to-one function from α into $Y\}$ is non-empty. As $\langle \mathscr{F}^2_\alpha : \alpha < \mathrm{hrtg}(Y)\rangle$ exists and $\mathrm{AC}_{\theta(Y)}$ holds, there is a sequence $\langle f_\alpha : \alpha < \mathrm{hrtg}(Y)\rangle$ with $f_\alpha \in \mathscr{F}^2_\alpha$. Define the function pr with domain $\{(\alpha,\zeta) : \alpha < \mathrm{hrtg}_\zeta < \theta(Y)\}$ by $\mathrm{pr}(\alpha,\zeta) = \sum\limits_{\varepsilon<\zeta} \aleph_\varepsilon + \alpha$, now $\mathrm{pr}(\alpha,\zeta) < \aleph_{\zeta+1} \le \mathrm{hrtg}(Y)$ so pr is one-to-one into $\mathrm{hrtg}(Y)$, also the range of pr is an initial segment of Ord, and $|\text{Rang}(\text{pr}| = \text{Dom}(\text{pr})$ as it is one-to-one and obviously $|\text{Dom}(\text{pr})| \ge \theta$; together pr is onto $\text{hrtg}(Y)$. We define $\langle y_\gamma : \gamma < \text{hrtg}(Y)\rangle$ by $y_{\text{pr}(\alpha,\zeta)} = f_\zeta(\alpha)$ for $\alpha < \aleph_\zeta < \text{hrtg}(Y)$; let $u = \{\gamma < \text{hrtg}(Y) : (\forall \beta < \gamma)(y_\gamma \neq y_\beta)\}$, so easily $\zeta < \delta(*) \Rightarrow \aleph_{\zeta+1} = |u \cap [\aleph_\zeta, \aleph_{\zeta+1})|$, hence $|u| = \text{hrtg}(Y)$, hence $\langle y_\gamma : \gamma \in \text{hrtg}(Y)\rangle$ exemplify $\Upsilon(Y) > \text{hrtg}(Y)$, contradiction. $\square_{11.3}$

{c3.12y}

Claim 11.4. *Assume [?]*

(a) $\langle \bar{\mathscr{F}}_D : D \in \text{ps-tcf-fil}(\bar\alpha)\rangle$ *is as in ?*

(b) $\bar D = \langle D_i : i < i(*) < \kappa\rangle \in {}^{\kappa >}\text{ps-tcf-fil}_\kappa(\bar\alpha)$

(c) *for* $\bar D$ *as above and* $\bar\beta \in \prod\limits_i \text{tcf}(\Pi\alpha, <_{D_i})$ *let* $\mathscr{F}_{D,\bar\beta} = \{\sup\{f_{\beta_i} : i < \ell g(\bar\beta)\} : \bar f \in \prod\limits_{i<\ell g(\bar\beta)} \mathscr{F}_{D_i,\beta_i}\}$ *where* $f = \sup\{f_{\beta_i} : i < \ell g(\bar\beta)\}$ *which means* $s \in Y \Rightarrow f(s) = \sup\{f_{\beta_i}(i) : i < \ell g(\bar\beta)\}$

(d) $\{\mathscr{F}_{\bar D,\bar\beta} : \bar D \in {}^{\kappa>}(\text{ps-tcf-fil}_\kappa(D))$ *and* $\bar\beta \in \prod\limits_{i<\ell g(\beta)} \text{tcf}(\Pi\bar\alpha, <_{D_i})\}$ *is cofinal*

(e) $\text{ps-cf}^\kappa(\Pi\bar\alpha) = \sup(\text{ps-pcf}_\kappa(\Pi\bar\alpha))$ *where we define* $\text{ps-cf}^\kappa(\Pi\bar\alpha) \le S$ *<u>when</u> ... ?*

{c13yajan}

Claim 11.5. *Assume*

(a) $D \in \text{Fil}^1_\kappa(Y), \kappa \ge \aleph_1$ *and* $\alpha_y > 1$ *for* $y \in Y$

(c) $\text{rk}_D(\bar\alpha) = \zeta = |\zeta|$

(d) $\text{cf}(\zeta) > \text{hrtg}(\text{Fil}^1_\kappa(Y))$.

1) For some $J \in \{J[f,D] : f \in {}^Y\text{Ord}\}$ *we have* $\zeta = \text{otp}(\{\gamma$*: there is* $\bar\beta \in \Pi\bar\alpha$ *such that* $\text{rk}_D(\bar\beta) = \gamma$ *and* $J[\bar\beta, D] = J\})$.
2) ? In (1) if $\text{dual}(I) \subseteq D_1 \in \text{Fil}^1_\kappa(Y)$ *<u>then</u>* $\text{rk}_{D_1}(\bar\alpha) = \zeta$ *and ?*
3) ? Moreover in (1) if $\bar\beta \in \Pi\bar\alpha$*,* $\text{rk}_D(\bar\beta) = \gamma, J[\bar\beta, D] = J$ *then* $\text{rk}_{D_1}(\bar\beta) \subseteq$ *??*

Proof. 1) For $\varepsilon < \zeta$ let $\mathscr{F}_\varepsilon = \{\bar\beta \in \Pi\bar\alpha$: $\text{rk}_D(\bar\beta) = \varepsilon\}$ so $\bar{\mathscr{F}} = \langle \mathscr{F}_\varepsilon : \varepsilon < \zeta\rangle$ exists and $\varepsilon < \zeta \Rightarrow \mathscr{F}_\varepsilon \neq \emptyset$ by xxxx and $\cup\{\mathscr{F}_\varepsilon : \varepsilon < \zeta\} = \Pi\bar\alpha$.

Let $\mathscr{F}_{\varepsilon,E} = \{\bar\beta \in \mathscr{F}_\varepsilon : J[\bar\beta, D] = \text{dual}(E)\}$ for $E \in \text{Fil}^1_\kappa(f))$ extending D and let $u_E = \{\varepsilon < \zeta : \mathscr{F}_{\varepsilon,E} \neq \emptyset\}$, so $\mathscr{F}_\varepsilon = \cup\bigcup\{\mathscr{F}_{\varepsilon,E} : E \in \text{Fil}^1_\kappa(D)\}$ and $\zeta = \cup\{u_E : D \subseteq E \in \text{Fil}^1_\kappa(Y)\}$. As $\text{cf}(\zeta) > \text{hrtg}(\text{Fil}^1_\kappa(Y))$ necessarily for some $E, |u_E| = \zeta$ but $u_E \subseteq \zeta = |\zeta|$ hence $\text{otp}(u_E) = \zeta$, so $\text{dual}(E)$ is as required.
2) By (3). ?
3) ? So J is from (1) and toward contradiction assume $\text{dual}(J) \subseteq D_1 \in \text{Fil}^1_\kappa(Y)$ and $\bar\alpha_1 \in \Pi\bar\alpha$, but $\text{rk}_{D_1}(\bar\alpha_1) \ge \zeta$; without loss of generality $y \in Y \Rightarrow \alpha_{1,y} > 0$ and $\text{rk}_{D_1}(\bar\alpha_1) = \zeta_1$. Now we choose $\mathscr{F}^1_\varepsilon, \mathscr{F}^1_{\varepsilon,E}, E_2$ as in the proof of part (1) starting with $\bar\alpha_1, \zeta_1$. $\square_{??}$

{c14y}

Claim 11.6. [DC] *1) If* $\aleph_0 < \kappa = \text{cf}(\mu) < \mu$ *then* $\text{rk}_{J^{\text{bd}}_\kappa}(\mu) > \mu^+$.

Proof. 1) Clearly J^{bd}_κ is a uniform κ-complete filter on κ. Let $\langle \mu_i : i < \kappa\rangle$ be increasing continuous with limit $\mu, \kappa < \mu_0$. For each $\alpha < \mu^+$ let

$$\mathscr{F}_\alpha = \{f : f \text{ a one-to-tone function from some subset of } \mu \text{ onto } \alpha\}$$

$\mathscr{G}_\alpha = \{g_\ell : g_\ell$ for some $f \in \mathscr{F}_\alpha\}$ where for $f \in \mathscr{F}_\alpha$ for some $\alpha < \mu^+$ we let g_f be defined by

$(*)_0$ $\mathrm{Dom}(g_f) = \kappa$ and for every $i < \kappa, g(i) = \mathrm{otp}(\{g(\varepsilon) : \varepsilon < \mu_i \cap \mathrm{Dom}(f)\})\}$
$(*)_1$ $\mathscr{F}_\alpha \neq \emptyset$ for $\alpha < \mu^+$
$(*)_2$ $\mathscr{G}_\alpha \neq \emptyset$ for $\alpha < \mu^+$
$(*)_3$ $\mathscr{G}_\alpha \subseteq \prod_{i<\kappa} \mu_i^+ \subseteq {}^\kappa\mu$.

[Why? As the set $\{f(\varepsilon) : \varepsilon \in \mu_i \cap \mathrm{Dom}(f)\}$ has cardinal $\leq \mu_i$, so have order type $< \mu_i^+$.]

$(*)_4$ if $\alpha_1 < \alpha_2$ and $g_2 \in \mathscr{G}_{\alpha_2}$ then for some $g_1 \in \mathscr{G}_{\alpha_1}$ we have $g_1 < g_2 \bmod J_\kappa^{\mathrm{bd}}$.

[Why? Let $g_2 = g_{f_2}$ so $\beta_1 \in \beta_2 = \mathrm{Rang}(f_2)$ so let $\beta_1 = f(\varepsilon_1)$ and i_1 be a $\min\{i < \kappa : \mu_i > \varepsilon_j\}$. Let $\mathscr{U} = \{\varepsilon \in \mathrm{Dom}(f_2) : f_2(\varepsilon) < \beta_1\}$ and $f_1 = f_2 {\restriction} \mathscr{U}$ and let $g_1 = g_{f_1}$, so clearly $g_1 \in \mathscr{G}_{\alpha_1}$. Now if $i \in i_1, \kappa 0$ then $\{f_1(\varepsilon) : \varepsilon \in \mu_1 \cap \mathrm{Dom}(g_{f_1})\} \subseteq B_1 \cap \{f_2(\varepsilon) : \varepsilon \in \mu_i \cap \mathrm{Dom}(f_2)\}$ and $\beta_1 \in \{g_2(\varepsilon) : \varepsilon \in \mu_i \cap \mathrm{Dom}(f_2)\}$, so clearly $g_{f_1}(i) < g_{f_2}(i)$.

So $g_1 < g_2 \bmod J_\kappa^{\mathrm{bd}}$ is as required.]

For $\alpha_* \in [\mu^+, \mu^{++})$ and we shall prove that $\mathrm{rk}_D(g) \geq \alpha_*$ for some $g \in {}^\kappa\mu$, this suffices.

As (α_*) there is $\bar{w}$ such that

$(*)$ (a) $\bar{u} = \langle w_i : i < \chi\rangle$
(b) $i < \chi \Rightarrow |w_i| = \mu$
(c) $\alpha_* = \cup\{w_i : i < \chi\}$.

As $\mathrm{cf}(\mu^+) = \chi$ we can choose $\bar{\alpha}$ such that

$(*)$ (a) $\bar{\alpha} = \langle \alpha_j : j < \chi\rangle$
(b) $\bar{\alpha}$ is increasing, $\alpha_j > \chi, \kappa$
(c) $\bar{\alpha}$ is with limit μ^+.

Now $y \in Y$ let

$(*)$ $w_y = \cup\{w_i : i \in y\}$
$(*)$ for $y \in Y$
(a) $|w_g| \leq \mu$
(b) ??

$\square$

§ 12. Private Appendix

We can add to [Sh:938, 2.6,2.7]

Claim 12.1. *The filter D_2 4-commutes with the filter D_1 (see [Sh:938, 3.1]) <u>when</u>:* {k17}

(a) $D_\ell \in \mathrm{Fil}_{\mathrm{cc}}(Y_\ell)$ *for* $\ell = 1, 2$

(b) D_1 *is* σ*-complete*

(c) *if* $J_1 \in \{J[f, D_1] : f \in {}^{Y_1}Ord\}$ *or just* J_1 *is a* σ*-complete ideal extending* $\mathrm{dual}(D_1)$ *then* $A \subseteq Y_1$ *but* $\mathrm{dual}(J_1) \in \{D_1 + A : A \in D_1^+\}$*; this follows from clause (b) + DC_σ* $\mathrm{VAC}_{\mathscr{P}(Y_1)}$ *when* D_1 *is* σ*-c.c., i.e. there is no sequence* $\langle A_i : i < \sigma\rangle$ *of a pairwise disjoint sets from* D_1^+

(d) DC_σ *and* AC_{Y_1}, AC_{Y_2}

(e) (α) D_1 *is* $\mathscr{P}(Y_2)$*-complete or just*

(β) *if* $\langle B_s : s \in A_1\rangle \in {}^A(J_2^+)$ *and* $A \in J_1^+, J_\ell \in \{J[f, D_\ell] : f\}$ *for* $\ell = 1, 2$ <u>*then*</u> *for some* $B_* \in J_2^+$ *and we have* $A_* \subseteq A, A_* \in J_1$ *we have* $s \in A_* \Rightarrow B_s \supseteq B_s$.

Proof. <u>Stage A</u>:

Let $A \in D_2$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ and $\bar{J}^1 = \langle J_t^1 : t \in Y_2\rangle$ where $J_t^1 \in \{J[f, D_1) : f \in {}^{Y_2}\mathrm{Ord}\}$ and $J_2 \in \{J[f, D_2) : f \in {}^{I_2}\mathrm{Ord}\}$, i.e. as in the assumption of $\boxplus_4$ of Definition [Sh:938, 2.1]. We should find A_*, B_* as there.

<u>Stage B</u>:

For each $t \in I_2$ there is $A_t \in D_1^+$ such that $J_t^1 =$ dual$(D_1 + A_t)$, hence as AC_{Y_2} holds such that $\langle A_t : t \in Y_2\rangle$ exist. Why? By clauses (b),(c) of the assumption.

<u>Stage C</u>:

Choice of B_*, A_*. Apply clause (d) of the assumption applied to $(J_2, \langle A_t : t \in I_2\rangle)$. $\square_{13.15}$

Remark 12.2. 1) We can weaken "D_1 is σ-complete, σ-c.c." to "D_2 is σ-complete, σ^+-c.c." when we have some normality conditions. {k19}
2) We can replace this by "any $J[f, D_1]$ is of the form $D_1 + A$ for some $A \in D_1^+$".

We can add in [Sh:938, §4]

Conclusion 12.3. [$\mathrm{AC}_{<\mu}$ and μ a limit singular cardinality] {k23}

Assume $\mu = \sup\{\kappa < \mu$: for some $\lambda \in [\kappa, \mu)$ on λ there is a κ-complete κ-c.c. filter D on $\lambda\}$. <u>Then</u> for every ordinal ζ for some $\kappa_* < \mu$, for every $\lambda \in [\kappa, \mu)$ and κ-complete κ-c.c. filter D on λ we have $\mathrm{rk}_D(\zeta) = \zeta$.

Proof. By 13.15 and [Sh:938, 4.1]. $\square_{12.3}$

We define $f : Y_1 \to \mathscr{P}(Y_2)$ by $f(s) = \{t \in Y_2 : s \in A_t\}$; as D_1 is $(P(Y_2))$-complete filters on Y_1 necessarily also J_2 is a $(\mathscr{P}(Y_2))$-complete ideal on Y_1 hence there is

- $Y_2^* \subseteq Y_2$ such that $A^* := \{y \in A_1 : f(y) = Y_2^*\}$ belongs to J_2^+.

Choose $s_* \in A^*$ so $Y_2^* = f(s) = \{t \in Y_2 : s \in A_t\}$.

§ 13. Private Appendix

Remark 13.1. pcf inventory (August 2009)
1) See [Sh:F663] lecture - [Sh:430, §6] is locality proved for $\mathrm{pcf}_{\theta\text{-com}}(-), \theta > |\mathfrak{a}|$.
2) See Rinot question [Sh:F893].
3) See the notes for Larson [Sh:F814] - on HOD.
4) Continue [?], see [Sh:F878].
5) Failed try to continue [Sh:460, §5B], [Sh:F563].
6) [Sh:F355] - on consistency - answer Gitik?
7) [Sh:F354] $\lambda = \sup(\lambda \cap \mathrm{pcf}(\mathfrak{a}))$ is weakly inaccessible.
8) Densities of basic product [Sh:F132], covered by paper with Moti?
9) [Sh:F50] to Shimoni.
10) Hopes rank for precipiousness?
11) Sort out? Y_n is well ordered, need $\mathrm{IND}_{\aleph_1}(\bar{D})$?
12) (09.10.19) A related question: let $\mathbf{x} = \langle (Y_n, D_n, h_n) : n < \omega \rangle$ is here $h_n : Y_n \to Y$ and D a filter on Y and we try to prove

- $(*)$ for every $f \in {}^Y\mathrm{Ord}$, for every large enough n we have $\mathrm{rk}_{D_n}(f \circ h_n) \subseteq \mathrm{rk}_D(f)$ or similarly for Depth.

13) (09.10.26, old thought) As we pass from cofinality to pseudo-cofinality, iterate this notion and then have strong dichotomies.
14) (09.11.15) Think of a problem where:

- (a) $\mathrm{Depth}({}^\omega(\aleph_n), \mathscr{D}_{\aleph_n})$ large given an answer.

15) Tasks (2010.1.08)

- (a) if $Y = \chi$, then we can replace $\mathrm{AC}_{\mathscr{P}(Y)}$ by DC_{χ^+}
- (b) replace Y by all $\mu < \theta(Y)$, just split to some ?
- (c) Definition $\text{dp-pcf}_\kappa(Y) = \{x : \lambda$ regular and there is a filter D such that $\lambda = \text{dp-tcf}(\pi\bar{\alpha}, <_D)\}$ where: $\text{dp-tcf}(\pi\bar{\alpha}, <_D)$ means there is an increasing cofinal of this length
- (d) nice results but no existence
- (e) given $\bar{\alpha}$, how much choice needed to find D with $\mathrm{dual}(D) = ([Z]^{<\kappa} + (Y \backslash Z)$ for some Z?
- (f) for a λ-sequence of length $\lambda, <_{D_1}$-increasing in ${}^Y\mathrm{Ord}$, is there $<_{D_2}$-lub for some $D_2 \supseteq D_1$?
- (g) smooth closed generating sequence: by $\mathrm{DC}_{|Y|}$?
- (h) generalize [Sh:460]
- (i) get bound or Depth $\aleph_{\omega_1}$
- (j) try for a dichotomy: with IND

{q4} **Discussion 13.2.** (2010.3.08) Why the question 13.4(1) help? similarly 13.4(2). So assume

- (a) $f_* \in {}^\theta(\mathrm{Reg} \cap \mu_1)$
- (b) D a non-principal ultrafilter on θ
- (c) $\mathrm{cf}(\prod_i f_*(i)/D) = \lambda^+$

(d) no $f/D < f_*/D$ satisfies (c), or do we use less?

(e) $\theta < \kappa, \langle \mu_1, \lambda \rangle \kappa^\theta$, probably assuming $2^\theta < \kappa$ maybe it is much less interesting though we may get more than in [Sh:460], <u>then</u> D is in $\mathbf{V}^{\mathbb{P}}, |\mathbb{P}| = 2^\theta$

(f) $\lambda_j \in$ Reg $\cap \mu_2 \backslash \mu_1 (j < \kappa)$

(g) $\lambda^+ =$ tcf$(\prod\limits_j \lambda_1, <_E)$

(h) for each $i < \theta, f_*(i)$ is inaccessible for any κ-complete filter/ideal on κ.

Without loss of generality $\bigwedge\limits_i \aleph_0 < f_*(i)$.

We can find $g_j \in \pi(f_*(i) \cap$ Reg) for $j < \kappa$ such that λ_j, cf$(\pi g_j, <_{D_1})$. Let $\mathfrak{a}_i = \{g_j(i) : j < \kappa\} \backslash (\kappa^\theta)^+, \mathbf{V} = V^\theta/D, \mathbf{j} : \mathbf{V} \to \mathbf{V}, \bar{\mathfrak{a}} = \langle \mathfrak{a}_i : i < \kappa \rangle, \mathscr{A} = \bar{a}/D$, so $\mathbf{V} \models$ "$(g_j/D) \in \mathscr{A}$" $\wedge$ ($\mathscr{A}$ has cardinality $\le \mathbf{j}(\kappa)) \cap (\mathscr{A}$ a set of regulars $> \mathbf{j}(\kappa))$".

So in $\mathbf{V}$ we have the basic pcf results $\langle \mathfrak{b}_{g/D}[\bar{a}/D : g/D \in \mathscr{A}\rangle, \langle f_\alpha^{g/D, \bar{\mathfrak{a}}/D} : \alpha \in g/D\rangle$ as in xxx.

Note

⊞ $\mathbf{V} \models$ "there is a division of κ to $\ll \kappa$ sets $\langle u_{i,\varepsilon} : \varepsilon < \varepsilon_1\rangle$, max pcf$\{g_j(i) : j \in u_{i,\varepsilon}\} < f_*(i)$"

⊞ in $\mathbf{V}, \mathscr{A}$ is listed by $\langle \lambda^*_\varepsilon : \varepsilon \in \kappa^\theta/D\rangle$

⊞ in $\mathbf{V}$ and $\kappa^\theta/D \in \mathbf{V}$ is linear order with $\{\mathbf{j}(j) : j < \kappa\}$ unbounded in it

⊞ if $\mathbf{V} \models g/D =$ tcf$(\prod\limits_{g \in I} g_a/D, <_{\mathscr{E}})$ <u>then</u> this is essentially true letting E be the filter on $\{a : \mathbf{V} \models a \in I\}, \lambda_a =$ cf$(\pi g_a, <_D), \lambda =$ cf$(\pi g, <_D)$ we have $\lambda =$ tcf$(\prod\limits_a \lambda_a, <_E)$ when the $\lambda_a > 2^\theta$.

Discussion 13.3. (2010.3.8) We return to the trying to improve [Sh:460].

{q2}

Question 13.4. Concerning [Sh:460], so say for $\mu >$ cf$(\mu)(= \aleph_0)$? λ is the first counterexample $> 2^\mu$ so cf$(\lambda) =$ cf(μ). Let $\theta < \kappa, D$ an ultrafilter on θ such that for some $f_\theta \in {}^\theta\lambda$, cf$(\prod\limits_{i \in \theta} f_\theta(i), <_D) = \lambda^+$.

1) Can we have "f_θ/D is the first f/D such that cf$(\prod\limits_i f(i); <_D) = \lambda^+$?

2) Or at least can we find $\bar{\mathfrak{a}}$ such that

(a) $\bar{\mathfrak{a}} = \langle \mathfrak{a}_i : i < \theta \rangle$

(b) $\bar{\mathfrak{a}} \in [\text{Reg } \cap \lambda]^{\le 2^\theta}$

(c) $f \in \pi \mathfrak{a}_i \Rightarrow$ cf$(\prod\limits_i f(i), <_D) = \lambda^+$ and

(d) $g \in {}^\theta\lambda \wedge$ cf$(\pi g, <_D) = \theta \Rightarrow \bigvee\limits_{f \in \pi \mathfrak{a}_i} (f/D < g/D)$.

3) Maybe λ is the first such that:

$(*)_1$ for arbitrarily large $\theta < \mu$ (regular $\theta < \mu$) there is $\mathfrak{a} \in [\text{Reg } \cap \lambda]^{\le \theta}$ bounded in $\lambda, \lambda \in$ pcf$(\mathfrak{a}), \mathfrak{b} \in [\mathfrak{a}]^{< \theta} \Rightarrow \lambda \notin$ pcf$(\mathfrak{b})$.

In the case clause (d) holds

{g23}

Claim 13.5. *(2010.3.08) We assume an axiom from* [Sh:835] *and prove RGCH in the depth version for $\mu >$ cf$(\mu) = \aleph_0$ strong limit and* AC$_\mu, \kappa < \mu \Rightarrow \theta(\mathscr{P}(\kappa)) < \mu$.

Alternative: (2010.3.08)
1) Assume $\mathrm{DC}_{<\mu}$ (and so $P(\kappa) < \mu$ for $\kappa < \mu$. Use the RGCH version with nice representation of $\mathrm{pcf}(\mathfrak{a})$, for the pseudo cofinality version.
2) Is ps-pcf(ps-pcf($\mathfrak{a}$)) = ps-pcf($\mathfrak{a}$)? So we have $\lambda_i =$ ps-tcf$_{\aleph_1}(\prod\limits_j \lambda_{i,j}, M_{D_1}), \lambda =$ ps-pcf$_{\aleph_1}(\pi\lambda_i, <_D)$. Yes (but as anyhow we use $\mathrm{pcf}_{\aleph_1-\mathrm{comp}}$, iterating $\omega \times \omega$ we are done).

Moved 2010.1.08 from 16.8, p.7:
2) $[\mathrm{AC}_{\mathscr{P}(Y)}]$ If D is κ-complete but not $(< \infty)$-complete then AC_κ.
2) So without loss of generality D is κ-complete not κ^+-complete hence there is a sequence $\bar{A} = \langle A_\alpha : \alpha < \kappa\rangle$ of members of D with $\cap\{A_\alpha : \alpha < \kappa\} \notin D$ and without loss of generality$\bar{A}$ is with no repetition. This implies $\kappa < \theta(\mathscr{P}(Y))$, but we have $\mathrm{AC}_{\mathscr{P}(Y)}$ hence we have AC_κ as promised.

* * *

Moved from pg.8:
For $\aleph_1$-complete ultrafilter we get more

{r31} **Claim 13.6.** *[true??] Let D be an $\aleph_1$-complete ultrafilter on Y. Then for any $f \in {}^Y(\mathrm{Ord}\backslash\{0\})$ we have $\mathrm{rk}_D(f) =$ ps-o-Depth$(\prod\limits_{t\in Y} f(t), <_D)$ and the supremum on the left is obtained.*

Proof. Obvious. $\square_{13.6}$

Question 13.7. 1) Can we prove parallel of the ZFC results?
2) (09.7.19) Is this not $\theta(\Pi\bar{\alpha}/D)$?

Moved from Anotated Content:

§(2A) Getting quasi-rank systems with $\mathrm{AC}_{<\mu}$, pg.7 (090909)?
[We start with pre-rank-system $\mathbf{p}$ and define rank trying to get a strict rank system using IND we get that the ranks are $< \infty$. Has to be read together with [Sh:938]. While this has to be checked we still use $\mathrm{AC}_{<\mu}, \mu = \sum\limits_n \kappa_n$. A new suggestion in f6.2, f6.3d, f6.9(5) has not been elaborated on.]

§3 Connection to IND, pg.13

§4 Appendix, pg.19
[We repeat [Sh:938, §5].]

NOTE: pg.9I - can't read the top of this page

{k10} **Discussion 13.8.** Whereas our orignal intention was to use $\mathrm{IND}(\mathbf{x})$, we actualy use only $\mathrm{IND}'(\mathbf{x})$, which is much better.

{k12} **Definition 13.9.** 1) $\mathrm{IND}'(\langle(Y_n, D_n) : n < \omega\rangle)$ means that if no $\bar{F} = \langle F_n : n < \omega\rangle$ is a witness against it which means:

(a) F_n is a two-place function from $I_{n+1} \cup \{\mathbf{x}\}$ into $\mathrm{dual}(D_n)$

(b) there are no $\bar{t}_n = \langle t_{n,\ell} : \ell < n\rangle \in I^1_{0,n}$ for $n < \omega$, stipulating $t_{n,n} = x$ we have $m < n \Rightarrow t_{n,m}t_{n-1,m} \notin F_m(t_{n,m+1}, t_{n+1,m})$.

2) Let $\mathrm{IND}''(\langle(Y_n, D_n) : n < \omega\rangle)$ means that there is no $\langle F_{m,n} : m, n < \omega\rangle$ a witness against it which means:

(a) $F_{m,n}$ is a two-place function from $I_n \cup \{\}$ into $\mathrm{dual}(D_n)$

(b) $u_{n,\varepsilon,\xi,t} \subseteq \{(\varepsilon_1, \xi_1 : \varepsilon_1 < xi_1 \le \zeta\}$ coming from $(\mathscr{F}_{n,\varepsilon}, \mathscr{F}_{n,\xi})$.

{k14}

Question 13.10. 1) If we try to prove 3.13 with choosing $\ll \sum\limits_n (I/n)$?

2) Try $\zeta_n =$ oDepth$({}^{\mathscr{I}_n}\zeta, <_n)$ is $\gg \zeta$. Really for every $\bar{\zeta} \in \prod\limits_{n<\omega} \xi_n$ we have $\bar{F}^{\bar{\zeta}}$ for the Y's witnessing failure of $\mathrm{IND}(\mathbf{x})$ can we combine to get a contradiction? We have the Z's colouring by large subsets of $Y_{0,n}$ with sub-additivity.

{k16}

Claim 13.11. [ZFC] *1) If $Y_n = \lambda, D_n = \{u \subseteq Y : Y \backslash u \le \kappa\}$ and $Y \to (\omega)^3_\kappa$ <u>then</u> $IND'(\langle(Y_n, D_n) : n < \omega)$.*
2) If $Y_n = \kappa_2, Y - D_n$-co-countable.

Discussion 13.12. We may wonder on relatives on 3.13. First, if instead ps-Depth we use Depth it seems that $\bigwedge\limits_n \mathrm{AC}_{I_n}$ is not necessary. Second, we may try to use ranks instead of depth.

* * *

Does looking at the proof of 3.13 give more?

{k18}

Definition 13.13. 1) We say $\bar{f}$ is an $(\mathbf{x}, \zeta)$-system or $(\bar{A}, \mathbf{x}, \zeta)$ is a system <u>when</u>

(a) $\mathbf{x} = \langle(Y_n, D_n : n < \omega\rangle, D_n$ a filter on Y_n

(b) ζ an ordinal

(c) $\bar{f} = \langle f_{n,\varepsilon} : n < \omega, \varepsilon \le \zeta)$

(d) $f_{n,\varepsilon} \in {}^{I_n}\zeta$ (with full choice without a more complicated

(e) $\varepsilon < \xi \le \zeta$ and $n < \omega$ then $f_{n,\varepsilon} <_{D_n} f_{n,\xi}$.

2) we say the pair $(\bar{t}, \bar{\varepsilon})$ solve the system $(\bar{A}, \mathbf{x}, \zeta)$ when

(a) $\bar{t} \in \prod\limits_{n<\omega} Y_n$

(b) $\bar{\varepsilon} = \langle\bar{\varepsilon}_n : n < \omega\rangle$ where $\bar{\varepsilon}_n = \langle\varepsilon_{n,\ell} : \ell \le n\rangle, \varepsilon_{n,\ell} \le \zeta$.

Remark 13.14. With little choice for $n < \omega, \varepsilon < \xi \le \varepsilon$ we have $\langle u_{n,\varepsilon,\xi,t} : t \in I_n\rangle$.
If D_{n+1} is λ^+_n-complete <u>then</u> ?

{k17}

Theorem 13.15. *[AC_{Y_n} for $n < \omega$.]*
Assume D_n is an $\aleph_1$-complete on Y_n for $n < \omega$ and $\mathrm{IND}(\langle D_n : n < \omega\rangle$ *then for every ζ, for some n we have* $\mathrm{rk}_{D_n}(\zeta) = \zeta$.

Definition 13.16. $\mathrm{AC}_{Y,2}$ where for every $\langle A_y : y \in Y\rangle$ there is $\langle B_y : y \in Y\rangle$ such that $Ay \ne \emptyset \Rightarrow B_y \ne \emptyset, |B_y| \le_* |Z|$.

Question 13.17. Interesting? Natural for a sequence $(\leq Z)$-complete filter, as in we can use $\langle \bigcap\limits_{a/\in By} : y \in Y\rangle$.

Proof. We choose g_n, Z_n as in the proof of 3.13 using the definition. $\square$

Remark 13.18. 1) In (5B), **??**(2) silly? We can find disjoint $Y_1 Y_2$ with $\mathrm{id}(Y_1) = \mathrm{id}(Y_2)$.
2) Definition **??**(2) line 2: $I \mapsto J$.

Discussion 13.19. Seemingly [Sh:835] connect well to [Sh:F955].

So ssume $\langle \lambda_i : i < \kappa\rangle$ is increasing with limit μ and that is we should deal with a game, where..?

* * *

§ 14. Private Appendix
Using pure Σ: July 2009

Definition 14.1. We say $\mathbf{s}$ is a frame <u>when</u> $\mathbf{s}$ consists of the following objects satisfying the following conditions: {m6}

(a) $\langle \kappa_i : i < \mathrm{cf}(\mu)\rangle$ is increasing with limit μ

(b) set $\mathbb{D}$

(c) $D_{\mathbf{d}}$ a filter on $I_{\mathbf{d}} = I[\mathbf{d}]$ for $\mathbf{d} \in \mathbb{D}$

(d) for $\mathbf{d} \in \mathbb{D}$
- (α) $\Sigma(\mathbf{d}) \subseteq \{(\mathbf{e}, h) : \mathbf{e} \in \mathbb{D}$ and h a function from $I_{\mathbf{e}}$ onto $I_{\mathbf{e}}$ such taht $D_{\mathbf{d}} = \{h''(A) : A \in D_{\mathbf{e}}\}$
- (β) $\Sigma_{\mathrm{pr}}(\mathbf{d}) \subseteq \Sigma(\mathbf{d})$, a set of so called pure extensions
- (γ) $\Sigma_{\mathrm{ap}}(\mathbf{d}) \subseteq \Sigma(\mathbf{d})$, a set of so called a-pure extensions such that $(\mathbf{e}, h) \in \Sigma_{\mathrm{ap}}(\mathbf{d}) \Rightarrow I_{\mathbf{e}} = I_{\mathbf{d}} \wedge h = \mathrm{id}_{I_{\mathbf{d}}}$
- (δ) $\mathbf{d} \in \Sigma_{\mathrm{pr}}(\mathbf{d}) \cap \Sigma_{\mathrm{ap}}(\mathbf{d})$
- (ε) transitivity of Σ? Σ_{pr}? Σ_{ap}?

(e) $\mathbf{j}$ is a function from $\mathbb{D}$ to $\mathrm{cf}(\mu)$ and $D_{\mathbf{d}}$ is $\kappa_{\mathbf{j}(\mathbf{d})}$-complete and $c \in \ell$ $\mathrm{par}(\mathbf{d}) \Rightarrow |S_\ell| < \kappa_{\mathbf{j}(\mathbf{d})}(?)$

(k) $\mathrm{par}(\mathbf{d})$ and for $p \in \mathrm{part}(\mathbf{d}), \bar{X}_p = \langle X_{p,s} : s \in S_p\rangle$ is a sequence of pairwise disjoint subsets of $I_{\mathbf{d}}$ with union $\in D_{\mathbf{d}}$ and $\langle \mathbf{e}_{p,s} : s \in S\rangle$ is such that $\mathbf{e}_{p,s} \in \mathbb{D}, I_{\mathbf{e}_{p,s}} = I_{\mathbf{d}}, D_{\mathbf{e}_{p,s}} = D_{\mathbf{d}} + X_{p,s}$ so $\mathbf{e}_{p,s} = \mathbf{d} + X_{p,s}$

(l)
- (α) if $\mathbf{d}_1 \in \Sigma_{\mathrm{pr}}(\mathbf{d}_0)$ and $\mathbf{d}_2 \in \Sigma_{\mathrm{ap}}(\mathbf{d}_0)$ then $\mathbf{d}_1 +_{\mathbf{d}_0} \mathbf{d}_2 = \mathbf{d}_1 +^{\mathbf{s}}_{\mathbf{d}_0} \mathbf{d}_2$ is a well defined member of $\mathbb{D}_{\mathbf{s}}$ and $\mathbf{d}_3 \in \Sigma_{\mathrm{pr}}(\mathbf{d}_2) \cap \Sigma_{\mathrm{ap}}(\mathbf{d}_1)$
- (β) above
- (γ) above if $\mathbf{e} \in \Sigma(\mathbf{d}_1) \cap \Sigma_1(\mathbf{d}_2)$ then $\mathbf{e} \in \Sigma(\mathbf{d})$.

Question 14.2. Maybe $\mathrm{cf}(\kappa)$ replaced by a linear order (which can have a pseudo cofinality)?

We now give examples

Definition/Claim 14.3. 1) Assume $\bar{\kappa} = \langle \kappa_n : n < \omega\rangle, \bar{J} = \langle J_n : n < \omega\rangle$, when J_n is a κ_n-complete ideal on I_n, and $\kappa_n < \kappa_{n+1}$ (or just $\kappa_n \le \kappa_{n+1}$)? We define $\mathbf{s} = \mathbf{s}_{\bar{\kappa},\bar{J}}$ and prove that $\mathbf{s}$ is a pre-system as follows (so $\mu = \mu_{\mathbf{s}}$, etc.) {m8}

(a) $\mu = \Sigma\kappa_n$ and κ is given

(b) $\mathbb{D}$ is the st of $\mathbf{d} : \mathbf{d} = (\eta, A) = (\eta_{\mathbf{d}}, A_{\mathbf{d}})$ and for some $m = m_{\mathbf{d}} \le n = n_{\mathbf{d}} < \omega$ we have
- (α) $\mathbf{F}_{\mathbf{d}} = \{\bar{F} : \bar{F} = \langle F_{m_1,n_1} : m_{\mathbf{d}} < m_1 \le n_1 \le n_{\mathbf{d}}\rangle = \langle F^{\mathbf{d}}_{m_1,n_1} : m_{\mathbf{d}} \le m_1 < n_1 \le n_{\mathbf{d}}\rangle$ and $F_{m_1,1} : \prod\limits_{\ell=m_1+1}^{n_1} I_{\eta(\ell)} \to J_{\eta(m_1)}$
- (β) $\eta = \langle n, n-1, \ldots, m\rangle$
- (γ) $I_{\mathbf{d}} = \prod\limits_{\ell=m}^{n} I_\ell$
- (δ) $D_{\mathbf{d}} = \{X \subseteq I_{\mathbf{d}}$: there are $X_\ell \in J_\ell$ for some $\bar{F} \in \mathbf{F}_{\bar{d}}$ for $\ell \in [m,n]$ such that $A \cap X \supseteq \{\rho \in I_{\mathbf{d}} : \rho(m_1) \notin F_{m_1,n_1}(\rho\restriction[m_1,n_1])$ whenever $m_{\mathbf{d}} \le m_1 < n_1 \le n_{\mathbf{d}}\}\}$

(ζ) $\emptyset \notin D_{\mathbf{d}}$ and nec $D_{\mathbf{d}}$ is κ_m-complete

(c) for $\mathbf{d} \in \mathbb{D}$

(α) let $\Sigma(\mathfrak{d})$ be the set of pairs $(\mathbf{e}, h)$ such that $\mathbf{e} \in \mathbb{D}, m_{\mathbf{e}} = m_{\mathbf{d}} \leq n_{\mathbf{d}} \leq n_{\mathbf{e}}, F^{\mathbf{e}}_{m_1,n_1} = F^{\mathbf{d}}_{m_1,n_1}$ when $n_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{d}}$ and h is $h(\rho) = \rho \restriction [m_{\mathbf{d}}, n_{\mathbf{d}}]$

(β) $\Sigma_{\rm pr}(\mathbf{d}) = \{(\mathbf{d}, h) \in \Sigma(\mathbf{d}) : F^{\mathbf{e}}_{m_1,n_1}$ is constantly $\emptyset$ when $n_{\mathbf{d}} < n_1$ (and $m_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{e}}$)

(γ) $\Sigma_{\rm ap}(\mathbf{d}) = \{(\mathbf{e}, h) \in \Sigma(\mathbf{d}) : h = {\rm id}_{I_{\mathbf{d}}}$ so $n_{\mathbf{e}} = n_{\mathbf{d}}$

(d) for $\mathfrak{d} \in \mathbb{D}_{\mathbf{s}}$ and $A \in D^+_{\mathbf{d}}$ let $\mathbf{e} = \mathbf{d} + A \in \mathbb{D}$ be defined naturally, it is $(\eta_{\mathbf{d}}, A_{\mathbf{d}} \cap A, \bar{F})$

(e) part ($\mathfrak{d}$) is the set of $p = ((X_{p,s}, \mathbf{e}_{p,s}) : s \in S\rangle$ such that: for some so called witness $\bar{G} = \ langleG_{m_1,n_1} : m_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{d}}\rangle, G_{m_1,n_1} : I_{[m_1+1,n_1]} \rightarrow \kappa_{m_1}$ with bounded range letting $S' = \{\langle \alpha_{m_1,n_1} : m_{\mathbf{d}} \leq m_1 < n_1 \leq n_{\mathbf{d}}\rangle : \alpha_{m_1,n_1} < \kappa_{m_1}\}$ and $A_{\bar{a}} = \{\rho \in I_{\mathbf{d}} : G_{m_1,n_1}(\rho \restriction [m_1+1, n_1]) = \alpha_{m_1,n_1}$ for $m_1 < n_1$ from $[m_{\mathbf{d}}, n_{\mathbf{d}}]$ <u>we have</u> $S_p = \{\bar{a} \in A' : \emptyset \in D_{\mathbf{d}} + A_{\bar{a}}\}$ and $\bar{e}_{\bar{d},p,s} = \mathbf{d} + A_{\bar{a}}$

(question): should we allow $|{\rm Rang}(G_{m_1,n_1})|$ be large, etc.?

(f) ${\rm part}(\mathbf{d}) = \{p \in {\rm par}(\mathbf{d}) : |S_p| < \kappa_{m_{\mathbf{d}}}\}$

(question: should we have ${\rm par}(\mathbf{d}) \subseteq \{(\mathbf{e}, d, p) : (\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and as above$\}$?

Discussion 14.4. (09.8.17) 1) Discuss (here?) to achieve our hope (dichotomy using [Sh:835]). We would like for every $\eta \in \mathbb{D}_{\mathbf{x}} = {\rm dec}_{<\omega}(\mathbf{O})$ to define what are η-objects which are a replacement for $(I_\eta)_{\rm Ord}$. Maybe we should repalce ${\rm dec}_{<\omega}(\theta)$ by closing $\mathbf{O}$ by ordered pairing,but first ignore this.

A natural try define when $x \in {\rm obj}(\eta)$ by induction on $\ell g(\eta)$.

If $\ell g(\eta) = 0$ then x is just an ordinal.

If $\ell g(\eta) = n + 1$ <u>then</u> x consists of a non-empty set $\mathscr{F} \in {}^{(I_{\eta(\theta)})}{\rm Ord}$, a set $A \in D^+_{\eta(\theta)}, A_B = \{t \in A : f(t) > 0\}$ (or $\langle A_f : f \in \mathscr{F}\rangle, A_g \in D^+_{\eta(\theta)}$?) and a fucntion which gives for every $f \in \mathscr{F}$ and $t \in A_f$ and object $x_{f,t} \in {\rm obj}(\langle \eta(1+\ell) : \ell < n\rangle)$. We have to: (A) define rank, (B) using DC criterion for the rank being an ordinal, (C) reprove [Sh:938] main Theorem.

2) (09.8.26) The example in [Sh:938, §0] can be pushed up: use $\lambda + \aleph_\omega$ ordinal addition, $(\lambda, {\rm rk}_J(\lambda) = \lambda$ for all relevant J's. Hence it seems there is no hope for $\mu = \aleph_\omega$ but there may be for $\mu = \beth_\omega$. At least combine $\mu = \beth_\omega, \theta(\mathscr{P}(\lambda_n)) < \mu_{n+1}, \mu = \sum\limits_{n<\omega} \lambda_n$ and ${\rm IND}(\langle \lambda_n : n < \omega\rangle)$ or try the proof of [Sh:460, §1].

{m10} **Claim/Definition 14.5.** Like 14.3 but $\bar{J} = \langle J_n : n \in \mathbf{O}\rangle$, FILL. Now $\eta_{\mathbf{d}}$ is a decreasing sequence of length $n_{\mathbf{d}} + 1$, so $D_{\mathbf{d}}$ is $\kappa_{\eta_{\mathbf{d}}(n_{\mathbf{d}})}$-complete and $\mathbf{e} \in \Sigma(\mathbf{d})$ implies $\eta_{\mathbf{e}}(n_{\mathbf{e}}) = \eta_{\mathbf{d}}(n_{\mathbf{d}})$, ${\rm Rang}(\eta_{\mathbf{d}}) \subseteq {\rm rang}(\eta_{\mathbf{e}})$.

{m7} **Convention 14.6.** We naturally let $\mathbf{s} = \langle \bar{\kappa}_{\mathbf{s}}, \mu_{\mathbf{s}}, \mathbb{D}_{\mathbf{s}}, {\rm par}(-,-), \ell{\rm par}(-,-)\rangle$ and $I_{\mathbf{s},\mathbf{d}}, D_{\mathbf{s},\mathbf{d}}, S_{\mathbf{s},p}, X_{\mathbf{s},p,s}, D_{\mathbf{s},p,2}$.

{m12} **Definition 14.7.** Given a frame $\mathbf{s}$ let ${\rm tru}(\mathbf{s})$ be the set of objects $\mathbf{t}$ consisting of:

(a) $\mathscr{T}_{\mathbf{t}}$ a set of finite sequences closed under initial segments

(b) $\mathbf{d}_{\mathbf{t},\rho} \in \mathbb{D}$ for $t \in \mathbf{T}$

(c) $\bar{h}_{\mathbf{t}} = \langle h^{\mathbf{t}}_{\rho,\varrho} : \rho \trianglelefteq \varrho \in \mathscr{T}_{\mathbf{t}}\rangle$

(*d*) for non-Δ-maxiam $\rho \in \mathscr{T}_{\mathbf{t}}, (\mathbf{d}^+_{\mathbf{t},\rho}, h^+_{\mathbf{t},\rho}) \in \Sigma_{\rm pr}(\mathbf{d}_{\mathbf{t},\rho})$ and $p_\rho \in \text{ par}(\mathbf{d}^+_{\mathbf{t},\rho})$ satisfied $\text{suc}_{\mathscr{T}_{\mathbf{t}}}(\rho) = \{\rho\hat{\ }\langle s\rangle : S \in S_{p_\rho}\}$ and $\mathbf{d}_{\mathbf{t},\rho\hat{\ }<s>} = \mathbf{e}_{\mathbf{d}_{\mathbf{t},\rho}}$ and $h_{\rho,rho\hat{\ }<s>} = h^+_{\mathbf{t}}$ for $\rho \lhd \varrho \in \mathscr{T}_{\mathbf{t}}, \ell g(\rho) = m, \ell g(\varrho) = n$ then $h_{\rho,\varrho} : I_{\mathbf{d}_{\mathbf{t},\varrho}} \to I_{\mathbf{d}_{\mathbf{t},\rho}}$ is $h_{\mathbf{t}}, h_{\rho_{m+1}} \circ \ldots \circ h_{\mathbf{t},\rho_n}$ where $h_{\mathbf{t},\rho_{\ell+1}} := h^+_{\mathbf{t},\rho_\ell}$ and $\rho_\ell = \varrho{\restriction}\ell$ for $\ell = m, \ldots, n-1$

(*e*) if $\rho \lhd \varrho \in \mathscr{T}_{\mathbf{t}}$ we have: $h_{\rho,\varrho}$ maps $A_{\mathbf{d}_\varrho}$ into $A_{\mathbf{d}_\rho}$
question: put this in Definition 14.1?

Definition 14.8. Given a candidate $\mathbf{s}$ we try to define a rank; (we may omit the subscript $\mathbf{s}$ as its value is fixed). {m16}

If $\mathbf{d} \in \mathbb{D}_{\mathbf{s}}$ and $f \in {}^{I[\mathbf{d}]}\text{Ord}$ we define $\text{rk}^{\rm tr}_{\mathbf{d}}(f) = \text{rk}^{\rm tr}_{\mathbf{d}}(f, \mathbf{s}) \in \text{Ord} \cup \{\infty\}$; or we may replace "tr" by 1 or omit it; by defining by induction on the ordinal ζ when $\text{rk}^{\rm tr}_{\mathbf{d}}(f) \supseteq \zeta$: it holds iff for every $\zeta_1 < \lambda$ there is a pair $(\mathbf{t}, \bar{g})$ such that

(*a*) $\mathbf{t} \in \text{ tree}(\mathbf{s})$ where $\mathscr{T}_{\mathbf{t}}$ is well founded, i.e. with no ω-branch
(*b*) $\mathbf{d}_{\mathbf{t}=<>} = \mathbf{d}$
(*c*) $\bar{g} = \langle g_\rho : \rho \in \text{ max}(\mathscr{T}_{\mathbf{t}})\rangle$
(*d*) $g_\rho : I_d[\mathbf{d}_{\mathbf{t},\rho}] \to \text{ Ord}$
(*e*) $g_\rho < f \circ h_{\mathbf{t},<>,\rho} \bmod D_{\mathbf{d}_{\mathbf{t},\rho}}$
(*f*) $\text{rk}^{\rm tr}_{\mathbf{d}_{\mathbf{t},\rho}}(g_\rho) \geq \zeta_1$.

The choice in **??** though more transparent than the following relative, need more use of choice.

Definition 14.9. Like 15.9 - FILL - $\text{rk}^2_{\mathbf{d}}(f)$, but maybe rk^1 is enough. {m18}
Check.

Claim 14.10. *Let* $\mathbf{s}$ *be a candidate and* $k = 0, 1$. {m21}
1) The rank $rk^k_{\mathbf{d}}(f)$ *for* $f \in {}^{I[\mathbf{d}]}Ord$ *is well defined* ($\in Ord \cup \{\infty\}$).
2) If $(\mathbf{d}_2, h) \in \Sigma_{pr}(\mathbf{d}_1)$ *and* $f_1 \in {}^{I[\mathbf{d}_1]}Ord$ *then* $rk^k_{\mathbf{d}_1}(f) = rk^k_{\mathbf{d}_2}(f \circ h)$.
3) If $\mathbf{d} \in \mathbb{D}_{\mathbf{s}}$ *and* $f \in {}^{I[\mathbf{d}]}Ord$ *and* $p \in \ell par(\mathbf{d})$ *then* $rk_{\mathbf{d}}(f) = min\{rk_{\mathbf{e}_{\mathbf{d},s}}(f) : s \in S_p\}$.

Proof. 1) Easy.
2) Use + on $\mathbb{D}$ - FILL.
3) By induction - FILL. □

Claim 14.11. *For a free?* $\mathbf{s}$ *the following condition (a),(b) are equivalent: and if* $\mathbf{s} = \mathbf{s}_{\bar{\kappa},\bar{J}}$ *from 14.3 we can add (c), and if* $\mathbf{s} = \mathbf{s}_{\bar{\kappa},\bar{J}}$ *is from 14.5 we can add clause* $(c)^+$*:* {m25}

(*a*) $\text{rk}_{\mathbf{d}}(f) = \infty$ *for some* $\mathbf{d} \in \mathbb{D}_{\mathbf{s}}$ *and* $f \in {}^{I[\mathbf{d}]}Ord$
(*b*) *there* $\mathbf{t} \in \text{tree}(\mathbf{s})$ *and* $\mathscr{Y} \subseteq \mathscr{T}_{\mathbf{t}}$ *such that* $(\forall \rho \in \lim_\omega(\mathscr{T}_{\mathbf{t}}))(\exists^\infty n)[\rho{\restriction}n \in \mathscr{Y})$ *and* $f_\rho \in {}^{I[\mathbf{d}_{\mathbf{t},\rho}]}\text{Ord}$ *for* $\eta \in \mathscr{Y}$ *such that for any* $\rho < \varrho$ *from* $\mathscr{Y}$ *we have* $f_\varrho < f_\rho \circ h^{\mathbf{t}}_{\rho,\varrho} \bmod D_{\mathbf{d}_{\mathbf{t},\varrho}}$
(*c*) $\neg\text{IND}(\bar{\kappa}, J)$ *when...?*

Definition 14.12. For $(\bar{\kappa}, \bar{J})$ as in 14.3 or 14.5 let $\text{IND}(\bar{\kappa}, J)$ mean that: {m26}

Case 1: Definition 14.3 for every $F_{m,n} : I_{[m+1,n]} \to J_m$ for $m < n < \omega$ there is $\eta \in \prod\limits_{\ell<n} I_\ell$ such that $m < n < \omega \Rightarrow \eta(\ell) \notin F_{m,n}(\eta{\restriction}[m+1, \eta])$.

Case 2: Definition 14.5

[copied] 1) Above $\bar{p}^1_{\bar{J}}$ is not well 0-founded iff: there are $\bar{\varepsilon}, \bar{f}$ such that

$\circledast_{\bar{\varepsilon},\bar{f}}$ (a) $\bar{\varepsilon} = \langle \varepsilon_i : i < \omega \rangle$ is increasing

(b) $\bar{f} = \langle f_{i,j} : i < j < \omega \rangle$

(c) $f_{i,j}$ is a function from $I_{\langle \varepsilon_j, \varepsilon_{j-1}, \ldots, \varepsilon_{i+1} \rangle}$ into J_{ε_i}

(d) for every $\bar{\alpha} \in \prod\limits_{i<\omega} \kappa_{\varepsilon_i}$ for some $i < j$ we have $\alpha_i \in f(\alpha_{n_j}, \alpha_{n_{j-1}}, \ldots, \alpha_{n_{i+1}})$.

Proof. FILL □

We quote [Sh:938]

Definition 14.13. Main Definition: We say that $\mathbf{p} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_{\mathbf{p}}, \text{rk}_{\mathbf{p}}, \Sigma_{\mathbf{p}}, \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{p}})$ is a weak (rank) 1-system when:

(a) μ is singular

(b) each $\mathbf{d} \in \mathbb{D}$ is (or just we can compute from it) a pair $(I, D) = (I_{\mathbf{d}}, D_{\mathbf{d}}) = (I[\mathbf{d}], D_{\mathbf{d}}) = (I_{\mathbf{p},\mathbf{d}}, D_{\mathbf{p},\mathbf{d}})$ such that:

  (α) $\theta(I_{\mathbf{d}}) < \mu$, on $\theta(-)$ see **??**

  (β) $D_{\mathbf{d}}$ is a filter on $I_{\mathbf{d}}$

(c) for each $\mathbf{d} \in \mathbb{D}$, a definition of a function $\text{rk}_{\mathbf{d}}(-)$ with domain ${}^{I[\mathbf{d}]}\text{Ord}$ and range $\subseteq$ Ord, that is $\text{rk}_{\mathbf{p},\mathbf{d}}(-)$ or $\text{rk}^{\mathbf{p}}_{\mathbf{d}}(-)$

(d) (α) Σ is a function with domain $\mathbb{D}$ such that $\Sigma(\mathbf{d}) \subseteq \mathbb{D}$

  (β) if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{e} \in \Sigma(\mathbf{d})$ then $I_{\mathbf{e}} = I_{\mathbf{d}}$ [natural to add $D_{\mathbf{d}} \subseteq D_{\mathbf{e}}$,

  this is not demanded but see **??**(2)]

(e) (α) $\mathbf{j}$ is a function from $\mathbb{D}$ onto $\text{cf}(\mu)$

  (β) let $\mathbb{D}_{\geq i} = \{\mathbf{d} \in \mathbb{D} : \mathbf{j}(\mathbf{d}) \geq i\}$ and $\mathbb{D}_i = \mathbb{D}_{\geq i} \backslash \mathbb{D}_{i+1}$

  (γ) $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow \mathbf{j}(\mathbf{e}) \geq \mathbf{j}(\mathbf{d})$

(f) for every $\sigma < \mu$ for some $i <$ cf(μ), if $\mathbf{d} \in \mathbb{D}_{\geq i}$, then $\mathbf{d}$ is $(\mathbf{p}, \leq \sigma)$-complete where:

  (∗) we say that $\mathbf{d}$ is $(\mathbf{p}, \leq X)$-complete (or $(\leq X)$-complete for $\mathbf{p}$) when: if $f \in {}^{I[\mathbf{d}]}\text{Ord}$ and $\zeta = \text{rk}_{\mathbf{d}}(f)$ and $\langle A_j : j \in X \rangle$ a partition[7] of $I_{\mathbf{d}}$, then for some $\mathbf{e} \in \Sigma(\mathbf{d})$ and $j < \sigma$ we have $A_j \in D_{\mathbf{e}}$ and $\zeta = \text{rk}_{\mathbf{e}}(f)$; so this is not the same as "$D_{\mathbf{d}}$ is $(\leq X)$-complete"; we define $(\mathbf{p}, |X|^+)$-complete, i.e. $(\mathbf{p}, < |X|^+)$-complete similarly

(g) no hole[8]: if $\text{rk}_{\mathbf{d}}(f) > \zeta$ then for some pair $(\mathbf{e}, g)$ we have: $\mathbf{e} \in \Sigma(\mathbf{d})$ and $g <_{D[\mathbf{e}]} f$ and $\text{rk}_{\mathbf{e}}(g) = \zeta$

(h) if $f = g + 1 \mod D_{\mathbf{d}}$ then $\text{rk}_{\mathbf{d}}(f) = \text{rk}_{\mathbf{d}}(g) + 1$

(i) if $f \leq g \mod D_{\mathbf{d}}$ then $\text{rk}_{\mathbf{d}}(f) \leq \text{rk}_{\mathbf{d}}(g)$.

{m30} **Definition 14.14.** We say $\mathbf{p}$ is a quasi rank ι-system when $\mathbf{p} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_{\mathbf{p}}, \text{rk}_{\mathbf{p}}, \Sigma_{\mathbf{p}}, \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{p}})$ satisfies Definition m4.3 of §3 of [Sh:938] if $\iota = 1$, Definition m4.4 of §3 of [Sh:938] if $\iota = 2$ except that the rank may be ∞; we write $\text{rk}_{\mathbf{d}}(f, \mathbf{d})$ for $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ and $f \in {}^{I[\mathbf{d}]}\text{Ord}$.

[7] as long as σ is a well ordered set it does not matter whether we use a partition or just a covering, i.e. $\cup\{A_j : j \in \sigma\} = I_{\mathbf{d}}$

[8] we may use another function Σ here, as in natural examples here we use $\Sigma(\mathbf{d}) = \{\mathbf{d}\}$ and not so in clause (f)

{m32}

Definition/Claim 14.15. For a frame $\mathbf{s}$ let $\mathbf{p}$ be the following quasi rank system:

- $\mu, \mathbb{D}, \Sigma, \mathbf{j}$ are as in Definition 14.1
- $\mathrm{rk}_{\mathbf{d}}(f)$ is as in Definition ?

{m34}

Claim 14.16. *1) If $(\bar\kappa, \bar J)$ is as in Definition 14.3 or 14.5 and* $\mathrm{IND}(\bar\kappa, J)$ *holds, see Definition 14.12 then $p_{\mathbf{s}(\bar\kappa, J)}$ is a rak system.*
2) Moreover it is a strict one.

Saharon copied. 1) As in the proof of e5.g of §4 of [Sh:938, §4,e5.g] or better see the proof of 15.17 except that we use 15.9 instead of 15.8 which simplify clause (f), but is cumbersome in other places.
2) We check Definition m4.3 of §3 of [Sh:938, §3,m4.3].

Clause (a): μ is singular.
As $\mu = \sum_n \kappa_n$ and $\kappa_n < \kappa_{n+1}$ this is obvious.

Clause (b): Let $\mathbf{d} \in \mathbb{D}, \eta = \eta_{\mathbf{d}}, J = J_n$ now clause (α) says $\theta(I_\eta) = \theta(|I_\eta|) = \kappa_{\eta(0)}, \kappa_{\eta(0)+1} < \mu$ so as for clause (β), "$D_{\mathbf{p}}$ is a filter on I_η", it holds by the choice of $\mathbf{p}$.

Clause (c): $\mathrm{rk}^{\mathbf{p}}_{\mathbf{d}}(f) = \mathrm{rk}_{\mathbf{d}}(f, \mathbf{p})$ is an ordinal as defined in 15.9.

Clause (d):
Clearly $\Sigma(\mathbf{d})$ is of the right form.

Clause (e):
On $\mathbf{j}$ - see 15.13(2)(c).

Clause (f):
We prove by induction on the ordinal ζ that:

$(*)$ if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{j}(\mathbf{d}) > \varepsilon$ and $A = \cup\{A_\alpha : \alpha < \kappa_\varepsilon\} \in D_{\mathbf{d}}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ and $A_\alpha \in D^+_{\mathbf{d}} \Rightarrow \mathrm{rk}_{\mathbf{d}+A_\alpha}(f) \geq \alpha$ then $\mathrm{rk}_{\mathbf{d}}(f) \geq \alpha$.

Now Definition 15.9 is tailored made for this.

Older version using 15.8 recheck:
For $\alpha = 0$ and α a limit ordinal this is obvious. For $\alpha = \beta + 1$ let $\mathscr{Y} = \{\alpha < \kappa_\varepsilon : A_\alpha \in D^+_{\mathbf{d}}\}$ and for $\alpha \in \mathscr{Y}$ let $\mathbf{n}_\alpha = \min\{n$: there is $(\mathbf{e}, h) \in \Sigma(\mathbf{d} + A_\alpha)$ such that $\mathrm{rk}_{\mathbf{e}}(f \circ h) \geq \beta$ and $\eta_{\mathbf{e}}(0) = n\}$. Clearly $\mathbf{n}_\alpha$ is well defined for $\alpha \in Y$, and let $w := \{n : \cup\{A_\alpha : \alpha \in \mathscr{Y}$ and $\mathbf{n}_\alpha = n\} \in D^+_{\mathbf{d}}\}$ and also the rest should be clear.

Clause (g): (no-hole)
By the Definition 15.8 or 15.8 of rk. Saharon 09.5.31 recheck.

Clause (h): $\mathrm{rk}_{\mathbf{d}}(f + 1) = \mathrm{rk}_{\mathbf{d}}(f) + 1$.
We prove by induction on the ordinal α that:

$(*)$ for every $\mathbf{d} \in \mathbb{D}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ we have $\mathrm{rk}_{\mathbf{d}}(f) \geq \alpha \Leftrightarrow \mathrm{rk}_{\mathbf{d}}(f+1) \geq \alpha+1$.

Clause (i): Obvious. □

{m37}

Question 14.17. (09.7.19) Assume little choice and $\mu_* = \min\{\mu : \text{IND}(\mu)\}$. So up to μ we can apply [Sh:835]. Now above it seemed that if $\alpha < \mu \Rightarrow \text{AC}_\alpha$ and μ is a limit cardinal, we can find bound above to $\text{rk}_{\mathbf{d}}$ hence to $\text{rk}_J(-)$ for J quite complete ideal.
1) Assume $\text{cf}(\mu_*) = \aleph_0$, we try to apply the above replacing $\text{AC}_{<\mu_*}$ by DC $+(\forall\alpha < \mu_*)(\neg\text{IND}(\mu))$. So the problem is, on the one hand, about [Sh:938, §3] with weaker form of choice (as in [Sh:835]) and on the other hand the right use of $\text{IND}(\mu_*)$ here.
2) What above μ is a successor?
3) Even with choice, the bound on rank does not give a bound on pp or $\text{tcf}(\mu^{\kappa_n}, <_D)$ well above $\theta(\mathscr{P}(\kappa_n))$ it gives with choice/without much choice - as can be done in §1.

Claim 14.18. *1) If $\langle f_\alpha : \alpha < \delta\rangle$ is $<_D$-increasing in $(\Pi, \bar\alpha, <_D)$ then $\text{rk}_D(\alpha) \ge \delta$.*
2) If $\langle f_\alpha : \alpha < \mu\rangle$ are $\ne_D$-distinct in $(\Pi\bar\alpha, <_D)$ and $\mu > \theta(\mathscr{P}(\ell g(\bar\alpha))$ then we can use [Sh:E38] *which continues* [Sh:497].
3) As in (1) devise μ to $\le \mathscr{P}(\kappa_n)$ on each for some $D_2 \supseteq D$ the sequence is increasing.

{m37}

Theorem 14.19. *1) If $\text{IND}(\langle\kappa_n : n < \omega\rangle$ <u>then</u> [?] - FILL.*
2) For $\aleph_\omega$ - [FILL].

The following information is not presently

{m40}

Claim 14.20. *1) Assume $(\bar\kappa, \bar J)$ is as in 14.3 and $n < \omega \Rightarrow |\mathscr{P}(I_n)| < \kappa_{n+1}$. <u>Then</u> for $\mathbf{s} = \mathbf{s}_{\bar\kappa,\bar J}$, for every $\mathbf{d} \in \mathbb{D}_{\mathbf{s}}$ we can find $A_\ell \in J_\ell^+$ for $\ell \in$ Rang$(\eta_{\mathbf{d}})$ such that $\prod\limits_{\ell\in m_{\mathbf{d}}}^{n(\mathbf{d})} A_\ell \in D_{\mathbf{d}}^+$ and $D_{\mathbf{d}} + \Pi A_\ell = D_{\mathbf{e}}$ for some $\mathbf{e}$ such that $I_{\mathbf{e}} = I_{\mathbf{d}}, A_{\mathbf{e}} = \prod\limits_\ell A_\ell$.*
2) Moreover, for every $p \in$ par$(\mathbf{d})$ there is a refinement q such that each $\mathbf{e}_{q,s} (s \in S_q)$ is of the form in (1).
3) In part (1) if $J_n = J_{\lambda_n}^{bd}$ where $\lambda_n =$ cf(λ_n) in$[\kappa_n, \kappa_{n+1})$ <u>then</u> in fact $D_{\mathbf{d}} + \Pi A_\ell$ is isomorphic to $D_{\mathbf{e}}$ where $\eta_{\mathbf{e}} = \eta_{\mathbf{d}}, A_\ell = I_{\mathbf{d}} = I_{\mathbf{e}}$.

Proof. FILL. □

§ 15. Connection to IND

§(2A) Getting quasi-rank system with $\text{AC}_{<\mu}$

{f6.2}

Remark 15.1. 1) Below we can concentrate on the case $\ell g(\bar J) = \omega, \langle\kappa_n : n < \omega\rangle$ increasing, even $2^{\kappa_n} < \kappa_{n+1}$ and $\kappa_n = \text{cf}(\kappa_n)$.
2) We like to use less choice say only DC not $\text{AC}_{<\mu}, \mu = \sum\limits_n \kappa_n$. This is not achieved for $\mathbf{q}^1_{\bar J}, \mathbf{q}^3_{\bar J}$, so it seems. So we may like to change [Sh:938, §3]. Consider $k = 2, 4$ in 15.13(2) to use.
3) (09.7.18) We may hope that if $J_n = [\kappa_n]^{\le\sigma}$ we need only, e.g. DC + $\text{AC}_{\mathscr{P}(\sigma)}$. But then we do not look at $J_{n+1} + A, |A| = \kappa_{n+1}$. So maybe have $\langle J^1_n, J^2_n : n < \omega\rangle$, see 15.14 or maybe have $J_{m,n}$ an ideal on $\kappa_n, J_{m,n} = [\kappa_n]^{\le\kappa_m}$, see 15.19.
4) (09.7.18) Try $\text{IND}_\kappa(\mu)$ or so $(\tau(\mathfrak{A})| = \kappa, |\mathfrak{A}| = \mu$, no ω-end-independent sequence or $\text{IND}\langle\mu_i, I_i : i < \kappa\rangle$ looking for $i_n < i_{n+1} < ldots\alpha_m \in \mu_m, \alpha_m \notin F(\alpha_{n+1}, \ldots, \alpha_m) \in I_{\alpha_m}$. Can we connect by Fodor?

5) (09.7.18) To define the ranks for $\mathbf{p}$ we better revise the pre-rank-system as follows. For every $\mathbf{d}$ we have $\Sigma_{\rm pr}(\mathbf{d}) = \Sigma^{\rm pr}_{\mathbf{p}}(\mathbf{d})$, the pure successors and $\Sigma_{\rm ap}(\mathbf{d}) = \Sigma^{\rm ap}_{\mathbf{p}}(\mathbf{d})$ the apure ones and we have interpolation. In the conclusion we try.

In clause (f), $\mathbf{p}$-completeness, we shall try to get $\mathbf{e} \in \Sigma_{\rm pr}(\mathbf{d})$.

In clause (i), also if $(\mathbf{e},h) \in \Sigma_{\rm pr}(\mathbf{d}), f \in {}^{I_{\mathbf{d}}}{\rm Ord}, g = f \circ h \in {}^{I_{\mathbf{e}}}{\rm Ord}$ then ${\rm rk}_{\mathbf{d}}(f) = {\rm rk}_{\mathbf{e}}(g)$.

In the definition of ${\rm rk}_{\mathbf{p}}$, **??**, $(\mathbf{e},h) \in \Sigma_{\rm pr}(\mathbf{d})$, we may instead of ${\rm rk}_{\mathbf{p},\zeta}(-,-)$ ask for a tree of pure extensions, but well founded tree.
5A) The natural case is $\bar J = \langle J_n : n < \omega\rangle, \mathbb{D} = \{\eta : \eta \text{ is } \langle n, n-1, \dots, m\rangle\}, \Sigma_{\rm pr}(\mathbf{d})$ is as there but $\eta_{\mathbf{e}} = \varrho\hat{\ }\eta_{\mathbf{d}}$ but on $I_{\varrho(f)}$ we use the original J. This fine to see that it fits. If $\mathbf{O}$ or κ larger, we allow "side extension of η" but min ${\rm Rang}(\eta)$ remains.
6) (09.7.18) But later we have preservation of ranks when we use isomorphic $\mathbf{p}$ or $\mathbf{p}$ restricted to "$\mathbf{d}$ and above". So if $J_n = J^{\rm bd}_{\kappa_n}, \kappa_n$ regular, $J^{\rm bd}_{\kappa_n}, J^{\rm bd}_{\kappa_n} + A$ are the same.
6A) Maybe legal partitions of $\prod\limits_\ell I_{\eta,\ell}$ is when $I_{\eta(\ell)}$ is divided to $< \kappa_{\eta(\ell)}$.

Definition 15.2. 1) Let $\bar J$ be called a candidate or δ-candidate when: {f6.3}

- (a) $\bar J = \langle J_\varepsilon : \varepsilon < \delta\rangle$, δ a limit ordinal
- (b) J_ε is an ideal on κ_ε
- (c) $\delta < \kappa_0$ and κ_ε is non-decreasing.

2) We say that $\bar J$ is a generalized candidate when for some $\mathbf{O}$:

- (a) $\mathbf{O}$ is a linear order with no last element
- (b) $J = \langle J_\varepsilon : \varepsilon \in \mathbf{O}\rangle$
- (c) J_ε is a $\aleph_1$-complete ideal on $I_\varepsilon :=$ ${\rm Dom}(J_\varepsilon) = \cup\{u : u \in J_\varepsilon\}$.

In some sense the simplest example is

Example 15.3. Let $\langle \kappa_n : n < \omega\rangle$ be an increasing sequence of ordinals, $J_n := [\kappa_n]^{\le \aleph_0}$. {f6.3d}

Discussion 15.4. (08.6.27) 1) We shall try to define a rank (from a p.r.s. or p.r.s.*) such that clause (j) of m4.6 of §3 of [Sh:938] follows. It seems that a necessary condition for the rank to be $< \infty$ we need ${\rm IND}(\mathbf{p})$. {f6.4}
2) Naturally we can define $\mathbf{p}$ from $\bar J$ and a reasonable condition is ${\rm IND}(\bar J)$ at least when $\ell g(\bar J) = \omega$.
3) We can below use generalized candidates.

Definition 15.5. 1) We say $\mathbf{p} = (\mathbb{D}, \Sigma, \mathbf{j})$ be a ι-p.r.s. (pre-rank-ι-system with $\iota = 1,2$; if $\iota = 2$ we may omit it) when in Definition m4.3 or m4.4 of §3 of [Sh:938, §3,m4.4] it satisfies clauses (a),(b),(d),(e) and we add in (d): {f6.5}

- $(*)$ Σ is transitive: if $(h_1, \mathbf{d}_1) \in \Sigma(\mathbf{d}_0)$ and $(h_2, \mathbf{d}_2) \in \Sigma(d_1)$ then $(h_2 \circ h_1, \mathbf{d}_2) \in \Sigma(\mathbf{d}_0)$

  [check where used].

2) We say $\mathbf{p}$ is a quasi rank ι-system when $\mathbf{p} = (\mathbb{D}, {\rm rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_{\mathbf{p}}, {\rm rk}_{\mathbf{p}}, \Sigma_{\mathbf{p}}, \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{p}})$ satisfies Definition m4.3 of §3 of [Sh:938] if $\iota = 1$, Definition m4.4 of §3 of [Sh:938] if $\iota = 2$ except that the rank may be ∞; we write ${\rm rk}_{\mathbf{d}}(f, \mathbf{d})$ for $\mathbf{d} \in \mathbb{D}_{\mathbf{P}}$ and $f \in {}^{I[\mathbf{d}]}{\rm Ord}$.

2A) Alternatively: $\mathrm{rk}_{\mathbf{p}}$ is defined as in 15.8 below [or 15.9].

{f6.6}

Convention 15.6. 1) $\mathbf{p}$ is a 2-p.r.s.
2) We usually omit the $\mathbf{p}$ when clear from the context, similarly for $\mathrm{rk}_{\mathbf{d}}(f, \mathbf{p})$ defined below.

{f6.6.3}

Remark 15.7. 1) We shall try to define rk. We shall try to prove mainly (f) [the version with $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$].

{f6.7}

Definition 15.8. For $\mathbf{p}$ a p.r.s., $\mathbf{d} \in \mathbb{D}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ we define $\mathrm{rk}_{\mathbf{d}}(f, \mathbf{p}) = \mathrm{rk}^0_{\mathbf{d}}(f, \mathbf{p})$ by defining when $\mathrm{rk}_{\mathbf{d}}(f, \mathbf{p}) \geq \alpha$ for an ordinal α by induction on α for all pairs $(\mathbf{d}, f)$; so $\mathrm{rk}^0_{\mathbf{d}}(f, \mathbf{p}) = \alpha$ when it is $\geq \alpha$ but not $\geq \alpha + 1$, and is ∞ otherwise; by monotonicity well defined.

$\underline{\alpha = 0}$: always.

$\underline{\alpha \text{ limit}}$: when $\mathrm{rk}^0_{\mathbf{d}}(f, \mathbf{p}) \geq \beta$ for every $\beta < \alpha$.

$\underline{\alpha = \beta + 1}$: when for some $(h, \mathbf{e}) \in \Sigma_{\mathbf{p}}(\mathbf{d})$ and $g \in {}^{I[\mathbf{e}]}\mathrm{Ord}$ we have $g <_{D_{\mathbf{e}}} f \circ h$ and $\mathrm{rk}^0_{\mathbf{e}}(g, \mathbf{p}) \geq \beta$.

{f6.8}

Definition 15.9. [Saharon 09.06.01: check that this definition satisfies additivity and $\mathrm{rk}(f + 1) = \mathrm{rk}(f) + 1$.

We define $\mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p})$ and $\mathrm{dp}^1_{\mathbf{d},\zeta}(f, \mathbf{p})$ from $\mathrm{Ord} \cup \{\infty\}$ for $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}, f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ by defining by induction on the ordinal ζ:

- (a) when $\mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p}) \geq \zeta$ and
- (b) when $\mathrm{dp}^1_{\mathbf{d},\zeta}(f, \mathbf{p}) \geq \xi$ for any ordinal ξ.

Arriving to ζ we let:

- $\mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p}) \geq \zeta$ $\underline{\text{iff}}$ for every $\zeta_1 < \zeta$ and $\xi < \infty$ there is $(h, \mathbf{e}) \in \Sigma(\mathbf{d})$ such that $\mathrm{rk}^1_{\mathbf{e}}(f \circ h, \mathbf{p}) \geq \zeta_1$ and $\mathrm{dp}^1_{\mathbf{e},\zeta_1}(f \circ h, \mathbf{p}) \geq \xi$
- we define by induction on $\xi < \infty$ when $\mathrm{dp}^1_{\mathbf{d},\zeta}(f, \mathbf{p}) \geq \xi$; it holds if $\mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p}) \geq \zeta$ and for every $\xi_1 < \xi$ and partition $\langle A_\varepsilon : \varepsilon < \varepsilon_* \rangle$ of $I_{\mathbf{d}}$ with $\varepsilon_* < \kappa_{\mathbf{j}(\mathbf{d})}$ parts, there is $(h, \mathbf{e}) \in \Sigma(\mathbf{d})$ such that $\mathrm{rk}^1_{\mathbf{e}}(f \circ h, \mathbf{p}) \geq \zeta$ and $\mathrm{dp}^1_{\mathbf{e},\zeta}(f \circ h, \mathbf{p}) \geq \xi_1$ and $I_{\mathbf{e}} = I_{\mathbf{d}}$ (Saharon 09.06.01: or use Σ_1.)

Remark 15.10. 1) In a $\underline{\text{variant}}$ we demand: and $I_{\mathbf{e}} = I_{\mathbf{d}} \wedge h = \mathrm{id}_{I[\mathbf{d}]}$.
2) By 15.9 we may derive a quasi rank system from a p.r.s., but we deal with the special case which seems most interesting.

{f6.8d}

Claim 15.11. *1) The rank in Definition 15.8, 15.9 are well defined.*
2) $\mathrm{rk}^0_{\mathbf{d}}(f, \mathbf{p}) \leq \mathrm{rk}^1_{\mathbf{d}}(f, \mathbf{p})$.

{f6.8g}

Discussion 15.12. (09.06.01) 1) We would like to use $\mathrm{AC}_{\mathscr{U}}$ for constant $\mathscr{U}$ or at most $\mathscr{U}$ depend on $\mathbf{0}$. By the amount of completeness we need (approaching μ), if we use $\mathrm{rk}^1_{\mathbf{d}}(-, \mathbf{f}^1_{\bar{J}})$ is it O.K.? Does it?

{f6.9}

Definition 15.13. 1) For $\ell = 1, 2$ and $\mathbf{p}$ a p.r.s. we say $\mathbf{p}$ is well ℓ-founded $\underline{\text{when}}$ $\mathrm{rk}^\ell_{\mathbf{d}}(f, \mathbf{p}) < \infty$ for every $\mathbf{d} \in \mathbb{D}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$.
2) Similarly for $\mathbf{p}$ a quasi rank system (so now $\mathrm{rk}_{\mathbf{d}}(f, \mathbf{p})$ is not as defined in Definition 15.9, but is from Definition 15.5(2)).

{f6.10}

Definition 15.14. For a candidate $\bar{J} = \langle J_\varepsilon : \varepsilon \in \delta \rangle, J_\varepsilon$ an ideal on κ_ε we define $\mathbf{p} = \mathbf{p}_{\bar{J}}$ as follows:

(a) $\mathbb{D}_{\mathbf{p}}$ is the set of $\mathbf{d} = (I, D) = (I_{\mathbf{d}}, D_{\mathbf{d}})$ such that for some $\eta = \eta_{\mathbf{d}}$ we have:

(α) η a non-empty decreasing sequence of ordinals $< \delta$

(β) $I = \prod\limits_{\ell < \ell g(\eta)} \kappa_{\eta(\ell)}$

(γ) $D = D_\eta + A_{\mathbf{d}}$ for some $(\bar\kappa, \eta)$-large subset of I_η which means

(δ) $A \subseteq I_\eta$ is $(\bar\kappa, \eta)$-large when $A = \prod\limits_{\ell<n} Y_\ell$ for some $Y_\ell \in [\kappa_{\eta(\ell)}]^{\kappa_{\eta(\ell)}}$ for $\ell < n$ and

(ε) let $u_{\mathbf{d}} = \text{ Rang}(\eta_{\mathbf{d}}), D_{u_{\mathbf{d}}} = D_\eta$

(ζ) $D = \{Y \subseteq \prod\limits_{\ell<n} \kappa_{\eta(\ell)}:$ there is a sequence $\langle Y_\ell : \ell \le \ell g(\eta)\rangle$ such that $Y_n = Y, Y_0 = \{<>\}$ and $\ell \le \ell g(\eta) \Rightarrow Y_\ell \subseteq \prod\limits_{m<\ell} \kappa_{\eta(m)}$ and $\ell < \ell g(\eta) \wedge \rho \in Y_\ell \Rightarrow \{\alpha < \kappa_{\eta(\ell)} : \rho\hat{}\langle\alpha\rangle \notin Y_{\ell+1}\} \in J_{\eta(\ell)}\}$

(b) $\Sigma(\mathbf{d}) = \{(h, \mathbf{e})$: for some ϱ we have $\eta_{\mathbf{e}} = \varrho\hat{}\eta_{\mathbf{d}} \in \mathbb{D}$ and $h : I_\nu \to I_\eta$ is defined by $h(\rho) = \langle\rho(\ell g(\varrho) + \ell) : \ell < \ell g(\eta)\rangle$ and h induces a mapping from $D_{\mathbf{e}}$ into $D_{\mathbf{d}}\}$

(c) $\mathbf{j}(\eta) = \eta(\ell g(\eta) - 1)$

(d) $\mu = \cup\{\kappa_\varepsilon : \varepsilon < \delta\}$.

Definition 15.15. 1) Similarly to 15.14 for a generalized candidate $\bar J = \langle J_\varepsilon : \varepsilon \in \mathbf{O}\rangle$. {f6.11}

2) For a candidate $\bar J = \langle J_n : n < \omega\rangle$ we define $\mathbf{p}^2_{\bar J} = (\mathbb{D}, \text{rk}, \Sigma, \mathbf{j}, \mu)$ as in 15.14 but:

$(a)'$ $\mathbb{D} = \{\mathbf{d} : \mathbf{d}$ as in clause (a) of Definition 15.14 but $\eta_{\mathbf{d}} = \langle n, n-1, \dots, m\rangle$ where $m \le n\}$

$(e)'$ rk is as defined in Definition 15.8.

3) We define $\mathbf{p}^{\ell+2}_{\bar J}$ as in part (1) or by **??** but replace clause $(a)(\delta)$ of **??** or part (1) by:

$(\delta)'$ $D_\eta = \{Y \subseteq \prod\limits_{\ell<n} \kappa_{\eta(\ell)}$: for some $Y_\ell \in J_\ell$ for $\ell < n$ we have $\prod\limits_{\ell<n} \kappa_{\eta(\ell)} \backslash \{\rho \in \prod\limits_{\ell<n} \kappa_{\eta(\ell)} : (\exists \ell < n)[\rho(\ell) \in Y_\ell]\}$.

4) For $\ell = 0, 1$ let $\mathbf{q}^{k,\ell}_{\bar J}$ be the $\mathbf{q}^k_{\bar J}$ expanded by $\text{rk}^\ell_{\mathbf{d}}(f, \mathbf{p}^k_{\bar J})$. If $\ell = 1$ we may omit it.

Claim 15.16. *1) Above $\bar p^1_{\bar J}$ is not well 0-founded* <u>*iff*</u>*: there are $\bar\varepsilon, \bar f$ such that* {f6.12}

$\circledast_{\bar\varepsilon, \bar f}$ (a) $\bar\varepsilon = \langle\varepsilon_i : i < \omega\rangle$ *is increasing*

(b) $\bar f = \langle f_{i,j} : i < j < \omega\rangle$

(c) $f_{i,j}$ *is a function from* $I_{\langle\varepsilon_j, \varepsilon_{j-1}, \dots, \varepsilon_{i+1}\rangle}$ *into* J_{ε_i}

(d) *for every* $\bar\alpha \in \prod\limits_{i<\omega} \kappa_{\varepsilon_i}$ *for some* $i < j$ *we have* $\alpha_i \in f(\alpha_{n_j}, \alpha_{n_{j-1}}, \dots, \alpha_{n_{i+1}})$.

2) Similarly for $\mathbf{p}^2_{\bar J}$ (i.e. $\delta = \omega$ we can above demand $\varepsilon_i = i$, so it is equivalent to $\neg\text{IND}\langle J_n : n < \omega\rangle$.

Proof. 1) As in [Sh:513].
2) Easy as we can add to a function dummy variables. $\square_{15.16}$

<u>Task</u>: 1) Prove $\mathbf{p}^2_{\bar J}$ satisfies clause (f) for $\text{rk} = \text{rk}^1_{\mathbf{p}}$ defined as in 15.8.
2) Check the $\text{rk}(f+1) = \text{ rk}(f) + 1$, but see below.

{f6.17}

Claim 15.17. *1) If $\bar J = \langle J_\varepsilon : \varepsilon \in \mathbf{O}\rangle$ is a generalized candidate and $k = 1,3$ then $\mathbf{p}^k_{\bar J}$ is a p.r.s. provided that "J_ε is $\theta(\mathbf{O})$-complete"(?)*
2) If $\bar J = \langle J_n : n < \omega\rangle$ is a candidate and $k = 2,4$ then $\mathbf{p}^k_{\bar J}$ is a p.r.s.
3) In part (1), $\mathbf{q}^k_{\bar J}$ is a quasi rank system.
4) Assume $\bar J = \langle J_n : n < \omega\rangle, J_n$ an ideal on $\kappa_n, \kappa^+_n < \kappa_{n+1}, \mu = \Sigma\kappa_n$. Then $\mathbf{q}^k_{\bar J}$ is a quasi rank system.

Proof. 1) As in the proof of e5.g of §4 of [Sh:938, §4,e5.g] or better see the proof of 15.17(?) except that we use 15.9 instead of 15.8 which simplify clause (f), but is cumbersome in other places.
2) We check Definition m4.3 of §3 of [Sh:938, §3,m4.3].

Clause (a): μ is singular.
As $\mu = \sum_n \kappa_n$ and $\kappa_n < \kappa_{n+1}$ this is obvious.

Clause (b): Let $\mathbf{d} \in \mathbb{D}, \eta = \eta_{\mathbf{d}}, J = J_n$ now clause (α) says $\theta(I_\eta) = \theta(|I_\eta|) = \kappa_{\eta(0)}, \kappa_{\eta(0)+1} < \mu$ so as for clause (β), "$D_{\mathbf{p}}$ is a filter on I_η", it holds by the choice of $\mathbf{p}$.

Clause (c): $\mathrm{rk}^{\mathbf{p}}_{\mathbf{d}}(f) = \mathrm{rk}_{\mathbf{d}}(f,\mathbf{p})$ is an ordinal as defined in 15.9.

Clause (d):
Clearly $\Sigma(\mathbf{d})$ is of the right form.

Clause (e):
On $\mathbf{j}$ - see 15.13(2)(c).

Clause (f):
We prove by induction on the ordinal ζ that:

$(*)$ if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{j}(\mathbf{d}) > \varepsilon$ and $A = \cup\{A_\alpha : \alpha < \kappa_\varepsilon\} \in D_{\mathbf{d}}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ and $A_\alpha \in D^+_{\mathbf{d}} \Rightarrow \mathrm{rk}_{\mathbf{d}+A_\alpha}(f) \ge \alpha$ then $\mathrm{rk}_{\mathbf{d}}(f) \ge \alpha$.

Now Definition 15.9 is tailored made for this.

Older version using 15.8 recheck:
For $\alpha = 0$ and α a limit ordinal this is obvious. For $\alpha = \beta + 1$ let $\mathscr{Y} = \{\alpha < \kappa_\varepsilon : A_\alpha \in D^+_{\mathbf{d}}\}$ and for $\alpha \in \mathscr{Y}$ let $\mathbf{n}_\alpha = \min\{n$: there is $(\mathbf{e}, h) \in \Sigma(\mathbf{d} + A_\alpha)$ such that $\mathrm{rk}_{\mathbf{e}}(f \circ h) \ge \beta$ and $\eta_{\mathbf{e}}(0) = n\}$. Clearly $\mathbf{n}_\alpha$ is well defined for $\alpha \in Y$, and let $w := \{n : \cup\{A_\alpha : \alpha \in \mathscr{Y}$ and $\mathbf{n}_\alpha = n\} \in D^+_{\mathbf{d}}\}$ and also the rest should be clear.

Clause (g): (no-hole)
By the Definition 15.8 or 15.8 of rk. Saharon 09.5.31 recheck.

Clause (h): $\mathrm{rk}_{\mathbf{d}}(f+1) = \mathrm{rk}_{\mathbf{d}}(f) + 1$.
We prove by induction on the ordinal α that:

$(*)$ for every $\mathbf{d} \in \mathbb{D}$ and $f \in {}^{I[\mathbf{d}]}\mathrm{Ord}$ we have $\mathrm{rk}_{\mathbf{d}}(f) \ge \alpha \Leftrightarrow \mathrm{rk}_{\mathbf{d}}(f+1) \ge \alpha + 1$.

Clause (i): Obvious. $\square_{15.17}$

{f6.19}

Claim 15.18. *Assume $\bar{J} = \langle J_n : n < \omega \rangle$ is a candidate and* $\mathrm{IND}(\bar{J})$.

Then $\mathbf{p}^2_{\bar{J}}$ is a strict rank system.

Proof. By 15.17 and the definition, it is a weak rank system. So we should prove the "strict", i.e. clause (j) of Definition m4.6 of §3 of [Sh:938] which we do by m4.16 of §3 of [Sh:938]. We use $\Sigma_1(\mathbf{d}) = \Sigma(\mathbf{d})$.

On $(*)_2$:

Given $\mathbf{d}$ we choose $j < \omega$ such that $j > \eta_{\mathbf{d}}(0)$ and assume $\mathbf{e} \in \mathbb{D}_{\geq j}$. □

§(2B) Revisiting

The simplest case below is: $\mathbf{x}$ consist $I_n = \kappa_n, \kappa_n < \kappa_{n+1}, J_{1,n} = [\kappa_n]^{<\theta}, J_{2,n} = [\kappa_n]^{<\kappa_n}$, $\mathrm{ind}(\mu,\theta), \mu = \Sigma\kappa_n, \mu$ minimal (or $\mu = \infty$) $\mathrm{ind}_{\mathbf{x}} :\in \mathrm{Ord}_* \cup \{\infty\}$.

For μ there are algebras on γ with no independent ω-sequence hence [Sh:835] and see §5 apply. But if using $\mathbf{x}$ we have a rank 2-system for which Theorem m4.13 of §3 of [**?**] apply (check!)

We may consider the pseudo version (using $\mathrm{comp}_\gamma(J)$. We have to sort out the amount of choice needed -seemingly.

Definition 15.19. We say that $\mathbf{x}$ is a ω-candidate when it consists of {k2}

(*a*) set I_n for $n < \omega$ (κ a cardinal and $\theta(< \kappa) = \kappa$

(*b*) ideal $J_{n,k}$ on I_n for $k < \omega, n < \omega$

(*c*) $J_{n,k} \subseteq J_{n,k+1}$

(*d*) κ_n.

Definition 15.20. For a 2-candidate $\mathbf{x}$ we define by induction on $i < \omega$ what is an $\mathbf{x}$-object $\mathfrak{c} = \mathbf{d}$ of depth i, such that {k5}

$(*)_\iota$ for some $n_{\mathbf{d}} < m_{\mathbf{d}} < \omega, \iota$ is an $\subseteq$-increasing sequence $\langle J_{\iota,k} : k < \omega \rangle$ of ideals on $I_{m_{\mathbf{d}},n_{\mathbf{d}}} = \Pi\{I_k : k \in [m,n)\}$.

The case $i = 0$:

$n_{\mathbf{d}} = m_{\mathbf{d}} + 1$ and let $h_{\mathbf{d}}$ be the one-to-one function from $I_{m_{\mathbf{d}}}$ onto $I_{\mathbf{m}_\iota,n_\iota}$ and $J_{\iota,k} \in \hat{h}_\iota(I_{m_\iota,k} + A_k)$ where $A_k \in J^+_{m_{\mathbf{d}}}$ and $A_k \supseteq A_{k+1}$ for $k < \omega$.

The case $i + 1$:

For some $k, \iota(1), \iota(2)$ we have

(*a*) $k \in (m_{\mathbf{d}}, n_{\mathbf{d}})$

(*b*) $\iota(\ell)$ is an i_ℓ-pair for some $i_\ell \leq i$ for $\ell = 1, 2$

(*c*) $m_{\mathbf{d}(1)} = m_\iota, n_{\mathbf{d}(1)} = k$

(*d*) $m_{\mathbf{d}(2)} = k, n_{\mathbf{d}(1)} = k$

(*e*) $m_{\mathbf{d}(2)} = k, n_{\mathbf{d}(2)} = n_\iota$

(*f*) there are $\langle A_{1,k}, A_{2,k} : k < \omega \rangle$ such that

  (α) $A_{\ell,k} \in J_{\iota(\ell),k+1}$

  (β) $B \in J_{\mathbf{d},k}$ iff $B \subseteq I_{m_{\mathbf{d}},n_{\mathbf{d}}}$ and for some $B_1 \in I_{\iota(1),k}$ we have $\eta \in A_{2,k} \subseteq I_{n_{\mathbf{d}(2)},n_{\mathbf{d}(2)}} \Rightarrow \{\nu \in I_{m_{\mathbf{d}(2)},n_{\mathbf{d}(2)}} : \eta \cup \nu \in B\} \in ?$

Remark 15.21. 1) Definition 15.20? seemingly does not behave transitively.
2) We may allow $n_{\mathbf{d}} = m_{\mathbf{d}}$.

{k7}

Definition 15.22. For $\mathbf{x}$ an ω-candidate, we define a p.c.s. $\mathbf{p} = \mathbf{p}^0_{\mathbf{x}}$ as follows:

(a) $\mathbb{D}_{\mathbf{p}} = \{\mathbf{d} : \mathbf{d}$ is an $\mathbf{x}$-object$\}$
(b) $\Sigma(\mathbf{d}_\iota) = \{\mathbf{d}$: for some $\mathbf{d}_2$ the triple $(\mathbf{d}, \mathbf{d}_{i_1}, \mathbf{d}_{i_2})$ is as in Definition 15.20
(c) $\mathbf{j}(\mathbf{d})$ is $m_{\mathbf{d}}$
(d) $\mu = \cup\{\kappa_n : n < \omega\}$.

{k9}

Claim 15.23. *If $\mathbf{x}$ is an ω-candidate then $p^0_{\mathbf{x}}$ is a quasi rank sytem.*

Proof. FILL. □

{k11}

Definition 15.24. 1) For an ω-candidate $\mathbf{x}$ we say it is well founded when the p.r.s. $\mathbf{p}^0_{\mathbf{x}}$ is well founded, e.g. $\mathbf{p}_{\mathbf{x}}$ is a weak rank system.
2) For a well founded.

{k13}

Claim 15.25. *If $\mathbf{x}$ is a well founded ω-candidate then $\mathbf{p}_{\mathbf{x}}$ is a strict rank system.*

Proof. Stage A: We have to check clause (1) from Definition m4.6 of §3 of [Sh:938].
So assume $\mathbf{d}, \zeta, \xi, f$ are as in ⊞ there. Choose $j < \omega$ such that $j > n_{\mathbf{d}}$ and toward contradiction assume $\mathbf{e}, g$ are as in ⊕ there.

Stage B: We find $(\mathbf{e}_1, g_1)$ satisfying ⊕ of clause (j) of m4.6 of §3 of [Sh:938] and $m_{\mathbf{e}_1} = n_{\mathbf{d}}$; note if we define as in [**?**](2) rather than as in 15.13(3), we would not need this step, but then we may have to reconsider the proof of (f) of Definition m4.3 of §3 of [Sh:938].

Stage C: We use $AC_{I[\mathbf{e}]}$ we continue as in 15.18 and in §4. But see footnote to $\bullet_3$ in ⊕ in clause (j) of m4.6 of §3 of [**?**]. $\square_{15.25}$

§ 16. Appendix: psuedo true cofinality

We repeat here [Sh:938, §5].

Pseudo PCF

We try to develop pcf theory with little choice. We deal only with $\aleph_1$-complete filters, and replace cofinality and other basic notions by pseudo ones, see below. This is quite reasonable as with choice there is no difference.

This section main result are **??**, existence of filters with pseudo-true-cofinality; 16.19, giving a parallel of $J_{<\lambda}[\alpha]$; and 1.6, on generators of $J^{[\bar{\alpha}]}_{<\lambda^+}$.

In the main case we may (in addition to ZF) assume DC + $AC_{\mathscr{P}(\mathscr{P}(Y))}$; this will be continued in [Sh:938].

{r1}

Hypothesis 16.1. ZF

{r2}

Definition 16.2. 1) We say that a partial order P is $(< \kappa)$-directed when every subset A of P of power $< \kappa$ has a common upper bound.
1A) Similarly P is $(\leq S)$-directed.
2) We say that a partial order P is pseudo $(< \kappa)$-directed when it is $(< \kappa)$-directed and moreover every subset $\cup\{P_\alpha : \alpha < \delta\}$ has a common upper bound when:

(a) if $\delta < \kappa$ is a limit ordinal
(b) $\bar{P} = \langle P_\alpha : \alpha < \delta\rangle$ is a sequence of non-empty subsets of P

(c) if $\alpha_1 < \alpha_2, p_1 \in P_{\alpha_1}$ and $p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$.

2A) For a partial order S we say that the partial order P is pseudo ($\leq S$)-directed when $\cup\{P_s : s \in S\}$ has a common upper bound whenever

(a) $\langle P_s : s \in S\rangle$ is a sequence
(b) $P_s \subseteq P$
(c) if $s <_S t$ and $f \in P_s, g \in P_t$ then $f <_p g$
(d) if $s \in S$ then P_s has a common upper bound (so if S has no minimal member this is redundant).

{r3}

Definition 16.3. We say that a partial (or quasi) order P has pseudo true cofinality δ when: δ is a limit ordinal and there is a sequence $\langle P_\alpha : \alpha < \delta\rangle$ such that

(a) $P_\alpha \subseteq P$ and $\delta = \sup\{\alpha < \delta : P_\alpha$ non-empty$\}$
(b) if $\alpha_1 < \alpha_2 < \delta, p_1 \in P_{\alpha_1}, p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$
(c) if $p \in P$ then for some $\alpha < \delta$ and $q \in P_\alpha$ we have $p \leq_P q$.

{r4}

Remark 16.4. 0) See 16.2(2) and 16.8(1).
1) We could replace δ by a partial order Q.
2) The most interesting case is in Definition 16.6.
3) We may in Definition 16.3 demand δ is a regular cardinal.
4) Usually in clause (a) without loss of generality $\bigwedge_\alpha P_\alpha \neq \emptyset$, as without loss of generality $\delta = \mathrm{cf}(\delta)$ using $P'_\alpha = P_{f(\alpha)}$ where $f(\alpha) =$ the α-th member of $\{\beta < \delta : P_\beta \neq \emptyset\}$. Why do we allow $P_\alpha = \emptyset$? as it is more natural in 16.17(1), but can usually ignore it.

{r5}

Example 16.5. Suppose we have a limit ordinal δ and a sequence $\langle A_\alpha : \alpha < \delta\rangle$ of sets with $\prod_{\alpha<\delta} A_\alpha = \emptyset$; moreover $u \subseteq \delta = \sup(u) \Rightarrow \prod_{\alpha\in u} A_\alpha = \emptyset$. Define a partial order P by:

(a) its set of elements is $\{(\alpha, a) : a \in A_\alpha$ and $\alpha < \delta\}$
(b) the order is $(\alpha_1, a_1) <_P (\alpha_2, a_2)$ iff $\alpha_1 < \alpha_2$ (and $a_\ell \in A_{\alpha_\ell}$ for $\ell = 1, 2$).

It seems very reasonable to say that P has true cofinality but there is no increasing cofinal sequence.

{r6}

Definition 16.6. 1) For a set Y and sequence $\bar\alpha = \langle \alpha_t : t \in Y\rangle$ of ordinals and cardinal κ we define

$$\text{ps-tcf-fil}_\kappa(\bar\alpha) = \{D : \ D \text{ a } \kappa\text{-complete filter on } Y \text{ such that } (\Pi\bar\alpha/D) \text{ has a pseudo true cofinality}\};$$

see below.
2) We say that $\Pi\bar\alpha/D$ or $(\Pi\bar\alpha, D)$ or $(\Pi\bar\alpha, <_D)$ has pseudo true cofinality γ when D is a filter on $Y = \mathrm{Dom}(\bar\alpha)$ and γ is a limit ordinal and the partial order $(\Pi\bar\alpha, <_D)$ essentially does[9], i.e., there is a sequence $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \gamma\rangle$ satisfying:

[9]so necessarily $\{s \in Y : \alpha_s > 0\}$ belongs to D but is not necessarily empty; if it is non-empty then $\Pi\bar\alpha = \emptyset$, so pedantically this is wrong, but we shall ignore this or assume $\bigwedge_t \alpha_t \neq 0$ when not said otherwise.

$\circledast_{\bar{\mathscr{F}}}$ (a) $\mathscr{F}_\beta \subseteq \{f \in {}^Y\mathrm{Ord} : f <_D \bar{\alpha}\}$

(b) $\mathscr{F}_\beta \neq 0$

(c) if $\beta_1 < \beta_2, f_1 \in \mathscr{F}_{\beta_1}$ and $f_2 \in \mathscr{F}_{\beta_2}$ then $f_1 < f_2 \bmod D$

(d) if $f \in {}^Y\mathrm{Ord}$ and $f < \bar{\alpha} \bmod D$ then for some $\beta < \gamma$ we have $g \in \mathscr{F}_\beta \Rightarrow f < g \bmod D$ (by clause (c) this is equivalent to: for some $\beta < \gamma$ and some $g \in \mathscr{F}_\beta$ we have $f \leq g \bmod D$).

3) $\text{ps-pcf}_\kappa(\bar{\alpha}) = \text{ps-pcf}_{\kappa\text{-comp}}(\bar{\alpha}) := \{\gamma$: there is a κ-complete filter D on Y such that $\Pi\bar{\alpha}/D$ has pseudo true cofinality γ and γ is minimal for $D\}$.
4) $\text{pcf-fil}_{\kappa,\gamma}(\bar{\alpha}) = \{D : D$ a κ-complete filter on Y such that $\Pi\bar{\alpha}/D$ has true cofinality $\gamma\}$.
5) In part (2) if γ is minimal we call it $\text{ps-tcf}(\Pi\bar{\alpha}, D)$ or simply $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$; note that it is a well defined (regular cardinal).

{r7}
Claim 16.7. *1) If $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, __then__ $(\Pi\bar{\alpha}, <_D)$ is pseudo $(< \lambda)$-directed.*
1A) If $\theta(S) < \lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ __then__ $(\Pi\bar{\alpha}, <_D)$ is pseudo $(\leq S)$-directed.
2) Similarly for any quasi order.
3) Assume AC_α for $\alpha < \lambda$. If $\mathrm{cf}(\alpha_t) \geq \lambda = \mathrm{cf}(\lambda)$ for $t \in Y$ then $(\Pi\bar{\alpha}, <_D)$ is λ-directed.
4) Assume $AC_{Y\times\lambda}$. If $\mathrm{cf}(\alpha_s) > \lambda$ for $s \in Y$ __then__ $(\Pi\bar{\alpha}, <_D)$ is pseudo λ^+-directed.

Proof. As in 16.8(1) below. $\square_{16.7}$

{r8}
Claim 16.8. *Let $\bar{\alpha} = \langle \alpha_s : s \in Y\rangle$ and D is a filter on Y.*
0) If $\Pi\bar{\alpha}/D$ has pseudo true cofinality __then__ $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is a regular cardinal; similarly for any partial order.
1) If $\Pi\bar{\alpha}/D$ has pseudo true cofinality γ_1 and true cofinality γ_2 __then__ $\mathrm{cf}(\gamma_1) = \mathrm{cf}(\gamma_2) = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, similarly for any partial order.
2) $\text{ps-pcf}_\kappa(\bar{\alpha})$ is a set of regular cardinals so if $\Pi\bar{\alpha}/D$ has pseudo true cofinality __then__ $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is γ where $\gamma = \mathrm{cf}(\gamma)$ and $\Pi\bar{\alpha}/D$ has pseudo cofinality γ.
3) Always $\text{ps-pcf}_\kappa(\bar{\alpha})$ has cardinality $< \theta(\{D : D$ a κ-complete filter on $Y\})$.
4) If $\bar{\beta} = \langle \beta_s : s \in Y\rangle \in {}^Y\mathrm{Ord}$ and $\{s : \beta_s = \alpha_s\} \in D$ __then__ $\text{ps-tcf}(\Pi\bar{\alpha}/D) = \text{ps-tcf}(\Pi\bar{\beta}/D)$ so one is well defined iff the other is.

Proof. 0) By the definitions.
1) Let $\langle \mathscr{F}^\ell_\beta : \beta < \gamma_\ell\rangle$ exemplify "$\Pi\bar{\alpha}/D$ has pseudo true cofinality γ_ℓ" for $\ell = 1, 2$. Now

(∗) if $\ell \in \{1, 2\}$ and $\beta_\ell < \gamma_\ell$ then for some $\beta_{3-\ell} < \gamma_{3-\ell}$ we have $g_1 \in \mathscr{F}^\ell_{\beta_\ell} \wedge g_2 \in \mathscr{F}^{3-\ell}_{\beta_{\ell-1}} \Rightarrow g_1 <_D g_2$.

[Why? Choose $g^\ell \in \mathscr{F}^\ell_{\beta_\ell+1}$, choose $\beta_{3-\ell} < \gamma_{3-1}$ and $g_{3-\ell} \in \mathscr{F}^{3-\ell}_{\beta_{3-\ell}}$ such that $g^\ell < g^{3-\ell} \bmod D$.]

Hence

(∗) $h_1 : \gamma_1 \to \gamma_2$ is well defined when
$$h_1(\beta_1) = \mathrm{Min}\{\beta_2 < \gamma_2 : (\forall g_1 \in \mathscr{F}^1_{\beta_1})(\forall g_2 \in \mathscr{F}^2_{\beta_2})(g_1 < g_2 \bmod D)\}.$$

Clearly h is non-decreasing and it is not eventually constant (as $\cup\{\mathscr{F}^1_\beta : \beta < \gamma_1\}$ is cofinal in $\Pi\bar{\alpha}/D$) and has range unbounded in γ_2 (similarly).

The rest should be clear.
2) Follows.
3),4) Easy. $\square_{16.8}$

Concerning [Sh:835]

{r9.yajan}

Claim 16.9. *The Existence of true cofinality filter* $[\kappa > \aleph_0 + \mathrm{DC} + \mathrm{AC}_{<\kappa}]$ *If*

(a) D *is a* κ*-complete filter on* Y

(b) $\bar{\alpha} \in {}^Y\mathrm{Ord}$

(c) $\delta := \mathrm{rk}_D(\bar{\alpha})$ *satisfies* $\mathrm{cf}(\delta) \geq \theta(\mathrm{Fil}^1_\kappa(Y))$, *see below.*

Then for some D' *we have*

(α) D' *is a* κ*-complete filter on* Y

(β) $D' \supseteq D$

(γ) $\Pi\bar{\alpha}/D'$ *has pseudo true cofinality, in fact,* $\text{ps-tcf}(\Pi\bar{\alpha}, <_D) = \mathrm{cf}(\mathrm{rk}_D(\bar{\alpha}))$.

Recall from [Sh:835]

{r9a}

Definition 16.10. 0) $\mathrm{Fil}^1_\kappa(Y) = \{D : D$ a κ-complete filter on $Y\}$ and if $D \in \mathrm{Fil}^1_\kappa(Y)$ then $\mathrm{Fil}^1_\kappa(D) = \{D' \in \mathrm{Fil}^1_\kappa(Y) : D \subseteq D'\}$.
1) $\mathrm{Fil}^4_\kappa(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are κ-complete filters on $Y\}$.
2) $J[f, D]$ where D is a filter on Y and $f \in {}^Y\mathrm{Ord}$ is $\{A \subseteq Y : A = \emptyset \mod D$ or $\mathrm{rk}_{D+A}(f) > \mathrm{rk}_D(f)\}$.

{r9b}

Remark 16.11. 1) On the Definition of pseudo $(< \kappa, 1+\gamma)$-complete D see [Sh:938, 1.13=0z.51]; we may consider changing the definition of $\mathrm{Fil}^1_\kappa(Y)$ to D is $\aleph_1$-complete and pseudo$(< \kappa, 1+\gamma))$-complete filter on Y.
2) Related to [Sh:835].

Proof. Proof of the Claim of **??**
Recall $\{y \in Y : \alpha_y = 0\} = \emptyset \mod D$ as $\mathrm{rk}_D(\langle \alpha_y : y \in Y\rangle) = \delta > 0$ but $f_1, f_2 \in {}^Y\mathrm{Ord} \wedge (f_1 = f_2 \mod D) \Rightarrow \mathrm{rk}_D(f_1) = \mathrm{rk}_D(f_2)$ hence without loss of generality $y \in Y \Rightarrow \alpha_y > 0$.

Let $\mathbb{D} = \{D' : D'$ is a filter on Y extending D which is κ-complete$\}$. So $\theta(\mathbb{D}) \leq \theta(\mathrm{Fil}^1_{\aleph_1}(Y)) \leq \mathrm{cf}(\delta)$. For any $\gamma < \mathrm{rk}_D(\bar{\alpha})$ and $D' \in \mathbb{D}$ let

$(*)_2$ (a) $\mathscr{F}_{\gamma,D'} = \{f \in \Pi\bar{\alpha} : \mathrm{rk}_D(f) = \gamma$ and D' is $\mathrm{dual}(J[f, D])\}$

(b) $\mathscr{F}_{D'} = \cup\{\mathscr{F}_{\gamma,D'} : \gamma < \mathrm{rk}_D(\bar{\alpha})\}$

(c) $\Xi_{\bar{\alpha},D'} = \{\gamma < \mathrm{rk}_D(\bar{\alpha}) : \mathscr{F}_{\gamma,D'} \neq \emptyset\}$

(d) $\mathscr{F}_\gamma = \cup\{\mathscr{F}_{\gamma,D''} : D'' \in \mathbb{D}\}$.

Now

$(*)_3$ if $\gamma < \mathrm{rk}_D(\bar{\alpha})$ then $\mathscr{F}_\gamma \neq \emptyset$.

[Why? By [Sh:938, 1.8(2)=z0.23(2)] there is $g \in {}^Y\mathrm{Ord}$ such that $g < f \mod D$ and $\mathrm{rk}_D(g) = \gamma$ and without loss of generality $g \in \Pi\bar{\alpha}$. Now let $D' = \mathrm{dual}(J[g, D])$, so $(D, D') \in \mathrm{Fil}^4_\kappa(Y), D' \in \mathbb{D}$ and $g \in \mathscr{F}_{\gamma,D'}$, see [Sh:938, 1.7(2)=z0.23(2)], Claim [Sh:835, 0.10(2)], here we use $\mathrm{AC}_{<\kappa}$.]

$(*)_4$ $\{\sup(\Xi_{\bar{\alpha},D'}) : D' \in \mathbb{D}$ and $\Xi_{\bar{\alpha},D'}$ is bounded in $\mathrm{rk}_D(\bar{\alpha})\}$ is a subset of $\mathrm{rk}_{D'}(\bar{\gamma})$ which has cardinality $< \theta(\mathbb{D}) \leq \theta(\mathrm{Fil}^1_\kappa(Y)) \leq \mathrm{cf}(\delta)$.

[Why? The function $D' \mapsto \sup(\Xi_{\bar\alpha,D'})$ witness this.]

$(*)_5$ the set in $(*)_4$ is bounded below $\mathrm{rk}_D(\bar\alpha)$ so let $\gamma(*) < \mathrm{rk}_D(\bar\alpha)$ be its supremum.

[Why? By $(*)_4$.]

$(*)_6$ there is $D' \in \mathbb{D}$ such that $\Xi_{\bar\alpha,D'}$ is unbounded in $(\Pi\bar\alpha, <_{D'})$.

[Why? Choose $\gamma < \mathrm{rk}_D(\bar\alpha)$ such that: $\gamma > \gamma(*)$. By $(*)_3$ there for some $f \in \mathscr{F}_{\gamma(*)}$ and $D' \in \mathbb{D}$ we have $f \in \mathscr{F}_{\gamma(*),D'}$ so by the choice of $\gamma(*)$ the set $\Xi_{\bar\alpha,D'}$ cannot be bounded in $\mathrm{rk}_D(\bar\alpha)$.]

$(*)_7$ if $\gamma_1 < \gamma_2$ are from $\Xi_{\bar\alpha,D'}$ and $f_1 \in \mathscr{F}_{\gamma_1,D'}, f_2 \in \mathscr{F}_{\gamma_2,D'}$ then $f_1 <_{D'} f_2$.

[Why? By [Sh:938, 1.7=z0.23], [Sh:835, 0.10(2)].]

Together we are done: by $(*)_6$ there is $D' \in \mathbb{D}$ such that $\Xi_{\bar\alpha,D'}$ is unbounded in $\mathrm{rk}_D(\bar\alpha)$. Let $\bar{\mathscr{F}} = \langle \mathscr{F}_{\gamma,D'} : \gamma \in \Xi_{\bar\alpha,D'}\rangle$ witness that $(\Pi\bar\alpha, <_{D'})$ has pseudo true cofinality, and so $\text{ps-tcf}(\Pi\bar\alpha, <_D) = \mathrm{cf}(\mathrm{otp}(\Xi_{\bar\alpha,D'})) = \mathrm{cf}(\mathrm{rk}_D(\bar\alpha))$, so we are done. $\square_{??}$

So we have

{r10}

Definition/Claim 16.12. 1) We say that $\delta = \text{ps-tcf}_{\bar D}(\bar\alpha)$, where δ is a limit ordinal when, for some set Y:

(a) $\bar\alpha \in {}^Y\mathrm{Ord}$
(b) $\bar D = (D_1, D_2)$
(c) $D_1 \subseteq D_2$ are $\aleph_1$-complete filters on Y
(d) $\mathrm{rk}_{D_1}(\bar\alpha) = \delta = \sup(\Xi_{\bar D,\bar\alpha})$ where $\Xi_{\bar D,\bar\alpha} = \{\gamma < \mathrm{rk}_{D_1}(\bar\alpha)$: for some $f < \bar\alpha$ mod D_1, we have $\mathrm{rk}_{D_1}(f) = \gamma$ and $D_2 = \mathrm{dual}(J[f, D_1]\}$.

2) If D_1 is $\aleph_1$-complete filter on $Y, \bar\alpha = \langle \alpha_t : t \in Y\rangle$ and $\mathrm{cf}(\alpha_t) \geq \theta(\mathrm{Fil}^1_{\aleph_1}(Y))$ for $t \in Y$ then for some $\aleph_1$-complete filter D_2 on Y extending D_1 we have $\text{ps-tcf}_{(D_1,D_2)}(\bar\alpha)$ is well defined.
3) Moreover in part (2) there is a definition giving for any $(Y, D_1, D_2, \bar\alpha)$ as there, a sequence $\langle \mathscr{F}_\gamma : \gamma < \delta\rangle$ exemplifying the value of $\text{ps-tcf}_{\bar D}(\bar\alpha)$.

Proof. Let $\delta := \mathrm{rk}_{D_1}(f)$, so by Claim 16.16 below $\mathrm{cf}(\delta) \geq \theta(\mathrm{Fil}^1_{\aleph_1}(Y))$ hence has Claim **??** above and its proof the conclusion holds: the proof is needed for "$\delta = \sup(\Xi_{\bar D,\alpha})$", noting observation 16.13 below. $\square_{16.12}$

{r10d}

Observation 16.13. 1) [DC] or just $[\mathrm{AC}_{\aleph_0}]$.

Assume D is an $\aleph_1$-complete filter on Y and $f, f_n \in {}^Y\mathrm{Ord}$ for $n < \omega$ and $f(t) = \sup\{f_n(t) : n < \omega\}$. Then $\mathrm{rk}_D(f) = \sup\{\mathrm{rk}_D(f_n) : n < \omega\}$.

Remark 16.14. Similarly for other amounts of completeness, see 16.18.

Proof. As $\mathrm{rk}_D(f) = \min\{\mathrm{rk}_{D+A_n}(f) : n < \omega\}$ if $\cup\{A_n : n < \omega\} \in D, A_n \in D^+$ by [Sh:71] or see [Sh:835, 1.9=z0.25]. $\square_{16.13}$

Remark 16.15. Also in [Sh:835, 1.9(2)=z0.25(2)] can use AC_I only, i.e. omit the assumption DC, a marginal point here.

{r11}

Claim 16.16. *$[\mathrm{AC}_{<\theta}]$ The ordinal δ has cofinality $\geq \theta$ when:*

$\circledast$ (a) $\delta = \mathrm{rk}_D(\bar\alpha)$

(b) $\bar{\alpha} = \langle \alpha_y : y \in Y \rangle \in {}^Y\mathrm{Ord}$
(c) *D is an $\aleph_1$-complete filter on Y*
(d) $y \in Y \Rightarrow \mathrm{cf}(\alpha_y) \geq \theta$.

Proof. Note that $y \in Y \Rightarrow \alpha_y > 0$. Toward contradiction assume $\mathrm{cf}(\delta) < \theta$ so δ has a cofinal subset C of cardinality $< \theta$. For each $\beta < \delta$ for some $f \in {}^Y\mathrm{Ord}$ we have $\mathrm{rk}_D(f) = \beta$ and $f <_D \bar{\alpha}$ and without loss of generality $f \in \prod_{y \in Y} \alpha_y$. By $\mathrm{AC}_{<\theta}$ there is a sequence $\langle f_\beta : \beta \in C \rangle$ such that $f_\beta \in \prod_{y \in Y} \alpha_y, f <_D \bar{\alpha}$ and $\mathrm{rk}_D(f_\beta) = \beta$. Define $g \in \prod_{y \in Y} \alpha_y$ by $g(y) = \cup\{f_\beta(y) : \beta \in C$ and $f_\beta(y) < \alpha_t\}$. By clause (d) we have $[y \in Y \Rightarrow g(y) < \alpha_y]$, so $g <_D \bar{\alpha}$, hence $\mathrm{rk}_D(\bar{g}) <\ \mathrm{rk}_D(\alpha)$ but by the choice of g we have $\beta \in C \Rightarrow f_\beta \leq_D g$ hence $\beta \in C \Rightarrow \beta =\ \mathrm{rk}_D(f_\beta) \leq\ \mathrm{rk}_D(g)$ hence $\delta = \sup(C) \leq\ \mathrm{rk}_D(g)$, contradiction. $\square_{16.16}$

Observation 16.17. 1) Assume $(\bar{\alpha}, D)$ satisfies {r12}

(a) D a filter on Y and $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ and each α_t is a limit ordinal
(b) $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \partial \rangle$ exemplify $\partial =$ ps-tcf$(\Pi\bar{\alpha}, <_D)$ so we demand just $\partial = \sup\{\beta < \partial : \mathscr{F}_\beta \neq \emptyset\}$
(c) $\mathscr{F}'_\beta = \{f \in \prod_{t \in Y} \alpha_t$: for some $g \in \mathscr{F}_\beta$ we have $f = g \bmod D\}$.

Then: $\langle \mathscr{F}'_\beta : \beta < \partial \rangle$ exemplify $\partial =$ ps-tcf$(\Pi\bar{\alpha}, <_D)$ that is

(α) $\bigcup_{\beta < \gamma} \mathscr{F}'_\beta$ is cofinal in $(\Pi\bar{\alpha}, <_D)$
(β) for every $\beta_1 < \beta_2 < \partial$ and $f_1 \in \mathscr{F}'_{\beta_1}$ and $f_2 \in \mathscr{F}'_{\beta_2}$ we have $f_1 \leq f_2$.

2) Similarly, if $D, \bar{\mathscr{F}}$ satisfies clauses (a),(b) above and D is $\aleph_1$-complete and $\partial = \mathrm{cf}(\partial) > \aleph_0$ then we can "correct" $\bar{\mathscr{F}}$ to make it $\aleph_0$-continuous that is $\langle \mathscr{F}''_\beta : \beta < \partial \rangle$ defined in $(c)_1 + (c)_2$ below satisfies $(\alpha) + (\beta)$ above and (γ) below and so is $\aleph_0$-continuous, (see below) where

$(c)_1$ if $\beta < \partial$ and $\mathrm{cf}(\beta) \neq \aleph_0$ then $\mathscr{F}''_\beta = \mathscr{F}'_\beta$
$(c)_2$ if $\beta < \partial$ and $\mathrm{cf}(\beta) = \aleph_0$ then $\mathscr{F}''_\beta = \{\sup\langle f_n : n < \omega \rangle$: for some increasing sequence $\langle \beta_n : n < \omega \rangle$ with limit β we have $n < \omega \Rightarrow f_n \in \mathscr{F}'_{\beta_n}\}$, see below

(γ) if $\beta < \partial$ and $\mathrm{cf}(\beta) = \aleph_0$ and $f_1, f_2 \in \mathscr{F}''_\beta$ then $f_1 = f_2 \bmod D$.

3) This applies to an increasing sequence $\langle \mathscr{F}_\beta : \beta < \delta \rangle, \mathscr{F}_\beta \subseteq {}^Y\mathrm{Ord}, \delta$ a limit ordinal.

Proof. Straightforward. $\square_{16.17}$

Definition 16.18. 0) If $f_n \in {}^Y\mathrm{Ord}$ for $n < \omega$, then $\sup\langle f_n : n < \omega \rangle$ is defined as the function f with domain Y such that $f(t) = \cup\{f_n(t) : n < \omega\}$. {r13}
1) We say $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \lambda \rangle$ exemplifying $\lambda =$ ps-tcf$(\Pi\bar{\alpha}, <_D)$ is weakly $\aleph_0$-continuous when:

if $\beta < \partial$, $\mathrm{cf}(\beta) = \aleph_0$ and $f \in \mathscr{F}_\beta$ then for some sequence $\langle (\beta_n, f_n) : n < \omega \rangle$ we have $\beta = \cup\{\beta_n : n < \omega\}, \beta_n < \beta_{n+1} < \beta, f_n \in \mathscr{F}_{\beta_\alpha}$ and $f = \sup\langle f_n : n < \omega \rangle$; so if D is $\aleph_1$-complete then $\{f/D : f \in \mathscr{F}_\beta\}$ is a singleton.
2) We say it is $\aleph_0$-continuous if we can replace the last "then" by "iff".

{r14}

Theorem 16.19. *The Canonical Filter Theorem Assume* DC *and* $AC_{\mathscr{P}(Y)}$.

Assume $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle \in {}^Y\text{Ord}$ *and* $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \theta(\mathscr{P}(Y))$ *and* $\partial \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ *hence is a regular cardinal. Then there is* $D = D^{\bar{\alpha}}_{\partial}$*, an* $\aleph_1$*-complete filter on* Y *such that* $\partial = \text{ps-tcf}(\Pi\bar{\alpha}/D)$ *and* $D \subseteq D'$ *for any other such* $D' \in \text{Fil}^1_{\aleph_1}(D)$.

{r14b}

Remark 16.20. 1) By **??** there are some such ∂.
2) We work to use just $AC_{\mathscr{P}(Y)}$ and not more.

Proof. Let

$\boxplus_1$ (a) $\mathbb{D} = \{D : D$ is an $\aleph_1$-complete filters on Y such that $(\Pi\bar{\alpha}/D)$ has pseudo true cofinality $\partial\}$,

(b) $D_* = \cap\{D : D \in \mathbb{D}\}$.

Now obviously

(c) D_* is an $\aleph_1$-complete filter on Y.

For $A \subseteq Y$ let $\mathbb{D}_A = \{D \in \mathbb{D} : A \notin D\}$ and let $\mathscr{P}_* = \{A \subseteq Y : \mathbb{D}_A \neq \emptyset\}$. As $AC_{\mathscr{P}(Y)}$ we can find $\langle D_A : A \in \mathscr{P}_* \rangle$ such that $D_A \in \mathbb{D}_A$ for $A \in \mathscr{P}_*$. Let $\mathbb{D}_* = \{D_A : A \in \mathscr{P}_*\}$, clearly

$\boxplus_2$ $D_* = \cap\{D : D \in \mathbb{D}_*\}$ and $\mathbb{D}_* \subseteq \mathbb{D}$ is non-empty.

As $AC_{\mathscr{P}_*}$ holds clearly

$(*)_0$ we can choose $\langle \bar{\mathscr{F}}^A : A \in \mathscr{P}_* \rangle$ such that $\bar{\mathscr{F}}_A$ exemplifies $D_A \in \mathbb{D}$ as in 16.17(1),(2), so in particular is $\aleph_0$-continuous.

For each $\beta < \partial$ let $\mathscr{F}^*_\beta = \cap\{\mathscr{F}^A_\beta : A \in \mathscr{P}_*\}$, now

$(*)_1$ $\mathscr{F}^*_\beta \subseteq \Pi\bar{\alpha}$.

[Why? As by 16.17(1)(c) we have $\mathscr{F}^A_\beta \subseteq \Pi\bar{\alpha}$ for each $A \in \mathscr{P}_*$.]

$(*)_2$ if $\beta_1 < \beta_2 < \partial, f_1 \in \mathscr{F}^*_{\beta_1}$ and $f_2 \in \mathscr{F}^*_{\beta_2}$ then $f_1 < f_2 \bmod D_*$.

[Why? As $A \in \mathscr{P}_* \Rightarrow f_1 <_{D_A} f_2$ by the choice of $\langle \mathscr{F}^*_\beta : \beta < \partial \rangle$, hence the set $\{t \in Y : f_1(t) < f_2(t)\}$ belongs to D_A for every $A \in \mathscr{P}_*$ hence by $\boxplus_2$ it belongs to D_* which means that $f_1 <_{D_*} f_2$ as required.]

$(*)_3$ if $f \in \Pi\bar{\alpha}$ then for some $\beta_f < \partial$ we have $f' \in \cup\{\mathscr{F}^*_\beta : \beta \in [\beta_f, \partial)\} \Rightarrow f < f' \bmod D_*$.

[Why? For each $A \in \mathscr{P}_*$ there are β, g such that $\beta < \partial, g \in \mathscr{F}^A_\beta$ and $f < g \bmod D$ hence $\beta' \in [\beta + 1, \partial) \wedge f' \in \mathscr{F}^A_{\beta'} \Rightarrow f < g < f' \bmod D_A$. Let β_A be the minimal such ordinal $\beta_A < \delta$. As $\text{cf}(\delta) \geq \theta(\mathscr{P}(Y)) \geq \theta(\mathscr{P}_*)$, clearly $\beta_* = \sup\{\beta_A + 1 : A \in \mathscr{P}_*\}$ is $< \delta$. So $A \in \mathscr{P}_* \wedge g \in \cup\{\mathscr{F}^*_\beta : \beta \in [\beta_*, \delta)) \Rightarrow f <_D g$. By $\boxplus_2$ the ordinal α_* is as required on α_ℓ.]

Moreover

$(*)_4$ there is a function $f \mapsto \beta_f$ in $(*)_3$.

[Why? As we can (and will) choose β_f as minimal β such that ...]

$(*)_5$ for every $\beta_* < \partial$ there is $\beta \in (\beta_*, \partial)$ such that $\mathscr{F}^*_\beta \neq \emptyset$.

[Why? We choose by induction on n, a sequence $\bar\beta_n = \langle \beta_{n,A} : A \in \mathscr{P}_*\rangle$ and a sequence $\bar f_n = \langle f_{n,A} : A \in \mathscr{P}_*\rangle$ and a function f_n such that

- (α) $\beta_n < \partial$ and $m < n \Rightarrow \beta_m < \beta_n$
- (β) $\beta_0 = \beta_*$ and for $n > 0$ we let $\beta_n = \sup\{\beta_{m,A} : m < n, A \in \mathscr{P}_*\}$
- (γ) $\beta_{n,A} \in (\beta_n, \partial)$ is minimal such that there is $f_{n,A} \in \mathscr{F}^A_{\beta_{n,A}}$ satisfying $n = m+1 \Rightarrow f_m < f_{\beta_n,A}$ mod D_A
- (δ) $\langle f_{n,A} : A \in \mathscr{P}_*\rangle$ is a sequence such that each $f_{n,A}$ are as in clause (γ)
- (ε) $f_n \in \Pi\bar\alpha$ is defined by $f_n(t) = \sup\{f_{m,A}(t) + 1 : A \in \mathscr{P}_*$ and $m < n\}$.

[Why can we carry the induction? Arriving to n first, f_n is well defined $\in \Pi\bar\alpha$ by clause (ε) as $\mathrm{cf}(\alpha_t) \geq \theta(\mathscr{P}_*)$ for $t \in Y$. Second by clause (γ), $\langle \beta_{n,A} : A \in \mathscr{P}_*\rangle$ is well defined. Third by clause (δ) we can choose $\langle f_{m,A} : A \in \mathscr{P}_*\rangle$ as $\mathrm{AC}_{\mathscr{P}_*}$.

Lastly, the inductive construction is possibly by DC.]

Let $\beta^* = \cup\{\beta_n : n < \omega\}$ and $f = \sup\langle f_n : n < \omega\rangle$. Easily $f \in \cap\{\mathscr{F}^A_{\beta^*} : A \in \mathscr{P}_*\}$ as each $\langle \mathscr{F}^A_\beta : \beta < \partial\rangle$ is $\aleph_0$-continuous.]

$(*)_6$ if $f \in \Pi\bar\alpha$ then for some $\beta < \gamma$ and $f' \in \mathscr{F}^*_\beta$ we have $f < f'$ mod D^*.

[Why? By $(*)_3 + (*)_4$.]

So we are done. $\square_{16.19}$

Definition 16.21. For $\bar\alpha \in {}^Y\mathrm{Ord}$ let $J^{\aleph_1\text{-comp}}_{<\lambda}(\bar\alpha) = \{X \subseteq Y$: $\text{ps-pcf}_{\aleph_1\text{-com}}(\bar\alpha \restriction X) \subseteq \lambda\}$ and $J^{\aleph_1\text{-comp}}_{\leq\lambda}$ is $J^{\aleph_2\text{-comp}}_{<\lambda^+}$. {r16.yajan}

Remark 16.22. In 1.3, see Definition 16.6(3). {r17}

On this and more see [Sh:F955].

§ 17. Appendix: Definition of Rank-System

Moved from pg.3:
We define a function H from $\Pi\bar{\alpha}$ into $\Pi\{\lambda_X : X \in D\}$ by:

(α) $(H(f))(X) = \text{ Min}\{\beta < \lambda_X$: if $f' \in \mathscr{F}^X_\beta$ then $f \le f'$ mod $D_X\}$.

We let

(β) $\check{D}$ be the following filter on the set $\check{Y} := D$:
$Z \in \check{D}$ iff $Z \subseteq D$ and $(\exists X \in D)[Z \supseteq \{X' \in D : X' \subseteq X\}]$.

Now

(γ) $\check{D}$ is an $\aleph_1$-complete filter on $\check{Y}$

(δ) if $f_1, f_2 \in \Pi\bar{\alpha}$ and $f_1 \le f_2$ mod D^*_1 then $H(f_1) \le H(f_2)$ mod $\check{D}$

(ε) $(\prod\limits_{t \in \check{Y}} \lambda_t, <_{\check{D}})$ is pseudo $(< \lambda^+)$-directed.

[Why? By claim 16.7, i.e. 16.7 of §5 of [Sh:938].]
Because by an assumption

(ζ) if $f_1, f_2 \in \mathscr{F}_\alpha$ and $\alpha < \delta$ then $H(f_1) = H(f_2)$ mod $\check{D}$.

Why? $f_1 = f_2$ mod D hence by ? we have $f_1 = f_2$ mod D^*_1 hence by (yyy), $H(f_1) = H(f_2)$ mod $\check{D}$. FILL
Now by $(\varepsilon) + (zzz)$ we are done proving (h).]

(i) $D \subseteq D^*_1$.

[Why? Because if $A \in D$ then $X_1 := A$ witness $A \in D$, as $X \in D \wedge X \subseteq X_1 \Rightarrow X \in D \wedge X \subseteq A \Rightarrow X \in D_X \wedge X \subseteq A \subseteq Y \Rightarrow A \in D_X$.]

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