

**PRESERVING OLD  $([\omega]^{N_0}, \supseteq^*)$  IS PROPER**  
**SH960**

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ABSTRACT. We give some sufficient and necessary conditions on a forcing notion  $\mathbb{Q}$  for preserving the forcing notion  $([\omega]^{N_0}, \supseteq^*)$  being proper. They cover many reasonable forcing notions.

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## ANOTATED CONTENT

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- {2b.1} [Under CH, if non-meagerness of  $(\omega_2)^{\mathbf{V}}$  is preserved then  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  is proper, (1.1). If  $\mathbf{V}$  fails to satisfy CH, then usually  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  is not proper after a forcing adding a new real and satisfying a relative of being proper, e.g. satisfies c.c.c. or is any true creature forcing.]
- §2 General sufficient conditions, pg. 10
- {3c.1} [If  $\mathbf{V}$  satisfies CH and  $\mathbb{Q}$  is c.c.c. then  $\Vdash_{\mathbb{Q}}$  “ $\mathbb{P}_{\mathcal{A}[\mathbf{V}]}$  is proper”, see in 2.1. In
- {3c.3} 2.3 we replace  $\mathcal{A}_*^{\mathbf{V}}$  by a forcing notion  $\mathbb{R}$  adding no  $\omega$ -sequence,  $\mathbb{Q}$  is c.c.c. even in  $\mathbf{V}^{\mathbb{P}}$ . Instead “ $\mathbb{Q}$  satisfies the c.c.c.” it suffices to demand  $\mathbb{Q}$  satisfies
- {3c.7} a weaker condition. Lastly, in 2.5 we prove some proper forcing does not preserve.]

## § 0. INTRODUCTION

{intro}

We investigate the question “ $\text{Pr}_1^+(\mathbb{Q}, \mathbb{R})$ ”, which means that the proper forcing  $\mathbb{Q}$  preserves that the (old)  $\mathbb{R}$  is proper for various  $\mathbb{R}$ 's. In what follows,  $B \subseteq^* A$  means  $|B \setminus A| < \aleph_0$ , and  $A \supseteq^* B$  means the same.

Recall:

{1a.0}

**Definition 0.1.** properness:

- (a) Assume that  $N \prec (\mathcal{H}(\chi), \in)$ ,  $\mathbb{P} \in N$  is a forcing notion and  $q \in \mathbb{P}$ . We say that  $q$  is  $(N, \mathbb{P})$ -generic iff for every dense  $D \subseteq \mathbb{P}$ , if  $D \in N$  then  $D \cap N$  is pre-dense above  $q$ .
- (b) A forcing notion  $\mathbb{P}$  is proper iff for every sufficiently large regular  $\chi$  and every countable  $N \prec (\mathcal{H}(\chi), \in)$ , if  $p, \mathbb{P} \in N$  then there is a condition  $q \in \mathbb{P}$ ,  $q \geq p$  such that  $q$  is  $(N, \mathbb{P})$ -generic.

Gitman proved that  $\text{Pr}_1^+(\mathbb{Q}, \mathbb{P}_{\mathcal{P}(\omega)[\mathbf{V}]})$  (see definition below, where  $\mathbb{P}_{\mathcal{P}(\omega)[\mathbf{V}]}$  is the forcing notion  $(\{A \in \mathbf{V} : A \subseteq \omega, |A| = \aleph_0\}, \supseteq^*)$ , when  $\mathbb{Q}$  is adding Cohen reals (or just Cohen subsets even  $> 2^{\aleph_0}$  many). But no other examples were known even Sacks forcing. Also for e.g.  $\mathbf{V} \models “V = L”$ , we did not know a forcing making it not proper.

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Let us state the problem and relatives. We are interested mainly in the case  $\mathbb{Q}$  is proper.

{1a.1}

**Definition 0.2.** 1) Let  $\text{Pr}_1(\mathbb{Q}, \mathbb{P})$  means:  $\mathbb{Q}, \mathbb{P}$  are forcing notions and  $\Vdash_{\mathbb{Q}} “\mathbb{P}$ , i.e.  $\mathbb{P}^{\mathbf{V}}$  is a proper forcing”.

1A) Let  $\text{Pr}_1^+(\mathbb{Q}, \mathbb{P})$  be defined similarly but adding “ $\mathbb{Q}$  is proper”.

2) For  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  let  $\mathbb{P}_{\mathcal{A}}$  be  $\mathcal{A} \setminus [\omega]^{<\aleph_0}$  ordered by  $\supseteq^*$ , inverse almost inclusion.

3) Let  $\mathcal{A}_* = \mathcal{A}_*[\mathbf{V}] = ([\omega]^{\aleph_0})^{\mathbf{V}}$ .

{1a.3}

**Observation 0.3.** A necessary condition for  $\text{Pr}_1(\mathbb{Q}, \mathbb{P})$  is:

(\*)<sub>1</sub> if  $\chi$  is regular and large enough,  $N \prec (\mathcal{H}(\chi), \in)$  is countable,  $\mathbb{Q}, \mathbb{P} \in N$ ,  $q_1 \in \mathbb{Q}$  is  $(N, \mathbb{Q})$ -generic and  $r_1 \in N \cap \mathbb{P}$  then we can find  $(q_2, r_2)$  such that:

- ⊙ (a)  $q_1 \leq_{\mathbb{Q}} q_2$
- (b)  $r_1 \leq_{\mathbb{P}} r_2$
- (c)  $q_2 \Vdash “r_2$  is  $(N[G_{\mathbb{Q}}], \mathbb{P})$ -generic”.

{1a.7}

**Definition 0.4.** 1) We define  $\text{Pr}^-(\mathbb{Q}, \mathbb{P}) = \text{Pr}_2(\mathbb{Q}, \mathbb{P})$  as the necessary condition from 0.3.

{1a.3}

2) Let  $\text{Pr}_3(\mathbb{Q}, \mathbb{P})$  mean that  $\mathbb{Q}, \mathbb{P}$  are forcing notions and for some  $\lambda$  and stationary  $S \subseteq [\lambda]^{\aleph_0}$  from  $\mathbf{V}$  we have  $\Vdash_{\mathbb{Q}} “\mathbb{P}$  is  $S$ -proper”, and note that  $S$  remains stationary of course.

3)  $\text{Pr}_4(\mathbb{Q}, \mathbb{P})$  is defined similarly but  $S \in \mathbf{V}^{\mathbb{Q}}$ , still  $S \subseteq ([\lambda]^{\aleph_0})^{\mathbf{V}}$ , so  $S$  is actually  $\dot{S}$ , a  $\mathbb{Q}$ -name.

4)  $\text{Pr}_5(\mathbb{Q}, \mathbb{P})$  is the statement (A) of 0.5(4) below.

{1a.10}

5) Let  $\text{Pr}_\ell^+(\mathbb{Q}, \mathbb{P})$  means  $\text{Pr}_\ell(\mathbb{Q}, \mathbb{P})$  and  $\mathbb{Q}$  is a proper forcing, for  $\ell = 2, 3, 4, 5$ .

{1a.10}

**Claim 0.5.** 1)  $\text{Pr}_2(\mathbb{Q}, \mathbb{P})$  means that for  $\lambda$  large enough, letting  $S = ([\lambda]^{N_0})^{\mathbf{V}}$ , we have  $\Vdash_{\mathbb{Q}}$  “ $\mathbb{P}$  is  $S$ -proper”.

2)  $\text{Pr}_1(\mathbb{Q}, \mathbb{P}) \Rightarrow \text{Pr}_2(\mathbb{Q}, \mathbb{P}) \Rightarrow \text{Pr}_3(\mathbb{Q}, \mathbb{P})$ ; similarly for  $\text{Pr}^+$ .

3) Also  $\text{Pr}_3(\mathbb{Q}, \mathbb{P}) \Rightarrow \text{Pr}_4(\mathbb{Q}, \mathbb{P}) \Rightarrow \text{Pr}_5(\mathbb{Q}, \mathbb{P})$ ; similarly for  $\text{Pr}^+$ .

4) If  $\mathbb{Q}, \mathbb{P}$  are forcing notions,  $\chi$  large enough and regular, then (A)  $\Leftrightarrow$  (B) where

(A) for some countable  $N \prec (\mathcal{H}(\chi), \in)$  and for some  $q \in \mathbb{Q}, p \in \mathbb{P}$  we have

(a)  $q$  is  $(N, \mathbb{Q})$ -generic

(b)  $q \Vdash_{\mathbb{Q}}$  “ $p$  is  $(N[G_{\mathbb{Q}}], \mathbb{P})$ -generic”

(B) for some  $q_* \in \mathbb{Q}, p_* \in \mathbb{P}$  we have  $\text{Pr}_4(\mathbb{Q}_{\geq q_*}, \mathbb{P}_{\geq p_*})$ .

*Proof.* Easy. □<sub>0.5</sub>

{z2}

*Notation 0.6.*  $<_{\chi}^*$  denotes a well ordering of  $\mathcal{H}(\chi)$ .

Recall (Balcar-Pelant-Simon [BPS80], or see, e.g. Blass [Bla])

{z5}

**Definition 0.7.**  $\mathfrak{h}$  is the following cardinal invariant, it is the minimal cardinality  $\chi$  (necessarily regular) such that forcing with  $\mathbb{P}_{\mathcal{A}^*}$  adds a new sequence of ordinals of length  $\chi$ .

{z8}

*Notation 0.8.* If  $\mathcal{T}$  is a tree, then  $\text{suc}_{\mathcal{T}}(p)$  is the set of immediate successors of  $p \in \mathcal{T}$  in the tree order.

§ 1. PROPERNESS OF  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  AND CH

{2b.4}

**Claim 1.1.** Assume  $\mathbf{V}_0 \models \text{CH}$ ,  $\mathbf{V}_1 \supseteq \mathbf{V}_0$ , e.g.  $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{Q}}$  and let  $\mathcal{A} = \mathcal{A}_*[\mathbf{V}_0]$ .

- (a) If  $\aleph_1^{\mathbf{V}_0}$  is a countable ordinal in  $\mathbf{V}_1$ , then  $\mathbf{V}_1 \models \text{“}\mathbb{P}_{\mathcal{A}} \text{ is proper”}$ .
- (b) If  $\aleph_1^{\mathbf{V}_0} = \aleph_1^{\mathbf{V}_1}$  and  $\mathbf{V}_1 \models \text{“}(\omega_2)^{\mathbf{V}_0} \text{ is non-meagre”}$ , then  $\mathbf{V}_1 \models \text{“}\mathbb{P}_{\mathcal{A}} \text{ is proper”}$ .

In both cases, if  $\mathbf{V}_1$  is a generic extension of  $\mathbf{V}_0$  by the forcing notion  $\mathbb{Q}$  then it means that  $\text{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathcal{A}})$  holds.

*Proof.* Assume that  $\mathbf{V}_1 \supseteq \mathbf{V}_0$ .

If  $\mathbf{V}_1 \models \text{“}\aleph_1^{\mathbf{V}_0} \text{ is countable”}$  then recalling  $\mathbf{V}_0 \models \text{CH}$  clearly  $\mathbf{V}_1 \models \text{“}\mathcal{A} \text{ is countable”}$  so we know that  $\mathbb{P}_{\mathcal{A}}$  is proper in  $\mathbf{V}_1$ , thus proving clause (a). So from now on we assume  $\aleph_1^{\mathbf{V}_0}$  is not collapsed.

In  $\mathbf{V}_0$  let  $\mathcal{T} = \omega_1^{>}(\omega_1)$  and choose a subset  $\mathcal{A}' \subseteq \mathcal{A}$  such that  $\mathcal{A}'$  is  $\subseteq^*$ -dense in  $\mathcal{A}$  and  $(\mathcal{A}', \supseteq^*)$  is tree-isomorphic to  $\mathcal{T}$ . Let  $\pi$  be the isomorphism between these trees<sup>1</sup>. Notice that all this is done in  $\mathbf{V}_0$  (recalling that  $\mathbf{V}_0 \models \text{CH}$ ). In  $\mathbf{V}_0$  there is a sequence  $\bar{\mathcal{T}} = \langle \mathcal{T}_\alpha : \alpha < \omega_1 \rangle$  which is  $\subseteq$ -increasing continuous with union  $\mathcal{T}$  and each  $\mathcal{T}_\alpha$  countable. Also there is  $\bar{C} = \langle C_\delta : \delta < \omega_1, \delta \text{ is a limit ordinal} \rangle \in \mathbf{V}_0$  such that  $C_\delta \subseteq \delta = \sup(C_\delta)$ ,  $\text{otp}(C_\delta) = \omega$ . Let  $\mathcal{T}'_\delta = \mathcal{T}_\delta \upharpoonright \{\eta \in \mathcal{T}_\delta : \text{lg}(\eta) \in C_\delta\}$ .

In  $\mathbf{V}_1$  choose a sufficiently large regular cardinal  $\chi$ , and let  $N \prec (\mathcal{H}(\chi), \in)$  be countable such that  $\mathcal{A}, \pi, \bar{\mathcal{T}} \in N$  and let  $\delta = \omega_1 \cap N$ , clearly  $\mathcal{T} \cap N = \mathcal{T}'_\delta$ . We have to prove the statement:

- (\*)<sub>0</sub> “for every  $p \in \mathbb{P}_{\mathcal{A}} \cap N$  there is  $q \in \mathbb{P}_{\mathcal{A}}$  above  $p$  which is  $(N, \mathbb{P}_{\mathcal{A}})$ -generic”.

As  $\mathbf{V}_0 \models \text{CH}$  and the density of  $\mathcal{A}'$  in  $\mathcal{A}$  and  $(\mathcal{A}', \supseteq^*)$  being isomorphic in  $\mathbf{V}_0$  by  $\pi$  to  $\mathcal{T}$  this is equivalent (in  $\mathbf{V}_1$ , of course) to:

- (\*)<sub>1</sub> for every  $\nu \in \mathcal{T} \cap N = \mathcal{T}'_\delta$  there is  $\eta \in \mathcal{T}$  which is  $(N, \mathcal{T})$ -generic and  $\nu \leq_{\mathcal{T}} \eta$ .

In  $\mathbf{V}_0$  we let  $\bar{S} = \langle S_\delta : \delta < \omega_1 \text{ a limit ordinal} \rangle$  where  $S_\delta = \{\bar{\nu} : \bar{\nu} = \langle \nu_n : n < \omega \rangle \text{ is } <_{\mathcal{T}}\text{-increasing, } \nu_n \in \mathcal{T}'_\delta, \text{ moreover } \text{lg}(\nu_n) \text{ is the } n\text{-th member of } C_\delta\}$ .

As  $(\forall \nu \in \mathcal{T}'_\delta)(\exists \rho)(\nu <_{\mathcal{T}} \rho \in \mathcal{T}'_\delta)$ , and  $[\bar{\nu} \in S_\delta \Rightarrow \text{there is a } <_{\mathcal{T}}\text{-upper bound } \rho \in \mathcal{T} \text{ of } \bar{\nu}, \text{ in } \mathbf{V}_0, \text{ of course}]$  recalling  $\mathcal{T}'_\delta, S_\delta \in \mathbf{V}_0$  clearly (\*)<sub>1</sub> is equivalent (in  $\mathbf{V}_1$ , of course) to

- (\*)<sub>2</sub> for every  $\nu \in \mathcal{T}'_\delta$  there is  $\bar{\nu} \in S_\delta$  such that  $\nu \in \text{Rang}(\bar{\nu})$  and  $\bar{\nu}$  induce a subset of  $\mathcal{T}'_\delta$  generic over  $N$  (i.e.  $(\forall A)[A \in N \text{ is a dense open subset of } \mathcal{T} \Rightarrow A \cap \{\nu_n : n < \omega\} \neq \emptyset]$ ).

Now a sufficient condition for (\*)<sub>2</sub> is

- (\*)<sub>3</sub>  $S_\delta$ , as a set of  $\omega$ -branches of the tree  $\mathcal{T}'_\delta$ , is non-meagre.

But in  $\mathbf{V}_0$ ,  $\mathcal{T}'_\delta$  and  $\omega^{> \omega}$  are isomorphic and  $S_\delta$  is the set of all  $\omega$ -branches of  $\mathcal{T}'_\delta$ , so by an assumption from part (b), (\*)<sub>3</sub> holds so we are done. □<sub>1.1</sub>

{2b.4}

**Discussion 1.2.** However, there can be  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  such that  $(\mathcal{A}, \subseteq^*)$  is a variation of Souslin tree.

<sup>1</sup>this is trivial as  $\mathbf{V}_0 \models \text{CH}$ , however always there is a dense tree with  $\mathfrak{h}$  levels by the celebrated theorem of Balcar-Pelant-Simon

{2b.7}

**Claim 1.3.** 1) We have  $\text{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$  when:

- (a)  $\aleph_1^{\mathbf{V}[\mathbb{Q}]} = \aleph_1$
- (b)  $\Vdash_{\mathbb{Q}} "|\lambda| = \aleph_1 \text{ where } \lambda = (2^{\aleph_0})^{\mathbf{V}}"$
- (c) moreover letting  $\langle u_i : i < \aleph_1 \rangle$  be a  $\mathbb{Q}$ -name of a  $\subseteq$ -increasing continuous sequence of countable subsets of  $\lambda$  with union  $\lambda$ , the  $\mathbb{Q}$ -name  $\mathcal{S} = \{i : u_i \in \mathbf{V}\}$  is forced to contain a club (of  $\aleph_1$ )
- (d) forcing with  $\mathbb{Q}$  preserves " $(\omega 2)^{\mathbf{V}}$  is non-meagre".

2) Assume the forcing notion  $\mathbb{Q}$  satisfies (a) + (d),  $\text{Pr}_4(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$  as witnessed by  $S$  and  $\mathbb{Q}$  is proper and  $\mathcal{S}$  is forced to be stationary.

Then the forcing notion  $\mathbb{Q} * \text{Levy}(\aleph_1, (|\mathbb{Q}|^{\aleph_0})^{\mathbf{V}}) * \mathbb{Q}_{\mathcal{S}}$  preserves " $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  is proper" where  $\mathbb{Q}_{\mathcal{S}}$  is the (well known) shooting of a club through the stationary subsets of  $\omega_1$  (to make clause (c) hold).

{2b.1} *Proof.* Like 1.1. □<sub>1.3</sub>

In what follows we prove that many forcing notions destroy properness. We need a preliminary concept.

{2b.13}

**Definition 1.4.** For  $\lambda > \kappa$  we say that a forcing notion  $\mathbb{Q}$  is  $(\lambda, \kappa)$ -newly proper (omitting  $\kappa$  means  $\kappa = \aleph_0$  and we define  $(\lambda, < \chi)$ -newly proper similarly) when: if  $\bar{N} = \langle (N_\eta, \nu_\eta) : \eta \in {}^\omega \lambda \rangle$  satisfies  $\otimes$  below and  $\mathbb{Q} \in N_{< \omega}$  and  $p \in \mathbb{Q} \cap N_{< \omega}$  then we can find  $q, \eta$  such that  $\boxtimes$  below holds where:

- $\otimes$  for some cardinal  $\chi > \lambda$ 
  - (a)  $N_\eta \prec (\mathcal{H}(\chi), \in, <_\chi^*)$  is countable
  - (b) if  $\nu \triangleleft \eta$  then  $N_\nu \prec N_\eta$
  - (c)  $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$  if  $\kappa = \aleph_0$  and  $N_{\eta_1}^\kappa \cap N_{\eta_2}^\kappa = N_{\eta_1 \cap \eta_2}^\kappa$  generally where  $N_\eta^\kappa := \cup \{v \in N_\eta : |v| \leq \kappa\}$
  - (d)  $\nu_\eta \in N_\eta \setminus \cup \{N_{\eta \upharpoonright m}^\kappa : m < \text{lg}(\eta)\}$  hence  $\nu_\eta \notin \cup \{N_\nu : \neg(\eta \trianglelefteq \nu) \text{ and } \nu \in {}^\omega \lambda\}$
  - (e)  $\nu_\eta \in {}^{\text{lg}(\eta)} \lambda$  and  $\ell < \text{lg}(\eta) \Rightarrow \nu_{\eta \upharpoonright \ell} \trianglelefteq \nu_\eta$
- $\boxtimes$  (a)  $p \leq_{\mathbb{Q}} q$
- (b)  $q \Vdash_{\mathbb{Q}} " \cup \{N_{\eta \upharpoonright n}[\mathbf{G}_{\mathbb{Q}}] : n < \omega\} \cap \mathbf{V} = \cup \{N_{\eta \upharpoonright n} : n < \omega\}"$
- (c)  $q \Vdash_{\mathbb{Q}} " \eta \in {}^\omega \lambda \text{ is new, i.e. } \eta \notin ({}^\omega \lambda)^{\mathbf{V}}"$
- (c)<sup>+</sup> moreover if  $\kappa > \aleph_0$  and  $\mathcal{T} \in \mathbf{V}$  is a sub-tree of  ${}^\omega \lambda$  of cardinality  $\leq \kappa$  then  $\eta \notin \lim(\mathcal{T)$ , i.e.  $\{\eta \upharpoonright n : n < \omega\} \notin \mathcal{T}$ .

{2b.14}

**Observation 1.5.** If  $\langle N_\eta : \eta \in {}^\omega \lambda \rangle$  satisfies clauses (a),(b),(c) of  $\otimes$  of Definition 1.4, then the following conditions are equivalent:

{2b.13}

{2b.13}

- <sub>1</sub> there is  $\langle \nu_\eta : \eta \in {}^\omega \lambda \rangle$  such that clauses (d),(e) of  $\otimes$  of Definition 1.4
- <sub>2</sub> if  $\eta \in {}^\omega \lambda$ , then  $N_\eta \cap \lambda \not\subseteq \cup \{N_{\eta \upharpoonright \ell} : \ell < \text{lg}(\eta)\}$ .

For a proper forcing notion adding a new real it is quite easy to be  $\aleph_1$ -newly proper; e.g.

{2b.19}

**Claim 1.6.** Assuming  $2^{\aleph_0} \geq \lambda = \text{cf}(\lambda) > \aleph_1$ , sufficient conditions for “ $\mathbb{Q}$  is  $\lambda$ -newly proper” are:

- (a)  $\mathbb{Q}$  is c.c.c. and adds a new real
- (b)  $\mathbb{Q}$  is Sacks forcing
- (c)  $\mathbb{Q}$  is a tree-like creature forcing in the sense of Roslanowski-Shelah [RoSh:470].

*Proof.* Easy; for clause (a) we use  $q = p$  for  $\boxplus$  of the definition noting that: if  $\eta \in {}^{\omega}>\lambda$  then  $p$  is  $(N_\eta, \mathbb{Q})$ -generic. For clauses (b),(c) we use fusion but in the  $n$ -th step use members of  $N_\eta \cap \mathbb{Q}$  for  $\eta \in {}^n\lambda$ , we get as many distinct  $\eta$ 's as we can.  $\square_{1.6}$

{2b.10}

**Theorem 1.7.** We have  $\Vdash_{\mathbb{Q}}$  “ $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  is not proper” when:

- (a)  $\mathbf{V} \models 2^{\aleph_0} \geq \aleph_2$
- (b)  $\lambda$  is regular,  $\aleph_2 \leq \lambda \leq 2^{\aleph_0}$  and<sup>2</sup>  $\alpha < \lambda \Rightarrow \text{cf}([\alpha]^{\aleph_0}, \subseteq) < \lambda$  hence (by [Sh:420]) there is a stationary  $\mathcal{U}_\alpha \subseteq [\alpha]^{\aleph_0}$  of cardinality  $< \lambda$
- (c)  $\mathfrak{h} < \lambda$
- (d) the forcing notion  $\mathbb{Q}$  adds at least one real and is  $\lambda$ -newly proper.

*Proof.* Let  $\chi$  be large enough and for transparency,  $x \in \mathcal{H}(\chi)$ .

By Rubin-Shelah [RuSh:117], see more [Sh:f, Ch.XI] in  $\mathbf{V}$  there is a sequence  $\langle N_\eta : \eta \in {}^{\omega}>\lambda \rangle$  such that:

- $\square_1$  (a)  $N_\eta \prec (\mathcal{H}(\chi), \in)$
- (b)  $\mathbb{Q}, x \in N_\eta$
- (c)  $N_\eta$  is countable
- (d)  $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$ .

Now for each  $\eta \in {}^{\omega}\lambda$  let  $N_\eta = \cup\{N_{\eta \upharpoonright k} : k < \omega\}$ ; we can easily add:

- (e) there is  $\mathcal{W}$  such that:
  - ( $\alpha$ )  $\mathcal{W}$  is a subtree of  ${}^{\omega}>\lambda$
  - ( $\beta$ )  $\langle \rangle \in \mathcal{W}$
  - ( $\gamma$ ) if  $\eta \in \mathcal{W}$  then  $(\exists^\lambda \alpha)(\eta \hat{\ } \langle \alpha \rangle \in \mathcal{W})$
  - ( $\delta$ ) if  $\eta \in \text{lim}(W)$  then  $\eta \in {}^{\omega}\lambda$  is increasing, and  $\text{sup}(N_\eta \cap \lambda) = \text{sup}(\text{Rang}(\eta))$

- ( $\varepsilon$ ) we can choose  $\nu_\eta \in N_\eta$  for  $\nu \in \mathcal{W}$  as in clauses (d),(e) of  $\otimes$  of 1.4. {2b.13}

By Balcar-Pelant-Simon [BPS80] there is  $\mathcal{T} \subseteq [\omega]^{\aleph_0}$  such that

- $\square_2$  ( $\alpha$ )  $(\mathcal{T}, \supseteq^*)$  is a tree with  $\mathfrak{h}$  levels ( $\mathfrak{h}$  is the cardinal invariant from 0.7, a regular cardinal  $\in [\aleph_1, 2^{\aleph_0}]$ ), the tree  $\mathcal{T}$  has a root and each node has  $2^{\aleph_0}$  many immediate successors, i.e.  $\mathcal{T}$  has splitting to  $2^{\aleph_0}$  {z5}
- ( $\beta$ )  $\mathcal{T}$  is dense in  $([\omega]^{\aleph_0}, \supseteq^*)$ , i.e. in  $\mathbb{P}_{\mathcal{P}(\omega)[\mathbf{V}]} = \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  recalling 0.2(2). {1a.1}

Choose  $\bar{h}$  such that

<sup>2</sup>If  $\lambda = \aleph_2$  the rest of clause (b) follows.

□<sub>3</sub>  $\bar{h} = \langle h_p : p \in \mathcal{T} \rangle$  satisfies:  $h_p$  is a one-to-one function from  $\text{succ}_{\mathcal{T}}(p)$  onto  $2^{\aleph_0} \setminus \{h_{p_0}(p_1) : p_0 <_{\mathcal{T}} p_1 <_{\mathcal{T}} p \text{ and } p_1 \in \text{succ}_{\mathcal{T}}(p_0)\}$ .

So without loss of generality

□<sub>4</sub>  $\mathcal{T} \in N_{< \rangle}, \mathfrak{h} \in N_{< \rangle}$  and  $\bar{h} \in N_{< \rangle}$ .

{2b.13} As  $\mathbb{Q}$  is  $\lambda$ -newly proper there are  $\eta, q$  as in □ of Definition 1.4. Let  $\mathbf{G} \subseteq \mathbb{Q}$  be generic over  $\mathbf{V}$  such that  $q \in \mathbf{G}$ , let  $\eta = \eta[G]$  and  $M_2 := N_{\eta[G]} := \cup\{N_{\eta \upharpoonright n}[\mathbf{G}] : n < \omega\}$ , so  $M_2 \prec (\mathcal{H}(\chi)^{\mathbf{V}[\mathbf{G}]}, \mathcal{H}(\chi)^{\mathbf{V}}, \in)$  is countable, pedantically  $(|M_2|, \mathcal{H}(\chi)^{\mathbf{V}} \cap |M_2|, \in \upharpoonright |M_2|) \prec (\mathcal{H}(\chi)^{\mathbf{V}[\mathbf{G}]}, \mathcal{H}(\chi)^{\mathbf{V}}, \in \upharpoonright \mathcal{H}(\chi)^{\mathbf{V}[\mathbf{G}]})$ .

{2b.13} By □ of 1.4, i.e. the choice of  $\eta, q$  as  $q \in \mathbf{G}$  we have  $M_1 = M_2 \cap \mathcal{H}(\chi)^{\mathbf{V}}$  is  $\cup\{N_{\eta \upharpoonright n} : n < \omega\}$ , and of course  $M_1 \prec (\mathcal{H}(\chi), \in)$ . Toward contradiction assume  $\mathbf{V}[\mathbf{G}] \models \text{“}\mathcal{P}_{\mathcal{A}_*[\mathbf{V}]}\text{ is proper”}$ , hence some  $p_* \in \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  is  $(M_2, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$ -generic. But  $\mathcal{T}$  is dense in  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  so without loss of generality  $p_* \in \mathcal{T}$  and  $p_*$  is  $(M_2, \mathcal{T})$ -generic.

Since  $\mathfrak{h} \in N_{< \rangle}$  and  $\mathfrak{h} < \lambda$ , without loss of generality  $\eta \in {}^{\omega} \lambda \Rightarrow N_{\eta} \cap \mathfrak{h} = N_{< \rangle} \cap \mathfrak{h}$ . For any  $\alpha < \lambda$  let

$$\mathcal{I}_{\alpha} = \{p \in \mathcal{T} : \text{for some } p_0 \in \mathcal{T} \text{ we have } p \in \text{succ}_{\mathcal{T}}(p_0) \text{ and } h_{p_0}(p) = \alpha\}$$

and letting  $\mathcal{I}_{\alpha}$  be the  $\alpha$ -th level of  $\mathcal{T}$  and let

$$\mathcal{I}_{\alpha}^+ = \{p \in \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]} : p \text{ is above some member of } \mathcal{I}_{\alpha}\}.$$

Now clearly (in  $\mathbf{V}$  and in  $\mathbf{V}[\mathbf{G}]$ ):

- (\*)<sub>1</sub> (a)  $\mathcal{I}_{\alpha}$  is a pre-dense subset of  $\mathcal{T}$  (and of  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ )
- (b)  $\mathcal{I}_{\alpha}^+$  is dense open decreasing with  $\alpha$
- (c) if  $p \in \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  then for every large enough  $\alpha < \lambda, p \notin \mathcal{I}_{\alpha}^+$
- (d) if  $p \in \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  and  $\alpha < \lambda$  then there is  $q \in \mathcal{I}_{\alpha}$  such that  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]} \models \text{“}p \leq q\text{”}$ .

Also clearly the sequence  $\langle \mathcal{I}_{\alpha} : \alpha < \lambda \rangle$  belongs to  $N_{\langle \rangle}$  hence if  $\alpha \in \lambda \cap N_{\eta[G]}$  then  $\mathcal{I}_{\alpha} \in N_{\eta[G]}$  and the set  $\{p \in \mathcal{T} \cap N_{\eta[G]} : p \leq_{\mathcal{T}} p_* \text{ and } p \in \mathcal{I}_{\alpha}\}$  is not empty.

Now

- (\*)<sub>2</sub> in  $\mathbf{V}[\mathbf{G}]$  the following functions  $h_{\bullet}, h_*$  are well defined
  - (a)  $\text{Dom}(p_{\bullet}) = \text{Dom}(h_*) = N_{< \rangle} \cap \mathfrak{h}$
  - (b)  $h_{\bullet}(\gamma)$  is the unique  $p \in N_{\eta[G]} \cap \mathcal{T}$  of level  $\gamma$  which is  $\leq_{\mathcal{T}} p_*$
  - (c) if  $\gamma < \mathfrak{h}$  then  $h_*(\gamma) = h_{\gamma+1}(h_{\bullet}(\gamma+1))$
- (\*)<sub>3</sub> if  $\alpha \in \mathfrak{h} \cap N_{\eta[G]}$  then  $h_*(\alpha) \in N_{\eta[G]} \cap \mathfrak{h} = N_{< \rangle} \cap \mathfrak{h}$

also by the choice of  $\bar{h}$  (and genericity) clearly

$$(*)_4 \text{ Rang}(h_*) \text{ is equal to } u := (2^{\aleph_0}) \cap N_{\eta[G]}.$$

Lastly,

$$(*)_5 \ h_* \in \mathbf{V}.$$

[Why? As its domain,  $N_{<>} \cap \mathfrak{h}$  belongs to  $\mathbf{V}$  and  $h_*(\gamma)$  is defined from  $\langle \mathcal{T}, \bar{h}, \gamma, p_* \rangle \in \mathbf{V}$  and  $\mathcal{T}$  is a tree.]

- (\*)<sub>6</sub> (a) from  $u := \lambda \cap N_{\eta[\mathbf{G}]}$  we can define  $\eta[\mathbf{G}]$   
 (b)  $u = \cup\{N_{\eta \upharpoonright n[\mathbf{G}]} \cap \lambda : n < \omega\}$ .

[Why? By the choice of  $\bar{N}$ .]

Together we get that  $\eta[\mathbf{G}] \in \mathbf{V}$ , contradiction. □<sub>1.7</sub>

**Claim 1.8.** *We have  $\neg \text{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$  when* {2b.23}

- (a)  $2^{\aleph_0} \geq \lambda = \text{cf}(\lambda) > \kappa = \mathfrak{h}$   
 (b)  $\alpha < \lambda \Rightarrow \text{cf}([\alpha]^{\leq \kappa}, \subseteq) < \lambda$   
 (c)  $\mathbb{Q}$  is  $(\lambda, \kappa)$ -newly proper.

*Proof.* Similar to 1.7. □<sub>1.8</sub>

**Conclusion 1.9.** *If  $\mathfrak{h} < 2^{\aleph_0}$  and  $\mathbb{Q}$  is a  $(\mathfrak{h}^+, \mathfrak{h})$ -newly proper then  $\neg \text{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$ .* {2b.10}  
{2b.26}

## § 2. GENERAL SUFFICIENT CONDITIONS

{general}

**Claim 2.1.** Assume  $\mathbf{V} \models \text{CH}$ .If  $\mathbb{Q}$  is c.c.c. then  $\text{Pr}_2(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$ .**Remark 2.2.** 1) This works replacing  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  by any  $\aleph_1$ -complete  $\mathbb{P}$  and strengthening the conclusions to  $\text{Pr}_1$ , see 2.3.

{3c.3}

{1a.7}

2) See Definition 0.4(1).

*Proof.* Let  $\mathbb{P} = \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ . Clearly it suffices to prove:(\*) if  $r \in \mathbb{P}$  and  $\Vdash_{\mathbb{Q}}$  “ $\mathcal{I}$  is a dense open subset of  $\mathbb{P}$ ” then there is  $r'$  such that:

- (a)  $r \leq_{\mathbb{P}} r'$
- (b)  $\Vdash_{\mathbb{Q}}$  “ $r' \in \mathcal{I} \subseteq \mathbb{P}$ ”.

Why (\*) holds? We try (all in  $\mathbf{V}$ ) to choose  $(r_\alpha, q_\alpha)$  by induction on  $\alpha < \omega_1$  but choosing  $q_\alpha$  together with  $r_{\alpha+1}$  such that:

- ⊗ (a)  $r_0 = r$
- (b)  $r_\alpha \in \mathbb{P}$  is  $\leq_{\mathbb{P}}$ -increasing
- (c)  $q_\alpha \in \mathbb{Q}$
- (d)  $q_\alpha, q_\beta$  are incompatible in  $\mathbb{Q}$  for  $\beta < \alpha$
- (e)  $q_\alpha \Vdash_{\mathbb{Q}}$  “ $r_{\alpha+1} \in \mathcal{I}$ ”.

We cannot succeed in carrying the induction  $\omega_1$  many steps because  $\mathbb{Q} \models \text{c.c.c.}$ For  $\alpha = 0$  no problem as only clause (a) is relevant.For  $\alpha$  limit - easy as  $\mathbb{P}$  is  $\aleph_1$ -complete (and the only relevant clause is (b)).For  $\alpha = \beta + 1$ , we first ask:Question: Is  $\langle q_\gamma : \gamma < \beta \rangle$  a maximal antichain of  $\mathbb{Q}$ ?If yes, then  $r_\beta$  is as required in (\*) on  $r'$ ; why? if  $\mathbf{G}_{\mathbb{Q}} \subseteq \mathbb{Q}$  is generic over  $\mathbf{V}$  to which  $r_\beta$  belongs, then for some  $\gamma < \beta$ ,  $q_\gamma \in \mathbf{G}_{\mathbb{Q}}$  hence  $r_{\gamma+1} \in \mathcal{I}[\mathbf{G}_{\mathbb{Q}}]$  but  $\mathcal{I}[\mathbf{G}_{\mathbb{Q}}]$  is a dense subset of  $\mathbb{P}$  and is open and  $r_{\gamma+1} \leq_{\mathbb{P}} r_\beta$  so  $r_\beta \in \mathcal{I}[\mathbf{G}_{\mathbb{Q}}]$ .If no, let  $q^\beta \in \mathbb{Q}$  be incompatible with  $q_\gamma$  for every  $\gamma < \beta$ . Recalling  $\Vdash_{\mathbb{Q}}$  “ $\mathcal{I}$  is dense and open” the set  $X_\beta = \{r \in \mathbb{P} : \text{for some } q, q^\beta \leq_{\mathbb{Q}} q \text{ and } q \Vdash_{\mathbb{Q}} “r \in \mathcal{I}”\}$  is a dense subset of  $\mathbb{P}$  hence there is a member of  $X_\beta$  above  $r_\beta$ , let  $r_\alpha$  be such member. By  $r_\alpha \in X_\beta$ , there is  $q, q^\beta \leq q$  such that  $q \Vdash_{\mathbb{Q}}$  “ $r_\alpha \in \mathcal{I}$ ”. So we choose  $q_\beta$  as such  $q$ , so we can carry the induction step.As said above we cannot carry the induction for all  $\alpha < \omega_1$  because then  $\{q_\alpha : \alpha < \omega_1\}$  contradicts “ $\mathbb{Q}$  satisfies the c.c.c.” So for some  $\alpha$  we cannot continue,  $\alpha$  is neither 0 nor limit hence for some  $\beta$ ,  $\alpha = \beta + 1$ . So the answer to the question is yes, hence we get the desired conclusion of (\*).  $\square_{2.1}$ {3c.3} We can weaken the demand on the second forcing (above, it is  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ ).**Claim 2.3.** If (A) then (B) where:

- (A) (a)  $\mathbb{P}, \mathbb{Q}$  are forcing notions
- (b)  $\mathbb{Q}$  is c.c.c. moreover  $\Vdash_{\mathbb{P}}$  “ $\mathbb{Q}$  is c.c.c.”
- (c) forcing with  $\mathbb{P}$  adds no new  $\omega$ -sequences,<sup>3</sup> from  $\lambda$

<sup>3</sup>if you assume  $\mathbb{P}$  is proper,  $\lambda = \aleph_0$  the proof may be easier to read

- (d)  $\mathbb{Q}$  has cardinality  $\leq \lambda$
- (B) (a) if  $\mathbb{P}$  is proper in  $\mathbf{V}$  then  $\text{Pr}_2(\mathbb{Q}, \mathbb{P})$
- (b) for every  $\mathbb{Q}$ -name  $\mathcal{J}$  of a dense open subset of  $\mathbb{P}$ , the set  $\mathcal{J}$  is dense and open in  $\mathbb{P}$  where:
- (\*)  $\mathcal{J} = \mathcal{J}_{\mathcal{J}}$  is the set of  $r \in \mathbb{P}$  such that some  $\bar{q}$  witnesses it, i.e. witness it belongs to  $\mathcal{J}$  which means:
- $\bar{q} = \langle q_\alpha : \alpha < \alpha_* \rangle$  is a maximal antichain of  $\mathbb{Q}$
  - for each  $\alpha < \alpha_*$ , the set  $\{r' \in \mathbb{P} : q_\alpha \Vdash "r' \in \mathcal{J}"\}$  is an open subset of  $\mathbb{P}$  dense above  $r$ .

*Proof.* First, we prove clause (b); so fix  $\mathcal{J}$  and  $\mathcal{J}$  as there. Let  $\langle q_\varepsilon : \varepsilon < \kappa := |\mathbb{Q}| \rangle$  list  $\mathbb{Q}$ .

For every  $r \in \mathbb{P}$  we define a sequence  $\eta_r$  of ordinals  $< \kappa \leq \lambda$  as follows:

- ⊗<sub>1</sub>  $\eta_r(\alpha)$  is the minimal ordinal  $\varepsilon < \kappa$  such that (so  $\ell g(\eta_r) = \alpha$  when there is no such  $\varepsilon$ ):
- (a)  $q_\varepsilon \Vdash "r \in \mathcal{J}"$
- (b) if  $\beta < \alpha$  then  $q_\varepsilon, q_{\eta_r(\beta)}$  are incompatible in  $\mathbb{Q}$ .

Now

- ⊗<sub>2</sub> (a)  $\eta_r$  is well defined
- (b)  $\ell g(\eta_r) < \omega_1$ .

[Why? Obviously  $\eta_r$  is a well defined sequence of ordinals, i.e. clause (a) and clause (b) holds because  $\mathbb{Q} \models \text{c.c.c.}$ ]

Note

- ⊗<sub>3</sub> if  $r_1 \leq_{\mathbb{P}} r_2$  then either  $\eta_{r_1} \trianglelefteq \eta_{r_2}$  or for some  $\alpha < \ell g(\eta_{r_1})$  we have

$$\eta_{r_1} \upharpoonright \alpha = \eta_{r_2} \upharpoonright \alpha$$

$$\eta_{r_1}(\alpha) > \eta_{r_2}(\alpha).$$

[Why? Think about the definition.]

For  $s \in \mathbb{P}$  let  $\eta'_s$  be  $\cap \{\eta_{s_1} : s \leq_{\mathbb{P}} s_1\}$ , i.e. the longest common initial segment of  $\{\eta_{s_1} : s \leq_{\mathbb{P}} s_1\}$ ; clearly  $s_1 \leq_{\mathbb{P}} s_2 \Rightarrow \eta'_{s_1} \trianglelefteq \eta'_{s_2}$ . So

- ⊗<sub>4</sub>  $\eta^* = \cup \{\eta'_s : s \in \mathbf{G}_{\mathbb{P}}\}$  is a  $\mathbb{P}$ -name of a sequence of ordinals  $< \kappa$  such that  $\langle q_{\eta^*(i)} : i < \ell g(\eta^*) \rangle$  is a sequence of pairwise incompatible members of  $\mathbb{Q}$ .

But by clause (A)(b) of the claim, forcing with  $\mathbb{P}$  preserve " $\mathbb{Q} \models \text{c.c.c.}$ ", so  $\ell g(\eta^*)$  is countable in  $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$ . By clause (A)(c) of the claim, forcing by  $\mathbb{P}$  adds no new  $\omega$ -sequences to  $\kappa = |\mathbb{Q}|$  (and  $\mathbb{Q}$  is infinite) and  $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$  has the same  $\aleph_1$  as  $\mathbf{V}$ , so

- ⊗<sub>5</sub>  $\eta^*$  is a sequence of countable length of ordinals  $< \kappa$  so is old.

Hence

- ⊗<sub>6</sub> the following set is dense open in  $\mathbb{P}$

$$\mathcal{J} = \{r \in \mathbb{P} : r \text{ forces in } \mathbb{P} \text{ that } \eta^* = \eta_r^* \text{ for some } \eta_r^* \in \mathbf{V}\}$$

As for clause (a), let  $\chi, N, q_1, r_1$  be as in the assumption of  $(*)_1$  of 0.3, so  $\mathbb{P}, \mathbb{Q} \in N$ . {1a.3}  
We have to find  $q_2, r_2$  as there.

Let  $q_2 = q_1$  and let  $r_2 \in \mathbb{P}$  be  $(N, \mathbb{P})$ -generic and above  $r_1$ , exists as  $\mathbb{P}$  is a proper forcing in  $\mathbf{V}$ .

We shall show that  $(r_2, q_2)$  is as required, i.e.  $q_2 \Vdash_{\mathbb{Q}} "r_2 \text{ is } (N[\mathbf{G}_{\mathbb{Q}}], \mathbb{P})\text{-generic}"$ . Let  $\mathbf{G}_{\mathbb{Q}} \subseteq \mathbb{Q}$  be generic over  $\mathbf{V}$  such that  $q_2 \in \mathbf{G}_{\mathbb{Q}}$  and we should prove that  $\mathbf{V}[\mathbf{G}_{\mathbb{Q}}] \models "r_2 \text{ is } (N[\mathbf{G}_{\mathbb{Q}}], \mathbb{P})\text{-generic}"$ . So let  $\mathcal{I} \in N[\mathbf{G}_{\mathbb{Q}}]$  be a dense open subset of  $\mathbb{P}$ , and we should prove that  $\mathbf{V}[\mathbf{G}_{\mathbb{Q}}] \models "\mathcal{I} \cap N[\mathbf{G}_{\mathbb{Q}}] \text{ is pre-dense above } r_2"$ .

It suffices to prove:

(\*) if  $r_2 \leq_{\mathbb{P}} r_3$  then  $r_3$  is compatible (in  $\mathbb{P}$ ) with some  $r \in \mathcal{I} \cap N$ .

So fix  $r_3 \in \mathbb{P}$ ; by the definition of  $N[\mathbf{G}_{\mathbb{Q}}]$  there is a  $\mathbb{Q}$ -name  $\mathcal{I}$  such that  $\mathcal{I} = \mathcal{I}[\mathbf{G}_{\mathbb{Q}}]$ , for some  $\mathcal{I} \in N$ ; without loss of generality  $\Vdash_{\mathbb{Q}} "\mathcal{I} \text{ is a dense open subset of } \mathbb{P}"$ . Let  $\mathcal{I} = \mathcal{I}_{\mathcal{I}} = \{r \in \mathbb{P} : r \text{ has an } \mathcal{I}\text{-witness } \bar{q}_* = \langle q_{\alpha}^* : \alpha < \alpha_* \rangle\}$ , see clause (B)(b) of the claim. Clearly  $\mathcal{I} \in N$  hence  $\mathcal{I} \cap N$  is pre-dense in  $\mathbb{P}$  over  $r_2$  hence also over  $r_3$  hence there are  $r_4, r_5 \in \mathbb{P}$  such that  $r_3 \leq_{\mathbb{P}} r_5, r_4 \leq_{\mathbb{P}} r_5$  and  $r_4 \in N \cap \mathcal{I}$ . By the definition of  $\mathcal{I}$  there is an  $\mathcal{I}$ -witness  $\bar{q}_* = \langle q_{\alpha}^* : \alpha < \alpha_* \rangle$  for  $r_4 \in \mathcal{I}$ .

But  $\mathcal{I}, r_4 \in N$  hence without loss of generality  $\bar{q}_* \in N$  and  $\bar{q}_*$  has countable length, so  $\{q_{\alpha}^* : \alpha < \alpha_*\} \subseteq N$ . As  $\bar{q}_*$  is a witness, necessarily it is a maximal antichain of  $\mathbb{Q}$  hence for some  $\alpha < \alpha_*$  we have  $q_{\alpha}^* \in \mathbf{G}_{\mathbb{Q}}$ , as  $\bar{q}_*$  is a witness for  $r_4 \in \mathcal{I}_{\mathcal{I}}$ , necessarily  $\mathcal{I}_1 = \{r \in \mathbb{P} : q_{\alpha}^* \Vdash_{\mathbb{Q}} "r \in \mathcal{I}"\}$  is an open subset of  $\mathbb{P}$  dense above  $r_4$ .

Clearly  $\mathcal{I}_1 \in N$  is an open subset of  $\mathbb{P}$ , dense above  $r_4$  and  $r_4 \leq_{\mathbb{P}} r_5$  hence  $\mathcal{I}_1 \cap N$  is pre-dense above  $r_5$  hence there are  $r_6 \leq_{\mathbb{P}} r_7$  from  $\mathbb{P}$  such that  $r_6 \in \mathcal{I}_1 \cap N$  and  $r_5 \leq_{\mathbb{P}} r_7$ .

Clearly  $r_6 \in \mathcal{I}[\mathbf{G}_{\mathbb{Q}}] \cap N$  and  $r_6$  is compatible with  $r_3$  in  $\mathbb{P}$ , so we are done proving  $r_2$  is  $(N[\mathbf{G}_{\mathbb{Q}}], \mathbb{P})$ -generic.

So we are done. □<sub>2.3</sub>

{3c.5}

{3c.3}

*Remark 2.4.* In 2.1, 2.3 we can replace “c.c.c.” by “strongly proper”.

But such  $\mathbb{Q}$  preserves “ $(\omega_2)^{\mathbf{V}}$ -non-meagre”.

{3c.7}

**Claim 2.5.** 1) *There is a proper forcing  $\mathbb{Q}$  which forces “ $\mathbb{P}_{\mathcal{A}^*}[\mathbf{V}]$  as a forcing notion is not proper”, (i.e.  $\neg \text{Pr}_1(\mathbb{Q}, \mathbb{P})$ ).*

{1a.10}

2) *Even (A) of 0.5(3) fails, i.e.  $\neg \text{Pr}_5(\mathbb{Q}, \mathbb{P}_{\mathcal{A}^*}[\mathbf{V}])$ .*

*Proof.* We use the proof of [Sh:f, Ch.17, Sec.2] and see references there. We repeat in short.

We use a finite iteration so let  $\mathbb{P}_0$  be the trivial forcing notion,  $\mathbb{P}_{k+1} = \mathbb{P}_k * \mathbb{Q}_k$  for  $k \leq 3$  and the  $\mathbb{P}_k$ -name  $\mathbb{Q}_k$  is defined below.

Step A:  $\mathbb{Q}_0 = \text{Levy}(\aleph_1, 2^{\aleph_0})$  so  $\Vdash_{\mathbb{Q}_0} \text{“CH”}$ .

Step B:  $\mathbb{Q}_1$  is Cohen forcing.

Step C: In  $\mathbf{V}^{\mathbb{P}_2}$ ,  $\mathbb{Q}_2$  in the Levy collapse of  $2^{2^{\aleph_0}}$  to  $\aleph_1$ , i.e.  $\mathbb{Q}_2 = \text{Levy}(\aleph_1, \beth_2)^{\mathbf{V}[\mathbb{P}_2]}$ .

Step D: Let  $\mathcal{T} = ((\omega_1 >) \omega_1)^{\mathbf{V}^{\mathbb{P}_1}} = ((\omega_1 >) \omega_1)^{\mathbf{V}^{\mathbb{P}_0}}$  be a tree, so we know that  $\lim_{\omega_1}(\mathcal{T})^{\mathbf{V}^{\mathbb{P}_1}} = \lim_{\omega_1}(\mathcal{T})^{\mathbf{V}^{\mathbb{P}_2}} = \lim_{\omega_1}(\mathcal{T})^{\mathbf{V}^{\mathbb{P}_3}}$  hence has cardinality  $\aleph_1$  in  $\mathbf{V}^{\mathbb{P}_3}$  and

$(*)_1$  in  $\mathbf{V}^{\mathbb{P}_1}$ ,  $\mathcal{T}$  is isomorphic to a dense subset of  $\mathbb{P}_{\mathcal{A}_*[\mathbb{P}_1]} = \mathbb{P}_{\mathcal{A}_*[\mathbb{P}_0]}$ .

So in  $\mathbf{V}^{\mathbb{P}_3}$  there is a list  $\langle \eta_\varepsilon^* : \varepsilon < \omega_1 \rangle$  of  $\lim_{\omega_1}(\mathcal{T})^{\mathbf{V}^{\mathbb{P}_1}}$  and let  $\langle \eta_\varepsilon^* \upharpoonright [\gamma_\varepsilon, \omega_1) : \varepsilon < \omega_1 \rangle$  be pairwise disjoint end segments so  $\gamma_\varepsilon < \omega_1$ ,  $\langle \gamma_\varepsilon : \varepsilon < \omega_1 \rangle \in \mathbf{V}^{\mathbb{P}_3}$  and  $\varepsilon_1 < \varepsilon_2 < \omega_1 \wedge \beta_1 \in [\gamma_{\varepsilon_1}, \omega_1) \wedge \beta_2 \in [\gamma_{\varepsilon_2}, \omega_1) \Rightarrow \eta_{\varepsilon_1}^* \upharpoonright \gamma_1 \neq \eta_{\varepsilon_2}^* \upharpoonright \gamma_2$ .

Step E: In  $\mathbf{V}^{\mathbb{P}_3}$  there is  $\mathbb{Q}_3$ , a c.c.c. forcing notion specializing  $\mathcal{T}$  in the sense of [Sh:74], i.e. there is  $h_* \in \mathbf{V}^{\mathbb{P}_4}$  such that  $h_* : \mathcal{T} \rightarrow \omega$ ,  $h_*$  is increasing in  $\mathcal{T}$  except being constant on each end segment  $\eta_\varepsilon^* \upharpoonright [\gamma_\varepsilon, \omega_1)$  for  $\varepsilon < \omega_1$ , i.e.  $\rho <_{\mathcal{T}} \nu \wedge h_*(\rho) = h_*(\nu) \Rightarrow (\exists \varepsilon)[\rho, \nu \in \{\eta_\varepsilon^* \upharpoonright \gamma : \gamma \in [\gamma_\varepsilon, \omega_1)\}]$ .

Now

$\boxtimes$  after forcing with  $\mathbb{P}_4 = \mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{Q}_2 * \mathbb{Q}_3$ , i.e. in  $\mathbf{V}^{\mathbb{P}_4}$  the forcing notion  $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$  is not proper, in fact it collapses  $\aleph_1$ .

Why? Recall  $(*)_1$  and note

$(*)_2$   $\mathcal{I}_n := \{\rho \in \mathcal{T} : (\forall \nu)(\rho \leq_{\mathcal{T}} \nu \rightarrow h_*(\nu) \neq n)\}$  is dense open in  $\mathcal{T}$

and trivially

$(*)_3$   $\bigcap_n \mathcal{I}_n = \emptyset$ ; in fact if  $\mathbf{G} \subseteq \mathcal{T}$  is generic, then:

- (A)  $\mathbf{G}$  is a branch of  $\mathcal{T}$  of order type  $\omega_1^{\mathbf{V}}$  let its name be  $\langle \rho_\gamma : \gamma < \omega_1 \rangle$
- (B) letting  $\gamma_n = \text{Min}\{\gamma < \omega_1 : \rho_\gamma \in \mathcal{I}_n\}$  we have  $\Vdash_{\mathcal{T}} \text{“}\{\gamma_n : n < \omega\} \text{ is unbounded in } \omega_1\text{”}$ .

$\square_{2.5}$

## § 3. PRIVATE APPENDIX

**Discussion 3.1.** 1) With CH, we still do not know what occurs for  $\mathbb{Q}$  proper not c.c.c.

A case which looks to point out where is the problem.

*Question 3.2.* Assume CH, and  $\prod_{\alpha < \omega_2} P_\alpha$ , CS product, each  $\mathbb{P}_\alpha$  adds a dense subset of  $\mathcal{T}_* = {}^\omega > (\omega_1)$  or  $F : \mathcal{T}_* \rightarrow \mathcal{T}_*, \eta <_{\mathcal{T}_*} F(\eta)$  (so the generic  $\eta \in \mathcal{T}'_\delta$  will satisfy  $(\exists^\infty \alpha < \delta)[F(\eta) \upharpoonright \alpha \triangleleft \eta]$ ).

2) Without CH, we know that every c.c.c. adding a real and may proper one destroy properness. We know that such forcing does not exist when  $\mathbf{V} = \mathbf{L}$  by xxx Boban Velichovic xxx Macia Groszek xxx.

Generally there are, but quotients but does this give examples here?

- {3c.1} 3) In 2.1, for  $\text{Pr}_2(\mathbb{Q}, \mathbb{P})$ , it seemed that CH is not necessary.  
 {3c.3} 4) Question: in 2.3, can we weaken “ $\mathbb{Q}$  is c.c.c.” to “ $\mathbb{Q}$  is  $\omega$ -proper”? (or less games...)  
 5) If  $\mathbb{Q}$  is nep,  $\mathbb{R}$  is  $\aleph_1$ -complete then  $\text{Pr}_5(\mathbb{Q}, \mathbb{P})$ .  
 {3c.3} 6) In 2.3, “ $\mathbb{Q}$  not adding a new  $\eta \in {}^\omega |Q|$ ”, not adding new  $\eta \in {}^\omega \text{Ord}$  are equivalent..Check use.

Moved 09.11.24: try to get “ $\text{Pr}(\mathbb{Q}, \mathbb{P}_{\mathcal{A}^*[\mathbf{V}]})$  fail” from newly  $(\lambda, *)$ -proper when  $\lambda \leq \mathfrak{h}$  (another direction: weaken the demand on  $\lambda$ )

Case 2:  $\mathfrak{h} \geq \lambda$ .

For  $\alpha < 2^{\aleph_0}$  we define  $T_\alpha$ , a  $\mathbb{P}_{\mathcal{A}^*[\mathbf{V}]}$ -name, equivalently a  $\mathcal{T}$ -name by:  $T_\alpha[\mathbf{G}] = p$  if for some  $p_0 \in \mathcal{T}, p \in \text{succ}_{\mathcal{T}}(p_0)$  we have  $h_{p_0}(p) = \alpha$ .

Let  $h(\alpha) = T_\alpha$  for  $\alpha < 2^{\aleph_0}$ .

So

$$q_* \Vdash “h \upharpoonright (2^{\aleph_0} \cap N_{\eta[\mathbf{G}]}) \text{ is a function from } 2^{\aleph_0} \cap N_{\eta[\mathbf{G}]} \text{ onto } 2^{\aleph_0} \cap N_{\eta[\mathbf{G}]}”.$$

Let  $h_{p_*}$  be the function  $p_* \in \mathcal{T}$  induce so  $h_{p_*}$  maps  $2^{\aleph_0} \cap N_{\eta[\mathbf{G}]}$  onto itself.

Fixing  $p_*$  in  $\mathbf{V}$  we define  $\text{rk}_{p_*} : {}^\omega > \lambda \rightarrow \text{Ord} \cup \{\infty\}$  by:

$\text{rk}_{p_*}(\eta) \geq \varepsilon$  iff for every  $\alpha \in N_\eta \cap {}^{\aleph_0}2$  and  $\zeta < \varepsilon$  there is  $\nu$  such that

- $\eta \triangleleft \nu \in {}^\omega > \lambda$
- $\text{rk}_{p_*}(\nu) \geq \zeta$
- $h_{p_*}(\alpha) \in N_\nu$ .

Clearly

- (\*)  $\text{rk}_{p_*}(\eta[\mathbf{G}] \upharpoonright n) = \infty$  for every  $\lambda$
- (\*) let  $\mathcal{S}_1 = \{\nu \in {}^\omega > \lambda : \text{rk}_{p_*}(\nu) = \infty\}$ .

So clearly

- (\*) (a)  $\mathcal{S}_1$  is a perfect subtree of  ${}^\omega > \lambda$  in  $\mathbf{V}$
- (b)  $\eta[\mathbf{G}]$  is a branch of  $\mathcal{S}_1$  [in  $\mathbf{V}[\mathbf{G}]$ ]

- (c) there is a perfect subtree  $\mathcal{S}_2$  of  $\mathcal{S}_1$  such that: if  $\eta \in \lim_\omega(\mathcal{S}_1)$  then  $h_{p_*}$  maps  $N_\eta \cap \aleph_0 2$  onto itself [in  $\mathbf{V}$ ]
- (d) we can demand moreover that  $p_*$  is  $(N_\eta, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$  generic absolutely in particular even for branches of  $\mathcal{S}_2$  not (in  $\mathbf{V}[\mathbf{G}]$ ) (recall  $|\mathcal{A}_*| = |\mathcal{T}| = 2^{\aleph_0}$ ), [used?]
- (\*) without loss of generality for some limit  $\delta_* < \lambda$ ,  $\text{cf}(\delta_*) = \aleph_0$  and for every  $\eta \in \mathcal{S}_2$ ,  $\sup\{\text{Rang}(\nu_{\eta \upharpoonright n}) : n < \omega\} = \delta_*$
- (\*) without loss of generality in  $\square$  we can add
  - (i)' we have  $\langle \beta_\eta : \eta \in \omega^{>\lambda} \rangle$  such that  $\beta_\eta \in \mathfrak{h} \cap N_\eta$  but  $\ell g(\eta) = n + 1 \Rightarrow \beta_\eta > \sup\{N_\rho \cap \mathfrak{h} : \rho \in \omega^{>}(\sup \text{Rang}(\nu_{\eta \upharpoonright n}))\}$
- (\*) let
  - (a)  $\beta_\varepsilon = \sup\{\beta_\eta : \eta \in \omega^{>\varepsilon}\}$  for  $\varepsilon < \lambda$  so  $\langle \beta_\varepsilon : \varepsilon < \lambda \rangle$  is increasing
  - (b)  $\beta_* = \sup\{\beta_\varepsilon : \varepsilon < \lambda\}$
  - (c)  $\gamma_\rho = \sup(N_\eta \cap \beta_*)$
- (\*) if  $\eta_1 \neq \eta_2 \in \lim(\mathcal{S}_2)$ , so  $\eta_1 \cap \eta_2 = \eta \in \omega^{>\lambda}$ , then  $p_*$  is  $(N_\eta, \mathcal{T})$ -generic
- (\*) for  $\rho \triangleleft \eta \in \lim(\mathcal{S}_2)$  let  $w_{\eta, \rho} = \{p \upharpoonright \gamma_\rho : p \in N_\eta \cap \mathcal{T}, \text{level}_{\mathcal{T}}(p) > \gamma_\rho\}$
- (\*) we can choose  $\bar{N}$  such that: if  $\eta, \eta_1, \eta_2$  as above and  $p \in w_{\eta_1, \eta} \cap w_{\eta_2, \eta}$  is contained in  $w_\eta^*$  (choose  $N_\eta^- \prec N_\eta^+$ , etc.)

Toward the case  $\mathfrak{h} \geq \lambda$  is similar:

Let  $\bar{M}^* = \langle M_\gamma^* : \gamma < \lambda \rangle$  is  $\prec$ -increasing continuous,  $\|M_\gamma^*\| < \mathfrak{h}$ ,  $\bar{M}^* \upharpoonright (\gamma + 1) \in M_{\gamma+1}^* \prec (\mathcal{H}(\chi), \in)$ . Use  $\langle C_\delta : \delta \in \mathfrak{S}_{\aleph_0}^\lambda \rangle$  guess clubs.

Choose  $\delta_* \in S_{\aleph_0}^\lambda$ . Choose countable  $N_1^* \prec (\mathcal{H}(\chi), \in)$ ,  $\{\bar{M}^*, \delta_*\} \in N_0$ . Choose similarly  $N_2^*$  such that  $N_1^* \subset N_2^*$ . Now by the game produce  $\langle (N_\eta^\ell, \nu_\eta) : \eta \in \omega^{>\lambda} \rangle$ .

Old Proof:

- ⊗ it suffices to prove that for  $(q, \mathcal{T}')$  as above
  - ⊠  $q \Vdash$  “for some  $\eta \in \lim_\omega(\mathcal{T}')$ , the tree  $\mathcal{T}_\eta := \mathcal{T} \cap N_\eta$  which has a set of levels  $u_\eta = N_\eta \cap \mathfrak{h}$ , has no  $u_\eta$ -branch with an upper bound in  $\mathcal{T}$ ”.

However

- (\*) without loss of generality for every  $\eta \in \lim_\omega(\mathcal{T}')$ , even new one,  $\gamma_\eta = \sup(N_\eta \cap \mathfrak{h})$  is the same  $\gamma_*$  (as we can replace); moreover in  $\square$  we can add
  - $\square^+$  (b) if  $\mathfrak{h} < \lambda$  and  $\eta \in \omega^\lambda$  and  $\alpha_1 < \alpha_2 < \lambda$  and  $t_1 \in N_{\eta \hat{\ } < \alpha_1} \cap \mathcal{T} \setminus N_\eta$ ,  $t_2 \in N_{\eta \hat{\ } < \alpha_2} \cap \mathcal{T} \setminus N_\eta$  then in  $\mathcal{T}$ ,  $t_1, t_2$  are  $<_I$ -incomparable
  - (b)' if  $\mathfrak{h} \geq \lambda$ , if  $(\alpha_1, \alpha_2, t_1, t_2)$  fails the above then  $t_1, t_2$  has a common  $<_{\mathcal{T}}$ -upper bound in  $N_\eta \cap \mathcal{T}$ .

Case 1:  $\mathfrak{h} < \lambda$

So  $\square$  holds, just use any new  $\eta \in \lim_\omega(\delta)$ .

Case 2: Not Case 1

We just weaken  $\square$ : add “except branches which are included in a branch of  $\mathcal{T}$  which belongs to  $N_\eta$ , i.e.  $N_{\eta \upharpoonright n}$  for some  $n < \omega$ ”.

**Discussion 3.3.** Second, how strong is that assumption  $\otimes$ ? Well, e.g. Cohen  $*$  Levy( $\aleph_1, 2^{\aleph_0}$ ) fail it but

$\boxplus$  any c.c.c. forcing is O.K.

Also we can weaken it

$\otimes$  if  $\eta$  in  $N_{< \omega}$  is a  $\mathbb{Q}$ -name of a new real, then for some perfect embedding  $h$  of  $\omega^{>2}$  into  $\omega^{>}(\omega_2)$  (so  $h(\eta) \triangleleft h(\eta^\wedge \langle \ell \rangle)$  for  $\ell = 1, 2, h(\eta^\wedge \langle 0 \rangle), h(\eta^1 \langle 1 \rangle)$ ) are  $\triangleleft$ -incomparable let  $\eta' = h(\eta)$ , i.e.  $\cup \{h(\eta \upharpoonright n) : n < \omega\}$  so  $\Vdash \eta' \in \text{lim}(\mathcal{S}')$  where  $\mathcal{S}' = \{\nu : \nu \triangleleft h(\rho) \text{ so } h \in \mathbf{V} \text{ for some } \rho \in \omega^{>2}\}$  and we demand: there is  $q \in \mathbb{Q}$  such that  $q$  is “ $(N_{\eta'}, \mathbb{Q})$ -generic”.

The proof is as before.

Moved 09.11.25 from the proof of 2.3 trying to prove  $\text{Pr}_1(\mathbb{Q}, \mathbb{R})$ :

(Problem: why  $N_2^*$  old - rethink)

So without loss of generality  $\mathbb{R} \subseteq \mathcal{H}_{< \aleph_1}(\lambda)$  and  $\lambda \geq \aleph_0$ . So toward contradiction assume  $q_* \in \mathbb{Q}$  and

(\*)<sub>1</sub>  $q_* \Vdash_{\mathbb{Q}}$  “ $\mathbb{R}$  is not proper”.

So letting  $\chi$  be large enough and

(\*)<sub>2</sub> (a)  $q_* \Vdash_{\mathbb{Q}}$  “ $N_1 \triangleleft (\mathcal{H}(\chi)^{\mathbf{V}[G]}, \mathcal{H}(\chi)^{\mathbf{V}}, \in)$  is countable hence  
 (b)  $N_0 = N_1 \upharpoonright \mathcal{H}(\chi)^{\mathbf{V}}$  is a countable elementary submodel of  $(\mathcal{H}(\chi), \in)$   
 (c) let  $N_2 = N_0 \upharpoonright \mathcal{H}_{< \aleph_1}(\lambda)$  even expanding by  $\mathbb{R}$ , note  $\mathbb{R} \cap N_0 = \mathbb{R} \cap N_2$   
and  $r_* \in N_1 \cap \mathbb{R}$  and no  $r'$  satisfying  $r' \leq_{\mathbb{R}} r_*$  is  $(N_1, \mathbb{R})$ -generic”.

Possibly increasing  $q_*$  without loss of generality

(\*)<sub>3</sub>  $r_* = r_*$  a member of  $\mathbb{R}$  as a forcing.

Similarly without loss of generality

(\*)<sub>4</sub> ??  $N_2 = N_2^*$ , i.e.  $N_2^*$  is an object from  $\mathbf{V}$  and  $(N_2, \mathbb{R}) \triangleleft (\mathcal{H}_{< \aleph_1}(\lambda), \in, \mathbb{R})$ .

So as  $\mathbb{R}$  is ( $< \aleph_1$ )-complete in  $\mathbf{V}$ , letting  $\text{Gen} = \{\mathbf{g} \subseteq N_2^* \cap \mathbb{R} : \mathbf{g} \text{ is directed and } r_* \in \mathbf{g}\}$  in  $\mathbf{V}$

(\*)<sub>5</sub> if  $r \in \mathbb{R}$  is  $\leq_{\mathbb{R}}$ -above  $r_*$  then  $\{r' \in \mathbb{R} \cap N_0^* : r' \leq_{\mathbb{R}} r\} \in \text{Gen}$ .

Hence

(\*)<sub>6</sub>  $q_* \Vdash_{\mathbb{Q}}$  “for no  $\mathbf{g} \in \text{Gen}$  for every  $\mathcal{I} \in N_0$  which is a dense open subset of  $\mathbb{R}$  do we have  $\mathbf{g} \cap \mathcal{I} \neq \emptyset$ ”.

By clause  $(\beta)$  of the claim which we have proved

(\*)<sub>7</sub>  $q_* \Vdash_{\mathbb{Q}}$  “for no  $\mathbf{g} \in \text{Gen}$  for every  $\mathcal{I} \in N_1 \cap \mathbf{V}$  which is a dense open subset of  $\mathbb{R}$  do we have  $\mathbf{g} \cap \mathcal{I} \neq \emptyset$ ”.

As  $\mathbb{Q}$  is c.c.c. (or as it is proper and we can increase  $q_*$ ) there is  $N_3$  such that

(\*)<sub>8</sub>  $N_3 \in \mathbf{V}, N_3 \triangleleft (\mathcal{H}(\chi), \in)$  is countable and  $q_* \Vdash$  “ $N_1 \subseteq N_3$ ”.

But obviously

- (\*)<sub>9</sub> if  $\mathcal{I} \in \mathbf{V}$  and  $q_* \Vdash \text{“}\mathcal{I} \notin N_2\text{”}$  is a dense open subset of  $\mathbb{R}$  then
- (a)  $\mathcal{I} \in N_3$
  - (b)  $\mathcal{I}$  is a dense open subset of  $\mathbb{R}$
  - (c)  $\mathcal{I} \cap N_2^*$  is a dense open subset of  $\mathbb{R} \cap N_2^*$ .

We can finish as

- (\*)<sub>10</sub> there is  $\mathbf{g} \in \text{Gen}$  such that: if  $\mathcal{I}$  satisfies (a),(b),(c) of (\*)<sub>3</sub> then  $\mathbf{g} \cap \mathcal{I} \neq \emptyset$ .

[Why? As  $N_3$  is countable.]

We get contradiction so we are done.

Moved from pg. 12:

Next

- ⊗<sub>8</sub>  $\mathcal{I}_2$  is dense open.

Clearly  $\mathcal{I}_*$  is open, but why is it dense? Let  $r \in \mathbb{P}$  and we shall find  $t \in \mathcal{I}_*$  such that  $r \leq_{\mathbb{P}} t$ ; so we can replace  $r$  by any  $r' \in \mathbb{P}$  above  $r$ . By ⊗<sub>6</sub> there is  $r_1 \in \mathcal{I}_1$  above  $r$  so  $\eta_{r_1}^*$  is well defined, so  $r_1 \Vdash \text{“}\eta^* = \eta_{r_1}^*\text{”}$ . Let  $\eta_* = \eta_{r_1}^*$ , so by the definition of  $\eta^*$  in ⊗<sub>4</sub> we have  $\eta_{r_1}' \trianglelefteq \eta_*$  and we claim that

- ⊗<sub>8.1</sub>  $\eta_{r_1}' = \eta_*$ .

[Why? If not then necessarily  $\eta_{p_1}' \triangleleft \eta_*$ , hence letting  $\xi = \ell g(\eta_{p_1}')$ , by the definition of  $\eta_{p_1}'$  there is  $p_2$  such that  $p_1 \leq_{\mathbb{P}} p_2$  and  $\eta_{p_2} \upharpoonright (\xi + 1) = \eta_* \upharpoonright (\xi + 1)$ , but necessarily  $\eta_{r_1}' \trianglelefteq \eta_{p_2}$  hence  $\eta_{p_1}' \triangleleft \eta_{p_2}$ . Hence the condition  $p_2, q_{\eta_*(\xi)}$  are compatible; let  $p_3 \in \mathbb{P}$  be a common upper bound of them so  $(\forall t)(p_3 \leq_{\mathbb{P}} t \Rightarrow q_{\eta_*(\xi)} \leq_{\mathbb{P}} t \Rightarrow \eta_* \upharpoonright (\xi + 1) \not\trianglelefteq \eta_t)$  hence  $p_3 \Vdash \text{“}\eta^* \neq \eta_*\text{”}$ , contradiction.]

We can conclude

- ⊗<sub>8.2</sub> if  $r_1 \leq t$  then  $\eta_* = \eta_t$ .

By ⊗<sub>8.1</sub> and the definition of  $\eta_{r_1}'$  we have  $r_1 \leq_{\mathbb{P}} t \Rightarrow \eta_* \trianglelefteq \eta_t$ . Let  $u = \{\eta_t(\ell g(\eta_*)) : t \in \mathbb{P} \text{ is above } r_1 \text{ and } \ell g(\eta_t) > \ell g(\eta_*)\}$ . If  $u$  is empty then  $r_1 \in \mathcal{I}_*$  by the definition of  $\mathcal{I}_*$ , so we are done proving ⊗<sub>8</sub>. If  $u$  is not empty, let  $r_2 \in \mathbb{P}$  be such that  $\ell g(\eta_{r_2}) > \ell g(\eta_*) \wedge \eta_{r_2}(\ell g(\eta_*)) = \min(u)$ . By the definition of  $\eta^*$  and of  $\langle \eta_t : t \in \mathbb{P} \rangle$  there is  $r_3 \in \mathbb{P}$  above  $r_2$  such that  $\ell g(\eta_{r_3}) > \ell g(\eta_*) \wedge \eta_{p_3}(\ell g(\eta_*)) < \min(u)$ , contradiction. So  $u = \emptyset$  and  $p_1$  is as required.

- ⊗<sub>9</sub> if  $r \in \mathcal{I}_*$  then  $\langle q_{\eta_r^*(\varepsilon)} : \varepsilon < \ell g(\eta_r^*) \rangle$  is a maximal antichain of  $\mathbb{Q}$ .

[Why? As in the proof of 2.1.]

Fixing  $r_* \in \mathcal{I}_* \subseteq \mathbb{P}$  and  $\alpha < \ell g(\eta_{r_*}^*)$  let

- (\*)<sub>0</sub>  $\mathcal{I}_{r_*, \alpha} = \{r \in \mathbb{P} : r_* \leq_{\mathbb{P}} r \text{ and } q_{\eta_{r_*}^*(\alpha)} \text{ forces (for } \Vdash_{\mathbb{Q}}) \text{ that } r \in \mathcal{I}\}$
- (\*)<sub>1</sub>  $\mathcal{I}_{r_*, \alpha}$  is dense in  $\mathbb{P}$  above  $r_*$ .

[Why? Assume  $\mathbb{P} \Vdash \text{“}r_* \leq r_1\text{”}$  so  $r_1 \Vdash_{\mathbb{P}} \text{“}\eta^*(\alpha) = \eta_{r_*}^*(\alpha)\text{”}$  hence for some  $r_2$  we have  $\mathbb{P} \Vdash \text{“}r_1 \leq r_2\text{”}$  and  $\eta^* \upharpoonright (\alpha + 1) \trianglelefteq \eta_{r_2}'$ , so by clause (a) of ⊗<sub>1</sub> we have  $q_{\eta_{r_*}^*(\alpha)} \Vdash_{\mathbb{Q}} \text{“}r_2 \in \mathcal{I}\text{”}$  hence  $r_2 \in \mathcal{I}_{r_*, \alpha}$  as required.]

So

- (\*)<sub>2</sub>  $\mathcal{I}_{r_*, \alpha}$  is a (dense and) open subset of  $\mathbb{P}_{\geq r_*}$  (i.e. above  $r_*$ ).

{3c.1}

[Why? As  $\Vdash_{\mathbb{Q}}$  “ $\mathcal{I}$  is an open subset”, for density use  $(*)_1$ .]

As forcing with  $\mathbb{P}$  adds no new  $\omega$ -sequence of ordinals  $< \lambda$  (by clause (A)(c) of the claim)

$(*)_3$   $\mathcal{I}_{r_*}^+ := \cap \{ \mathcal{I}_{r_*, \alpha} : \alpha < \text{lg}(\eta_{r_*}^*) \}$  is dense open in  $\mathbb{P}$  above  $r_*$ .

[Why? Let  $\mathcal{I}_{r_*, \alpha}^*$  be a maximal antichain  $\subseteq \mathcal{I}_{r_*, \alpha}$  for  $\alpha < \text{lg}(\eta_{r_*}^*)$  let  $f$  be the  $\mathbb{P}$ -name of  $\{(\alpha, q) : \alpha < \text{lg}(\eta_{r_*}^*) \text{ and } q \in \mathcal{I}_{r_*, \alpha}^* \cap \dot{G}_{\mathbb{P}}\}$  so  $r_* \Vdash$  “ $f$  a function from  $\text{lg}(\eta_{r_*}^*)$  to  $\mathbb{P}$ ” hence  $r_* \Vdash$  “ $f \in \mathbf{V}$ ”.]

{3c.1} Clearly by the definition (recalling  $\otimes_7$ , as in the proof of 2.1)

$(*)_4$  if  $r \in \mathcal{I}_{r_*}^+$  then  $\Vdash_{\mathbb{Q}}$  “ $r \in \mathcal{I}$ ”.

As  $\cap \{ \mathcal{I}_{r_*}^+ : r_* \in \mathcal{I} \}$  is dense open in  $\mathbb{P}$  we are done proving clause (B)(b) of the claim.

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