Abstract. We give some sufficient and necessary conditions on a forcing notion $\mathcal{Q}$ for preserving the forcing notion $([\omega]^{\aleph_0}, \supseteq)$ is proper. They cover many reasonable forcing notions.
§0 Introduction, pg.3

{l.a.1} [I.e. Definition 0.1, we define the problem and some variants.]

§1 Properness of $\mathbb{P}_{\mathfrak{A}[\mathcal{V}]}$ and CH, pg.5

{2b.1} [Under CH, if non-meagerness of $(\omega^2)^V$ is preserved then $\mathbb{P}_{\mathfrak{A}[\mathcal{V}]}$ is proper, (1.1). If $\mathcal{V}$ fail CH, then usually $\mathbb{P}_{\mathfrak{A}[\mathcal{V}]}$ is not proper after a forcing adding a new real and satisfying a relative of being proper, e.g. satisfies c.c.c. or is any true creature forcing.]

§2 General sufficient conditions, pg. 10

{3c.3} [If $\mathcal{V}$ satisfies CH and $\mathbb{Q}$ is c.c.c. then $\Vdash_{\mathbb{Q}} "\mathbb{P}_{\mathfrak{A}[\mathcal{V}]}$ is proper", in 2.1. In 2.3 we replace $\mathfrak{A}[\mathcal{V}]$ by a forcing notion $\mathbb{R}$ adding no $\omega$-sequence, $\mathbb{Q}$ is c.c.c. even in $\mathcal{V}^{\mathbb{P}}$. Instead "$\mathbb{Q}$ satisfies the c.c.c." it suffices to demand $\mathbb{Q}$ satisfy a weaker condition. Lastly, in 2.5 we prove some proper forcing does not preserve.]

{3c.7}
§ 0. Introduction

We investigate the question “Pr$_1^+(Q, R)$”, which means that the proper forcing $Q$ preserves that the (old) $R$ is proper for various $R$’s.

Gitman proved that Pr$_1^+(Q, P_{\mathcal{P}(\omega)}|\mathcal{V}|)$ (see definition below, where $P_{\mathcal{P}(\omega)}|\mathcal{V}|$ is the forcing notion $\{A \in \mathcal{V} : A \subseteq \omega, |A| = \aleph_0, \supseteq^*\}$), of course $A \supseteq^* B$ means $B \subseteq^* A$ when $Q$ is adding Cohen (or Cohen even $\aleph_0$). But no other examples were known even Sacks forcing. Also for e.g. $V = “V = L”$, we did not know a forcing making it not proper.

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Let us state the problem and relatives. We are interested mainly in the case $Q$ is proper.

Definition 0.1. 1) Let $Pr_1(Q, P)$ means: $Q, P$ are forcing notions, $Q$ is proper and $\vdash_Q “P$ is a proper forcing”. 1A) Let $Pr_1^+(P, Q)$ be defined similarly but adding “$Q$ is proper”.
2) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $P_{\mathcal{A}}$ be $\mathcal{A}[\omega]<\aleph_0$ ordered by $\supseteq^*$, inverse almost inclusion.
3) Let $\mathcal{A}_*$ = $\mathcal{A}_*[\mathcal{V}] = ([\omega]^{\aleph_0})^\mathcal{V}$.

Observation 0.2. A necessary condition for $Pr_1(Q, P)$ is:

(*) if $\chi$ large enough, $N < (\mathcal{H}(\chi), \in)$ is countable, $Q, P \in N, q_1 \in Q$ is $N, Q$-generic and $r_1 \in N \cap P$ then we can find $(q_2, r_2)$ such that:

- (a) $q_1 \leq_Q q_2$
- (b) $r_1 \leq_R r_2$
- (c) $q_2 \vdash “r_2$ is $(N[G_Q], P)$-generic”.

Definition 0.3. 1) We define $Pr_1^-(Q, P) = Pr_2(Q, P)$ as the necessary condition from 0.2.
2) Let $Pr_3(Q, P)$ mean that $Q, P$ are forcing notions and for some $\lambda$ and stationary $S \subseteq [\lambda]^{\aleph_0}$ from $V$ we have $\vdash_Q “P$ is $S$-proper”, and $S$ is stationary of course.
3) $Pr_4(Q, P)$ is defined similarly but $S \subseteq V^\mathcal{V}$, still $S \subseteq ([\lambda]^{\aleph_0})^\mathcal{V}$, so $S$ is actually $S$, a $\mathcal{Q}$-name.
4) $Pr_5(Q, P)$ is the statement (A) of 0.4(4) below.
5) Let $Pr_7(Q, P)$ means $Pr_5(Q, P)$ and $Q$ is a proper forcing, for $\ell = 2, 3, 4, 5$.

Claim 0.4. 1) $Pr_2(Q, P)$ means that for $\lambda$ large enough, letting $S = ([\lambda]^{\aleph_0})^\mathcal{V}$, we have $\vdash_Q “P$ is $S$-proper”.
2) $Pr_1(Q_1, P) \Rightarrow Pr_2(Q, P) \Rightarrow Pr_3(Q, P)$; similarly for $Pr^+$.
3) Also $Pr_5(Q, P) \Rightarrow Pr_4(Q, P) \Rightarrow Pr_5(Q, P)$; similarly for $Pr^+$.
4) If $Q, P$ are forcing notions, $\chi$ large enough, then (A) $\iff$ (B) where

- (A) for some countable $N < (\mathcal{H}(\chi), \in)$ and for some $q \in Q, p \in P$ we have
  - (a) $q$ is $(N, Q)$-generic
  - (b) $q \vdash_Q “p$ is $(N[G_Q], P)$-generic”
- (B) for some $q_* \in Q, p_* \in P$ we have $Pr_2(Q_{\geq q_*}, P_{\geq p_*})$.

Proof. Easy.

Notation 0.5. $<^*$ denote a well ordering of $\mathcal{H}(\chi)$.
Recall (Balcar-Pelant-Simon [BPS80], or see, e.g. Blass [Bla])

**Definition 0.6.** \( h \) is the following cardinal invariant, it is the minimal cardinality \( \lambda \) (necessarily regular) such that forcing with \( \mathbb{P}_{\mathcal{A}} \) add a new sequence of ordinals of length \( \chi \).
§ 1. Properness of $\mathcal{P}_{\mathcal{A}_0}[\mathcal{V}]$ and CH

Claim 1.1. 1) Assume $V_0 \models CH$, $V_1 \supseteq V_0$, e.g. $V_1 = V_0^\mathcal{A}_0$ and let $\mathcal{A} = \mathcal{A}_0[\mathcal{V}_0]$.

Recalling Definition 0.1(3), we have $V_1 \models \text{"$\mathcal{P}_{\mathcal{A}}$ is proper"}$, i.e. $\text{Pr}_1(Q, \mathcal{P}_{\mathcal{A}})$ when $V_1 \models \text{"if $N_{V_0}$ is not collapsed then $\omega_2$ is non-meagre"}.$

Proof. Let $V_1 = V_0[G]$, where $G$ is a subset of $Q$ generic over $V_0$.

If $V_1 \models \text{"$N_{V_0}$ is countable"}$ then recalling $V_0 \models CH$ clearly $V_1 \models \text{"$\mathcal{A}$ is countable"}$ so we know that $\mathcal{P}_{\mathcal{A}}$ is proper in $V_1$. So from now on we assume $N_{V_0}$ is not collapsed.

Second\(^1\) in $V_0$, there is a dense $\mathcal{A}' \subseteq \mathcal{A}$ downward dense in it by $\pi$, which under $\subseteq^*$ is a tree isomorphic to $\mathcal{T} = \omega_1^\omega(\omega_1)$. In $V_0$ there is a sequence $\mathcal{T} = (\mathcal{T}_\alpha : \alpha < \omega_1)$ which is $\subseteq$-increasing union with union $\mathcal{T}$ and each $\mathcal{T}_\alpha$ countable. Also there is $\mathcal{C} = (C_\delta : \delta < \omega_1 \text{ limit}) \in V_0$ such that $C_\delta \subseteq \delta = \sup(C_\delta)$, otp$(C_\delta) = \omega$.

Let $\mathcal{T}'_\delta = \mathcal{T}_\delta|\{\eta \in \mathcal{T}_\delta : \ellg(\eta) \in C_\delta\}$. In $V_1$ let $N < (\mathcal{H}(\chi), \subseteq^*)$ be countable such that $\mathcal{A}'$, $\pi$, $\mathcal{T} \in N$ and let $\delta = \omega_1 \cap N$, clearly $\mathcal{T} \cap N = \mathcal{T}_\delta$. We have to prove the statements

\[ (*)_0 \text{ "for every } p \in \mathcal{P}_{\mathcal{A}} \cap N \text{ there is } q \in \mathcal{P}_{\mathcal{A}} \text{ above } p \text{ which is $(N, \mathcal{P}_{\mathcal{A}})$-generic."} \]

As $V_0 \models CH$ and the density of $\mathcal{A}'$ in $\mathcal{A}$ and $(\mathcal{A}', \subseteq^*)$ being isomorphic in $V_0$ by $\pi$ to $\mathcal{T}$ this is equivalent (in $V_1$, of course) to

\[ (*)_1 \text{ for every } \nu \in \mathcal{T} \cap N = \mathcal{T}_\delta \text{ there is } \eta \in \mathcal{T} \text{ which is $(N, \mathcal{T})$-generic and } \nu \leq^* \mathcal{T} \eta. \]

In $V_0$ we let $\bar{S} = (S_\delta : \delta < \omega_1 \text{ a limit ordinal})$ where $S_\delta = \{\bar{\nu} : \bar{\nu} = \langle \nu_n : n < \omega \rangle \text{ is } <_{\mathcal{T}}$-increasing, $\nu_n \in \mathcal{T}_\delta$, moreover $\ellg(\nu_n)$ is the $n$-th member of $C_\delta\}$.\(^2\)

As $(\forall \nu \in \mathcal{T}_\delta)(\exists \rho)(\nu <_{\mathcal{T}} \rho \in \mathcal{T}_\delta')$, and $[\bar{\nu} \in \mathcal{S} \Rightarrow$ there is a $<_{\mathcal{T}}$-upper bound $\rho \in \mathcal{T}$ of $\bar{\nu}$, in $V_0$, of course] recalling $\mathcal{S}_\delta$, $\mathcal{S}_\delta \in V_0$ clearly $(*)_1$ is equivalent (in $V_1$, of course) to

\[ (*)_2 \text{ for every } \nu \in \mathcal{T}'_\delta \text{ there is } \bar{\nu} \in \mathcal{S}_\delta \text{ such that } \nu \in \text{Rang}(\bar{\nu}) \text{ and } \bar{\nu} \text{ induce a subset of } \mathcal{S}_\delta \text{ generic over } N \text{ (i.e. $(\forall A)|A \in N \text{ is a dense open subset of } \mathcal{T} \Rightarrow A \cap \{\nu_n : n < \omega \} \neq \emptyset$).} \]

Now a sufficient condition for $(*)_2$ is

\[ (*)_3 \text{ $\mathcal{S}_\delta$, as a set of } \omega \text{-branches of the tree } \mathcal{T}'_\delta, \text{ is non-meagre.} \]

But in $V_0$, $\mathcal{T}'_\delta$ and $\omega^\omega$ are isomorphic and $\mathcal{S}_\delta$ is the set of all $\omega$-branches of $\mathcal{T}'_\delta$, so by an assumption $(*)_3$ holds so we are done. \( \square_{1.1} \)

Discussion 1.2. However, there can be $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $(\mathcal{A}, \subseteq^*)$ is a variation of Souslin tree.

Claim 1.3. 1) We have $\text{Pr}_1(Q, \mathcal{P}_{\mathcal{A}_0}[\mathcal{V}_1])$ when:

\[ (a) \mathcal{P}^{|Q|=\omega_1}[\mathcal{V}] = N_1 \]

\[ (b) \models_Q \text{"$|\lambda| = N_1$ where $\lambda = (2^{|\mathcal{V}_0|})^N"} \]

\( \text{1this is trivial as } V_0 \models CH, \text{ always there is a dense tree with } h \text{ levels by the celebrated theorem of Balcar-Pelant-Simon} \)
(c) moreover letting $\langle a_i : i \leq \aleph_1 \rangle$ be a $\mathbb{Q}$-name of a $\subseteq$-increasing continuous sequence of countable subsets of $\lambda$ with union $\lambda$, the $\mathbb{Q}$-name $S = \{ i : a_i \in V \}$ is forced to contain a club (of $\aleph_1$)

(d) forcing with $\mathbb{Q}$ preserves "$(\exists \mathbb{V})$ is non-meagre".

2) Assume the forcing notion $\mathbb{Q}$ satisfies (a) + (d), $\mathbf{P}_\text{tr}(\mathbb{Q}, \mathbb{P}_{\text{str}}[\mathbb{V}])$ as witnessed by $S$ and $\mathbb{Q}$ is proper and $\mathcal{S}$ is forced to be stationary.

Then the forcing notion $\mathbb{Q} \ast \text{Levy}(\aleph_1, (\mathbb{Q}^\aleph_0)^\mathbb{V}) \ast \mathbb{Q}_S$ preserves $\mathbb{P}_{\text{str}}[\mathbb{V}]$ is proper" where $\mathbb{Q}_S$ is the (well known) shooting of a club through the stationary subsets of $\omega_1$ (to make clause (c) hold).

\[\square_{1.3}\]

\textbf{Theorem 1.4.} We have $\Vdash_{\mathbb{Q}} "\mathbb{P}_{\text{str}}[\mathbb{V}]$ is not proper" when:

(a) $V = \mathcal{P}^{\omega_1}_\mathbb{V} \geq \aleph_2$

(b) $\lambda \leq \aleph_2$ or just $\lambda$ is regular, $\aleph_2 \leq \lambda \leq \mathcal{P}^{\omega_1}_\mathbb{V}$ and $\alpha \leq \lambda \Rightarrow \text{cf}[\alpha]^\mathbb{V}, \subseteq \lambda$ hence (by [Sh:420]) there is a stationary $\mathcal{Z}_\alpha \subseteq [\alpha]^{\aleph_1}$ of cardinality $\lambda$

(c) $\eta \leq \lambda$

(d) the forcing notion $\mathbb{Q}$ adds at least one real and is $\lambda$-newly proper, see Definition 1.5 below.

{[2b.13]}

Before proving 1.4

\textbf{Definition 1.5.} For $\lambda > \kappa$ we say that a forcing notion $\mathbb{Q}$ is $(\lambda, \kappa)$-newly proper (omitting $\kappa$ means $\kappa = \aleph_0$ and we define newly $(\lambda, \lambda)$-proper similarly) when: if $N = ((N_\eta, \nu_\eta) : \eta \in \mathcal{C}^\lambda)$ satisfies $\otimes$ below and $\mathbb{Q} \in N_{<\omega}$ and $p \in \mathbb{Q} \cap N_{<\omega}$ then we can find $q, y$ such that $\boxtimes$ below holds where:

\(\otimes\) for some cardinal $\chi > \lambda$

(a) $N_\eta \prec (\mathbb{H}(\chi), \in, <^\chi_\eta)$ is countable

(b) if $\nu \not\in \eta$ then $N_\nu \prec N_\eta$

(c) $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$ if $\kappa = \aleph_0$ and $N_{\eta_1}^\kappa \cap N_{\eta_2}^\kappa = N_{\eta_1 \cap \eta_2}^\kappa$ generally where $N_\eta^\kappa := \{ v \in N_\eta^\kappa : |v| \leq \kappa \}$

(d) $\nu_\eta \in N_\eta \setminus \bigcup \{ N_{\eta_m}^\kappa : m < \ell \eta(\eta) \}$ hence $\nu_\eta \not\in \bigcup \{ N_\nu : \neg (\eta \leq \nu) \}$ and $\nu_\eta \not\in \bigcup \{ N_\nu : \neg (\eta \leq \nu) \}$

\(\boxtimes\) (a) $p \leq q$

(b) $q \Vdash_{\mathbb{Q}} "\bigcup \{ N_{\eta[n]}^\mathbb{Q} : n < \omega \} \cap V = \bigcup \{ N_{\eta[n]}^\mathbb{Q} : n < \omega \}"$

(c) $q \Vdash_{\mathbb{Q}} "\eta \in \text{"} \lambda \text{" is new, i.e. } \eta \not\in \text{"} \omega \text{"} \mathbb{V}^\mathbb{V}"

(c) moreover if $\kappa > \aleph_0$ and $\mathcal{T} \in V$ is a sub-tree of $\mathcal{C}^\lambda$ of cardinality $\leq \kappa$ then $\eta \not\in \text{lim}(\mathcal{T})$, i.e. $\{ \eta[n] : n < \omega \} \not\in \mathcal{T}$.

For a proper forcing notion adding a new real it is quite easy to be $\aleph_1$-newly proper; e.g.

\textbf{Claim 1.6.} Assuming $\mathcal{P}^\aleph_0 \geq \lambda = \text{cf}(\lambda) > \aleph_1$, sufficient conditions for $\mathbb{Q}$ is $\lambda$-newly proper are:

(a) $\mathbb{Q}$ is c.c.c. and add a new real
(b) \( Q \) is Sacks forcing
(c) \( Q \) is a tree-like creature forcing in the sense of Roslanowski-Shelah [RoSh:470].

Proof. Easy; for clause (a) we use \( q = p \) for \( \square \) of the definition. For clauses (b),(c) we use fusion but in the \( n \)-th step use members of \( N_\eta \cap Q \) for \( \eta \in \omega^\omega \), we get as many distinct \( \eta \)'s as we can. \[ \square_{1.6} \]

Proof of 1.4 Let \( \chi \) be large enough and for transparency, \( x \in \mathcal{H}(\chi) \).
By Rubin-Shelah [RuSh:117] in \( V \) there are sequences \( \langle N_\eta : \eta \in \omega^\omega \rangle ; \langle \nu_\eta : \eta \in \omega^\omega \rangle \) such that:

\[ \square_1 \]
(a) \( N_\eta \prec (\mathcal{H}(\chi), \in) \)
(b) \( Q, x \in N_\eta \)
(c) \( N_\eta \) is countable
(d) \( N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2} \)
(e) \( \nu_\eta \in f_\eta(\nu)^{\omega^\omega} \)
(f) \( \nu_\eta \in N_\eta \)
(g) if \( \eta_1 \in \omega^\omega(\lambda) \) and \( \neg(\eta_1 \subseteq \eta_1) \) then \( \nu_\eta \notin N_{\eta_1} \)
(h) if \( \eta_1, \ell_1 = \nu_{\eta_1}(\ell_2) \Rightarrow \ell_1 = \ell_2 \land \eta_1|(\ell_1 + 1) = \eta_2|(\ell_2 + 1) \).

Now for each \( \eta \in \omega^\omega \) let \( N_\eta = \{ N_\eta[k] : k < \omega \} \); we can add:

(i) if \( \ell_\eta(n) = n + 1 \) then \( \nu_\eta(n) > \text{sup}(N_\eta[n] \cap \lambda) \) and even \( \nu_\eta(n) > \text{sup}(N_\rho \cap \lambda : \rho \in \omega^\omega(\nu_\eta(n))) \)
(j) if \( \eta \in \omega^\omega \) is increasing, then \( \text{sup}(N_\eta \cap \lambda) = \text{sup}(\text{Rang}(\eta)) \).

Why is this sufficient? By Balcar-Pelant-Simon [BPS80] there is \( \mathcal{T} \subseteq [\omega]^{\omega_0} \) such that

\[ \square_2 \]
(a) \( (\mathcal{T}, \ast, \supseteq) \) is a tree with \( h \) levels (\( h \) is the cardinal invariant from 0.6, a regular cardinal \( \in [\kappa_1, 2^{\aleph_0}] \)), the tree \( \mathcal{T} \) is with a root and each node has \( 2^{\aleph_0} \) many immediate successors, i.e. \( \mathcal{T} \) has splitting to \( 2^{\aleph_0} \)
(b) \( \mathcal{T} \) is dense in \( (\omega]^{\omega_0}, \supseteq^* \) i.e. in \( \mathbb{P}_{\mathcal{T}(\omega)V} = \mathbb{P}_{\mathcal{R}[*][V]} \)
(recalling 0.1(2)).

Choose \( h \) such that

\[ \square_3 \]
\( \tilde{h} = (h_p : p \in \mathcal{T}) \) satisfies: \( h_p \) is one to one from \( \text{suc}_\mathcal{T}(p) \) onto \( 2^{\aleph_0} \setminus \{ h_{p_0}(p_1) : p_0 < \mathcal{T} p_1 < \mathcal{T} p \text{ and } p_1 \in \text{suc}_\mathcal{T}(p_0) \} \).

So without loss of generality

\[ \square_4 \]
\( \mathcal{T} \in N_{<\omega} \) and \( \tilde{h} \in N_{<\omega} \).

As \( Q \) is newly \( \lambda \)-newly proper there are \( \eta, q \) as in \( \square \) of Definition 1.5. Let \( G \subseteq Q \) be generic over \( V \) such that \( G \prec Q \), let \( \eta = \eta[G] \) and \( M_2 := N_{\eta[G]} := \{ N_{\eta[n]}[G] : n < \omega \} \), so \( M_2 \prec (\mathcal{H}(\chi)^V \setminus \mathcal{H}(\chi)^V, \in) \) is countable, pedantically \( (\mathcal{M}_2, \mathcal{H}(\chi)^V \cap |M_2|, \in |M_2|) \prec (\mathcal{H}(\chi)^V \setminus \mathcal{H}(\chi)^V, \in |\mathcal{H}(\chi)^V|) \).

By \( \square \) of 1.6 as \( q \in G \) we have \( M_1 = M_2 \cap \mathcal{H}(\chi)^V \) is \( \cup \{ N_{\eta[n]} : n < \omega \} \), and of course \( M_1 \prec (\mathcal{H}(\chi), \in) \). Toward contradiction assume \( V[G] = \mathcal{P}_{\mathcal{T}(\mathcal{R})}[V] \) is
proper", hence some \( p_\ast \in \mathbb{P}_{\mathcal{M}}[\mathbf{V}] \) is \((M_2, \mathbb{P}_{\mathcal{M}}[\mathbf{V}])\)-generic. But \( \mathcal{F} \) is dense in \( \mathbb{P}_{\mathcal{M}}[\mathbf{V}] \) so without loss of generality \( p_\ast \in \mathcal{F} \) and \( p_\ast \) is \((M_2, \mathcal{F})\)-generic.

Clearly \( h \in N_{<\lambda} \) or we may demand this, so without loss of generality \( \eta \in \omega > \lambda \Rightarrow N_\eta \cap h = N_{<\lambda} \cap h \). For any \( \alpha < \lambda \) let

\[ \mathcal{I}_\alpha = \{ p \in \mathcal{F} : \text{ for some } p_0 \in \mathcal{F} \text{ we have } p \in \text{ suc}_\mathcal{F}(p_0) \text{ and } h_{p_0}(p) = \alpha \} \]

and letting \( \mathcal{I}_\alpha^+ \) be the \( \alpha \)-th level of \( \mathcal{F} \)

\[ \mathcal{I}_\alpha^+ = \{ p \in \mathbb{P}_{\mathcal{M}}[\mathbf{V}] : p \text{ is above some member of } \mathcal{I}_\alpha \}. \]

Now clearly (in \( \mathbf{V} \) and in \( \mathbf{V}[\mathbf{G}] \)):

\[ (*)_1 (a) \] \( \mathcal{I}_\alpha \) is a pre-dense subset of \( \mathcal{F} \) (and of \( \mathbb{P}_{\mathcal{M}}[\mathbf{V}] \))

\[ (b) \] \( \mathcal{I}_\alpha^+ \) is dense open decreasing with \( \alpha \)

\[ (c) \] if \( p \in \mathbb{P}_{\mathcal{M}}[\mathbf{V}] \) then for every large enough \( \alpha < \lambda, p \notin \mathcal{I}_\alpha^+ \)

\[ (d) \] if \( p \in \mathbb{P}_{\mathcal{M}}[\mathbf{V}] \) and \( \alpha < \lambda \) then there is \( q \in \mathcal{I}_\alpha \) such that

\[ \mathbb{P}_{\mathcal{M}}[\mathbf{V}] \models \text{"} p \leq q \text{"}. \]

Also if \( \alpha \in \lambda \cap N_{\gamma}\mathbf{G} \) then \( \mathcal{I}_\alpha \in N_{\gamma}\mathbf{G} \) and the set \( \{ p \in \mathcal{F} \cap N_{\gamma}\mathbf{G} : p \leq \mathcal{F} p_\ast \} \) is not empty, let \( p_\ast \) be in it and let its level in \( \mathcal{F} \) be \( \gamma_\ast \).

Now

\[ (*)_2 \text{ if } \alpha \in h \cap N_{\gamma}\mathbf{G} \text{ then } \gamma_\ast \in N_{\gamma}\mathbf{G} \cap h = N_{<\lambda} \cap h \] hence

\[ (*)_3 \text{ in } \mathbf{V}[\mathbf{G}] \text{ the following function } h_\ast \text{ is well defined} \]

\[ (a) \text{ Dom}(h_\ast) = N_{<\lambda} \cap h \]

\[ (b) h_\ast(\gamma) \text{ is the unique } p \in N_{\gamma}\mathbf{G} \cap \mathcal{F} \text{ of level } \gamma \text{ which is } \leq \mathcal{F} p_\ast. \]

also by the choice of \( h \) (and genericity) clearly

\[ (*)_4 \text{ Rang}(h_\ast) \text{ is equal to } u := (2^{\aleph_0}) \cap N_{\gamma}\mathbf{G}. \]

Lastly,

\[ (*)_5 h_\ast \in \mathbf{V}. \]

[Why? As its domain, \( N_{<\lambda} \cap h \) belongs to \( \mathbf{V} \) and \( h_\ast(\gamma) \) is defined from \( \langle \mathcal{F}, \gamma, p_\ast \rangle \in \mathbf{V} \text{ and } \mathcal{F} \text{ is a tree.} \]

\[ (*)_6 \text{ from } u := \lambda \cap N_{\gamma}\mathbf{G} \text{ we can define } \eta|\mathbf{G} \]

\[ (a) u = \cup\{N_{\gamma}\mathbf{G} \cap \lambda : n < \omega \}. \]

[Why? By the choice of \( N_\cdot \).

Together we get that \( \eta|\mathbf{G} \in \mathbf{V} \), contradiction. \( \square \)_{1.4}]

\[ \{2b.23\} \text{ Claim 1.7. We have } -\text{Pr}_1(Q, \mathbb{P}_{\mathcal{M}}[\mathbf{V}]) \text{ when} \]

\[ (a) 2^{\aleph_0} \geq \lambda = cf(\lambda) > \kappa = h \]

\[ (b) \alpha < \lambda \Rightarrow cf([\alpha]^{\leq \kappa}, \subseteq) \leq \kappa ) < \lambda \]

\[ (c) Q \text{ is } (\lambda, \kappa)-\text{newly proper.} \]

\[ \{2b.10\} \text{ Proof. Similar to 1.4.} \]
Conclusion 1.8. If $\eta < 2^{\aleph_0}$ and $Q$ is a $(\eta^+, \eta)$-newly proper then $\neg \text{Pr}_1(Q, \mathbb{P}_{\mathcal{V}}(V))$. 
\section{General sufficient conditions}

\begin{claim}
Assume CH, i.e. \( V \models CH \).
If \( Q \) is c.c.c. then \( \Pr_2(Q, \mathbb{P}_{\mathcal{A}^\ast}(V)) \).
\end{claim}

\begin{remark}
1) This works replacing \( \mathbb{P}_{\mathcal{A}^\ast}(V) \) by any \( \aleph_1 \)-complete \( \mathbb{P} \) and strengthening the conclusions to \( \Pr_1 \), see 2.3.
2) See Definition 0.3(1).
\end{remark}

\begin{proof}
Let \( \mathbb{P} = \mathbb{P}_{\mathcal{A}^\ast}(V) \). The point is
\begin{itemize}
\item[(\ast)] if \( r \in \mathbb{P} \) and \( \Vdash Q \) \text{" is a dense open subset of } \mathbb{P} \text{" then there is } r' \text{ such that: }
\begin{itemize}
\item[(a)] \( r \leq r' \)
\item[(b)] \( \Vdash Q \) \text{" } r' \text{ } \in \mathcal{J} \subseteq \mathbb{P} \).
\end{itemize}
\end{itemize}
Why (\ast) holds? We try (all in \( V \)) to choose \( (r_\alpha, q_\alpha) \) by induction on \( \alpha < \omega_1 \) but choosing \( q_\alpha \) together with \( r_\alpha + 1 \) such that:
\begin{itemize}
\item[(\oplus)] \( (a) \) \( r_0 = r \)
\item[(b)] \( r_\alpha \in \mathbb{P} \) is \( \leq \)-increasing
\item[(c)] \( q_\alpha \in Q \)
\item[(d)] \( q_\alpha, q_\beta \) are incompatible in \( Q \) for \( \beta < \alpha \)
\item[(e)] \( q_\alpha \Vdash Q \) \text{" } r_\alpha + 1 \in \mathcal{J} \).
\end{itemize}

We cannot succeed because \( Q \models \text{ c.c.c.} \).
For \( \alpha = 0 \) no problem as only clause (a) is relevant.
For a limit - easy as \( \mathbb{P} \) is \( \aleph_1 \)-complete (and the only relevant clause is (b)).
For \( \alpha = \beta + 1 \), we first ask:

\begin{question}
Is \( \langle q_\gamma : \gamma < \beta \rangle \) a maximal antichain of \( Q \)?
\end{question}

If yes, then \( r_\beta \) as required: if \( G_Q \subseteq Q \) is generic over \( V \) then for some \( \gamma < \beta \), \( q_\gamma \in G_Q \) hence \( r_\gamma + 1 \in \mathcal{J}[G_Q] \) but \( \mathcal{J}[G_Q] \) is a dense subset of \( \mathbb{P} \) and is open and \( r_{\gamma + 1} \leq r_\beta \) so \( r_\beta \in \mathcal{J}[G_Q] \).

If no, let \( q_\beta \in Q \) be incompatible with \( q_\gamma \) for every \( \gamma < \beta \). Recalling \( \Vdash Q \) \text{" } \mathcal{J} \text{ is dense and open} \text{" the set } X_\beta = \{ r \in \mathbb{P} : \text{ for some } q, q_\beta \leq q \text{ and } q \Vdash \text{" } r \in \mathcal{J} \text{"} \} \text{ is a dense subset of } \mathbb{P} \text{ hence there is a member of } X_\beta \text{ above } r_\beta, \text{ let } r_\alpha \text{ be such member. By } r_\alpha \in X_\beta, \text{ there is } q, q^3 \leq q \text{ such that } q \Vdash \text{" } r_\alpha \in \mathcal{J} \text{"}. \text{ So we choose } q_\beta \text{ as such } q, \text{ so we can carry the induction step.}

As said above we cannot carry the induction for all \( \alpha < \omega_1 \) because then \( \{ q_\alpha : \alpha < \omega_1 \} \) contradicts \text{" } Q \text{ satisfies the c.c.c.} \text{" So for some } \alpha \text{ we cannot continue, } \alpha \text{ is neither 0 no limit hence for some } \beta, \alpha = \beta + 1. \text{ So the answer to the question is yes, hence we get the desired conclusion of (\ast).}
So (\ast) indeed holds and this is clearly enough.
\end{proof}

We can weaken the demand on the second forcing (above \( \mathbb{P}_{\mathcal{A}^\ast}(V) \)).

\begin{claim}
If (A) then (B) where:
\begin{itemize}
\item[(A)] \( \mathbb{P}, Q \) are forcing notions
\item[(B)] \( Q \) is c.c.c. moreover \( \Vdash_Q \) \text{" } Q \text{ is c.c.c. } \)
\end{itemize}
\end{claim}
PRESERVING OLD $(\omega|^\aleph_0, \geq)$ IS PROPER

\[ (c) \text{ forcing with } P \text{ and no new } \omega\text{-sequences,}^2 \text{ from } \lambda \]
\[ (d) \text{ } Q \text{ has cardinality } \leq \lambda \]

(B) \( (a) \text{ if } P \text{ is proper in } V \text{ then } Pr_2(Q, P) \)
\[ (b) \text{ for every } Q\text{-name } \mathscr{I} \text{ of a dense open subset of } P, \text{ the set } \mathscr{J} = \{ r \in P : \models_Q \text{ “} r \in \mathscr{I} \text{”} \} \text{ is dense and open.} \]

Proof. First we prove clause (b); so fix \( \mathscr{I} \) and \( \mathscr{J} \) as there. Let \( \langle q_\varepsilon : \varepsilon < \kappa := |Q| \rangle \) list \( Q \).

For every \( r \in P \) we define a sequence \( \eta_r \) of ordinals < \( \lambda \) as follows:
\[ \text{(a) } q_\varepsilon \models “r \in \mathscr{J}” \]
\[ \text{(b) if } \beta < \alpha \text{ then } q_\varepsilon, q_{\eta_r(\beta)} \text{ are incompatible in } Q” \].

Now
\[ \text{(a) } \eta_r(\alpha) \text{ is well defined} \]
\[ \text{(b) } \ell g(\eta_r) < \omega_1. \]

[Why? As \( Q \models \text{ c.c.c.} \)]

Note
\[ \text{(a) } \text{if } r_1 \leq_P r_2 \text{ then either } \eta_{r_1} \leq \eta_{r_2} \text{ or for some } \alpha < \ell g(\eta_{r_1}) \text{ we have} \]
\[ \eta_{r_1}|\alpha = \eta_{r_2}|\alpha \]
\[ \eta_{r_1}(\alpha) > \eta_{r_2}(\alpha). \]

[Why? Think about the definition.]

For \( s \in P \) let \( \eta'_s \) be \( \cap\{ \eta_{s_1} : s \leq_P s_1 \} \), i.e. the longest common initial segment of \( \{ \eta_{s_1} : s \leq_P s_1 \} \); clearly \( s_1 \leq_P s_2 \Rightarrow \eta'_{s_1} \leq \eta'_{s_2} \). So
\[ \eta^* = \cup \{ \eta'_s : s \in G_P \} \text{ is an } P\text{-name of a sequence of pairwise incompatible members of } Q. \]

But by clause (A)(b) of the claim, forcing with \( P \) preserve “\( Q \models \text{ c.c.c.} \), so \( \ell g(\eta^*) \) is countable in \( V[G_P] \). But by clause (A)(c) of the claim, forcing by \( P \) adds no new \( \omega\)-sequences to \( \kappa = |Q| \) (and \( Q \) is infinite) and \( V[G_P] \) has the same \( \aleph_1 \) as \( V \), so
\[ \text{(a) } \eta^* \text{ is a sequence of countable length of ordinals } \lambda \text{ so is old.} \]

Hence
\[ \text{(a) } \text{the following set is dense open in } P \]
\[ \mathscr{J} = \{ r \in P : r \text{ forces } (\models_P) \text{ that } \eta^*_r = \eta^*_t \text{ for some } \eta^*_r \in V \} \]

Let us define:
\[ \text{(a) } \mathscr{J}_s = \{ s \in \mathscr{J} : \eta_s = \eta^*_s \text{ and moreover } s \leq_P t \Rightarrow \eta_t = \eta^*_t = \eta_s \}. \]

Next

\[ ^2 \text{If you assume } Q, P \text{ are proper, } \lambda = \aleph_0 \text{ the proof may be easier to read} \]
Clearly $\mathcal{F}_*\,$ is open, but why is it dense? Let $p \in \mathbb{P}$ and we shall find $t \in \mathcal{F}_*$ such that $p \leq t$; so we can replace $p$ by any $p' \in \mathbb{P}$ above $p$. By $@_8$ there is $p_1 \in \mathcal{F}$ above $p$ so $\eta_{p_1}^\ast$ is well defined, so $p_1 \Vdash \{ \eta^\ast = \eta_{p_1}^\ast \}$. Let $\eta_* = \eta_{p_1}^\ast$, so by the definition of $\eta^\ast$ in $@_3$ we have $\eta_{p_1}^\ast \leq \eta_*$ and let

$@_{8.1} \eta_{p_1}^\ast = \eta_*$. 

[Why? If not then necessarily $\eta_{p_1}^\ast < \eta_*$ hence letting $\xi = \ell g(\eta_{p_1}^\ast)$, by the definition of $\eta_{p_1}^\ast$ there is $p_2$ such that $p_1 \leq p_2$ and $\eta_{p_2}^\ast(\xi + 1) = \eta_*(\xi + 1)$, but necessarily $\eta_{p_1}^\ast < \eta_{p_2}^\ast$. Hence the condition $p_2, q_{\eta_*}(\xi)$ are compatible; let $p_3 \in \mathbb{P}$ be a common upper bound of them so $(\forall t)(p_3 \leq p \Rightarrow q_{\eta_*}(\xi) \leq p \Rightarrow \eta_*(\xi + 1) \not\in \eta_*)$ hence $p_3 \Vdash \{ \eta^\ast \neq \eta_* \}$, contradiction.

We can conclude]

$@_{8.2}$ if $p_1 \leq t$ then $\eta_* \leq \eta_*$.

Let $u = \{ \eta_\ell(\ell g(\eta_*)) : t \in \mathbb{P} \mbox{ is above } \mathbb{P} \}$. If $u$ is empty then $p_1 \in \mathcal{F}_*$ by the definition of $\mathcal{F}_*$ so we are done proving $@_8$. If $u$ is not empty, let $p_2 \in \mathbb{P}$ be such that $\ell g(\eta_{p_2}^\ast) > \ell g(\eta_*) \land \eta_{p_2}(\ell g(\eta_*)) = \min(u)$. By the definition of $\eta^\ast$ and of $(\eta_t : t \in \mathbb{P})$ there is $p_3 \in \mathbb{P}$ above $p_2$ such that $\ell g(\eta_{p_3}) > \ell g(\eta_*) \land \eta_{p_3}(\ell g(\eta_*)) < \min(u)$, contradiction. So $u = \emptyset$ and $p_1$ is as required.

$@_9$ if $r \in \mathcal{F}$ then $(q_{\eta^\ast}(\varepsilon) : \varepsilon < \ell g(\eta_*))$ is a maximal antichain of $\mathcal{Q}$.

{3c.1} [Why? As in the proof of 2.1.]

Fixing $r_* \in \mathcal{F}_* \subseteq \mathbb{P}$ and $\alpha < \ell g(\eta_*^\ast)$ let

$(*)_0 \ \mathcal{F}_{r_*, \alpha} = \{ r \in \mathbb{P} : r_* \leq r \mbox{ and } q_{\eta_*^\ast}(\alpha) \mbox{ forces } (\forall \mathcal{Q}) \mbox{ that } r \in \mathcal{F} \}$

$(*)_1 \ \mathcal{F}_{r_*, \alpha} \mbox{ is dense in } \mathbb{P} \mbox{ above } r_*$. 

[Why? Assume $\mathbb{P} \models \{ \forall r_* \leq r_1 \} \mbox{ so } r_1 \Vdash \eta^\ast(\alpha) = \eta_{p_1}^\ast(\alpha)$ hence for some $r_2$ we have $\mathbb{P} \models \{ \forall r_* \leq r_2 \} \mbox{ and } \eta^\ast(\alpha + 1) \leq \eta_{p_2}^\ast$, so by clause (a) of $@_1$ we have $q_{\eta_*^\ast}(\alpha) \Vdash \{ \forall r_* \leq r_2 \in \mathcal{F} \} \mbox{ hence } r_2 \in \mathcal{F}_{r_*, \alpha} \mbox{ as required.}

So

$(*)_2 \ \mathcal{F}_{r_*, \alpha} \mbox{ is a (dense and) open subset of } \mathbb{P}_{\geq r_*} \mbox{ (i.e. above } r_*)$.

[Why? As $\forall \mathcal{Q} \{ \mathcal{F} \mbox{ is an open subset} \}$, for density use $(*)_1$.]

As forcing with $\mathcal{P}$ add no new $\omega$-sequence of ordinals $< \lambda$ (by clause (A)(c) of the claim)

$(*)_3 \ \mathcal{F}_{r_*}^+ := \cap \{ \mathcal{F}_{r_*, \alpha} : \alpha < \ell g(\eta_*^\ast) \}$ is dense open in $\mathbb{P}$ above $r_*$. 

[Why? Let $\mathcal{F}_{r_*}^+$ be a maximal antichain $\subseteq \mathcal{F}_{r_*, \alpha}$ for $\alpha < \ell g(\eta_*^\ast)$ let $f$ be the $\mathbb{P}$-name of $\{ (\alpha, q) : \alpha < \ell g(\eta_*^\ast) \mbox{ and } q \in \mathcal{F}_{r_*, \alpha} \cap G_{\mathbb{P}} \} \mbox{ so } r_* \Vdash \{ f \mbox{ a function from } \ell g(\eta_*^\ast) \mbox{ to } \mathbb{P} \mbox{ hence } r_* \upharpoonright f \in \mathcal{V} \}$.]

{3c.1} Clearly by the definition (recalling $@_7$, as in the proof of 2.1)

$(*)_4 \mbox{ if } r \in \mathcal{F}_{r_*}^+ \mbox{ then } \forall \mathcal{Q} \{ r \in \mathcal{F} \}$. 


As $\cap \{ J^+_r : r_\ast \in J \}$ is dense open in $P$ we are done proving clause (b) of the claim.  

For clause (a), let $\chi, N, q_1, r_1$ be as in the assumption of $(*)_1$ of 0.2, so $P, Q \in N$. We have to find $q_2, r_2$. 

Let $q_2 = q_1$ and let $r_2 \in P$ be $(N, P)$-generic and above $r_1$, exists as $P$ is a proper forcing in $V$. So let $G \subseteq Q$ be a subset of $\check{Q}$ generic over $V$ such that $q_2 = q_1 \in G$. 

Now if $N[G] \models \text{“} J \text{ is a dense open subset of } P \text{”, then by the definition of } N[G] \text{ for some } \check{Q}\text{-name } J \text{ from } N \text{ of a dense open subset of } P \text{ we have } J[G] = J. \text{ By clause (}\beta\text{) the set } J = \{ r \in P : \exists q < Q \text{ and } J \models r \in J \} \text{ is dense and open in } P \text{ and clearly } \in N \text{ hence } r_2 \models \text{“} J \cap G_2 \neq \emptyset \text{”}. 

So we are done. 

□

Remark 2.4. In 2.1, 2.3 we can replace “c.c.c.” by “strongly proper”.

But such $Q$ preserves “($\ast_2$)$V$-non-meagre”.

Claim 2.5. 1) There is a proper forcing $Q$ which forces $P_{\check{\alpha}} \models [V] \text{ as a forcing notion is not proper”, (i.e. } \models \neg P_{\check{\alpha}}(Q, P)).

2) Even (A) of 0.4(3) fail, i.e. $\models \neg P_3(Q, P_{\check{\alpha}}(V))$.

Proof. We use the proof of [Sh:f, Ch.17,Sec.2] and see references there. We repeat in short.

We use a finite iteration so let $P_0$ be the trivial forcing notion, $P_{k+1} = P_k \ast Q_k$ for $k \leq 3$ and $P(q)$-name $Q_k$ is defined below.

Step A: $Q_0 = \text{Levy}(\aleph_1, 2^{\aleph_0}) \text{ so } P_{Q_0} \text{ “CH”}.$

Step B: $Q_1$ is Cohen forcing.

Step C: In $V^{\aleph_2}, Q_2$ in the Levy collapse of $2^{\aleph_0}$ to $\aleph_1$, i.e. $Q_2 = \text{Levy}(\aleph_1, \aleph_2)^{V|^{\aleph_2}}.$

Step D: Let $\mathcal{T} = ((\omega_1, \omega_1)^{V|^{\aleph_1}})(\omega_1)^{V|^{\aleph_0}} \in [V] \text{ a tree, so we know that } \lim_{\omega_1} (\mathcal{T})^{V|^{\aleph_1}} = \lim_{\omega_1} \omega_1 (\mathcal{T})^{V|^{\aleph_2}} = \lim_{\omega_1} (\mathcal{T})^{V|^{\aleph_3}} \text{ and }$

$(*)_1 \text{ in } V^{\aleph_3}, \mathcal{T} \text{ is isomorphic to a dense subset of } P_{\check{\alpha}}(P_3) = P_{\check{\alpha}}(V).$

So in $V^{\aleph_3}$ there is a list $\langle \mathcal{H}^2 : \varepsilon < \omega_1 \rangle$ of $\lim_{\omega_1} (\mathcal{T})^{V|^{\aleph_3}} \text{. Let } \langle \eta^\varepsilon : \varepsilon < \omega_1 \rangle \text{ be pairwise disjoint end segments.}$

Step E: In $V^{\aleph_3}$ there is $Q_3$, a c.c.c. forcing notion specializing $\mathcal{T}$ in the sense of [Sh:74], i.e. there is $h_* \in V^{\aleph_3}$ such that $h_* : \mathcal{T} \rightarrow \omega, h$ is increasing in $\mathcal{T}$ except on the end segments $\eta^\varepsilon_* (\gamma, \omega_1)$ for $\varepsilon < \omega_1$, i.e. $\rho \in \mathcal{T}, \nu \in h_* (\rho) 
\varepsilon < \omega_1, \nu \uparrow h_* (\rho) \Rightarrow (\exists \varepsilon) \rho, \nu \in \{ \eta^\varepsilon_* : \gamma \in H, \gamma_1 \}$. 

Now

$\exists$ after forcing with $P_4 = Q_0 \ast Q_1 \ast Q_2 \ast Q_3$, i.e. in $V^{\aleph_4}$ the forcing notion $P_{\check{\alpha}}(V)$ is not proper, in fact it collapses $\aleph_1$.

Why? Recall $(*)_1$ and note

$(*)_2 \mathcal{J}_n := \{ \rho \in \mathcal{T} : (\forall \nu)(\rho \leq \mathcal{T} \nu \implies h_{\ast}(\nu) \neq n) \}$ is dense open in $\mathcal{T}$

and trivially
\((*)_3 \bigcap_n \mathcal{I}_n = \emptyset\); in fact if \(G \subseteq \mathcal{F}\) is generic, then

\((A)\) \(G\) is a branch of \(\mathcal{F}\) of order type \(\omega_1^\gamma\) let its name be \((\rho_\gamma : \gamma < \omega_1)\)

\((B)\) letting \(\bar{\gamma}_n = \text{Min}\{\gamma < \omega_2 : \rho_\gamma \in \mathcal{I}_n\}\) we have \(\models_{\mathcal{F}} \{\bar{\gamma}_n : n < \omega\}\) is unbounded in \(\omega_1\).

\(\square_{2.5}\)
§ 3. Private Appendix

§3

Discussion 3.1. 1) With CH, we still do not know what occurs for Q proper not c.c.c.
A case which looks to point out where is the problem.

Question 3.2. Assume CH, and \( \prod_{\alpha < \omega} P_\alpha \), CS product, each \( P_\alpha \) adds a dense subset of \( T = \omega^\omega \) or \( F : \mathcal{T} \to T, \eta < \mathcal{T}, F(\eta) \) (so the generic \( \eta \in \mathcal{T} \) will satisfy \( (\exists^\infty \alpha < \delta)[F(\eta) \upharpoonright \alpha] \sim \eta \).

2) Without CH, we know that every c.c.c. adding a real and may proper one destroy properness. We know that such forcing does not exist when \( V = L \) by xxx Boban Velichovic xxx Macia Groszek xxx.

Generally there are, but quotients but does this give examples here?

3) In 2.1, for \( \text{Pr}_2(Q, P) \), it seemed that CH is not necessary.

4) Question: in 2.3, can we weaken "Q is c.c.c." to "Q is \( \omega \)-proper"? (or less games...)

5) If Q is nep, R is \( \aleph_1 \)-complete then \( \text{Pr}_3(Q, P) \).

6) In 2.3, "Q not adding a new \( \eta \in \omega \mid Q \)" , not adding new \( \eta \in \text{Ord} \) are equivalent..Check use.

Moved 09.11.24: try to get "\( \text{Pr}(Q, P_{\mathcal{A}}, \mathcal{V}) \) fail" from newly \( \lambda > \) (another direction: weaken the demand on \( \lambda \))

Case 2: \( h \geq \lambda \).

For \( \alpha < \omega_1 \) we define \( T_\alpha \), a \( P_{\mathcal{A}}[\mathcal{V}] \)-name, equivalently a \( \mathcal{T} \)-name by: \( \tau_\alpha[G] = p \)
if for some \( p_\alpha \in \mathcal{T}, p \in \text{suc}_*(p_\alpha) \) we have \( \eta_{p_\alpha}(p) = \alpha \).

Let \( b(\alpha) = \tau_\alpha \) for \( \alpha < \omega_1 \).
So

\[ q_* \models \phi(2^{\omega_1} \cap N_{\eta}[G]) \text{ is a function from } 2^{\omega_1} \cap N_{\eta}[G] \text{ onto } 2^{\omega_1} \cap N_{\eta}[G] \].

Let \( h_p \) be the function \( p \in \mathcal{T} \) induce so \( h_p \) maps \( 2^{\omega_1} \cap N_{\eta}[G] \) onto itself.

Fixing \( p_\alpha \) in \( \mathcal{V} \) we define \( rkp_\alpha : \omega^\omega \lambda \to \text{Ord} \cup \{ \infty \} \) by:
\[ rkp_\alpha(\eta) \geq \epsilon \text{ if for every } \alpha < N_{\eta} \cap \omega_1 \text{ and } \zeta < \epsilon \text{ there is } \nu \text{ such that} \]
\[ \bullet \eta < \nu \in \omega^\omega \lambda \]
\[ \bullet rkp_\alpha(\nu) \geq \zeta \]
\[ \bullet h_\alpha(\alpha) \in N_{\nu}. \]

Clearly
\[ (\ast) \ rkp_\alpha(\eta[G] \upharpoonright n) = \infty \text{ for every } \lambda \]
\[ (\ast) \text{ let } \mathcal{T}_1 = \{ \nu \in \omega^\omega \lambda : rkp_\alpha(\nu) = \infty \}. \]

So clearly
\[ (\ast) (a) \ T_1 \text{ is a perfect subtree of } \omega^\omega \lambda \text{ in } \mathcal{V} \]
\[ (b) \eta[G] \text{ is a branch of } \mathcal{T}_1 \text{ in } [G[\mathcal{V}]] \]
(c) there is a perfect subtree $\mathcal{S}_2$ of $\mathcal{S}_1$ such that: if $\eta \in \lim_\omega(\mathcal{S}_1)$ then $h_{p*}$ maps $N_\eta \cap \check{R}_0$ onto itself [in $V$]

(d) we can demand moreover that $p_*$ is $(N_\eta, P_\mathcal{W}[\check{V}])$ generic absolutely in particular even for branches of $\mathcal{S}_2$ not (in $V[\check{V}]$).

(\ast) without loss of generality for some limit $\delta < \lambda$, $\text{cf}(\delta) = \aleph_0$ and for every $\eta \in \mathcal{S}_2$, $\sup\{\text{Rang}(\nu_n^{\eta}): n < \omega\} = \delta$.

(\ast) without loss of generality in $\square$ we can add

(i) we have $\langle \beta_\eta : \eta \in \omega > \lambda \rangle$ such that $\beta_\eta \in h \cap N_\eta$ but $\ell g(\eta) = n + 1 \Rightarrow \beta_\eta > \sup\{N_\rho \cap h : \rho \in \omega > (\sup \text{Rang}(\nu_n^{\eta}))\}$

(\ast) let

(a) $\beta_\varepsilon = \sup\{\beta_\eta : \eta \in \omega > \varepsilon\}$ for $\varepsilon < \lambda$ so $\langle \beta_\varepsilon : \varepsilon < \lambda\rangle$ is increasing

(b) $\beta_* = \sup\{\beta_\varepsilon : \varepsilon < \lambda\}$

(c) $\gamma_\rho = \sup(N_\rho \cap \beta_\rho)$

(\ast) if $\eta_1 \neq \eta_2 \in \lim(\mathcal{S}_2)$, so $\eta_1 \cap \eta_2 = \eta \in \omega > \lambda$, then $p_*$ is $(N_\eta, \mathcal{S}_2)$-generic

(\ast) for $\rho < \eta \in \lim(\mathcal{S}_2)$ let $w_{\eta, \rho} = \{\rho|\gamma_\rho : p \in N_\eta \cap \mathcal{S}_2, \text{level}_\mathcal{S}(p) > \gamma_\rho\}$

(\ast) we can choose $\check{N}$ such that: if $\eta, \eta_1, \eta_2$ as above and $p \in w_{\eta_1, \eta} \cap w_{\eta_2, \eta}$ is contained in $w_\eta^*$ (choose $N_\eta^* \prec N_\eta^*$, etc.)

Toward the case $h \geq \lambda$ is similar:

Let $M^* = \langle M^*_\gamma : \gamma < \lambda \rangle$ is $\langle \text{<}, \text{<} \rangle$-increasing continuous, $\|M^*_\gamma\| \ast h, \check{M}^*|(\gamma + 1) \in M^*_{\gamma+1} = \langle \mathcal{M}(\check{c}), \check{c}\rangle$. Use $(C_\delta : \delta \in S_0^h)$ guess clubs.

Choose $\delta_0 \in S_0^h$. Choose countable $N_1^* \prec (\mathcal{M}(\check{c}), \check{c}), \langle M^*, \delta_0 \rangle \in N_0$. Choose similarly $N_2^*$ such that $N_1^* \subset N_2^*$.

Now by the game produce $\langle (N^*_\eta, \nu_\eta) : \eta \in \omega > \lambda\rangle$.

Old Proof:

\* it suffices to prove that for $(q, \mathcal{T})$ as above

\* $q \vdash \text{"for some } \eta \in \lim_\omega(\mathcal{T})\text{, the tree } \mathcal{T}_\eta := \mathcal{T} \cap N_\eta \text{ which has a set of levels } u_\eta = N_\eta \cap h \text{, has no } u_\eta\text{-branch with an upper bound in } \mathcal{T}^\omega\text{"}.$

However

(\ast) without loss of generality for every $\eta \in \lim_\omega(\mathcal{T})$, even new one, $\gamma_\eta = \sup(N_\eta \cap h)$ is the same $\gamma_\eta$ (as we can replace); moreover in $\square$ we can add

\dagger (b) if $h < \lambda$ and $\eta \in \omega > \lambda$ and $\alpha_1 < \alpha_2 < \lambda$ and $t_1 \in N_\eta^{<\alpha_1} \cap \mathcal{T} \cap N_\eta$, $t_2 \in N_\eta^{<\alpha_2} \cap \mathcal{T} \cap N_\eta$ then in $\mathcal{T}, t_1, t_2$ are $<_t$-incomparable

(b)' if $h \geq \lambda$, if $(\alpha_1, \alpha_2, t_1, t_2)$ fails the above then $t_1, t_2$ has a common $<_t$-upper bound in $N_\eta \cap \mathcal{T}$.

Case 1: $h < \lambda$

So $\exists h$ holds, just use any new $\eta \in \lim_\omega(\delta)$.

Case 2: Not Case 1

We just weaken $\exists$: add “except branches which are included in a branch of $\mathcal{T}$ which belongs to $N_\eta$, i.e. $N_\eta|n$ for some $n < \omega^\ast$.”
**Discussion 3.3.** Second, how strong is that assumption ⊗? Well, e.g. Cohen * Levy($\aleph_1, 2^\aleph_0$) fail it but

\[ \Box \] any c.c.c. forcing is O.K.

Also we can weaken it

\[ \otimes \] if $\eta$ in $N_{<\eta}$ is a $Q$-name of a new real, then for some perfect embedding $h$ of $^{>\aleph_0}2$ into $^{>\aleph_0}(\omega_2)$ (so $b(h) \in h(\eta^{-1}(\ell))$ for $\ell = 1, 2, h(\eta^{-1}(0)), h(\eta^{-1}(1))$) are $\prec$-incomparable let $\eta' = h(\eta)$, i.e. $\cup\{h(\eta(n) : n < \omega) \} \models \eta' \in \lim(\mathcal{F})$ where $\mathcal{F}' = \{ \nu : \nu \in h(\rho) \text{ so } h \in V \text{ for some } \rho \in ^{>\aleph_0}2 \}$ and we demand: there is $q \in Q$ such that $q$ is “($N_{\eta'}, Q$)-generic”.

The proof is as before.

Moved 09.11.25 from the proof of 2.3 trying to prove $Pr_1(Q, \mathbb{R})$: (Problem: why $N_2^2$ old - rethink)

So without loss of generality $\mathbb{R} \subseteq \mathcal{H}_{<\aleph_1}(\lambda)$ and $\lambda \geq \aleph_0$. So toward contradiction assume $q_* \in Q$ and

\[ (*)_1 \quad q_* \models_{Q} \text{“$\mathbb{R}$ is not proper”}. \]

So letting $\chi$ be large enough and

\[ (*)_2 \]

\[ (a) \quad q_* \models_{Q} \text{“$N_1 \prec (H(\chi)^V, \mathcal{H}(\chi)^V, \in)$ is countable hence } \]

\[ (b) \quad N_0 = N_1 \mid H(\chi)^V \text{ is a countable elementary submodel of } (H(\chi), \in) \]

\[ (c) \quad \text{let } N_2 = N_0 \mid H_{<\aleph_1}(\lambda) \text{ even expanding by } \mathbb{R}, \text{ note } \mathbb{R} \cap N_0 = \mathbb{R} \cap N_2 \]

and $r_* \in N_1 \cap \mathbb{R}$ and no $r'$ satisfying $r' \leq_{\mathbb{R}} r_*$ is ($N_1, \mathbb{R}$)-generic”.

Possibly increasing $q_*$ without loss of generality

\[ (*)_3 \quad r_* = r_* \text{ a member of } \mathbb{R} \text{ as a forcing.} \]

Similarly without loss of generality

\[ (*)_4 \]

\[ ?? \quad N_2 = N_2^2, \text{ i.e. } N_2^2 \text{ is an object from } V \text{ and } (N_2, \mathbb{R}) \prec (H_{<\aleph_1}(\lambda), \in, \mathbb{R}). \]

So as $\mathbb{R}$ is ($<\aleph_1$)-complete in $V$, letting $Gen = \{ g \subseteq N_2^2 \cap \mathbb{R} : g \text{ is directed and } r_* \in g \} \subseteq V$

\[ (*)_5 \quad \text{if } r \in \mathbb{R} \text{ is } \leq_{\mathbb{R}} \text{-above } r_* \text{ then } \{ r' \in \mathbb{R} \cap N_0^2 : r' \leq_{\mathbb{R}} r \} \subseteq Gen. \]

Hence

\[ (*)_6 \quad q_* \models_{Q} \text{“for no } g \in Gen \text{ for every } \mathcal{I} \in N_0 \text{ which is a dense open subset of } \mathbb{R} \text{ do we have } g \cap \mathcal{I} \neq \emptyset”. \]

By clause (β) of the claim which we have proved

\[ (*)_7 \quad q_* \models_{Q} \text{“for no } g \in Gen \text{ for every } \mathcal{I} \in N_1 \cap V \text{ which is a dense open subset of } \mathbb{R} \text{ do we have } g \cap \mathcal{I} \neq \emptyset”. \]

As $Q$ is c.c.c. (or as it is proper and we can increase $q_*$) there is $N_3$ such that

\[ (*)_8 \quad N_3 \in V, N_3 \prec (H(\chi), \in) \text{ is countable and } q_* \models \text{“$\aleph_1 \subseteq N_3$”}. \]

But obviously
\[\ast\] if \(I \in V\) and \(q \models "I \notin N_2"\) is a dense open subset of \(\mathbb{R}\) then

\(\ast\) \(I \in N_3\)

\(\ast\) \(I\) is a dense open subset of \(\mathbb{R}\)

\(\ast\) \(I \cap N_2^*\) is a dense open subset of \(\mathbb{R} \cap N_2^*\).

We can finish as

\(\ast\) \((\ast)_{10}\) there is \(g \in \text{Gen}\) such that: if \(I\) satisfies (a),(b),(c) of \((\ast)_3\) then \(g \cap I \neq \emptyset\).

[Why? As \(N_3\) is countable.]

We get contradiction so we are done.

**References**


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