

PRESERVING OLD $([\omega]^{N_0}, \supseteq^*)$ IS PROPER
SH960

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ABSTRACT. We give some sufficient and necessary conditions on a forcing notion \mathbb{Q} for preserving the forcing notion $([\omega]^{N_0}, \supseteq^*)$ being proper. They cover many reasonable forcing notions.

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ANOTATED CONTENT

- §0 Introduction, pg.3
- {1a.1} [I.e. Definition 0.2, we define the problem and some variants.]
- §1 Properness of $\mathbb{P}_{\mathcal{A}[\mathbf{V}]}$ and CH, pg.5
- {2b.1} [Under CH, if non-meagerness of $(\omega_2)^{\mathbf{V}}$ is preserved then $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ is proper, (1.1). If \mathbf{V} fails to satisfy CH, then usually $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ is not proper after a forcing adding a new real and satisfying a relative of being proper, e.g. satisfies c.c.c. or is any true creature forcing.]
- §2 General sufficient conditions, pg. 10
- {3c.1} [If \mathbf{V} satisfies CH and \mathbb{Q} is c.c.c. then $\Vdash_{\mathbb{Q}}$ “ $\mathbb{P}_{\mathcal{A}[\mathbf{V}]}$ is proper”, see in 2.1. In
- {3c.3} 2.3 we replace $\mathcal{A}_*^{\mathbf{V}}$ by a forcing notion \mathbb{R} adding no ω -sequence, \mathbb{Q} is c.c.c. even in $\mathbf{V}^{\mathbb{P}}$. Instead “ \mathbb{Q} satisfies the c.c.c.” it suffices to demand \mathbb{Q} satisfies
- {3c.7} a weaker condition. Lastly, in 2.5 we prove some proper forcing does not preserve.]

§ 0. INTRODUCTION

{intro}

We investigate the question “ $\text{Pr}_1^+(\mathbb{Q}, \mathbb{R})$ ”, which means that the proper forcing \mathbb{Q} preserves that the (old) \mathbb{R} is proper for various \mathbb{R} 's. In what follows, $B \subseteq^* A$ means $|B \setminus A| < \aleph_0$, and $A \supseteq^* B$ means the same.

Recall:

{1a.0}

Definition 0.1. properness:

- (a) Assume that $N \prec (\mathcal{H}(\chi), \in), \mathbb{P} \in N$ is a forcing notion and $q \in \mathbb{P}$. We say that q is (N, \mathbb{P}) -generic iff for every dense $D \subseteq \mathbb{P}$, if $D \in N$ then $D \cap N$ is pre-dense above q .
- (b) A forcing notion \mathbb{P} is proper iff for every sufficiently large regular χ and every countable $N \prec (\mathcal{H}(\chi), \in)$, if $p, \mathbb{P} \in N$ then there is a condition $q \in \mathbb{P}, q \geq p$ such that q is (N, \mathbb{P}) -generic.

Gitman proved that $\text{Pr}_1^+(\mathbb{Q}, \mathbb{P}_{\mathcal{P}(\omega)[\mathbf{V}]})$ (see definition below, where $\mathbb{P}_{\mathcal{P}(\omega)[\mathbf{V}]}$ is the forcing notion $(\{A \in \mathbf{V} : A \subseteq \omega, |A| = \aleph_0\}, \supseteq^*)$, when \mathbb{Q} is adding Cohen reals (or just Cohen subsets even $> 2^{\aleph_0}$ many). But no other examples were known even Sacks forcing. Also for e.g. $\mathbf{V} \models “V = L”$, we did not know a forcing making it not proper.

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Let us state the problem and relatives. We are interested mainly in the case \mathbb{Q} is proper.

{1a.1}

Definition 0.2. 1) Let $\text{Pr}_1(\mathbb{Q}, \mathbb{P})$ means: \mathbb{Q}, \mathbb{P} are forcing notions and $\Vdash_{\mathbb{Q}} “\mathbb{P}$, i.e. $\mathbb{P}^{\mathbf{V}}$ is a proper forcing”.

1A) Let $\text{Pr}_1^+(\mathbb{Q}, \mathbb{P})$ be defined similarly but adding “ \mathbb{Q} is proper”.

2) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\mathbb{P}_{\mathcal{A}}$ be $\mathcal{A} \setminus [\omega]^{<\aleph_0}$ ordered by \supseteq^* , inverse almost inclusion.

3) Let $\mathcal{A}_* = \mathcal{A}_*[\mathbf{V}] = ([\omega]^{\aleph_0})^{\mathbf{V}}$.

{1a.3}

Observation 0.3. A necessary condition for $\text{Pr}_1(\mathbb{Q}, \mathbb{P})$ is:

(*)₁ if χ is regular and large enough, $N \prec (\mathcal{H}(\chi), \in)$ is countable, $\mathbb{Q}, \mathbb{P} \in N, q_1 \in \mathbb{Q}$ is (N, \mathbb{Q}) -generic and $r_1 \in N \cap \mathbb{P}$ then we can find (q_2, r_2) such that:

- ⊙ (a) $q_1 \leq_{\mathbb{Q}} q_2$
- (b) $r_1 \leq_{\mathbb{P}} r_2$
- (c) $q_2 \Vdash “r_2$ is $(N[G_{\mathbb{Q}}], \mathbb{P})$ -generic”.

{1a.7}

Definition 0.4. 1) We define $\text{Pr}^-(\mathbb{Q}, \mathbb{P}) = \text{Pr}_2(\mathbb{Q}, \mathbb{P})$ as the necessary condition from 0.3.

{1a.3}

2) Let $\text{Pr}_3(\mathbb{Q}, \mathbb{P})$ mean that \mathbb{Q}, \mathbb{P} are forcing notions and for some λ and stationary $S \subseteq [\lambda]^{\aleph_0}$ from \mathbf{V} we have $\Vdash_{\mathbb{Q}} “\mathbb{P}$ is S -proper”, and note that S remains stationary of course.

3) $\text{Pr}_4(\mathbb{Q}, \mathbb{P})$ is defined similarly but $S \in \mathbf{V}^{\mathbb{Q}}$, still $S \subseteq ([\lambda]^{\aleph_0})^{\mathbf{V}}$, so S is actually \dot{S} , a \mathbb{Q} -name.

4) $\text{Pr}_5(\mathbb{Q}, \mathbb{P})$ is the statement (A) of 0.5(4) below.

{1a.10}

5) Let $\text{Pr}_\ell^+(\mathbb{Q}, \mathbb{P})$ means $\text{Pr}_\ell(\mathbb{Q}, \mathbb{P})$ and \mathbb{Q} is a proper forcing, for $\ell = 2, 3, 4, 5$.

{1a.10}

Claim 0.5. 1) $\text{Pr}_2(\mathbb{Q}, \mathbb{P})$ means that for λ large enough, letting $S = ([\lambda]^{\aleph_0})^{\mathbf{V}}$, we have $\Vdash_{\mathbb{Q}}$ “ \mathbb{P} is S -proper”.

2) $\text{Pr}_1(\mathbb{Q}, \mathbb{P}) \Rightarrow \text{Pr}_2(\mathbb{Q}, \mathbb{P}) \Rightarrow \text{Pr}_3(\mathbb{Q}, \mathbb{P})$; similarly for Pr^+ .

3) Also $\text{Pr}_3(\mathbb{Q}, \mathbb{P}) \Rightarrow \text{Pr}_4(\mathbb{Q}, \mathbb{P}) \Rightarrow \text{Pr}_5(\mathbb{Q}, \mathbb{P})$; similarly for Pr^+ .

4) If \mathbb{Q}, \mathbb{P} are forcing notions, χ large enough and regular, then (A) \Leftrightarrow (B) where

(A) for some countable $N \prec (\mathcal{H}(\chi), \in)$ and for some $q \in \mathbb{Q}, p \in \mathbb{P}$ we have

(a) q is (N, \mathbb{Q}) -generic

(b) $q \Vdash_{\mathbb{Q}}$ “ p is $(N[G_{\mathbb{Q}}], \mathbb{P})$ -generic”

(B) for some $q_* \in \mathbb{Q}, p_* \in \mathbb{P}$ we have $\text{Pr}_4(\mathbb{Q}_{\geq q_*}, \mathbb{P}_{\geq p_*})$.

Proof. Easy. □_{0.5}

{z2}

Notation 0.6. $<_{\chi}^*$ denotes a well ordering of $\mathcal{H}(\chi)$.

Recall (Balcar-Pelant-Simon [BPS80], or see, e.g. Blass [Bla])

{z5}

Definition 0.7. \mathfrak{h} is the following cardinal invariant, it is the minimal cardinality χ (necessarily regular) such that forcing with $\mathbb{P}_{\mathcal{A}^*}$ adds a new sequence of ordinals of length χ .

{z8}

Notation 0.8. If \mathcal{T} is a tree, then $\text{suc}_{\mathcal{T}}(p)$ is the set of immediate successors of $p \in \mathcal{T}$ in the tree order.

§ 1. PROPERNESS OF $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ AND CH

{2b.4}

Claim 1.1. Assume $\mathbf{V}_0 \models \text{CH}$, $\mathbf{V}_1 \supseteq \mathbf{V}_0$, e.g. $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{Q}}$ and let $\mathcal{A} = \mathcal{A}_*[\mathbf{V}_0]$.

- (a) If $\aleph_1^{\mathbf{V}_0}$ is a countable ordinal in \mathbf{V}_1 , then $\mathbf{V}_1 \models \text{“}\mathbb{P}_{\mathcal{A}} \text{ is proper”}$.
- (b) If $\aleph_1^{\mathbf{V}_0} = \aleph_1^{\mathbf{V}_1}$ and $\mathbf{V}_1 \models \text{“}(\omega_2)^{\mathbf{V}_0} \text{ is non-meagre”}$, then $\mathbf{V}_1 \models \text{“}\mathbb{P}_{\mathcal{A}} \text{ is proper”}$.

In both cases, if \mathbf{V}_1 is a generic extension of \mathbf{V}_0 by the forcing notion \mathbb{Q} then it means that $\text{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathcal{A}})$ holds.

Proof. Assume that $\mathbf{V}_1 \supseteq \mathbf{V}_0$.

If $\mathbf{V}_1 \models \text{“}\aleph_1^{\mathbf{V}_0} \text{ is countable”}$ then recalling $\mathbf{V}_0 \models \text{CH}$ clearly $\mathbf{V}_1 \models \text{“}\mathcal{A} \text{ is countable”}$ so we know that $\mathbb{P}_{\mathcal{A}}$ is proper in \mathbf{V}_1 , thus proving clause (a). So from now on we assume $\aleph_1^{\mathbf{V}_0}$ is not collapsed.

In \mathbf{V}_0 let $\mathcal{T} = \omega_1^{>}(\omega_1)$ and choose a subset $\mathcal{A}' \subseteq \mathcal{A}$ such that \mathcal{A}' is \subseteq^* -dense in \mathcal{A} and $(\mathcal{A}', \supseteq^*)$ is tree-isomorphic to \mathcal{T} . Let π be the isomorphism between these trees¹. Notice that all this is done in \mathbf{V}_0 (recalling that $\mathbf{V}_0 \models \text{CH}$). In \mathbf{V}_0 there is a sequence $\bar{\mathcal{T}} = \langle \mathcal{T}_\alpha : \alpha < \omega_1 \rangle$ which is \subseteq -increasing continuous with union \mathcal{T} and each \mathcal{T}_α countable. Also there is $\bar{C} = \langle C_\delta : \delta < \omega_1, \delta \text{ is a limit ordinal} \rangle \in \mathbf{V}_0$ such that $C_\delta \subseteq \delta = \sup(C_\delta)$, $\text{otp}(C_\delta) = \omega$. Let $\mathcal{T}'_\delta = \mathcal{T}_\delta \upharpoonright \{\eta \in \mathcal{T}_\delta : \text{lg}(\eta) \in C_\delta\}$.

In \mathbf{V}_1 choose a sufficiently large regular cardinal χ , and let $N \prec (\mathcal{H}(\chi), \in)$ be countable such that $\mathcal{A}, \pi, \bar{\mathcal{T}} \in N$ and let $\delta = \omega_1 \cap N$, clearly $\mathcal{T} \cap N = \mathcal{T}'_\delta$. We have to prove the statement:

- (*)₀ “for every $p \in \mathbb{P}_{\mathcal{A}} \cap N$ there is $q \in \mathbb{P}_{\mathcal{A}}$ above p which is $(N, \mathbb{P}_{\mathcal{A}})$ -generic”.

As $\mathbf{V}_0 \models \text{CH}$ and the density of \mathcal{A}' in \mathcal{A} and $(\mathcal{A}', \supseteq^*)$ being isomorphic in \mathbf{V}_0 by π to \mathcal{T} this is equivalent (in \mathbf{V}_1 , of course) to:

- (*)₁ for every $\nu \in \mathcal{T} \cap N = \mathcal{T}'_\delta$ there is $\eta \in \mathcal{T}$ which is (N, \mathcal{T}) -generic and $\nu \leq_{\mathcal{T}} \eta$.

In \mathbf{V}_0 we let $\bar{S} = \langle S_\delta : \delta < \omega_1 \text{ a limit ordinal} \rangle$ where $S_\delta = \{\bar{\nu} : \bar{\nu} = \langle \nu_n : n < \omega \rangle \text{ is } <_{\mathcal{T}}\text{-increasing, } \nu_n \in \mathcal{T}'_\delta, \text{ moreover } \text{lg}(\nu_n) \text{ is the } n\text{-th member of } C_\delta\}$.

As $(\forall \nu \in \mathcal{T}'_\delta)(\exists \rho)(\nu <_{\mathcal{T}} \rho \in \mathcal{T}'_\delta)$, and $[\bar{\nu} \in S_\delta \Rightarrow \text{there is a } <_{\mathcal{T}}\text{-upper bound } \rho \in \mathcal{T} \text{ of } \bar{\nu}, \text{ in } \mathbf{V}_0, \text{ of course}]$ recalling $\mathcal{T}'_\delta, S_\delta \in \mathbf{V}_0$ clearly (*₁) is equivalent (in \mathbf{V}_1 , of course) to

- (*)₂ for every $\nu \in \mathcal{T}'_\delta$ there is $\bar{\nu} \in S_\delta$ such that $\nu \in \text{Rang}(\bar{\nu})$ and $\bar{\nu}$ induce a subset of \mathcal{T}'_δ generic over N (i.e. $(\forall A)[A \in N \text{ is a dense open subset of } \mathcal{T} \Rightarrow A \cap \{\nu_n : n < \omega\} \neq \emptyset]$).

Now a sufficient condition for (*₂) is

- (*)₃ S_δ , as a set of ω -branches of the tree \mathcal{T}'_δ , is non-meagre.

But in \mathbf{V}_0 , \mathcal{T}'_δ and $\omega^{> \omega}$ are isomorphic and S_δ is the set of all ω -branches of \mathcal{T}'_δ , so by an assumption from part (b), (*₃) holds so we are done. □_{1.1}

{2b.4}

Discussion 1.2. However, there can be $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $(\mathcal{A}, \subseteq^*)$ is a variation of Souslin tree.

¹this is trivial as $\mathbf{V}_0 \models \text{CH}$, however always there is a dense tree with \mathfrak{h} levels by the celebrated theorem of Balcar-Pelant-Simon

{2b.7}

Claim 1.3. 1) We have $\text{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$ when:

- (a) $\aleph_1^{\mathbf{V}[\mathbb{Q}]} = \aleph_1$
- (b) $\Vdash_{\mathbb{Q}} "|\lambda| = \aleph_1 \text{ where } \lambda = (2^{\aleph_0})^{\mathbf{V}}"$
- (c) moreover letting $\langle u_i : i < \aleph_1 \rangle$ be a \mathbb{Q} -name of a \subseteq -increasing continuous sequence of countable subsets of λ with union λ , the \mathbb{Q} -name $\mathcal{S} = \{i : u_i \in \mathbf{V}\}$ is forced to contain a club (of \aleph_1)
- (d) forcing with \mathbb{Q} preserves " $(\omega 2)^{\mathbf{V}}$ is non-meagre".

2) Assume the forcing notion \mathbb{Q} satisfies (a) + (d), $\text{Pr}_4(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$ as witnessed by S and \mathbb{Q} is proper and \mathcal{S} is forced to be stationary.

Then the forcing notion $\mathbb{Q} * \text{Levy}(\aleph_1, (|\mathbb{Q}|^{\aleph_0})^{\mathbf{V}}) * \mathbb{Q}_{\mathcal{S}}$ preserves " $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ is proper" where $\mathbb{Q}_{\mathcal{S}}$ is the (well known) shooting of a club through the stationary subsets of ω_1 (to make clause (c) hold).

{2b.1} *Proof.* Like 1.1. □_{1.3}

In what follows we prove that many forcing notions destroy properness. We need a preliminary concept.

{2b.13}

Definition 1.4. For $\lambda > \kappa$ we say that a forcing notion \mathbb{Q} is (λ, κ) -newly proper (omitting κ means $\kappa = \aleph_0$ and we define $(\lambda, < \chi)$ -newly proper similarly) when: if $\bar{N} = \langle (N_\eta, \nu_\eta) : \eta \in {}^{\omega >} \lambda \rangle$ satisfies \otimes below and $\mathbb{Q} \in N_{< \omega}$ and $p \in \mathbb{Q} \cap N_{< \omega}$ then we can find q, η such that \boxtimes below holds where:

- \otimes for some cardinal $\chi > \lambda$
 - (a) $N_\eta \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ is countable
 - (b) if $\nu \triangleleft \eta$ then $N_\nu \prec N_\eta$
 - (c) $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$ if $\kappa = \aleph_0$ and $N_{\eta_1}^\kappa \cap N_{\eta_2}^\kappa = N_{\eta_1 \cap \eta_2}^\kappa$ generally where $N_\eta^\kappa := \cup \{v \in N_\eta : |v| \leq \kappa\}$
 - (d) $\nu_\eta \in N_\eta \setminus \cup \{N_{\eta \upharpoonright m}^\kappa : m < \text{lg}(\eta)\}$ hence $\nu_\eta \notin \cup \{N_\nu : \neg(\eta \trianglelefteq \nu) \text{ and } \nu \in {}^{\omega >} \lambda\}$
 - (e) $\nu_\eta \in {}^{\text{lg}(\eta)} \lambda$ and $\ell < \text{lg}(\eta) \Rightarrow \nu_{\eta \upharpoonright \ell} \trianglelefteq \nu_\eta$
- \boxtimes (a) $p \leq_{\mathbb{Q}} q$
- (b) $q \Vdash_{\mathbb{Q}} " \cup \{N_{\eta \upharpoonright n}[\mathbf{G}_{\mathbb{Q}}] : n < \omega\} \cap \mathbf{V} = \cup \{N_{\eta \upharpoonright n} : n < \omega\}"$
- (c) $q \Vdash_{\mathbb{Q}} " \eta \in {}^\omega \lambda \text{ is new, i.e. } \eta \notin ({}^\omega \lambda)^{\mathbf{V}}"$
- (c)⁺ moreover if $\kappa > \aleph_0$ and $\mathcal{T} \in \mathbf{V}$ is a sub-tree of ${}^{\omega >} \lambda$ of cardinality $\leq \kappa$ then $\eta \notin \lim(\mathcal{T)$, i.e. $\{\eta \upharpoonright n : n < \omega\} \notin \mathcal{T}$.

{2b.14}

Observation 1.5. If $\langle N_\eta : \eta \in {}^{\omega >} \lambda \rangle$ satisfies clauses (a),(b),(c) of \otimes of Definition 1.4, then the following conditions are equivalent:

{2b.13}

{2b.13}

- ₁ there is $\langle \nu_\eta : \eta \in {}^{\omega >} \lambda \rangle$ such that clauses (d),(e) of \otimes of Definition 1.4
- ₂ if $\eta \in {}^{\omega >} \lambda$, then $N_\eta \cap \lambda \not\subseteq \cup \{N_{\eta \upharpoonright \ell} : \ell < \text{lg}(\eta)\}$.

For a proper forcing notion adding a new real it is quite easy to be \aleph_1 -newly proper; e.g.

{2b.19}

Claim 1.6. Assuming $2^{\aleph_0} \geq \lambda = \text{cf}(\lambda) > \aleph_1$, sufficient conditions for “ \mathbb{Q} is λ -newly proper” are:

- (a) \mathbb{Q} is c.c.c. and adds a new real
- (b) \mathbb{Q} is Sacks forcing
- (c) \mathbb{Q} is a tree-like creature forcing in the sense of Roslanowski-Shelah [RoSh:470].

Proof. Easy; for clause (a) we use $q = p$ for \boxplus of the definition noting that: if $\eta \in {}^{\omega}>\lambda$ then p is (N_η, \mathbb{Q}) -generic. For clauses (b),(c) we use fusion but in the n -th step use members of $N_\eta \cap \mathbb{Q}$ for $\eta \in {}^n\lambda$, we get as many distinct η 's as we can. $\square_{1.6}$

{2b.10}

Theorem 1.7. We have $\Vdash_{\mathbb{Q}}$ “ $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ is not proper” when:

- (a) $\mathbf{V} \models 2^{\aleph_0} \geq \aleph_2$
- (b) λ is regular, $\aleph_2 \leq \lambda \leq 2^{\aleph_0}$ and² $\alpha < \lambda \Rightarrow \text{cf}([\alpha]^{\aleph_0}, \subseteq) < \lambda$ hence (by [Sh:420]) there is a stationary $\mathcal{U}_\alpha \subseteq [\alpha]^{\aleph_0}$ of cardinality $< \lambda$
- (c) $\mathfrak{h} < \lambda$
- (d) the forcing notion \mathbb{Q} adds at least one real and is λ -newly proper.

Proof. Let χ be large enough and for transparency, $x \in \mathcal{H}(\chi)$.

By Rubin-Shelah [RuSh:117], see more [Sh:f, Ch.XI] in \mathbf{V} there is a sequence $\langle N_\eta : \eta \in {}^{\omega}>\lambda \rangle$ such that:

- \square_1 (a) $N_\eta \prec (\mathcal{H}(\chi), \in)$
- (b) $\mathbb{Q}, x \in N_\eta$
- (c) N_η is countable
- (d) $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$.

Now for each $\eta \in {}^\omega\lambda$ let $N_\eta = \cup\{N_{\eta \upharpoonright k} : k < \omega\}$; we can easily add:

- (e) there is \mathcal{W} such that:
 - (α) \mathcal{W} is a subtree of ${}^{\omega}>\lambda$
 - (β) $\langle \rangle \in \mathcal{W}$
 - (γ) if $\eta \in \mathcal{W}$ then $(\exists^\lambda \alpha)(\eta \hat{\ } \langle \alpha \rangle \in \mathcal{W})$
 - (δ) if $\eta \in \text{lim}(W)$ then $\eta \in {}^\omega\lambda$ is increasing, and $\text{sup}(N_\eta \cap \lambda) = \text{sup}(\text{Rang}(\eta))$

- (ε) we can choose $\nu_\eta \in N_\eta$ for $\nu \in \mathcal{W}$ as in clauses (d),(e) of \otimes of 1.4. {2b.13}

By Balcar-Pelant-Simon [BPS80] there is $\mathcal{T} \subseteq [\omega]^{\aleph_0}$ such that

- \square_2 (α) $(\mathcal{T}, \supseteq^*)$ is a tree with \mathfrak{h} levels (\mathfrak{h} is the cardinal invariant from 0.7, a regular cardinal $\in [\aleph_1, 2^{\aleph_0}]$), the tree \mathcal{T} has a root and each node has 2^{\aleph_0} many immediate successors, i.e. \mathcal{T} has splitting to 2^{\aleph_0} {z5}
- (β) \mathcal{T} is dense in $([\omega]^{\aleph_0}, \supseteq^*)$, i.e. in $\mathbb{P}_{\mathcal{P}(\omega)[\mathbf{V}]} = \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ recalling 0.2(2). {1a.1}

Choose \bar{h} such that

²If $\lambda = \aleph_2$ the rest of clause (b) follows.

□₃ $\bar{h} = \langle h_p : p \in \mathcal{T} \rangle$ satisfies: h_p is a one-to-one function from $\text{succ}_{\mathcal{T}}(p)$ onto $2^{\aleph_0} \setminus \{h_{p_0}(p_1) : p_0 <_{\mathcal{T}} p_1 <_{\mathcal{T}} p \text{ and } p_1 \in \text{succ}_{\mathcal{T}}(p_0)\}$.

So without loss of generality

□₄ $\mathcal{T} \in N_{< \rangle}, \mathfrak{h} \in N_{< \rangle}$ and $\bar{h} \in N_{< \rangle}$.

{2b.13} As \mathbb{Q} is λ -newly proper there are η, q as in □ of Definition 1.4. Let $\mathbf{G} \subseteq \mathbb{Q}$ be generic over \mathbf{V} such that $q \in \mathbf{G}$, let $\eta = \eta[G]$ and $M_2 := N_{\eta[G]} := \cup\{N_{\eta \upharpoonright n}[\mathbf{G}] : n < \omega\}$, so $M_2 \prec (\mathcal{H}(\chi)^{\mathbf{V}[\mathbf{G}]}, \mathcal{H}(\chi)^{\mathbf{V}}, \in)$ is countable, pedantically $(|M_2|, \mathcal{H}(\chi)^{\mathbf{V}} \cap |M_2|, \in \upharpoonright |M_2|) \prec (\mathcal{H}(\chi)^{\mathbf{V}[\mathbf{G}]}, \mathcal{H}(\chi)^{\mathbf{V}}, \in \upharpoonright \mathcal{H}(\chi)^{\mathbf{V}[\mathbf{G}]})$.

{2b.13} By □ of 1.4, i.e. the choice of η, q as $q \in \mathbf{G}$ we have $M_1 = M_2 \cap \mathcal{H}(\chi)^{\mathbf{V}}$ is $\cup\{N_{\eta \upharpoonright n} : n < \omega\}$, and of course $M_1 \prec (\mathcal{H}(\chi), \in)$. Toward contradiction assume $\mathbf{V}[\mathbf{G}] \models \text{“}\mathcal{P}_{\mathcal{A}_*[\mathbf{V}]}\text{ is proper”}$, hence some $p_* \in \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ is $(M_2, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$ -generic. But \mathcal{T} is dense in $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ so without loss of generality $p_* \in \mathcal{T}$ and p_* is (M_2, \mathcal{T}) -generic.

Since $\mathfrak{h} \in N_{< \rangle}$ and $\mathfrak{h} < \lambda$, without loss of generality $\eta \in {}^{\omega} \lambda \Rightarrow N_{\eta} \cap \mathfrak{h} = N_{< \rangle} \cap \mathfrak{h}$. For any $\alpha < \lambda$ let

$$\mathcal{I}_{\alpha} = \{p \in \mathcal{T} : \text{for some } p_0 \in \mathcal{T} \text{ we have } p \in \text{succ}_{\mathcal{T}}(p_0) \text{ and } h_{p_0}(p) = \alpha\}$$

and letting \mathcal{I}_{α} be the α -th level of \mathcal{T} and let

$$\mathcal{I}_{\alpha}^+ = \{p \in \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]} : p \text{ is above some member of } \mathcal{I}_{\alpha}\}.$$

Now clearly (in \mathbf{V} and in $\mathbf{V}[\mathbf{G}]$):

- (*)₁ (a) \mathcal{I}_{α} is a pre-dense subset of \mathcal{T} (and of $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$)
- (b) \mathcal{I}_{α}^+ is dense open decreasing with α
- (c) if $p \in \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ then for every large enough $\alpha < \lambda, p \notin \mathcal{I}_{\alpha}^+$
- (d) if $p \in \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ and $\alpha < \lambda$ then there is $q \in \mathcal{I}_{\alpha}$ such that $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]} \models \text{“}p \leq q\text{”}$.

Also clearly the sequence $\langle \mathcal{I}_{\alpha} : \alpha < \lambda \rangle$ belongs to $N_{\langle \rangle}$ hence if $\alpha \in \lambda \cap N_{\eta[G]}$ then $\mathcal{I}_{\alpha} \in N_{\eta[G]}$ and the set $\{p \in \mathcal{T} \cap N_{\eta[G]} : p \leq_{\mathcal{T}} p_* \text{ and } p \in \mathcal{I}_{\alpha}\}$ is not empty.

Now

- (*)₂ in $\mathbf{V}[\mathbf{G}]$ the following functions h_{\bullet}, h_* are well defined
 - (a) $\text{Dom}(p_{\bullet}) = \text{Dom}(h_*) = N_{< \rangle} \cap \mathfrak{h}$
 - (b) $h_{\bullet}(\gamma)$ is the unique $p \in N_{\eta[G]} \cap \mathcal{T}$ of level γ which is $\leq_{\mathcal{T}} p_*$
 - (c) if $\gamma < \mathfrak{h}$ then $h_*(\gamma) = h_{\gamma+1}(h_{\bullet}(\gamma+1))$
- (*)₃ if $\alpha \in \mathfrak{h} \cap N_{\eta[G]}$ then $h_*(\alpha) \in N_{\eta[G]} \cap \mathfrak{h} = N_{< \rangle} \cap \mathfrak{h}$

also by the choice of \bar{h} (and genericity) clearly

$$(*)_4 \text{Rang}(h_*) \text{ is equal to } u := (2^{\aleph_0}) \cap N_{\eta[G]}.$$

Lastly,

$$(*)_5 h_* \in \mathbf{V}.$$

[Why? As its domain, $N_{<>} \cap \mathfrak{h}$ belongs to \mathbf{V} and $h_*(\gamma)$ is defined from $\langle \mathcal{T}, \bar{h}, \gamma, p_* \rangle \in \mathbf{V}$ and \mathcal{T} is a tree.]

- (*)₆ (a) from $u := \lambda \cap N_{\eta[\mathbf{G}]}$ we can define $\eta[\mathbf{G}]$
 (b) $u = \cup\{N_{\eta \upharpoonright n[\mathbf{G}]} \cap \lambda : n < \omega\}$.

[Why? By the choice of \bar{N} .]

Together we get that $\eta[\mathbf{G}] \in \mathbf{V}$, contradiction. □_{1.7}

Claim 1.8. *We have $\neg \text{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$ when* {2b.23}

- (a) $2^{N_0} \geq \lambda = \text{cf}(\lambda) > \kappa = \mathfrak{h}$
 (b) $\alpha < \lambda \Rightarrow \text{cf}([\alpha]^{\leq \kappa}, \subseteq) < \lambda$
 (c) \mathbb{Q} is (λ, κ) -newly proper.

Proof. Similar to 1.7. □_{1.8}

Conclusion 1.9. *If $\mathfrak{h} < 2^{N_0}$ and \mathbb{Q} is a $(\mathfrak{h}^+, \mathfrak{h})$ -newly proper then $\neg \text{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$.* {2b.10}
{2b.26}

§ 2. GENERAL SUFFICIENT CONDITIONS

{general}

Claim 2.1. Assume $\mathbf{V} \models \text{CH}$.If \mathbb{Q} is c.c.c. then $\text{Pr}_2(\mathbb{Q}, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$.**Remark 2.2.** 1) This works replacing $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ by any \aleph_1 -complete \mathbb{P} and strengthening the conclusions to Pr_1 , see 2.3.

{3c.3}

{1a.7}

2) See Definition 0.4(1).

Proof. Let $\mathbb{P} = \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$. Clearly it suffices to prove:(*) if $r \in \mathbb{P}$ and $\Vdash_{\mathbb{Q}}$ “ \mathcal{I} is a dense open subset of \mathbb{P} ” then there is r' such that:

- (a) $r \leq_{\mathbb{P}} r'$
- (b) $\Vdash_{\mathbb{Q}}$ “ $r' \in \mathcal{I} \subseteq \mathbb{P}$ ”.

Why (*) holds? We try (all in \mathbf{V}) to choose (r_α, q_α) by induction on $\alpha < \omega_1$ but choosing q_α together with $r_{\alpha+1}$ such that:

- ⊗ (a) $r_0 = r$
- (b) $r_\alpha \in \mathbb{P}$ is $\leq_{\mathbb{P}}$ -increasing
- (c) $q_\alpha \in \mathbb{Q}$
- (d) q_α, q_β are incompatible in \mathbb{Q} for $\beta < \alpha$
- (e) $q_\alpha \Vdash_{\mathbb{Q}}$ “ $r_{\alpha+1} \in \mathcal{I}$ ”.

We cannot succeed in carrying the induction ω_1 many steps because $\mathbb{Q} \models \text{c.c.c.}$ For $\alpha = 0$ no problem as only clause (a) is relevant.For α limit - easy as \mathbb{P} is \aleph_1 -complete (and the only relevant clause is (b)).For $\alpha = \beta + 1$, we first ask:Question: Is $\langle q_\gamma : \gamma < \beta \rangle$ a maximal antichain of \mathbb{Q} ?If yes, then r_β is as required in (*) on r' ; why? if $\mathbf{G}_{\mathbb{Q}} \subseteq \mathbb{Q}$ is generic over \mathbf{V} to which r_β belongs, then for some $\gamma < \beta$, $q_\gamma \in \mathbf{G}_{\mathbb{Q}}$ hence $r_{\gamma+1} \in \mathcal{I}[\mathbf{G}_{\mathbb{Q}}]$ but $\mathcal{I}[\mathbf{G}_{\mathbb{Q}}]$ is a dense subset of \mathbb{P} and is open and $r_{\gamma+1} \leq_{\mathbb{P}} r_\beta$ so $r_\beta \in \mathcal{I}[\mathbf{G}_{\mathbb{Q}}]$.If no, let $q^\beta \in \mathbb{Q}$ be incompatible with q_γ for every $\gamma < \beta$. Recalling $\Vdash_{\mathbb{Q}}$ “ \mathcal{I} is dense and open” the set $X_\beta = \{r \in \mathbb{P} : \text{for some } q, q^\beta \leq_{\mathbb{Q}} q \text{ and } q \Vdash_{\mathbb{Q}} “r \in \mathcal{I}”\}$ is a dense subset of \mathbb{P} hence there is a member of X_β above r_β , let r_α be such member. By $r_\alpha \in X_\beta$, there is $q, q^\beta \leq q$ such that $q \Vdash_{\mathbb{Q}}$ “ $r_\alpha \in \mathcal{I}$ ”. So we choose q_β as such q , so we can carry the induction step.As said above we cannot carry the induction for all $\alpha < \omega_1$ because then $\{q_\alpha : \alpha < \omega_1\}$ contradicts “ \mathbb{Q} satisfies the c.c.c.” So for some α we cannot continue, α is neither 0 nor limit hence for some $\beta, \alpha = \beta + 1$. So the answer to the question is yes, hence we get the desired conclusion of (*). $\square_{2.1}$ {3c.3} We can weaken the demand on the second forcing (above, it is $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$).**Claim 2.3.** If (A) then (B) where:

- (A) (a) \mathbb{P}, \mathbb{Q} are forcing notions
- (b) \mathbb{Q} is c.c.c. moreover $\Vdash_{\mathbb{P}}$ “ \mathbb{Q} is c.c.c.”
- (c) forcing with \mathbb{P} adds no new ω -sequences,³ from λ

³if you assume \mathbb{P} is proper, $\lambda = \aleph_0$ the proof may be easier to read

- (d) \mathbb{Q} has cardinality $\leq \lambda$
- (B) (a) if \mathbb{P} is proper in \mathbf{V} then $\text{Pr}_2(\mathbb{Q}, \mathbb{P})$
- (b) for every \mathbb{Q} -name \mathcal{J} of a dense open subset of \mathbb{P} , the set \mathcal{J} is dense and open in \mathbb{P} where:
- (*) $\mathcal{J} = \mathcal{J}_{\mathcal{J}}$ is the set of $r \in \mathbb{P}$ such that some \bar{q} witnesses it, i.e. witness it belongs to \mathcal{J} which means:
- $\bar{q} = \langle q_\alpha : \alpha < \alpha_* \rangle$ is a maximal antichain of \mathbb{Q}
 - for each $\alpha < \alpha_*$, the set $\{r' \in \mathbb{P} : q_\alpha \Vdash "r' \in \mathcal{J}"\}$ is an open subset of \mathbb{P} dense above r .

Proof. First, we prove clause (b); so fix \mathcal{J} and \mathcal{J} as there. Let $\langle q_\varepsilon : \varepsilon < \kappa := |\mathbb{Q}| \rangle$ list \mathbb{Q} .

For every $r \in \mathbb{P}$ we define a sequence η_r of ordinals $< \kappa \leq \lambda$ as follows:

- ⊗₁ $\eta_r(\alpha)$ is the minimal ordinal $\varepsilon < \kappa$ such that (so $\ell g(\eta_r) = \alpha$ when there is no such ε):
- (a) $q_\varepsilon \Vdash "r \in \mathcal{J}"$
- (b) if $\beta < \alpha$ then $q_\varepsilon, q_{\eta_r(\beta)}$ are incompatible in \mathbb{Q} .

Now

- ⊗₂ (a) η_r is well defined
- (b) $\ell g(\eta_r) < \omega_1$.

[Why? Obviously η_r is a well defined sequence of ordinals, i.e. clause (a) and clause (b) holds because $\mathbb{Q} \models \text{c.c.c.}$]

Note

- ⊗₃ if $r_1 \leq_{\mathbb{P}} r_2$ then either $\eta_{r_1} \trianglelefteq \eta_{r_2}$ or for some $\alpha < \ell g(\eta_{r_1})$ we have

$$\eta_{r_1} \upharpoonright \alpha = \eta_{r_2} \upharpoonright \alpha$$

$$\eta_{r_1}(\alpha) > \eta_{r_2}(\alpha).$$

[Why? Think about the definition.]

For $s \in \mathbb{P}$ let η'_s be $\cap \{\eta_{s_1} : s \leq_{\mathbb{P}} s_1\}$, i.e. the longest common initial segment of $\{\eta_{s_1} : s \leq_{\mathbb{P}} s_1\}$; clearly $s_1 \leq_{\mathbb{P}} s_2 \Rightarrow \eta'_{s_1} \trianglelefteq \eta'_{s_2}$. So

- ⊗₄ $\eta^* = \cup \{\eta'_s : s \in \mathbf{G}_{\mathbb{P}}\}$ is a \mathbb{P} -name of a sequence of ordinals $< \kappa$ such that $\langle q_{\eta^*(i)} : i < \ell g(\eta^*) \rangle$ is a sequence of pairwise incompatible members of \mathbb{Q} .

But by clause (A)(b) of the claim, forcing with \mathbb{P} preserve " $\mathbb{Q} \models \text{c.c.c.}$ ", so $\ell g(\eta^*)$ is countable in $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$. By clause (A)(c) of the claim, forcing by \mathbb{P} adds no new ω -sequences to $\kappa = |\mathbb{Q}|$ (and \mathbb{Q} is infinite) and $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$ has the same \aleph_1 as \mathbf{V} , so

- ⊗₅ η^* is a sequence of countable length of ordinals $< \kappa$ so is old.

Hence

- ⊗₆ the following set is dense open in \mathbb{P}

$$\mathcal{J} = \{r \in \mathbb{P} : r \text{ forces in } \mathbb{P} \text{ that } \eta^* = \eta_r^* \text{ for some } \eta_r^* \in \mathbf{V}\}$$

As for clause (a), let χ, N, q_1, r_1 be as in the assumption of $(*)_1$ of 0.3, so $\mathbb{P}, \mathbb{Q} \in N$. {1a.3}
We have to find q_2, r_2 as there.

Let $q_2 = q_1$ and let $r_2 \in \mathbb{P}$ be (N, \mathbb{P}) -generic and above r_1 , exists as \mathbb{P} is a proper forcing in \mathbf{V} .

We shall show that (r_2, q_2) is as required, i.e. $q_2 \Vdash_{\mathbb{Q}} "r_2 \text{ is } (N[\mathbf{G}_{\mathbb{Q}}], \mathbb{P})\text{-generic}"$. Let $\mathbf{G}_{\mathbb{Q}} \subseteq \mathbb{Q}$ be generic over \mathbf{V} such that $q_2 \in \mathbf{G}_{\mathbb{Q}}$ and we should prove that $\mathbf{V}[\mathbf{G}_{\mathbb{Q}}] \models "r_2 \text{ is } (N[\mathbf{G}_{\mathbb{Q}}], \mathbb{P})\text{-generic}"$. So let $\mathcal{I} \in N[\mathbf{G}_{\mathbb{Q}}]$ be a dense open subset of \mathbb{P} , and we should prove that $\mathbf{V}[\mathbf{G}_{\mathbb{Q}}] \models "\mathcal{I} \cap N[\mathbf{G}_{\mathbb{Q}}] \text{ is pre-dense above } r_2"$.

It suffices to prove:

(*) if $r_2 \leq_{\mathbb{P}} r_3$ then r_3 is compatible (in \mathbb{P}) with some $r \in \mathcal{I} \cap N$.

So fix $r_3 \in \mathbb{P}$; by the definition of $N[\mathbf{G}_{\mathbb{Q}}]$ there is a \mathbb{Q} -name \mathcal{I} such that $\mathcal{I} = \mathcal{I}[\mathbf{G}_{\mathbb{Q}}]$, for some $\mathcal{I} \in N$; without loss of generality $\Vdash_{\mathbb{Q}} "\mathcal{I} \text{ is a dense open subset of } \mathbb{P}"$. Let $\mathcal{I} = \mathcal{I}_{\mathcal{I}} = \{r \in \mathbb{P} : r \text{ has an } \mathcal{I}\text{-witness } \bar{q}_* = \langle q_{\alpha}^* : \alpha < \alpha_* \rangle\}$, see clause (B)(b) of the claim. Clearly $\mathcal{I} \in N$ hence $\mathcal{I} \cap N$ is pre-dense in \mathbb{P} over r_2 hence also over r_3 hence there are $r_4, r_5 \in \mathbb{P}$ such that $r_3 \leq_{\mathbb{P}} r_5, r_4 \leq_{\mathbb{P}} r_5$ and $r_4 \in N \cap \mathcal{I}$. By the definition of \mathcal{I} there is an \mathcal{I} -witness $\bar{q}_* = \langle q_{\alpha}^* : \alpha < \alpha_* \rangle$ for $r_4 \in \mathcal{I}$.

But $\mathcal{I}, r_4 \in N$ hence without loss of generality $\bar{q}_* \in N$ and \bar{q}_* has countable length, so $\{q_{\alpha}^* : \alpha < \alpha_*\} \subseteq N$. As \bar{q}_* is a witness, necessarily it is a maximal antichain of \mathbb{Q} hence for some $\alpha < \alpha_*$ we have $q_{\alpha}^* \in \mathbf{G}_{\mathbb{Q}}$, as \bar{q}_* is a witness for $r_4 \in \mathcal{I}_{\mathcal{I}}$, necessarily $\mathcal{I}_1 = \{r \in \mathbb{P} : q_{\alpha}^* \Vdash_{\mathbb{Q}} "r \in \mathcal{I}"\}$ is an open subset of \mathbb{P} dense above r_4 .

Clearly $\mathcal{I}_1 \in N$ is an open subset of \mathbb{P} , dense above r_4 and $r_4 \leq_{\mathbb{P}} r_5$ hence $\mathcal{I}_1 \cap N$ is pre-dense above r_5 hence there are $r_6 \leq_{\mathbb{P}} r_7$ from \mathbb{P} such that $r_6 \in \mathcal{I}_1 \cap N$ and $r_5 \leq_{\mathbb{P}} r_7$.

Clearly $r_6 \in \mathcal{I}[\mathbf{G}_{\mathbb{Q}}] \cap N$ and r_6 is compatible with r_3 in \mathbb{P} , so we are done proving r_2 is $(N[\mathbf{G}_{\mathbb{Q}}], \mathbb{P})$ -generic.

So we are done. □_{2.3}

{3c.5}

{3c.3}

Remark 2.4. In 2.1, 2.3 we can replace “c.c.c.” by “strongly proper”.

But such \mathbb{Q} preserves “ $(\omega_2)^{\mathbf{V}}$ -non-meagre”.

{3c.7}

Claim 2.5. 1) *There is a proper forcing \mathbb{Q} which forces “ $\mathbb{P}_{\mathcal{A}^*}[\mathbf{V}]$ as a forcing notion is not proper”, (i.e. $\neg \text{Pr}_1(\mathbb{Q}, \mathbb{P})$).*

{1a.10}

2) *Even (A) of 0.5(3) fails, i.e. $\neg \text{Pr}_5(\mathbb{Q}, \mathbb{P}_{\mathcal{A}^*}[\mathbf{V}])$.*

Proof. We use the proof of [Sh:f, Ch.17, Sec.2] and see references there. We repeat in short.

We use a finite iteration so let \mathbb{P}_0 be the trivial forcing notion, $\mathbb{P}_{k+1} = \mathbb{P}_k * \mathbb{Q}_k$ for $k \leq 3$ and the \mathbb{P}_k -name \mathbb{Q}_k is defined below.

Step A: $\mathbb{Q}_0 = \text{Levy}(\aleph_1, 2^{\aleph_0})$ so $\Vdash_{\mathbb{Q}_0}$ “CH”.

Step B: \mathbb{Q}_1 is Cohen forcing.

Step C: In $\mathbf{V}^{\mathbb{P}_2}$, \mathbb{Q}_2 in the Levy collapse of $2^{2^{\aleph_0}}$ to \aleph_1 , i.e. $\mathbb{Q}_2 = \text{Levy}(\aleph_1, \beth_2)^{\mathbf{V}[\mathbb{P}_2]}$.

Step D: Let $\mathcal{T} = ((\omega_1 >) \omega_1)^{\mathbf{V}^{\mathbb{P}_1}} = ((\omega_1 >) \omega_1)^{\mathbf{V}^{\mathbb{P}_0}}$ be a tree, so we know that $\lim_{\omega_1}(\mathcal{T})^{\mathbf{V}^{\mathbb{P}_1}} = \lim_{\omega_1}(\mathcal{T})^{\mathbf{V}^{\mathbb{P}_2}} = \lim_{\omega_1}(\mathcal{T})^{\mathbf{V}^{\mathbb{P}_3}}$ hence has cardinality \aleph_1 in $\mathbf{V}^{\mathbb{P}_3}$ and

$(*)_1$ in $\mathbf{V}^{\mathbb{P}_1}$, \mathcal{T} is isomorphic to a dense subset of $\mathbb{P}_{\mathcal{A}_*[\mathbb{P}_1]} = \mathbb{P}_{\mathcal{A}_*[\mathbb{P}_0]}$.

So in $\mathbf{V}^{\mathbb{P}_3}$ there is a list $\langle \eta_\varepsilon^* : \varepsilon < \omega_1 \rangle$ of $\lim_{\omega_1}(\mathcal{T})^{\mathbf{V}^{\mathbb{P}_1}}$ and let $\langle \eta_\varepsilon^* \upharpoonright [\gamma_\varepsilon, \omega_1) : \varepsilon < \omega_1 \rangle$ be pairwise disjoint end segments so $\gamma_\varepsilon < \omega_1$, $\langle \gamma_\varepsilon : \varepsilon < \omega_1 \rangle \in \mathbf{V}^{\mathbb{P}_3}$ and $\varepsilon_1 < \varepsilon_2 < \omega_1 \wedge \beta_1 \in [\gamma_{\varepsilon_1}, \omega_1) \wedge \beta_2 \in [\gamma_{\varepsilon_2}, \omega_1) \Rightarrow \eta_{\varepsilon_1}^* \upharpoonright \gamma_1 \neq \eta_{\varepsilon_2}^* \upharpoonright \gamma_2$.

Step E: In $\mathbf{V}^{\mathbb{P}_3}$ there is \mathbb{Q}_3 , a c.c.c. forcing notion specializing \mathcal{T} in the sense of [Sh:74], i.e. there is $h_* \in \mathbf{V}^{\mathbb{P}_4}$ such that $h_* : \mathcal{T} \rightarrow \omega$, h_* is increasing in \mathcal{T} except being constant on each end segment $\eta_\varepsilon^* \upharpoonright [\gamma_\varepsilon, \omega_1)$ for $\varepsilon < \omega_1$, i.e. $\rho <_{\mathcal{T}} \nu \wedge h_*(\rho) = h_*(\nu) \Rightarrow (\exists \varepsilon)[\rho, \nu \in \{\eta_\varepsilon^* \upharpoonright \gamma : \gamma \in [\gamma_\varepsilon, \omega_1)\}]$.

Now

\boxtimes after forcing with $\mathbb{P}_4 = \mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{Q}_2 * \mathbb{Q}_3$, i.e. in $\mathbf{V}^{\mathbb{P}_4}$ the forcing notion $\mathbb{P}_{\mathcal{A}_*[\mathbf{V}]}$ is not proper, in fact it collapses \aleph_1 .

Why? Recall $(*)_1$ and note

$(*)_2$ $\mathcal{I}_n := \{\rho \in \mathcal{T} : (\forall \nu)(\rho \leq_{\mathcal{T}} \nu \rightarrow h_*(\nu) \neq n)\}$ is dense open in \mathcal{T}

and trivially

$(*)_3$ $\bigcap_n \mathcal{I}_n = \emptyset$; in fact if $\mathbf{G} \subseteq \mathcal{T}$ is generic, then:

- (A) \mathbf{G} is a branch of \mathcal{T} of order type $\omega_1^{\mathbf{V}}$ let its name be $\langle \rho_\gamma : \gamma < \omega_1 \rangle$
- (B) letting $\gamma_n = \text{Min}\{\gamma < \omega_1 : \rho_\gamma \in \mathcal{I}_n\}$ we have $\Vdash_{\mathcal{T}} \text{“}\{\gamma_n : n < \omega\} \text{ is unbounded in } \omega_1\text{”}$.

$\square_{2.5}$

§ 3. PRIVATE APPENDIX

Discussion 3.1. 1) With CH, we still do not know what occurs for \mathbb{Q} proper not c.c.c.

A case which looks to point out where is the problem.

Question 3.2. Assume CH, and $\prod_{\alpha < \omega_2} P_\alpha$, CS product, each \mathbb{P}_α adds a dense subset of $\mathcal{T}_* = {}^\omega > (\omega_1)$ or $F : \mathcal{T}_* \rightarrow \mathcal{T}_*, \eta <_{\mathcal{T}_*} F(\eta)$ (so the generic $\eta \in \mathcal{T}'_\delta$ will satisfy $(\exists^\infty \alpha < \delta)[F(\eta) \upharpoonright \alpha \triangleleft \eta]$).

2) Without CH, we know that every c.c.c. adding a real and may proper one destroy properness. We know that such forcing does not exist when $\mathbf{V} = \mathbf{L}$ by xxx Boban Velichovic xxx Macia Groszek xxx.

Generally there are, but quotients but does this give examples here?

- {3c.1} 3) In 2.1, for $\text{Pr}_2(\mathbb{Q}, \mathbb{P})$, it seemed that CH is not necessary.
 {3c.3} 4) Question: in 2.3, can we weaken “ \mathbb{Q} is c.c.c.” to “ \mathbb{Q} is ω -proper”? (or less games...)
 5) If \mathbb{Q} is nep, \mathbb{R} is \aleph_1 -complete then $\text{Pr}_5(\mathbb{Q}, \mathbb{P})$.
 {3c.3} 6) In 2.3, “ \mathbb{Q} not adding a new $\eta \in {}^\omega |Q|$ ”, not adding new $\eta \in {}^\omega \text{Ord}$ are equivalent..Check use.

Moved 09.11.24: try to get “ $\text{Pr}(\mathbb{Q}, \mathbb{P}_{\mathcal{A}^*[\mathbf{V}]})$ fail” from newly $(\lambda, *)$ -proper when $\lambda \leq \mathfrak{h}$ (another direction: weaken the demand on λ)

Case 2: $\mathfrak{h} \geq \lambda$.

For $\alpha < 2^{\aleph_0}$ we define T_α , a $\mathbb{P}_{\mathcal{A}^*[\mathbf{V}]}$ -name, equivalently a \mathcal{T} -name by: $T_\alpha[\mathbf{G}] = p$ if for some $p_0 \in \mathcal{T}, p \in \text{succ}_{\mathcal{T}}(p_0)$ we have $h_{p_0}(p) = \alpha$.

Let $h(\alpha) = T_\alpha$ for $\alpha < 2^{\aleph_0}$.

So

$$q_* \Vdash “h \upharpoonright (2^{\aleph_0} \cap N_{\eta[\mathbf{G}]}) \text{ is a function from } 2^{\aleph_0} \cap N_{\eta[\mathbf{G}]} \text{ onto } 2^{\aleph_0} \cap N_{\eta[\mathbf{G}]}.”.$$

Let h_{p_*} be the function $p_* \in \mathcal{T}$ induce so h_{p_*} maps $2^{\aleph_0} \cap N_{\eta[\mathbf{G}]}$ onto itself.

Fixing p_* in \mathbf{V} we define $\text{rk}_{p_*} : {}^\omega > \lambda \rightarrow \text{Ord} \cup \{\infty\}$ by:

$\text{rk}_{p_*}(\eta) \geq \varepsilon$ iff for every $\alpha \in N_\eta \cap {}^{\aleph_0}2$ and $\zeta < \varepsilon$ there is ν such that

- $\eta \triangleleft \nu \in {}^\omega > \lambda$
- $\text{rk}_{p_*}(\nu) \geq \zeta$
- $h_{p_*}(\alpha) \in N_\nu$.

Clearly

- (*) $\text{rk}_{p_*}(\eta[\mathbf{G}] \upharpoonright n) = \infty$ for every λ
- (*) let $\mathcal{S}_1 = \{\nu \in {}^\omega > \lambda : \text{rk}_{p_*}(\nu) = \infty\}$.

So clearly

- (*) (a) \mathcal{S}_1 is a perfect subtree of ${}^\omega > \lambda$ in \mathbf{V}
- (b) $\eta[\mathbf{G}]$ is a branch of \mathcal{S}_1 [in $\mathbf{V}[\mathbf{G}]$]

- (c) there is a perfect subtree \mathcal{S}_2 of \mathcal{S}_1 such that: if $\eta \in \lim_\omega(\mathcal{S}_1)$ then h_{p_*} maps $N_\eta \cap \aleph_0 2$ onto itself [in \mathbf{V}]
- (d) we can demand moreover that p_* is $(N_\eta, \mathbb{P}_{\mathcal{A}_*[\mathbf{V}]})$ generic absolutely in particular even for branches of \mathcal{S}_2 not (in $\mathbf{V}[\mathbf{G}]$) (recall $|\mathcal{A}_*| = |\mathcal{T}| = 2^{\aleph_0}$), [used?]
- (*) without loss of generality for some limit $\delta_* < \lambda$, $\text{cf}(\delta_*) = \aleph_0$ and for every $\eta \in \mathcal{S}_2$, $\sup\{\text{Rang}(\nu_{\eta \upharpoonright n}) : n < \omega\} = \delta_*$
- (*) without loss of generality in \square we can add
 - (i)' we have $\langle \beta_\eta : \eta \in \omega^{>\lambda} \rangle$ such that $\beta_\eta \in \mathfrak{h} \cap N_\eta$ but $\ell g(\eta) = n + 1 \Rightarrow \beta_\eta > \sup\{N_\rho \cap \mathfrak{h} : \rho \in \omega^{>}(\sup \text{Rang}(\nu_{\eta \upharpoonright n}))\}$
- (*) let
 - (a) $\beta_\varepsilon = \sup\{\beta_\eta : \eta \in \omega^{>\varepsilon}\}$ for $\varepsilon < \lambda$ so $\langle \beta_\varepsilon : \varepsilon < \lambda \rangle$ is increasing
 - (b) $\beta_* = \sup\{\beta_\varepsilon : \varepsilon < \lambda\}$
 - (c) $\gamma_\rho = \sup(N_\eta \cap \beta_*)$
- (*) if $\eta_1 \neq \eta_2 \in \lim(\mathcal{S}_2)$, so $\eta_1 \cap \eta_2 = \eta \in \omega^{>\lambda}$, then p_* is (N_η, \mathcal{T}) -generic
- (*) for $\rho \triangleleft \eta \in \lim(\mathcal{S}_2)$ let $w_{\eta, \rho} = \{p \upharpoonright \gamma_\rho : p \in N_\eta \cap \mathcal{T}, \text{level}_{\mathcal{T}}(p) > \gamma_\rho\}$
- (*) we can choose \bar{N} such that: if η, η_1, η_2 as above and $p \in w_{\eta_1, \eta} \cap w_{\eta_2, \eta}$ is contained in w_η^* (choose $N_\eta^- \prec N_\eta^+$, etc.)

Toward the case $\mathfrak{h} \geq \lambda$ is similar:

Let $\bar{M}^* = \langle M_\gamma^* : \gamma < \lambda \rangle$ is \prec -increasing continuous, $\|M_\gamma^*\| < \mathfrak{h}$, $\bar{M}^* \upharpoonright (\gamma + 1) \in M_{\gamma+1}^* \prec (\mathcal{H}(\chi), \in)$. Use $\langle C_\delta : \delta \in \mathfrak{S}_{\aleph_0}^\lambda \rangle$ guess clubs.

Choose $\delta_* \in S_{\aleph_0}^\lambda$. Choose countable $N_1^* \prec (\mathcal{H}(\chi), \in)$, $\{\bar{M}^*, \delta_*\} \in N_0$. Choose similarly N_2^* such that $N_1^* \subset N_2^*$. Now by the game produce $\langle (N_\eta^\ell, \nu_\eta) : \eta \in \omega^{>\lambda} \rangle$.

Old Proof:

- ⊗ it suffices to prove that for (q, \mathcal{T}') as above
 - ⊠ $q \Vdash$ “for some $\eta \in \lim_\omega(\mathcal{T}')$, the tree $\mathcal{T}_\eta := \mathcal{T} \cap N_\eta$ which has a set of levels $u_\eta = N_\eta \cap \mathfrak{h}$, has no u_η -branch with an upper bound in \mathcal{T} ”.

However

- (*) without loss of generality for every $\eta \in \lim_\omega(\mathcal{T}')$, even new one, $\gamma_\eta = \sup(N_\eta \cap \mathfrak{h})$ is the same γ_* (as we can replace); moreover in \square we can add
 - \square^+ (b) if $\mathfrak{h} < \lambda$ and $\eta \in \omega^\lambda$ and $\alpha_1 < \alpha_2 < \lambda$ and $t_1 \in N_{\eta \hat{\ } < \alpha_1} \cap \mathcal{T} \setminus N_\eta$, $t_2 \in N_{\eta \hat{\ } < \alpha_2} \cap \mathcal{T} \setminus N_\eta$ then in \mathcal{T} , t_1, t_2 are $<_I$ -incomparable
 - (b)' if $\mathfrak{h} \geq \lambda$, if $(\alpha_1, \alpha_2, t_1, t_2)$ fails the above then t_1, t_2 has a common $<_{\mathcal{T}}$ -upper bound in $N_\eta \cap \mathcal{T}$.

Case 1: $\mathfrak{h} < \lambda$

So \square holds, just use any new $\eta \in \lim_\omega(\delta)$.

Case 2: Not Case 1

We just weaken \square : add “except branches which are included in a branch of \mathcal{T} which belongs to N_η , i.e. $N_{\eta \upharpoonright n}$ for some $n < \omega$ ”.

Discussion 3.3. Second, how strong is that assumption \otimes ? Well, e.g. Cohen $*$ Levy($\aleph_1, 2^{\aleph_0}$) fail it but

\boxplus any c.c.c. forcing is O.K.

Also we can weaken it

\otimes if η in $N_{<>}$ is a \mathbb{Q} -name of a new real, then for some perfect embedding h of $\omega^{>2}$ into $\omega^{>}(\omega_2)$ (so $h(\eta) \triangleleft h(\eta^\wedge \langle \ell \rangle)$ for $\ell = 1, 2, h(\eta^\wedge \langle 0 \rangle), h(\eta^1 \langle 1 \rangle)$ are \triangleleft -incomparable let $\eta' = h(\eta)$, i.e. $\cup\{h(\eta \upharpoonright n) : n < \omega\}$ so $\Vdash \eta' \in \text{lim}(\mathcal{S}')$ where $\mathcal{S}' = \{\nu : \nu \triangleleft h(\rho) \text{ so } h \in \mathbf{V} \text{ for some } \rho \in \omega^{>2}\}$ and we demand: there is $q \in \mathbb{Q}$ such that q is “ $(N_{\eta'}, \mathbb{Q})$ -generic”.

The proof is as before.

Moved 09.11.25 from the proof of 2.3 trying to prove $\text{Pr}_1(\mathbb{Q}, \mathbb{R})$:

(Problem: why N_2^* old - rethink)

So without loss of generality $\mathbb{R} \subseteq \mathcal{H}_{<\aleph_1}(\lambda)$ and $\lambda \geq \aleph_0$. So toward contradiction assume $q_* \in \mathbb{Q}$ and

(*)₁ $q_* \Vdash_{\mathbb{Q}}$ “ \mathbb{R} is not proper”.

So letting χ be large enough and

(*)₂ (a) $q_* \Vdash_{\mathbb{Q}}$ “ $N_1 \triangleleft (\mathcal{H}(\chi)^{\mathbf{V}[G]}, \mathcal{H}(\chi)^{\mathbf{V}}, \in)$ is countable hence
 (b) $N_0 = N_1 \upharpoonright \mathcal{H}(\chi)^{\mathbf{V}}$ is a countable elementary submodel of $(\mathcal{H}(\chi), \in)$
 (c) let $N_2 = N_0 \upharpoonright \mathcal{H}_{<\aleph_1}(\lambda)$ even expanding by \mathbb{R} , note $\mathbb{R} \cap N_0 = \mathbb{R} \cap N_2$
and $r_* \in N_1 \cap \mathbb{R}$ and no r' satisfying $r' \leq_{\mathbb{R}} r_*$ is (N_1, \mathbb{R}) -generic”.

Possibly increasing q_* without loss of generality

(*)₃ $r_* = r_*$ a member of \mathbb{R} as a forcing.

Similarly without loss of generality

(*)₄ ?? $N_2 = N_2^*$, i.e. N_2^* is an object from \mathbf{V} and $(N_2, \mathbb{R}) \triangleleft (\mathcal{H}_{<\aleph_1}(\lambda), \in, \mathbb{R})$.

So as \mathbb{R} is ($< \aleph_1$)-complete in \mathbf{V} , letting $\text{Gen} = \{\mathbf{g} \subseteq N_2^* \cap \mathbb{R} : \mathbf{g} \text{ is directed and } r_* \in \mathbf{g}\}$ in \mathbf{V}

(*)₅ if $r \in \mathbb{R}$ is $\leq_{\mathbb{R}}$ -above r_* then $\{r' \in \mathbb{R} \cap N_0^* : r' \leq_{\mathbb{R}} r\} \in \text{Gen}$.

Hence

(*)₆ $q_* \Vdash_{\mathbb{Q}}$ “for no $\mathbf{g} \in \text{Gen}$ for every $\mathcal{I} \in N_0$ which is a dense open subset of \mathbb{R} do we have $\mathbf{g} \cap \mathcal{I} \neq \emptyset$ ”.

By clause (β) of the claim which we have proved

(*)₇ $q_* \Vdash_{\mathbb{Q}}$ “for no $\mathbf{g} \in \text{Gen}$ for every $\mathcal{I} \in N_1 \cap \mathbf{V}$ which is a dense open subset of \mathbb{R} do we have $\mathbf{g} \cap \mathcal{I} \neq \emptyset$ ”.

As \mathbb{Q} is c.c.c. (or as it is proper and we can increase q_*) there is N_3 such that

(*)₈ $N_3 \in \mathbf{V}, N_3 \triangleleft (\mathcal{H}(\chi), \in)$ is countable and $q_* \Vdash$ “ $N_1 \subseteq N_3$ ”.

But obviously

- (*)₉ if $\mathcal{I} \in \mathbf{V}$ and $q_* \Vdash \text{“}\mathcal{I} \notin N_2\text{”}$ is a dense open subset of \mathbb{R} then
- (a) $\mathcal{I} \in N_3$
 - (b) \mathcal{I} is a dense open subset of \mathbb{R}
 - (c) $\mathcal{I} \cap N_2^*$ is a dense open subset of $\mathbb{R} \cap N_2^*$.

We can finish as

- (*)₁₀ there is $\mathbf{g} \in \text{Gen}$ such that: if \mathcal{I} satisfies (a),(b),(c) of (*)₃ then $\mathbf{g} \cap \mathcal{I} \neq \emptyset$.

[Why? As N_3 is countable.]

We get contradiction so we are done.

Moved from pg. 12:

Next

- ⊗₈ \mathcal{I}_2 is dense open.

Clearly \mathcal{I}_* is open, but why is it dense? Let $r \in \mathbb{P}$ and we shall find $t \in \mathcal{I}_*$ such that $r \leq_{\mathbb{P}} t$; so we can replace r by any $r' \in \mathbb{P}$ above r . By ⊗₆ there is $r_1 \in \mathcal{I}_1$ above r so $\eta_{r_1}^*$ is well defined, so $r_1 \Vdash \text{“}\eta^* = \eta_{r_1}^*\text{”}$. Let $\eta_* = \eta_{r_1}^*$, so by the definition of η^* in ⊗₄ we have $\eta_{r_1}' \trianglelefteq \eta_*$ and we claim that

- ⊗_{8.1} $\eta_{r_1}' = \eta_*$.

[Why? If not then necessarily $\eta_{p_1}' \triangleleft \eta_*$, hence letting $\xi = \ell g(\eta_{p_1}')$, by the definition of η_{p_1}' there is p_2 such that $p_1 \leq_{\mathbb{P}} p_2$ and $\eta_{p_2} \upharpoonright (\xi + 1) = \eta_* \upharpoonright (\xi + 1)$, but necessarily $\eta_{r_1}' \trianglelefteq \eta_{p_2}$ hence $\eta_{p_1}' \triangleleft \eta_{p_2}$. Hence the condition $p_2, q_{\eta_*(\xi)}$ are compatible; let $p_3 \in \mathbb{P}$ be a common upper bound of them so $(\forall t)(p_3 \leq_{\mathbb{P}} t \Rightarrow q_{\eta_*(\xi)} \leq_{\mathbb{P}} t \Rightarrow \eta_* \upharpoonright (\xi + 1) \not\trianglelefteq \eta_t)$ hence $p_3 \Vdash \text{“}\eta^* \neq \eta_*\text{”}$, contradiction.]

We can conclude

- ⊗_{8.2} if $r_1 \leq t$ then $\eta_* = \eta_t$.

By ⊗_{8.1} and the definition of η_{r_1}' we have $r_1 \leq_{\mathbb{P}} t \Rightarrow \eta_* \trianglelefteq \eta_t$. Let $u = \{\eta_t(\ell g(\eta_*)) : t \in \mathbb{P} \text{ is above } r_1 \text{ and } \ell g(\eta_t) > \ell g(\eta_*)\}$. If u is empty then $r_1 \in \mathcal{I}_*$ by the definition of \mathcal{I}_* , so we are done proving ⊗₈. If u is not empty, let $r_2 \in \mathbb{P}$ be such that $\ell g(\eta_{r_2}) > \ell g(\eta_*) \wedge \eta_{r_2}(\ell g(\eta_*)) = \min(u)$. By the definition of η^* and of $\langle \eta_t : t \in \mathbb{P} \rangle$ there is $r_3 \in \mathbb{P}$ above r_2 such that $\ell g(\eta_{r_3}) > \ell g(\eta_*) \wedge \eta_{p_3}(\ell g(\eta_*)) < \min(u)$, contradiction. So $u = \emptyset$ and p_1 is as required.

- ⊗₉ if $r \in \mathcal{I}_*$ then $\langle q_{\eta_r^*(\varepsilon)} : \varepsilon < \ell g(\eta_r^*) \rangle$ is a maximal antichain of \mathbb{Q} .

[Why? As in the proof of 2.1.]

Fixing $r_* \in \mathcal{I}_* \subseteq \mathbb{P}$ and $\alpha < \ell g(\eta_{r_*}^*)$ let

- (*)₀ $\mathcal{I}_{r_*, \alpha} = \{r \in \mathbb{P} : r_* \leq_{\mathbb{P}} r \text{ and } q_{\eta_{r_*}^*(\alpha)} \text{ forces (for } \Vdash_{\mathbb{Q}}) \text{ that } r \in \mathcal{I}\}$
- (*)₁ $\mathcal{I}_{r_*, \alpha}$ is dense in \mathbb{P} above r_* .

[Why? Assume $\mathbb{P} \Vdash \text{“}r_* \leq r_1\text{”}$ so $r_1 \Vdash_{\mathbb{P}} \text{“}\eta^*(\alpha) = \eta_{r_*}^*(\alpha)\text{”}$ hence for some r_2 we have $\mathbb{P} \Vdash \text{“}r_1 \leq r_2\text{”}$ and $\eta^* \upharpoonright (\alpha + 1) \trianglelefteq \eta_{r_2}'$, so by clause (a) of ⊗₁ we have $q_{\eta_{r_*}^*(\alpha)} \Vdash_{\mathbb{Q}} \text{“}r_2 \in \mathcal{I}\text{”}$ hence $r_2 \in \mathcal{I}_{r_*, \alpha}$ as required.]

So

- (*)₂ $\mathcal{I}_{r_*, \alpha}$ is a (dense and) open subset of $\mathbb{P}_{\geq r_*}$ (i.e. above r_*).

{3c.1}

[Why? As $\Vdash_{\mathbb{Q}}$ “ \mathcal{I} is an open subset”, for density use $(*)_1$.]

As forcing with \mathbb{P} adds no new ω -sequence of ordinals $< \lambda$ (by clause (A)(c) of the claim)

$(*)_3$ $\mathcal{I}_{r_*}^+ := \cap \{ \mathcal{I}_{r_*, \alpha} : \alpha < \text{lg}(\eta_{r_*}^*) \}$ is dense open in \mathbb{P} above r_* .

[Why? Let $\mathcal{I}_{r_*, \alpha}^*$ be a maximal antichain $\subseteq \mathcal{I}_{r_*, \alpha}$ for $\alpha < \text{lg}(\eta_{r_*}^*)$ let f be the \mathbb{P} -name of $\{(\alpha, q) : \alpha < \text{lg}(\eta_{r_*}^*) \text{ and } q \in \mathcal{I}_{r_*, \alpha}^* \cap \dot{G}_{\mathbb{P}}\}$ so $r_* \Vdash$ “ f a function from $\text{lg}(\eta_{r_*}^*)$ to \mathbb{P} ” hence $r_* \Vdash$ “ $f \in \mathbf{V}$ ”.]

{3c.1} Clearly by the definition (recalling \otimes_7 , as in the proof of 2.1)

$(*)_4$ if $r \in \mathcal{I}_{r_*}^+$ then $\Vdash_{\mathbb{Q}}$ “ $r \in \mathcal{I}$ ”.

As $\cap \{ \mathcal{I}_{r_*}^+ : r_* \in \mathcal{I} \}$ is dense open in \mathbb{P} we are done proving clause (B)(b) of the claim.

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