Abstract. We continue our investigation on pcf with weak forms of the axiom of choice. Characteristically, we assume DC + $\mathcal{P}(Y)$ when looking at $\prod_{s \in Y} \delta_s$. We get more parallels of pcf theorems.

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 [We continue [Sh:938, §5] to try to generalize the pcf theory for \(\aleph_1\)-complete filters \(D\) on \(Y\) assuming only DC + AC\(_\varpi(Y)\). So this is similar to [Sh:b, ChXII]. We suggest to replace cofinality by pseudo cofinality. In particular we get the existence of a sequence of generators, get a bound to Reg \(\cap\) \(pp(\mu)\setminus\mu_0\), the size of \(\text{Reg} \cap \mu \setminus \mu_0\) using a no-hole claim and existence of lub (unlike [Sh:835]).]
 §2 Composition and generating sequences for pseudo pcf, pg.16
 [We deal with pseudo true cofinality of \(\prod_{i \in Z} \prod_{j \in Y_i} \lambda_{i,j}\), also with the degenerated case in which each \(\langle \lambda_{i,j} : j \in Y_i \rangle\) is constant. We then use it to clarify the state of generating sequences; see 2.1, 2.2, 2.4, 2.6, 2.12, 2.13.]
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 [We get that several measures of \(*\mu/D\) are essentially equal.]
 §(3B) Depth of reduced powers of ordinals, pg.31
 [Using the independence property for a sequence of filters we can bound the relevant depth. This generalizes [Sh:460] or really [Sh:513, §3].]
 §(3C) Bounds on the Depth, pg.37
 [We start by basic properties dealing with the No-Hole Claim (1.13(1)) and dependence on \(\langle |\alpha_s| : s \in Y \rangle/D\) only (3.23). We give a bound for \(\lambda^{+\alpha(1)}/D\) (in Theorem 3.24, 3.26).]
§ 0. Introduction

In the first section we deal with generalizing the pcf theory in the direction started in [Sh:938, §5] trying to understand the pseudo true cofinality of small products of regular cardinals. The difference with earlier works is that here we assume $\text{AC}_\mathcal{U}$ for any set $\mathcal{U}$ of power $\leq |\mathcal{P}(\mathcal{P}(Y))|$ or, actually working harder, just $\leq |\mathcal{P}(Y)|$ when analyzing $\prod_{t \in Y} \alpha_t$, whereas in [Sh:497] we assumed $\text{AC}_{\text{sup}(\alpha, t \in Y)}$ and in [Sh:835] we have (in addition to $\text{AC}_{\mathcal{P}(\mathcal{P}(Y))}$) assumptions like $\text{“}[\sup\{\alpha_t : t \in Y\}]^{\aleph_0}$ is well ordered". In [Sh:938, §1-§4] we assume only $\text{AC}_{\aleph_0} + \text{DC}$ and consider $\aleph_1$-complete filters on $\mu$ but in the characteristic case $\mu$ is a limit of measurable cardinals.

Note that generally in this work, though we try occasionally not to use DC, it will not be a real loss to assume it all the time. More specifically, we prove the existence of a minimal $\aleph_1$-complete filter $D$ on $Y$ such that $\lambda = \text{ps-tcf}(\Pi\alpha_\delta, <_D)$ assuming $\text{AC}_{\mathcal{P}(Y)}$ and (of course) DC and $\alpha_\delta$ of large enough cofinality. We then prove the existence of one generator, that is, of $X \subseteq Y$ such that $\nu^\text{comp}_{\lambda, \alpha}(\lambda) = \nu^\text{comp}_{\lambda, \alpha} + X$, see 1.6 and even in (1.8) the parallel of the existence of a $<_D$-hub for an $<_D$-increasing sequence $<_{\mathcal{P}}(\alpha : \alpha < \lambda)$, generalize the no-hole claim in 1.13, and give a bound on $pp$ for non-fix points (in 1.11).

In §2 we further investigate true cofinality. In Claim 2.2, assuming $\text{AC}_\lambda$ and $D$ an $\aleph_1$-complete filter on $Y$, we start from $\text{ps-tcf}(\Pi\alpha_\delta, <_D)$, dividing by $\text{eq}(\alpha) = \{(s, t) : \alpha_s = \alpha_t\}$. We also prove the composition Theorem 2.6: it tells us when $\text{ps-tcf}(\bigcap_{i \in Y} \text{ps-tcf}(\bigcap_{j \in j} \lambda_{i,j}, <_{D_i}), <_E)$ is equal to $\text{ps-tcf}(\bigcap_{s \in S} \lambda_{s}, <_{D_s})$.

We then prove the pcf closure conclusion: giving a sufficient condition for the operation $\text{ps-pcf}_{\aleph_1, \text{comp}}$ to be idempotent. Lastly, we revisit the generating sequence.

In §3(A) we measure $\prod_{t \in Y} g(t)$ modulo a filter $D$ on $Y$ for $g \in Y(\text{Ord}\setminus\{0\})$ in three ways and show they are almost equal in 3.2. The price is that we replace (true) cofinality by pseudo (true) cofinality, which is inevitable. We try to sort out the "almost equal" in 3.5 - 3.7.

In §3(B) we prove a relative of [Sh:513, §3]; again dealing with depth (instead of rank as in [Sh:938]) adding some information even under ZFC. Assuming that the sequence $\langle D_n : n < \omega \rangle$ of filters has the independence property (IND), see Definition 3.12, with $D_0$ a filter on $Y_n$ we can bound the depth of $\langle \langle \nu_\lambda, \zeta, <_{D_n} \rangle, \zeta \rangle$, for every $\zeta$ for many $n$'s, see 3.13. Of course, we can generalize this to $\langle D_s : s \in S \rangle$. This is incomparable with the results of [Sh:938, §4]. See a continuation of [Sh:835] in [Sh:1005].

Note that the assumptions like $\text{IND}(D)$ are complementary to ones used in [Sh:835] to get considerable information. Our original hope was to arrive to a dichotomy. The first possibility will say that one of the versions of an axiom suggested in [Sh:835] holds, which means "for some suitable algebra", there is no independent $\omega$-sequence; in this case [Sh:835] tells us much. The second possibility will be a case of IND, and then we try to show that there is a rank system in the sense of [Sh:938]. But presently for this we need too much choice. The dichotomy we succeed to prove is with small $\alpha$-Depth in one side, the results of [Sh:835] on the other side. It would be better to have ps-Depth in the first side.

Question 0.1. $[\text{DC} + \text{AC}_{\mathcal{P}(Y)}]$
Assume
(a) \( \bar{\alpha} \in Y \text{ Ord} \)
(b) \( \text{cf}(\alpha_t) \geq \text{hrtg}(\mathcal{P}(Y)) \text{ for every } t \in Y \)
(c) \( \lambda_t \in \text{pcf}_t(\bar{\alpha}) \text{ for } t \in Z \), in fact, \( \lambda_t = \text{ps-tcf}(\Pi t : t \in Y) \)
(d) \( \lambda = \text{ps-tcf}_\text{comp}(\bar{\alpha}) \text{ for } t \in Z \)
(e) \( (a \in \text{pairwise disjoint}) \)
(A) Now does \( \lambda \in \text{ps-pcf}_\text{comp}(\bar{\alpha}) \)? (See 2.6.)
(B) Can we say something on \( D_\lambda \) from [Sh:938, 5.9] improved in 1.3?

Question 0.2. How well can we generalize the RGCH, see [Sh:460] and [Sh:829]; the above may be relevant; see [Sh:938] and here in §(3C).

Recall

Notation 0.3. 1) For any set \( X \) let \( \text{hrtg}(X) = \min{\alpha : \alpha \text{ an ordinal such that there is no function from } X \text{ onto } \alpha} \).
2) \( A \leq_{qu} B \text{ means that either } A = \emptyset \text{ or there is a function from } A \text{ onto } B. \)

Central in this work is

Definition 0.4. For a quasi order \( P \) we say \( P \) has pseudo-true-cofinality \( \lambda \) or “\( \lambda \) is the pseudo true cofinality of \( P \)” when \( \lambda \) is a regular cardinal and \( \lambda \) is a pseudo true cofinality of \( P \) which means that there is a sequence \( \mathcal{F} \) such that:
(a) \( \mathcal{F} = \langle \mathcal{F}_\alpha : \alpha < \lambda \rangle \)
(b) \( \mathcal{F}_\alpha \subseteq P \)
(c) if \( \alpha_1 < \alpha_2, p_1 \in \mathcal{F}_{\alpha_1} \text{ and } p_2 \in \mathcal{F}_{\alpha_2}, \text{ then } p_1 \leq_{P} p_2 \)
(d) if \( q \in \mathcal{F} \text{ then for some } \alpha < \lambda \text{ and } p \in \mathcal{F}_\alpha \text{ we have } q <_{P} p \)
(e) \( \lambda = \sup{\alpha < \lambda : \mathcal{F}_\alpha \neq \emptyset} \).

We may consider replacing AC\(_A\) by more refined version, AC\(_{A,B}\) defined below (e.g. in 1.1, 2.6) but we have not dealt with it systematically.

Definition 0.5. 1) AC\(_{A,B}\) means: if \( \langle X_a : a \in A \rangle \) is a sequence of non-empty sets then there is a sequence \( \langle Y_a : a \in A \rangle \) such that \( Y_a \subseteq X_a \) is not empty and \( Y_a \leq_{qu} B. \)
2) AC\(_{A,<\kappa}\), AC\(_{A,\leq B}\) are defined similarly but \( |Y_a| < \kappa, |Y_a| \leq |B| \) respectively in the end.

Observation 0.6. 1) We have AC\(_A\) if AC\(_{A,1}\).
2) AC\(_{A,B}\) fails if \( B = \emptyset. \)
3) If AC\(_{A,B}\) and \( |A_1| \leq |A| \) and \( B \leq_{qu} B_1 \) then AC\(_{A,B_1}\).
§ 1. On pseudo true cofinality

We continue [Sh:938, §5]. Below we improve [Sh:938, 5.19] by omitting DC from the assumptions but first we observe

Claim 1.1. Assume AC$_Z$.
1) We have $\theta \geq \text{hrtg}(Z)$ when $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ and $\theta \in \text{ps-pcf}(\Pi\bar{\alpha})$ and $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Z)$.
2) We have $\text{cf}(\text{rk}_D(\alpha)) \geq \text{hrtg}(Z)$ when $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ and $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Z)$.

Remark 1.2. We can weaken the assumption $\text{cf}(\alpha_t) \geq \text{hrtg}(Z)$ by using the ideal of small cofinality, cf $- \text{id}_\theta(\bar{\alpha})$, see [Sh:1005, 1.1=Lc2]. This can be done systematically in this work.

Proof. 1) If we have AC$_\alpha$ for every $\alpha < \text{hrtg}(Z)$ then we can use [Sh:938, 5.7(4)] but we do not assume this. In general let $D$ be a filter on $Y$ such that $\theta = \text{ps-tcf}(\Pi\bar{\alpha}, \prec_D)$, exists as we are assuming $\theta \in \text{ps-pcf}(\Pi\bar{\alpha})$. Let $\mathcal{F} = \langle \mathcal{F}_\alpha : \alpha < \theta \rangle$ witness $\theta = \text{ps-tcf}(\Pi\bar{\alpha}, \prec_D)$, i.e. as in [Sh:938, 5.6(2)] or see 0.4 here; note $t \in Y \Rightarrow \alpha_t > 0$, as we are assuming $\mathcal{F}_\alpha \subseteq \Pi\bar{\alpha}$ for some $\alpha < \theta$; also if $\Pi\bar{\alpha}$ is non-empty then we can assume $\mathcal{F}_\alpha \neq \emptyset$ for every $\alpha < \theta$.

To contradict assumption $\theta < \text{hrtg}(Z)$, let $g \in Y(\text{Ord})$ be a function $h$ from $Z$ onto $\theta$, so the sequence $\langle \mathcal{F}_{h(z)} : z \in Z \rangle$ is well defined. As we are assuming AC$_Z$, there is a sequence $\langle f_z : z \in Z \rangle$ such that $f_z \in \mathcal{F}_{h(z)}$ for $z \in Z$. Now define $g \in Y(\text{Ord})$ by $g(s) = \bigcup \{f_z(s) : z \in Z\}$; clearly $g$ exists and $g \leq \bar{\alpha}$. But for each $s \in Y$, the set $\{f_z(s) : z \in Z\}$ is a subset of $\alpha_s$ of cardinality $\leq \theta < \text{hrtg}(Z)$ hence $\text{cf}(\alpha_s)$ hence $g(s) < \alpha_s$. Together $g \in \Pi\bar{\alpha}$ is a $\prec_D$-upper bound of $\bigcup \{\mathcal{F}_\varepsilon : \varepsilon < \theta\}$, contradiction to the choice of $\mathcal{F}$.

2) Otherwise let $\theta = \text{cf}(\text{rk}_D(\bar{\alpha}))$ so $\theta < \text{hrtg}(Z), \langle \alpha_z : \varepsilon < \theta \rangle$ be increasing with limit $\text{rk}_D(\bar{\alpha})$ and again let $g$ be a function from $Z$ onto $\theta$. As AC$_Z$ holds, we can find $\langle f_z : z \in Z \rangle$ such that for every $z \in Z$ we have $\text{rk}_D(f_z) \geq \alpha_{h(z)}$ and $f_z \prec_D \bar{\alpha}$ and without loss of generality $f_z \in \Pi\bar{\alpha}$. Let $f \in \Pi\bar{\alpha}$ be defined by $f(t) = \sup \{f_{h(z)}(t) : z \in Z\}$ so $\text{rk}_D(f) \geq \sup \{\alpha_z : z \in Z\} = \text{rk}_D(\bar{\alpha}) > \text{rk}_D(f)$, contradiction.

Theorem 1.3. The Canonical Filter Theorem Assume AC$_{\mathcal{P}(Y)}$.

Assume $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle \in Y(\text{Ord})$ and $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathcal{P}(Y))$ and $\partial \in \text{ps-pcf}_{\text{p-comp}}(\bar{\alpha})$ hence is a regular cardinal. Then there is $D = D^\partial_\alpha$, an $\aleph_1$-complete filter on $Y$ such that $\partial = \text{ps-tcf}(\Pi\bar{\alpha}/D)$ and $D \subseteq D'$ for any other such $D' \in \text{Fil}^1_{\aleph_1}(D)$.

Remark 1.4. 1) By [Sh:938, 5.9] there are some such $\partial$ if DC holds.
2) We work more to use just AC$_{\mathcal{P}(Y)}$ and not more.
3) If $\kappa > \aleph_0$ we can replace “$\aleph_1$-complete” by “$\kappa$-complete”.
4) If we waive “$\partial$ regular” so just $\partial$, an ordinal, is a pseudo true cofinality of $(\Pi\bar{\alpha}, \prec_D)$ for $D \subseteq D^\partial_{\aleph_1}(Y)$, exemplified by $\mathcal{F}D, D \neq \emptyset$ the proof gives some $\partial', \text{cf}(\partial') = \text{cf}(\partial)$ and $\mathcal{F}$ witnessing $(\Pi\bar{\alpha}, \prec_{D'})$ has pseudo true cofinality $\partial'$ where $D_\alpha = \bigcap \{D : D \in \mathcal{D}\}$ for $\mathcal{D}$ as below.

Proof. Note that by 1.1
$\exists_1 \partial \geq \text{hrtg}(\mathcal{P}(Y))$. 

Let

$\exists_2 (a) \quad \mathcal{D} = \{D : D \text{ is an } \aleph_1\text{-complete filter on } Y \text{ such that } (\Pi\alpha/D) \text{ has pseudo true cofinality } \partial\}$,

(b) \quad D_\ast = \bigcap\{D : D \in \mathcal{D}\}.

Now obviously

$\exists_3 (a) \quad \mathcal{D} \text{ is non-empty}

(b) \quad D_\ast \text{ is an } \aleph_1\text{-complete filter on } Y.$

For $A \subseteq Y$ let $\mathcal{D}_A = \{D \in \mathcal{D} : (Y\setminus A) \notin D\}$ and let $\mathcal{P}_\ast = \{A \subseteq Y : \mathcal{D}_A \neq \emptyset\}$, equivalently $\mathcal{P}_\ast = \{A \subseteq Y : A \neq \emptyset \text{ mod } D \text{ for some } D \in \mathcal{D}\}$. As $\text{AC}_{\mathcal{P}(Y)}$ holds also $\text{AC}_{\mathcal{P}_\ast}$ holds hence we can find $\langle D_A : A \in \mathcal{P}_\ast \rangle$ such that $D_A \in \mathcal{D}_A$ for $A \in \mathcal{P}_\ast$.

Let $\mathcal{D}_\ast = \{D_A : A \in \mathcal{P}_\ast\}$, clearly

$\exists_4 (a) \quad D_\ast = \bigcap\{D : D \in \mathcal{D}_\ast\}$

(b) \quad $\mathcal{D}_\ast \subseteq \mathcal{D}$ is non-empty.

As $\text{AC}_{\mathcal{P}_\ast}$ holds clearly

(*1) we can choose $\langle \mathcal{F}^A : A \in \mathcal{P}_\ast \rangle$ such that $\mathcal{F}^A$ exemplifies $D_A \in \mathcal{D}$ as in [Sh:938, 5.17,(1),(2)], so in particular $\mathcal{F}^A$ is $\aleph_0$-continuous and without loss of generality $\mathcal{F}^\alpha \neq \emptyset$, $\mathcal{F}^\alpha \subseteq \Pi\alpha$ for every $\alpha < \partial$.

For each $\beta < \partial$ let

(*2) $\mathcal{F}^\beta_1 = \{f = \langle f_A : A \in \mathcal{P}_\ast \rangle : f \text{ satisfies } A \in \mathcal{P}_\ast \Rightarrow f_A \in \mathcal{F}^A\}$

(*3) for $f \in \mathcal{F}^\beta_1$ let $\sup\{f_A : A \in \mathcal{P}_\ast\}$ be the function $f \in Y\text{Ord}$ defined by $f(y) = \sup\{f_A(y) : A \in \mathcal{P}_\ast\}$

(*4) $\mathcal{F}^\beta_\delta = \{\sup\{f_A : A \in \mathcal{P}_\ast\} : f = \langle f_A : A \in \mathcal{P}_\ast \rangle \text{ belongs to } \mathcal{F}^\beta_1\}.$

Now

(*5) (a) \quad $\langle \mathcal{F}^\beta_\delta : \beta < \partial \rangle$ is well defined, i.e. exist

(b) \quad $\mathcal{F}^\beta_\delta \subseteq \Pi\bar{\alpha}$.

[Why? Clause (a) holds by the definitions, clause (b) holds as $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathcal{P}(Y)).$]

(*6) $\mathcal{F}^\beta_\delta \neq \emptyset$ for $\beta < \partial$.

[Why? As for $\beta < \lambda$, the sequence $\langle \mathcal{F}^A : A \in \mathcal{P}_\ast \rangle$ is well defined (as $\langle \mathcal{F}^A : A \in \mathcal{P}_\ast \rangle$ is) and $A \in \mathcal{P}_\ast \Rightarrow \mathcal{F}^A \neq \emptyset$, so we can use $\text{AC}_{\mathcal{P}(Y)}$ to deduce $\mathcal{F}^A_\beta \neq \emptyset.$]

Define

(*7) (a) \quad for $f \in \Pi\bar{\alpha}$ and $A \in \mathcal{P}_\ast$ let $\beta_A(f) = \min\{\beta < \partial : f < g \text{ mod } D_A \text{ for every } g \in \mathcal{F}^A_\beta\}$

(b) \quad for $f \in \Pi\bar{\alpha}$ let $\beta(f) = \sup\{\beta_A(f) : A \in \mathcal{P}_\ast\}.$

Now

(*8) (a) \quad for $A \in \mathcal{P}_\ast$ and $f \in \Pi\bar{\alpha}$, the ordinal $\beta_A(f) < \partial$ is well defined
(b) for $f \in \Pi\alpha$ the sequence $\langle \beta_A(f) : A \in \mathcal{P}_* \rangle$ is well defined.

[Why? Clause (a) holds because $\langle \mathcal{F}_1^A : \gamma < \partial \rangle$ is cofinal in $(\Pi, \alpha, <_{D_A})$, clause (b) holds by $(\ast)_7(a)$.]

\[\textbf{(\ast)}_9\] (a) for $f \in \Pi\alpha$ the ordinal $\beta(f)$ is well defined and $\langle \partial \rangle$
(b) if $f \leq g$ are from $\Pi\alpha$ then $\beta(f) \leq \beta(g)$.

[Why? For clause (a), first, $\beta(f)$ is well defined and $\leq \partial$ by $(\ast)_8$ and the definition of $\beta(f)$ in $(\ast)_7(b)$. Second, recalling that $\partial$ is regular $\Rightarrow hrtg(\mathcal{P}(Y)) \geq hrtg(\mathcal{P}_*)$ clearly $\beta(f) < \partial$. Clause (b) is obvious.]

Now

\[\textbf{(\ast)}_{10}\] (a) if $A \in \mathcal{P}_*$, $\gamma < \partial$ and $f \in \mathcal{F}_1^A$ then $\beta_A(f) > \gamma$
(b) if $\gamma < \partial$ and $f \in \mathcal{F}_1^A$ then $\beta(f) > \gamma$.

[Why? Clause (a) holds because $\beta < \gamma$ and $g \in \mathcal{F}_1^A \Rightarrow g < f \mod D_A$ and $\beta = \gamma \Rightarrow f \in \mathcal{F}_1^A \land f \notin f \mod D_A$. Clause (b) holds because for some $\langle f_B : B \in \mathcal{P}_* \rangle \in \Pi[\mathcal{F}_1^B : B \in \mathcal{P}_*]$ we have $f = \sup\{f_B : B \in \mathcal{P}_*\}$ hence $B \in \mathcal{P}_* \Rightarrow f_B \leq f$ hence in particular $f_A \leq f$; now recalling $\beta(f_A) > \gamma$ by clause (a) it follows that $\beta(f) > \gamma$.]

\[\textbf{(\ast)}_{11}\] (a) for $\xi < \partial$ let $\gamma_\xi = \min\{\beta(f) : f \in \mathcal{F}_1^\xi\}$
(b) for $\xi < \partial$ let $\mathcal{F}_2^\xi = \{f \in \mathcal{F}_1^\xi : \beta(f) = \gamma_\xi\}$

\[\textbf{(\ast)}_{12}\] (a) $\langle(\gamma_\xi, \mathcal{F}_2^\xi) : \xi < \partial\rangle$ is well defined, i.e. exists
(b) if $\xi < \partial$ then $\xi < \gamma_\xi < \partial$.

[Why? $\gamma_\xi$ is the minimum of a set of ordinals which is non-empty by $(\ast)_6$ and $\subseteq \partial$, by $(\ast)_9(a)$, and all members are $\geq \gamma$ by $(\ast)_{10}(b)$.]

\[\textbf{(\ast)}_{13}\] for $\xi < \partial$ we have $\mathcal{F}_2^\xi \subseteq \Pi\alpha$ and $\mathcal{F}_2^\xi \neq \emptyset$.

[Why? By $(\ast)_{11}$ as $\mathcal{F}_1^\xi \neq \emptyset$ and $\mathcal{F}_1^\xi \subseteq \Pi\alpha$.]

\[\textbf{(\ast)}_{14}\] we try to define $\beta_\varepsilon < \partial$ by induction on the ordinal $\varepsilon < \partial$

- $\varepsilon = 0$: $\beta_0 = 0$
- $\varepsilon$ limit: $\beta_\varepsilon = \cup\{\beta_\zeta : \zeta < \varepsilon\}$
- $\varepsilon = \zeta + 1$: $\beta_\varepsilon = \gamma_\beta_\zeta$

\[\textbf{(\ast)}_{15}\] (a) if $\varepsilon < \partial$ then $\beta_\varepsilon < \partial$ is well defined $\geq \varepsilon$
(b) if $\zeta < \varepsilon$ is well defined then $\beta_\zeta < \beta_\varepsilon$.

[Why? Clause (a) holds as $\partial$ is a regular cardinal so the case $\varepsilon$ limit is O.K., the case $\varepsilon = \zeta + 1$ holds by $(\ast)_{12}(b)$. As for clause (b) we prove this by induction on $\varepsilon$; for $\varepsilon = 0$ this is empty, for $\varepsilon$ a limit ordinal use the induction hypothesis and the choice of $\beta_\varepsilon$ in $(\ast)_{14}$ and for $\varepsilon = \zeta + 1$, clearly by $(\ast)_{12}(b)$ and the choice of $\gamma_\varepsilon$ in $(\ast)_{14}$ we have $\beta_\varepsilon < \beta_\zeta$ and use the induction hypothesis.]
Why? Recall that \( \beta_A(f) \) for \( A \in P_\kappa \) and \( \beta(f) \) are well defined ordinals \( \kappa \) and \( A \in P_\kappa \Rightarrow \beta_A(f) \leq \beta(f) \). Now let \( \zeta < \kappa \) be such that \( \beta(f) < \beta_\zeta \), exists as we can prove by induction on \( \varepsilon \) (using (4) in (b)) that \( \beta_\varepsilon \geq \varepsilon \). As \( F_A \) is increasing for \( A \in P_\kappa \) clearly \( A \in P_\kappa \& g \in F_\beta_\kappa \Rightarrow f < g \) mod \( D_\kappa \). So by the definition of \( F_\beta_\kappa \) we have \( A \in P_\kappa \& g \in F_\beta_\kappa \Rightarrow f < g \) mod \( D_\kappa \). Hence \( g \in F_\beta_\kappa \Rightarrow f < g \) mod \( D_\kappa \). As \( \mathcal{F}_\beta_\kappa \subseteq \mathcal{F}_\beta_\kappa \) we are done.]

\[ (*){17} \] if \( \xi < \beta < \varepsilon \) and \( f \in \mathcal{F}_\varepsilon \) for some \( \varepsilon \in \mathcal{F}_\xi \) then \( f < g \) mod \( D_\varepsilon \).

Why? As in the proof of (4) but now \( \beta(f) = \gamma_\kappa \).

Together by \( (*){13} + (*){16} + (*){17} \) the sequence \( \langle \mathcal{F}_\beta_\kappa : \varepsilon < \beta \rangle \) as required. \( \square_{1.3} \)

A central definition here is

\[ [\text{Definition 1.5}] \]

1) For \( \alpha \in \mathcal{Y} \) let \( J_{\lambda_\alpha}^{\text{comp}}(\alpha) = \{ X \subseteq Y : \text{ps-pcf}_{\lambda_\alpha}(\alpha \upharpoonright X) \subseteq \lambda \} \). So for \( X \subseteq Y, X \notin J_{\lambda_\alpha}^{\text{comp}}(\alpha) \) if there is an \( \lambda_\alpha \)-complete filter \( D \) on \( Y \) such that \( X \neq \emptyset \) mod \( D \) and ps-tcf(\( \Pi_\alpha, <D \)) is well defined \( \geq \lambda \) iff there is an \( \lambda_\alpha \)-complete filter \( D \) on \( Y \) such that ps-tcf(\( \Pi_\alpha, <D \)) is well defined \( \geq \lambda \) and \( X \subseteq D \).

2) \( J_{\lambda}^{\text{comp}} \) is \( J_{\alpha}^{\text{comp}} \) and we can use a set \( \alpha \) of ordinals instead of \( \alpha \).

\[ [\text{Claim 1.6}] \]

The Generator Existence Claim

Let \( \alpha \in \mathcal{Y} \setminus \{0\} \).

1) \( J_{\lambda}^{\text{comp}}(\alpha) \) is an \( \lambda_\alpha \)-complete ideal on \( Y \) for any cardinal \( \lambda \) except that it may be \( \mathcal{Y}(Y) \).

2) \( |AC_{\mathcal{Y}(Y)}| \) Assume \( t \in \mathcal{Y} \Rightarrow \text{cf}(\alpha(t)) \geq \text{hrtg}(\mathcal{Y}(Y)) \). If \( \lambda \in \text{ps-pcf}_{\lambda}^{\text{comp}}(\alpha) \) then for some \( X \subseteq Y \) we have

\[ (A) \quad J_{\lambda}^{\text{comp}}(\alpha) = J_{\lambda}^{\text{comp}}(\alpha) + X \]

\[ (B) \quad \lambda = \text{ps-tcf}(\Pi_\alpha, <J_{\lambda}^{\text{comp}}(\alpha)) \text{ where } J_{\lambda}^{\text{comp}}(\alpha) = J_{\lambda}^{\text{comp}}(\alpha) + (Y \setminus X) \]

\[ (C) \quad \lambda \notin \text{ps-pcf}_{\lambda}^{\text{comp}}(\alpha \upharpoonright \{Y \setminus X\}) \].

\[ [\text{Remark 1.7}] \]

1) Recall that if \( AC_{\mathcal{Y}(Y)} \) then without loss of generality \( AC_{\lambda_0} \) holds.

Why? Otherwise by \( AC_{\mathcal{Y}(Y)} \) we have \( Y \) is well ordered and \( AC_Y \) hence \( |Y| = n \) for some \( n < \omega \) and in this case our claims are obvious, e.g. 1.6(2), 1.8.

2) Note that \( J_{\lambda}^{\text{comp}}(\alpha) \) is a well defined ideal in 1.6(2)(B) though \( X \) is not uniquely determined.

3) Note that if \( \theta = \text{ps-tcf}(\Pi_\alpha, <D) \) and \( X \in D^+ \) then \( \theta = \text{ps-tcf}(\Pi_\alpha \setminus X) \).

\[ [\text{Proof}] \]

1) Clearly \( J_{\lambda}^{\text{comp}}(\alpha) \) is a \( \subseteq \)-downward closed subset of \( \mathcal{Y}(Y) \). If the desired conclusion fails, then we can find a sequence \( \langle A_\alpha : n < \omega \rangle \) of members of \( J_{\lambda}^{\text{comp}}(\alpha) \) such that their union \( A := \bigcup\{ A_\alpha : n < \omega \} \) does not belong to it. As \( A \notin J_{\lambda}^{\text{comp}}(\alpha) \), by the definition there is an \( \lambda_\alpha \)-complete filter \( D \) on \( Y \) such that \( A \neq \emptyset \) mod \( D \) and ps-tcf(\( \Pi_\alpha, <D \)) is well defined, so let it be \( \mu = \text{cf}(\mu) \geq \lambda \) and let \( \langle \mathcal{F}_\alpha : \alpha < \lambda \rangle \) exemplify it.

As \( D \) is \( \lambda \)-complete and \( A = \bigcup\{ A_\alpha : n < \omega \} \neq \emptyset \) mod \( D \) for necessarily \( n, A_\alpha \neq \emptyset \) mod \( D \) but then this witness \( A_\alpha \notin J_{\lambda}^{\text{comp}}(\alpha) \), contradiction.

2) Recall \( \lambda \) is a regular cardinal by [Sh:938, 5.8(0)] and \( \lambda \geq \text{hrtg}(\mathcal{Y}(Y)) \) by 1.1.

Let \( D = D^\kappa_\beta \) be as in [Sh:938, 5.19] when DC holds, and as in 1.3 in general, i.e. \( \Pi_\alpha / D \) has pseudo true cofinality \( \lambda \) and \( D \) contains any other such \( \lambda \)-complete
filter on $Y$. Now if $X \in D^+$ then $\lambda = \text{ps-tcf}_{\mathfrak{N}_1 \text{-comp}}(\mathfrak{a} \upharpoonright X, <_{(D^+ \cap \mathcal{P}(X))})$ hence $X \notin j^{\mathfrak{N}_1 \text{-comp}}_{\lambda}[\mathfrak{a}]$, so

\[(*)_1 \quad X \in j^{\mathfrak{N}_1 \text{-comp}}_{\lambda}[\mathfrak{a}] \Rightarrow X = 0 \text{ mod } D.
\]

A major point is

\[(*)_2 \quad \text{some } X \in D \text{ belongs to } j^{\mathfrak{N}_1 \text{-comp}}_{\lambda}[\mathfrak{a}].\]

Why $(*)_2$? The proof will take awhile; assume that not, we have $AC_{\mathcal{P}(Y)}$ hence $AC_D$, so we can find $((\mathcal{F}^X, D_X, \lambda_X) : X \in D)$ such that:

(a) $\lambda_X$ is a regular cardinal $\geq \lambda^+$, i.e. $> \lambda$

(b) $D_X$ is an $\aleph_1$-complete filter on $Y$ such that $X \in D_X$ and $\lambda_X = \text{ps-tcf}(\Pi_{\lambda} <_{D_X})$

(c) $\mathcal{F}^X = \langle \mathcal{F}^X_\alpha : \alpha < \lambda_X \rangle$ exemplifies that $\lambda_X = \text{ps-tcf}(\Pi_{\lambda} <_{D_X})$

(d) moreover $\mathcal{F}^X$ is as in [Sh:938, 5.17(2)], that is, it is $\aleph_0$-continuous and $\alpha < \lambda_X \Rightarrow \mathcal{F}^X_\alpha \neq \emptyset$. Let

(e) $D_1^* = \{ A \subseteq Y \colon \text{for some } X_1 \in D \text{ we have } X \in D \land X \subseteq X_1 \Rightarrow A \in D_X \}$. Clearly

(f) $D_1^*$ is an $\aleph_1$-complete filter on $Y$ extending $D$.

Why? First, clearly $D_1^* \subseteq \mathcal{P}(Y)$ and $\emptyset \notin D_1^*$ as $X \in D \Rightarrow \emptyset \notin D_X$. Second, if $A \in D$ then $X \in D \land X \subseteq A \Rightarrow A \in D_X$ by clause (b) hence choosing $X_1 = A$ the demand for "$A \in D_1^*$" holds so indeed $D \subseteq D_1^*$. Third, assume $\hat{A} = \langle A_n : n < \omega \rangle$ and "$A_n \in D_1^*$" for $n < \omega$, then for each $A_n$ there is a witness $X_n \in D$, so by $AC_{\aleph_0}$, recalling 1.7, there is an $\omega$-sequence $\langle X_n : n < \omega \rangle$ with $X_n$ witnessing $A_n \in D_1^*$. Then $X = \cap \{ X_n : n < \omega \}$ belongs to $D$ and witness that $A := \cap \{ A_n : n < \omega \} \in D_1^*$ because every $D_X$ is $\aleph_1$-complete. Fourth, if $A \subseteq B \subseteq Y$ and $A \in D_1^*$, then some $X_1$ witness $A \in D_1^*$, i.e. $X \in D \land X \subseteq X_1 \Rightarrow A \in D_X$; but then $X_1$ witness also $B \in D_1^*$.

(g) assume $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ is $<_{D^*}$-increasing in $\Pi_{\lambda}$, i.e. $\alpha < \lambda \Rightarrow \mathcal{F}_\alpha \subseteq \Pi_{\lambda}$ and $\alpha_1 < \alpha_2 \land f_1 \in \mathcal{F}_{\alpha_1} \land f_2 \in \mathcal{F}_{\alpha_2} \Rightarrow f_1 <_{D^*} f_2$ and $\mathcal{F}_\alpha \neq \emptyset$ for every or at least unboundedly many $\alpha < \lambda$ then $\bigcup_{\alpha < \lambda} \mathcal{F}_\alpha$ has a common $<_{D_1^*}$-upper bound.

Why? For each $X \in D$ recall $(\Pi_{\lambda}, <_{D_X})$ has true cofinality $\lambda_X$ which is regular $> \lambda$ hence by [Sh:938, 5.7(1A)] is pseudo $\lambda^+$-directed hence there is a common $<_{D_X}$-upper bounded $h_X$ of $\cup \{ \mathcal{F}_\alpha : \alpha < \lambda \}$. As we have $AC_{\mathcal{P}(Y)}$ we can find a sequence $\langle h_X : X \in D \rangle$ with each $h_X$ as above. Define $h \in \Pi_{\lambda}$ by $h(t) = \sup \{ h_X(t) : X \in D \}$, it belongs to $\Pi_{\lambda}$ as we are assuming $t \in Y \Rightarrow \text{cf}(\alpha_1) > \text{hrtg}(\mathcal{P}(Y)) > \text{hrtg}(D)$. So $h \in \Pi_{\lambda}$ is a $<_{D_X}$-upper bound of $\cup \{ \mathcal{F}_\alpha : \alpha < \lambda \}$ for every $X \in D$, hence by the choice of $D_1^*$ it is a $<_{D_1^*}$-upper bound of $\cup \{ \mathcal{F}_\alpha : \alpha < \lambda \}$. But by the choice of $D$ in the beginning of the proof we have $\lambda = \text{ps-tcf}(\Pi_{\lambda}, <_{\mathfrak{a}})$ so there is a sequence $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ witnessing it. By clause (f) we have $D \subseteq D_1^*$ so clearly $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ is also $<_{D_1^*}$-increasing hence we can apply clause (g) to
the sequence \( \langle \hat{h}_n : \alpha < \lambda \rangle \) and got a \( <_{DF} \)-upper bound \( f \in \Pi \hat{\alpha} \), contradiction to the choice of \( \langle \hat{h}_n : \alpha < \lambda \rangle \) recalling 0.4(d) because \( D \subseteq D^*_1 \), contradiction. So (2.2) really holds.

Choose \( X \) as in (2.2), now

\[
(*)_3 \quad D = \text{dual}(J_{<\lambda}^{\text{comp}}[\hat{\alpha}] + (Y \setminus X)).
\]

[Why? The inclusion \( \supseteq \) holds by (1.8) and (2.2), i.e. the choice of \( X \) as a member of \( D \). Now for every \( Z \subseteq X \) which does not belong to \( J_{<\lambda}^{\text{comp}}[\hat{\alpha}] \), by the definition of \( J_{<\lambda}^{\text{comp}}[\hat{\alpha}] \) there is an \( \mathfrak{N} \)-complete filter \( D_2 \) on \( Y \) to which \( Z \) belongs such that \( \theta := \text{ps-cf}(\Pi \hat{\alpha}, <_D) \) is well defined and \( \geq \lambda \). But \( \theta \geq \lambda^+ \) is impossible as we know that \( Z \subseteq X \in J_{<\lambda}^{\text{comp}}[\hat{\alpha}] \), so necessarily \( \theta = \lambda \), hence by the choice of \( D \) by using 1.3 we have \( D \subseteq D^* \), hence \( Z \not\in \emptyset \mod D \). Together we are done.]

\[
(*)_4 \quad \lambda = \text{ps-tcf}(\Pi \hat{\alpha}, <_{J_{<\lambda}^{\text{comp}}} \), see clause (B) of the conclusion of 1.6(2).
\]

[Why? By (2.3), the choice of \( J_{<\lambda}^{\text{comp}}[\hat{\alpha}] \) and as \( \lambda = \text{ps-tcf}(\Pi \hat{\alpha}, <_D) \) by the choice of \( D \).]

\[
(*)_5 \quad \lambda \not\in \text{ps-pcf}_{\mathfrak{N}}(\mathfrak{A}, \uparrow_{\lambda} \downarrow (Y \setminus X)).
\]

[Why? Otherwise there is an \( \mathfrak{N} \)-complete filter \( D' \) on \( Y \) such that \( Y \setminus X \in D' \) and \( \lambda = \text{ps-tcf}(\Pi \hat{\alpha}, <_{D'}) \). But this contradicts the choice of \( D \) by using 1.3.]

So \( X \) is as required in the desired conclusion of 1.6(2): clause (D) by (2.4), clause (C) by (2.5) and clause (A) follows. Note that the notation \( J_{<\lambda}^{\text{comp}}[\hat{\alpha}] \) is justified, as if \( X' \) satisfies the requirements on \( X \) then \( X' = X \mod J_{<\lambda}^{\text{comp}}[\hat{\alpha}] \). □

\{
\}

Concluding 1.8. \([\text{AC}(\mathcal{P}^\mathfrak{N}(Y))] \) Assume \( \hat{\alpha} \in Y \) Ord and each \( \alpha \), a limit ordinal of cofinality \( \geq \text{hrtg}(\mathcal{P}^\mathfrak{N}(Y)) \) and \( \text{ps} - \text{pcf}_{\mathfrak{N}}(\mathfrak{A}, \uparrow_{\lambda} \downarrow \emptyset) \) is not C-empty.

1) If \( t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\text{Fil}_{\mathfrak{N}}^1(Y)) \) then there is a function \( h \) such that:

\begin{itemize}
  \item \( h \) is the domain of \( \mathcal{P}(Y) \)
  \item \( \text{Rang}(h) \) includes \( \text{ps} - \text{pcf}_{\mathfrak{N}}(\mathfrak{A}, \uparrow_{\lambda} \downarrow \emptyset) \) and is included in \( \text{ps} - \text{pcf}_{\mathfrak{N}}(\mathfrak{A}, \uparrow_{\lambda} \downarrow \emptyset) \cup \{0\} \cup \{ \mu : \mu = \text{sup}(\mu \cap \text{ps} - \text{pcf}_{\mathfrak{N}}(\mathfrak{A}, \uparrow_{\lambda} \downarrow \emptyset)) \} \), also \( \text{Rang}(h) \) includes \( \{ \text{cf}(\alpha_t) : t \in Y \} \), but see \( \#_5 \)
  \item \( A \subseteq B \subseteq Y \Rightarrow h(A) \leq h(B) \) and \( h(A) = 0 \Leftrightarrow A = \emptyset \)
  \item \( h(A) = \min(\lambda : A \in J_{<\lambda}^{\text{comp}}[\hat{\alpha}] \setminus \emptyset \setminus \emptyset) \)
  \item \( \text{if } h(A) = \lambda \text{ and } \text{cf}(\lambda) > \aleph_0 \text{ then } \lambda \text{ is regular and } h(A) \in \text{ps} - \text{pcf}_{\mathfrak{N}}(\mathfrak{A}, <_{\lambda} \downarrow \emptyset), i.e. for } \text{some } \mathfrak{N} \text{-complete filter } D \text{ on } Y \text{ we have } A \in D \text{ and } \text{ps-tcf}(\Pi \hat{\alpha}, <_D) = \lambda \)
  \item \( \text{the set } \text{ps} - \text{pcf}_{\mathfrak{N}}(\mathfrak{A}, \uparrow_{\lambda} \downarrow \emptyset) \text{ has cardinality } \text{hrtg}(\mathcal{P}(Y)) \)
  \item \( \text{if } h(A) = \lambda \text{ and } \text{cf}(\lambda) = \aleph_0 \text{ then we can find a sequence } \langle A_n : n < \omega \rangle \text{ such that } A = \cup\{A_n : n < \omega\} \text{ and } h(A_n) < \lambda \text{ for } n < \omega \)
  \item \( J_{<\lambda}^{\text{comp}}[\hat{\alpha}] = \{ A \subseteq Y : h(A) < \lambda \} \text{ when } \text{cf}(\lambda) > \aleph_0 \)
  \item \( \text{if } \text{cf}(\text{otp}(\text{ps} - \text{pcf}_{\mathfrak{N}}(\mathfrak{A}, \uparrow_{\lambda} \downarrow \emptyset))) > \aleph_0 \text{ then } \text{ps} - \text{pcf}_{\mathfrak{N}}(\mathfrak{A}, \uparrow_{\lambda} \downarrow \emptyset) \text{ has a last member.} \)
\end{itemize}

2) Without the extra assumption of part (1), still there is \( h \) such that:

\begin{itemize}
  \item \( h \) is a function with domain \( \mathcal{P}(Y) \)
\end{itemize}
• the range of \( h \) is \( \text{ps-pcf}_{\aleph_1} \)-\( \text{comp} (\bar{\alpha}) \). \\
• \( \text{cf}(\mu) = \aleph_0 \) or just \( \text{cf}(\mu) < \text{hrtg}(\text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha})) \) and \( J_{\aleph_0}^{\text{comp}}[\bar{\alpha}] \neq \text{sup}((J_{\aleph_0}^{\text{comp}}[\bar{\alpha}] : \chi < \mu)) \)

3) The set \( \mathcal{c} = \text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha}) \) satisfies \( \mathcal{c} \leq_{qu} \mathcal{P}(Y) \). If also \( AC_{\alpha} \) holds for \( \alpha < \text{hrtg}(\mathcal{P}(Y)) \) or just \( AC_{\text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha})} \) then we can find a sequence \( \{ X_\lambda : \lambda \in \mathcal{c} \} \) of subsets of \( Y \) such that for every cardinality \( \mu, J_{\aleph_0}^{\text{comp}}[\bar{\alpha}] \) is the \( \aleph_1 \)-complete ideal on \( Y \) generated by \( \{ X_\lambda : \lambda < \mu \text{ and } \lambda \in \text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha}) \} \).

**Proof.** 1) Let \( \Theta = \text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha}) \). We define the function \( h \) from \( \mathcal{P}(Y) \) into \( \Theta^+ \) which is defined as the closure of \( \Theta \cup \{ 0 \} \), i.e. \( \Theta \cup \{ \mu : \mu = \sup(\mu \cap \Theta) \} \), by \( h(X) = \text{Min}\{ \lambda \in \Theta^+ : X \in J_{\lambda}^{\aleph_1} \text{-comp} (\bar{\alpha}) \} \). It is well defined as \( \text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha}) \) is a set, that is as \( \mu_\ast = \text{hrtg}(\Pi \bar{\alpha}) \) is well defined and so \( J_{\aleph_0}^{\text{comp}}[\bar{\alpha}] = \mathcal{P}(Y) \) (see [Sh:938, 5.8(2)]), non-empty by an assumption and \( J_{\aleph_0}^{\text{comp}}[\bar{\alpha}] = \mathcal{P}(Y) \) when \( \lambda \geq \text{sup}(\text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha})) \). This function \( h \), its range is included in \( \Theta^+ \), but \( \text{otp}(\Theta^+) \leq \text{otp}(\Theta) + 1 \); also clearly \( \bullet_1 \) of the conclusion holds. Also if \( \lambda \in \Theta \) and \( X \) is as in \( 1.6(2) \) then \( h(X) = \lambda \); so \( h \) is a function from \( \mathcal{P}(Y) \) into \( \Theta^+ \) and its range include \( \Theta \) hence \( |\Theta| < \text{hrtg}(\mathcal{P}(Y)) \) so \( \bullet_2 \) first clause holds; the second clause of \( \bullet_2 \) holds as trivially \( h(\emptyset) = 0 \) and the definition of \( \Theta^+ \) and the third clause by \( t \in Y \Rightarrow h(\{ t \}) = \text{cf}(\alpha_t) \) holds. Now first by \( 1.1 \) we have \( \theta \in \Theta \Rightarrow \theta \geq \text{hrtg}(\mathcal{P}(Y)) \), hence \( \theta \in \Theta \Rightarrow \theta \geq \text{sup}(\Theta \cap \Theta) \) so the range of \( h \) is as required in \( \bullet_2 \).

Second, if \( \lambda \in \Theta^+ \) and \( \text{cf}(\lambda) = \aleph_0 \) then clearly \( \lambda \in \Theta \setminus \Theta \) and we can find an increasing sequence \( \langle \lambda_n : n < \omega \rangle \) of members of \( \text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha}) \) with limit \( \lambda \). For each \( n \) there is \( X_n \in J_{\aleph_0}^{\text{comp}}[\bar{\alpha}] \setminus J_{\lambda_n}^{\text{comp}}[\bar{\alpha}] \) by \( 1.6(2) \), but \( AC_{\aleph_0} \) holds, see \( 1.7 \) hence such a sequence \( \langle X_n : n < \omega \rangle \) exists. Easily \( A := \cup \{ X_n : n < \omega \} \in \mathcal{P}(Y) \) satisfies \( h(A) = \lambda \) hence \( \lambda \in \text{Rang}(h) \). Third, if \( \lambda = \text{sup}(\text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha})) \) and \( \text{cf}(\lambda) > \aleph_0 \), then \( \bigcup J_{\alpha}^{\text{comp}}[\bar{\alpha}] \neq \mathcal{P}(Y) \) because \( Y \) does not belong to the union while \( J_{\aleph_0}^{\text{comp}} (\bar{\alpha}) = \mathcal{P}(Y) \), so \( h(Y) = \lambda \).

Fourth, assume \( \lambda = h(A), \lambda \notin \text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha}) \) and \( \text{cf}(\lambda) > \aleph_0 \), we can find \( \langle \lambda_i : i < \text{cf}(\lambda) \rangle \), an increasing sequence with limit \( \lambda \), but by the definition of \( h \) necessarily \( \lambda \cap \text{ps-pcf}_{\aleph_1} \text{-comp} (\bar{\alpha}) \) is an unbound subset of \( \lambda \) so without loss of generality all are members of \( \text{ps-pcf}_{\aleph_1} \text{-comp} (\Pi \bar{\alpha}) \). Now \( \langle J_i := J_{\lambda_i}^{\text{comp}}[\bar{\alpha}] : i < \text{cf}(\lambda) \rangle \) is a \( \subseteq \)-increasing sequence of \( \aleph_1 \)-complete ideals on \( Y \), no choice is needed, and by our present assumption \( \aleph_0 < \text{cf}(\lambda) \) hence the union \( J := \cup \{ J_i : i < \text{cf}(\lambda) \} \) is an \( \aleph_1 \)-complete ideal on \( Y \) and obviously \( A \notin J \). So also \( D_1 = \text{dual}(J) + A \) is an \( \aleph_1 \)-complete filter hence by [Sh:938, 5.9] (recalling the extra assumption \( t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\text{Fil}_{\aleph_0}^\lambda(Y)) \))
for some $\aleph_1$-complete filter $D_2$ extending $D_1$ we have $\mu = \text{ps-tcf}(\Pi \alpha, \langle D \rangle)$ is well defined, so by 1.6(2) we have some $D_2 \cap J_{\leq \mu}^{\aleph_1-\text{comp}}[\alpha] \neq \emptyset$ but $\emptyset = D_2 \cap J_1 = D_2 \cap J_{\leq \lambda_1}^{\aleph_1-\text{comp}}[\alpha]$ hence $\mu \geq \lambda_1$. Hence $\mu > \lambda_1$ for every $i < \text{cf}(\lambda)$ but $\lambda$ is singular
so $\mu > \lambda$ and $\mu \in \text{ps-pcf}_{\aleph_1-\text{comp}}(\alpha)$. Hence $\chi := \min(\text{ps-pcf}_{\aleph_1-\text{comp}}(\alpha) \setminus \lambda)$ is well defined and $J_{\leq \chi}^{\aleph_1-\text{comp}}[\alpha] = J$ trivially $\chi \geq \lambda$, but as $\chi$ is regular while $\lambda$ is singular clearly $\chi > \lambda$. But as $h(A) = \lambda < \chi$ we get that $A \in J_{\leq \chi}^{\aleph_1-\text{comp}}[\alpha]$, contradiction to the definition of $h$.

So we have proved $\bullet$, the fifth clause of the conclusion. The other clauses follow from the properties of $h$.

2) Similar proof.

3) We define a function $g$ with domain $\mathcal{P}(Y)$ by $g(A) = \min\{\lambda : A \in J_{\leq \chi}^{\aleph_1-\text{comp}}[\alpha]\}$. This function is well defined as if $\lambda = hrtg(\Pi \alpha)$ then $A \subseteq Y \Rightarrow A \in J_{\leq \chi}^{\aleph_1-\text{comp}}[\alpha]$; and the cardinals are well ordered. Also $\alpha \subseteq \text{Rang}(h)$ because if $\lambda \in \alpha$, then by 1.6(2) we are done recalling that we are assuming $\text{AC}_{\mathcal{P}(Y)}$.

So clearly $\alpha \subseteq \text{Rang}(Y)$ so as $\alpha$ is a set of cardinals, clearly $\text{otp}(\alpha) < hrtg(\mathcal{P}(Y))$ hence $|\alpha| < hrtg(\mathcal{P}(Y))$.

For the second sentence in 1.8(3) by the last sentence it suffices to assume $\text{AC}_\alpha$. For $\lambda \in \alpha$ let $\mathcal{P}_\lambda = \{X \subseteq Y : X$ as in 1.6(2)$\}$, so $\mathcal{P}_\lambda \neq \emptyset$. By $\text{AC}_\alpha$ there is a sequence $(X_\lambda : \lambda \in \alpha) \subseteq \prod \mathcal{P}_\lambda$. For $\lambda \in \alpha$, let $J^0_\lambda$ be the $\aleph_1$-complete ideal on $Y$ generated by $\{X_\mu : \mu \in \alpha \cap \lambda\}$, so by the definitions of $\mathcal{P}_\lambda$ we have $\mu < \lambda \land \mu \in \alpha \Rightarrow X_\mu \in J_{\leq \mu}^{\aleph_1-\text{comp}}[\alpha]$, also $J_{\leq \chi}^{\aleph_1-\text{comp}}(\alpha)$ is $\aleph_1$-complete hence $\lambda \in \alpha \Rightarrow J^0_\lambda \subseteq J_{\leq \chi}^{\aleph_1-\text{comp}}[\alpha]$.

If for every $\lambda$ equality holds we are done, otherwise there is a minimal counterexample and use 1.6(2).

\begin{definition}
Assume $\text{cf}(\mu) < hrtg(Y)$ and $\mu$ is singular of uncountable cofinality limit of regulars. We let

$$\begin{align*}
(a) \quad \text{pp}_Y^{+}(\mu) &= \sup\{\lambda : \text{for some } \bar{\alpha}, D \text{ we have} \}
\begin{align*}
(a) &\quad \lambda = \text{ps-tcf}(\Pi \bar{\alpha}, \langle D \rangle), \\
(b) &\quad D \text{ is an } \aleph_1 \text{-complete filter on } Y \\
(c) &\quad \bar{\alpha} = (\alpha_t : t \in Y), \text{ each } \alpha_t \text{ regular} \\
(d) &\quad \mu = \lim_D \bar{\alpha}
\end{align*}
\begin{align*}
(b) &\quad \text{pp}_Y^{+}(\mu) = \sup\{\lambda^+: \lambda \in \text{as above}\}. \\
(c) &\quad \text{similarly } \text{pp}_{\kappa-\text{comp}}(\mathcal{P}(Y)) \cup \text{pp}_Y^{+}(\mu) \text{ restricting ourselves to } \kappa-\text{complete filters } D; \text{ similarly for other properties} \\
(d) &\quad \text{we can replace } Y \text{ by an } \aleph_1-\text{complete filter } D \text{ on } Y, \text{ this means we fix } D \text{ but not } \bar{\alpha} \text{ above.}
\end{align*}
$$
\end{definition}

\begin{remark}
1) of course, if we consider sets $Y$ such that $\text{AC}_Y$ may fail, it is natural to omit the regularity demands, so $\bar{\alpha}$ is just a sequence of ordinals.

2) We may use $\bar{\alpha}$ a sequence of cardinals, not necessarily regular; see §3.
\end{remark}

\begin{conclusion}
Assume $\theta = hrtg(\mathcal{P}(Y)) < \mu, \mu$ is as in Definition 1.9, $\mu_0 < \mu$ and $\bar{\alpha} \in Y(\text{Reg } \cap \mu^+_0) \cap \text{ps-pcf}_{\aleph_1-\text{comp}}(\alpha) \neq \emptyset \Rightarrow \text{ps-pcf}_{\aleph_1-\text{comp}}(\alpha) \subseteq \mu$. If $\sigma = |\text{Reg } \cap \mu^+_0| < \mu$ and $\kappa = |\text{Reg } \cap \text{pp}_Y^{+}(\mu) \setminus \mu_0|$ then $\kappa < hrtg(\theta \times Y, \sigma)$.

\begin{remark}
In the ZFC parallel the assumption on $\mu_0 < \mu$ is not necessary.
\end{remark}
Proof. Obvious by Definition [Sh:938, 5.6] noting Conclusion 1.8 above and 1.13 below. That is, letting $\Xi := \text{Reg} \cap \text{pp}^Y(\mu) \backslash \mu_0$ so $|\Xi| = \kappa$ and $\Lambda = \text{Reg} \cap \mu^\lambda \mu_0$, for every $\alpha \in Y \Lambda$ by Definition 1.9 the set $\text{ps-pcf}_{\lambda_1}^\mu(\alpha)$ is a subset of $\text{Reg} \cap \text{pp}^Y(\mu) \backslash \mu_0$, and by claim 1.8 it is a set of cardinality $< \text{hrtg}(\mathcal{P}(Y))$. By Definition 1.9 and Claim 1.13 below we have $\Xi = \bigcup \{ \text{ps}_\tau \text{pcf}_{\lambda_1}^\mu(\alpha) : \alpha \in Y \Lambda \}$. Clearly there is a function $h$ with domain $\text{hrtg}(\mathcal{P}(Y)) \times Y \sigma$ such that $\varepsilon < \text{hrtg}(\mathcal{P}(Y)) \wedge \alpha \in Y \sigma$ $\Rightarrow (h(\varepsilon, \alpha)$ is the $\varepsilon$-th member of $\text{ps-pcf}_{\lambda_1}^\mu(\alpha)$ if there is one, $\min(\Lambda)$ otherwise). So $h$ is a function from $\text{hrtg}(\mathcal{P}(Y)) \times Y \sigma$ onto a set including $\Xi$ which has cardinality $\kappa$, so we are done. \hfill $\square_{1.11}$

Claim 1.13. The No Hole Claim/DC
1) If $\alpha \in Y \text{Ord}$ and $\lambda_2 \in \text{ps-pcf}_{\lambda_1}^\mu(\alpha)$, for transparency $t \in Y \Rightarrow \alpha_t > 0$ and $\text{hrtg}(\mathcal{P}(Y)) \leq \lambda_1 = \text{cf}(\lambda_1) < \lambda_2$, then for some $\alpha' \in \Pi \alpha$ we have $\lambda_1 = \text{ps-pcf}_{\lambda_1}^\mu(\alpha')$.
2) In part (1), if in addition $AC_Y$ then without loss of generality $\alpha' \in Y \text{Reg}$.
3) If in addition $AC_{\mathcal{P}(Y)} + AC_{< \kappa}$ then even witnessed by the same filter (on $Y$).

Proof. 1) Let $D$ be an $\aleph_1$-complete filter on $Y$ such that $\lambda_2 = \text{ps-tcf}(\Pi \alpha, <_D)$, let $\langle \mathcal{F}_\alpha : \alpha < \lambda_2 \rangle$ exemplify this.

First assume $\text{hrtg}(\text{Fil}_{\lambda_2}^1(Y)) \leq \lambda_1$, clearly $f \in \mathcal{F}_\alpha \Rightarrow \text{rk}_D(f) \geq \alpha$ for every $\alpha < \lambda_2$, hence in particular for $\alpha = \lambda_1$ hence there is $f \in Y \text{Ord}$ such that $\text{rk}_D(f) = \lambda_1$ and now use [Sh:938, 5.9] but there we change the filter $D$, (extend it), so is O.K. for part (1). In general, i.e. without the extra assumption $\text{hrtg}(\text{Fil}_{\lambda_2}^1(Y)) \leq \lambda_1$, use 1.14(1),(2) below.

2) Easy, too.

3) Similarly using 1.14(3) below. \hfill $\square_{1.13}$

Claim 1.14. Assume $D \in \text{Fil}_{\lambda_1}^2(Y), \kappa > \aleph_0, \mathcal{F}_\alpha \subseteq Y \text{Ord}$ non-empty for $\alpha < \delta$ and $\mathcal{F} = \langle \mathcal{F}_\alpha : \alpha < \delta \rangle$ is $<_D$-increasing, $\delta$ a limit ordinal.
1) [DC] There is $f^* \in \Pi \mathcal{F}$ which satisfies $f \in \bigcup \{ \mathcal{F}_\alpha : \alpha < \lambda_1 \} \Rightarrow f <_D f^*$ but there is no such $f^{**} \in \Pi \mathcal{F}$ satisfying $f^{**} <_D f$.
2) [AC, $\kappa$] For $f^*$ as above, let $D_1 = D_{f^*} := \{ Y \setminus A : A \not\in \omega \text{ mod } D \text{ or } A \in D^+ \}$ and there is $f^* \in Y \text{Ord}$ such that $f^{**} <_D f^*$ and $f \in \bigcup \{ \mathcal{F}_\alpha : \alpha < \lambda_1 \} \Rightarrow f < D_{f^*} f^{**}$. Now $D_1$ is a $\kappa$-complete filter and $\emptyset \notin D_1, D_1$ extends $D$ and if $\text{cf}(\delta) = \text{hrtg}(\mathcal{P}(Y))$ then $\langle \mathcal{F}_\alpha : \alpha < \delta \rangle$ witness that $f^*$ is a $<_D$-exact upper bound of $\mathcal{F}$ hence ( $\prod_{\omega \in Y} f(y), <_D$ has pseudo-true-cofinality $\text{cf}(\delta)$).
3) [DC + AC $\kappa$, $\kappa$] $\text{AC}_{\mathcal{P}(Y)}$.
If $\text{cf}(\delta) = \text{hrtg}(\mathcal{P}(Y))$ then there is $f' \in Y \text{Ord}$ which is an $<_D$-exact upper bound of $\mathcal{F}$, i.e. $f <_D f' \Rightarrow (\exists \alpha < \delta)(\exists g \in \mathcal{F}_\alpha)[f < g \text{ mod } D] \text{ and } f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \Rightarrow f < D f'$.

Proof. 1) If not then by DC we can find $f = \{ f_n : n < \omega \}$ such that:

(a) $f_n \in Y \text{Ord}$
(b) $f_{n+1} < f_n \text{ mod } D$
(c) if $f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha$ and $n < \omega$ then $f < f_n \text{ mod } D$.

So $A_n = \{ t \in Y : f_{n+1}(t) < f_n(t) \} \in D$ hence $\cap \{ A_n : n < \omega \} \in D$, contradiction.
2) First, clearly $D_1 \subseteq \mathcal{P}(Y)$ and by the assumption $\emptyset \notin D_1$. Second, if $f^\ast$ witness $A \in D_1$ and $A \cap B \subseteq Y$ then $f^\ast$ witness $B \notin D_1$.

Third, we prove $D_1$ is closed under intersection of $< \kappa$ members, so assume $\zeta < \kappa$ and $A = \{ A_\varepsilon : \varepsilon < \zeta \}$ is a sequence of members of $D_1$. Let $A := \cap \{ A_\varepsilon : \varepsilon < \zeta \}$, $B_\varepsilon = Y \setminus A_\varepsilon$ for $\varepsilon < \zeta$ and $B'_\varepsilon = B_\varepsilon \cup \{ B_\xi : \xi < \varepsilon \}$ and $B = \cup \{ B_\varepsilon : \varepsilon < \zeta \}$. Clearly $B = Y \setminus A, A \subseteq Y$ and $(B'_\varepsilon : \varepsilon < \zeta)$ is a sequence of pairwise disjoint subsets of $Y$ with union $B$. But $AC_\zeta$ holds and $\varepsilon < \zeta \Rightarrow A_\varepsilon \in D_1$ hence we can find $(f^\ast_\varepsilon : \varepsilon < \zeta)$ such that $f^\ast_\varepsilon \in Y$ Ord and if $A_\varepsilon \notin D$ then $f^\ast_\varepsilon$ witness $A_\varepsilon \in D_1$. Let $f^\ast \in Y$ Ord be defined by $f^\ast(t) = f^\ast_\varepsilon(t)$ if $t \in B'_\varepsilon$ or $\varepsilon = 0 \wedge t \in Y \setminus B$; easily $B'_\varepsilon \in D^+ \wedge f \in \bigcup \mathcal{F}_\alpha \Rightarrow f < f^\ast_\varepsilon = f^\ast$ mod $(D + B'_\varepsilon)$ but $B = \cup \{ B'_\varepsilon : \varepsilon < \zeta \}$ and $D$ is $\kappa$-complete hence $f \in \bigcup \mathcal{F}_\alpha \Rightarrow f < f^\ast$ mod$(D + B)$. So as $A = Y \setminus B$ clearly $f^\ast$ witness $A = \bigcap \{ A_\varepsilon : \varepsilon \in D_1 \}$ so $D_1$ is indeed $\kappa$-complete.

Lastly, assume $cf(\delta) \geq \hrtg(\mathcal{P}(Y))$ and we shall show that $f^\ast$ is an exact upper bound of $\mathcal{F}$ modulo $D_1$. So assume $f^\ast \in Y$ Ord and $f^\ast < f^\ast$ mod $D_1$ and we shall prove that there are $\alpha < \delta$ and $f \in \mathcal{F}_\alpha$ such that $f^\ast \leq f$ mod $D_1$.

Let $\mathcal{A} = \{ A \in D_1^+ : \text{there is } f \in \bigcup \mathcal{F}_\alpha \text{ such that } f^\ast \leq f \text{ mod}(D + A) \}$, yes, not $D_1$!

Case 1: For every $B \in D_1^+$ there is $A \in \mathcal{A}, A \subseteq B$.

For every $A \in \mathcal{A}$ let $\alpha_A = \min \{ \beta : \text{there is } f \in \mathcal{F}_\beta \text{ such that } f^\ast \leq f \text{ mod}(D + A) \}$.

So the sequence $(\alpha_A : A \in \mathcal{A})$ is well defined.

Let $\alpha(\ast) = \sup \{ \alpha_A + 1 : A \in \mathcal{A} \}$, it is $\beta$ as $cf(\delta) \geq \hrtg(\mathcal{P}(Y)) \geq \hrtg(\mathcal{A})$.

Choose $f \in \mathcal{F}_\alpha(\ast)$ and let $B_f := \{ t \in Y : f^\ast(t) \geq f(t) \}$. Now if $A \in \mathcal{A}$ (so $A \in D_1^+$) and $f^\ast \leq f$ mod $(D + A)$; without loss of generality $f^\ast \in \mathcal{F}_\alpha_A$ hence $f^\ast \leq f$ mod $D$ recalling $\alpha_A < \alpha(\ast)$, then $A \notin B_f$ as otherwise $f^\ast \leq f < f^\ast$ mod $(D + A)$. So $B_f$ contains no $A \in \mathcal{A}$ hence necessarily $B_f$ is $\emptyset$ mod $D_1$ by the case assumption; this means that $f^\ast \leq f$ mod $D_1$. So recalling $f \in \mathcal{F}_\alpha(\ast) \subseteq \bigcup \mathcal{F}_\alpha$, we have “$f$ is as required” thus finishing the proof of “$f^\ast$ is an exact upper bound of $\mathcal{F}$ modulo $D$”.

Case 2: $B \in D_1^+$ and there is no $A \in \mathcal{A}$ such that $A \subseteq B$.

For $f \in \bigcup \mathcal{F}_\alpha$ let $B_f = \{ t \in B : f(t) < f^\ast(t) \}$ and for $\alpha < \delta$ we define $\mathcal{B}_\alpha = \{ B_f : f \in \mathcal{F}_\alpha \}$ and we define a partial function $h$ from $\mathcal{P}(Y)$ into $\delta$ by $h(A) = \sup \{ \alpha < \delta : A \in \mathcal{B}_\alpha \}$. As $cf(\delta) \geq \hrtg(\mathcal{P}(Y))$ necessarily $\alpha(\ast) = \sup(\delta \cap \operatorname{Rang}(h))$ is $\delta$. Choose $g \in \mathcal{F}_\alpha(\ast + 1)$, hence $u := \{ \alpha : \alpha \in (\alpha(\ast), \delta) \}$ and $B_u \in \mathcal{B}_\alpha$ is an unbounded subset of $\delta$.

Let $A = B \cap B_u$, now if $A \in D^+$ then $\alpha \in u \Rightarrow \bigvee_{f \in \mathcal{F}_\alpha} f < f^\ast \text{ mod}(D + A)$ but $\mathcal{F}$ is $<D$-increasing and $\delta = \sup(u)$ hence $f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \Rightarrow f < f^\ast \text{ mod}(D + A)$ hence by the definition of $D_1$, $f^\ast$ witness that $Y \setminus A \in D_1$, hence $A = \emptyset$ mod $D_1$.

As $B \in D_1^+$ and $A = B \cap B_u$ it follows that $B \setminus B_u \in D_1^+$ and by the choice of $\mathcal{A}$ the set $B \setminus B_u$ belongs to $\mathcal{A}$. But $B \setminus B_u \subseteq B$ by its definition so we get a contradiction to the case assumption.
3) By [Sh:938, 5.12] without loss of generality $\mathcal{F}$ is $\aleph_0$-continuous. For every $A \in D^+$ the assumptions hold even if we replace $D$ by $D + A$ and so there are $D_1, f^*$ as in part (2), we are allowed to use part (1) as we have DC and part (2) as we have $\text{AC}_{<\kappa}$. As we are assuming $\text{AC}_{\mathcal{P}(Y)}$ there is a sequence $\langle (D_A, f_A) : A \in D^+ \rangle$ such that:

\[
\begin{align*}
(\ast) \quad & (a) \quad D_A \text{ is a $\kappa$-complete filter extending } D + A \\
& (b) \quad f_A \in Y \text{Ord is a $<_{D_A}$-exact upper bound of } \mathcal{F}.
\end{align*}
\]

Recall $|A| \leq_{\text{qu}} |B|$ is defined as: $A$ is empty or there is a function from $B$ onto $A$. Of course, this implies $\text{hrtg}(A) \leq \text{hrtg}(B)$.

Let $\mathcal{U} = \langle \mathcal{U}_t : t \in Y \rangle$ be defined by $\mathcal{U}_t = \{ f_A(t) : A \in D^+ \} \cup \{ \text{sup} \{ f(t) : f \in t \} \}$ hence $t \in Y \Rightarrow 0 < |\mathcal{U}_t| \leq_{\text{qu}} |\mathcal{P}(Y)|$ even uniformly so there is a sequence $\langle h_t : t \in Y \rangle$ such that $h_t$ is a function from $\mathcal{P}(Y)$ onto $\mathcal{U}_t$ hence $| \prod_{t \in Y} \mathcal{U}_t | \leq_{\text{qu}} |\mathcal{P}(Y) \times Y| \leq_{\text{qu}} |\mathcal{P}(Y) \times Y|$ but $\text{AC}_{\mathcal{P}(Y)}$ holds hence $Y$ can be well ordered however without loss of generality $Y$ is infinite hence $|Y \times Y| = Y$, so $| \prod_{t \in Y} \mathcal{U}_t | \leq_{\text{qu}} |\mathcal{P}(Y)|$.

Now for every $g \in \mathcal{G}$ the sequence $\langle \{ t \in Y : g(t) \leq f(t) \} : f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \rangle$ is a $\subseteq$-increasing sequence of subsets of $\mathcal{P}(Y)$, but $\text{hrtg}(\mathcal{G}) \leq \text{cf}(\delta)$ hence the sequence is eventually constant and let $\alpha(g) < \delta$ be the minimal $\alpha$ such that

\[
(\ast)_g \quad \langle \forall \beta \rangle \langle \alpha \leq \beta < \delta \rangle \Rightarrow \langle \{ t \in Y : g(t) \leq f(t) \} : f \in \bigcup_{\gamma < \beta} \mathcal{F}_\gamma \rangle = \langle \{ t \in Y : g(t) \leq f(t) \} : f \in \bigcup_{\gamma < \beta} \mathcal{F}_\gamma \rangle.
\]

But recalling $\text{hrtg}(\mathcal{G}) \leq \text{cf}(\delta)$, the ordinal $\alpha(\ast) := \text{sup} \{ \alpha(g) : g \in \mathcal{G} \}$ is $< \delta$. Now choose $f^* \in \mathcal{F}_{\alpha(\ast) + 1}$ and define $g^* \in \prod_{t \in Y} \mathcal{U}_t$ by $g^*(t) = \text{min}(\mathcal{U}_t \setminus f^*(t))$, well defined as $\text{sup} \{ f(t) : t \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \} \in \mathcal{U}_t$. It is easy to check that $g^*$ is as required. \hfill \Box_{1.14}

**Observation 1.15.** 1) Let $D$ be a filter on $Y$.

If $D$ is $\kappa$-complete for every $\kappa$ then for every $f \in Y \text{Ord}$ and $A \in D^+$ there is $B \subseteq A$ from $D^+$ such that $f|B$ is constant.

2) If $\bar{\alpha} = \langle \alpha_s : s \in Y \rangle$ and $X_\varepsilon \subseteq Y$ for $\varepsilon < \alpha < \kappa$ and $X = \bigcup_\varepsilon X_\varepsilon$ then $\text{ps} - \text{pcf}_{\kappa - \text{comp}}(\bar{\alpha} \upharpoonright X) = \bigcup_\varepsilon \text{ps} - \text{pcf}_{\kappa - \text{comp}}(\bar{\alpha} \upharpoonright X_\varepsilon)$.

**Remark 1.16.** 1) Note that 1.15(1) is not empty; its assumptions hold when $Y$ is an infinite set such that: for every $X \subseteq Y$, $|X| < \kappa \vee |Y \setminus X| < \kappa$ and $D = \{ X \subseteq Y : |Y \setminus X| \not< \kappa \}$.

**Proof.** Straightforward. \hfill \Box_{1.15}
§ 2. Composition and generating sequence for pseudo pcf

How much choice suffice to show \( \lambda = ps - \text{tcf}(\prod_{i,j \in Y} \lambda_{i,j} / D) \) when \( \lambda \) is the pseudo true equality of \( \prod_{j \in Y_i} \lambda_{i,j} <_{D_i} \) for \( i \in Z \) where \( Z = \{ i : (i, j) \in Y \} \) and \( Y_i = \{ (i, j) : i \in Z, j \in Y_i \} \) and \( \lambda = ps - \text{tcf}(\prod_{i \in Z} \lambda_i, <_{E}) \)? This is 2.6, the parallel of \( [\text{Sh} : \text{Ch.II}, 1.10, \text{pg.12}] \).

\{e1\}

Claim 2.1. If \( \bowtie \) below holds then for some partition \( (Y_1, Y_2) \) of \( Y \) and club \( E \) of \( \lambda \) we have

\[ \oplus (a) \quad \text{if } Y_1 \in D^+ \text{ and } f,g \in \cup\{ \mathcal{F}_\alpha : \alpha \geq \min(E) \} \text{ then } f = g \mod (D + Y_1) \]

\[ (b) \quad \text{if } Y_2 \in D^+ \text{ then } (\mathcal{F}_\alpha : \alpha \in E) \text{ is } <_{D + Y_2} \text{-increasing} \]

where

\[ \bowtie (a) \quad \lambda \text{ is regular } \geq hrtg(\mathcal{F}(Y)) \]

\[ (b) \quad \mathcal{F}_\alpha \subseteq Y \text{Ord for } \alpha < \lambda \text{ is non-empty} \]

\[ (c) \quad D \text{ is an } \aleph_1 \text{-complete filter on } Y \]

\[ (d) \quad \text{if } \alpha_1 < \alpha_2 < \lambda \text{ and } f_1 \in \mathcal{F}_\alpha \text{ for } \ell = 1, 2 \text{ then } f_1 \leq f_2 \mod D. \]

Proof. For \( Z \in D^+ \) let

\[ (\ast)_1 \quad (a) \quad S_Z = \{ (\alpha, \beta) : \alpha \leq \beta < \lambda \text{ and for some } f \in \mathcal{F}_\alpha \text{ and } g \in \mathcal{F}_\beta \text{ we have } f < g \mod (D + Z) \} \]

\[ (b) \quad S_Z^+ = \{ (\alpha, \beta) : \alpha \leq \beta < \lambda \text{ and for every } f \in \mathcal{F}_\alpha \text{ and } g \in \mathcal{F}_\beta \text{ we have } f < g \mod (D + Z) \}. \]

Note

\[ (\ast)_2 \quad (a) \quad \text{if } \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \text{ and } (\alpha_2, \alpha_3) \in S_Z \text{ then } (\alpha_1, \alpha_4) \in S_Z \]

\[ (b) \quad \text{similarly for } S_Z^+ \]

\[ (c) \quad \text{if } \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \text{ and } (\alpha_1 \neq \alpha_2) \wedge (\alpha_3 \neq \alpha_4) \text{ and } \forall \alpha_2 \in S_Z \]

\[ \text{then } (\alpha_1, \alpha_4) \in S_Z^+ \]

\[ (d) \quad S_Z \subseteq S_Z^+. \]

[Why? By the definitions.]

Let

\[ (\ast)_3 \quad J := \{ Z \subseteq Y : Z \in \text{dual}(D) \text{ or } Z \in D^+ \text{ and } (\forall \alpha < \lambda)(\exists \beta)((\alpha, \beta) \in S_Z^+) \}. \]

Next

\[ (\ast)_4 \quad (a) \quad J \text{ is an } \aleph_1 \text{-complete ideal on } Y \]

\[ (b) \quad \text{if } D \text{ is } \kappa \text{-complete then } J \text{ is } \kappa \text{-complete}^1 \]

\[ (c) \quad J = \{ Z \subseteq Y : Z \in \text{dual}(D) \text{ or } Z \in D^+ \text{ and } (\forall \alpha < \lambda)(\exists \beta) \]

\[ ((\alpha, \beta) \in S_Z) \}. \]

[Why? For clauses (a),(b) check and for clause (c) recall (\ast)_2(c).]

Let

\[ ^1 \text{not used; note that } AC_\kappa \text{ holds in the non-trivial case as } AC_{\mathcal{P}(Y)} \text{ holds, see 1.15} \]
\[ (*)_5 \] (a) for \( Z \in J^+ \) let \( \alpha(Z) = \min\{ \alpha < \lambda : \text{for no } \beta \in (\alpha, \lambda) \text{ do we have} \ (\alpha, \beta) \in S_Z \} \]
\[ (*)_6 \] (a) for \( Z \in J^+ \) we have \( \alpha(Z) < \lambda \)
\( (b) \alpha(*) < \lambda. \]

[Why? Clause (a) by the definition of the ideal \( J \), and clause (b) as \( \lambda = \text{cf}(\lambda) \geq \text{hrtg}(\mathcal{P}(Y)) \).]

Let \[ (*)_7 \] (a) for \( Z \in D^+ \) let \( f_Z : \lambda \to \lambda + 1 \) be defined by \( f_Z(\alpha) = \text{Min}\{ \beta : (\alpha, \beta) \in S_Z^{+} \text{ or } \beta = \lambda \} \)
\[ (b) \ f_\ast : \lambda \to \lambda \] be defined by: \( f_\ast(\alpha) = \sup\{ f_Z(\alpha) : Z \in D^+ \cap J \} \)
\[ (c) \ E = \{ \delta : \delta \text{ a limit ordinal} < \lambda \text{ such that } \alpha < \delta \Rightarrow f_\ast(\alpha) < \delta \}\setminus\alpha(*) \]

Hence

\[ (*)_8 \] (a) if \( Z \in D^+ \cap J \) then \( f_Z \) is indeed a function from \( \lambda \) to \( \lambda \)
\[ (b) \ f_\ast \text{ is indeed a function from } \lambda \text{ to } \lambda \]
\[ (c) \ f_\ast \text{ is non-decreasing} \]
\[ (d) \ E \text{ is a club of } \lambda. \]

[Why? Clause (a) by the definition of \( J \) and of \( f_\ast \) and clause (b) as \( \lambda = \text{cf}(\lambda) \geq \text{hrtg}(\mathcal{P}(Y)) \) and clause (c) by \( (*)_2 \) and clause (d) follows from (b)+(c).]

\[ (*)_9 \] Let \( \alpha_0 = \min(E), \alpha_1 = \min(E \setminus (\alpha_0 + 1)) \) choose \( f_0 \in \mathcal{F}_{\alpha_0}, f_1 \in \mathcal{F}_{\alpha_1} \) and let \( Y_1 = \{ y \in Y : f_0(y) = f_1(y) \} \) and \( Y_2 = Y \setminus Y_1 \)
\[ (*)_{10} \ (Y_1, Y_2, E) \text{ are as required.} \]

[Why? Think.]

\[ \square_{2.1} \]

Claim 2.2. We have \( \lambda = \text{ps} - \text{tcf}(\Pi \check{\alpha}_1, _D_1) = \text{ps} - \text{tcf}(\Pi \check{\alpha}_2, _D_2) \), this means also that one of them is well defined iff the other is, when

\[ (a) \ \check{\alpha} \in Y \text{ Ord and } t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Y) \]
\[ (b) \ E \text{ is the equivalence relation on } Y \text{ such that } sE \Leftrightarrow \alpha_s = \alpha_t \]
\[ (c) D \text{ is a filter on } X \]
\[ (d) Y_1 = Y/E \]
\[ (e) D_1 = \{ Z \subseteq Y/E : \cup\{ X : X \in Z \} \in D \}, \text{ so a filter on } Y_1 \]
\[ (f) \check{\alpha}_1 = (\alpha_{1,y_1} : y_1 \in Y_1) \text{ where } y_1 = y/E \Rightarrow \alpha_{1,y_1} = \alpha_y. \]

Remark 2.3. We can for the “only if” direction in 2.2 weaken the demand on \( \text{cf}(\alpha_t) \) to \( \text{cf}(\alpha_t) \geq \text{hrtg}(t/E) \).

Proof. The claim means

\[ (*) \ \lambda = \text{ps} - \text{tcf}(\Pi \check{\alpha}_1, _D_1) \text{ if and only if } \lambda = \text{ps} - \text{tcf}(\Pi \check{\alpha}_2, _D_2). \]
First “only if” direction holds by 2.4.

Second, for the “if direction”, assume that $\psi = \text{pcf}(\Pi_{\alpha_1}, < D)$ is well defined and call it $\lambda_1$. Let $(\mathcal{F}_{1, \alpha} : \alpha < \lambda)$ witness this, for $f \in \mathcal{F}_{1, \alpha}$ let $f[0] \in Y \text{Ord}$ be defined by $f[0](s) = f(s/E)$ and let $\mathcal{F}_0 = \{f[0] : f \in \mathcal{F}_{1, \alpha}\}$. It is easy to check that $(\mathcal{F}_0 : \alpha < \lambda)$ witness $\lambda_1 = \text{pcf}(\Pi_a, D)$ recalling $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Y)$ by clause (d), so we have proved also the “if” implication. 

By the following claims we do not really lose by using $a \subseteq \text{Reg}$ instead $\alpha \in Y \text{Ord}$ as by 2.5 below, without loss of generality $\alpha_t = \text{cf}(\alpha_t)$ (when AC$_Y$) and by 2.2.

{e23} 

Claim 2.4. Assume $\alpha \in Y \text{Ord}$, $D \in \text{Fil}(Y)$ and $\lambda = \text{psi}(\Pi_{\alpha}, < D)$ so $\lambda$ is regular, and $y \in Y \Rightarrow \alpha_y < \lambda$.

If $(\mathcal{F}_0 : \alpha < \lambda)$ witness $\lambda = \text{psi}(\Pi_{\alpha}, < D)$ and $y \in Y \Rightarrow \text{cf}(\alpha_y) \geq \text{hrtg}(Y)$ and $\lambda \geq \text{hrtg}(Y)$ then for some $c$:

(a) $e \in \text{eq}(Y) = \{e : e \text{ an equivalence relation on } Y\}$

(b) the sequence $\mathcal{F}_e = (\mathcal{F}_{e, \alpha} : \alpha < \lambda)$ witness $\text{psi}(\alpha_{y/e} : y \in Y/e), D/e)$ where

(c) $\alpha_{y/e} = \alpha_y, D/e = \{A/e : \alpha \in D\}$ where $A/e = \{y/e : y \in A\}$ and $\mathcal{F}_{e, \alpha} = \{f^{i[\ast]} : f \in \mathcal{F}_{a}, f^{i[\ast]} : Y/e \rightarrow \text{Ord} \text{ is defined by } f^{i[\ast]}(t/e) = \sup\{g(s) : s \in \text{hrtg}(y/e) \} \leq \text{hrtg}(Y)\}$

(d) $e = \{(s_1, s_2) : \alpha_{s_1} = \alpha_{s_2}\}$. 

Proof. Let $e = \text{eq}(\alpha) = \{(y_1, y_2) : y_1, y_2 \in Y, y_1, y_2 \in Y \text{ and } \alpha_{y_1} = \alpha_{y_2}\}$. For each $f \in \Pi_{\alpha}$ let the function $f^{i[\ast]} \in \Pi_{\alpha}$ be defined by $f^{i[\ast]}(y) = \sup\{f(z) : z \in y/e\}$. Clearly $f^{i[\ast]}$ is a function from $\prod_{y \in Y} (\alpha_y + 1)$ and it belongs to $\Pi_{\alpha}$ as $y \in Y \Rightarrow \text{cf}(\alpha_y) \geq \text{hrtg}(Y) \geq \text{hrtg}(y/E)$. Let $H : \lambda \rightarrow \lambda$ be: $H(\alpha) = \min\{\beta \in \lambda : \beta > \alpha \}$ and there are $f_1 \in \mathcal{F}_a$ and $f_2 \in \mathcal{F}_b$ such that $f^{i[\ast]}_1 \leq f^{i[\ast]}_2 \bmod D$, well defined as $\mathcal{F}$ is cofinal in $(\Pi_{\alpha, < D})$. We choose $\alpha_t < \lambda$ by induction on $t$ by: $\alpha_t = \cup \{H(\alpha_j) + 1 : j < t\}$. So $\alpha_0 = 0$ and $(\alpha_t : t < \lambda)$ is increasing continuous. Let $\mathcal{F}_{t} = \{f^{i[\ast]} : f \in \mathcal{F}_{\alpha_t}, \text{ and there is } y \in \mathcal{F}_{H(\alpha_t)} = \mathcal{F}_{\alpha_{t+1}} \text{ such that } f^{i[\ast]} < f \bmod D\}$.

So

(*)$_1$ $\mathcal{F}_t \subseteq \{f \in \Pi_{\alpha} : \text{eq}(\alpha) \text{ refine } \text{eq}(f)\}$.

[By the choice of $\mathcal{F}_t$ and of $e$]

(*)$_2$ $\mathcal{F}_t$ is non-empty.

[Why? By the choice of $H(\alpha_i)$]

(*)$_3$ if $(i(1) < i(2) < \lambda$ and $h \in \mathcal{F}_{t}$ for $t = 1, 2$ then $h_1 < h_2 \bmod D$.

[Why? For $\ell = 1, 2$ let $g \in \mathcal{F}_{H(\alpha_{i(1)})}$ be such that $h \in f^{i[\ast]}_\ell < g \bmod D$, exists by the definition of $\mathcal{F}_{i[\ell]}$. But $H(\alpha_{i(1)}) < \alpha_{i(1)+1} \leq \alpha_{i(2)}$ hence $g_1 \leq f_2 \bmod D$ so together $h_1 = f^{i[\ast]}_1 < g_1 \leq f_2 \leq f^{i[\ast]}_2 = h_2 \bmod D$ hence we are done.]

(*)$_4$ $\bigcup_{i \leq \lambda} \mathcal{F}_i$ is cofinal in $(\Pi_{\alpha}, < D)$. 


5.6. \[
\text{Why? Exists by clause (h) of the assumption and AC.}
\]

Now for \( \beta < \lambda \)

Proof. Straightforward.

Lastly, let \( \mathcal{F}_t = \{ f/e : e \in \mathcal{F}_t \} \) where \( f/e \in Y/\text{Ord} \), is defined by \( (f/e)(y/e) = f(y) \), clearly well defined.

\[\square_{2.4}\]

Claim 2.5. Assume \( AC_Y \) and \( \bar{\alpha}_t = \langle \alpha^t_y : y \in Y \rangle \in Y/\text{Ord} \) for \( \ell = 1, 2 \). If \( y \in Y \Rightarrow cf(\alpha^t_y) = cf(\alpha^t_z) \) then \( \lambda = ps-tcf(\Pi\bar{\alpha}_t, <_D) \) iff \( \lambda = ps - tcf(\Pi\bar{\alpha}_t, <_D) \).

Proof. Straightforward.

Now we come to the heart of the matter

\[\square_{2.5}\]

Theorem 2.6. \( The \ Composition \ Theorem \) \[Assume \ AC_Z \ and \ \kappa \geq \aleph_0\]

We have \( \lambda = ps-tcf(\prod \lambda_{i,j}, <_D) \) and \( D \) is a \( \kappa \)-complete filter on \( Y \) when:

(a) \( E \) is a \( \kappa \)-complete filter on \( Z \)

(b) \( \langle \lambda_i : i \in Z \rangle \) is a sequence of regular cardinals

(c) \( \lambda = ps-tcf(\prod \lambda_{i,j}, <_E) \)

(d) \( Y = \{ Y_i : i \in Z \} \)

(e) \( D = \{ D_i : i \in Z \} \)

(f) \( D_i \) is a \( \kappa \)-complete filter on \( Y_i \)

(g) \( \lambda = \langle \lambda_{i,j} : i \in Z, j \in Y_i \rangle \) is a sequence of regular cardinals (or just limit ordinals)

(h) \( \lambda_i = ps-tcf(\prod j \in Y_i \lambda_{i,j}, <_{D_i}) \)

(i) \( Y = \{ (i,j) : j \in Y_i \text{ and } i \in Z \} \)

(j) \( D = \{ A \subseteq Y : \text{for some } B \in E \text{ we have } i \in B \Rightarrow \{ j : (i,j) \in A \} \in D_i \} \).

Proof.

\( (*)_0 \) \( D \) is a \( \kappa \)-complete filter on \( Y \).

[Why? Straightforward (and do not need any choice).]

Let \( \langle \mathcal{F}_i, \alpha : \alpha < \lambda_i, i \in Z \rangle \) be such that

\( (*)_1 \) (a) \( \mathcal{F}_i = \langle \mathcal{F}_i, \alpha : \alpha < \lambda_i \rangle \) witness \( \lambda_i = ps - tcf(\prod j \in Y_i \lambda_{i,j}, <_{D_i}) \)

(b) \( \mathcal{F}_i, \alpha \neq \emptyset \).

[Why? Exists by clause (h) of the assumption and \( AC_Z \), for clause (b) recall [Sh:938, 5.6].]

By clause (c) of the assumption let \( \mathcal{G} \) be such that

\( (*)_2 \) (a) \( \mathcal{G} = \langle \mathcal{G}_\beta : \beta < \lambda \rangle \) witness \( \lambda = ps-tcf(\prod i \in Z \lambda_i, <_E) \)

(b) \( \mathcal{G}_\beta \neq \emptyset \) for \( \beta < \lambda \).

Now for \( \beta < \lambda \) let

\( (*)_3 \) \( \mathcal{F}_\beta \) := \( \{ f : f \in \prod (i,j) \in Y \lambda_{i,j} \text{ and for some } g \in \mathcal{G}_\beta \text{ and } \bar{h} = \langle h_i : i \in Z \rangle \in \prod (i,j) \in Y \text{ we have } (i,j) \in Y \Rightarrow f((i,j)) = h_i(j) \} \)

\( (*)_4 \) the sequence \( \langle \mathcal{F}_\beta : \beta < \lambda \rangle \) is well defined (so exists).
\[ \text{Why? Obviously.} \]

[\( \ast \)] if \( \beta_1 < \beta_2, f_1 \in \mathcal{F}_{\beta_1} \) and \( f_2 \in \mathcal{F}_{\beta_2} \) then \( f_1 \prec_D f_2 \).

[\( \ast \)] Let \( g_{\ell} \in \mathcal{F}_{\beta_{\ell}} \) and \( \bar{h}_{\ell} = \langle h^1_{\ell} : i \in Z \rangle \in \prod_{i \in Z} \mathcal{F}_{i, g_{i}(i)} \), witness \( f_{\ell} \in \mathcal{F}_{\beta_{\ell}} \) for \( \ell = 1, 2 \). As \( \beta_1 < \beta_2 \) by \( \ast \) we have \( B := \{ i \in Z : g_1(i) < g_2(i) \} \in E \). For each \( i \in B \) we know that \( g_1(i) < g_2(i) < \lambda_i \) and so \( h_1^1 \in \mathcal{F}_{i, g_1(i)}, h_2^1 \in \mathcal{F}_{i, g_2(i)} \); hence recalling the choice of \( (\mathcal{F}_{i, \alpha} : \alpha < \lambda_i) \), see \( \ast \), we have \( A_i \in D_i \) where for every \( i \in Z \) we let \( A_i := \{ j \in Y_i : h_1^1(j) < h_2^1(j) \} \). As \( h_1, h_2 \) exists clearly \( (A_i : i \in Z) \) exist hence \( A = \{ (i, j) : i \in B \text{ and } j \in A_i \} \) is a well defined subset of \( Y \) and it belongs to \( D \) by the definition of \( D \).

Lastly, \( (i, j) \in A \Rightarrow f_1((i, j)) f_2((i, j)) \), shown above; so by the definition of \( D \) we are done.

[\( \ast \)] for every \( \beta < \lambda \) the set \( \mathcal{F}_{\beta} \) is non-empty.

[\( \ast \)] Recall \( \mathcal{G}_{\beta} \neq \emptyset \) by \( \ast \) \( \neq \emptyset \) and let \( g \in \mathcal{G}_{\beta} \). As \( \langle \mathcal{F}_{i, g(i)} : i \in Z \rangle \) is a sequence of non-empty sets (recalling \( \ast \)), and we are assuming \( AC_Z \) there is a sequence \( \langle h_i : i \in Z \rangle \in \prod_{i \in Z} \mathcal{F}_{i, g(i)} \). Let \( f \) be the function with domain \( Y \) defined by \( f((i, j)) = h_i(j) \); so \( g, h \) witness \( f \in \mathcal{F}_{\beta}, \) so \( \mathcal{F}_{\beta} \neq \emptyset \) as required.

[\( \ast \)] if \( f^*_i \in \prod_{(i,j) \in Y_i} \lambda_{i,j} \) for some \( \beta < \lambda \) and \( f \in \mathcal{F}_{\beta} \) we have \( f^*_i < f \text{ mod } D \).

[\( \ast \)] We define \( \bar{f} = \langle f^*_i : i \in Z \rangle \) as follows: \( f^*_i \) is the function with domain \( Y_i \) such that

\[ j \in Y_i \Rightarrow f^*_i(j) = f((i, j)). \]

Clearly \( \bar{f} \) is well defined and for each \( i, f^*_i \in \prod_{j \in Y_i} \lambda_{i,j} \) hence by \( \ast \) \( \neq \) for some \( \alpha < \lambda_i \) and \( h \in \mathcal{F}_{i, \alpha} \) we have \( f^*_i < h \text{ mod } D_i \) and let \( \alpha_i \) be the first such \( \alpha \) so \( \langle \alpha_i : i \in Z \rangle \) exists.

By the choice of \( \langle \mathcal{G}_{\beta} : \beta < \lambda \rangle \) there are \( \beta < \lambda \) and \( g \in \mathcal{G}_{\beta} \) such that \( \langle \alpha_i : i \in Z \rangle < g \text{ mod } E \) hence \( A := \{ i \in Z : \alpha_i < g(i) \} \) belongs to \( E \). So \( \langle \mathcal{F}_{i, g(i)} : i \in Z \rangle \) is a (well defined) sequence of non-empty sets hence recalling \( AC_Z \) there is a sequence \( h = \langle h_i : i \in Z \rangle \in \prod_{i \in Z} \mathcal{F}_{i, g(i)} \). By the property of \( \langle \mathcal{F}_{i, \alpha} : \alpha < \lambda_i \rangle \) and the choice of \( h \) recalling the definition of \( A \), we have \( i \in A \Rightarrow f^*_i < h_i \text{ mod } D_i \), exists as \( \langle h_i : i \in Z \rangle \) exist.

Lastly, let \( f \in \prod_{(i,j) \in Y} \lambda_{i,j} \) be defined by \( f((i, j)) = h_i(j) \). Easily \( g, \bar{h} \) witness that \( f \in \mathcal{F}_{\beta}, \) and by the definition of \( D \), recalling \( A \in E \) and the choice of \( \bar{h} \) we have \( f^*_i < f \text{ mod } D \), so we are done.]

Together we are done proving the theorem. \( \square \)

\{es\}

**Conclusion 2.7.** The pcf closure conclusion Assume \( AC_{\mathcal{P}(a)} \). We have \( c = \text{ps-pcf}_{\aleph_1 - \text{comp}}(\mathcal{C}) \) when:

\begin{itemize}
  \item [(a)] \( \mathcal{C} \) a set of regular cardinals, non-empty
  \item [(b)] \( \text{hrg}(\mathcal{P}(a)) \leq \text{min}(a) \)
  \item [(c)] \( c = \text{ps-pcf}_{\aleph_1 - \text{comp}}(\mathcal{C}) \).
\end{itemize}
\textbf{Definition 2.8.} Let a set $\mathfrak{a}$ of regular cardinals.

1) We say $b = \langle b_{\lambda} : \lambda \in \mathfrak{c} \rangle$ is a generating sequence for $\mathfrak{a}$ when:

   (a) $b_{\lambda} \subseteq \mathfrak{a} \subseteq \mathfrak{c} \subseteq \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{a})$

   (b) $J_{<\lambda}[a] = J_{\mathfrak{c}\lambda}[a] + b_{\lambda}$ for every $\lambda \in \mathfrak{c}$, hence for every cardinal $\lambda$ we have $J_{<\lambda}[a]$ is the $\mathfrak{R}^1_{\lambda}$-complete ideal on $\mathfrak{a}$ generated by $\{b_\theta : \theta \in \text{pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{a})$ and $\theta < \lambda\}$. \hfill \{e10\}

2) We say $\mathcal{F}$ is a witness for $\tilde{b} = \langle b_{\lambda} : \lambda \in \mathfrak{c} \subseteq \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{a}) \rangle$ when:

   (a) $\mathcal{F} = \langle \tilde{\mathcal{F}}_{\lambda} : \lambda \in \mathfrak{c} \rangle$

   (b) $\tilde{\mathcal{F}}_{\lambda} = (\tilde{\mathcal{F}}_{\lambda, \alpha} : \alpha < \lambda)$ witness $\lambda = \text{ps-tcf}(\Pi\{b_{\theta} : \theta \in \mathfrak{c}_{<\lambda}[a]\})$. \hfill \{e11\}

3) Above $\tilde{b}$ is closed when $b_{\lambda} = a \cap \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(b_{\lambda})$; if $a$ is not mentioned it means $a = \mathfrak{c}$.

3A) Above $\tilde{b}$ is smooth when $\theta \in b_{\lambda} \Rightarrow b_{\theta} \subseteq b_{\lambda}$.

4) We say above $\tilde{b}$ is full when $\mathfrak{c} = \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{a})$.

\textbf{Remark 2.9.} 1) Note that 1.8 gives sufficient conditions for the existence of $\tilde{b}$ as in 2.8.1 which is full.

2) Of course, Definition 2.8 is interesting particularly when $a = \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{a})$. \hfill \{e7\}

\textbf{Theorem 2.10.} Assume $\text{AC}_\mathfrak{c}$ and $\text{AC}_{\mathcal{P}(\mathfrak{a})}$. Then $\mathfrak{c} = \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{a})$ has a full closed generating sequence for $\mathfrak{R}^1_{\lambda}$-complete filters (see below) when:

(a) $\mathfrak{a}$ is a set of regular cardinals

(b) $\text{hrtg}(\mathcal{P}(\mathfrak{a})) < \min(\mathfrak{a})$

(c) $\mathfrak{c} = \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{a})$.

\textbf{Proof.} Proof of 2.10

\hspace{1cm} (+) $1. \mathfrak{c} = \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{c})$.

[Why? By 2.7 using $\text{AC}_{\mathcal{P}(\mathfrak{a})}$.]

\hspace{1cm} (+) $2$ there is a generating sequence $\langle b_{\lambda} : \lambda \in \mathfrak{c} \rangle$ for $\mathfrak{a}$. \hfill \{955\}

\textbf{Proof.} Note that $\mathfrak{c}$ is non-empty because $a \subseteq \mathfrak{c}$.

Assume $\lambda \in \text{ps-pcf}_{\mathfrak{R}^1_{\lambda\text{-comp}}}(\mathfrak{c})$, hence there is $E$ an $\mathfrak{R}^1_{\lambda}$-complete filter on $\mathfrak{c}$ such that $\lambda = \text{ps-tcf}(\Pi\{\mathfrak{c}, <_{E}\})$. As we have $\text{AC}_{\mathcal{P}(\mathfrak{a})}$ by 1.3 (as the $D$ there is unique) there is a sequence $\langle D_\theta : \theta \in \mathfrak{c} \rangle$, $D_\theta$ an $\mathfrak{R}^1_{\lambda}$-complete filter on $\mathfrak{a}$ such that $\theta = \text{ps-tcf}(\Pi\{\mathfrak{a}, <_{D_\theta}\})$, also by 1.8 there is a function $h$ from $\mathcal{P}(\mathfrak{a})$ onto $\mathfrak{c}$, let $E_1 = \{S \subseteq \mathcal{P}(\mathfrak{a}) : \{\theta \in \mathfrak{c} : h^{-1}(\theta) \subseteq S\} \subseteq E\}$. By claim 2.2, the "if" direction with $\mathcal{P}(Y)$ here standing for $Y$ there, we have $\lambda = \text{ps-tcf}(\Pi\{h(b) : b \in \mathcal{P}(\mathfrak{a}), <_{E_1}\})$ and $E_1$ is an $\mathfrak{R}^1_{\lambda}$-complete filter on $\mathcal{P}(\mathfrak{a})$.

Now we apply Theorem 2.6 with $E_1$, $\langle D_{h(b)} : b \in \mathcal{P}(\mathfrak{a}), \lambda, \langle h(b) : b \in \mathcal{P}(\mathfrak{a})\rangle, \langle \theta : \theta \in \mathfrak{a} \rangle \rangle$ here standing for $E$, $\langle D_i : i \in Z, \lambda, \langle \lambda_i : i \in Z\rangle, \langle \lambda_{i,j} : j \in Y_i \rangle \rangle$ for every $j \in Z$ (constant here). We get a filter $D_1$ on $Y = \{(b, \theta) : b \in \mathcal{P}(\mathfrak{a}), \theta \in \mathfrak{a}\}$ such that $\lambda = \text{ps-tcf}(\Pi\{\theta : (b, \theta) \in Y\}, <_{D_1})$.

Now $|Y| = |\mathcal{P}(\mathfrak{a})|$ as $\mathfrak{a}$ can be well ordered (hence $\mathfrak{c}_0 \leq |\mathfrak{a}|$ or $\mathfrak{a}$ finite and all is trivial) so applying 2.2 again we get an $\mathfrak{R}^1_{\lambda}$-complete filter $D$ on $\mathfrak{a}$ such that $\lambda = \text{ps-tcf}(\Pi\{\mathfrak{c}, <_{D}\})$, so we are done. \hfill \{955\}
([Why? By 1.8(3) using also \(AC_c\).])

\((*)_3\) let \(b^*_\lambda = \text{ps-pcf}_{\aleph_1-\text{com}}(b_\lambda)\) for \(\lambda \in \mathcal{C}\).

Now

\((*)_4\)

\((a)\) \(\bar{b}^* = \langle b^*_\lambda : \lambda \in \mathcal{C} \rangle\) is well defined

\((b)\) \(b_\lambda \subseteq b^*_\lambda \subseteq \mathcal{C}\)

\((c)\) \(b^*_\lambda = \text{ps-pcf}_{\aleph_1-\text{com}}(b_\lambda)\)

\((d)\) \(\lambda = \max(b^*_\lambda)\)

\((e)\) \(\lambda \notin \text{pcf}(\mathcal{C} \setminus b^*_\lambda)\).

([Why? First, \(\bar{b}^*\) is well defined as \(\bar{b} = \langle b_\lambda : \lambda \in \mathcal{C} \rangle\) is well defined. Second, \(b_\lambda \subseteq b^*_\lambda\) by the choice of \(b^*_\lambda\) and \(b^*_\lambda \subseteq \mathcal{C}\) as \(b_\lambda \subseteq \mathcal{C}\) hence \(b^*_\lambda = \text{ps-pcf}_{\aleph_1-\text{com}}(b_\lambda)\subseteq \text{ps-pcf}_{\aleph_1-\text{com}}(\mathcal{C}) = \mathcal{C}\), the last equality by 2.7. Third, \(b^*_\lambda = \text{ps-pcf}_{\aleph_1-\text{com}}(b_\lambda)\) by Conclusion 2.7, it is easy to check that its assumption holds recalling \(\lambda \subseteq \mathcal{A}\). Fourth, \(\lambda \in b^*_\lambda\) as \(J_{\lambda, \mathcal{A}}|a|\) witness \(\lambda \in \text{ps-pcf}_{\aleph_1-\text{com}}(b_\lambda) = b^*_\lambda\) and \(\max(b^*_\lambda) = \lambda\) by \((*)_2\) recalling Definition 2.8. Lastly, note that \(\text{ps-pcf}_{\aleph_1-\text{comp}}(a) = \text{ps-pcf}_{\aleph_1-\text{comp}}(b_\lambda) \cup \text{ps-pcf}_{\aleph_1-\text{comp}}(a \setminus b_\lambda)\) by 1.15(2) hence \(\mu, \nu \in b^*_\lambda \Rightarrow \mu \in \text{ps-pcf}_{\aleph_1-\text{comp}}(a \setminus b_\lambda)\) and \(\nu \notin \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathcal{C} \setminus b^*_\lambda)\) by 2.7 it follows that \(\lambda \in \text{pcf}(a \setminus b^*_\lambda)\) which contradict 1.8(3) and 1.6(2) so \(\lambda \notin \text{ps-pcf}_{\aleph_1-\text{comp}}(\mathcal{C} \setminus b^*_\lambda)\) that is, clause (e) holds.]

We can now choose \(\mathcal{F}\) such that

\((*)_5\)

\((a)\) \(\mathcal{F} = \langle \mathcal{F}_\lambda : \lambda \in \mathcal{C} \rangle\)

\((b)\) \(\mathcal{F}_\lambda = \langle \mathcal{F}_\lambda, \alpha : \alpha < \lambda \rangle\)

\((c)\) \(\mathcal{F}_\lambda\) witness \(\lambda = \text{ps-tcf}(\Pi \mathcal{A}, < J_{\lambda, \mathcal{A}}|a|)\)

\((d)\) if \(\lambda \in \mathcal{A}, \alpha < \lambda\) and \(f \in \mathcal{F}_\lambda, \alpha\) then \(f(\lambda) = \alpha\).

([Why? For each \(\lambda\) there is such \(\mathcal{F}\) as \(\lambda = \text{ps-tcf}(\Pi \mathcal{A}, < J_{\lambda, \mathcal{A}}|a|)\). But we are assuming \(AC_c\) and for clause (d) it is easy; in fact it is enough to use \(AC_{\mathcal{F}(a)}\) and \(\lambda\) as in 2.7, getting \(\langle \mathcal{F}_b : b \in \mathcal{F}(a) \rangle, \mathcal{F}_b\) witness \(h(b) = \text{ps-tcf}(\Pi \mathcal{A}, < J_{\lambda, \mathcal{A}}|a|)\) and putting \(\langle \mathcal{F}_b : b \in h^{-1}(\{\lambda\}) \rangle\) together for each \(\lambda \in \mathcal{C}\).]

\((*)_6\)

\((a)\) for \(\lambda \in \mathcal{C}\) and \(f \in \Pi \mathcal{B}_\lambda\) let \(f^{[\lambda]} \in \Pi \mathcal{B}_\lambda^{\langle \lambda \rangle}\) be defined by: \(f^{[\lambda]}(\theta) = \min\{\alpha < \lambda : f(\theta) = \min(\mathcal{F}_\lambda, \alpha)\text{ for every } g \in \mathcal{F}_\lambda, \alpha\}\) we have \(f|\mathcal{B}_\lambda \leq (g|\mathcal{B}_\lambda)\mod J_{\lambda, \mathcal{A}}(\mathcal{B}_\lambda)\)

\((b)\) for \(\lambda \in \mathcal{C}\) and \(\lambda < \alpha < \lambda\) let \(\mathcal{F}_\lambda^{\lambda, \alpha} = \{f^{[\lambda]}(\{\theta\}) : f \in \mathcal{F}_\lambda, \alpha\}\).

Now

\((*)_7\)

\((a)\) \(f^{[\lambda]}|\mathcal{B}_\lambda \geq f\) for \(f \in \Pi \mathcal{B}_\lambda, \lambda \in \mathcal{C}\)

\((b)\) \(\langle \mathcal{F}_\lambda^{\lambda, \alpha} : \lambda \in \mathcal{C}, \alpha < \lambda\rangle\) is well defined (hence exist)

\((c)\) \(\mathcal{F}_\lambda^{\lambda, \alpha} \subseteq \Pi \mathcal{B}_\lambda\).

([Why? Obvious, e.g. for clause (a) note that \(\theta \in \mathcal{A} \Rightarrow \{\theta\} \in (J_{\lambda, \mathcal{A}}|\mathcal{B}_\lambda|)^{+}\).]

\((*)_8\) let \(J_\lambda\) be the \(\aleph_1\)-complete ideal on \(b^*_\lambda\) generated by \(\{b^*_\theta \cap b^*_\lambda : \theta \in \mathcal{C} \cap \lambda\}\)

\((*)_9\) \(J_\lambda \subseteq J_{\lambda, \mathcal{A}}^{\aleph_1-\text{comp}}[b^*_\lambda]\).
Why? As for \( \theta_0, \ldots, \theta_n \in \gamma \cap \lambda \) by 1.15(2) we have \( \text{ps} \prec \text{pcf}_{\lambda_1} \cap \text{comp}(\bigcup \{ b^*_\theta : n < \omega \}) = \bigcup \{ \text{ps} \prec \text{pcf}_{\lambda_1} \cap \text{comp}(b^*_\theta) : n < \omega \} \in J_{\lambda_1^{\langle \lambda \rangle}} \cap \text{comp}[\epsilon]. \]

\( \odot_1 \) if \( \lambda \in \gamma \) and \( \alpha_1 < \alpha_2 < \lambda \) and \( f_\ell \in F_{\lambda, \alpha} \) for \( \ell = 1, 2 \) then \( f_1^{(\lambda_1)} \leq f_2^{(\lambda_1)} \mod J_{\lambda_1}. \)

Why? Let \( a_* = \{ \theta \in b_\lambda : f_1(\theta) \geq f_2(\theta) \} \), hence by the assumption on \( \langle F_{\lambda, \alpha} : \alpha < \lambda \rangle \) we have \( a_* \in J_{\lambda_1^{\langle \lambda \rangle}} \cap \text{comp}[\epsilon] \), hence we can find a sequence \( \{ \theta_n : n < n \leq \omega \} \) such that \( \theta_n \in \gamma \cap \lambda \) and \( a_* \subseteq \bigcup \{ b_\theta : n < n \} \) hence \( c_* := \text{ps} \prec \text{pcf}_{\lambda_1} \cap \text{comp}(a_*) \subseteq \bigcup \{ b_\theta : n < n \} \in J_{\lambda_1}. \) So it suffices to prove \( f_1^{(\lambda_1)} (b^*_\lambda, c_*) \leq f_2^{(\lambda_1)} (b^*_\lambda, c_*) \), so let \( \theta \in b^*_\lambda \setminus \bigcup b^*_\theta, \) by \( \odot_4(d) \) we have \( \theta \leq \lambda, \) let \( \alpha := f_2^{(\lambda_1)} (\theta) \), so by the definition of \( f_2^{(\lambda_1)} (\theta) \) we have \( (v \in \gamma \cap F_{\beta, \alpha}) (f_2[b_\lambda] \leq (g[b_\lambda] \mod J_{\alpha \lambda} [b_\lambda]). \) But \( a_* \subseteq \bigcup b^*_\theta \) and \( n < \omega \) \( \implies \) \( \theta \notin b^*_\theta = \text{ps} \prec \text{pcf}_{\lambda_1} \cap \text{comp}(b_\theta) \) hence by 1.15(2) we have \( \theta \notin \text{ps} \prec \text{pcf}_{\lambda_1} \cap \text{comp}(b_\theta) \), hence \( \bigcup b_\theta \in J_{\lambda_1} \cap \text{comp}[\epsilon] \) hence \( a_* \in J_{\lambda_1} \cap \text{comp}[\epsilon]. \) So (first inequality by the previous sentence and the choice of \( a_* \), second by the earlier sentence)

\[
(f_1(b_\lambda) \leq f_2(b_\lambda) \leq (g(b_\lambda) \mod J_{\alpha \lambda} [b_\lambda])
\]

hence by the definition of \( f_1^{(\lambda_1)}, f_2^{(\lambda_1)}(\theta) \leq \alpha = f_2^{(\lambda_1)}(\theta). \) So we are done.

\( \odot_2 \) if \( \lambda \in \gamma \) and \( g \in \Pi b_\lambda^* \) then for some \( \alpha < \lambda \) and \( f \in F_{\lambda, \alpha} \) we have \( g < f \mod J_{\lambda}. \)

Why? We choose \( (h_\theta : \theta \in b^*_\lambda) \) such that \( h_\theta \in F_{\theta, \beta} \) for each \( \theta \in b^*_\lambda; \) this is possible as we are assuming AC and \( b^*_\lambda \subseteq \gamma \). Let \( h_1 \in \Pi b^*_\lambda \) be defined by \( h_1(\kappa) = \sup \{ h_\theta(\kappa) : \kappa \in b_\theta \in b^*_\lambda \} \) for \( \kappa \in b^*_\lambda, \) the result is \( < \kappa \) because the supremum is \( \leq b_\theta \) \( \gamma \cap \lambda \) and \( \kappa \geq \min(b^*_\lambda) \geq \min(\epsilon) = \min(\epsilon) \geq h_{\text{rg}}(\gamma(\theta)). \)

Hence there are \( \alpha < \lambda \) and \( h_2 \in F_{\lambda, \alpha} \) such that \( h_1 \leq h_2 \mod J_{\lambda \alpha}[\alpha]. \) Now \( f := h_2^{(\lambda_1)} \in \Pi b_\lambda^* \) recalling \( \odot_4(d) \) \( a \) is as required, in particular \( f \in F_{\lambda, \alpha}^* \).

\( \odot_3 \) the sequence \( \langle F_{\lambda, \alpha} : \alpha < \lambda \rangle \) witness \( \lambda \) is \( \text{ps} \prec \text{tcf}(\Pi b_\lambda^*, <_{\lambda 1}). \)

Why? In \( \odot_4(d) \) \( (b, c) + \odot_1 + \odot_2 \).

\( \odot_4 \) if \( \lambda \in \gamma \) then \( J_{\lambda 1} = J_{\lambda 1^{\langle \lambda \rangle}} \cap \text{comp}[\epsilon]. \)

Why? By \( \odot_4(d), \odot_8, \odot_9 \) and \( \odot_3 \).

So

\( \odot_5 \) \( b^* = \{ b^*_\lambda : \lambda \in \gamma \} \) is a generating sequence for \( \epsilon. \)

Why? By \( \odot_4(d), \odot_8, \) recalling that \( \lambda \notin \text{ps} \prec \text{pcf}_{\lambda_1} \cap \text{comp}(c) \setminus b^*_\lambda \) by \( \odot_4(c). \) \( \square \)

Remark 2.11. Clearly \( b^* \) is full and closed, but what about smooth? Is this necessary for generalizing [Sh:460]?

Discussion 2.12. Naturally the definition now of \( \gamma \) as in 2.8(2) for \( \Pi a \) is more involved where \( \gamma = \langle \gamma_{\lambda} : \lambda \in \text{ps-pcf}_{\lambda_1}(a) \rangle, \) \( \gamma_{\lambda} = \langle \gamma_{\lambda, \alpha} : \alpha < \lambda \rangle \) exemplifies \( \text{ps-tcf}(\Pi a, J_{\lambda, \alpha}). \)
Claim 2.13. \([\text{DC} + AC_{\aleph_0}]\) Assume

\(\text{a set of regular cardinals}\)
\(\kappa\) is regular > \(\aleph_0\)
\(\epsilon = \text{ps-pcf}_{\kappa-\text{comp}}(a)\)
\(\min(a) \geq \text{hrtg}(\mathcal{P}(\epsilon))\) or at least \(\geq \text{hrtg}(\epsilon)\)
\(\mathcal{F} = \langle \mathcal{F}_\lambda : \lambda \in \mathcal{C} \rangle, \mathcal{F}_\lambda = (\mathcal{F}_{\lambda, \alpha} : \alpha < \lambda)\) witness 2 \(\lambda = \text{ps-tcf}(\Pi a, <_{\text{comp}}[a])\).

Then

\(\exists f \in \Pi a\) for some \(g \in \Pi c\), if \(g \leq g_1 \in \Pi c\) and \(h \in \Pi \{\mathcal{F}_{\lambda, g_1(\lambda)} : \lambda \in c\}\) then \((\exists \alpha \in |c|^{\leq \kappa})(f < \sup\{h_\lambda : \lambda \in \alpha\})\).

Proof. Let \(f \in \Pi a\). For each \(\lambda \in \text{ps-pcf}_{\kappa-\text{com}}(a)\) let \(\alpha_{f, \lambda} = \min\{\alpha < \lambda : f < g\) mod \(J_{=\lambda}[a]\) for every \(g \in \mathcal{F}_{\lambda, h}\)\}, so clearly each \(\alpha_f\) is well defined hence \(\alpha = \langle \alpha_{f, \lambda} : \lambda \in \text{ps-pcf}_{\kappa-\text{com}}(a)\rangle\) exists. So \(g = \langle \alpha_{f, \lambda} : \lambda \in \mathcal{C} \rangle \in \Pi c\) is well defined. Assume \(g_1 \in \Pi c\) and \(g \leq g_1\). Let \(h_\lambda : \lambda \in c\) be any sequence from \(\prod \mathcal{F}_{\lambda, g_1(\lambda)}\); at least one exists when AC holds but this is not needed here. Let \(a_{f, \lambda} = \{\theta : a : f(\theta) < h_\lambda(\theta)\}\) so \(a_{f, \lambda} : \lambda \in c\) exists and we claim that for some \(\alpha \in |c|^{< \kappa}\) we have \(a = \cup\{a_{f, \lambda} : \lambda \in \alpha\}\). Otherwise let \(J\) be the \(\kappa\)-complete ideal on \(a\) generated by \(\{a_{f, \lambda} : \lambda \in c\}\), it is a \(\kappa\)-complete ideal. So by [Sh:938, 5.9=r9], applicable by our assumptions, there is a \(\kappa\)-complete ideal \(J_1\) on \(a\) extending \(J\) such that \(\lambda_\alpha = \text{ps-tcf}(\Pi a, <_{\text{comp}}[a])\) is well defined. So \(\lambda_\alpha \in c\) and \(a_{f, \mathcal{C}} \in J_1\), easy contradiction. \(\Box\)

Claim 2.14. \([\text{AC}_{\aleph_0}]\) We can uniformly define 3 a \(\aleph_0\)-continuous witness for \(\lambda = \text{ps-pcf}_{\kappa-\text{com}}(\Pi a, <_D)\) where:

\(\text{a set of regular cardinals}\)
\(\epsilon = \text{ps-pcf}_{\kappa-\text{com}}(a)\)
\(\min(a) \geq \text{hrtg}(\mathcal{P}(\epsilon))\) or at least \(\geq \text{hrtg}(\epsilon)\)
\(\mathcal{F} = \langle \mathcal{F}_a : a \in S \rangle\) satisfies each \(\mathcal{F}_a\) is a witness for \(\lambda = \text{ps-pcf}_{\kappa-\text{com}}(\Pi a, <_D)\)
\(\text{if } a \in S \text{ then } \mathcal{F}_a\) is \(\aleph_0\)-continuous and \(f_1, f_2 \in \mathcal{F}_{a, \alpha} \Rightarrow f_1 = f_2 \mod D\).

Proof.

\(\forall t \in Y\) \(\text{hrtg}(S \times S)\) is \(\leq \lambda\) and \(\leq \text{cf}(\alpha_t)\) for \(t \in Y\).

Why? As \(\lambda, \text{cf}(\alpha_t)\) are regular cardinals.

For \(a, b \in S\) let

\(\epsilon_1(1) a \in Y\) \(\text{Ord}\)
\(\text{is a limit ordinal with } cf(\alpha_t) \geq \text{hrtg}(S)\)
\(\epsilon_2(1) \mathcal{F} = \langle \mathcal{F}_a : a \in S \rangle\) satisfies each \(\mathcal{F}_a\) is a witness for \(\lambda = \text{ps-pcf}_{\kappa-\text{com}}(\Pi a, <_D)\)
\(\epsilon_1(2) a \in S\) then \(\mathcal{F}_a\) is \(\aleph_0\)-continuous and \(f_1, f_2 \in \mathcal{F}_{a, \alpha} \Rightarrow f_1 = f_2 \mod D\).

\(\epsilon_2(2) g_{a,b}\) is well defined.

\(\forall a, b \in S\) let

\(\epsilon_1(3) a \in Y\) \(\text{Ord}\)
\(\text{if } a < b\) then \(f_1, f_2 \in \mathcal{F}_{a, \alpha}\) we have \(f_1 < f_2 \mod D\)
\(\text{define } g_{a,b} : \lambda \rightarrow \lambda\) by \(g_{a,b}(\alpha) = \min\{\beta < \lambda : \text{there are } f_1, f_2 \in \mathcal{F}_{a, \alpha}\)
\(\text{and } f_1 < f_2 \mod D\) such that \(f_1 < f_2 \mod D\).

\(\epsilon_2(3) g_{a,b}\) is well defined.

2So we are assuming it is well defined, now if \(\text{AC}_{\mathcal{P}(\mathcal{Y})}\) such \(\mathcal{F}\) exists.

3Of course, mere existence is already given by the assumptions.
Question 2.15. 1) Can we in 2.5 get smoothness? 
2) If 2.10 does it suffice to assume AC_{\mathcal{F}(a)} (and omit AC_{\mathcal{F}}) and we can conclude that \( e \in \text{ps} - \text{pcf}_{\mathcal{N}_1 - \text{comp}}(\mathcal{E}) \) has a full closed generating sequence.

We may try to repeat the proof of 2.10, only in the proof of (\ast)_5 we use claim 2.16 below.

Claim 2.16. In 2.10 we can add "\( \mathbb{b} \) is weakly smooth" which means \( \theta \in \mathbb{b}_\lambda \Rightarrow \theta \notin \text{ps} - \text{pcf}_{\mathcal{N}_1 - \text{comp}}(\mathcal{E} \setminus \mathbb{b}_\lambda) \).

Proof. Let \( \mathbb{b} = \{ \mathbb{b}_\lambda : \lambda \in \mathcal{E} \} \) be a full closed generating sequence.

We choose \( \mathbb{b}_\lambda \) by induction on \( \lambda \in \mathcal{E} \) such that

\[
\begin{align*}
(\ast)_1 \quad (a) & \quad J_{\lambda, \lambda}[a] = J_{\lambda, \lambda}^{\mathcal{N}_1 - \text{comp}}[a] + \mathbb{b}_\lambda^1 \\
& \quad (b) \quad \text{ps} - \text{pcf}_{\mathcal{N}_1 - \text{comp}}(\mathbb{b}_\lambda^1) = \mathbb{b}_\lambda^1 \\
& \quad (c) \quad \max(\mathbb{b}_\lambda^1) = \lambda \\
& \quad (d) \quad \text{if } \theta \in \mathbb{b}_\lambda^1 \text{ then } b_\lambda^1 \supseteq b_\theta \text{ mod } J_{\lambda, \lambda}[a], \text{ i.e. } \mathbb{b}_\lambda^1 \setminus \mathbb{b}_\theta \in J_{\lambda, \lambda}^{\mathcal{N}_1 - \text{comp}}[a].
\end{align*}
\]
Arriving to \( \lambda \) let \( \mathfrak{d}_\lambda = \{ \theta \in b_\lambda : b_\lambda \setminus b_\lambda^{\downarrow} \not\in J^{\beta \downarrow}_{<\theta} \} \), \( \mathfrak{d}_\lambda^{\downarrow} = \text{ps} - \text{pcf}_{\mathfrak{b}_\lambda - \text{comp}}(\mathfrak{d}_\lambda) \).

Now

\[ \text{(*)}_2 \text{ ps} - \text{pcf}_{\mathfrak{b}_\lambda - \text{comp}}(\mathfrak{d}_\lambda) \subseteq b_\lambda \cap \lambda. \]

[Why? \( \subseteq b_\lambda \) is obvious; recalling \( b_\lambda^{\downarrow} = \text{ps} - \text{pcf}(b_\lambda^{\downarrow} \cap \alpha) \) because \( b \) is closed. If “\( \nsubseteq \lambda \)” recall \( \mathfrak{d}_\lambda^{\downarrow} = \text{ps} - \text{pcf}_{\mathfrak{b}_\lambda - \text{comp}}(\mathfrak{d}_\lambda) \), now \( \mathfrak{d}_\lambda \subseteq b_\lambda \) hence \( \mathfrak{d}_\lambda^{\downarrow} \subseteq \text{ps} - \text{pcf}_{\mathfrak{b}_\lambda - \text{comp}}(b_\lambda) \subseteq \lambda^+ \). So the only problematic case is \( \lambda \in \mathfrak{d}_\lambda^{\downarrow} = \text{ps} - \text{pcf}_{\mathfrak{b}_\lambda - \text{comp}}(\mathfrak{d}_\lambda) \). But then, \( \mathfrak{d}_\lambda \subseteq \text{ps} - \text{pcf}_{\mathfrak{b}_\lambda - \text{comp}}(c \setminus b_\lambda) \) by the definition of \( \mathfrak{d}_\lambda \) hence by the composition theorem we have \( \lambda \in \text{ps} - \text{pcf}_{\mathfrak{b}_\lambda - \text{comp}}(c \setminus b_\lambda) \), contradicting an assumption on \( b \).]

\[ \text{(*)}_3 \text{ there is a countable } \epsilon_\lambda \subseteq \mathfrak{d}_\lambda^{\downarrow} \text{ such that } \mathfrak{d}_\lambda^{\downarrow} \subseteq \bigcup \{ b_\sigma^{\downarrow} : \sigma \in \epsilon_\lambda \}. \]

[Why? Should be clear.]

Lastly, let \( b_\lambda^{\downarrow} = \bigcup \{ b_\sigma^{\downarrow} : \sigma \in \epsilon_\lambda \} \cup b_\lambda \) and check. \( \square \text{2.16} \)
§ 3. Measuring Reduced Products

§ 3(A). On \( \text{ps-}T_D(g) \).

Now we consider some ways to measure the size of \( \kappa \mu/D \) and show that they essentially are equal; see Discussion 3.9 below.

**Definition 3.1.** Let \( \check{\alpha} = ⟨\alpha_t : t ∈ Y⟩ \in \text{Ord} \) be such that \( t ∈ Y ⇒ α_t > 0 \).
1) For \( D \) a filter on \( Y \) let \( \text{ps-}T_D(\check{\alpha}) = \sup \{ \text{hrtg}(F) : F \) is a family of non-empty subsets of \( Π\check{\alpha} \) such that for every \( \mathcal{F}_1 \neq \mathcal{F}_2 \) from \( F \) we have \( f_1 ∈ \mathcal{F}_1 \land f_2 ∈ \mathcal{F}_2 ⇒ f_1 \neq D f_2 \}, \) recalling \( f_1 \neq D f_2 \) means \( \{ t ∈ Y : f_1(t) \neq D f_2(t) \} \in D \).
2) Let \( \text{ps-}T_{κ-\text{comp}}(\check{\alpha}) = \sup \{ \text{hrtg}(F) : \) for some \( κ \)-complete filter \( D \) on \( Y \), \( F \) is as above for \( D \).\)
3) If we allow \( α_t = 0 \) just replace \( Π\check{\alpha} \) by \( Π^*\check{\alpha} := \{ f : f ∈ \prod_t (α_t + 1) \) and \( t : f(t) = α_t \} = ∅ \mod D \}.

**Theorem 3.2.** \([DC + AC_\mathbb{P}(Y)/] \) Assume that \( D \) is a \( κ \)-complete filter on \( Y \) and \( κ > ∅_0 \) and \( g ∈ Y \) (Ord \( \setminus \{0\} \)), if \( g \) is constantly \( \check{\alpha} \) we may write \( \check{\alpha} \). The following cardinals are equal or at least \( λ_1, λ_2, λ_3 \) are Fil \( _κ(D) \)-almost equal which means:

for \( \ell_1, \ell_2 ∈ \{1, 2, 3\} \) we have \( λ_{\ell_1} ≤ \text{sal} Fil_{κ}(D) \) \( λ_{\ell_2} \) which means if \( \alpha < λ_{\ell_1} \) then \( α \) is included in the union of \( S \) sets each of order type \( < λ_{\ell_2} \):

\[
\begin{align*}
(a) & \quad λ_1 = \sup \{ |\text{rk}(D_1)|^+ : D_1 ∈ \text{Fil}_κ(D) \} \\
(b) & \quad λ_2 = \sup \{ λ^+ : \text{there are } D_1 ∈ \text{Fil}_κ(D) \) and \( a < D_1 \)-increasing sequence \( \{ \mathcal{F}_α : α < λ \} \) such that \( \mathcal{F}_α ⊆ \prod_{t ∈ Y} g(t) \) is non-empty \} \\
(c) & \quad λ_3 = \sup \{ \text{ps-}T_{D_1}(g) : D_1 ∈ \text{Fil}_κ(D) \}.
\end{align*}
\]

**Remark 3.3.** 1) Recall that for \( D \) a \( κ \)-complete filter on \( Y \) we let Fil \( _κ(D) = \{ E : E \) is a \( κ \)-complete filter on \( Y \) extending \( D \} \).
2) The conclusion gives slightly less than equality of \( λ_1, λ_2, λ_3 \).
3) See 3.10(6) below, by it \( λ_2 = \text{ps-}\text{Depth}^+(\kappa \mu/D) \) recalling 3.10(5).
4) We may replace \( κ \)-complete by \( (≤ Z) \)-complete if \( ∅_0 < |Z| \).
5) Compare with Definition 3.10.
6) Note that those cardinals are \( ≤ \text{hrtg}(Π^*g) \), see 3.1(3).

**Proof.** Stage A: \( λ_1 ≤ \text{sal} Fil_{κ}(D) λ_2, λ_3 \).

Why? Let \( α < λ_1 \), so by clause (a) there is \( D_1 ∈ \text{Fil}_κ(D) \) such that \( \text{rk}(D_1, g) ≥ α \).

Let \( X_{D_2} = \{ α < χ : \) some \( f ∈ \prod_{t ∈ Y} g(t) \) satisfies \( D_2 = \text{dual}(J[f, D_1]) \) and \( α = \text{rk}_{D_1}(f) \} \), for any \( D_2 ∈ \text{Fil}_κ(D_1) \). By [Sh:938, 1.11(5)] we have \( χ = \bigcup \{ X_{D_2} : D_2 ∈ \text{Fil}_κ(D_1) \} \).

Now

\( ⊚ D_2 ∈ \text{Fil}_κ(D_1) ⇒ |\text{otp}(X_{D_2})| < λ_2, λ_3 \); this is enough.

---

\(^4\text{recall dual}(J[f, D_1]) = \{ X ⊆ Y : X ∈ D_1 \) or \( \text{rk}_{D_1 +(X \setminus Y)}(f) > \text{rk}_{D_1}(f) \}.\)
Why does this hold? Letting \( F_{D_i} = \{ f \in \mathcal{F}_{D_i} : \text{rk}_{D_i}(f) = \text{rk}(D_i) \} \), it follows by [Sh:938, 1.11(2)] we have: if \( i < j \) and \( j \in X_{D_j} \wedge f \in F_{D_i} \wedge g \notin F_{D_i} \Rightarrow f < g \mod D_2 \) so by the definitions of \( \lambda_2, \lambda_3 \) we have \( \text{otp}(X_{D_2}) < \lambda_2, \lambda_3 \).

Stage B: \( \lambda_2 \leq_{\text{Fil}_{\kappa}(D)} \lambda_1, \lambda_3 \), moreover \( \lambda_2 \leq \lambda_1, \lambda_3 \).

Why? Let \( \chi < \lambda_2 \) and let \( D_1 \) and \( \{ \mathcal{F}_{\alpha} : \alpha < \chi \} \) exemplify \( \chi < \lambda_2 \). Let \( \gamma_\alpha = \min\{\text{rk}_{D_i}(f) : f \in \mathcal{F}_{\alpha} \} \) so easily \( \alpha < \beta < \chi \Rightarrow \gamma_\alpha < \gamma_\beta \) hence \( \text{rk}(D_1) \geq \chi \).

So \( \chi < \lambda_1 \) by the definition of \( \lambda_1 \) and as for \( \chi < \lambda_3 \) this holds by Definition 3.1(2) as \( \alpha < \beta \) for \( \mathcal{F}_\alpha \wedge g \in \mathcal{F}_\beta \Rightarrow f < g \mod D_1 \Rightarrow f \neq g \mod D_1 \) as \( \chi^+ = \text{hrtg}(\chi) \leq \lambda_3 \).

Stage C: \( \lambda_3 \leq_{\text{Fil}_{\kappa}(D)} \lambda_1, \lambda_2 \).

Why? Let \( \chi < \lambda_3 \). Let \( \{ \mathcal{F}_\alpha : \alpha < \chi \} \) exemplify \( \chi < \lambda_3 \). For each \( \alpha < \chi \) let \( D_\alpha = \{ \text{dual}(J[f,D]) : f \in \mathcal{F}_\alpha \} \) so a non-empty subset of \( \text{Fil}_{\kappa}(Y) \). Now for every \( D_1 \in D_\ast := \{ D_\alpha : \alpha < \chi \} \) let \( X_{D_1} = \{ \alpha < \chi : D_1 \in D_\alpha \} \) and for \( \alpha \in X_{D_1} \) let \( \zeta_{D_1,\alpha} = \min\{\text{rk}_D(f) : f \in \mathcal{F}_{\alpha} \} \) and \( D_1 = \text{dual}(J[f,D]) \) and let \( \mathcal{F}_{D_1,\alpha} = \{ f \in \mathcal{F}_\alpha : \alpha < \lambda_2 \leq \lambda_1 \} \leq \zeta_{D_1,\alpha} \) then \( f < g \mod D_1 \).

[Why? For clause (a), if \( \alpha \neq \beta \) then \( f \in \mathcal{F}_{\alpha,\beta}, g \in \mathcal{F}_{\alpha,\beta} \) then \( f \neq g \mod D_1 \) hence by [Sh:938, 1.11(2)] we have \( \zeta_{D_1,\alpha} \neq \zeta_{D_1,\beta} \). For clause (b), it follows by the choices of \( D_\alpha \). Lastly, clause (c) follows by [Sh:938, 1.11(2)].]

Hence by (clause (c))

\[
\text{(d) otp}(X_{D_1}) \leq \lambda_2 \text{ and is } \leq \text{rk}_{D_1}(g) \text{ for } D_1 \in \text{Fil}_{\kappa}(D). \]

Together clause (d) shows that \( D \in D_\ast \Rightarrow |X_D| < \lambda_1, \lambda_2 \) so by clause (b), \( \lambda_3 \leq_{\text{Fil}_{\kappa}(D)} \lambda_1, \lambda_2 \) hence we are done. \( \square_{3.2} \)

Observation 3.4. If \( D \) is a filter on \( Y \) and \( \hat{\alpha} \in (\text{Ord}\setminus\{0\}) \) then \( \text{ps} - T_D(\hat{\alpha}) = \sup\{\lambda^+ : \text{there is a sequence } \{ \mathcal{F}_\alpha : \alpha < \lambda \} \text{ such that } \mathcal{F}_\alpha \subseteq \Pi \alpha, \mathcal{F}_\alpha \neq \emptyset \text{ and } \alpha \neq \beta \wedge f_1 \subseteq \mathcal{F}_\alpha \wedge f_2 \subseteq \mathcal{F}_\beta \Rightarrow f_1 \neq f_2 \} \).

Proof. Clearly the new definition gives a cardinal \( \leq \text{ps} - T_D(\hat{\alpha}) \). For the other inequality assume \( \lambda < \text{ps} - T_D(\hat{\alpha}) \) so there is \( \mathcal{F} \) as such that \( \lambda < \text{hrtg}(\mathcal{F}) \).

As \( \lambda < \text{hrtg}(\mathcal{F}) \) there is a function \( h \) from \( \mathcal{F} \) onto \( \lambda \). For \( \alpha < \lambda \) define \( \mathcal{F}_\alpha' = \cup\{ \mathcal{F} : \mathcal{F} \subseteq \mathcal{F}_\alpha \wedge h(\mathcal{F}) = \alpha \} \). So \( \mathcal{F}_\alpha : \alpha < \lambda \) exists and is as required. \( \square_{3.4} \)

Concerning Theorem 3.2 we may wonder “when does \( \lambda_1, \lambda_2 \) being \( S\)-almost equal implies they are equal”. We consider a variant this time for sets (or powers, not just cardinals).

Definition 3.5. 1) We say “the power of \( \mathcal{U}_1 \) is \( S\)-almost smaller than the power of \( \mathcal{U}_2 \)”, or write \( |\mathcal{U}_1| \leq |\mathcal{U}_2| \mod S \) or \( |\mathcal{U}_1| \leq_{\text{Min}}^S |\mathcal{U}_2| \mod S \) when we can find a sequence \( \{ u_{1,s} : s \in S \} \) such that \( \mathcal{U}_1 = \cup\{ u_{1,s} : s \in S \} \) and \( s \in S \Rightarrow |\mathcal{U}_{1,s}| \leq |\mathcal{U}_2| \).
2) We say the power $|\mathcal{V}_1| , |\mathcal{V}_2|$ are $S$-almost equal (or $|\mathcal{V}_1| = |\mathcal{V}_2|$ mod $S$ or $|\mathcal{V}_1| = |\mathcal{V}_2|$ when $|\mathcal{V}_1| \leq_{S} |\mathcal{V}_2|$).
3) Let $|\mathcal{V}_1| \leq_{<S} |\mathcal{V}_2|$ be defined naturally.
4) In particular this applies to cardinals.
5) Let $|\mathcal{V}_1| \leq_{S} |\mathcal{V}_2|$ means there is a sequence $\langle u_{1,s} : s \in S \rangle$ with union $\mathcal{V}_1$ such that $s \in S \Rightarrow |\mathcal{V}_1| < |\mathcal{V}_2|$.
6) Let $|\mathcal{V}_1| \leq_{<S} |\mathcal{V}_2|$ means that if $|\mathcal{V}| < |\mathcal{V}_1|$ then $|\mathcal{V}| < |\mathcal{V}_1| < |\mathcal{V}_2|$. 

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**Observation 3.6.** 1) If $|\mathcal{V}_1| \leq_{S} |\mathcal{V}_2|$ and $S \neq \emptyset$ then $|\mathcal{V}_1| \leq_{S} |\mathcal{V}_2|$.
2) If $\lambda_1 \leq \lambda_2$ and $S \neq \emptyset$ then $\lambda_1 \leq_{S} \lambda_2$.
3) If $\lambda_2 = \lambda_1^+$ and cf($\lambda_2$) < hrtg($S$) then the power of $\lambda_2$ is $S$-almost smaller than $S$.

**Proof.** Immediate. $\square_{3.6}$

**Observation 3.7.** 1) The cardinals $\lambda_1, \lambda_2$ are equal when $\lambda_1 = |\mathcal{V}_1| = S$ and $\lambda_1, \lambda_2$ are limit cardinals > hrtg($\mathcal{P}(S)$).
2) The cardinals $\lambda_1, \lambda_2$ are equal when $\lambda_1 = |\mathcal{V}_1| = S$ and $\lambda_1, \lambda_2$ are limit cardinals > hrtg($\mathcal{P}(S)$).
3) If $\lambda_1 \leq_{S} \lambda_2$ and $\vartheta = hrtg(\mathcal{P}(S))$ then $\lambda_1 \leq_{S} \lambda_2$.
4) If $\lambda_1 \leq_{S} \lambda_2$ and $\vartheta$ then $\lambda_1 \leq \lambda_2$.
5) If $\lambda_1 \leq_{S} \lambda_2$ and $\vartheta$ then $\lambda_1 \leq \lambda_2^+$. 

**Proof.** 1) Otherwise, let $\vartheta = hrtg(\mathcal{P}(S))$, without loss of generality $\lambda_2 < \lambda_1$ and by part (3) we have $\lambda_1 \leq_{S} \lambda_2$ and by part (4) we have $\lambda_1 \leq \lambda_2$ contradiction.
2) Otherwise letting $\vartheta = hrtg(\mathcal{P}(S))$ without loss of generality $\lambda_2 < \lambda_1$ and by part (3) we have $\lambda_1 \leq_{S} \lambda_2$ but $\vartheta < \lambda_2$ is assume and $\lambda_2^+ < \lambda_1 < \lambda_2$ as $\lambda_2$ is a limit cardinal so together we get contradiction to part (5).
3) If (1) witness $\lambda_1 \leq_{S} \lambda_2$, let $\omega = \{ \omega < \lambda_1 : \text{for no } \beta < \lambda \text{ do we have } \forall s \in S \langle \alpha \in u_s \equiv \beta \in u_s \rangle \rangle \text{ so clearly } \omega < hrtg(\mathcal{P}(S)) = \vartheta \text{ and for } \omega \in u_\omega \text{ then } \mu_\omega = \{ \beta < \lambda_1 : \forall s \in S \langle \alpha \in u_\omega \equiv \beta \in u_\omega \rangle \rangle \text{ so } \omega_\alpha : \alpha \in u_\omega \text{ witness } \lambda_1 \leq_{S} \lambda_2 \text{ hence } \lambda_1 \leq_{S} \lambda_2^+.
4,5) Let $\sigma < \theta$ be such that $\lambda_1 \leq_{S} \lambda_2$ and let $\langle u_\epsilon : \epsilon < \sigma \rangle$ witness $\lambda_1 \leq_{S} \lambda_2$, that is $|u_\epsilon| \leq \lambda_2$ for $\epsilon < \sigma$ and $U\{u_\epsilon : \epsilon < \sigma \} = \lambda_1$.

For part (4), if $\lambda_2 < \lambda_1$, then we have $\epsilon < \sigma \Rightarrow |u_\epsilon| < \lambda_1$, but cf($\lambda_1$) > $\sigma$ hence $|U\{u_\epsilon : \epsilon < \sigma \}| < \lambda_1$, contradiction.

For part (5) for $\epsilon < \sigma$, let $u_\epsilon' = u_\epsilon \setminus U\{u_\zeta : \zeta < \epsilon \}$ and so otp($u_\epsilon'$) < otp($u_\epsilon$) < $|u_\epsilon| + \lambda_2^+$ so easily $|\lambda_1| = |\cup\{u_\epsilon : \epsilon < \sigma \}| = |\cup\{u_\epsilon : \epsilon < \sigma \}| < \sigma \cdot \lambda_2^+ \leq \lambda_2 \cdot \lambda_2^+ = \lambda_2^+$. $\square_{3.7}$

Similarly

**Observation 3.8.** 1) If $\lambda_1 \leq_{S} \lambda_2$ and $\vartheta = hrtg(\mathcal{P}(S))$ then $\lambda_1 <_{<\vartheta} \lambda_2$.
2) If $\lambda_1 <_{<\vartheta} \lambda_2$ and $\vartheta = hrtg(\mathcal{P}(S))$ then $\lambda_1 <_{<\vartheta} \lambda_2$.
3) If $\lambda_1 <_{<\vartheta} \lambda_2$ and $\vartheta = hrtg(\mathcal{P}(S))$ then $\lambda_1 <_{<\vartheta} \lambda_2$.
4) If $\lambda_1 <_{<\vartheta} \lambda_2$ and $\vartheta = hrtg(\mathcal{P}(S))$ then $\lambda_1 <_{<\vartheta} \lambda_2$.
5) If $\lambda_1 <_{<\vartheta} \lambda_2$ and $\vartheta = hrtg(\mathcal{P}(S))$ then $\lambda_1 <_{<\vartheta} \lambda_2$.
6) If $\lambda_1 <_{<\vartheta} \lambda_2$ and $\vartheta = hrtg(\mathcal{P}(S))$ then $\lambda_1 <_{<\vartheta} \lambda_2$.

**Proof.** Similar, e.g.
1) Like the proof of 3.7(3). $\square_{3.8}$
Discussion 3.9. 1) We like to measure \( (Y, \mu) / D \) in some ways and show their equivalence, as was done in ZFC. Natural candidates are:

(A) \( \text{pp}_D(\mu) \): say of length of increasing sequence \( \bar{P} \) (not \( \bar{p} ! \), i.e. sets) ordered by \( \prec \)

(B) \( \text{pp}_Y(\mu) = \sup \{ \text{pp}_D^+(\mu) : D \text{ an } \aleph_1 \text{-complete filter on } Y \} \)

(C) As in 3.1.

2) We may measure \( Y, \mu \) by considering all \( \partial \)-complete filters.

3) We may be more lenient in defining “same cardinality”. E.g.

(A) we define when sets have similar powers say by divisions to \( \mathcal{P}(\mathcal{P}(Y)) \) sets we measure \( (Y, \mu) / \approx_{\mathcal{P}(\mathcal{P}(Y))} \) where \( \approx_B \) is the following equivalence relation on sets:

\[ X \approx_B Y \text{ when we can find sequences } \langle X_b : b \in B \rangle, \langle Y_b : b \in B \rangle \]

such that:

(a) \( X = \bigcup \{ X_b : b \in B \} \)

(b) \( Y = \bigcup \{ Y_b : b \in B \} \)

(c) \( |X_b| = |Y_b| \)

(B) we may demand more: the \( \langle X_b : b \in B \rangle \) are pairwise disjoint and the \( \langle Y_b : b \in B \rangle \) are pairwise disjoint

(C) we may demand less: e.g.

\[ (c), \ |X_\beta| \leq |Y_\beta| \leq |X_\alpha| \]

and/or

\[ (\forall b \in B)(\exists c \in B) \bigl( |X_b| \leq |Y_c| \bigr) \text{ and } (\forall b \in B)(\exists c \in B) \bigl( |Y_b| \leq |X_c| \bigr). \]

Note that some of the main results of [Sh:835] can be expressed this way.

(D) \( \text{rk-sup}_{Y, \partial}(\mu) = \text{rk-sup} \{ \text{rk}_D(\mu) : D \text{ is } \partial \text{-complete filters on } Y \} \)

(E) for each non-empty \( X \subseteq Y, \mu \) let

\[ \text{sp}_1^1(X) = \{ (D, J) : D \text{ an } \aleph_1 \text{-complete filter on } Y, J = J[f, D], \alpha = \text{rk}_D(f) \text{ and } f \in X \} \]

\[ \text{sp}_1(X) = \bigcup \{ \text{sp}_1^1(X) : \alpha \} \]

(F) question: If \( \{ \text{sp}(X_s) : s \in S \} \) is constant, can we bound \( J \)?

(G) \( X, Y \) are called connected when \( \text{sp}(X_1), \text{sp}(X_2) \) are non-disjoint or equal.

4) We hope to prove, at least sometimes \( \gamma := Y(\check{Y}, \mu) \leq \text{pp}_\mu(\mu) \) that is we like to imitate [Sh:835] without the choice axioms on \( \alpha \). So there is \( f = \langle f_\alpha : \alpha < \delta \rangle \) witnessing \( \gamma < Y(\check{Y}, \mu) \). We define \( u = u_f = \{ \alpha : \text{there is no } \bar{\beta} < \omega | \alpha \text{ such that } (\forall t \in Y)(f_\alpha(t) \in \{ f_\beta(t) : n < \omega \}) \} \). You may say that \( u_f \) is the set of \( \alpha < \delta \) such that \( f_\alpha \) is “really novel”.

By DC this is O.K., i.e.

\[ \exists \check{1} \text{ for every } \alpha < \delta \text{ there is } \check{\beta} < \omega | (u_f \cap \alpha) \text{ such that } (\forall t \in Y)(f_\alpha(t)) = \{ f_\beta(t) : n < \omega \}. \]
Next for \( \alpha \in u_f \) we can define \( D_{f,\alpha} \), the \( \aleph_1 \)-complete filter on \( Y \) generated by 
\[
\{ t \in Y : f_\beta(t) = f_\alpha(t) \} : \beta < \alpha \}.
\]
So clearly \( \alpha \neq \beta \in u_f \land D_{f,\alpha} = D_{f,\beta} \Rightarrow f_\alpha \neq D_{f,\beta} \).
\\( f_\beta \). Now for each pair \( \bar{D} = (D_1, D_2) \in \text{Fil}_Y \) (i.e. for the \( \aleph_1 \)-complete case) let
\[
\Lambda_{f,\bar{D}} = \{ \alpha \in u_f : D_{f,\alpha} = D_1 \land J[f_\alpha, D_1] \} = \text{dual}(D_2).
\]
So \( \gamma \) is the union of \( \leq P(\mathcal{P}(Y)) \)-sets (as \( |Y| = |Y| \times |Y| \), well ordered.
\[
\text{Remark 3.11. Note that 1.14 can be phrased using this definition.}
\]

**Definition 3.10.** 1) Let \( \text{suc}_X(\alpha) \) be the first ordinal \( \beta \) such that we cannot find a sequence \( \langle \mathcal{U}_x : x \in X \rangle \) of subsets of \( \beta \), each of order type \( < \alpha \) such that \( \beta = \cup \{ \mathcal{U}_x : x \in X \} \).
2) We define \( \text{suc}_X^{[\varepsilon]}(\alpha) \) by induction on \( \varepsilon \) naturally: if \( \varepsilon = 0 \) it is \( \alpha \), if \( \varepsilon = \zeta + 1 \) it is \( \text{suc}_X^{[\zeta]}(\alpha) \) and if \( \varepsilon \) is a limit ordinal then it is \( \cup \{ \text{suc}_X^{[\zeta]}(\alpha) : \zeta < \varepsilon \} \).
3) For a quasi-order \( P \) let the pseudo ordinal depth of \( P \), denoted by \( \text{ps-o-Depth}(P) \) be \( \sup \gamma \) there is a \( \mu \)-increasing sequence \( \langle X_\alpha : \alpha < \gamma \rangle \) of non-empty subsets of \( P \).
4) \( \alpha \)-Depth\( (P) \) is defined similarly demanding \( |X_\alpha| = 1 \) for \( \alpha < \gamma \),
5) Omitting the “ordinal” means \( \gamma \) is replaced by \( |\gamma| \); similarly in the other variants including omitting the letter \( o \) in ps-o-Depth.
6) Let ps-o-Depth\( ^+(P) = \sup \{ \gamma + 1 : \text{there is an increasing sequence } \langle X_\alpha : \alpha < \gamma \rangle \) of non-empty subsets of \( P \)\).
7) For \( D \) a filter on \( Y \) and \( \alpha \in Y \setminus \{ \text{Ord} \setminus \{ 0 \} \} \) let ps-o-Depth\( _0^+(\alpha) = \text{ps-o-Depth}^+(\Pi \alpha, < D) \). Similarly for the other variants and we may allow \( \alpha_0 = 0 \) as in 3.1(3).
8) Let ps-o-Depth\( _0^+(\alpha) \) be the cardinality of ps-o-Depth\( _0^+(\alpha) \).

**Remark 3.11.** Note that 1.14 can be phrased using this definition.

**Definition 3.12.** 0) We say \( x \) is a filter \( \omega \)-sequence when \( x = \langle (Y_n, D_n) : n < \omega \rangle \) is such that \( D_n \) is a filter on \( Y_n \) for each \( n < \omega \); we may omit \( Y_n \) as it is \( \cup \{ Y : Y \in D \} \) and may write \( D \) if \( \bigwedge D_n = D \).
1) Let IND\( (x) \)\( x \) has the independence property, mean that for every sequence \( \vec{F} = \{ F_{m,n} : m < n < \omega \} \) from \( \text{alg}(x) \), see below, there is \( \vec{t} \in \prod_{n<\omega} Y_n \) such that \( m < n < \omega \) \( \Rightarrow t_m \notin F_{m,n}(\vec{t}((m,n)]) \). Let NIND\( (x) \) be the negation.
2) Let \( \text{alg}(x) \) be the set of sequence \( \{F_{n,m} : m < n < \omega \} \) such that \( F_{m,n} : \\
\prod_{\ell = m+1}^{n} Y_{\ell} \to \text{dual}(D_{n}) \).

3) We say \( x \) is \( \kappa \)-complete when each \( D_{x,n} \) is a \( \kappa \)-complete filter.

{\text{k6}}

**Theorem 3.13.** Assume \( \text{IND}(x) \) where \( x = \langle (Y_{n}, D_{n}) : n < \omega \rangle \) as in Definition 3.12. \( D_{n} \) is \( \kappa_{n} \)-complete, \( \kappa_{n} \geq \aleph_{1} \).

1) \( \text{[DC + ACY}_{n} \text{ for } n < \omega] \) For every ordinal \( \zeta \), for infinitely many \( n \)'s ps-\( \text{Depth}(\langle Y_{n}, D_{n} \rangle, <_{D_{n}}) \leq \zeta \).

2) \( \text{[DC]} \) For every ordinal \( \zeta \) for infinitely many \( n \), \( \text{o-Depth}((\langle Y_{n} \rangle, <_{D_{n}}) \leq \zeta \), equivalently there is no \( <_{D_{n}} \)-increasing sequence of length \( \omega + 1 \).

{\text{k7}}

**Remark 3.14.** 0) Note that the present results are incomparable with [Sh:938, §4] - the loss is using depth instead of rank and possibly using “pseudo”.

1) [Assume AC\( \kappa_{n} \)] If for every \( n \) we have \( \text{rk}_{\kappa_{n}}(\zeta) > \sup_{\text{Fil}_{n}(Y_{n})}(\langle \zeta \rangle) \) then for some \( D_{n} \in \text{Fil}_{n}(Y_{n}) \) for \( n < \omega \) we have \( \text{NIND}(\langle Y_{n}, D_{n}^{n} \rangle) : n < \omega \) (Why? By [Sh:938, 5.9]). But we do not know much on the \( D_{n} \)'s.

2) This theorem applies to e.g. \( \zeta = \aleph_{\omega}, Y_{n} = \aleph_{n}, D_{n} = \text{dual}(\langle J_{n}^{bd} \rangle) \). So even in ZFC, it tells us things not covered by [Sh:513, §3]. So it also tells us that it is easy by forcing to get, e.g. \( \text{NIND}(\langle \langle \aleph_{n+1}, \text{dual}(J_{n+1}^{bd}) \rangle \rangle : n < \omega) \), see 3.19. Note that Depth and pcf are closely connected but only for sequences of length \( \geq \text{hrtg}(\mathcal{P}(Y)) \) and see 3.19 below.

3) If we assume \( \text{IND}(\langle Y_{\eta(n)}, D_{\eta(n)} : n < \omega \rangle) \) for every increasing \( \eta \in \omega^{\omega} \), which is quite reasonable then in Theorem 3.13 we can strengthen the conclusion, replacing “for infinitely many \( n \)’s” by “for every \( n < \omega \) large enough”.

4) Note that 3.13(2) is complimentary to [Sh:835].

{\text{k8}}

**Observation 3.15.** 1) If \( x \) is a filter \( \omega \)-sequence, \( x \) is \( \aleph_{1} \)-complete and \( n_{x} < \omega \) and \( \text{IND}(x)[n_{x}, \omega] \), then \( \text{IND}(x) \).

2) If \( x \) is a filter \( \omega \)-sequence and \( \text{IND}(x) \) and \( \eta \in \omega^{\omega} \) is increasing and \( y = \langle Y_{x,\eta(n)}, D_{x,\eta(n)} : n < \omega \rangle \) then \( y \) is a filter \( \omega \)-sequence and \( \text{IND}(y) \).

**Proof.** 1) Let \( \hat{F} = \langle F_{n,m} : n < m < \omega \rangle \in \text{alg}(x) \), so \( \langle F_{n,m} : n \in [n_{x}, \omega] \) and \( m \in (n, \omega) \rangle \) belongs to \( \text{alg}(x)[n_{x}, \omega] \) hence by the assumption “\( \text{IND}(x)[n_{x}, \omega] \)” there is \( t = \langle t_{n} : n \in [n_{x}, \omega] \rangle \in \prod_{\eta \in \mathcal{P}(Y)} Y_{\eta} \) such that \( t_{n} \notin F_{n,m}(\langle \ell \rangle(n+1, m)) \) when \( n_{x} \leq n < \omega \). Now by downward induction on \( n < n_{x} \) we choose \( t_{n} \in Y_{\eta} \) such that \( t_{n} \notin F_{n,m}(\langle \ell \rangle(n+1, m)) \) for \( m \in [n+1, \omega) \). This is possible as the countable union of members of \( \text{dual}(D_{n}) \) is not equal to \( Y_{n} \). We can carry the induction and \( \langle t_{n} : n < \omega \rangle \) is as required to verify \( \text{IND}(x) \).

2) Let \( \hat{F} = \langle F_{i,j} : i < j < \omega \rangle \in \text{alg}(y) \). For \( m < n \) we define \( F_{m,n}^{'(m,n)} \) as the following function from \( \prod_{k=m+1}^{n} Y_{\eta,k} \) into \( \text{dual}(D_{x,m}) \) by

- if \( i < j, m = \eta(i), n = \eta(j) \) and \( s = \langle s_{k} : k \in (m, n] \rangle \in \prod_{k=m+1}^{n} Y_{\eta,k} \) then \( F_{m,n}^{'(m,n)}(s) = F_{i,j}(\langle \eta_{[i+k]} : k \in [1, j-i] \rangle) \)

- if there are no such \( i, j \) then \( F_{m,n} \) is constantly \( \emptyset \).

As \( \text{IND}(x) \) holds there is \( \bar{t} = \langle t_{\eta(k)} : k < \omega \rangle \in \prod_{\eta} Y_{\eta(n)} = \prod_{\eta} Y_{\eta,n} \) is necessarily as required. \( \square_{3.15} \)
Proof. Proof of Theorem 3.13

We concentrate on proving part (1), part (2) is easier, (i.e. below each $F_{n, \varepsilon}$ is a singleton hence so is $G_{m, n, \varepsilon}$ so there is no need to use AC$_{Y_n}$.

Assume this fails. So for some $n_0 < \omega$ for every $n \in [n_0, \omega)$ there is a counterexample. As AC$_{\aleph_0}$ holds we can find a sequence $\langle F_n : n \in [n_0, \omega) \rangle$ such that:

- for $n \in [n_0, \omega)$
  - $F_n = \langle F_{n, \varepsilon} : \varepsilon \leq \zeta \rangle$ is non-empty
  - $\langle F_{n, \varepsilon} \rangle$ is a $<_{D_n}$-increasing sequence of sets, i.e. $\varepsilon_1 < \varepsilon_2 \leq \zeta \land f_1 \in F_{n, \varepsilon_1} \land f_2 \in F_{n, \varepsilon_2} \Rightarrow f_1 <_{D_n} f_2$.

Now by AC$_{\aleph_0}$ we can choose $\langle f_n : n \in [n_0, \omega) \rangle$ such that $f_n \in F_{n, \varepsilon}$ for $n \in [n_0, \omega)$.

(*) without loss of generality $n_0 = 0$.

[Why? As $x[n_0, \omega]$ satisfies the assumptions on $x$ by 3.15(2).]

Now

1. for $m \leq n < \omega$ let $Y^0_{m, n} = \prod_{\ell=m}^{n-1} Y_{\ell}$ and for $m, n < \omega$ let $Y^1_{m, n} := \cup \{ Y^0_{k, n} : k \in [m, n] \}$ so $Y^0_{m, n} = \emptyset = Y^1_{m, n}$ if $m > n$ and $Y^0_{m, n} = \{ < > \} = Y^1_{m, n}$ if $m = n$; so if $\eta \in Y^0_{m+1, n}$ and $s \in Y_m, t \in Y_{n+1}$ we define $\langle s \rangle \hat{\eta} = \eta \in Y^0_{m, n}$ and $\eta \hat{\langle t \rangle} \in Y^1_{m+1, n+1}$ naturally

2. for $m \leq n$ let $G^1_{m, n}$ be the set of functions $g$ such that:
   - $g$ is a function from $Y^1_{m, n}$ into $\zeta + 1$
   - $\langle \rangle \neq \eta \in Y^1_{m, n} \Rightarrow g(\eta) < \zeta$
   - if $k \in [m, n]$ and $\eta \in Y^0_{k+1, n}$ then the sequence $\langle g(\hat{s}) \hat{\eta} : s \in Y_k \rangle$ belongs to $F_{k, g(\eta)}$

3. $G^1_{m, n, \varepsilon} := \{ g \in G^1_{m, n} : g(\hat{\langle \rangle}) = \varepsilon \}$ for $\varepsilon \leq \zeta$ and $m \leq n < \omega$.

Now the sets $G^1_{m, n}$ are non-trivial, i.e.

4. if $m \leq n$ and $\varepsilon \leq \zeta$ then $G^1_{m, n, \varepsilon} \neq \emptyset$.

[Why? We prove it by induction on $n$; first if $n = m$ this is trivial because the unique function $g$ with domain $\{ < > \}$ and value $\varepsilon$ belongs to $G^1_{m, n, \varepsilon}$. Next, if $m < n$ we choose $f \in F_{n-1, \varepsilon}$ hence the sequence $\langle G^1_{m, n-1, f(s)} : s \in Y_{n-1} \rangle$ is well defined and by the induction hypothesis each set in the sequence is non-empty. As AC$_{Y_{n-1}}$ holds there is a sequence $\langle g_s : s \in Y_{n-1} \rangle$ such that $s \in Y_{n-1} \Rightarrow g_s \in G^1_{m, n-1, f(s)}$.

Now define $g$ as the function with domain $Y^1_{m, n}$:

$$g(\hat{\langle \rangle}) = \varepsilon$$

$$g(\hat{\nu}(s)) = g_s(\nu)$$

for $\nu \in Y^1_{m, n-1}$ and $s \in Y_n$.

It is easy to check that $g \in G^1_{m, n, \varepsilon}$ indeed so 4 holds.]
\[\exists 5 \text{ if } g, h \in Y_{m,n}^1 \text{ and } g(\langle \rangle) < h(\langle \rangle) \text{ then there is an } (m, n)\text{-witness } Z \text{ for } (h, g) \text{ which means (just being an } (m, n)\text{-witness means we omit clause (d))}: \]

(a) \( Z \subseteq Y_{m,n}^1 \) is closed under initial segments, i.e. if \( \eta \in Y_{k,n}^0 \cap Z \) and \( m \leq k < \ell \leq n \) then \( \eta[\ell, n) \in Y_{\ell,n}^0 \cap Z \)

(b) \( \langle \rangle \in Z \)

(c) if \( \eta \in Z \cap Y_{k+1,n}^0, m \leq k < n \) then \( \{ s \in Y_k : s \rangle \eta \in Z \} \in D_k \)

(d) if \( \eta \in Z \) then \( g(\eta) < h(\eta) \).

[Why? By induction on \( n \), similarly to the proof of \( \exists 4 \).]

\[\exists 6 (a) \text{ we can find } \bar{g} = \langle g_n : n < \omega \rangle \text{ such that } g_n \in \mathcal{G}_{0,n,k}^1 \text{ for } n < \omega \]

(b) for \( \bar{g} \) as above for \( n < \omega, s \in Y_n \) let \( g_{n+1,s} \in \mathcal{G}_{0,n}^1 \) be defined by
\[g_{n+1,s}(\nu) = g_{n+1}(\nu^\langle s \rangle) \text{ for } \nu \in Y_{0,n} \]

[Why? Clause (a) by \( \exists 4 \) as AC\( \kappa_0 \) holds, clause (b) is obvious by the definitions in \( \exists 2 + \exists 3 \).]

We fix \( \bar{g} \) as in \( \exists 6(a) \) for the rest of the proof.

\[\exists 7 \text{ There is } \langle Z_{n,s} : s \in Y_n \rangle : n < \omega \text{ such that } Z_{n,s} \text{ witness } (g_n, g_{n+1,s}) \text{ for } n < \omega, s \in Y_n. \]

[Why? For a given \( n < \omega, s \in Y_n \) we know that \( g_{n+1}(\langle s \rangle) < \zeta = g_n(\langle \rangle) \) hence \( Z_{n,s} \) as required exists by \( \exists 5 \). By AC\( Y_0 \) for each \( n \) a sequence \( \langle Z_{n,s} : s \in Y_n \rangle \) as required exists, and by AC\( \kappa_0 \) we are done.]

\[\exists 8 Z_n := \{ \langle \rangle \} \cup \{ \nu^\langle s \rangle : s \in Y_{n-1}, \nu \in Z_{n-1,s} \} \text{ is a } (0, n)\text{-witness.} \]

[Why? By our definitions.]

\[\exists 9 \text{ there is } \bar{F} \text{ such that:} \]

(a) \( \bar{F} = \{ F_{m,n} : m < n < \omega \} \)

(b) \( F_{m,n} : Y_{m+1,n+1}^1 \rightarrow \text{dual}(D_m) \)

(c) \( F_{m,n}(\nu) = \{ s \in Y_n : \nu^\langle s \rangle \notin Z_{n-1} \} \) when \( \nu \in Z_n \) and is \( \emptyset \) otherwise.

[Why? As clauses (a),(b),(c) define \( \bar{F} \).]

\[\exists 10 \text{ } \bar{F} \text{ witness } \text{IND}((Y_n, D_n) : n < \omega )) \text{ fail.} \]

[Why? Clearly \( \bar{F} = \langle F_{m,n} : m < n < \omega \rangle \) has the right form.

So toward contradiction assume \( t = \langle t_n : n < \omega \rangle \in \prod_{n<\omega} Y_n \) is such that

\[\ast1 \text{ } m < n < \omega \Rightarrow t_m \notin F_{m,n}(\bar{f}(\langle m,n \rangle)). \]

Now

\[\ast2 \text{ } t|[m,n) \in Z_n \text{ for } m \leq n < \omega. \]

[Why? For each \( n \), we prove this by downward induction on \( m \). If \( m = n \) then \( t|[m,n) = \emptyset \) but \( \emptyset \in Z_n \) by its definition. If \( m < n \) and \( t|[m+1,n) \in Z_n \) then \( t_m \notin F_{m,n-1}(\bar{f}([m,n])) \) by \( \ast1 \) so \( t|[m,n) = (t_m)^\langle \bar{f}([m+1,n]) \rangle \in Z_n \) holds by clause \( \exists 9(c) \).

\[\ast3 \text{ } g_{n+1}(\bar{f}([m,n])) < g_n(\bar{f}([m,n])). \]
[Why? Note that $Z_{n,t_n}$ is a witness for $(g_n,g_{n+1,t_n})$ by $\mathbb{B}_7$. So by $\mathbb{B}_5$ (see clause (d) there) we have $\eta \in Z_{n,t_n} \Rightarrow g_{n+1,t_n}(\eta) < g_n(\eta)$. But $m < n \Rightarrow \ell[m,n] \in Z_{n+1} \Rightarrow \ell[m,n] \in Z_{n,t_n}$, the first implication by $(*)_2$, the second implication by the definition of $Z_{n+1}$ in $\mathbb{B}_8$. Hence by $\mathbb{B}_6(b)$ and the last sentence, and by the sentence before last $g_{n+1}(\ell[m,n]) = g_{n+1}(\ell[m,n]) < g_n(\ell[m,n])$ as required. So $(*)_3$ holds indeed.]

So for each $m < \omega$ the sequence $(g_n(\ell[m,n]) : n \in [m,\omega))$ is a decreasing sequence of ordinals, contradiction. Hence there is no $\ell$ as above, so indeed $\mathbb{B}_{10}$ holds. But $\mathbb{B}_{10}$ contradicts an assumption, so we are done. \qed

Remark 3.16. 1) Note that in the proof of 3.13 there was no use of completeness demands, still natural to assume $\aleph_1$-completeness because: if $D'_\lambda$ is the $\aleph_1$-completion of $D_\lambda$ then IND($D'_\lambda : n < \omega$) is equivalent to IND($D_\lambda : n < \omega$).

2) Recall that by [Sh:513, 2.7], if $\text{pp}(\aleph_\omega) > \aleph_\omega$ then for every $\lambda > \aleph_\omega$ for infinitely many $n < \omega$ we have $(\forall \mu < \lambda)(\text{cf}(\mu) = \aleph_\omega \Rightarrow \text{pp}(\mu) \leq \lambda)$.

3) Concerning 3.17 below recall that:

(A) if $Y_n$ is a regular cardinal, $D_n$ witness $Y_n$ is a measurable cardinal, then clause (a) of 3.17 holds, but [Sh:938, §4] says more

(B) if $\mu = \mu^+ \mu$ and $\mathbb{P}_\mu$ is the Levy collapse a measurable cardinal $\lambda > \mu$ to be $\mu^+$ with $D$ a normal ultrafilter on $\lambda$, then $\mathbb{P}_\mu$ “the filter which $D$ generates is as required in (b) with $\mu$ in the role of $Z_n$”, by Jech-Magidor-Mitchell-Prikry [JMMP80].

So we can force that $n < \omega \Rightarrow Y_n = \aleph_{2n}$.

4) So

(a) if $\text{pp}(\aleph_\omega) > \aleph_\omega$ and $\aleph_\omega$ divides $\delta, \text{cf}(\delta) < \aleph_\omega$ and $\delta < \aleph_\delta$ then $\text{pp}(\aleph_\delta) < \aleph_\delta$.

(b) we can replace $\aleph_\omega$ by any singular $\mu < \aleph_\mu$

(c) if, e.g. $\delta_n < \lambda_n = \aleph_\delta, \delta_n < \delta_{n+1}$ and $\text{cf}(\delta_n) < \aleph_\delta$ for $n < \omega$, then, except for at most one $n, \text{pp}(\aleph_\lambda) < \aleph_\lambda$.

5) We had thought that maybe: if $\mu$ is singular and $\text{pp}(\mu) \geq \aleph_\mu$ then some case of IND follows. Why? Because by [Sh:513, 2.8] this holds if $\mu < \aleph_\mu$ provided that $\mu = \aleph_\delta \land |\delta|^{\aleph_0} < \mu$, (even getting IND($(\text{dual}(J^\mu) : n < \omega$)) for some increasing sequence $(\lambda_n : n < \omega$) of regular cardinals $< \mu$ with limit $\mu$ if $\text{cf}(\mu) = \aleph_\delta$ and $\subseteq \{\lambda^+ : \lambda \in E\}$ for any pre-given club $E$ of $\mu$ if $\text{cf}(\delta) > \aleph_0$). If only $\mu = \aleph_\delta \land |\delta| < \mu$ then in [Sh:513] we get a weaker version of IND.

Claim 3.17. [DC] For $x = (Y_n, D_n : n < \omega$) with each $D_n$ being an $\aleph_1$-complete filter on $Y_n$, each of the following is a sufficient condition for IND$(x)$, letting $Y(< n) := \prod_{m < n} Y_m$ and for $m < n$, let $Z_{m,n} = \{t : t$ is a function from $n-1 \prod_{\ell=m+1}^n Y_\ell$ into $Y_m\}$ and let $Z_n = \prod_{m < n} Z_{m,n}$.

(a) $D_n$ is a $(\leq Z_n)$-complete ultrafilter

(b) $D_n$ is a $(\preceq Z_n)$-complete filter
• for each $n$ in the following game $\mathcal{O}_{X,n}$ the non-empty player has a winning strategy. A play last $\omega$-moves. In the $k$-th move the empty player chooses $A_k \in D_n$ and $(X_k^t : t \in Z_n)$, a partition of $A_k$ and the non-empty player chooses $t_k \in Z_n$. In the end the non-empty player wins the play if $\bigcap_{k<\omega} X_k^t$ is non-empty.

(c) like clause (b) but in the second part the non-empty player instead $S_k \subseteq Z_n$ satisfying $|S_k| \leq X |S|$ and every $D_{X,n}$ is $(\leq S)$-complete, $S$ is infinite.

(d) if $m < n < \omega$ then $D_m$ is $(\leq \prod_{k=m+1}^n Y_k)$-complete.

Proof. Straightforward. E.g.

Clause (b):

Let $(\mathbf{st}_n : n < \omega)$ be such that $\mathbf{st}_n$ is a winning strategy of the non-empty player in the game $\mathcal{O}_{X,n}$.

Let $\mathcal{F} = (F_{m,n} : m < n < \omega) \in \text{alg}(X)$ and we should find a member of $\prod_n Y_n$ as required in Definition 3.12(2). We now, by induction on $i < \omega$, choose the following objects satisfying the following condition:

\[ (*)_i \quad (a) \quad \text{for } k < m \text{ and } j < i, G_{j,k,m} \text{ is a function from } \prod_{\ell=k+1}^m Y_\ell \text{ into } Y_k \]

\[ (b)(a) \quad \text{for } m < \omega, ((X_{j,m}, t_{j,m}) : j < i) \text{ is an initial segment of a play of the game } \mathcal{O}_{X,m} \text{ in which the non-empty player uses the strategy } \mathbf{st}_m; \]

\[ (\beta) \quad \text{we have } X_{j,m} = (X_{j,m}, t : t \in Z_m) \text{ so } X_{j,m} \subseteq Y_m \]

\[ (\gamma) \quad t_{j,m} = (t_{j,k,m} : k < m) \text{ and } t_{j,k,m} \in Z_k \]

\[ (\delta) \quad X_{j,m} = \bigcap_{k<m} X_{j,k,m} \text{, see clause (e) when } t = (t_k : k < m) \in Z_m, \bigcap_{k<m} t_k \in Z_{k,m} \]

\[ (c)(\alpha) \quad Y_{j,m} = Y_m \text{ if } j = 0 \]

\[ (\beta) \quad Y_{j,m} \text{ is } \bigcap \{X_{i,m,k} : t \in \{j + 1, \ldots, m\}\} \subseteq Y_m \text{ if } j \in (0, i) \]

\[ (d)(\alpha) \quad \text{if } j = 0 < i \text{ then } G_{j,k,m} \text{ is } F_{k,m} \]

\[ (\beta) \quad \text{if } j \in (0, i) \text{ then } G_{j,k,m} \text{ is defined by: for } (y_{k+1}, \ldots, y_m) \in \prod_{\ell=k+1}^m Y_\ell \]

\[ \text{ we have } G_{j,k,m}((y_{k+1}, \ldots, y_m)) = G_{j-1,k,m}((y_{k+1}, \ldots, y_{m+1})) \]

\[ \text{ for any } y_{m+1} \in Y_{j,m+1} \text{ (so the value does not depend on } y_{m+1}) \]

\[ (e) \quad \text{for } k < m \text{ and } t \in Z_{k,m} \text{ let } X_{j,k,m} \text{ be } \{y \in Y_m : (y_{k+1}, \ldots, y_{m-1}) \in \prod_{\ell=k+1}^{m-1} Y_\ell \text{ then } G_{j,k,m}(y_{k+1}, \ldots, y_{m-1}, y) = (y_{k+1}, \ldots, y_{m-1}) \}. \]

---

So the $Y_k$’s are not well ordered! But, on the one hand, if $\alpha < \text{hrtg}(Y_n) = D_n$ is $|\alpha|^\omega$-complete then $\alpha^\omega / D_n \simeq \alpha$. On the other hand, if $D_n$ is $\aleph_1$-complete and $\alpha Y_n / D \simeq \alpha$ then $D$ projects onto a uniform $\aleph_1$-complete filter on some $\mu \leq \alpha$ and those projections cover the ultra-power.
Clearly (*)$_0$ holds empty.

For $i \geq 1$, let $j = i - 1$ clearly $\langle Y_{j,m} : m < \omega \rangle$ is well defined by clause (c), hence we can define $\langle X_{j,k,m,t} : t \in Z_{k,m} \rangle$ by clause (e) and let $X_{j,m,t} = \cap \{ X_{j,k,m,t} : k < m \}$ when $t = (t_k : k < m)$.

So $X_{j,m} = \langle X_{j,m,t} : t \in Z_m \rangle$ is a legal $j$-move of the empty player in the game $\mathcal{X}_m$, so we can use $\text{st}_m$ to define $t_{j,m} = (t_{j,k,m} : k < m)$ as the $j$-th move of the non-empty player.

Lastly, the function $G_{j,k,m}$ is well defined. Having carried the induction, for each $m$ clearly $\langle X_{j,m,t} : j < \omega \rangle$ is a play of the game $\mathcal{X}_m$ in which the non-empty player uses the strategy $\text{st}_m$ hence win in the play, so $\cap \{ X_{j,m,t_{j,m}} : j < \omega \}$ is non-empty so by AC$_{\aleph_0}$ we can choose $\bar{y} = \langle y_m : m < \omega \rangle$ such that $y_m \in \cap \{ X_{j,m,t_{j,m}} : j < \omega \}$.

It is easy to see that $\bar{y}$ is as required in Definition 3.12(2). \hfill \Box_{3.17}

Conclusion 3.18. [DC] Assume $(\kappa_n : n)$ is increasing and $\kappa_n$ is measurable as witnessed by the ultrafilter $D_n$ or just $D_n$ is a uniform$^6$ $\mathcal{T}(\mathcal{P}(\kappa_n))$-complete ultrafilter on $\kappa_n$.

Then for every ordinal $\zeta$, for every large enough $n$ we have $\alpha$-Depth$_{D_n}^+$ $(\zeta) \leq \zeta$.

Proof. By 3.17 we know that $\text{IND}(\langle D_n : n < \omega \rangle)$ and by 3.13(2) we get the desired conclusion. \hfill \Box_{3.18}

Claim 3.19. (ZFC for simplicity). If (A) then (B) where

(A) (a) $\lambda_n = \text{cf}(\lambda_n)$ and $\langle \lambda_n^{< \lambda_n} < \lambda_{n+1} \rangle$ and $\mu = \Sigma \{ \lambda_n : n < \omega \}$ and $\lambda = \mu^+$

(b) $\mathbb{P}_n$ is the natural $\lambda_n$-complete $\lambda_n^+$-c.c. forcing adding $\langle f_{n,\alpha} : \alpha < \lambda \rangle$ of members of $\lambda_n (\lambda_n)$, $<\theta_n$-increasing

(c) $\mathbb{P}$ is the product $\prod_n \mathbb{P}_n$ with full support

(B) in $\mathbb{V}^\mathbb{P}$ we have NIND$(\langle \text{dual}(J_{\lambda_n}^\text{bd}) : n < \omega \rangle)$ and a cardinal $\theta$ is not collapsed if $\theta \notin (\mu^+, \mu^{\kappa_n})$.

Proof. So $p \in \mathbb{P}_n$ if $p$ is a function from some $u \in [\lambda^+]^{< \lambda_n}$ into $\cup \{ \zeta (\lambda_n) : \zeta < \lambda_n \}$, ordered by $\mathbb{P}_n \models \lnot p \leq q$ iff $\alpha \in \text{Dom}(q) \Rightarrow \alpha \in \text{Dom}(q) \land p(\alpha) \leq q(\alpha)$. Now use 3.13. \hfill \Box_{3.19}

§ 3(C). Bounds on the Depth. We continue 3.2. We try to get a bound for singulars of uncountable cofinality say for the depth, recalling that depth, rank and ps-$T_D$ are closely related.

Hypothesis 3.20. $D$ an $\aleph_1$-complete filter on a set $Y$.

Remark 3.21. Some results do not need the $\aleph_1$-completeness.

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$^6$Recall $\mathcal{T}(A) = \min \{ \theta : \text{there is no one-to-one function from } \theta \text{ into } A \}$. 

---

(955)
Claim 3.22. Assume $\alpha \in Y$Ord.

1) [DC] (No-hole-Depth) If $\zeta + 1 \leq \text{ps-o-Depth}_D^+(\alpha)$ then for some $\beta \in Y$Ord, we have $\beta \leq \alpha \mod D$ and $\zeta + 1 = \text{ps-o-Depth}_D^+(\beta)$.
2) In Definition 3.1 we may allow $F \subseteq Y$Ord such that $g \in F \Rightarrow g \prec f \mod D$.
3) If $\beta \in Y$Ord and $\alpha = \beta \mod D$ then $\text{ps-o-Depth}_D^+(\alpha) = \text{ps-o-Depth}_D^+(\beta)$.
4) If $\{y \in Y : \alpha_y = 0\} \in D^+$ then $\text{ps-o-Depth}_D^+(\alpha) = 1$.
5) Similarly for the other versions of depth from Definition 3.10.

Proof. 1) By DC without loss of generality there is no $\beta < D \alpha$ such that $\zeta + 1 \leq \text{ps-o-Depth}_D^+(\beta)$. Without loss of generality $\alpha$ itself fails the desired conclusion hence $\zeta + 1 < \text{ps-o-Depth}_D^+(\beta)$. By parts (3),(4) without loss of generality $s \in Y \Rightarrow \alpha_s > 0$. As $\zeta + 1 < \text{ps-o-Depth}_D^+(\alpha)$ there is a $<D$-increasing sequence $\{F_\varepsilon : \varepsilon < \zeta + 1\}$ with $F_\varepsilon$ a non-empty subset of $\Pi\alpha$. Now any $\beta \in F_\varepsilon$, $\zeta + 1 \leq \text{ps-o-Depth}_D^+(\beta)$ as witnessed by $\langle F_\varepsilon : \varepsilon < \zeta \rangle$, recalling part (2); contradicting the extra assumption on $\alpha$ (being $<D$-minimal such that...).
2) Let $F_\varepsilon = \{f^{[\alpha]} : f \in F_\varepsilon\}$ where $f^{[\alpha]}(s)$ is $f(s)$ if $f(s) < \alpha_s$ and is zero otherwise.
3), 4) Obvious.
5) Similarly. \hfill $\square_{3.22}$

Claim 3.23. [DC + AC] If $\alpha, \beta \in Y$Ord and $D$ is a filter on $Y$ and $s \in Y \Rightarrow |\alpha_s| = |\beta_s|$ then $\text{ps}\text{-}T_D(\alpha) = \text{ps}\text{-}T_D(\beta)$.

Proof. Straightforward. \hfill $\square_{3.23}$

Assuming full choice the following is a relative of Galvin-Hajnal theorem.

Theorem 3.24. [DC + AC] Assume $\alpha(1) < \alpha(2) < \lambda^+$, $\text{ps-o-Depth}_D^+(\lambda) \leq \lambda^{+\alpha(1)}$ and $\xi = \text{hrtg}(Y\alpha(2)/D)$. Then $\text{ps-o-Depth}_D^+(\lambda^{+\alpha(2)}) \leq \lambda^{+\alpha(1)+\xi}$.

Proof. Let $\Lambda = \{\mu : \lambda^{+\alpha(1)} < \mu \leq \lambda^{+\alpha(1)+\xi}\}$ and for every $\mu \in \Lambda$ let

$(*)_1 \quad F_\mu = \{f : f \in Y \{\lambda^{+\alpha} : \alpha < \alpha(2)\} \text{ and } \mu = \text{ps-Depth}_D^+(f)\}$

$(*)_2$ obviously $\langle F_\mu : \mu \in \Lambda \rangle$ is a sequence of pairwise disjoint subsets of $Y\alpha(2)$ closed under equality modulo $D$.

By the no-hole-depth claim 3.22(1) above we have

$(*)_3$ if $\mu_1 < \mu_2$ are from $\Lambda$ and $f_2 \in F_{\mu_2}$ then for some $f_1 \in F_{\mu_1}$ we have $f_1 < f_2 \mod D$

$(*)_4 \quad \xi > \sup\{\zeta + 1 : F(\lambda^{+\alpha(1)+\xi}) \neq \emptyset\}$ implies the conclusion.

Lastly, as $\xi = \text{hrtg}(Y\alpha(2)/D)$ we are done. \hfill $\square_{3.24}$

Remark 3.25. 0) Compare this with conclusion 1.11.
1) We may like to lower $\xi$ to $\text{ps-Depth}_D^+(\alpha(2))$, toward this let

$(*)_1 \quad F_\mu = \{f \in F_\mu : \text{there is no } g \in F_\mu \text{ such that } g < f \mod D\}$

By DC

$(*)_2 \quad \text{if } f \in F_\mu \text{ then there is } g \in F_\mu \text{ such that } g \leq_D f \mod D$.

2) Still the sequence of those $F_\mu$ is not $<D$-increasing.

Instead of counting cardinals we can count regular cardinals.
Theorem 3.26. \( \textit{(DC+AC_Y)} \) The number of regular cardinals in the interval 
\( (\lambda^{+\alpha(1)}, \text{ps-depth}^+_D(\lambda^{+\alpha(2)}) \) is at most \( \text{hrtg}(Y^{\alpha(2)}/D) \) when:

(a) \( \alpha(1) < \alpha(2) < \lambda^+ \)
(b) \( \kappa > \aleph_0 \)
(c) \( D \) is a \( \kappa \)-complete filter on \( Y \)
(d) \( \lambda^{+\alpha(1)} = \text{ps-Depth}_D(\lambda) \).

Proof. Straightforward, using the No-Hole Claim 1.13. \( \square_{3.26} \)
§ 4. Private Appendix

When ready §3D will be moved to the paper or to a new one.


Discussion 4.1. (2013.2.12) More try to continue [Sh:386] with games for $D \in \text{Fil}^1(Y)$ giving rank to $2^\theta < \kappa$, function from $Y$. 

Theorem 4.2. Assume $\text{DC} + AC_{<\mu}$. If $\mu$ is strong limit (i.e. $\chi < \mu \Rightarrow 2^\chi < \mu$ and $\mu$ uncountable) then for every $\lambda \geq \mu$ for some $\kappa < \mu$ we have: if $\xi < \mu, \chi < \lambda, D$ is a $\kappa$-complete filter on $\xi$ then $\text{Depth}_D(\lambda) \leq \lambda$, that is, $\text{depth}(\xi, \lambda, <, D) \leq \lambda$.

Theorem 4.3. The second composition theorem. Assume $AC_Z$ we have $\lambda < \text{Depth}^+(\prod_{i \in Z} P_i, <_D)$ when:

(a) $E$ is a filter on $Z$
(b) $\langle P_i : i \in Z \rangle$ is a sequence of partial orders
(c) $\lambda < \text{Depth}^-(\prod_{i \in Z} \lambda_i, M_D)$
(d) $\lambda_u < \text{Depth}^+(P_i)$
(e) $<_D$ is the following partial orders on $P = \prod_{i \in Z} P_i : f <_D g \Leftrightarrow \{i \in Z : f(i) < P_i g(i)\} \in E$.

Proof.

(a) $E$ is a $\kappa$-complete filter on $Z$
(b) $\langle Y_i : i \in Z \rangle$ is a sequence of regular cardinals
(c) $\lambda = \text{ps-tcf}(\prod_{i \in Z} \lambda_i, <_E)$
(d) $\bar{Y} = \langle Y_i : i \in Z \rangle$
(e) $\bar{D} = \langle D_i : i \in Z \rangle$
(f) $D_i$ is a $\kappa$-complete filter on $Y_i$
(g) $\bar{P} = \langle P_{i,j} : i \in Z, j \in Y_i \rangle$ is a sequence of regular cardinals (or just limit ordinals)
(h) $\lambda_i = \text{ps-tcf}(\prod_{j \in Y_i} P_{i,j}, <_{D_i})$
(i) $Y = \{(i,j) : j \in Y_i$ and $i \in Z\}$
(j) $D = \{A \subseteq Y :$ for some $B \in E$ we have $i \in B \Rightarrow \{j : (i,j) \in A\} \in D_i\}$.

Proof.

($\ast)_0$ $D$ is a $\kappa$-complete filter on $Y$.

[Why? Straightforward (and do not need any choice).]

Let $\langle \mathcal{F}_{i,\alpha} : \alpha < \lambda_i, i \in Z \rangle$ be such that

($\ast)_1$ (a) $\mathcal{F}_i = \langle \mathcal{F}_{i,\alpha} : \alpha < \lambda_i \rangle$ witness $\lambda_i = \text{ps-tcf}(P_i)$
(b) \( \mathcal{F}_{i,\alpha} \neq \emptyset \).

[Why? Exists by clause (d) of the assumption, for clause (b) recall [Sh:938, 5.6].]

By clause (c) of the assumption let \( \mathcal{G} \) be such that

\[ (*)_2 \quad (a) \quad \mathcal{G} = \{ \mathcal{G}_\beta : \beta < \lambda \} \text{ witness } \lambda = \text{ps-tcf}(\prod_{i \in Z} \lambda_i, <_E) \]

(b) \( \mathcal{G}_\beta \neq \emptyset \) for \( \beta < \lambda \).

Now for \( \beta < \lambda \) let

\[ (*)_3 \quad \mathcal{F}_\beta = \{ f : f \in \prod_{i \in Z} P_i \text{ and for some } g \in \mathcal{G}_\beta \text{ we have } i \in Z \Rightarrow f(i) = \mathcal{F}_{i,g(i)} \} \]

(*) the sequence \( \langle \mathcal{F}_\beta : \beta < \lambda \rangle \) is well defined (so exists).

[Why? Obviously.]

\[ (*)_5 \quad \text{if } \beta_1 < \beta_2, f_1 \in \mathcal{F}_{\beta_1} \text{ and } f_2 \in \mathcal{F}_{\beta_2} \text{ then } f_1 <_D f_2. \]

[Why? Let \( g_\ell \in \mathcal{G}_{\beta_\ell} \), witness \( f_\ell \in \mathcal{F}_{\beta_\ell} \) for \( \ell = 1, 2 \). As \( \beta_1 < \beta_2 \) by \( (*)_2 \) we have \( B := \{ i \in Z : g_1(i) < g_2(i) \} \in E \). For each \( i \in B \) we know that \( g_1(i) < g_2(i) < \lambda_i \) and as \( f_1(i) \in \mathcal{F}_{i,g_1(i)}, f_2(i) \in \mathcal{F}_{i,g_2(i)} \); hence recalling the choice of \( \langle \mathcal{F}_{i,\alpha} : \alpha < \lambda_i \rangle \), see \( (*)_1 \), we have \( f_1(i) <_{P_i} f_2(i) \). As \( B \in E \) and \( f_1, f_2 \in \prod_{i \in Z} P_i \), it follows that \( f_1 <_D f_2 \).]

\[ (*)_6 \quad \text{for every } \beta < \lambda \text{ the set } \mathcal{F}_\beta \text{ is non-empty.} \]

[Why? Recall \( \mathcal{G}_\beta \neq \emptyset \) by \( (*)_2(b) \) and let \( g \in \mathcal{G}_\beta \). As \( \langle \mathcal{F}_{i,g(i)} : i \in Z \rangle \) is a sequence of non-empty sets (recalling \( (*)_2(b) \)), and we are assuming ACZ there is a function \( g \in \prod_{i \in Z} \mathcal{F}_{i,g(i)} \) so \( g \in \mathcal{G}_\beta \), so \( \mathcal{F}_\beta \neq \emptyset \) as required.]

\[ (*)_7 \quad \text{if } f_* \in \prod_{i \in Z} P_i \text{ then for some } \beta < \lambda \text{ and } f \in \mathcal{F}_\beta \text{ we have } f_* < f \mod D. \]

[Why? For each \( i \in Z \) let \( \alpha_i = \min \{ \alpha : \beta < \lambda_i \} \); there is \( g \in \mathcal{F}_\alpha \) such that \( f_*(i) <_{P_i} g \), clearly well defined so \( \bar{a} = \langle \alpha_i : i \in Z \rangle \) exists. By the choice of \( \mathcal{G} \) there are \( \beta < \lambda \) and \( g \in \mathcal{G}_\beta \) such that \( \bar{a} \in g \). Recalling \( \mathcal{F}_\beta \neq \emptyset \) choose \( f \in \mathcal{F}_\gamma \), it is as required.]

Together we are done proving the theorem. \( \square \)

\[ \{ h10 \} \]

Conclusion 4.4. The third composition theorem: assume ACZ and \( \kappa \geq \lambda \).

We have \( \lambda < \text{Depth}^+(\prod_{(i,j) \in Y} P_{i,j}, <_D) \) and \( D \) is a \( \kappa \)-complete filter on \( Y \) when?

Proof. Combine the proof of 2.6 and 4.3. \( \square_{4.4} \)

ADD 39A? NOT SENT
§5. Private Appendix

§(3D) Concluding Remarks, pg.??

[Comments to [Sh:938].]

Old Proof of 2.2 moved from pgs.18,19:

First for the “only if” direction, assume \( ps \)-\( tcf(\Pi \alpha, <_D) \) is well defined and call it \( \lambda \).

Let \( \mathcal{F} = (\mathcal{F}_\alpha : \alpha < \lambda) \) witness \( \lambda = ps-tcf(\alpha, <_D) \). For \( f \in \bigcup \{ \mathcal{F}_\alpha : \alpha < \lambda \} \) let \( f^{[*]} \in Y \text{Ord} \) be defined by \( f^{[*]}(s) = \sup\{ f(t) : t \in s/E \} \). Clearly \( f^{[*]} \in \Pi \alpha \) as \( t \in Y \Rightarrow cf(\alpha_t) \geq hrtg(Y) \) by clause (a) of the assumption. Let \( \mathcal{F}_\alpha^{[*]} = \{ f^{[*]} : f \in \mathcal{F}_\alpha \} \) for \( \alpha < \lambda \) so \( (\mathcal{F}_\alpha^{[*]} : \alpha < \lambda) \) exists and \( \mathcal{F}_\alpha^{[*]} \subseteq \Pi \alpha \). Also \( f_1 \in \mathcal{F}_{\alpha_1} \wedge f_2 \in \mathcal{F}_{\alpha_2} \wedge \alpha_1 < \alpha_2 < \lambda \Rightarrow f_1 <^D f_2 \Rightarrow f_1 \leq_D f_2 \Rightarrow f^{[*]} \leq_D f^{[*]} \) hence \( \alpha_1 < \alpha_2 < \lambda \wedge f_2 \in \mathcal{F}_\delta^{[*]} \wedge f_2 \in \mathcal{F}_\lambda^{[*]} \) means \( f^{[*]} \in \bigcup \{ \mathcal{F}_\alpha^{[*]} : \alpha < \lambda \} \).

Now apply 2.1, getting \( Y_1', Y_2' \) as there, but by the choice of \( \mathcal{F} \) necessarily \( Y_1' = \emptyset \mod D \). Hence for some club \( E \) of \( \lambda, (\mathcal{F}_\alpha^{[*]} : \alpha \in E) \) is \( <_D \)-increasing cofinal in \( \Pi \alpha \).

Lastly, for \( f \in \bigcup \{ \mathcal{F}_\alpha^{[*]} : \alpha \in E \} \) let \( f^{[*][*]} \in \bigcup \{ \mathcal{F}_\alpha^{[*]} : \alpha < \lambda \} \) be defined by \( f^{[*][*]}(t/E) = f(t) \), well defined as \( f(t/E) \) is constant. Let \( \mathcal{F}_\alpha^{[*][*]} = \{ f^{[*][*]} : f \in \mathcal{F}_\alpha^{[*]} \} \) for \( \alpha \in E \). Easily \( (\mathcal{F}_\alpha^{[*][*]} : \alpha < E) \) witness the desired conclusions, that is, \( ps-tcf(\Pi \alpha_1, <_{D_1}) \) is well defined and equal to \( \lambda \), so we have proved the “only if” implication.


Those are comments to [Sh:938].

Definition 5.1. We say \((\Pi \alpha, <_{D_1})\) has weak \( \kappa \)-true cofinality \( \delta \), omitting \( \kappa \) means \( \kappa = \aleph_0 \), if there is some witness or \((\mathcal{D}, f)\)-witness \( \mathcal{F} \) which means:

(a) \( \mathcal{D} \subseteq \{ D : D \text{ an } \kappa \text{-complete filter on } Y \text{ extending } D \} \) is not empty
(b) \( D_{\alpha} = \cap\{ D : D \in \mathcal{D} \} \)
(c) \( \mathcal{F} = (\mathcal{F}_{D_{\alpha}} : D \in \mathcal{D}, \alpha < \delta) \)
(d) \( (\mathcal{F}_{D_{\alpha}} : \alpha < \delta) \) witness \((\Pi \alpha, <_D)\) has pseudo-true-cofinality \( \delta \).

Definition 5.2. \( \delta = \text{wtfc}_\kappa(\Pi \alpha, <_{D_1}) \) means \((\Pi \alpha, <_{D_1})\) has weak \( \kappa \)-true cofinality \( \delta \) and \( \delta \) is minimal (hence a regular cardinal).

Discussion 5.3. 1) Why do not ask \( \delta \) to be regular always? We may consider a sequence of \( \delta \)’s and as in id - cf, \( \alpha \) in [Sh:1005].
2) Can we (ZF + DC + AC \_\_\_) prove [Sh:460], using \( ps - T_D(\alpha) \)? Use [Sh:460, §1].
3) Can we generalize the proof of [Sh:829, §1] using \( ps - T_D(f) \)? We get \( \lambda \) is \( ps - T_D(\prod_{i < \kappa} A_i), \kappa < \mu \) as witnessed by \((\mathcal{F}_\alpha^{[*]} : \alpha < \lambda)\), but toward contradiction we have \( D_{\alpha} \in \text{Fil}^1_{\kappa_{\alpha}}(\kappa_{\alpha+1}) \).

Remark 5.4. For \( D \in \text{Fil}^1_{\kappa_{\alpha}}(Y), ps - T_D(f) \) is closely related to \( \sup\{ ps - T_{D_1}(f) \} \). \( D_1 \) is a filter on some \( \theta < hrtg(Y) \) such that \( D_2 \leq_{RK} D \) so natural to define \( ps - T_D \).
Definition 5.5. 1) Assume $\mathcal{D}_1$ is a set of filters and let $\text{prj}(\mathcal{D}_1)$ be

$$\{D_2 : \text{ for some } D_1 \in \mathcal{D}_2, \mu < \text{hrtg}(\text{Dom}(D_2)) \text{ and } h : \text{Dom}(D_1) \to \mu \text{ we have } D_2 = h(D_1)\}.$$ 

2) Let $\text{ps} - T_D(\bar{\alpha}) = \sup\{\text{ps} - T_D(\bar{\alpha}) : D \in \mathcal{D}\}.

Claim 5.6. Let $\mathcal{D}_1$ be a set of $\aleph_1$-complete filters, $\mathcal{D}_2 = \text{pry}(\mathcal{D}_1)$, Then the following cardinals are $S$-almost equivalent where $S = \text{Fil}_{\aleph_1}(\mathcal{D}_1) = \cup\{\text{Fil}_{\aleph_1}(D_1) : D_1 \in \mathcal{D}_1\}$:

(a) $\text{ps} - T_{\mathcal{D}_1}(\bar{\alpha})$
(b) $\text{ps} - T_{\mathcal{D}_1}(\bar{\alpha})$
(c) FILL.
§ 6. ON RGCH WITH LITTLE CHOICE

If we assume (ZF + DC of course and) $\text{Ax}_4$ can we prove a theorem parallel to the RGCH from [Sh:460]? See [Sh:1005]. We like to prove such a result just that assuming DC; so if we have enough cases of IND, we use [Sh:955, §(3B)] if not, assume for every $\kappa$ we have $p$ more or less as in [Sh:938, 3.1], i.e. omitting the ranks such that $(\forall \lambda) \text{Ax}_0^{\lambda, \mu, \kappa, 0}$ all $D \in D_p$ are $\mu_1$-complete. We try to repeat.

So trying to imitate, e.g. [Sh:829] in the main case we have $d \in \mathbb{P}, \bar{\alpha} \in \mathcal{Y}(d)_{\alpha}$. Without loss of generality $(\forall \ell \in Y_\mathbb{D})[(\alpha_\ell, 1)]$ is as required, using the induction hypothesis.

For $s \in Y_d$, using $\text{cf} : [\alpha]^{\kappa}(P) \rightarrow [\alpha]^{\kappa}(P)$ which exists by $\text{Ax}_0^{\lambda, \ldots}$ we have $(f_\mathbb{E}, Y, \beta) : \beta < \alpha_\mathbb{E} : \mathbb{E} \in D_{\geq 1}, \beta, y \in \text{Fil}_{\alpha_{(1,p)}}(D_{\mathbb{E}}))$ such that: if $d \in D_{\geq 1}, s \in Y_d, f \in \mathcal{Y}[e](\alpha_s)$ then for some set $(\langle y_i, \beta_i \rangle : i < \zeta_p < \kappa(i, p)), \bigwedge_{t \in \mathbb{Y}_a} f(t) = f_{\mathbb{E}, y, \beta_i}(s)$.

Why? Given $(e, f)$ if there is no such sequence, we can find a filter $\kappa(i_1, p)$-complete filter on $Y_\mathbb{E}$ such that...

But we need more: given $f = \langle f_s : s \in Y_\mathbb{E} \rangle, f_s \in \mathcal{Y}[e]\alpha_s$ and we like to consider all $f_s$ simultaneously, say find $(\langle y_{i_1}, \beta_{s_1} \rangle : s \in Y_s, i < i_1)$ as above.

If we have $d \in D_p \Rightarrow \text{AC}_{Y_\mathbb{D}}$ this can be done. So the status of $\text{Ax}_0^{\lambda}$ change: given $p$ we say? If $(\forall x A_\mu^{\lambda, \ldots})$ fix. If not, then for some $\lambda(*)$ we have $i < \text{cf}(\mu) \Rightarrow \neg \text{Ax}_0^{\lambda, \kappa(i_1, p), \kappa}$. (can determine the other cases).

We get

\begin{equation}
(*) \text{ if } \theta < \mu_p \text{ then } I = [\lambda]^{<\theta} \text{ and } D_n = \text{dual}(I) \text{ then } \text{IND}(\langle I, D : n \rangle).
\end{equation}

**Question 1.** Can we use $([\lambda]^{n(i,p)}, I_{\lambda, \kappa(i,p)}) : n < \omega$? Can we avoid using $\langle \text{AC}_{\kappa(i,p)} : i < \text{cf}(\mu) \rangle$? Given $f = \langle f_\alpha : s \in Y_d \rangle$ we can consider $Y_s = Y_d \times Y_e$ and for every sequence $x = \langle (y_s, f_s) : s \in Y_d \rangle, f_s \in \mathcal{Y}[e](\alpha_s)$ let $\alpha_x = \{ (s, t)(Y_d \times Y_e) : f_s(t) = f_s(t) \}$.

Now we may look at $(R \text{ not too large})$

\[ D^* = \{ Z \subseteq Y_s : \text{ there is } \langle x_r : r \in P \rangle \text{ such that } Y \setminus Z \subseteq \bigcup_{r \in R} A_x_r \}. \]

So $D_R$ is a $\kappa_p(i_1, p)$-complete filter.

Let $D^*_R,s$ be the projection of $D^*_R$ to $\{ s \} \in Y_e$. Clearly it is the filter defined by $(\alpha_s, f_s)$.

Recall [Sh:835, 2.2].

**Definition 6.2.** We say $\text{Ax}_0^{\lambda, \kappa, \mu}$ when some $\text{cf}$ exemplifies it which means:

\begin{enumerate}
  \item $\text{cf} : [\alpha]^{<\kappa} \rightarrow [\alpha]^{<\mu}$
  \item $u \subseteq \text{cf}(u)$
  \item $u_1 \subseteq u_2 \Rightarrow \text{cf}(u_1) \subseteq \text{cf}(u_2)$
  \item $\alpha_n : n < \omega \in ^*\alpha$ such that $\alpha_n \notin \text{cf}(\alpha_k) : k > n$.
\end{enumerate}

**Definition 6.3.** We say $x$ is a filter system (as in [Sh:938, 3.1], add $\kappa_p.d, \text{Rep}_{\kappa(d,p)}(D_p)$ but no $\text{rk}$

\begin{enumerate}
  \item $\mu$ is singular
\end{enumerate}
Claim 6.5. Assume \( F \subseteq \sup_{\kappa} \kappa \) and:

(a) \( f \in \mathcal{F} \) represents \( \beta \)
(b) \( D_d \) is a filter on \( Y_d \)
(c) \( \kappa_i = \kappa_{p_i} = \kappa(i, p) \) is a cardinal < \( \mu \)
(d) \( \kappa < \kappa \) on \( \kappa \)
(e) \( \kappa \) is strong limit
(f) \( \kappa < \kappa \) on \( \kappa \)

\( \beta \) and \( \delta \) each \( d \in D \) is (or just we can compute from it) a pair \( (Y, D) = (Y_d, D_d) \) such that:

(a) \( \hrtg(Y_d) < \mu \), on \( \hrtg(-) \) see Definition ??
(b) \( D_d \) is a filter on \( Y_d \)
(c) \( \alpha_i < \kappa_{p_i} < \kappa_{p_i} \)
(d) \( \forall \sigma < \mu \) \( |\exists i < cf(\mu)| (\sigma < \kappa_{p}, \sigma \) a filter on \( \sigma \)
(e) \( \mu \) is strong limit
(f) \( \kappa < \kappa \) on \( \kappa \)

\( \Sigma \) is a function with domain \( D \) such that \( \Sigma(d) \subseteq D \)
\( \alpha_i < \kappa_{p_i} < \kappa_{p_i} \)
\( \kappa \) is \( \kappa \)-complete filter on \( \kappa \)
\( \theta \) is \( \theta \)-complete filter on \( \theta \)

Definition 6.4. Let \( \mathcal{A}_{\alpha, p}^0 \) means that: there is a function \( \mathcal{I} \) satisfying (a)-(c) of 6.2 and:

(a) \( \mathcal{I} \) is a function from domain \( D \) onto \( cf(\mu) \)
(b) \( \mathcal{I} \) is \( \mathcal{I} \)-complete on \( \mathcal{I} \)
(c) \( \mathcal{I} \) is \( \mathcal{I} \)-complete on \( \mathcal{I} \)
(d) \( \mathcal{I} \) is \( \mathcal{I} \)-complete on \( \mathcal{I} \)

Claim 6.5. Assume \( \mathcal{A}_{\alpha, p}^0 \subseteq \mathcal{A}_{\alpha, \bar{\alpha}} \), \( D \) a filter on \( Y \) and \( \alpha \in \gamma(\alpha + 1) \).
Then \( p_s - o - \text{Depth}_D(\bar{\alpha}) \leq S o - \text{Depth}_D(\bar{\alpha}) \).

\( \mathcal{A}_{\alpha, p}^0 \subseteq \mathcal{A}_{\alpha, \bar{\alpha}} \)
\( \Sigma(d) \subseteq D \)
\( \alpha \) is \( \alpha \)-complete on \( \alpha \)

Proof. Let \( \mathcal{I} \) witness \( \mathcal{A}_{\alpha, p}^0 \), and assume \( u \in [\alpha]^{\text{chr}(\mu)} \) \( \Rightarrow \mathcal{I}(\alpha) \in [\alpha]^{< \mu} \). Let \( \kappa = \sup\{ |\mathcal{I}(\alpha)| : \alpha \in [\alpha]^{\text{chr}(\mu)} \} \). For transparency as \( 0 \notin \text{Rang}(\bar{\alpha}) \), assume \( \beta \notin p_s - o - \text{Depth}_D(\bar{\alpha}) \), so there is a sequence \( \{ \mathcal{F}_\beta : \beta < \beta \} \) witnessing it so \( f \in \mathcal{F}_\beta \Rightarrow f < \alpha \).

For each \( \beta < \beta \), \( \mathcal{F}_\beta \subseteq \mathcal{F}_\beta \), \( f \in \mathcal{F} := \cup \{ \mathcal{F}_\beta : \beta < \beta \} \), there is \( y \in \text{Rep}_\alpha(D) \) which represents \( f \) which means:

(a) \( y \equiv (Y, D, A, h) \)
(b) \( h \) is a function with domain \( A \) such that: \( h(t) = \text{otp}(f(t) \cap \mathcal{I}(f(s)) \mid s \in A) \)
(c) \( f \) is a function from domain \( A \) such that:

\[ \text{if } f_1, f_2 \in \mathcal{F}_\beta \text{ are represented by } y \text{ then } f_1[A] = f_2[A] \]

Now

\[ |\text{Rep}_\alpha(D)| = |D \times Y \kappa| \]
\[ \text{for } y \in \text{Rep}_\alpha(D) \text{ let } \mathcal{A}_y = \{ \beta < \beta : \text{ there is } f \in \mathcal{F}_\beta \text{ represented by } y \} \]
\[ \{ \mathcal{A}_y : y \in \text{Rep}(D, \kappa) \} \text{ is well defined} \]
\[ \beta_\ast = \bigcup \{ \mathcal{Y}_y : y \in \text{Rep}(D, \kappa) \} \]
for \( y \in \text{Rep}_\kappa(D) \) and \( \alpha \in \mathcal{Y}_y \) let \( g_{y, \beta} \) is the unique member of \( \Pi_\ast \) such that: if \( f \in \mathcal{F}_\beta \) is represented by \( y \) then \( g_{y, \beta} | A_y = f | A_y \) and \( g_{y, \beta}(t) = 0 \) for \( t \in Y \setminus A_y \).
\( \langle g_{y, \beta} : \beta \in \mathcal{Y}_y \rangle \) is \( \triangleleft_D \)-increasing sequence in \( \Pi_\ast \).

\[ \square \]

**Claim 6.6.** Assume \( D_\ast \) is a \( \kappa \)-complete filter on \( Y, \kappa \geq \aleph_1 \) and \( \text{Ax}_D^0, \gamma \leq \kappa \leq \kappa \) so \( \gamma \) acts as an ordinal and \( \mu = \chi \) and \( S = \text{Fil}^1(D_\ast, \gamma), \) so \( \gamma \) fixes the order type of \( c\ell(\{f(s) : s \in Y\}) \) and \( \mathbb{D} = \{\text{dual}(J[f, D]) : f \in Y^{\text{Ord}}\} . \)

The following cardinals are \( S \)-almost equal for \( \bar{\alpha} \in Y^{\text{Ord}} \)

\[ \begin{align*}
(a) & \ 0 - \text{Depth}_{\mathcal{D}}(\bar{\alpha}) \\
(b) & \ ps - 0 - \text{Depth}(\bar{\alpha}) \\
(c) & \ ps - \text{T}_{\mathcal{D}}(\bar{\alpha}) \\
(d) & \sup \{\text{rk}_{\mathcal{D}}(\bar{\alpha}) + 1 : D \in \mathbb{D} \}.
\end{align*} \]

**Proof.** FILL.

\[ \square \]

**Theorem 6.7.** Let \( p = (\mathbb{D}, \mu, \ldots) \) be a filter system and \( \langle \forall \alpha \rangle (\forall \infty, i < \text{cf}(\mu)) \langle \text{Ax}_D^0, \kappa_{<i}, \delta_i > \rangle \).

Assume further \( \text{AC}_{\kappa(i, p)} \) for \( i < \text{cf}(\mu) \). For \( d \in \mathbb{D}_p \) let \( \ast \) obey??

For every \( \alpha \) (question: or \( \lambda ? \) ?) such that \( \text{Ax}_D^0, \kappa_{<i}, \delta_i > \) there is \( i < \text{cf}(\mu_p) \) such that: if \( d \in \mathbb{D}_\alpha \) then the following as \( \text{Rep}_{\kappa(d, p)}(D_d) \)-almost equal

\[ \begin{align*}
(a) & \ \alpha \\
(b) & \ o - \text{Depth}_{D_d}(\alpha) \\
(c) & \ ps - o - \text{Depth}_{D_d}(\alpha) \\
(d) & \ ps - \text{T}_{D_d}(\alpha) \\
(e) & \ \text{rk}_{D_d}(\alpha).
\end{align*} \]

**Remark 6.8.**
1) For (b),(c) their being almost equal we already know, see \( \S(3A) \).
2) Use \( \text{rk}_d \) or \( \text{rk}_{D_d} \)? Presently, \( \text{rk}_d \).

**Proof.**

**Case 1:** \( \alpha < \mu \)

Obvious.

**Case 2:** \( \alpha < \mu^+ \)

Easy.

**Case 3:** \( \alpha \geq \mu^+ \) and for \( d \in \mathbb{D} \) and \( \bar{\alpha} \in Y^{[d]} \alpha \) do we have \( \alpha < ps - 0 - \text{Depth}(\bar{\alpha}) \).

Easy by the definitions.

**Case 4:** as ab there are \( d \in \mathbb{D} \) and \( \bar{\alpha} \in Y^{[d]} \alpha \).

Choose \( \langle g_{\ast}^\alpha : \varepsilon < \alpha \rangle \) witness \( \alpha < o - \text{Depth}_{\mathcal{D}}(\alpha) \) or more: such that \( J[g_{\ast}^\alpha, D_\alpha] \)

is constant; \( D_\alpha \) the dual.

For \( s \in Y_d \) clearly \( i(s) = \min\{i < \text{cf}(\mu) : \text{for } \alpha_i, i \text{ is as required in the claim}\} \).

Clearly \( i(s) < \text{cf}(\mu) \) is well defined by the induction hypothesis

\[ (*) \text{ without loss of generality for some } i_0, A = \{s \in Y_d : i(s) = i_0\} \in D_d. \]

[Why? See Definition ??, clause \( (*) \)??]

We choose \( i_1 \in (i_0, \text{cf}(\mu)) \) such that
(§) FILL.

Now let $e \in D_{\geq 1}$ and $\beta_e < 0 - \text{Depth}^+(\alpha)$ and let $(f_\beta : \beta < \beta_e)$ witness this.

Define $(f_{\beta,s} : \beta < \beta_e, s \in Y_\delta)$ with $f_{\beta,s}$ the function from $Y_e$ into $\alpha_s$ defined by $f_{\beta,s}(t) = g_{f_\beta(t)}(s)$.

let $\langle \xi_{\beta,s} : \beta < \beta_e, s \in Y_\delta \rangle$ be defined by

- $\xi_{\beta,s} = \text{rk}_{D_e}(f_{\beta,s})$.

Now

(§) $\xi_{\beta,s} < \alpha_s$.

Lastly, let $\langle \xi_\beta : \beta < \beta_e \rangle$ be defined by

- $\xi_\beta = \text{rk}_{D_\delta}(\eta_\beta)$ where $\eta_\beta = \langle \xi_{\beta,s} : s \in Y_\delta \rangle$.

As $\text{rk}_{D_\delta}(\alpha) = \alpha$ and $\langle \xi_{\beta,s} : s \in Y_\delta \rangle <_{D_\delta} \alpha$ we have

(§) $\xi_\beta \leq \alpha$ (or $\xi_\beta < \alpha$).

Now for each $\xi \leq \alpha$ let

(§) $u_\xi = \{ \beta < \beta_e : \xi_\beta = \xi \}$.

It suffices (check formulation) to prove

$\prod [u_\xi] < \text{hrg}(\text{Fil}^\alpha_\eta(D_\delta) \times \text{Fil}^\beta_\alpha(D_\delta))$.

Why? For every $\beta < \beta_e$ let $x^\beta_\delta = (J(\xi_{\beta,s}) : s \in Y_\delta), D_\delta), x^\beta_\delta = J([g^\beta_{f_\beta,s}(t) : t \in D_\delta), D_\delta]) : s \in Y_\delta)$. Let $x^\beta_\delta = J(f^\beta_{\delta,s}, D_\delta) : t \in Y_e)$.

Now

- if $\beta_1 < \beta_1 < \beta_e$ and $\langle \xi_{\beta_1}, x^\beta_\delta_{\beta_1} \rangle = (\xi_{\beta_2}, x^\beta_\delta_{\beta_2})$ then $\xi_1 = \xi$.

[The delicate point: how much should $i_1$ or comp(e) be above $d$? or too similar to

[Sh:938, §2].]

* * *

Let $J = J([\xi_{\beta,s} : s \in D_\delta], D_\delta), J_\delta = J([g^\delta_{f_\delta,s}(t) : t \in D_\delta])$.

First, note that as $\xi_{\beta_1} = \xi_{\beta_2}$, clearly $A = \{ s \in Y_\delta : \xi_{\beta_1} = \xi_{\beta_2} \} = Y_\delta$ mod $J$.

Also for every $s \in A$ we have $B_s := \{ t \in Y_e : g_\delta_{f_\delta,s}(t) := g^\delta_{f_\delta,s}(t) \} = Y_e$ mod $J$.

Is $i_1$ large enough?

* * *

- $A_{\beta_1, \beta_2} = \{ t \in Y_e : f_{\beta_1}(t) < f_{\beta_2}(t) \} = Y_e$ mod $D_e$
- for $t \in Y_\delta : A^t_{\beta_1, \beta_2} = \{ s \in Y_\delta : g_{f_\delta,t}(s) < g_{f_{\beta_2,t}}(s) \}$.

So

- $A_{\beta_1, \beta_2} = Y_e$ mod $D_e$
- $A^t_{\beta_1, \beta_2} = Y_e$ mod $D_\delta$ for $t \in A_{\beta_1, \beta_2}$.

As $\text{hrg}(D_\delta) < \text{comp}(D_\delta)$ by the choice of $i_2$ and "$e \in D_{\geq 1}$", for some $A_s \in D_\delta$ we have
• $B_* = \{ t \in Y_* : A_{\beta_1, \beta_2}^t = A_* \} \neq \emptyset \text{ mod } J$ where $J = J[f_{\beta_1}, D_e] = J[f_{\beta_2}, D_e]$.

Hence

• for every $s \in A_*$, $t \in B_*$ we have $g_{f_{\beta_1}(t)}(s) < g_{f_{\beta_2}(t)}(s)$. \hfill \square

* * *
§ 7. **Private Appendix**

**Bounds**

Saharon: check with [Sh:F1039]

Moved from pg.2:

§ 4 **Bounds**

§(4B) **Minimality for ps-0-Depth**

[We define “$f$ is $(Y, D, \gamma)$-ps-0-Depth$^{(+)}$-minimal and variants (clarify which we deal with). Note existence and how it commutes with $(D + A_i : i < \partial) \mapsto \langle D + \bigcup_{i<\partial} A_i \rangle$.]

§(4C) **Depth is regular and obtained**

[A main claim is that: $f \in Y^{\text{Ord}}, (Y, D, \lambda^+)$-minimal then $\{y : f(y) \text{ is regular}\} \in D^+$ (see 7.8), existence 7.10.]

§(4D) **Weakly inaccessible (to [Sh:F1039])**

[We like to show that if $\aleph_0 < \text{cf}(\mu) < \mu$ and $\mu$ is not the accumulation point of the class of inaccessible cardinals then there is no (weakly) inaccessible cardinals $\in (\mu, \text{pp}^{+}_{\text{S1-com}}(\mu))$. This will be the main result of this section. In [Sh:F1039] we shall get a similar theorem with somewhat different assumptions.]

§ 5 **Try to immitate [Sh:460], pg. 28 [to [Sh:F1039]? till the end?], pg.29**

[Check carefully.]

§ 6 **Absoluteness for non-well founded ultra-powers, pg.36**

§ 7 **More pcf with little choice, a try, pg.39**

§(7A) **Semi-filter**

[Is it helpful to use semi-filters in [Sh:938, §3, §4]?]

§(7B) **Games and Rank, pg.40**

[This is an alternative to the present [Sh:F1039] using games and forcing.]

§(7C) **Various**

[In 11.1, 11.2 we show that investigating ps-tcf it is enough to consider $Y$ a cardinal. In 11.3 we note $\text{AC}_{\text{hrtg}(Y)=\text{hrtg}(Y)}$ successor. In ?? we (?) check.]

In 11.6 we show $\aleph_0 < \kappa = \text{cf}(\mu) < \mu \Rightarrow \text{rk}_{J^*}(\mu) > \mu^+$. In 11.5 we use pigeon $\perp$ hull for $J[f, D], \text{ nec?}]

§ 8 **More, pg.42-44**
§ 7(A). Replacing $\operatorname{rank}_D$ by $\operatorname{Depth}_D$ - [FILL].

In ZFC we know that, e.g. for $\mu$ singular strong limit of uncountably cofinality, if $\lambda \in (\mu, 2^\mu]$ is weakly inaccessible then weakly inaccessible are unbounded below $\mu$. We like to prove such results with little choice, for this we look at the minimal case.

{c23}

**Definition 7.1.** 1) We say $f \in \gamma \text{Ord}$ is $(Y, D, \gamma)$-ps-o-Depth$^+$-minimal when (may omit $Y, D$ in this section), $\gamma \leq \text{ps-o-depth}^+_D(f)$ but for no $g \in \gamma \text{Ord}$ satisfying $g < f \mod D$ do we have $\gamma \leq \text{ps-depth}_D(f)$.

2) Similarly for other variants.

{c25}

**Claim 7.2.** 1) If $\gamma \leq \text{ps-o-Depth}_D^+(f)$ where $f \in \gamma \text{Ord}$ then $\text{ps-o-Depth}_D^+(f)$ is such that $g \leq f \mod D$.

2) Similarly for other variants.

{c26}

**Claim 7.3.** If $f = g + 1 \in \gamma \text{Ord}$ then $\text{ps-o-Depth}_D^+(f) = \bigcup \{\alpha + 2 : \alpha < \text{ps-o-depth}_D^+(g)\}$.

{c27}

**Claim 7.4.** 1) If $f$ is $(Y, D, \lambda)$-ps-depth-minimal and $\lambda$ is a limit ordinal then $\{y \in Y : f(y) \text{ limit} \} \in D^+$.

2) If $\gamma = \delta + 1, \delta$ a limit ordinal and $f$ is $(Y, D, \gamma)$-ps-minimal, then $\{y \in Y : f(y) \text{ a limit ordinal} \} \neq \emptyset \mod D$.

**Proof.** Fill more? \(\square\)

{c29}

**Definition 7.5.** Let $f \in \gamma \text{Ord}$.

1) Let $J_{\text{ps-o-depth}}[f, D] = \{A \subseteq Y : A = \emptyset \mod D \text{ or } A \in D^+ \text{ but } \text{ps-o-depth}_D(f) < \text{ps-o-depth}_D^+\}(f)$.

2) Similarly for other variants, but we write $\text{ps-o-depth}(+)$.

{c31}

**Claim 7.6.** 1) $\gamma \leq \text{ps-o-Depth}_{D+A_\ell}(f)$ for $\ell = 1, 2$ then $\gamma \leq \text{ps-depth}_{D+A_\ell}\cup_{A_\ell}(f)$.

2) $[\text{AC}_\sigma]$ If $D$ is $(\leq \partial)$-complete and $\gamma \leq \text{ps-Depth}_{D+A}(f)$ for $i < \partial$ and $A = \bigcup\{A_i : i < \sigma\}$ then $\gamma \leq \text{ps-o-Depth}_{D+A}(f)$.

3) $[\text{AC}_\kappa]$ If $D$ is $\kappa$-complete and $f$ is $\text{ps-Depth}$-minimal then $J_{\text{ps-o-depth}}[f, D]$ is $\kappa$-complete ideal disjoint to $D$.

**Proof.** FILL \(\square\)

§ 7(B). Depth is regular and obtained.

Recall

{c35}

**Definition 7.7.** We call $\lambda$ inaccessible when $\lambda$ is regular uncountable limit cardinal.

{c37}

**Claim 7.8.** $[\text{AC}_Y]$ Assume $\lambda$ is regular and $f \in Y \text{Ord}$ is $(Y, D, \lambda)$-ps-o-depth-minimal. Then $\{y \in Y : f(y) \text{ is regular} \} \neq \emptyset \mod D$.

**Remark 7.9.** The assumption is equivalent to $(Y, D, \lambda + 1)$-ps-o-Depth$^+$-minimal.

**Proof.** Assume that not, so without loss of generality

1. \((*)_1\) $f(y)$ is not regular for $y \in Y$
2. \((*)_2\) $f(y) > 0$ for $y \in Y$. 

(955)
Claim 7.10. \[ DC + AC \] If \( \text{cf}(\lambda) = \lambda > \text{hrtg}(\mathcal{P}(Y)) \) and \( \lambda^+ < \text{ps-Depth}^+_D(f) \) and \( \lambda \) is regular \( \geq \text{hrtg}(\mathcal{P}(Y)) \) then there is \( f_1 \) such that

\begin{enumerate}
  \item \( f_1 \in Y \text{ Reg} \)
  \item \( f_1 \leq f \mod D \)
  \item \( f_1 \) is \( \langle Y, D, \lambda^+ \rangle - \text{ps-Depth}^+\text{-minimal.} \)
\end{enumerate}
Remark 7.11. Use just 7.8 and the existence of minimality. So we can replace “regular” by any property which satisfies a parallel statement.

Proof. We try to choose \( (f_n, y_n, \iota_m) \) by induction on \( n \) such that

\[ \begin{aligned} 
(a) & \quad f_n \in Y \text{ Ord} \\
(b) & \quad f_0 = f \\
(c) & \quad n = m + 1 \Rightarrow f_n \leq f_m \\
(d) & \quad Y_n = \{ y \in Y : f_n(y) \text{ is regular}\} \cup \{ Y_m : m < n \} \text{ and } \iota_m = 1 \\
(e) & \quad i_n = 1 \text{ if } \iota_n \neq 2 \text{ if } Y_n \in D^+ \text{ and } f_n \text{ is } (Y, D, \lambda^+) - \text{ps-Depth}^+ - \text{minimal} \\
(f) & \quad \text{if } m < n \text{ and } \iota_m = 1 \text{ then } f_n|Y_m = f_m|Y_m \\
(g) & \quad \text{if } n = m + 1, Y_m \in D^+ \text{ and } \iota_m = 2 \text{ then } f_n < f_m \text{ mod } (D + Y_m) \\
(h) & \quad \text{if } n = m + 1, Z_m = Y\backslash Y_m \cup \{ Y_k : k < n \text{ and } \iota_k = 1 \} \in D^+ \text{ then } f_n < f_m \text{ mod } (D + Z_m). 
\end{aligned} \]

Each step is O.K. (for (h) by 7.6) and so by DC we can carry the inductive choice. In this case a \( D \) is \( \aleph_1 \)-complete, let \( Z \in D \) be the set of \( y \)'s such that all the relevant inequalities mentioned hold. As \( \{ f_n(y) : n < \omega \} \) is not decreasing, for some \( m, y \in Y_m \land \iota_m = 1 \), so \( Z' := \cup \{ Y_m : m < \omega, \iota_m = 1 \} \in D \) and let \( g \in Y \text{ Ord be } g|y_m = f_m|y_m \) if \( \iota_m = 1, g(y) = \aleph_1 \) otherwise.

Easily \( g \) is as required by 7.6. Check. \( \square \)

Conclusion 7.12. In 7.8 we can weaken the assumption - FILL.

Remark 7.13. The point is that we do not have to change the filter, hence the demand on “\( \lambda \) large enough is weaker”.

\[ \begin{aligned} 
(3D) & \quad \text{Weakly inaccessible (to [Sh:F1039]?)} \\
\end{aligned} \]

Claim 7.14. void\(\square + AC_{\mathcal{P}(Y)}(?)\]

Assume \( f \in Y \text{ Ord is } (Y, D, \lambda^+) - \text{ps-Depth}^+ - \text{minimal. If } \lambda \text{ is weakly inaccessible then } \{ t : f(t) \text{ is weakly inaccessible} \} \in D^+.\)

Proof. Let

\[ \begin{aligned} 
(*)_1 & \quad (a) \quad Y_0 = \{ t : f(t) = 0 \} \\
(b) & \quad Y_1 = \{ t : f(t) \text{ a successor ordinal} \} \\
(c) & \quad Y_2 = \{ t : f(t) \text{ a limit ordinal of cofinality } < f(t) \} \\
(d) & \quad Y_3 = \{ t : f(t) \text{ is regular cardinal which is a successor} \} \\
(e) & \quad Y_4 = \{ t : f(t) \text{ is weakly inaccessible} \}. 
\end{aligned} \]

Obviously

\[ \begin{aligned} 
(*)_2 & \quad \{ Y_\ell : \ell \leq 4 \} \text{ is a partition of } Y. 
\end{aligned} \]

By 7.6 without loss of generality

\[ \begin{aligned} 
(*)_3 & \quad (a) \quad \ell(*) \leq 4 \text{ and } Y_\ell(*) \in D \\
(b) & \quad \text{moreover } Y_\ell(*) = Y. 
\end{aligned} \]
The cases \( \ell(*) < 3 \) are easily discarded.

Also if \( \ell(*) = 4 \) then the desired conclusion holds, so we can assume \( \ell(*) = 3 \) and eventually will get a contradiction.

Choose \( f_\alpha \) such that

\( (\ast) \quad f_\alpha \in Y \text{ Card such that } (f_\alpha(y))^+ = f(y) \text{ for } y \in Y. \)

By the assumption on \( f \) we can find

\( (\ast) \quad \mathcal{F} = \langle \mathcal{F}_\alpha : \alpha < \lambda \rangle \text{ is } _D \text{-increasing, } \mathcal{F}_\alpha \subseteq \Pi f \text{ is non-empty.} \)

\( \square 7.14 \)

Discussion 7.15. By AC\( \forall \) and 3.1(?) we can get \( \lambda \leq \text{ps-Depth}_D(f_\ast) \). But does this suffice? Or can we do the regular for ps-Depth-minimal?

\%\%\{3E\} \quad \text{Higher rank (to [Sh:F1039]?)}

1) We like to repeat [Sh:g, V,VI], but there are some different points; fix \( \kappa = cf(\kappa) > \aleph_0 \), e.g. \( \aleph_1 \).

First, suppose that we have AC\( _{\varphi ^* (\kappa )} \), \( k \) large enough and \( \mathcal{H}(\chi ) \) we choose and we know that \( \text{rk}_E^k(f, E) < \infty \) for \( f \in ^* \chi \), does this imply the same for \( f \in ^* \text{Ord} \)? The remedy we take here is DC\( _{\kappa ^+} \). It is enough to use \( \text{rk}_E^{5/4} (f, E) \), so the “antagonist” can chose any “legal filter”.

2) Fix \( E = E^k \). Now if \( \lambda \) is regular (or less?) we can find \( \text{rk}_E^k(f_0, E) = \text{rk}_E^k(f_0, E) = \lambda \) or just \( \text{rk}_k^0 (f, E) \geq \lambda \). So for every \( \alpha < \lambda, E_\alpha := \{ E : E \geq E_\alpha \} \) and for some \( g < E(1, E) \) \( f_0 \) we have \( \text{rk}_E^k(g, E) = \text{rk}_E^k(g, E) = \alpha \). Hence for some \( E_1 \geq E_0 \), the set \( \mathcal{U} := \{ \alpha : E_1 \in E_\alpha \} \) is unbounded in \( \lambda \) (and has order type \( \lambda \)). For \( \alpha \in \mathcal{U} \) let \( \mathcal{F}_\alpha = \{ f < f_0 : \text{rk}_E^k(g, E) = \text{rk}_E^k(g, E) = \alpha \} \). So \( \langle \mathcal{F}_\alpha : \alpha \in \mathcal{U} \rangle \) is \( \kappa \)-increasing and let \( f_1 \leq f_0 \) be a \( \langle E(E_\alpha) \rangle \)-lub.

Hence (forgetting \( f_0 \)) we have \( \text{rk}_E^k(f_1, E) = \text{rk}_E^k \). Suppose we force by \( \mathcal{P} = \{ (E_\ast) : D \in E_\ast \} \) getting \( G, D[G] \) what is \( 3\pi f_1 / D \) [Maybe better: what is \( \text{hrtg}(\Pi f / D) \) for \( f' \in \Pi f_1 \)?]

Clearly \( \lambda > \lambda^* \). Toward contradiction assume \( \lambda_1 = \lambda_2 = \text{cf}(\lambda) \) or just \( \lambda_2 \geq \text{Suc}_{\text{rl}(\kappa)}(\lambda_1) \) \( \lambda_1 > \lambda, E \not\vDash \) “\( \text{hrtg}(\Pi f_1 / D) > \lambda_1 \) say \( F \) witness this. Hence for \( \alpha < \lambda_2 \) the following set is non-empty

\[ D_{\alpha, \beta} = \{ D : (E_1, D_1), D \not\vDash (\exists g \in (\pi f_1)^Y) (F(g)) = \alpha \text{ and } \text{rk}_k^k(g, E^k) = \text{rk}_E^k(g, E) = \beta \}. \]

So for some \( E_2 = (E_1)[D_2] \), the set \( \mathcal{U}_2 = \{ \alpha < \lambda_1 : (\exists \beta) D_2 \in D_{\alpha, \beta} \} \) has order type \( \lambda_1 \).

Let \( \beta_\alpha = \min \{ \beta : D_2 \in D_{\alpha, \beta} \} \) for \( \alpha \in u_2 \). Let \( \mathcal{F}^2_\alpha = \{ g \in (\pi f_1)^Y : D_2 \not\vDash F(g) = \alpha, \text{rk}_E^k(g, E) = \beta_\alpha \}. \)

Again \( \langle \mathcal{F}^2_\alpha : \alpha \in \mathcal{U}_2 \rangle \) is increasing.

3) Similarly with \( \text{rk}_E^k(f_0) = \lambda \) forcing with \( (D^+, \supseteq) \).

4) Now go back to [Sh:460]. The above is just going back to [Sh:386], [Sh:333], an avenue I had tried and failed, but why?

5) Instead of DC\( _{\kappa ^+} \) we may consider a definition of a filter on \( [\lambda]^\kappa \) with \( \theta \geq 2\mathfrak{d}(\kappa) \) or so; we do not use real sets just definitions of the sets used. Now to prove in the game \( \mathcal{O}_\kappa (\lambda) \) the protagonist wins, we use \( \chi \) such that \( A \subseteq \lambda, |A| \rightarrow K[\varphi] \rightarrow \chi \rightarrow (\lambda)^{<\omega} \).
§ 8. A try on immitating [Sh:460]

**Theorem 8.2.** For every $\lambda$ there is $n \leq \omega$ such that for no set $a \subseteq \lambda \cap \text{Reg} \setminus \mu_n$ of cardinality $< \mu$ and $\mu_n$-complete ideal $I$ on $a$ do we have $\text{ps-tcf}(\Pi a, <_I)$ is a well defined (regular) cardinal $\geq \lambda$, when :

(a) $\langle \mu_n : n < \omega \rangle$ is increasing with limit $\mu$

(b) $\text{AC}_{\mu_n}$

(c) $\text{AC}_{\mathcal{P}(\mu_n)}$

(d) $\text{DC}$?

(e) $\text{hrtg}(\mathcal{P}(\mu_n)) < \mu_{n+1}$ moreover $\text{hrtg}(\text{Fil}^4(\mu_n)) < \mu_{n+1}$.

**Proof.** We prove this by induction on $\lambda$; there is such $n$ let $n(\lambda)$ be the minimal such $\lambda$.

Case 1: $\lambda < \mu$

Easy: even for $n = 0$, as if $\kappa = \text{cf}(\kappa) > \mu$ and $a \subseteq \text{Reg} \cap \lambda$ so trivially $|a| < \mu$ and $I$ is $\aleph_1$-complete ideal and $\langle P_\alpha : \alpha < \lambda \rangle$ is witness to $\lambda = \text{ps-tcf}(\Pi a, <_I)$ then $\lambda_* < \text{hrtg}(\mathcal{P}(\mathcal{P}(\sup(a))))$ (can use less)?.

Case 2: $\lambda = \mu$

Let $n = 1$ and use the $\aleph_1$-completeness to get that without loss of generality $a$ is bounded in $\lambda$ and use the proof of Case 1.

Case 3: $\text{cf}(\lambda) > \aleph_0, \lambda > \mu$

We let $\langle \lambda_\nu : \nu < \text{cf}(\lambda) \rangle$ be an increasing sequence of cardinals $< \lambda$ with limit $\lambda$ so $\nu \mapsto n(\lambda_\nu)$ is a function from $\text{cf}(\lambda)$ to $\omega$ hence for some $n_1$ we have $\lambda = \sup\{\lambda_\nu : n(\lambda_\nu) = n_1\}$.

Let $n_2$ be such that $\text{cf}(\lambda) < \mu \Rightarrow \text{cf}(\lambda) < \mu_n$. Now $\max\{n_1, n_2\}$ can serve.

Case 4: $\lambda_1 = \lambda^+ \text{ or sup}(\lambda \cap \text{Reg}) < \lambda$.

Easy.

Case 5: $\text{cf}(\lambda) = \aleph_0$ and $\lambda > \mu$ and $\lambda = \sup(\lambda \cap \text{Reg})$.

Toward contradiction assume this fails. We first choose $a_1, D_1$ such that

$(\ast)_1 (a)$ $a_1 \subseteq \text{Reg} \cap \lambda$ of cardinality $< \mu$

(b) $D_1$ as $\aleph_1$-complete filter on $a_1$

such that

(c) $\lambda_1 = \text{ps-tcf}(\Pi a_1, <_{D_1})$ is well defined and $\geq \lambda$ hence $> \lambda$.

Without loss of generality

$(\ast)_2 (a)$ $a_1 \cap \mu^{++} = \emptyset$ and $\sup(a_1) < \lambda$ and $n_3 = n(\ast) \geq \max\{n(\theta) : \theta \in a_1\} < \omega$

(b) $\sup(ga_1) < \lambda_0 < \lambda$

(c) $(a_1) < \mu_{n_3} < \mu$. 
[Why? By AC, AC₁, and AC₃, etc. and Theorem xxx using clauses (x), (y) of the assumption.]

□₁ (a) \( a₂ \subseteq \text{Reg} \cap \lambda |λₙ^+ \)
(b) \(|a₂| < μ\)
(c) \(D₂\) is a \( μₙ^{(s)}\)-complete filter on \(a\)
(d) \(\text{ps-tcf}(Πa₂, <D₂)\) is well defined \(\geq λ\) hence \(> λ\).

As \(\text{hrtg}(\text{Fil}^{|a₁|}) < μₙ₃\) and \(\min(a₁) > \min(\text{Reg} \setminus μ)\) by 1.13, the no-hole claim, we know

□₂ for every \(κ \in a₂\) there is a sequence \(\bar{λ}_κ = (λ_{κ, θ} : θ ∈ a₁)\) such that
(a) \(λ_{κ, θ} \in \text{Reg} \cap θ|μ\)
(b) \(κ = \text{ps-tcf}(Πa₂, <D₂)\).

As we assume \(AC_{a₂}\), recalling \(|a₂| < μ\)

□₃ (a) there is a sequence \((\bar{λ}_κ : κ ∈ a₂)\) as above
(b) \(a₃ = (a₃, θ : θ ∈ a - 1)\) where \(a₃, θ = \{λ_{κ, θ} : κ \in a₂\} ∈ [\text{Reg} \cap θ]_{≤a₂}\)
(c) let \(λ₀ = (λ_{κ, θ} : κ \in a₂)\).

By the choice of \(n₂\), etc. and Theorem xxx using clauses (x), (y) of the assumption

□₄ for each \(θ ∈ a₁\) there is a set \(ℱ₀ \subseteq \{b ≤ a₃, θ : \text{sup}(\text{ps-pcf}_κ, ✷b) ≤ λ\}\) of cardinality \(< μₙ₃\) with union \(a₂\)

□₅ here is \((ℱ₀ : θ ∈ a₁)\) as above.

[Why? By ACₙ₃ because ACₙ₃₁]

□₆ there is \(A \in D₂^+\) such that \((∀θ ∈ a₁)(∃B ∈ ℱ₀)(A \subseteq B)\).

[Why? \(ℱ := \cup\{ℱ₀ : θ ∈ a₁\}\) is a set of cardinality \(≤ μₙ₃\) as we have \(|ℱ₀| \leq μ₂\) and ACₙ₁ holds and \(|a₁| ≤ μ₁\) and \(n₁ < n₂\). Define an equivalence relation \(e\) on \(\{κ₁ \in κ₂ \text{ iff } (∀A ∈ ℱ) (κ₁ ∈ A ⇔ κ₂ ∈ A)\). So the function \(κ ↦ \{A ∈ ℱ : κ ∈ A\}\) witness that \(|a₂/e| < \text{hrtg}(ℱ(ℱ₀)) ≤ \text{hrtg}(ℱ(μₙ₂))\). But \(D₂\) is \(μ₁ₙ₃\)-complete and \(μₙ₃ > θ(ℱ(μₙ₂))\), so we are done.]

□₇ (a) without loss of generality \(A ∈ ℱ₀ \land θ ∈ a₁ ⇒ A ∈ \{a₂, θ\}\).

[Why? By x.)

□₈ \(θ ≥ \sup(\text{ps-pcf}_κ, ✷(a₃, θ))\) for \(θ ∈ a₁\).

[Why? By □₇ and the assumption on ℱ₀.]

□₉ let \(D₃\) be the following filter on \(Y = a₂ × a₁\)
\[D₂ × D₁ := \{A \subseteq a₂ × a₁ : \{κ ∈ a₂, θ : \{θ ∈ a₁ : (κ, θ) ∈ A\} ∈ D₁\} ∈ D₂\}\]
Discussion 8.3. We try to continue below but §5 seems to solve it another way.

Discussion 8.4. We try to analyze the remaining cases. If we add $|\mathcal{P}(\mu_n)| < \mu$ for $n < \omega$ by forcing without loss of generality

- $\text{otp}(a_\ell) = \delta_\ell = \text{cf}(\delta_\ell)$
- $D_\ell = \text{dual}(\mathcal{J}_{<\lambda}[\mathcal{A}_\ell])$
- $E = \{b_2 \times b_1: b_\ell \subseteq a_\ell, |b_0| < \delta_\ell \text{ for } \ell = 1, 2\}$

So let

- $\sigma \in a_1 \Rightarrow \epsilon_\sigma := \text{ps-pcf}_{\mathcal{A}_2}(\{\lambda, \sigma: \lambda \in a_2\}) \subseteq N$
- $\mathcal{F} = \Pi_{\mathcal{E}_\sigma}$
- $\bar{\alpha} \in \mathcal{F} \Rightarrow \mathcal{D}_\alpha = \text{ps-pcf}_{\mathcal{A}_1}(\bar{\alpha}, <_{\mathcal{F}_{\mathcal{A}_1}})$ define naturally
- $\mathcal{D}_\alpha = \cup\{\bar{\alpha}: \alpha \in \mathcal{F}\}$
- $\langle A_\chi: \chi \in \text{ps-pcf}_{\mathcal{A}_1}(\bar{\lambda})\rangle$.

So

- $\chi \in \mathcal{D} \Rightarrow (v_{\bar{\alpha}_2} \in a_2)(v_{\bar{\alpha}_1})[(\kappa, \sigma) \notin A_\chi]$.

By forcing without loss of generality

- $|\epsilon_\sigma| = \delta_2$.

Question 8.5. Assume $a$ is the disjoint union of $\langle a_\varepsilon : \varepsilon < \theta, a \subseteq \text{Reg}\setminus\mu, |a| < \mu$.

Do we have $\text{ps-pcf}_{\mathcal{A}_1}(\mathcal{A}) = \cup\{\text{ps-pcf}_{\mathcal{A}_1}(\bigcup_{\varepsilon < \theta} d_\varepsilon, a_\varepsilon \subseteq (\text{ps-pcf}_{\mathcal{A}_1}(a_\varepsilon))\}$ countable?

This is a consequence of the existence of smooth closed generating sequences; but does it exist here?

Question 8.6. Does it help to collapse $2^{|a_1|}$ and so find as an ultrafilter $E_\ast$ on $a_1$ such that $V^{a_1}/E_\ast$ has standard $\mathbb{N}$, etc.?
§ 9. Absoluteness for non-well founded ultra-powers

Question 9.1. (to [Sh:F1039])

This may be used in §5 to imitate [Sh:460]. Here we try to avoid using the "smooth closed generating sequences".

Check: What does this give directly?

Hypothesis 9.2.

(a) $AC_{\kappa(\ast)}, \kappa(\ast) > \aleph_0$

(b) $D_{\ast}$ is a uniform $\aleph_1$-complete ultrafilter on $\kappa(\ast) = \kappa(\ast)$

(c) $\mathbb{P}$ a forcing notion, $D$ a $\mathbb{P}$-name of an ultrafilter on $\mathcal{P}(\kappa)^V$ extending $D_{\ast}$, $G \subseteq \mathbb{P}$ generic over $V$, in $V[G]; D|D| = D_{\ast}$

(d) $W = W^{\kappa(\ast)} / D_{\ast}$, so in general not well founded, computed in $V[G]$

(e) $j = j_G$ is the canonical embedding of $V$ into $W$.

Remark 9.3. 1) We may demand $\mathcal{P}(\mathbb{P})$ well ordered and $AC_{\mathcal{P}(\mathbb{P})}$ holds.

2) Natural to choose $\mathbb{P} = \{ \langle D : D \mbox{ and } \aleph_1 \mbox{-complete filter on } \kappa \mbox{ extending } D_{\ast} \rangle, \geq \}$.

Claim 9.4. If $\mathfrak{H}_0 + \mathfrak{H}_1$ then $\oplus$ when

$\mathfrak{H}_1 (a)$

- $\kappa_1 < \kappa_2 < \kappa_3$
- $\kappa_2 = |(\kappa_2)^{\kappa_1}/D_2|$ can we use less? $AC_{\mu}$
- $V$ satisfies enough for Theorem gxxx with $(\kappa_2, \kappa_3)$ here standing for $(\kappa, |Y|)$ there

(b) $D_1$ is an $\aleph_1$-complete ultrafilter on $\kappa_1$

(c) $W$ is $V^{\kappa_1}/D_1$, i.e. $(V_j \in)^{\mathfrak{H}_1}$

(d) $j$ is the canonical elementary embedding of $V$ into $W$

(e) $W \models \{ a \mbox{ is a set of regular cardinals } > j(\mu) \mbox{ of cardinality } \leq j(\kappa_3) \}$

(f) $W \models \langle A_0 : \theta \in c \rangle$ where $\varepsilon = \text{ps-pcf}_{\kappa_2-cord}(a)$ is a generating force $\varepsilon$

(not just $\{ a \}$ as in gxxx)

(g) $Y = \{ \theta : W \models \langle \theta \in a \rangle \}$

(h) for $\theta \in Y$ let $I_0$ be $\{ \alpha : W \models \alpha < \theta \}$ linearly ordered by $< W$ so $\langle I_0 : \theta \in Y \rangle$ exists in $V$

(i) $\bar{\lambda} = \langle \lambda_0 : \theta \in Y \rangle$ where $\lambda_0 = \text{cf}(I_0)$

(j) $J = \{ \theta : Y : W \models \theta \in b \} : W \models \{ b \subseteq a \mbox{ have cardinality } \leq j(\kappa_2) \}$

(k) $J^0 = \{ Z \subseteq Y : (\exists u)(Z \subseteq u \in J) \}$

(l) $J_0$ is $\{ \theta : W \models \theta \in u \} : W \models \{ u \in J_0 \{ c \} \}$

(m) $J^0_0 = \{ W : (\exists u)(W \subseteq u \in J_0) \}$

$\oplus (a)$ $Y$ is of cardinality $(\kappa_3)^{\kappa_1}/D_2$ a cardinal

(b) if $Z \in J$ then $|Z| \leq (\kappa_2)^{\kappa_1}/D_1$ (is this well ordered?) no real harm assuming yes; similarly $\bar{Y}$

(c) the following are equivalent

- $Z \subseteq W$ has cardinality $\leq \kappa_2$
- for some $u \in W, W \models \langle |u| \leq j(\kappa_2) \rangle$ and $Z \subseteq \{ a : W \models \langle a \in u \rangle \}$

(d) $J^+$ is an ideal of subsets of $Y$, in fact in $[Y]^{\leq \kappa_2}$
(e) \( \lambda_\theta = \text{ps-tcf}(\Pi \lambda, J_\theta^+) \) for \( \theta \in \mathcal{M} \)

(f) \( J_\theta^+ = \{ A : A \subseteq \bigcup_{\sigma \in b} [A_\sigma] \leq \kappa_2 \text{ and } (\forall \sigma \in b)(\sigma < _w \theta) \} \)

(g) \( \text{ps-pcf}_{\kappa_2^+}^{\text{con}}(\lambda_\theta : \theta \in \mathfrak{c}) \) = \{ \lambda_\theta : \theta \in \mathfrak{c} \}.

{k2d}

Remark 9.5.

1) Applying this in \( \S 4 \) we let \( a, \kappa_2, \kappa_3 \) stand for \( \{ \lambda_{\kappa, \theta} : \kappa \in a_2 \} : \theta \in a_1 \} / D, j(\mu_{[a_2]}^\kappa) \) there.

2) Well the problem may come from undefinable Dedekind cuts in \( (Y, <_W Y) \).

However \( a_1 = (\theta_i : i < \kappa_1), a_2 = (\theta_i : \varepsilon < \kappa_2) \) let \( D_2 \) be a \( \kappa_2 \)-complete filter on \( a_2 \) such that \( \lambda_\varepsilon = \text{ps-tcf}(\Pi a_2, <_{D_2}) \) is too large. So we use \( \lambda^* = (\lambda_{\varepsilon, i} : \varepsilon < \kappa_2) \in \text{Reg} \cap (\theta_i), W \models "a, i.e. } Y = (\theta_i : i \in j(\kappa_1)), j(D_2) \) is a \( j(\kappa_2^+ \kappa) \)-complete filter on \( j(\kappa_2) \).

We may wonder: what filter does \( j(D_2) \) induce on \( (\lambda_{\varepsilon, i} : i < \kappa_1) / D_2 : \varepsilon < \kappa_1 \) (from the outside)?

Exactly \( D_2 \) by the completeness.

Proof. Clause (a): Straight

Clause (b): Follows from clause (c).

Clause (c): If \( Z \subseteq W \) and \( |Z| \leq \kappa_2 \) (in \( V \)) this member of \( Z \) has the form \( f(D_1) \) with \( f \in (\kappa V, V) \), so by AC_{\kappa_2} we can find a sequence \( (f_i : i < i(*) \leq \kappa_2) \) such that \( Z = \{ f_i(D_1) : i < i(*) \} \). For \( \varepsilon < \kappa_1 \) let \( Z_\varepsilon = \{ f_i(\varepsilon) : i < i(*) \} \) so \( Z_\varepsilon = \varepsilon < \kappa_1 \in \kappa_1 V \) hence \( Z^* = (Z_\varepsilon : \varepsilon < \kappa_1) / D_2 \in W \).

As \( W \models "|Z_\varepsilon| \leq \kappa_2 \" \) for \( \varepsilon < \kappa_2 \) by the relevant version of Los theorem (quote use AC_{\kappa_1}) we have \( W \models "|Z^*| \leq j(\kappa_2)\)" and obviously \( i < i(*) \Rightarrow f_i \in \prod_{\varepsilon < \kappa_1} Z_\varepsilon \Rightarrow W \models "f_i(D_1) \in Z^*\). So we have proved one direction. The other is even easier. \( \square \)

{k7}

Observation 9.6. [AC_{\kappa_2}]

Los theorem holds and so \( j \) is an elementary embedding.

Claim 9.7. If \( \theta = \theta^a / D_\ast \) (in \( V \)) then for every \( w \subseteq W \) the following are equivalent

- \( \{ a : W \models a \in w \} \leq \theta \)
- for some \( w \in W \) we have \( W \models "|w| \leq j(\theta)\)" and \( u \subseteq \{ a : W \models "a \in w \} \).

Proof. See above. \( \square \)

{m10}

Claim 9.8. If \( \alpha \) is a linear order of cofinality \( \theta > \alpha \) then \( \{ j(s) : s \in I \} \) is a cofinal subset of \( I^W = I[W] \) the linear order with set of elements \( \{ a : W \models "a \in I \} \) and \( I^W = "a < b" \) iff \( W \models "(I \models a < b)\)".

{m13}

Claim 9.9. (Also 3.10!) 1) If \( W \models "I \) is the linear order \( (a, \kappa) \in \text{Reg} \setminus \mu \)" then in \( V[G] \), tcf(\( I^W \)) \( \in \text{Reg} \). 2) Moreover if \( I = f / D_\ast, f : \kappa_\ast \rightarrow \text{the class of regular cardinals} \) then for some \( p \in G \) and \( \lambda \in \text{Reg} \setminus \mu \) we have \( \lambda = \text{ps-tcf}(\Pi f, <_{D_\ast}) \).

{m15}

Claim 9.10. If \( \theta \) is a regular cardinal \( > |P| + \kappa \) and \( \bar{u} = (u_\alpha : \alpha < \theta) \) is a sequence of non-empty subsets of \( \text{Ord}^W \) and \( a \in u_\alpha \land b \in u_\beta \land \alpha < \beta \Rightarrow W \models a < b \) then \( \bar{u} \) has an l.u.b, i.e. there is \( a_\ast \) such that

- \( a_\ast \in \text{Ord}^W \)
Proof. By xxx. \hfill \{m17\}

Claim 9.11. (Like 9.5(2).) \hfill \{m19\}

Claim 9.12. A sufficient condition for $W \models \theta \in (f/D) \cap (\text{Reg} \setminus \text{j}(\mu)) \Rightarrow \chi < \lambda$ is: $\mathbb{P} = (D_{**}, \supseteq)$ and $(f, D_{**})$ or niceness (check!).
§ 10. More pcf with little choice: a try

Question 10.1. (To [Sh:F1039])

§(7A) Introductory Remark

Discussion 10.2. We observe [Sh:938, §3.§4] works if we demand just that $D_d$ a semi-filter. Then we replace measurable by the chosen win in a cut and choose game. Third, ?

Lastly, let the chosen choose few instead?

Definition 10.3. We say $D$ is a semi-filter on $Y$ when:

(a) $D \subseteq \mathcal{P}(Y)$
(b) if $A \subseteq B \subseteq Y$ and $A \in D$ then $B \in D$
(c) $\emptyset \notin D$ and $Y \in D$.

Claim 10.4. If in [Sh:938, Def.3.1(b)(β)] we weaken the demand "$D_d$ is a filter on $Y_d$" to "$D_d$ is a semi-filter on $Y_d$" still all the claims (and definitions) in [Sh:938, §3,§4] works.

§(7B) Games and Rank

Definition 10.5. We say $x$ is appropriate when:

(a) $x = (\kappa, \theta, \sigma, D_{x,1}, D_{x,2}) = (\kappa_x, \theta_x, \sigma_x, D_{x,1}, D_{x,2})$
(b) $\kappa > \theta > \sigma$ are cardinals
(c) $D_{x,1} \subseteq D_{x,2}$ are filters on $\kappa$.

Definition 10.6. 1) We say $x$ is large when the chooser has a winning strategy in the game $\mathcal{G}_x$ defined below.
2) The game $\mathcal{G}_x$ between the player cutter and chooser last $\omega$ moves in the $n$-th move a set $A_{n+1} \in D_{x,2}$ is chosen, letting $A_0 = \kappa$. In the $n$-th move the cutter chooses $\zeta_n < \theta$ and $f_n : A_n \rightarrow \alpha_n$, and the chooser chooses $w_n \in [\zeta_n]^{<(1+\sigma)}$ and let $A_{n+1} = \{ \alpha \in A_n : f_n(\alpha) \in w_n \}$.

In the end the chooser wins iff $\cap \{A_n : n < \omega\} \in D_{x,1}^+$. For the rest of this section

Hypothesis 10.7. We assume $x$ is large and $st$ is a winning strategy for the chooser and $\sigma_x = 1$.

Definition 10.8. 1) $P = \text{pos}(x, st)$ is the set of finite initial segments of a play of the game $\mathcal{G}_x$ during which the chooser uses the strategy $G$; we denote such initial segments by $s$ and $A_s$ is $A_n$ for the maximal $n < \omega$ such that it is well defined.
2) For $s, t \in P$ let $s \leq t$ iff $s$ is an initial segment of $t$.
3) Let $P_{2s} = \{ t \in P : s \leq t \}$.

Definition 10.9. 1) For $s \in P$ let $D_s = D_{x,st,s} = \{ A \subseteq \kappa_x : \text{for no } t \text{ do we have } s \leq t \land A \cap A_t = \emptyset \text{ mod } D_{x,2} \}$.
2) We define $\text{rk}_s(f) \in \text{Ord} \cup \{\infty\}$ by defining when $\text{rk}_s(f) = \alpha$ for $s \in P, f \in {}^\omega \text{Ord}$ and $\alpha \in \text{Ord}$ (and let $\text{rk}_s(f) = {}^* \alpha$ when below $t = s$ is O.K.)
\(\text{Claim 10.10.} \ 1) \text{ For } s \in P \text{ and } f \in {}^s\text{Ord}, \text{ exactly one } \alpha \in \text{Ord} \cup \{\infty\} \text{ we have } \text{rk}_s(f) = \alpha. \)

\(\text{2) Assume } f, g \in {}^s\text{Ord} \text{ and } s \in P. \text{ If } f = g \mod (D_{x,K} + A_s) \text{ then } \text{rk}_s(f) = \text{rk}_s(g) \text{ and if } f \leq g \mod (D_{x,2} + A_s) \text{ then } \text{rk}_s(f) \leq \text{rk}_s(g). \)

\(\text{3) } [\text{DC}] \text{ For } s \in P \text{ and } f \in {}^s\text{Ord} \text{ we have } \text{rk}_s(f) \in \text{Ord}. \)

\text{Proof. Easy.} \quad \Box \{g17\}

\text{Claim 10.11.} \text{ If } \zeta = \text{rk}_s(f) \text{ and } h : A_s \to \theta \text{ then for some } \varepsilon < \theta \text{ and } t \in P_{> s} \text{ we have } \text{rk}_t(f) = \zeta \text{ and } h|A_t \text{ is constant}. \)

\text{Proof. Without loss of generality } \text{rk}_s(f) = {}^s\zeta. \quad \Box
\section*{11. Various}

\section*{Definition 11.1.} Assume $D$ is a filter on $Y$.
1) Let $\text{oq}(Y) = \text{oq}(Y,D) = \{ f : a function from $Y$ onto some ordinal\}.
2) For $f \in \text{oq}(Y)$ let $e_f = \{(y_1,y_2) : y_1 \in Y, y_2 \in Y \text{ and } f(y_1) = f(y_2)\}$.
3) Let $\text{oq}(Y) = \{ e_f : f \in \text{oq}(Y,d)\}$.
4) For $h \in \text{oq}(Y,D)$ let $D/h$ be $\{ x \subseteq \text{Rang}(h) : h^{-1}(X) \in D\}$, a filter on $\text{Rang}(f)$ which necessarily is an ordinal $< \text{hrtg}(Y)$.
5) For $f \in Y$ let $\partial_f$ be the following function:
\begin{enumerate}
\item $\text{Dom}(\partial_f) = \text{otp}(\text{Rang}(f))$
\item $\partial_f(i) = \alpha$ iff $(\exists y)(y \in Y \wedge f(y) = \alpha \wedge i = \text{ otp}(f(y)) \cap \text{Rang}(f))$.
\end{enumerate}
6) For $f \in \text{Y} \text{Ord}$ let $\partial_f$ be the following function:
\begin{enumerate}
\item $\text{Dom}(\partial_f) = Y$
\item $\partial_f(y) = \text{ otp}(f(y) \cap \text{Rang}(f)) \in \text{oq}(Y,d)$.
\end{enumerate}
7) Assume $D \in \text{Fil}_1(Y)$ and $\bar{f} = (f_\alpha : \alpha < \alpha(\ast))$ is a $<_D$-increasing sequence of members of $\text{Y} \text{Ord}$
\begin{enumerate}
\item we let $\bar{u} = (u_{f,h} : h \in \text{oq}(Y,D))$ where $u_{f,g} = \{ \alpha < \alpha(\ast) : h_{f,g} = h\}$
\item $f_\alpha^{[h]} = (g_{f,a} : \alpha \in u_{f,h})$ is $<_D$-increasing.
\end{enumerate}

\section*{Claim 11.2.} Assume $D \in \text{Fil}_1$.\\
1) Assume $f = (f_\alpha : \alpha < \delta)$ is a $<_D$-increasing sequence of members of $\text{Y} \text{Ord}$
\begin{enumerate}
\item $\langle u_{f,h} : h \in \text{oq}(Y,D) \rangle$ is a partition of Y
\item $\text{cf}(\delta) \geq \text{hrtg}(\text{oq}(Y,D)) \text{ then for some } h \in \text{oq}(Y,D) \text{ the set } u_{f,h} \text{ is an unbounded subset of } \delta$
\item for $h \in \text{oq}(Y)$ the sequence $\langle g_{f,a} : \alpha \in u_{f,h} \rangle$ is a $<_D/h$-increasing sequence of members of $\text{Dom}(h) \text{Ord}$
\item in (b); if $\delta = |\delta| \text{ then for some } h \in \text{oq}(Y) \text{ the set } u_{f,h} \text{ has order-type } \delta$.
\end{enumerate}
2) For $\alpha \in \text{Y} \text{Ord}$ for every regular $\lambda \geq \text{hrtg}(Y)$ we have
\begin{enumerate}
\item $\lambda \in \text{ps-tcf}_{\text{c-com}}(\bar{a}) \text{ iff } \lambda \in \text{ps-tcf}_{\text{c-com}}(g_{\bar{a}})$
\item $\lambda \in \text{dp-tcf}_{\text{c-com}}(\bar{a}) \text{ iff } \lambda \in \text{dp-tcf}_{\text{c-com}}(g_{\bar{a}}) \text{ recalling } \text{dp-tcf}_{\text{c-com}}(\bar{a}) = \{ \lambda : \text{for some } D \in \text{Fil}_1(Y), \lambda = \text{tcf}(\Pi \bar{a}, D), \text{ equivalently there is a cofinal sequence of members of } \Pi \bar{a}\}$.
\end{enumerate}

\section*{Observation 11.3.} If $\text{AC}_{\text{hrtg}(Y)} \text{ then } \text{hrtg}(Y)$ is a successor cardinal.

\begin{proof}
Toward contradiction assume $\text{hrtg}(Y)$ is a limit cardinal say $\aleph_\delta(\ast)$.

For $\alpha < \text{hrtg}(Y)$ let $\mathcal{F}_\alpha = \{ g : g \text{ a function from } Y \text{ onto } \alpha \}$, by the definition of $\text{hrtg}(Y)$ it is non-empty, hence by $\text{AC}_\alpha$ the set $\mathcal{F}_\alpha = \{ f : f \text{ a one-to-one function from } \alpha \text{ into } Y \}$ is non-empty. As $\langle \mathcal{F}_\alpha : \alpha < \text{hrtg}(Y) \rangle$ exists and $\text{AC}_\theta(Y)$ holds, there is a sequence $\langle f_\alpha : \alpha < \text{hrtg}(Y) \rangle$ with $f_\alpha \in \mathcal{F}_\alpha$. Define the function $\text{pr}$ with domain $\{ (\alpha, \zeta) : \alpha < \text{hrtg}(Y) \}$ by $\text{pr}(\alpha, \zeta) = \sum_{\xi < \zeta} \aleph_\xi + \alpha$, now $\text{pr}(\alpha, \zeta) < \aleph_{\zeta+1} \leq \text{hrtg}(Y)$ so $\text{pr}$ is one-to-one into $\text{hrtg}(Y)$, also the range of $\text{pr}$ is an initial segment
of Ord, and |Rang(pr) = Dom(pr) as it is one-to-one and obviously |Dom(pr)| ≥ θ; together pr is onto hrtg(Y). We define \( y_\gamma : \gamma < hrtg(Y) \) by \( y_{pr(\alpha, \zeta)} = f_\zeta(\alpha) \) for \( \alpha < \aleph_\zeta < hrtg(Y) \); let \( u = \{ \gamma < hrtg(Y) : (\forall \beta < \gamma) (y_\gamma \neq y_\beta) \} \), so easily \( \zeta < \delta(\ast) \Rightarrow \aleph_{\gamma+1} = |u \cap [\aleph_\zeta, \aleph_{\gamma+1}]| \), hence \( |u| = hrtg(Y) \), hence \( \langle y_\gamma : \gamma < hrtg(Y) \rangle \) exemplify \( Y > hrtg(Y) \), contradiction.

\[ \square_{11.3} \]

Claim 11.4. Assume \[ \{? \]

(a) \( \langle F_D : D \in ps-tcf-fil(\bar{\alpha}) \rangle \) is as in \[ ? \]

(b) \( \bar{D} = \langle D_i : i < i(\ast) \rangle \) is a \( \kappa \)-complete filter \( \bar{\alpha} \)

(c) for \( \alpha \) as above and \( \beta \in \prod_{n} tcf(\Pi_\alpha, < D_\alpha) \) let \( F_{D, \beta} = \{ f_\beta : i < \ell(\beta) \} \) such that \( f(s) = \sup \{ f_\beta(i) : i < \ell(\bar{\beta}) \} \)

(d) \( \{ F_{D, \beta} : D \in ps-tcf-fil(\bar{\alpha}) \) and \( \beta \in \prod_{n} tcf(\Pi_\alpha, < D_\alpha) \) is cofinal

(e) \( ps-cf^\kappa(\Pi_\alpha) = sup(ps-pcf_\kappa(\Pi_\alpha)) \) where we define \( ps-cf^\kappa(\Pi_\alpha) \leq S \) when ...

\[ \{c13yajan \}

Claim 11.5. Assume

(a) \( D \in Fil_\kappa(Y), \kappa \geq \aleph_1 \) and \( \alpha_y > 1 \) for \( y \in Y \)

(c) \( rk_D(\bar{\alpha}) = \zeta = |\zeta| \)

(d) \( cf(\zeta) \leq hrtg(Fil_\kappa(Y)) \).

1) For some \( J \in \{ J[f, D] : f \in Y, Ord \} \) we have \( \zeta = \text{opt}(\{ \gamma : \text{there is } \beta \in \Pi_\alpha \text{ such that } rk_D(\bar{\beta}) = \gamma \text{ and } J[\beta, D] = J \}) \).

2) \( ? \) In (1) if dual(D) \( \subseteq D_1 \in Fil_\kappa(Y) \) then \( nk(D_1, \bar{\alpha}) = \zeta \) and \( ? \)

3) \( ? \) Moreover in (1) if \( \beta \in \Pi_\alpha \), \( nk(\bar{\beta}) = \gamma \), \( J[\beta, D] = J \), then \( nk(D_1, \bar{\beta}) \leq ?? \)

Proof. 1) For \( \epsilon < \zeta \) let \( F_\epsilon = \{ \beta \in \Pi_\alpha : nk(\bar{\beta}) = \epsilon \} \) so \( F = \langle F_\epsilon : \epsilon < \zeta \rangle \) exists and \( \epsilon < \zeta \Rightarrow F_\epsilon \neq \emptyset \) by xxxxxx and \( \cup \{ F_\epsilon : \epsilon < \zeta \} = \Pi_\alpha \).

Let \( E \subseteq D \) such that dual(D) \( \subseteq D_1 \in Fil_\kappa(Y) \) extending \( D \) and let \( u_E = \{ \epsilon < \zeta : \exists F_\epsilon \neq \emptyset \} \), so \( \epsilon \subseteq \cup \{ F_\epsilon : E \in Fil_\kappa(Y) \}) \). As \( cf(\zeta) \geq hrtg(Fil_\kappa(Y)) \) necessarily for some \( E, \text{opt}(u_E) = \zeta \) but \( u_E \subseteq \zeta \leq |\zeta| \) hence \( \text{opt}(u_E) = \zeta \), so dual(E) as is required.

2) By (3). \( ? \)

3) \( ? \) So \( J \) is from (1) and toward contradiction assume dual(J) \( \subseteq D_1 \in Fil_\kappa(Y) \) and \( \alpha_1 \in \Pi_\alpha \), but \( nk(D_1, \alpha) > \zeta \); without loss of generality \( y \in Y \Rightarrow \alpha_1, y > 0 \) and \( nk(D_1, \alpha_1) = \zeta_1 \). Now we choose \( F_1, F_1^*, E_2 \) as in the proof of part (1) starting with \( \alpha_1, \zeta_1 \).

\[ \square_{?} \]

Claim 11.6. \[ [DC] \] 1) If \( \aleph_0 < \kappa = cf(\mu) < \mu \) then \( nk(\mu, \kappa) > \mu^+ \).

Proof. 1) Clearly \( J^{(\kappa)} \) is a uniform \( \kappa \)-complete filter on \( \kappa \). Let \( \langle \mu_i : i < \kappa \rangle \) be increasing continuous with limit \( \lim \mu_i < \mu_0 \). For each \( \alpha < \mu^+ \) let

\[ \mathcal{F}_\alpha = \{ f : f \text{ a one-to-one function from some subset of } \mu \text{ onto } \alpha \} \]

\[ \mathcal{G}_\alpha = \{ g_f : g_f \text{ for some } f \in \mathcal{F}_\alpha \} \text{ where for } f \in \mathcal{F}_\alpha \text{ for some } \alpha < \mu^+ \text{ we let } g_f \text{ be defined by} \]

\[ \{c14y \}
\[ (*)_0 \text{ Dom}(g_f) = \kappa \text{ and for every } i < \kappa, g(i) = \text{ otp}\{g(\varepsilon) : \varepsilon < \mu_i \cap \text{ Dom}(f)\}\]

\[ (*)_1 \mathcal{F}_\alpha \neq \emptyset \text{ for } \alpha < \mu^+ \]

\[ (*)_2 \mathcal{G}_\alpha \neq \emptyset \text{ for } \alpha < \mu^+ \]

\[ (*)_3 \mathcal{G}_\alpha \subseteq \prod_{i<\kappa} \mu_i^+ \subseteq ^*\mu. \]

[Why? As the set \{f(\varepsilon) : \varepsilon \in \mu_i \cap \text{ Dom}(f)\} has cardinal \leq \mu_i, so have order type < \mu_i^+.]

\[ (*)_4 \text{ if } \alpha_1 < \alpha_2 \text{ and } g_2 \in \mathcal{G}_{\alpha_2} \text{ then for some } g_1 \in \mathcal{G}_{\alpha_1} \text{ we have } g_1 < g_2 \mod J^*_\kappa. \]

[Why? Let \( g_2 = g_{f_2} \) so \( \beta_1 \in \beta_2 = \text{ Rang}(f_2) \) so let \( \beta_1 = f(\varepsilon_1) \) and \( i_1 \) be a min\{i < \kappa : \mu_i > \varepsilon_j\}. Let \( \mathcal{W} = \{\varepsilon \in \text{ Dom}(f_2) : f_2(\varepsilon) < \beta_1\} \) and \( f_1 = f_2|\mathcal{W} \) and let \( g_1 = g_{f_1} \), so clearly \( g_1 \in \mathcal{G}_{\alpha_1} \). Now if \( i \in i_1, \kappa 0 \) then \( \{f_1(\varepsilon) : \varepsilon \in \mu_i \cap \text{ Dom}(g_{f_1})\} \subseteq B_1 \cap \{f_2(\varepsilon) : \varepsilon \in \mu_i \cap \text{ Dom}(f_2)\} \) and \( \beta_1 \in \{g_2(\varepsilon) : \varepsilon \in \mu_i \cap \text{ Dom}(f_2)\} \), so clearly \( g_{f_1}(i) < g_{f_2}(i) \).

So \( g_1 < g_2 \mod J^*_\kappa \) is as required.]

For \( \alpha_* \in [\mu^+, \mu^++1) \) and we shall prove that \( \text{ rk}_D(g) \geq \alpha_* \) for some \( g \in {}^\kappa \mu \), this suffices.

As \((\alpha_*)\) there is \( \bar{w} \) such that

\[ (*)_5 \text{ (a) } \bar{u} = \langle w_i : i < \chi \rangle \]

\[ (*)_6 \text{ (b) } i < \chi \Rightarrow |w_i| = \mu \]

\[ (*)_7 \text{ (c) } \alpha_* = \cup \{w_i : i < \chi \}. \]

As \( \text{ cf}(\mu^+) = \chi \) we can choose \( \bar{\alpha} \) such that

\[ (*)_8 \text{ (a) } \bar{\alpha} = \langle \alpha_j : j < \chi \rangle \]

\[ (*)_9 \text{ (b) } \bar{\alpha} \text{ is increasing, } \alpha_j > \chi, \kappa \]

\[ (*)_{10} \text{ (c) } \bar{\alpha} \text{ is with limit } \mu^+. \]

Now \( y \in Y \) let

\[ (*)_11 \text{ (a) } |w_y| \leq \mu \]

\[ (*)_12 \text{ (b) } ?? \]

\[ \square \]
§ 12. Private Appendix

We can add to [Sh:938, 2.6.2.7]

Claim 12.1. The filter \( D_2 \) \( \mathcal{A} \)-commutes with the filter \( D_1 \) (see [Sh:938, 3.1]) when:

(a) \( D_\ell \in \text{Fil}_c(Y_\ell) \) for \( \ell = 1, 2 \)
(b) \( D_1 \) is \( \sigma \)-complete
(c) if \( J_1 \in \{ J[f, D_1] : f \in Y_\ell \text{ Ord} \} \) or just \( J_1 \) is a \( \sigma \)-complete ideal extending dual\( (D_1) \) then \( A \subseteq Y_1 \) but dual\( (J_1) \in \{ D_1 + A : A \in D_1^+ \} \); this follows from clause (b) + DC\( _\sigma \) VAC\( _{\psi(Y_1)} \) when \( D_1 \) is \( \sigma \)-c.c., i.e. there is no sequence \( \langle A_i : i < \sigma \rangle \) of a pairwise disjoint sets from \( D_1^+ \)
(d) DC\( _\sigma \) and AC\( _Y_1, AC_Y_2 \)
(e) (\( \alpha \)) \( D_1 \) is \( \mathcal{P}(Y_2) \)-complete or just
(\( \beta \)) if \( \langle B_s : s \in A_1 \rangle \in A(J^+_2) \) and \( A \in J^+_1, J_\ell \in \{ J[f, D_\ell] : f \} \) for \( \ell = 1, 2 \) then for some \( B_s \in J^+_2 \) and we have \( A_\ast \subseteq A, A_\ast \in J_1 \) we have \( s \in A_\ast \Rightarrow B_s \supseteq B_s \).

Proof. Stage A:

Let \( A \in D_2 \) and \( B = \langle B_s : s \in A \rangle \in A(D_2) \) and \( J^1 = \langle J^1_\ell : \ell \in Y_2 \rangle \) where \( J^1_\ell \in \{ J[f, D_1] : f \in Y_2 \text{ Ord} \} \) and \( J_2 = \{ J[f, D_2] : f \in Y_2 \text{ Ord} \}, \) i.e. as in the assumption of Definition [Sh:938, 2.1]. We should find \( A_\ast, B_s \) as there.

Stage B:

For each \( \ell \in I_2 \) there is \( A_\ell \in D_1^+ \) such that \( J^1_\ell = \text{dual}(D_1 + A_\ell) \), hence as AC\( _Y_2 \) holds such that \( \langle A_\ell : \ell \in Y_2 \rangle \) exist. Why? By clauses (b),(c) of the assumption.

Stage C:

Choice of \( B_s, A_\ast. \) Apply clause (d) of the assumption applied to \( \langle J_2, \langle A_\ell : \ell \in I_2 \rangle \rangle \).

Remark 12.2. 1) We can weaken “\( D_1 \) is \( \sigma \)-complete, \( \sigma \)-c.c.” to “\( D_2 \) is \( \sigma \)-complete, \( \sigma^+ \)-c.c.” when we have some normality conditions.
2) We can replace this by “any \( J[f, D_1] \) is of the form \( D_1 + A \) for some \( A \in D_1^+ \).

We can add in [Sh:938, §4]

Conclusion 12.3. \([AC_{\ell\mu} \text{ and } \mu \text{ a limit singular cardinality}]\)

Assume \( \mu = \sup(\kappa < \mu : \text{ for some } \lambda < \kappa, \mu \text{ on } \lambda \text{ there is a } \kappa \)-complete \( \kappa \)-c.c. filter \( D \) on \( \lambda \). Then for every ordinal \( \zeta \) for some \( \kappa_\ast < \mu \), for every \( \lambda \in [\kappa, \mu) \) and \( \kappa \)-complete \( \kappa \)-c.c. filter \( D \) on \( \lambda \) we have \( \text{rk}_D(\zeta) = \zeta \).

Proof. By 13.15 and [Sh:938, 4.1].

We define \( f : Y_1 \to \mathcal{P}(Y_2) \) by \( f(s) = \{ t \in Y_2 : s \in A_t \} \); as \( D_1 \) is \( (\mathcal{P}(Y_2)) \)-complete filters on \( Y_1 \) necessarily also \( J_2 \) is a \( (\mathcal{P}(Y_2)) \)-complete ideal on \( Y_1 \) hence there is

- \( Y_2^* \subseteq Y_2 \) such that \( A^* := \{ y \in A_1 : f(y) = Y_2^* \} \) belongs to \( J_2^* \).

Choose \( s^* \in A^* \) so \( Y_2^* = f(s) = \{ t \in Y_2 : s \in A_t \} \).
§ 13. Private Appendix

Remark 13.1. pcf inventory (August 2009)
1) See [Sh:F663] lecture - [Sh:430, §6] is locality proved for pcf_{θ-com}(−, θ > |a|).
2) See Rinot question [Sh:F893].
3) See the notes for Larson [Sh:F814] - on HOD.
4) Continue [?], see [Sh:F878].
5) Failed try to continue [Sh:460, §5B], [Sh:F563].
6) [Sh:F355] - on consistency - answer Gitik?
7) [Sh:F354] λ = sup(λ ∩ pcf(a)) is weakly inaccessible.
8) Densities of basic product [Sh:F132], covered by paper with Moti?
9) [Sh:F50] to Shimoni.
10) Hopes rank for precipiousness?
11) Sort out?
12) (09.10.19) A related question: let x = ⟨⟨Y_n, D_n, h_n⟩ : n < ω⟩ is here h_n : Y_n → Y and D a filter on Y and we try to prove

(∗) for every f ∈ Y_{Ord}, for every large enough n we have rk_{D_n}(f ◦ h_n) ≤ rk_D(f) or similarly for Depth.

13) (09.10.26, old thought) As we pass from cofinality to pseudo-cofinality, iterate this notion and then have strong dichotomies.
14) (09.11.15) Think of a problem where:

(a) Depth(α(8_n), P_n) large given an answer.

15) Tasks (2010.1.08)

(a) if Y = χ, then we can replace AC_{P(Y)} by DC_{χ^+}
(b) replace Y by all µ < θ(Y), just split to some ?
(c) Definition dp-pcf_{θ}(Y) = {x : λ regular and there is a filter D such that λ = dp-tcf(πα, <_D)} where: dp-tcf(πα, <_D) means there is an increasing cofinal of this length
(d) nice results but no existence
(e) given α, how much choice needed to find D with dual(D) = ([Z]<κ + (Y \ Z) for some Z?
(f) for a λ-sequence of length λ, <_{D_1}-increasing in Y_{Ord}, is there <_{D_2}-lub for some D_2 ≥ D_1?
(g) smooth closed generating sequence: by DC_{|Y|}?
(h) generalize [Sh:460]
(i) get bound or Depth 8_{ω_1}
(j) try for a dichotomy: with IND


(a) f_* ∈ θ^+(Reg ∩ µ_1)
(b) D a non-principal ultrafilter on θ
(c) cf(Π_i f_*(i)/D) = λ^+
(d) no $f/D < f_\ast/D$ satisfies (c), or do we use less?

(e) $\theta < \kappa, (\mu_1, \lambda)\kappa^\theta$, probably assuming $2^\theta < \kappa$ maybe it is much less interesting though we may get more than in [Sh:460], then $D$ is in $V^\theta$, $|\mathcal{P}| = 2^\theta$.

(f) $\lambda_j \in \text{Reg} \cap \mu_2/\mu_1(j < \kappa)$

(g) $\lambda^+ = \text{pcf}(\prod_j \lambda_j, < \kappa)$

(h) for each $i < \theta, f_\ast(i)$ is inaccessible for any $\kappa$-complete filter/ideal on $\kappa$.

Without loss of generality $\bigwedge_i \mathcal{U}_i < f_\ast(i)$.

We can find $g_j \in \pi(f_\ast(i) \cap \text{Reg})$ for $j < \kappa$ such that $\lambda_j, \text{cf}(\pi g_j, < D_i)$. Let $a_i = \{g_j(i) : j < \kappa\} \{\kappa^\theta\}^\ast, V = V^\theta/D, j : V \to V, \bar{a} = \langle a_i : i < \kappa\rangle, \mathcal{A} = \bar{a}/D$, so $V \models "(g_j/D) \in \mathcal{A} minimise \lambda^+ \land (\mathcal{A} \text{ has cardinality} \leq \text{p}(\kappa))"$.

So in $V$ we have the basic $\text{pcf}$ results $\langle b_{g/D}(\bar{a}/D) : g/D \in \mathcal{A} \rangle, \langle f_{\mathcal{A}/D, a/D} : a \in g/D \rangle$ as in $\mathcal{X}$.

Note

- $V \models "\text{there is a division of } \kappa \text{ to } \kappa \text{ sets } \langle u_{a, \varepsilon} : \varepsilon \in \varepsilon_1 \rangle, \text{max } \text{pcf}(g_j(i) : j \in u_{a, \varepsilon})"$.
- $V, \mathcal{A}$ is listed by $\langle \lambda^+ : \varepsilon \in \kappa^\theta/D\rangle$.
- In $V$ and $\kappa^\theta/D \in V$ is linear order with $\{j(j) : j < \kappa\}$ unbounded in it.
- If $V \models g/D = \text{pcf}(\prod_{g \in I} a_g/D, < \mathcal{A})$ then this is essentially true letting $\mathcal{E}$ be the filter on $\{a : V \models a \in I\}, \lambda_a = \text{cf}(\pi a_g, < D), \lambda = \text{cf}(\pi g, < D)$ we have $\lambda = \text{pcf}(\prod a_g, \lambda_a)$ when the $\lambda_a > 2^\theta$.

Discussion 13.3. (2010.3.8) We return to the trying to improve [Sh:460].

Question 13.4. Concerning [Sh:460], so say for $\mu > \text{cf}(\mu)(= \aleph_0)$? $\lambda$ is the first counterexample $> 2^\theta$ so $\text{cf}(\lambda) = \text{cf}(\mu)$. Let $\theta < \kappa, D$ an ultrafilter on $\theta$ such that for some $f_\theta \in \theta, \text{cf}(\prod_{i \in \theta} f_\theta(i), < D) = \lambda^+$.

1) Can we have $\"f_\theta/D\text{ is the first } f/D\text{ such that } \text{cf}(\prod_{i} f(i); < D) = \lambda^+\"$?

2) Or at least can we find $\bar{a}$ such that

- (a) $\bar{a} = \langle a_i : i < \theta \rangle$
- (b) $\bar{a} \in [\text{Reg} \cap \lambda]^{< \theta}$
- (c) $f \in \pi a_i \Rightarrow \text{cf}(\prod_{i} f(i), < D) = \lambda^+$ and
- (d) $g \in \theta \lambda \land \text{cf}(\pi g, < D) = \theta \Rightarrow \bigvee_{f \in \pi a_i} (f/D < g/D)$.

3) Maybe $\lambda$ is the first such that:

- ($\ast_1)$ for arbitrarily large $\theta < \mu$ (regular $\theta < \mu$) there is $a \in [\text{Reg} \cap \lambda]^{< \theta}$ bounded in $\lambda, \lambda \in \text{pcf}(a), b \in [a]^{< \theta} \Rightarrow \lambda \not\in \text{pcf}(b)$.

In the case clause (d) holds

Claim 13.5. (2010.3.08) We assume an axiom from [Sh:835] and prove $RGC\text{H in}$ the depth version for $\mu > \text{cf}(\mu) = \aleph_0$ strong limit and $AC_\mu, \kappa < \mu \Rightarrow \theta(\mathcal{P}(\kappa)) < \mu$. 

\{g23\}
Alternative: (2010.3.08)
1) Assume DC_{<\mu} (and so \( P(\kappa) < \mu \) for \( \kappa < \mu \)). Use the RGCH version with nice representation of pcf(\( \kappa \)), for the pseudo cofinality version.
2) Is ps-pcf(ps-pcf(\( \kappa \))) = ps-pcf(\( \kappa \))? So we have \( \lambda_i = \text{ps-tcf}_{\text{comp}}(\prod_{\lambda_i,j} M_{D_i}), \lambda = \text{ps-pcf}_{\text{comp}}(\pi \lambda_i, < D) \). Yes (but as anyhow we use pcf_{\text{comp}} iterating \( \omega \times \omega \) we are done).

Moved 2010.1.08 from 16.8, p.7:
2) \[\text{AC}_{P(\kappa)}(\text{Y})\] If \( D \) is \( \kappa \)-complete but not \((<\infty)\)-complete then AC_{\kappa}.

2) So without loss of generality \( D \) is \( \kappa \)-complete not \( \kappa^+ \)-complete hence there is a sequence \( \bar{A} = \langle A_\alpha : \alpha < \kappa \rangle \) of members of \( D \) with \( \cap\{A_\alpha : \alpha < \kappa\} \notin D \) and without loss of generality \( \bar{A} \) is with no repetition. This implies \( \kappa < \theta(\mathcal{P}(\text{Y})) \), but we have AC_{\mathcal{P}(\text{Y})} hence we have AC_{\kappa} as promised.

\[\ast\ \ast\ \ast\]

Moved from pg.8:

For \( \aleph_1 \)-complete ultrafilter we get more

Claim 13.6. [true??] Let \( D \) be an \( \aleph_1 \)-complete ultrafilter on \( \text{Y} \). Then for any \( f \in \prod(\text{Ord}\setminus\{0\}) \) we have \( \text{rk}_D(f) = \text{ps-o-Depth}(\prod_{t \in \text{Y}} f(t), < D) \) and the supremum on the left is obtained.

Proof. Obvious. \( \square \)

Question 13.7. 1) Can we prove parallel of the ZFC results?
2) (09.7.19) Is this not \( \theta(\mathcal{P}(\text{Y})) \)?

Moved from Anotated Content:

\[\text{(2A)}\] Getting quasi-rank systems with AC_{<\mu}, pg.7 (090909)?

[We start with pre-rank-system \( p \) and define rank trying to get a strict rank system using IND we get that the ranks are \( < \infty \). Has to be read together with [Sh:938]. While this has to be checked we still use AC_{<\mu}, \mu = \sum_n \kappa_n.

A new suggestion in f6.2, f6.3d, f6.9(5) has not been elaborated on.]

\[\text{(5A)}\] Connection to IND, pg.13

\[\text{(4A)}\] Appendix, pg.19

[We repeat [Sh:938, \$5].]

NOTE: pg.91 - can't read the top of this page

Discussion 13.8. Whereas our original intention was to use IND(\( \text{x} \)), we actually use only IND'(\( \text{x} \)), which is much better.

Definition 13.9. 1) IND'(\( \langle Y_n, D_n \rangle : n < \omega \)) means that if no \( \bar{F} = \langle F_n : n < \omega \rangle \) is a witness against it which means:

(a) \( F_n \) is a two-place function from \( I_{\alpha+1} \cup \{x\} \) into dual(\( D_n \))
(b) there are no \( \bar{t}_n = (t_{n,\ell} : \ell < n) \in I^1_{0,n} \) for \( n < \omega \), stipulating \( t_{n,n} = \bar{x} \) we have \( m < n \Rightarrow t_{n,m}^{\bar{t}_n,m} \notin F_m(t_{n,m+1}, \bar{t}_{n+1,m}) \).

2) Let \( \text{IND}'((Y_n, D_n) : n < \omega) \) means that there is no \( \{ F_{m,n} : m, n < \omega \} \) a witness against it which means:

(a) \( F_{m,n} \) is a two-place function from \( I_n \cup \{ \} \) into dual\( (D_n) \)

(b) \( u_{n,\varepsilon,\xi,\ell} \subseteq \{(\varepsilon_1, \xi_1 : \varepsilon_1 < \xi_1 \leq \xi) \} \) coming from \( (\mathcal{F}_{n,\epsilon}, \mathcal{F}_{n,\xi}) \).

**Claim 13.11.** \( \text{[ZFC]} \) 1) If we try to prove 3.13 with choosing \( \bar{y} \)’s witnessing failure of \( \text{IND}(\bar{x}) \) can we combine to get a contradiction? We have the \( Z \)’s colouring by large subsets of \( Y_{0,n} \) with sub-additivity.

2) If \( Y_n = \kappa_n, Y - D_n \)-co-countable.

**Discussion 13.12.** We may wonder on relatives on 3.13. First, if instead \( \text{ps-Depth} \) we use \( \text{Depth} \) it seems that \( \text{ranks instead of depth.} \)

Does looking at the proof of 3.13 give more?

**Definition 13.13.** 1) We say \( \bar{f} \) is an \( (\bar{A}, \bar{x}, \bar{\zeta}) \)-system or \( (\bar{A}, \bar{x}, \bar{\zeta}) \) is a system when

\[
\begin{align*}
& (a) \bar{x} = \langle (Y_n, D_n : n < \omega), D_n \rangle \text{ a filter on } Y_n \\
& (b) \bar{\zeta} \text{ an ordinal} \\
& (c) \bar{f} = (f_{n,\varepsilon} : n < \omega, \varepsilon \leq \zeta) \\
& (d) f_{n,\varepsilon} \subseteq I^n_\xi (\text{with full choice without a more complicated}) \\
& (e) \varepsilon < \xi \leq \zeta \text{ and } n < \omega \text{ then } f_{n,\varepsilon} \leq_{D_n} f_{n,\xi}. \\
\end{align*}
\]

2) we say the pair \( (\bar{t}, \bar{e}) \) solve the system \( (\bar{A}, \bar{x}, \bar{\zeta}) \) when

\[
\begin{align*}
& (a) \bar{t} \in \prod_{n<\omega} Y_n \\
& (b) \bar{e} = \langle \varepsilon_n : n < \omega \rangle \text{ where } \varepsilon_n = \langle \varepsilon_{n,\ell} : \ell \leq n \rangle, \varepsilon_{n,\ell} \leq \xi.
\end{align*}
\]

**Remark 13.14.** With little choice for \( n < \omega, \varepsilon < \xi \leq \varepsilon \) we have \( \langle u_{n,\varepsilon,\zeta,\ell} : t \in I_n \rangle \).

If \( D_n+1 \) is \( \lambda_\xi^+ \)-complete then ?

**Theorem 13.15.** \( \text{[AC}_{Y_n} \text{ for } n < \omega.] \)

Assume \( D_n \) is an \( \aleph_\xi \)-complete on \( Y_n \) for \( n < \omega \) and \( \text{IND}((D_n : n < \omega) \text{ then for every } \zeta, \text{ for some } n \text{ we have } \text{rk}_{D_n}(\zeta) = \zeta.} \)

**Definition 13.16.** \( \text{AC}_{Y^2} \) where for every \( \langle A_y : y \in Y \rangle \) there is \( \langle B_y : y \in Y \rangle \) such that \( A_\emptyset \neq \emptyset \Rightarrow B_\emptyset \neq \emptyset, |B_y| < |Z| \).
Question 13.17. Interesting? Natural for a sequence \((\leq Z)\)-complete filter, as in we can use \(\bigcap_{a \in B} : y \in Y\).

Proof. We choose \(g_n, Z_n\) as in the proof of 3.13 using the definition. □

Remark 13.18. 1) In (5B), ??(2) silly? We can find disjoint \(Y_1, Y_2\) with \(\text{id}(Y_1) = \text{id}(Y_2)\).
2) Definition ??(2) line 2: \(I \mapsto J\).

Discussion 13.19. Seemingly [Sh:835] connect well to [Sh:F955]. So assume \(\langle \lambda_i : i < \kappa \rangle\) is increasing with limit \(\mu\) and that is we should deal with a game, where..?

* * *

(955)
\section{Private Appendix}

Using pure $\Sigma$: July 2009

\begin{itemize}
    \item[(a)] $\langle \kappa_i : i < \text{cf}(\mu) \rangle$ is increasing with limit $\mu$
    \item[(b)] set $D$
    \item[(c)] $D_d$ a filter on $I_d = I[\bar{d}]$ for $d \in D$
    \item[(d)] for $d \in D$
        \begin{itemize}
            \item[(a)] $\Sigma(d) \subseteq \{(e, h) : e \in D$ and $h$ a function from $I_e$ onto $I_\bar{e}$ such that $D_d = \{h''(A) : A \in D_\bar{e}\}$
            \item[(b)] $\Sigma_{pr}(d) \subseteq \Sigma(d)$, a set of so called pure extensions
            \item[(c)] $\Sigma_{ap}(d) \subseteq \Sigma(d)$, a set of so called $\alpha$-pure extensions such that $(e, h) \in \Sigma_{ap}(d) \Rightarrow I_e = I_\bar{d} \land h = \text{id}_{I_d}$
            \item[(d)] $d \in \Sigma_{pr}(d) \cap \Sigma_{ap}(d)$
            \item[(e)] transitivity of $\Sigma$? $\Sigma_{pr}$? $\Sigma_{ap}$?
            \item[(f)] $\Sigma$ is a function from $D$ to $\text{cf}(\mu)$ and $D_d$ is $\kappa_d$-complete and $e \in \ell \text{ par}(d) \Rightarrow |S_\ell| < \kappa_d(d)$?
        \end{itemize}
    \item[(k)] $\text{par}(d)$ and for $p \in \text{part}(d)$, $\bar{X}_p = \{X_{p,s} : s \in S_p\}$ is a sequence of pairwise disjoint subsets of $I_d$ with union $\in D_d$ and $(e_{p,s} \cdot s \in S)$ is such that $e_{p,s} \in D, e_{p,s} = D_d + X_p.s$ so $e_{p,s} = d + X_p.s$
    \item[(l)] if $d_1 \in \Sigma_{pr}(d_0)$ and $d_2 \in \Sigma_{ap}(d_0)$ then $d_1 + d_2 = d_1 + d_2$ is a well defined member of $D_d$ and $d_3 \in \Sigma_{pr}(d_2) \cap \Sigma_{ap}(d_1)$
        \begin{itemize}
            \item[(a)] above
            \item[(b)] above if $e \in \Sigma(d_1) \cap \Sigma_1(d_2)$ then $e \in \Sigma(d)$.
        \end{itemize}
\end{itemize}

\textbf{Question 14.2.} Maybe $\text{cf}(\kappa)$ replaced by a linear order (which can have a pseudo cofinality)?

We now give examples

\begin{itemize}
    \item[(a)] $\mu = \Sigma \kappa_n$ and $\kappa$ is given
    \item[(b)] $D$ is the set of $d : d = (\eta, A) = (\eta_d, A_d)$ and for some $m = m_d \leq n = n_d < \omega$
        \begin{itemize}
            \item[(a)] $F_d = \{F : F = (F_{m_d, n_1} : m_d < m_1 \leq n_1 \leq n_d) \in F_{m_d, n_1} : m_d \leq m_1 < n_1 \leq n_d\}$ and $F_{m_1, :} \bigcap_{\ell = m_1 + 1}^n I_{q(\ell)} \rightarrow J_{q(m_1)}$
            \item[(b)] $\eta = \langle n, n - 1, \ldots, m\rangle$
            \item[(c)] $I_d = \prod_{\ell = 1}^m I_{\ell}$
            \item[(d)] $D_d = \{X \subseteq I_d :$ there are $X_\ell \in J_\ell$ for some $\bar{F} \in \mathcal{F}_d$ for $\ell \in [m, n]$ such that $A \cap X \subseteq \{\rho \in I_d : \rho(m_1) \notin F_{m_1, n_1}(\rho[m_1, n_1]) \text{ whenever } m_d \leq m_1 < n_1 \leq n_d\}\}$.
        \end{itemize}
\end{itemize}
\( \emptyset \notin D_d \) and nec \( D_d \) is \( \kappa_m \)-complete

(c) for \( d \in \mathbb{D} \)

(\( \alpha \)) let \( \Sigma(d) \) be the set of pairs \( (e, h) \) such that \( e \in \mathbb{D}, n_e = m_d \leq n_d \leq n_e, F_{m_1, n_1}^e = F_{m_1, n_1}^d \) when \( n_d \leq m_1 < n_1 \leq n_d \) and \( h = h(\rho) = \rho | [m_d, n_d) \)

(\( \beta \)) \( \Sigma_{d'}(d) = \{ (d, h) \in \Sigma(d) : F_{m_1, n_1}^e \) is constantly \( \emptyset \) when \( n_d < n_1 \) and \( m_d \leq m_1 < n_1 \leq n_d \)

(\( \gamma \)) \( \Sigma_{d, n_d}(d) = \{ (e, h) \in \Sigma(d) : h = id_{I_d} \) so \( n_e = n_d \)

(\( d \)) for \( d \in \mathbb{D}_s \) and \( A \in D_d^+ \) let \( \mathbf{e} = d + A \in \mathbb{D} \) be defined naturally, it is \( \langle \eta_1, A_d \cap A, F \rangle \)

(\( e \)) part \( \langle \rangle \) is the set of \( p = (X_{p, s}, e_{p, s}) : s \in S \) such that: for some so-called witness \( G = \text{langle}G_{m_1, n_1} : m_d \leq m_1 < n_1 \leq n_d, G_{m_1, n_1} : I_{m_1+1, n_1} \rightarrow \kappa_{n_1} \) with bounded range letting \( S_1 = \{ (\alpha_{m_1, n_1} : m_d \leq m_1 < n_1 \leq n_d) :\alpha_{m_1, n_1} < \kappa_{m_1} \) and \( A_a = \{ \rho \in I_d \rightarrow G_{m_1, n_1}(\rho)[m_1 + 1, n_1] = \alpha_{m_1, n_1} \) for \( m_1 < n_1 \) from \( [m_d, n_d) \) we have \( S_p = \{ \bar{a} \in A' : \emptyset \in D_d + A_a \} \) and \( e_{d, p, s} = d + A_a \)

(\( question \)): should we allow \( \text{Range}(G_{m_1, n_1}) \) to be large, etc.?

(\( f \)) \( \text{part}(d) = \{ p \in \text{par}(d) : |S_p| < \kappa_{m_d} \}

(\( question \): should we have \( \text{par}(d) \subseteq \{ (e, d, p) : (e, h) \in \Sigma(d) \) and as above?\)

**Discussion 14.4.** (09.8.17) 1) Discuss (here?) to achieve our hope (dichotomy using \[ Sh:835 \]). We would like for every \( \eta \in \mathbb{D}_x = \text{dec}_{<\omega}(\mathbb{O}) \) to define what are \( \eta \)-objects which are a replacement for \( (I_n)_{\text{Ord}} \). Maybe we should replace \( \text{dec}_{<\omega}(\theta) \) by closing \( \mathbb{O} \) by ordered pairing, but first ignore this.

A natural try define when \( x \in \text{obj}(\eta) \) by induction on \( \ell g(\eta) \).

If \( \ell g(\eta) = 0 \) then \( x \) is just an ordinal.

If \( \ell g(\eta) = n + 1 \) then \( x \) consists of a non-empty set \( \mathcal{F} \in (I_n(\sigma))_{\text{Ord}} \), a set \( A \in D_{\eta(\sigma)}^+, A_\beta = \{ \ell \in A : \ell(t) > 0 \} \) (or \( A_f : f \in \mathcal{F}, A_\beta \in D_{\eta(\sigma)}^{+\alpha} \) and a function which gives for every \( f \in \mathcal{F} \) and \( \ell \in A_f \) and object \( x_f \), \( \xi \in \text{obj}(\eta(n + 1) : \ell < n_i) \).

We have to: (A) define rank, (B) using DC criterion for the rank being an ordinal, (C) reprove \[ Sh:938 \] main Theorem.

2) (09.8.26) The example in \[ Sh:938, \angle 0 \] can be pushed up: use \( \lambda + \kappa_{\omega} \), ordinal addition, \( (\lambda, \kappa, \lambda(\lambda) = \lambda \) for all relevant \( J \)'s. Hence it seems there is no hope for \( \mu = \kappa_{\omega} \) but there may be for \( \mu = \kappa_{\omega} \). At least combine \( \mu = \kappa_{\omega}, \theta(\mathcal{P}(\lambda_n)) < \mu_{n+1}, \mu = \sum_{n \in \omega} \lambda_n \) and \( \text{Ind}(\lambda_n : n : \omega) \) or try the proof of \[ Sh:460, \angle 1 \].

**Claim/Definition 14.5.** Like 14.3 but \( \bar{J} = \langle J_n : n \in \mathbb{O} \rangle \), FILL. Now \( \eta_d \) is a decreasing sequence of length \( n_d + 1 \), so \( D_d \) is \( \kappa_{n_d(n_d)} \)-complete and \( e \in \Sigma(d) \) implies \( \eta_d(n_e) = \eta_d(n_d), \text{Rang}(\eta_d) \subseteq \text{Rang}(\eta_e) \).

**Convention 14.6.** We naturally let \( s = \langle \bar{\kappa}_s, \mu_s, D_s, \text{par}(\ell, \gamma) \rangle \) and \( I_{s, \text{d}}, D_{s, \text{d}}, S_{s, \text{d}}, X_{s, \text{d}, \text{p}}, D_{s, \text{p}, \text{d}}, S_{s, \text{p}, \text{d}}, \).

**Definition 14.7.** Given a frame \( s \) let \( \text{true}(s) \) be the set of objects \( t \) consisting of:

(\( a \)) \( \mathcal{R} \) a set of finite sequences closed under initial segments

(\( b \)) \( \mathbf{d}_{t, \rho} \in \mathbb{D} \) for \( t \in \mathcal{T} \)

(\( c \)) \( h_t = \{ h^t_{\rho, \rho} : \rho \leq \rho \in \mathcal{R} \} \)
Like 15.9 - FILL - rk

\{use of choice.

The choice in Case 1 may replace "tr" by 1 or omit it; by definition by induction on the ordinal \( \zeta \)

Definition 14.12. For \((r, k)\) as in 14.3 or 14.5 let \( \text{IND}(r, k) \) mean that:

Case 1: Definition 14.3 for every \( F_{m,n} : I_{m+1,n} \to J_m \) for \( m < n < \omega \) there is \( \eta \in \prod \{ I_{\ell} \} \) such that \( m < n < \omega \Rightarrow \eta(\ell) \notin F_{m,n}(\eta | [m + 1, \eta]) \).
Case 2: Definition 14.5

[copied] 1) Above \( p^*_j \) is not well 0-founded \( \iff \) there are \( \bar{e}, \bar{f} \) such that

\( \otimes_{\bar{e}, \bar{f}} \)

(a) \( \bar{e} = (e_i : i < \omega) \) is increasing
(b) \( \bar{f} = (f_{i,j} : i < j < \omega) \)
(c) \( f_{i,j} \) is a function from \( I(\bar{e}_i, \bar{e}_{i-1}, \ldots, \bar{e}_{i+j}) \) into \( J_{\bar{e}_i} \)
(d) for every \( \alpha \in \prod_{i<\omega} \kappa_{\bar{e}_i} \) for some \( i < j \) we have \( \alpha_i \in f(\alpha_{n_j}, \alpha_{n_{j-1}}, \ldots, \alpha_{n_{i+1}}) \).

\( \Box \)

Proof. FILL

We quote [Sh:938]

Definition 14.13. Main Definition: We say that \( p = (\mathbb{D}, \text{rk}, \Sigma, \mu) = (\mathbb{D}_p, \text{rk}_p, \Sigma_p, \mu_p) \) is a weak (rank) 1-system when:

(a) \( \mu \) is singular
(b) each \( d \in \mathbb{D} \) is (or just we can compute from it) a pair \( (I, D) = (I_d, D_d) = (I|d|, D|d|) \) such that:
   (\( \alpha \)) \( \theta(I_d) < \mu \), on \( \theta(\cdot) \) see ??
   (\( \beta \)) \( D_d \) is a filter on \( I_d \)
   (\( \gamma \)) for each \( d \in \mathbb{D} \), a definition of a function \( \text{rk}_d(\cdot) \) with domain \( I|d| \) Ord and range \( \subseteq \) Ord, that is \( \text{rk}_p d(\cdot) \) or \( \text{rk}_d^p(\cdot) \)
   (\( \delta \)) \( \Sigma \) is a function with domain \( \mathbb{D} \) such that \( \Sigma(d) \subseteq \mathbb{D} \)
   (\( \gamma \)) if \( d \in \mathbb{D} \) and \( e \in \Sigma(d) \) then \( I_e = I_d \) [natural to add \( D_d \subseteq D_e \),
   this is not demanded but see ??(2)]
(c) \( \alpha \) \( j \) is a function from \( \mathbb{D} \) onto \( \text{cf}(\mu) \)
(b) \( \gamma \) \( e \in \Sigma(d) \) \( \Rightarrow \) \( j(e) \geq j(d) \)
(f) for every \( \sigma < \mu \) for some \( i < \text{cf}(\mu) \), if \( d \in \mathbb{D}_{\geq i} \), then \( d \) is \( (p, \leq \sigma) \)-complete where:
   (\( \gamma \)) \( we \ say \ that \ d \ is \ (p, \leq X) \)-complete \( (or \ (\leq X)) \)-complete for \( p \) when: \( f \in I|d| \) Ord and \( \zeta = \text{rk}_d(f) \) and \( \langle A_j : j \in X \rangle \) a partition\(^7\) of \( I_d \), then for some \( e \in \Sigma(d) \) and \( j < \sigma \) we have \( A_j \in D_e \) and \( \zeta = \text{rk}_e(f) \); so this is not the same as \( "D_d \) is \( (\leq X) \)-complete\(^6\); we define \( (p, |X|) \)-complete, i.e. \( (p, < |X|) \)-complete similarly
(g) \( (p, \leq X) \)-complete \( \iff \) if \( \text{rk}_d(f) > \zeta \) then for some pair \( (e, g) \) we have: \( e \in \Sigma(d) \) and \( g < D|e| f \) and \( \text{rk}_e(g) = \zeta \)

\( m30 \)

Definition 14.14. We say \( p \) is a quasi rank \( \iota \)-system when \( p = (\mathbb{D}, \text{rk}, \Sigma, j, \mu) = (\mathbb{D}_p, \text{rk}_p, \Sigma_p, j_p, \mu_p) \) satisfies Definition \( m4.3 \) of \( \S 3 \) of [Sh:938] if \( \iota = 1 \), Definition \( m4.4 \) of \( \S 3 \) of [Sh:938] if \( \iota = 2 \) except that the rank may be \( \infty \); we write \( \text{rk}_d(f, d) \) for \( d \in \mathbb{D}_p \) and \( f \in I|d| \) Ord.

\(^7\)as long as \( \sigma \) is a well ordered set it does not matter whether we use a partition or just a covering, i.e. \( \cup \{ A_j : j \in \sigma \} = I_d \)
\(^6\)we may use another function \( \Sigma \) here, as in natural examples here we use \( \Sigma(d) = \{ d \} \) and not so in clause (f)
Definition/Claim 14.15. For a frame $s$ let $p$ be the following quasi rank system:

- $\mu, D, \Sigma, j$ are as in Definition 14.1
- $\text{rk}_d(f)$ is as in Definition ?

Claim 14.16. 1) If $(\bar{\kappa}, \bar{J})$ is as in Definition 14.3 or 14.5 and $\text{IND}(\bar{\kappa}, J)$ holds, see Definition 14.12 then $p_{\bar{\kappa}(\kappa, J)}$ is a rak system.

2) Moreover it is a strict one.

Saharon copied. 1) As in the proof of e5.g of §4 of [Sh:938, §4,e5,g] or better see the proof of 15.17 except that we use 15.9 instead of 15.8 which simplify clause (f), but is cumbersome in other places.

2) We check Definition m4.3 of §3 of [Sh:938, §3,m4.3].

Clause (a): $\mu$ is singular.

As $\mu = \sum_n \kappa_n$ and $\kappa_n < \kappa_{n+1}$ this is obvious.

Clause (b): Let $d \in \mathcal{D}, \eta = \eta_d, J = J_n$ now clause (a) says $\theta(I_\eta) = \theta(|I_\eta|) = \kappa_{\eta(0)}\times \kappa_{\eta(0)+1} < \mu$ so as for clause (b), “$D_p$ is a filter on $I_\eta$”, it holds by the choice of $p$.

Clause (c): $\text{rk}_d^p(f) = \text{rk}_d(f, p)$ is an ordinal as defined in 15.9.

Clause (d):

Clearly $\Sigma(d)$ is of the right form.

Clause (e):

On $j$ - see 15.13(2)(c).

Clause (f):

We prove by induction on the ordinal $\zeta$ that:

(∗) if $d \in \mathcal{D}$ and $j(d) > \varepsilon$ and $A = \cup\{A_\alpha : \alpha < \kappa_\varepsilon\} \in D_d$ and $f \in I[d]\text{Ord}$ and $A_\alpha \in D_\alpha^+ \Rightarrow \text{rk}_d+A_\alpha(f) \geq \alpha \text{ then } \text{rk}_d(f) \geq \alpha$.

Now Definition 15.9 is tailored made for this.

Older version using 15.8 recheck:

For $\alpha = 0$ and $\alpha$ a limit ordinal this is obvious. For $\alpha = \beta + 1$ let $\mathcal{Y} = \{\alpha < \kappa_\varepsilon : A_\alpha \in D_\alpha^+\}$ and for $\alpha \in \mathcal{Y}$ let $n_\alpha = \min\{n : \text{ there is } (e, h) \in \Sigma(d + A_\alpha) \text{ such that } \text{rk}_e(f \circ h) \geq \beta \text{ and } \eta_\alpha(0) = n\}$. Clearly $n_\alpha$ is well defined for $\alpha \in Y$, and let $w := \{n : \cup\{A_\alpha : \alpha \in \mathcal{Y} \text{ and } n_\alpha = n\} \in D_\alpha^+\}$ and also the rest should be clear.

Clause (g): (no-hole)

By the Definition 15.8 or 15.8 of rk. Saharon 09.5.31 recheck.

Clause (h): $\text{rk}_d(f + 1) = \text{rk}_d(f) + 1$.

We prove by induction on the ordinal $\alpha$ that:

(∗) for every $d \in \mathcal{D}$ and $f \in I[d]\text{Ord}$ we have $\text{rk}_d(f) \geq \alpha \iff \text{rk}_d(f + 1) \geq \alpha + 1$.

Clause (i): Obvious. □
Question 14.17. (09.7.19) Assume little choice and \( \mu_* = \min\{\mu : \text{IND}(\mu)\} \). So up to \( \mu \) we can apply [Sh:835]. Now above it seemed that if \( \alpha < \mu \Rightarrow AC_{\alpha} \) and \( \mu \) is a limit cardinal, we can find bound above to \( \text{rk}_d \) hence to \( \text{rk}_J(-) \) for \( J \) quite complete ideal.

1) Assume \( \text{cf}(\mu_*) = \aleph_0 \), we try to apply the above replacing \( AC_{<\mu} \) by \( DC + (\forall \alpha < \mu_*)(-\text{IND}(\mu)) \). So the problem is, on the one hand, about [Sh:938, §3] with weaker form of choice (as in [Sh:835]) and on the other hand the right use of \( \text{IND}(\mu_*) \) here.

2) What above \( \mu \) is a successor?

3) Even with choice, the bound on rank does not give a bound on pp or \( tcf(\mu^{< \kappa}, < \delta) \) well above \( \theta(\mathcal{P}(\kappa_\alpha)) \) it gives with choice/without much choice - as can be done in §1.

Claim 14.18. 1) If \( (f_n : \alpha < \delta) \) is \( <_D \)-increasing in \( (\Pi, \in, <_D) \) then \( \text{rk}_D(\alpha) \geq \delta \).

2) If \( (f_n : \alpha < \mu) \) are \( \not\equiv_D \)-distinct in \( (\Pi, \in, <_D) \) and \( \mu > \theta(\mathcal{P}(\ell g(\check{\alpha}))) \) then we can use [Sh:E38] which continues [Sh:497].

3) As in (1) devise \( m \) to be \( \leq \mathcal{P}(\kappa_n) \) on each for some \( D_2 \supseteq D \) the sequence is increasing.

The following information is not presently

Theorem 14.19. 1) If \( \text{IND}(\{\kappa_n : n < \omega\}) \) then \([?]\) - FILL.

2) For \( \aleph_\omega - [\text{FILL}] \).

§ 15. Connection to IND

Remark 15.1. 1) Below we can concentrate on the case \( \ell g(\check{J}) = \omega, (\kappa_n : n < \omega) \) increasing, even \( 2^{< \kappa} < \kappa_{n+1} \) and \( \kappa_n = \text{cf}(\kappa_n) \).

2) We like to use less choice say only DC and \( AC_{<\mu} \mu = \sum_n \kappa_n \). This is not achieved for \( q_1, q_2 \), so it seems. So we may like to change [Sh:938, §3]. Consider \( k = 2, 4 \) in 15.13(2) to use.

3) (09.7.18) We may hope that if \( J_n = [\kappa_n]^{< \sigma} \) we need only, e.g. DC + \( AC_{\mathcal{P}(\sigma)} \). But then we do not look at \( J_{n+1} + A, |A| = \kappa_{n+1} \). So maybe have \( (J_1^1, J_2^2 : n < \omega) \), see 15.14 or maybe have \( J_{m,n} \) an ideal on \( \kappa_n, J_{m,n} = [\kappa_n]^{< \lambda_m}, \) see 15.19.

4) (09.7.18) Try \( \text{IND}_\kappa(\mu) \) or so \( (\tau(\mathcal{X})) = \kappa, [\mathcal{X}] = \mu, \) no \( \omega \)-end-independent sequence or \( \text{IND}(\mu, I_i : i < \kappa) \) looking for \( i_n < i_{n+1} < \text{ldot} s e_m \in \mu_m, \alpha_m \notin F(\alpha_{n+1}, \ldots, \alpha_m) \in I_{\alpha_m} \). Can we connect by Fodor?
5) (09.7.18) To define the ranks for \( p \) we better revise the pre-rank-system as follows. For every \( d \) we have \( \Sigma_p(d) = \Sigma_p^p(d) \), the pure successors and \( \Sigma_{ap}(d) = \Sigma_{ap}^p(d) \) the pure ones and we have interpolation. In the conclusion we try.

In clause \((f)\), \( p \)-completeness, we shall try to get \( e \in \Sigma_p(d) \).

In clause \((i)\), also if \((e, h) \in \Sigma_p(d), f \in ^1d \text{Ord}, g = f \circ h \in ^1e \text{Ord} \) then \( \text{rk}_d(f) = \text{rk}_e(g) \).

In the definition of \( \text{rk}_p \), we may instead of \( \text{rk}_{p,c}((-,-)) \) ask for a tree of pure extensions, but well founded tree.

5A) The natural case is \( J = (J_n : n < \omega), D = \{ \eta : \eta \text{ is } (n, n-1, \ldots, m) \}, \Sigma_p(d) \) is as there but \( \eta_0 = \eta \) and on \( I_{\eta(f)} \) we use the original \( J \). This fine to see that it fits. If \( O \) or \( \kappa \) larger, we allow “side extension of \( \eta \)” but \( \min \text{Rang}(\eta) \) remains.

6) (09.7.18) But later we have preservation of ranks when we use isomorphic \( p \) or \( p \) restricted to “\( d \) and above”. So if \( J_n = J_d^{\text{bd}}, \kappa_n \) regular, \( J_d^{\text{bd}}, J_d^{\text{bd}} + A \) are the same.

6A) Maybe legal partitions of \( \prod \Sigma_{\text{h},\ell} \) is when \( I_{\eta(\ell)} \) is divided to \( < \kappa_{\eta(\ell)} \).

\[ \text{Definition 15.2.} \]

1) Let \( \tilde{J} \) be called a candidate or \( \delta \)-candidate when:

\( (a) \) \( \tilde{J} = \{ J_{\varepsilon} : \varepsilon < \delta \}, \delta \) a limit ordinal

\( (b) \) \( J_{\varepsilon} \) is an ideal on \( \kappa_{\varepsilon} \)

\( (c) \) \( \delta < \kappa_0 \) and \( \kappa_\varepsilon \) is non-decreasing.

2) We say that \( \tilde{J} \) is a generalized candidate when for some \( O \):

\( (a) \) \( O \) is a linear order with no last element

\( (b) \) \( J = \{ J_{\varepsilon} : \varepsilon \in O \} \)

\( (c) \) \( J_{\varepsilon} \) is a \( \kappa_{\varepsilon} \)-complete ideal on \( I_{\varepsilon} := \text{Dom}(J_\varepsilon) = \cup \{ u : u \in J_{\varepsilon} \} \).

In some sense the simplest example is

\[ \text{Example 15.3.} \]

Let \( \langle \kappa_n : n < \omega \rangle \) be an increasing sequence of ordinals, \( J_n := [\kappa_n]_{< \kappa_0} \).

\[ \text{Discussion 15.4.} \]

(08.6.27) 1) We shall try to define a rank (from a p.r.s. or p.r.s.*) such that clause \((j)\) of m4.6 of §3 of [Sh:938] follows. It seems that a necessary condition for the rank to be \( < \omega \) we need \( \text{IND}(p) \).

2) Naturally we can define \( p \) from \( J \) and a reasonable condition is \( \text{IND}(\tilde{J}) \) at least when \( \ell_{\eta}(J) = \omega \).

3) We can below use generalized candidates.

\[ \text{Definition 15.5.} \]

1) We say \( p = (D, \Sigma, j) \) be a \( \iota \)-p.r.s. (pre-rank-\( \iota \)-system with \( \iota = 1, 2 \); if \( \iota = 2 \) we may omit it) \text{ when} in Definition m4.3 or m4.4 of §3 of [Sh:938, §3,m4.4] it satisfies clauses \((a),(b),(d),(v)\) and we add in \( (d)\):

\( (*) \) \( \Sigma \) is transitive: if \( (h_1, d_1) \in \Sigma(d_0) \) and \( (h_2, d_2) \in \Sigma(d_1) \) then \( (h_2 \circ h_1, d_2) \in \Sigma(d_0) \)

[check where used].

2) We say \( p \) is a quasi rank \( \iota \)-system \text{ when} \( p = (D, \text{rk}, \Sigma, j, \mu) = (D_p, \text{rk}_p, \Sigma_p, j_p, \mu_p) \) satisfies Definition m4.3 of §3 of [Sh:938] if \( \iota = 1, \) Definition m4.4 of §3 of [Sh:938] if \( \iota = 2 \) except that the rank may be \( \infty \); we write \( \text{rk}_d(f, d) \) for \( d \in D_p \) and \( f \in ^I[d] \text{Ord} \).
2A) Alternatively: \( \text{rk}_p \) is defined as in 15.8 below [or 15.9].

**Convention 15.6.**
1) \( p \) is a 2-p.r.s.
2) We usually omit the \( p \) when clear from the context, similarly for \( \text{rk}_d(f, p) \) defined below.

**Remark 15.7.**
1) We shall try to define \( \text{rk} \). We shall try to prove mainly (f) [the version with \( (e, h) \in D(d) \)].

**Definition 15.8.**
For \( p \) a p.r.s., \( d \in D \) and \( f \in I^d[\text{Ord}] \) we define \( \text{rk}_d(f, p) = \text{rk}_d^0(f, p) \) by defining when \( \text{rk}_d(f, p) \geq \alpha \) for an ordinal \( \alpha \) by induction on \( \alpha \) for all pairs \( (d, f) \); so \( \text{rk}_d^0(f, p) = \alpha \) when it is \( \geq \alpha \) but not \( \geq \alpha + 1 \), and is \( \alpha \) otherwise; by monotonicity well defined.

\[ \text{rk}_d^0(f, p) = \alpha \text{ by monotonicity well defined.} \]

\[ \text{pairs } \left( \begin{array}{c} \text{Definition 15.8.} \end{array} \right) \]

1) We shall try to define \( \text{rk} \). We shall try to prove mainly (f) [the version with \( (e, h) \in D(d) \)].

**Remark 15.10.**
1) In a variant we demand: and \( J_d = I_d \land h = \text{id}_{I(d)} \).
2) By 15.9 we may derive a quasi rank system from a p.r.s., but we deal with the special case which seems most interesting.

**Claim 15.11.**
1) The rank in Definition 15.8, 15.9 are well defined.

**Discussion 15.12.**
(09.06.01) 1) We would like to use AC\( \mathcal{W} \) for constant \( \mathcal{W} \) or at most \( \mathcal{W} \) depend on \( 0 \). By the amount of completeness we need (approaching \( \mu \)), if we use \( \text{rk}_d^0(-, f_1) \) is it O.K.? Does it?

**Definition 15.13.**
1) For \( \ell = 1, 2 \) and \( p \) a p.r.s. we say \( p \) is well \( \ell \)-founded when \( \text{rk}_d^0(f, p) < \infty \) for every \( d \in D \) and \( f \in I^d[\text{Ord}] \).
2) Similarly for \( p \) a quasi rank system (so now \( \text{rk}_d(f, p) \) is not as defined in Definition 15.9, but is from Definition 15.5(2)).

**Definition 15.14.** For a candidate \( J = (J_\varepsilon : \varepsilon \in \delta) \), \( J_\varepsilon \) an ideal on \( \kappa_\varepsilon \) we define \( p = p_J \) as follows:
(a) $D_\eta$ is the set of $\mathbf{d} = (I, D) = (I_\eta, D_\eta)$ such that for some $\eta = \eta_\mathbf{d}$ we have:

(1) $\eta$ a non-empty decreasing sequence of ordinals $< \delta$

(2) $I = \prod_{\ell < \ell(g(\eta))} \kappa_{\eta(\ell)}$

(3) $D = D_\eta + A_\mathbf{d}$ for some $(k, \eta)$-large subset of $I_\eta$ which means

(4) $A \subseteq I_\eta$ is $(k, \eta)$-large when $A = \prod_{\ell < n} Y_\ell$ for some $Y_\ell \in [\kappa_{\eta(\ell)}]^m_{\kappa(\ell)}$ for

$\ell < n$ and

(5) let $u_\mathbf{d} = \text{Rang}(\eta_\mathbf{d}), D_{u_\mathbf{d}} = D_\eta$

(6) $D = \{ Y \subseteq \prod_{\ell < \ell(g(\eta))} \kappa_{\eta(\ell)} :$ there is a sequence $\langle Y_\ell : \ell \leq \ell(g(\eta)) \rangle$

such that $Y_n = Y, Y_0 = \langle \rangle$ and $\ell \leq \ell(g(\eta)) \Rightarrow Y_\ell \subseteq \prod_{m < \ell} \kappa_{\eta(m)}$

and $\ell < \ell(g(\eta)) \land \rho \in Y_\ell \Rightarrow \{ \alpha < \kappa_{\eta(\ell)} : \rho \vdash (\alpha) \not\in Y_{\ell + 1} \} \in J_{\eta(\ell)}$}

(b) $\Sigma(d) = \{ (h, e) :$ for some $g$ we have $\eta_e = g^{\kappa_\mathbf{d}} \in D$ and $h : I_\nu \rightarrow I_\eta$ is defined by $h(\rho) = \langle \rho(\ell(g) + \ell) : \ell < \ell(g(\eta)) \rangle$ and $h$ induces a mapping from $D_\eta$ into $D_\mathbf{d} \}$

(c) $\delta(\eta) = \eta(\ell(g(\eta)) - 1)

(d) $\mu = \cup \{ \kappa_\varepsilon : \varepsilon < \delta \}$.

**Definition 15.15.** 1) Similarly to 15.14 for a generalized candidate $\bar{J} = \langle J_\varepsilon : \varepsilon \in O \rangle$.

2) For a candidate $\bar{J} = \langle J_\varepsilon : n < \omega \rangle$ we define $\mathbf{p}_J^2 = (\mathbb{D}, \text{rk}, \Sigma, j, \mu)$ as in 15.14 but:

(a) $\mathbb{D} = \{ \mathbf{d} : \mathbf{d}$ as in clause (a) of Definition 15.14 but $\eta_\mathbf{d} = \langle n, n - 1, \ldots, \rangle$

where $m \leq n \} \quad \{\text{f6.11}\}

(c) $\delta(\eta) = \eta(\ell(g(\eta)) - 1)

(d) $\mu = \cup \{ \kappa_\varepsilon : \varepsilon \in \delta \}$

3) We define $\mathbf{p}_J^{k+2}$ as in part (1) or by ?? but replace clause (a)(δ) of ?? or part (1) by:

(δ) $D_\eta = \{ Y \subseteq \prod_{\ell < \ell(g(\eta))} \kappa_{\eta(\ell)} :$ for some $Y_\ell \in J_\varepsilon$ for $\ell < n$ we have $\prod_{\ell < n} \kappa_{\eta(\ell)} \setminus \{ \rho < \prod_{\ell < n} \kappa_{\eta(\ell)}(\exists \ell < n)[\rho(\ell) \in Y_\ell] \} \} \quad \{\text{f6.12}\}

4) For $\ell = 0, 1$ let $\mathbf{q}_J^{\ell} \tilde{=} \mathbf{q}_J^{k+2}$ be the $\mathbf{q}_J^{k+2}$ expanded by $\text{rk}_\mathbf{d}(J, \mathbf{p}_J^{k+2})$. If $\ell = 1$ we may omit it.

**Claim 15.16.** 1) Above $\mathbf{p}_J^1$ is not well 0-founded iff there are $\varepsilon, \bar{f}$ such that

(α) $\varepsilon = (\varepsilon_i : i < \omega)$ is increasing

(δ) $f_i$ is a function from $I_{(\varepsilon_i, \varepsilon_{i-1}, \ldots, \varepsilon_{i+1})}$ into $J_{\varepsilon_i}$

(δ) for all $\bar{a} \in \prod_{i < \omega} \kappa_{\varepsilon_i}$ for some $i < j$ we have $\alpha_i \in f(\alpha_{j,j}, \alpha_{j-1,j}, \ldots, \alpha_{j+1,j})$.

2) Similarly for $\mathbf{p}_J^2$ (i.e. $\delta = \omega$ we can above demand $\varepsilon_i = i$, so it is equivalent to $\neg \text{IND}(J_\varepsilon : n < \omega)$.

**Proof.** 1) As in [Sh513].

2) Easy as we can add to a function dummy variables. \[\square_{15.16}\]

**Task:** 1) Prove $\mathbf{p}_J^2$ satisfies clause (f) for $\text{rk} = \text{rk}_\mathbf{d}^1$ defined as in 15.8.

2) Check the $\text{rk}(f + 1) = \text{rk}(f) + 1$, but see below.
Claim 15.17. 1) If \( J = \langle J_\varepsilon : \varepsilon \in O \rangle \) is a generalized candidate and \( k = 1, 3 \) then \( p_k^J \) is a p.r.s. provided that "\( J_\varepsilon \) is \( \theta(O) \)-complete"(?)

2) If \( J = \langle J_n : n < \omega \rangle \) is a candidate and \( k = 2, 4 \) then \( p_k^J \) is a p.r.s.

3) In part (1), \( q_k^J \) is a quasi rank system.

4) Assume \( J = \langle J_n : n < \omega \rangle \), \( J_n \) an ideal on \( \kappa_n, \kappa_n^+ < \kappa_n^+ + 1 \), \( \mu = \Sigma \kappa_n \). Then \( q_k^J \) is a quasi rank system.

Proof. 1) As in the proof of e5.g of §4 of [Sh:938, §4,e5.g] or better see the proof of 15.17(?) except that we use 15.9 instead of 15.8 which simplify clause (f), but is cumbersome in other places.

2) We check Definition m4.3 of §3 of [Sh:938, §3,m4.3].

Clause (a): \( \mu \) is singular.

As \( \mu = \sum_n \kappa_n \) and \( \kappa_n < \kappa_n^+ + 1 \) this is obvious.

Clause (b): Let \( d \in D, \eta = \eta_d, J = J_n \) now clause (a) says \( \theta(I_\eta) = \theta(|I_\eta|) = \kappa_\eta(0), \kappa_\eta(0) + 1 < \mu \) so as for clause (b), "\( D_p \) is a filter on \( I_\eta \)", it holds by the choice of \( p \).

Clause (c): \( rk_d^P(f) = rk_d(f, p) \) is an ordinal as defined in 15.9.

Clause (d):

Clearly \( \Sigma(d) \) is of the right form.

Clause (e):

On \( j \) - see 15.13(2)(c).

Clause (f):

We prove by induction on the ordinal \( \zeta \) that:

\[ (\ast) \text{ if } d \in D \text{ and } j(d) > \varepsilon \text{ and } A = \bigcup \{ A_\alpha : \alpha < \kappa_\varepsilon \} \in D_d \text{ and } f \in I[d] \text{Ord and } A_\alpha \in D_d^+ \Rightarrow rk_{d+A_\alpha}(f) \geq \alpha \text{ then } rk_d(f) \geq \alpha. \]

Now Definition 15.9 is tailored made for this.

Older version using 15.8 recheck:

For \( \alpha = 0 \) and \( \alpha \) a limit ordinal this is obvious. For \( \alpha = \beta + 1 \) let \( \mathcal{Y} = \{ \alpha < \kappa_\varepsilon : A_\alpha \in D_d^+ \} \) and for \( \alpha \in \mathcal{Y} \) let \( n_\alpha = \min \{ n : \alpha \in \mathcal{Y} \text{ and } n \} \). Clearly \( n_\alpha \) is well defined for \( \alpha \in Y \), and let \( w := \{ n : \alpha \in \mathcal{Y} \text{ and } n_\alpha = n \} \in D_d^+ \) and also the rest should be clear.

Clause (g): (no-hole)

By the Definition 15.8 or 15.8 of rk. Saharon 09.5.31 recheck.

Clause (h): \( rk_d(f + 1) = rk_d(f) + 1. \)

We prove by induction on the ordinal \( \alpha \) that:

\[ (\ast) \text{ for every } d \in D \text{ and } f \in I[d] \text{Ord we have } rk_d(f) \geq \alpha \Leftrightarrow rk_d(f + 1) \geq \alpha + 1. \]

Clause (i): Obvious. \( \Box_{15.17} \)
Claim 15.18. Assume $J = \langle J_n : n < \omega \rangle$ is a candidate and $\text{IND}(J)$.

Then $p_J^J$ is a strict rank system.

Proof. By 15.17 and the definition, it is a weak rank system. So we should prove the “strict”, i.e. clause (j) of Definition m4.6 of §3 of [Sh:938] which we do by m4.16 of §3 of [Sh:938]. We use $\Sigma_1(d) = \Sigma(d)$.

On $\langle \rangle_2$:
Given $d$ we choose $j < \omega$ such that $j > \eta_d(0)$ and assume $e \in D_{\geq j}$.

§2. Revisiting

The simplest case below is: $x$ consist $I_n = \kappa_n, \kappa_n < \kappa_{n+1}, J_{1,n} = [\kappa_n]^{< \theta}, J_{2,n} = [\kappa_n]^\kappa, \gamma, \mu = \Sigma\kappa_n, \mu$ minimal (or $\mu = \infty$) ind$\in$ : $\in$ Ord $\cup \{\infty\}$.

For $\mu$ there are algebras on $\gamma$ with no independent $\omega$-sequence hence [Sh:835] and see §5 apply. But if using $x$ we have a rank 2-system for which Theorem m4.13 of §3 of [?] apply (check!)

We may consider the pseudo version (using comp$_\gamma(J)$). We have to sort out the amount of choice needed –seemingly.

Definition 15.19. We say that $x$ is a $\omega$-candidate when it consists of

(a) set $I_n$ for $n < \omega$ ($\kappa$ a cardinal and $\theta(< \kappa) = \kappa$
(b) ideal $J_{n,k}$ on $I_n$ for $k < \omega, n < \omega$
(c) $J_{n,k} \subseteq J_{n,k+1}$
(d) $\kappa_n$.

Definition 15.20. For a 2-candidate $x$ we define by induction on $i < \omega$ what is an $x$-object $\epsilon = d$ of depth $i$, such that

(*) for some $n_d < m_d < \omega, i$ is an $\subseteq$-increasing sequence $\langle J_{i,k} : k < \omega \rangle$ of ideals on $I_{m_d,n_d} = \Pi\{I_k : k \in [m, n]\}$.

The case $i = 0$:

$n_d = m_d + 1$ and let $h_d$ be the one-to-one function from $I_{m_d}$ onto $I_{m,n}$ and $J_{i,k} \in h_d(I_{m,n} + A_k)$ where $A_k \in J_{m_d,n_d}$ and $A_k \supseteq A_{k+1}$ for $k < \omega$.

The case $i + 1$:

For some $k, i(1,2)$ we have

(a) $k \in (m_d,n_d)$
(b) $i(\ell)$ is an $i_{\ell}$-pair for some $i_{\ell} \leq i$ for $\ell = 1, 2$
(c) $m_d(1) = m_i, n_d(1) = k$
(d) $m_d(2) = k, n_d(1) = k$
(e) $m_d(2) = k, n_d(2) = n_i$
(f) there are $\langle A_{1,k}, A_{2,k} : k < \omega \rangle$ such that

(α) $A_{i,k} \in J_{i,\ell(k+1) + 1}$
(β) $B \in J_{d,k}$ iff $B \subseteq I_{m_d,n_d}$ and for some $B_1 \in I_{i(1,k)}$ we have $\eta \in A_{2,k} \subseteq I_{n_d(2), n_d(2)} \Rightarrow \{\nu \in I_{m_d(2), n_d(2)} : \eta \cup \nu \in B\} \in ?$
Remark 15.21. 1) Definition 15.20? seemingly does not behave transitively.
2) We may allow $n_d = m_d$.

Definition 15.22. For $x$ an $\omega$-candidate, we define a p.c.s. $p = p^0_x$ as follows:

(a) $D_p = \{d : d$ is an $x$-object$\}$
(b) $\Sigma(d_i) = \{d :$ for some $d_2$ the triple $(d, d_i, d_2)$ is as in Definition 15.20
(c) $j(d)$ is $m_d$
(d) $\mu = \cup\{\kappa_n : n < \omega\}$.

Claim 15.23. If $x$ is an $\omega$-candidate then $p^0_x$ is a quasi rank system.

Proof. FILL.

Definition 15.24.
1) For an $\omega$-candidate $x$ we say it is well founded when the p.r.s. $p_x^0$ is well founded, e.g. $p^0_x$ is a weak rank system.
2) For a well founded.

Claim 15.25. If $x$ is a well founded $\omega$-candidate then $p_x$ is a strict rank system.

Proof. Stage A: We have to check clause (1) from Definition m4.6 of § 3 of [Sh:938].
So assume $d, \zeta, \xi, f$ are as in $\oplus$ there. Choose $j < \omega$ such that $j > n_d$ and toward contradiction assume $e, g$ are as in $\oplus$ there.
Stage B: We find $(e_1, g_1)$ satisfying $\oplus$ of clause (j) of m4.6 of § 3 of [Sh:938] and $m_{e_1} = m_d$; note if we define as in $[?]$(2) rather than as in 15.13(3), we would not need this step, but then we may have to reconsider the proof of (f) of Definition m4.3 of § 3 of [Sh:938].
Stage C: We use AC$[\mathcal{I} e]$ we continue as in 15.18 and in § 4. But see footnote to $\bullet_3$ in $\oplus$ in clause (j) of m4.6 of § 3 of $[?]$.

§ 16. APPENDIX: PSEUDO TRUE COFINALITY

We repeat here [Sh:938, § 5].

Pseudo PCF

We try to develop pcf theory with little choice. We deal only with $\aleph_1$-complete filters, and replace cofinality and other basic notions by pseudo ones, see below. This is quite reasonable as with choice there is no difference.
This section main result are $??$, existence of filters with pseudo-true-cofinality; 16.19, giving a parallel of $J_{<\lambda}[\alpha]$; and 1.6, on generators of $J_{<\alpha}^{<\lambda}[\alpha]$. In the main case we may (in addition to ZF) assume DC + AC$[\mathcal{P}(\mathcal{P}(Y))]$; this will be continued in [Sh:938].

Hypothesis 16.1. ZF

Definition 16.2. 1) We say that a partial order $P$ is ($< \kappa$)-directed when every subset $A$ of $P$ of power $< \kappa$ has a common upper bound.
1A) Similarly $P$ is ($\leq S$)-directed.
2) We say that a partial order $P$ is pseudo ($< \kappa$)-directed when it is ($< \kappa$)-directed and moreover every subset $\cup\{P_\alpha : \alpha < \delta\}$ has a common upper bound when:
(a) if $\delta < \kappa$ is a limit ordinal
(b) $P = \langle P_\alpha : \alpha < \delta \rangle$ is a sequence of non-empty subsets of $P$
(c) if $\alpha_1 < \alpha_2, p_1 \in P_{\alpha_1}$ and $p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$.

2A) For a partial order $S$ we say that the partial order $P$ is pseudo $(\leq S)$-directed when $\cup \{ P_s : s \in S \}$ has a common upper bound whenever

\begin{enumerate}
  
  \item $\langle P_s : s \in S \rangle$ is a sequence
  \item $P_s \subseteq P$
  \item if $s <_S t$ and $f \in P_s, g \in P_t$ then $f <_P g$
  \item if $s \in S$ then $P_s$ has a common upper bound (so if $S$ has no minimal member this is redundant).
\end{enumerate}

**Definition 16.3.** We say that a partial (or quasi) order $P$ has pseudo true cofinality $\delta$ when: $\delta$ is a limit ordinal and there is a sequence $\langle P_\alpha : \alpha < \delta \rangle$ such that

\begin{enumerate}
  
  \item $P_\alpha \subseteq P$ and $\delta = \sup \{ \alpha < \delta : P_\alpha \text{ non-empty} \}$
  \item if $\alpha_1 < \alpha_2 < \delta, p_1 \in P_{\alpha_1}, p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$
  \item if $p \in P$ then for some $\alpha < \delta$ and $q \in P_\alpha$ we have $p \leq_P q$.
\end{enumerate}

**Remark 16.4.** 0) See 16.2(2) and 16.8(1).
1) We could replace $\delta$ by a partial order $Q$.
2) The most interesting case is in Definition 16.6.
3) We may in Definition 16.3 demand $\delta$ is a regular cardinal.
4) Usually in clause (a) without loss of generality $\bigwedge \alpha P_\alpha \neq \emptyset$, as without loss of generality $\delta = \text{cf}(\delta)$ using $P_\alpha^* = P_{f(\alpha)}$ where $f(\alpha) =$ the $\alpha$-th member of $\{ \beta < \delta : P_\beta \neq \emptyset \}$. Why do we allow $P_\alpha = \emptyset$? as it is more natural in 16.17(1), but can usually ignore it.

**Example 16.5.** Suppose we have a limit ordinal $\delta$ and a sequence $\langle A_\alpha : \alpha < \delta \rangle$ of sets with $\prod_{\alpha < \delta} A_\alpha = \emptyset$; moreover $u \subseteq \delta = \sup(u) \Rightarrow \prod_{\alpha \in u} A_\alpha = \emptyset$. Define a partial order $P$ by:

\begin{enumerate}
  
  \item its set of elements is $\{ (\alpha, a) : a \in A_\alpha \text{ and } \alpha < \delta \}$
  \item the order is $(\alpha_1, a_1) <_P (\alpha_2, a_2)$ iff $\alpha_1 < \alpha_2$ and $a_\ell \in A_{\alpha_\ell}$ for $\ell = 1, 2$.
\end{enumerate}

It seems very reasonable to say that $P$ has true cofinality but there is no increasing cofinal sequence.

**Definition 16.6.** 1) For a set $Y$ and sequence $\langle \alpha_t : t \in Y \rangle$ of ordinals and cardinal $\kappa$ we define $\text{ps-}\text{tcf-fill}_\kappa(\langle \alpha \rangle) = \{ D : D \text{ a } \kappa\text{-complete filter on } Y \text{ such that } (\Pi\alpha/D) \text{ has a pseudo true cofinality} \}$.

see below.

2) We say that $\Pi\alpha/D$ or $(\Pi\alpha, D)$ or $(\Pi\alpha, <_D)$ has pseudo true cofinality $\gamma$ when $D$ is a filter on $Y = \text{Dom}(\langle \alpha \rangle)$ and $\gamma$ is a limit ordinal and the partial order $(\Pi\alpha, <_D)$ essentially does\(^9\) i.e., there is a sequence $\mathcal{F} = \langle \mathcal{F}_\beta : \beta < \gamma \rangle$ satisfying:

\(^9\)so necessarily $\{ s \in Y : \alpha_s > 0 \}$ belongs to $D$ but is not necessarily empty; if it is non-empty then $\Pi\alpha = \emptyset$, so pedantically this is wrong, but we shall ignore this or assume $\bigwedge \alpha \alpha_t \neq 0$ when not said otherwise.
\[ \text{Claim 16.7.} \] 1) If \( \gamma = \text{ps-tcf}(\Pi\lambda, \langle < \rangle) \), then \( (\Pi\lambda, \langle < \rangle) \) is pseudo \((< \lambda)\)-directed.
2) Similarly for any quasi order.
3) Assume \( AC_\alpha \) for \( \alpha < \lambda \). If \( cf(\alpha_t) = \gamma \) for \( t \in Y \) then \( (\Pi\lambda, \langle < \rangle) \) is \( \lambda \)-directed.
4) Assume \( AC_{Y \lambda} \). If \( cf(\alpha_s) > \gamma \) for \( s \in Y \) then \( (\Pi\lambda, \langle < \rangle) \) is pseudo \( \lambda^+ \)-directed.

\text{Proof.} As in 16.8(1) below. \( \square_{16.7} \)

\[ \text{Claim 16.8.} \] Let \( \alpha = (\alpha_s : s \in Y) \) and \( D \) is a filter on \( Y \).
0) If \( \Pi\lambda / D \) has pseudo true cofinality then \( \text{ps-tcf}(\Pi\lambda, \langle < \rangle) \) is a regular cardinal; similarly for any partial order.
1) If \( \Pi\lambda / D \) has pseudo true cofinality \( \gamma_1 \) and true cofinality \( \gamma_2 \) then \( cf(\gamma_1) = cf(\gamma_2) = \text{ps-tcf}(\Pi\lambda, \langle < \rangle) \), similarly for any partial order.
2) \( \text{ps-tcf}(\alpha) \) is a set of regular cardinals so if \( \Pi\lambda / D \) has pseudo true cofinality then \( \text{ps-tcf}(\Pi\lambda, \langle < \rangle) \) is \( \gamma \) where \( \gamma = cf(\gamma) \) and \( \Pi\lambda / D \) has pseudo cofinality \( \gamma \).
3) Always \( \text{ps-tcf}(\alpha) \) has cardinality \( \theta(\{D : D a \kappa\text{-complete filter on } Y\}) \).
4) If \( \beta = (\beta_s : s \in Y) \in Y \text{Ord} \) and \( \{s : \beta_s = \alpha_s\} \in D \) then \( \text{ps-tcf}(\Pi\lambda / D) = \text{ps-tcf}(\Pi\lambda / D) \) so one is well defined iff the other is.

\text{Proof.} 0) By the definitions.
1) Let \( (\mathcal{F}_\beta : \beta < \gamma_t) \) exemplify “\( \Pi\lambda / D \) has pseudo true cofinality \( \gamma_t \)” for \( t = 1, 2 \).

Now

\[ \text{(*) if } \ell \in \{1, 2\} \text{ and } \beta_\ell < \gamma_\ell \text{ then for some } \beta_{3-\ell} < \gamma_{3-\ell} \text{ we have } g_1 \in \mathcal{F}_{\beta_1}^\ell \wedge g_2 \in \mathcal{F}_{\beta_2}^{3-\ell} \Rightarrow g_1 <_D g_2. \]

[Why? Choose \( g_\ell \in \mathcal{F}_{\beta_{3-\ell}+1}^\ell \), choose \( \beta_{3-\ell} < \gamma_{3-\ell} \) and \( g_{3-\ell} \in \mathcal{F}_{\beta_{3-\ell}}^{3-\ell} \) such that \( g_\ell < g_{3-\ell} \mod D \).]

Hence

\[ \text{(*) } h_1 : \gamma_1 \rightarrow \gamma_2 \text{ is well defined when } h_1(\beta_1) = \text{Min}\{\beta_2 < \gamma_2 : (\forall g_1 \in \mathcal{F}_{\beta_1}^1)(\forall g_2 \in \mathcal{F}_{\beta_2}^2)(g_1 < g_2 \mod D)\}. \]

Clearly \( h \) is non-decreasing and it is not eventually constant (as \( \cup \{\mathcal{F}_\beta^1 : \beta < \gamma_1\} \) is cofinal in \( \Pi\lambda / D \) and has range unbounded in \( \gamma_2 \) (similarly).
The rest should be clear.

2) Follows.

3), 4) Easy.

Claim 16.9. The Existence of true cofinality filter \( \kappa > R_0 + DC + AC_{<\kappa} \)

(a) \( D \) is a \( \kappa \)-complete filter on \( Y \)

(b) \( \bar{\alpha} \in Y \)

(c) \( \delta := rk_D(\bar{\alpha}) \) satisfies \( cf(\delta) \geq \theta(\text{Fil}_1^k(Y)) \), see below.

Then for some \( D' \) we have

(a) \( D' \) is a \( \kappa \)-complete filter on \( Y \)

(b) \( D' \supseteq D \)

(\( \gamma \)) \( \Pi \bar{\alpha}/D' \) has pseudo true cofinality, in fact, \( ps-\text{tcf}(\Pi \bar{\alpha}, < D) = cf(rk_D(\bar{\alpha})) \).

Recall from [Sh:835]

Definition 16.10. 0) \( \text{Fil}_1^k(Y) = \{D : D \) a \( \kappa \)-complete filter on \( Y \} \) and if \( D \in \text{Fil}_1^k(Y) \) then \( \text{Fil}_1^k(D) = \{D' \in \text{Fil}_1^k(Y) : D \subseteq D' \} \).

1) \( \text{Fil}_1^k(Y) = \{(D_1, D_2) : D_1 \subseteq D_2 \) are \( \kappa \)-complete filters on \( Y \} \).

2) \( J[f, D] \) where \( D \) is a filter on \( Y \) and \( f \in \text{Ord} \) is \( \{A \subseteq Y : A = 0 \) mod \( D \) or \( rk_{D+A}(f) > rk_D(f) \} \).

Remark 16.11. 1) On the Definition of pseudo \( (< \kappa, 1+\gamma) \)-complete \( D \) see [Sh:938, 1.13=0z.51]; we may consider changing the definition of \( \text{Fil}_1^k(Y) \) to \( D \) is \( \kappa \)-complete and pseudo\((< \kappa, 1+\gamma) \)-complete filter on \( Y \).

2) Related to [Sh:835].

Proof. Proof of the Claim of ??

Recall \( \{y \in Y : \alpha_y = 0\} = \emptyset \) mod \( D \) as \( rk_D((\alpha_y : y \in Y)) = \delta > 0 \) but \( f_1, f_2 \in Y \) Ord \( \land (f_1 = f_2 \) mod \( D \) \( \Rightarrow rk_D(f_1) = rk_D(f_2) \) hence without loss of generality \( y \in Y \Rightarrow \alpha_y > 0 \).

Let \( D = \{D' : D' \) is a filter on \( Y \) extending \( D \) which is \( \kappa \)-complete\}. So \( \theta(D) \leq \theta(\text{Fil}_1^k(Y)) \leq cf(\delta) \).

For any \( \gamma < rk_D(\bar{\alpha}) \) and \( D' \in D \) let

\[ (*)_2 \quad (a) \quad \mathcal{F}_{\gamma, D'} = \{f \in \Pi \bar{\alpha} : rk_D(f) = \gamma \) and \( D' \) is dual\(J[f, D])\}

\[ (*)_2 \quad (b) \quad \mathcal{F}_{D'} = \cup\{\mathcal{F}_{\gamma, D'} : \gamma < rk_D(\bar{\alpha})\}

\[ (*)_2 \quad (c) \quad \Xi_{\bar{\alpha}, D'} = \{\gamma < rk_D(\bar{\alpha}) : \mathcal{F}_{\gamma, D'} \neq \emptyset\}

\[ (*)_2 \quad (d) \quad \mathcal{F}_\gamma = \cup\{\mathcal{F}_{\gamma, D'} : D' \in D\} \).

Now

\[ (*)_3 \quad \text{if } \gamma < rk_D(\bar{\alpha}) \text{ then } \mathcal{F}_\gamma \neq \emptyset. \]

[Why? By [Sh:938, 1.8(2)=z0.23(2)] there is \( g \in Y \) Ord such that \( g < f \) mod \( D \) and \( rk_D(g) = \gamma \) and without loss of generality \( g \in \Pi \bar{\alpha} \). Now let \( D' = \text{dual}(J[g, D]) \), so \( (D, D' \in \text{Fil}_1^k(Y), D' \in D \) and \( g \in \mathcal{F}_{\gamma, D'} \), see [Sh:938, 1.7(2)=z0.23(2)], Claim [Sh:835, 0.10(2)], here we use \( AC_{<\kappa} \).

\[ (*)_4 \quad \{\sup(\Xi_{\bar{\alpha}, D'}) : D' \in D \) and \( \Xi_{\bar{\alpha}, D'} \) is bounded in \( rk_D(\bar{\alpha}) \} \) is a subset of \( rk_{D'}(\bar{\gamma}) \) which has cardinality \( < \theta(D) \leq \theta(\text{Fil}_1^k(Y)) \leq cf(\delta) \).}
(955)  revision:2014-05-02
modified:2014-05-19

1) \[DC\] or just \[AC\]
Claim
□
[Sh:71] or see [Sh:835, 1.9=\[z0.25\]].
1) We say that
Definition/Claim 16.12.
□
\[\text{done.}\]

As \(rk\)
Proof.
16.14
16.15
Remark
\[\text{sup(\(\Xi_{\alpha,D'}\)) witness this.}\]
\[(\ast)_5\] the set in \((\ast)_4\) is bounded below \(rk_D(\bar{\alpha})\) so let \(\gamma(\ast) < rk_D(\bar{\alpha})\) be its supremum.

[Why? By \((\ast)_4\).]
\[(\ast)_6\] there is \(D' \in \mathbb{D}\) such that \(\Xi_{\bar{\alpha},D'}\) is unbounded in \((\Pi\bar{\alpha}, <_{D'}).\)

[Why? Choose \(\gamma < rk_D(\bar{\alpha})\) such that: \(\gamma > \gamma(\ast)\). By \((\ast)_3\) there for some \(f \in \mathcal{F}_{\gamma(\ast)}\) and \(D' \in \mathbb{D}\) we have \(f \in \mathcal{F}_{\gamma(\ast),D'}\) so by the choice of \(\gamma(\ast)\) the set \(\Xi_{\alpha,D'}\) cannot be bounded in \(rk_D(\bar{\alpha})\).

\[(\ast)_7\] if \(\gamma_1 < \gamma_2\) are from \(\Xi_{\bar{\alpha},D'}\) and \(f_1 \in \mathcal{F}_{\gamma_1,D'}, f_2 \in \mathcal{F}_{\gamma_2,D'}\) then \(f_1 <_{D'} f_2.\)

[Why? By [Sh:938, 1.7=\[z0.23\]], [Sh:835, 0.10(2)].]
Together we are done: by \((\ast)_6\) there is \(D' \in \mathbb{D}\) such that \(\Xi_{\bar{\alpha},D'}\) is unbounded in \(rk_D(\bar{\alpha})\). Let \(\mathcal{F} = \langle \mathcal{F}_{\gamma,D'} : \gamma \in \Xi_{\bar{\alpha},D'}\rangle\) witness that \((\Pi\bar{\alpha}, <_{D'})\) has pseudo true cofinality, and so \(ps\text{-}tcf(\Pi\bar{\alpha}, <_D) = cf(otp(\Xi_{\bar{\alpha},D'})) = cf(rk_D(\bar{\alpha}))\), so we are done. \(\square\)

So we have

\{r10\}

Definition/Claim 16.12. 1) We say that \(\delta = ps\text{-}tcf_D(\bar{\alpha})\), where \(\delta\) is a limit ordinal when, for some set \(Y:\)
\[
(a) \ \bar{\alpha} \in \text{Ord} \\
(b) \ D = (D_1,D_2) \\
(c) \ D_1 \subseteq D_2 \text{ are } \aleph_1\text{-complete filters on } Y \\
(d) \ rk_{D_1}(\bar{\alpha}) = \delta = sup(\Xi_{\bar{\alpha},D}) \text{ where } \Xi_{\bar{\alpha},D} = \{\gamma < rk_{D_1}(\bar{\alpha}) : \text{for some } f < \bar{\alpha} \mod D_1 \text{, we have } rk_{D_1}(f) = \gamma \text{ and } D_2 = dual(J[f,D_1])\}.
\]

2) If \(D_1\) is \(\aleph_1\text{-complete filter on } Y, \bar{\alpha} = \langle \alpha_t : t \in Y\rangle\) and \(cf(\alpha_t) \geq \theta(\text{Fil}_{\aleph_1}(Y))\) for \(t \in Y\) then for some \(\aleph_1\text{-complete filter } D_2\) on \(Y\) extending \(D_1\) we have \(ps\text{-}tcf(D_1,D_2,\bar{\alpha})\) is well defined.

3) Moreover in part (2) there is a definition giving for any \((Y,D_1,D_2,\bar{\alpha})\) as there, a sequence \(\langle \mathcal{F}_{\gamma} : \gamma < \delta \rangle\) exemplifying the value of \(ps\text{-}tcf_D(\bar{\alpha})\).

Proof. Let \(\delta := rk_{D_1}(f)\), so by Claim 16.16 below \(cf(\delta) \geq \theta(\text{Fil}_{\aleph_1}(Y))\) hence has Claim ?? above and its proof the conclusion holds: the proof is needed for “\(\delta = sup(\Xi_{\bar{\alpha},D})\)” noting observation 16.13 below. \(\square\)

\{r10d\}

Observation 16.13. 1) \([DC]\) or just \([AC_{\aleph_0}]\).
Assume \(D\) is an \(\aleph_1\text{-complete filter on } Y\) and \(f, f_n \in Y\text{ Ord for } n < \omega\) and \(f(\bar{\alpha}) = sup\{f_n(\bar{\alpha}) : n < \omega\} \text{. Then } rk_D(f) = sup\{rk_D(f_n) : n < \omega\}.\)


Proof. As \(rk_D(f) = min\{rk_{D,A_n}(f) : n < \omega\}\) if \(\cup\{A_n : n < \omega\} \in D, A_n \in D^+\) by [Sh:71] or see [Sh:835, 1.9=\[z0.25\]]. \(\square\)

Remark 16.15. Also in [Sh:835, 1.9(2)=\[z0.25(2)\]] can use \(AC_I\) only, i.e. omit the assumption \(DC\), a marginal point here.

\{r11\}

Claim 16.16. \([AC_{<\theta}]\) The ordinal \(\delta\) has cofinality \(\geq \theta\) when:
\[
\oplus (a) \ \delta = rk_D(\bar{\alpha})
\]
(b) \( \bar{\alpha} = (\alpha_t : t \in Y) \in Y\) Ord
(c) \( D \) is an \( \aleph_1 \)-complete filter on \( Y \)
(d) \( y \in Y \Rightarrow cf(\alpha_y) \geq \theta \).

Proof. Note that \( y \in Y \Rightarrow \alpha_y > 0 \). Toward contradiction assume \( cf(\delta) < \theta \) so \( \delta \) has a cofinal subset \( C \) of cardinality \( < \theta \). For each \( \beta < \delta \) for some \( f \in Y \) Ord we have \( \text{rk}_D(f) = \beta \) and \( f <_D \bar{\alpha} \) and without loss of generality \( f \in \{ y \in Y \mid \alpha_y \} \). By AC\( \mathbb{N}_\theta \) there is a sequence \( (f_\beta : \beta \in C) \) such that \( f_\beta \in \{ y \in Y \mid \alpha_y \} \) and \( \text{rk}_D(f_\beta) = \beta \).

Define \( g \in \prod_{y \in Y} ^\alpha \) by \( g(y) = \cup \{ f_\beta(y) : \beta \in C \text{ and } f_\beta(y) \in \alpha_t \} \). By clause (d) we have \( y \in Y \Rightarrow g(y) \in \alpha_y \), so \( g <_D \bar{\alpha} \), hence \( \text{rk}_D(g) < \text{rk}_D(\bar{\alpha}) \) but by the choice of \( g \) we have \( \beta \in C \Rightarrow f_\beta \leq f_D \) which \( \beta \in C \Rightarrow \beta = \text{rk}_D(f_\beta) \leq \text{rk}_D(g) \) hence \( \delta = \text{sup}(C) \leq \text{rk}_D(g) \), contradiction. \( \square \)

Observe 16.17. 1) Assume \( (\bar{\alpha}, D) \) satisfies

\[
(a) \quad D \text{ a filter on } Y \text{ and } \bar{\alpha} = (\alpha_t : t \in Y) \text{ and each } \alpha_t \text{ is a limit ordinal}
\]
\[
(b) \quad \mathcal{F} = (\mathcal{F}_\beta : \beta < \partial) \text{ exemplify } \partial = \text{ps-tcf}(\Pi \bar{\alpha}, <_D) \text{ so we demand just}
\]
\[
\partial = \text{sup}(\beta < \partial : \mathcal{F}_\beta \neq \emptyset)
\]
\[
(c) \quad \mathcal{F}_\beta' = \{ f \in \prod_{t \in Y} ^\alpha : \text{ for some } g \in \mathcal{F}_\beta \text{ we have } f = g \bmod D \}.
\]

Then: \( (\mathcal{F}_\beta : \beta < \partial) \) exemplify \( \partial = \text{ps-tcf}(\Pi \bar{\alpha}, <_D) \) that is

\[
(a) \quad \bigcup \mathcal{F}_\beta \text{ is cofinal in } (\Pi \bar{\alpha}, <_D)
\]
\[
(b) \quad \text{for every } \beta_1 < \beta_2 < \partial \text{ and } f_1 \in \mathcal{F}_{\beta_1} \text{ and } f_2 \in \mathcal{F}_{\beta_2} \text{ we have } f_1 \leq f_2.
\]

2) Similarly, if \( D, \mathcal{F} \) satisfies clauses (a),(b) above and \( D \) is \( \aleph_1 \)-complete and \( \partial = \text{cf}(\partial) > \aleph_0 \) then we can “correct” \( \mathcal{F} \) to make it \( \aleph_0 \)-continuous that is \( (\mathcal{F}_\beta : \beta < \partial) \) defined in \( (c)_1 + (c)_2 \) below satisfies (a) + (b) above and (c) below and so is \( \aleph_0 \)-continuous, (see below) where

\[
(c)_1 \text{ if } \beta < \partial \text{ and } \text{cf}(\beta) \neq \aleph_0 \text{ then } \mathcal{F}_\beta'' = \mathcal{F}_\beta'
\]
\[
(c)_2 \text{ if } \beta < \partial \text{ and } \text{cf}(\beta) = \aleph_0 \text{ then } \mathcal{F}_\beta'' = \{ \text{sup}(f_n : n < \omega) : \text{ for some increasing sequence } \beta_n : n < \omega \text{ with limit } \beta \text{ we have } n < \omega \Rightarrow f_n \in \mathcal{F}_{\beta_n} \}, \text{ see below}
\]

\[
(\gamma) \text{ if } \beta < \partial \text{ and } \text{cf}(\beta) = \aleph_0 \text{ and } f_1, f_2 \in \mathcal{F}_\beta'' \text{ then } f_1 = f_2 \bmod D.
\]

3) This applies to an increasing sequence \( (\mathcal{F}_\beta : \beta < \delta), \mathcal{F}_\beta \subseteq Y \) Ord, \( \delta \) a limit ordinal.

Proof. Straightforward. \( \square \)

Definition 16.18. 0) If \( f_n \in Y \) Ord for \( n < \omega \), then \( \text{sup}(f_n : n < \omega) \) is defined as the function \( f \) with domain \( Y \) such that \( f(t) = \cup \{ f_n(t) : n < \omega \} \).

1) We say \( \mathcal{F} = (\mathcal{F}_\beta : \beta < \lambda) \) exemplifying \( \lambda = \text{ps-tcf}(\Pi \bar{\alpha}, <_D) \) is weakly \( \aleph_0 \)-continuous when:

if \( \beta < \partial, \text{cf}(\beta) = \aleph_0 \) and \( f \in \mathcal{F}_\beta \) then for some sequence \( (\beta_n, f_n) : n < \omega \) we have \( \beta = \cup \{ \beta_n : n < \omega \}, \beta_n < \beta_{n+1} < \beta, f_n \in \mathcal{F}_{\beta_n} \) and \( f = \text{sup}(f_n : n < \omega) \); so if \( D \) is \( \aleph_1 \)-complete then \( \{ f/D : f \in \mathcal{F}_\beta \} \) is a singleton.

2) We say it is \( \aleph_0 \)-continuous if we can replace the last “then” by “iff”.}
Theorem 16.19. The Canonical Filter Theorem Assume DC and AC_\(\mathbb{P}(Y)\).
Assume \(\vec{\alpha} = \langle \alpha_t : t \in Y \rangle \in Y^{\text{Ord}}\) and \(t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \theta(\mathbb{P}(Y))\) and \(\partial \in \text{ps-pcf}_{\mathcal{N}_1, \text{comp}}(\vec{\alpha})\) hence is a regular cardinal. Then there is \(D = D^2_\partial\), an \(\mathcal{N}_1\)-complete filter on \(Y\) such that \(\partial = \text{ps-tcf}(\Pi \vec{\alpha}/D)\) and \(D \subseteq D'\) for any other such \(D' \in \text{Fil}^1_{\mathcal{N}_1}(D)\).

Remark 16.20. 1) By ?? there are some such \(\partial\).
2) We work to use just AC_\(\mathbb{P}(Y)\) and not more.

Proof. Let

\(\mathcal{B}_1\) (a) \(D = \{D : D\text{ is an }\mathcal{N}_1\text{-complete filters on }Y\text{ such that }\Pi \vec{\alpha}/D\}\) has pseudo true cofinality \(\partial\),

(b) \(D_* = \cap\{D : D \in \mathcal{B}\}\).

Now obviously

(c) \(D_*\) is an \(\mathcal{N}_1\)-complete filter on \(Y\).

For \(A \subseteq Y\) let \(\mathcal{D}_A = \{D \in D : A \notin D\}\) and let \(\mathcal{P}_* = \{A \subseteq Y : D_A \neq \emptyset\}\).

As AC_\(\mathbb{P}(Y)\) we can find \(\langle D_A : A \in \mathcal{P}_* \rangle\) such that \(D_A \in \mathcal{D}_A\) for \(A \in \mathcal{P}_*\). Let \(\mathcal{D}_* = \{D_A : A \in \mathcal{P}_*\}\), clearly

\(\mathcal{B}_2\) \(D_* = \cap\{D : D \in \mathcal{D}_*\}\) and \(\mathcal{D}_* \subseteq \mathcal{B}\) is non-empty.

As AC_\(\mathbb{P}_*\) holds clearly

\((*)_0\) we can choose \(\langle \mathcal{F}_A^* : A \in \mathcal{P}_* \rangle\) such that \(\mathcal{F}_A^*\) exemplifies \(D_A \in \mathcal{D}_*\) as in 16.17(1),(2), so in particular is \(\mathcal{N}_0\)-continuous.

For each \(\beta < \partial\) let \(\mathcal{F}_\beta^* = \cap\{\mathcal{F}_A^* : A \in \mathcal{P}_*\}\), now

\((*)_1\) \(\mathcal{F}_\beta^* \subseteq \Pi \vec{\alpha}\).

[Why? As by 16.17(1)(c) we have \(\mathcal{F}_A^* \subseteq \Pi \vec{\alpha}\) for each \(A \in \mathcal{P}_*\).]

\((*)_2\) if \(\beta_1 < \beta_2 < \partial, f_1 \in \mathcal{F}_{\beta_1}^*\) and \(f_2 \in \mathcal{F}_{\beta_2}^*\), then \(f_1 < f_2\) mod \(D_*\).

[Why? As \(A \in \mathcal{P}_* \Rightarrow f_1 < f_2\) by the choice of \(\langle \mathcal{F}_\beta^* : \beta < \partial\rangle\), hence the set \(\{t \in Y : f_1(t) < f_2(t)\}\) belongs to \(D_A\) for every \(A \in \mathcal{P}_*\), hence by \(\mathcal{B}_2\) it belongs to \(D_*\) which means that \(f_1 < f_2\) as required.]

\((*)_3\) if \(f \in \Pi \vec{\alpha}\) then for some \(\beta_f < \partial\) we have \(f' \in \cup\{\mathcal{F}_\beta^* : \beta \in [\beta_f, \partial)\} \Rightarrow f < f'\)

mod \(D_*\).

[Why? For each \(A \in \mathcal{P}_*\) there are \(\beta, g\) such that \(\beta < \partial, g \in \mathcal{F}_\beta^*\) and \(f < g\)

mod \(D\) hence \(f' \in [\beta + 1, \partial) \wedge f' \in \mathcal{F}_\beta^*\) \(\Rightarrow f < g < f'\) mod \(D_A\). Let \(\beta_A\)

be the minimal such ordinal \(\beta_A < \delta\). As \(\text{cf}(\delta) \geq \theta(\mathbb{P}(Y)) \geq \theta(\mathcal{P}_*),\) clearly \(\beta_\delta = \sup\{\beta_A + 1 : A \in \mathcal{P}_*\}\) is \(< \delta\). So \(A \in \mathcal{P}_* \wedge g \in \cup\{\mathcal{F}_\beta^* : \beta \in [\beta_A, \delta]\} \Rightarrow f < g\).

By \(\mathcal{B}_2\) the ordinal \(\alpha_*\) is as required on \(\alpha_t\).]

Moreover

\((*)_4\) there is a function \(f \mapsto \beta_f\) in \((*)_3\).

[Why? As we can (and will) choose \(\beta_f\) as minimal \(\beta\) such that ...]

\((*)_5\) for every \(\beta_* < \partial\) there is \(\beta \in (\beta_*, \partial)\) such that \(\mathcal{F}_\beta^* \neq \emptyset\).
Why? We choose by induction on \( n \), a sequence \( \beta_n = \langle \beta_n, A : A \in \mathcal{P}_* \rangle \) and a function \( f_n \) such that

- \( \alpha \), \( \beta_n < \partial \) and \( m < n \Rightarrow \beta_m < \beta_n \)
- \( \beta_0 = \beta_* \) and for \( n > 0 \) we let \( \beta_n = \sup \{ \beta_{m,A} : m < n, A \in \mathcal{P}_* \} \)
- \( \gamma \), \( \beta_n,A \in (\beta_n, \partial) \) is minimal such that there is \( f_{n,A} \in \mathcal{F}_{\beta_n,A} \) satisfying \( n = m+1 \Rightarrow f_m < f_{\beta_n,A} \mod D_A \)

Why can we carry the induction? Arriving to \( n \) first, \( f_n \) is well defined by clause (\( \varepsilon \)) as \( cf(\alpha_t) \geq \theta(\mathcal{P}_*) \) for \( t \in Y \). Second by clause (\( \gamma \)), \( \langle \beta_n, A : A \in \mathcal{P}_* \rangle \) is well defined. Third by clause (\( \delta \)) we can choose \( \langle f_{m,A} : A \in \mathcal{P}_* \rangle \) as \( AC_{\mathcal{P}_*} \).

Lastly, the inductive construction is possibly by DC.

Let \( \beta_* = \cup \{ \beta_n : n < \omega \} \) and \( f = \sup \{ f_n : n < \omega \} \). Easily \( f \in \cap \{ \mathcal{F}_{\beta} : \beta < \partial \} \) is \( \aleph_0 \)-continuous.

(*) \( \gamma \) if \( f \in \Pi(\alpha) \) then for some \( \beta < \gamma \) and \( f' \in \mathcal{F}_{\beta} \) we have \( f < f' \mod D^* \).

Why? By (\( \gamma \)) and (\( \delta \)).

\[ \Box_{16.19} \]

**Definition 16.21.** For \( \bar{\alpha} \in Y \) Ord let \( J^{J_1_{\text{comp}}}_{<\lambda}(\bar{\alpha}) = \{ X \subseteq Y : \text{ps-pcf}_{J_1_{\text{comp}}} (\bar{\alpha} | X) \subseteq \lambda \} \) and \( J^{J_{\text{comp}}}_{<\lambda} \) is \( J^{J_1_{\text{comp}}}_{<\lambda} \).

**Remark 16.22.** In 1.3, see Definition 16.6(3).

On this and more see [Sh:F955].
§ 17. **APPENDIX: DEFINITION OF RANK-SYSTEM**

Moved from pg. 3:

We define a function $H$ from $\Pi\bar{\alpha}$ into $\Pi\{\lambda_X : X \in D\}$ by:

1. $(H(f))(X) = \min\{\beta < \lambda_X : \text{if } f' \in \mathcal{F}_\beta^X \text{ then } f \leq f' \text{ mod } D_X\}$.

We let

2. $\hat{D}$ be the following filter on the set $\hat{Y} := D$:
   
   $$Z \in \hat{D} \iff Z \subseteq D \text{ and } (\exists X \in D)[Z \supseteq \{X' \in D : X' \subseteq X\}]$$

Now

3. $\hat{D}$ is an $\aleph_1$-complete filter on $\hat{Y}$.
   
4. If $f_1, f_2 \in \Pi\alpha$ and $f_1 \leq f_2 \text{ mod } D^*_1$ then $H(f_1) \leq H(f_2) \text{ mod } \hat{D}$.

      (\epsilon) \quad (\prod_{i \in Y} \lambda_i, <_D^1) \text{ is pseudo } (<_\lambda^+)-\text{directed.}

Why? By claim 16.7, i.e. 16.7 of §5 of [Sh:938].]

Because by an assumption

5. If $f_1, f_2 \in \mathcal{F}_\alpha$ and $\alpha < \delta$ then $H(f_1) = H(f_2) \text{ mod } \hat{D}$.

Why? $f_1 = f_2 \text{ mod } D$ hence by $\ast$ we have $f_1 = f_2 \text{ mod } D^*_1$ hence by (yyy), $H(f_1) = H(f_2) \text{ mod } \hat{D}$.

Now by (\epsilon) + (zzz) we are done proving (h).

6. $D \subseteq D^*_1$.

Why? Because if $A \in D$ then $X_1 := A$ witness $A \in D$, as $X \in D \land X \subseteq X_1 \Rightarrow X \in D \land X \subseteq A \Rightarrow X \in D_X \land X \subseteq A \Rightarrow Y \Rightarrow A \in D_X$.

REFERENCES


[Sh:F132], *Densities for box products*.


[Sh:F354], *Looking at huge pcf*.


More on the revised GCH.

Pcf: the advanced pcf theorems.

On Choice and HODx.


Rinot question.


PCF with little choice.


Bounds on pcf with weak choice using ranks and normal filters.


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