PRESERVING OLD $([\omega]^{\aleph_0}, \supseteq)$ IS PROPER
SH960

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Abstract. We give some sufficient and necessary conditions on a forcing notion $\mathbb{Q}$ for preserving the forcing notion $([\omega]^{\aleph_0}, \supseteq)$ is proper. They cover many reasonable forcing notions.

modified:2013-10-06

(960) revision:2013-10-05

Date: October 5, 2013.

2010 *Mathematics Subject Classification.* Primary: 03E35; Secondary: 03E50.

Key words and phrases. set theory, forcing, proper forcing, preservation.

Research supported by the United States-Israel Binational Science Foundation (Grants No. 2002323 and 2006108) and the NSF. The author thanks Alice Leonhardt for the beautiful typing. First typed Dec. 19, 2007.

Anotated Content

modified:2013-10-06

(960) revision:2013-10-05

§ 0. Introduction

{intro}

We investigate the question "$\mathrm{Pr}_1^+(\mathbb{Q}, \mathbb{R})$", which means that the proper forcing $\mathbb{Q}$ preserves that the (old) $\mathbb{R}$ is proper for various $\mathbb{R}$'s.

Gitman proved that $\mathrm{Pr}_1^+(\mathbb{Q}, P_{\mathscr{P}(\omega)[\mathbf{V}]})$ (see definition below, <u>where</u> $\mathbb{P}_{\mathscr{P}(\omega)^{\mathbf{V}}} = \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ is the forcing notion $(\{A \in \mathbf{V} : A \subseteq \omega, |A| = \aleph_0, \supseteq^*\})$, of course $A \supseteq^* B$ means $B \subseteq^* A$) <u>when</u> $\mathbb{Q}$ is adding Cohen (or Cohen even $> 2^{\aleph_0}$). But no other examples were known even Sacks forcing. Also for e.g. $\mathbf{V} \models$ "$V = L$", we did not know a forcing making it not proper.

We thank Victoria Gitman for asking us the question and Otmar Spinas and Haim Horowitz for some comments.

Let us state the problem and relatives. We are interested mainly in the case $\mathbb{Q}$ is proper.

Definition 0.1. 1) Let $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P})$ means: $\mathbb{Q}, \mathbb{P}$ are forcing notions, $\mathbb{Q}$ is proper and $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$, i.e. $\mathbb{P}^{\mathbf{V}}$ is a proper forcing". {1a.1}
1A) Let $\mathrm{Pr}_1^+(\mathbb{P}, \mathbb{Q})$ be defined similarly but adding "$\mathbb{Q}$ is proper".
2) For $\mathscr{A} \subseteq \mathscr{P}(\omega)$ let $\mathbb{P}_{\mathscr{A}}$ be $\mathscr{A} \backslash [\omega]^{<\aleph_0}$ ordered by $\supseteq^*$, inverse almost inclusion.
3) Let $\mathscr{A}_* = \mathscr{A}_*[\mathbf{V}] = ([\omega]^{\aleph_0})^{\mathbf{V}}$.

Observation 0.2. *A necessary condition for* $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P})$ *is:* {1a.3}

- $(*)_1$ *if* χ *large enough,* $N \prec (\mathscr{H}(\chi), \in)$ *is countable,* $\mathbb{Q}, \mathbb{P} \in N, q_1 \in \mathbb{Q}$ *is* $(N, \mathbb{Q})$*-generic and* $r_1 \in N \cap \mathbb{P}$ <u>*then*</u> *we can find* (q_2, r_2) *such that:*
  - $\odot$ (a) $q_1 \leq_{\mathbb{Q}} q_2$
    - (b) $r_1 \leq_{\mathbb{R}} r_2$
    - (c) $q_2 \Vdash$ "r_2 *is* $(N[\underset{\sim}{G}_{\mathbb{Q}}], \mathbb{P})$*-generic".*

Definition 0.3. 1) We define $\mathrm{Pr}^-(\mathbb{Q}, \mathbb{P}) = \mathrm{Pr}_2(\mathbb{Q}, \mathbb{P})$ as the necessary condition from 0.2. {1a.7} {1a.3}
2) Let $\mathrm{Pr}_3(\mathbb{Q}, \mathbb{P})$ mean that $\mathbb{Q}, \mathbb{P}$ are forcing notions and for some λ and stationary $S \subseteq [\lambda]^{\aleph_0}$ from $\mathbf{V}$ we have $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ is S-proper", and S is stationary of course.
3) $\mathrm{Pr}_4(\mathbb{Q}, \mathbb{P})$ is defined similarly but $S \in \mathbf{V}^{\mathbb{Q}}$, still $S \subseteq ([\lambda]^{\aleph_0})^{\mathbf{V}}$, so S is actually $\underset{\sim}{S}$, a $\mathbb{Q}$-name.
4) $\mathrm{Pr}_5(\mathbb{Q}, \mathbb{P})$ is the statement (A) of 0.4(4) below. {1a.10}
5) Let $\mathrm{Pr}_\ell^+(\mathbb{Q}, \mathbb{P})$ means $\mathrm{Pr}_\ell(\mathbb{Q}, \mathbb{P})$ and $\mathbb{Q}$ is a proper forcing, for $\ell = 2, 3, 4, 5$.

Claim 0.4. *1)* $\mathrm{Pr}_2(\mathbb{Q}, \mathbb{P})$ *means that for* λ *large enough, letting* $S = ([\lambda]^{\aleph_0})^{\mathbf{V}}$*, we have* $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ *is* S*-proper".* {1a.10}
2) $\mathrm{Pr}_1(\mathbb{Q}_1, \mathbb{P}) \Rightarrow \mathrm{Pr}_2(\mathbb{Q}, \mathbb{P}) \Rightarrow \mathrm{Pr}_3(\mathbb{Q}, \mathbb{P})$*; similarly for* Pr^+.
3) Also $\mathrm{Pr}_3(\mathbb{Q}, \mathbb{P}) \Rightarrow \mathrm{Pr}_4(\mathbb{Q}, \mathbb{P}) \Rightarrow \mathrm{Pr}_5(\mathbb{Q}, \mathbb{P})$*; similarly for* Pr^+.
4) If $\mathbb{Q}, \mathbb{P}$ *are forcing notions,* χ *large enough,* <u>*then*</u> *(A)* $\Leftrightarrow$ *(B) where*

- (A) *for some countable* $N \prec (\mathscr{H}(\chi), \in)$ *and for some* $q \in \mathbb{Q}, p \in \mathbb{P}$ *we have*
  - (a) q *is* $(N, \mathbb{Q})$*-generic*
  - (b) $q \Vdash_{\mathbb{Q}}$ "p *is* $(N[\underset{\sim}{G}_{\mathbb{Q}}], \mathbb{P})$*-generic"*
- (B) *for some* $q_* \in \mathbb{Q}, p_* \in \mathbb{P}$ *we have* $\mathrm{Pr}_2(\mathbb{Q}_{\geq q_*}, \mathbb{P}_{\geq p_*})$.

Proof. Easy. $\square_{0.4}$

Notation 0.5. $<^*_\chi$ denote a well ordering of $\mathscr{H}(\chi)$. {z2}

modified:2013-10-06
(960) revision:2013-10-05

Recall (Balcar-Pelant-Simon [BPS80], or see, e.g. Blass [Bla])

Definition 0.6. $\mathfrak{h}$ is the following cardinal invariant, it is the minimal cardinality λ (necessarily regular) such that forcing with $\mathbb{P}_{\mathscr{A}_*}$ add a new sequence of ordinals of length χ. {z5}

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modified:2013-10-06

§ 1. Properness of $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ and CH

{prop} {2b.1}

Claim 1.1. *1) Assume* $\mathbf{V}_0 \models$ CH, $\mathbf{V}_1 \supseteq \mathbf{V}_0$, *e.g.* $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{Q}}$ *and let* $\mathscr{A} = \mathscr{A}_*[\mathbf{V}_0]$.

Recalling Definition 0.1(3), we have $\mathbf{V}_1 \models$ *"$\mathbb{P}_{\mathscr{A}}$ is proper", i.e.* $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathscr{A}})$ *<u>when</u>* {1a.1} $\mathbf{V}_1 \models$ *"if $\aleph_1^{\mathbf{V}_0}$ is not collapsed then $({}^{\omega}2)^{\mathbf{V}_0}$ is non-meagre".*

Proof. Let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}]$, where $\mathbf{G}$ is a subset of $\mathbb{Q}$ generic over $\mathbf{V}_0$.

If $\mathbf{V}_1 \models$ "$\aleph_1^{\mathbf{V}_0}$ is countable" then recalling $\mathbf{V}_0 \models$ CH clearly $\mathbf{V}_1 \models$ "$\mathscr{A}$ is countable" so we know that $\mathbb{P}_{\mathscr{A}}$ is proper in $\mathbf{V}_1$. So from now on we assume $\aleph_1^{\mathbf{V}_0}$ is not collapsed.

Second[1] in $\mathbf{V}_0$, there is a dense $\mathscr{A}' \subseteq \mathscr{A}$ downward dense in it by π, which under $\subseteq^*$ is a tree isomorphic to $\mathscr{T} = {}^{\omega_1 >}(\omega_1)$. In $\mathbf{V}_0$ there is a sequence $\bar{\mathscr{T}} = \langle \mathscr{T}_\alpha : \alpha < \omega_1 \rangle$ which is $\subseteq$-increasing continuous with union $\mathscr{T}$ and each $\mathscr{T}_\alpha$ countable. Also there is $\bar{C} = \langle C_\delta : \delta < \omega_1 \text{ limit} \rangle \in \mathbf{V}_0$ such that $C_\delta \subseteq \delta = \sup(C_\delta)$, $\mathrm{otp}(C_\delta) = \omega$. Let $\mathscr{T}'_\delta = \mathscr{T}_\delta {\restriction} \{\eta \in \mathscr{T}_\delta : \ell g(\eta) \in C_\delta\}$.

In $\mathbf{V}_1$ let $N \prec (\mathscr{H}(\chi), \in)$ be countable such that $\mathscr{A}, \pi, \bar{\mathscr{T}} \in N$ and let $\delta = \omega_1 \cap N$, clearly $\mathscr{T} \cap N = \mathscr{T}_\delta$. We have to prove the statements

$(*)_0$ "for every $p \in \mathbb{P}_{\mathscr{A}} \cap N$ there is $q \in \mathbb{P}_{\mathscr{A}}$ above p which is $(N, \mathbb{P}_{\mathscr{A}})$-generic".

As $\mathbf{V}_0 \models$ CH and the density of $\mathscr{A}'$ in $\mathscr{A}$ and $(\mathscr{A}', \supseteq^*)$ being isomorphic in $\mathbf{V}_0$ by π to $\mathscr{T}$ this is equivalent (in $\mathbf{V}_1$, of course) to

$(*)_1$ for every $\nu \in \mathscr{T} \cap N = \mathscr{T}_\delta$ there is $\eta \in \mathscr{T}$ which is $(N, \mathscr{T})$-generic and $\nu \le_{\mathscr{T}} \eta$.

In $\mathbf{V}_0$ we let $\bar{S} = \langle S_\delta : \delta < \omega_1 \text{ a limit ordinal} \rangle$ where $S_\delta = \{\bar{\nu} : \bar{\nu} = \langle \nu_n : n < \omega \rangle$ is $<_{\mathscr{T}}$-increasing, $\nu_n \in \mathscr{T}'_\delta$, moreover $\ell g(\nu_n)$ is the n-th member of $C_\delta\}$.

As $(\forall \nu \in \mathscr{T}_\delta)(\exists \rho)(\nu <_{\mathscr{T}} \rho \in \mathscr{T}'_\delta)$, and $[\bar{\nu} \in S_\delta \Rightarrow$ there is a $<_I$-upper bound $\rho \in \mathscr{T}$ of $\bar{\nu}$, in $\mathbf{V}_0$, of course$]$ recalling $\mathscr{T}_\delta, S_\delta \in \mathbf{V}_0$ clearly $(*)_1$ is equivalent (in $\mathbf{V}_1$, of course) to

$(*)_2$ for every $\nu \in \mathscr{T}'_\delta$ there is $\bar{\nu} \in S_\delta$ such that $\nu \in \mathrm{Rang}(\bar{\nu})$ and $\bar{\nu}$ induce a subset of $\mathscr{T}_\delta$ generic over N (i.e. $(\forall A)[A \in N$ is a dense open subset of $\mathscr{T} \Rightarrow A \cap \{\nu_n : n < \omega\} \ne \emptyset$.

Now a sufficient condition for $(*)_2$ is

$(*)_3$ S_δ, as a set of ω-branches of the tree $\mathscr{T}'_\delta$, is non-meagre.

But in $\mathbf{V}_0$, $\mathscr{T}'_\delta$ and ${}^{\omega >}\omega$ are isomorphic and S_δ is the set of all ω-branches of $\mathscr{T}'_\delta$, so by an assumption $(*)_3$ holds so we are done. $\square_{1.1}$

{2b.4}

Discussion 1.2. However, there can be $\mathscr{A} \subseteq \mathscr{P}(\omega)$ such that $(\mathscr{A}, \subseteq^*)$ is a variation of Souslin tree.

{2b.7}

Claim 1.3. *1) We have* $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$ *<u>when</u>:*

(*a*) $\aleph_1^{\mathbf{V}[\mathbb{Q}]} = \aleph_1$

(*b*) $\Vdash_{\mathbb{Q}}$ "$|\lambda| = \aleph_1$ *where* $\lambda = (2^{\aleph_0})^{\mathbf{V}}$

[1] this is trivial as $\mathbf{V}_0 \models$ CH, always there is a dense tree with $\mathfrak{h}$ levels by the celebrated theorem of Balcar-Pelant-Simon

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(c) *moreover letting* $\langle \underset{\sim}{u}_i : i < \aleph_1\rangle$ *be a* $\mathbb{Q}$*-name of a* $\subseteq$*-increasing continuous sequence of countable subsets of* λ *with union* λ*, the* $\mathbb{Q}$*-name* $\underset{\sim}{S} = \{i : u_i \in \mathbf{V}\}$ *is forced to contain a club (of* $\aleph_1$*)*

(d) *forcing with* $\mathbb{Q}$ *preserves "*$({}^\omega 2)^{\mathbf{V}}$ *is non-meagre".*

2) Assume the forcing notion $\mathbb{Q}$ *satisfies (a) + (d),* $\mathrm{Pr}_4(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$ *as witnessed by* S *and* $\mathbb{Q}$ *is proper and* $\underset{\sim}{S}$ *is forced to be stationary.*

<u>Then</u> *the forcing notion* $\mathbb{Q} * \mathrm{Levy}(\aleph_1, (|\mathbb{Q}|^{\aleph_0})^{\mathbf{V}}) * \mathbb{Q}_{\underset{\sim}{S}}$ *preserves "*$\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ *is proper" where* $\mathbb{Q}_S$ *is the (well known) shooting of a club through the stationary subsets of* ω_1 *(to make clause (c) hold).*

{2b.1} {2b.10} *Proof.* Like 1.1. $\square_{1.3}$

Theorem 1.4. *We have* $\Vdash_{\mathbb{Q}}$ *"*$\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ *is not proper"* <u>*when*</u>*:*

(a) $\mathbf{V} \models 2^{\aleph_0} \geq \aleph_2$

(b) $\lambda = \aleph_2$ *or just* λ *is regular,* $\aleph_2 \leq \lambda \leq 2^{\aleph_0}$ *and* $\alpha < \lambda \Rightarrow \mathrm{cf}([\alpha]^{\aleph_0}, \subseteq) < \lambda$ *hence (by* [Sh:420]*) there is a stationary* $\mathscr{U}_\alpha \subseteq [\alpha]^{\aleph_0}$ *of cardinality* $< \lambda$

(c) $\mathfrak{h} < \lambda$

(d) *the forcing notion* $\mathbb{Q}$ *adds at least one real and is* λ*-newly proper, see Definition 1.5 below.*

{2b.13}

{2b.10} {2b.13} Before proving 1.4

Definition 1.5. For $\lambda > \kappa$ we say that a forcing notion $\mathbb{Q}$ is (λ, κ)-newly proper (omitting κ means $\kappa = \aleph_0$ and we define newly $(\lambda, < \chi)$-proper similarly) when: if $\bar N = \langle (N_\eta, \nu_\eta) : \eta \in {}^{\omega >}\lambda\rangle$ satisfies $\circledast$ below and $\mathbb{Q} \in N_{<>}$ and $p \in \mathbb{Q} \cap N_{<>}$ <u>then</u> we can find $q, \underset{\sim}{\eta}$ such that $\boxtimes$ below holds where:

$\circledast$ for some cardinal $\chi > \lambda$

- (a) $N_\eta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is countable
- (b) if $\nu \triangleleft \eta$ then $N_\nu \prec N_\eta$
- (c) $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$ if $\kappa = \aleph_0$ and $N^\kappa_{\eta_1} \cap N^\kappa_{\eta_2} = N^\kappa_{\eta_1 \cap \eta_2}$ generally where $N^\kappa_\eta := \cup\{v \in N^\kappa_\eta : |v| \leq \kappa\}$
- (d) $\nu_\eta \in N_\eta \setminus \cup \{N^\kappa_{\eta \restriction m} : m < \ell g(\eta)\}$ hence $\nu_\eta \notin \cup\{N_\nu : \neg(\eta \trianglelefteq \nu)$ and $\nu \in {}^{\omega >}\lambda\}$
- (e) $\nu_\eta \in {}^{\ell g(\eta)}\lambda$ and $\ell < \ell g(\eta) \Rightarrow \nu_{\eta \restriction \ell} \trianglelefteq \nu_\eta$

$\boxtimes$ (a) $p \leq_{\mathbb{Q}} q$

- (b) $q \Vdash_{\mathbb{Q}}$ "$\cup \{N_{\underset{\sim}{\eta} \restriction n}[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}] : n < \omega\} \cap \mathbf{V} = \cup\{N_{\underset{\sim}{\eta} \restriction n} : n < \omega\}$"
- (c) $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in {}^\omega\lambda$ is new, i.e. $\underset{\sim}{\eta} \notin ({}^\omega\lambda)^{\mathbf{V}}$"
- $(c)^+$ moreover if $\kappa > \aleph_0$ and $\mathscr{T} \in \mathbf{V}$ is a sub-tree of ${}^{\omega >}\lambda$ of cardinality $\leq \kappa$ then $\underset{\sim}{\eta} \notin \lim(\mathscr{T})$, i.e. $\{\underset{\sim}{\eta} \restriction n : n < \omega\} \notin \mathscr{T}$.

For a proper forcing notion adding a new real it is quite easy to be $\aleph_1$-newly proper; e.g.

{2b.19}

Claim 1.6. *Assuming* $2^{\aleph_0} \geq \lambda = \mathrm{cf}(\lambda) > \aleph_1$*, sufficient conditions for "*$\mathbb{Q}$ *is* λ*-newly proper are:*

(a) $\mathbb{Q}$ *is c.c.c. and add a new real*

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(b) $\mathbb{Q}$ *is Sacks forcing*

(c) $\mathbb{Q}$ *is a tree-like creature forcing in the sense of Roslanowski-Shelah* [RoSh:470].

Proof. Easy; for clause (a) we use $q = p$ for $\boxplus$ of the definition. For clauses (b),(c) we use fusion but in the n-th step use members of $N_\eta \cap \mathbb{Q}$ for $\eta \in {}^n\lambda$, we get as many distinct η's as we can. $\square_{1.6}$

Proof. Proof of 1.4 Let χ be large enough and for transparency, $x \in \mathscr{H}(\chi)$.

By Rubin-Shelah [RuSh:117] in $\mathbf{V}$ there are sequences $\langle N_\eta : \eta \in {}^{\omega>}\lambda\rangle; \langle \nu_\eta : \eta \in {}^{\omega>}\lambda\rangle$ such that:

$\boxdot_1$ (a) $N_\eta \prec (\mathscr{H}(\chi), \in)$

(b) $\mathbb{Q}, x \in N_\eta$

(c) N_η is countable

(d) $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$

(e) $\nu_\eta \in {}^{\ell g(\eta)}({}^\omega 2)$

(f) $\nu_\eta \in N_\eta$

(g) if $\eta_1 \in {}^{\omega>}(\lambda)$ and $\neg(\eta \trianglelefteq \eta_1)$ then $\nu_\eta \notin N_{\eta_1}$

(h) $\nu_{\eta_1}(\ell_1) = \nu_{\eta_2}(\ell_2) \Rightarrow \ell_1 = \ell_2 \wedge \eta_1{\restriction}(\ell_1+1) = \eta_2{\restriction}(\ell_2+1)$.

Now for each $\eta \in {}^\omega\lambda$ let $N_\eta = \cup\{N_{\eta\restriction k} : k < \omega\}$; we can add:

(i) if $\ell g(\eta) = n+1$ then $\nu_\eta(n) > \sup(N_{\eta\restriction n} \cap \lambda)$ and even $\nu_\eta(n) > \sup\{N_\rho \cap \lambda : \rho \in {}^{\omega>}(\nu_\eta(n))\}$

(j) if $\eta \in {}^\omega\lambda$ is increasing, then $\sup(N_\eta \cap \lambda) = \sup(\mathrm{Rang}(\eta))$.

Why is this sufficient? By Balcar-Pelant-Simon [BPS80] there is $\mathscr{T} \subseteq [\omega]^{\aleph_0}$ such that

$\boxdot_2$ (α) $(\mathscr{T}, {}^*\supseteq)$ is a tree with $\mathfrak{h}$ levels ($\mathfrak{h}$ is the cardinal invariant from 0.6, a regular cardinal $\in [\aleph_1, 2^{\aleph_0}]$), the tree $\mathscr{T}$ is with a root and each node has $2^{\aleph_0}$ many immediate successors, i.e. $\mathscr{T}$ has splitting to $2^{\aleph_0}$) {z5}

(β) $\mathscr{T}$ is dense in $([\omega]^{\aleph_0}, \supseteq^*)$, i.e. in $\mathbb{P}_{\mathscr{P}(\omega)^{\mathbf{V}}} = \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ recalling 0.1(2). {1a.1}

Choose $\bar{h}$ such that

$\boxdot_3$ $\bar{h} = \langle h_p : p \in \mathscr{T}\rangle$ satisfies: h_p is one to one from $\mathrm{suc}_{\mathscr{T}}(p)$ onto $2^{\aleph_0} \backslash \{h_{p_0}(p_1) : p_0 <_{\mathscr{T}} p_1 <_{\mathscr{T}} p$ and $p_1 \in \mathrm{suc}_{\mathscr{T}}(p_0)\}$.

So without loss of generality

$\boxdot_4$ $\mathscr{T} \in N_{<>}$ and $\bar{h} \in N_{<>}$.

As $\mathbb{Q}$ is newly λ-newly proper there are $\underset{\sim}{\eta}, q$ as in $\boxtimes$ of Definition 1.5. Let $\mathbf{G} \subseteq \mathbb{Q}$ be generic over $\mathbf{V}$ such that $q \in \mathbf{G}$, let $\eta = \underset{\sim}{\eta}[G]$ and $M_2 := N_{\underset{\sim}{\eta}[G]} := \cup\{N_{\eta\restriction n}[\mathbf{G}] : n < \omega\}$, so $M_2 \prec (\mathscr{H}(\chi)^{\mathbf{V}[\mathbf{G}]}, \mathscr{H}(\chi)^{\mathbf{V}}, \in)$ is countable, pedantically $(|M_2|, \mathscr{H}(\chi)^{\mathbf{V}} \cap |M_2|, \in {\restriction}|M_2|) \prec (\mathscr{H}(\chi)^{\mathbf{V}[\mathbf{G}]}, \mathscr{H}(\chi)^{\mathbf{V}}, \in {\restriction}\mathscr{H}(\chi)^{\mathbf{V}[\mathbf{G}]})$. {2b.13}

By $\boxtimes$ of 1.6 as $q \in \mathbf{G}$ we have $M_1 = M_2 \cap \mathscr{H}(\chi)^{\mathbf{V}}$ is $\cup\{N_{\eta\restriction n} : n < \omega\}$, and of course $M_1 \prec (\mathscr{H}(\chi), \in)$. Toward contradiction assume $\mathbf{V}[\mathbf{G}] \models$ "$\mathscr{P}_{\mathscr{A}_*[\mathbf{V}]}$ is {2b.19}

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proper", hence some $p_* \in \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ is $(M_2, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$-generic. But $\mathscr{T}$ is dense in $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ so without loss of generality $p_* \in \mathscr{T}$ and p_* is $(M_2, \mathscr{T})$-generic.

Clearly $\mathfrak{h} \in N_{<>}$ or we may demand this, so without loss of generality $\eta \in {}^{\omega>}\lambda \Rightarrow N_\eta \cap \mathfrak{h} = N_{<>} \cap \mathfrak{h}$. For any $\alpha < \lambda$ let

$$\mathscr{I}_\alpha = \{p \in \mathscr{T} : \text{ for some } p_0 \in \mathscr{T} \text{ we have } p \in \operatorname{suc}_{\mathscr{T}}(p_0) \text{ and } h_{p_0}(p) = \alpha\}$$

and letting $\mathscr{T}_\alpha$ be the α-th level of $\mathscr{T}$

$$\mathscr{I}^+_\alpha = \{p \in \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]} : p \text{ is above some member of } \mathscr{T}_\alpha\}.$$

Now clearly (in $\mathbf{V}$ and in $\mathbf{V}[\mathbf{G}]$):

$(*)_1$ (a) $\mathscr{I}_\alpha$ is a pre-dense subset of $\mathscr{T}$ (and of $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$)
- (b) $\mathscr{I}^+_\alpha$ is dense open decreasing with α
- (c) if $p \in \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ <u>then</u> for every large enough $\alpha < \lambda, p \notin \mathscr{I}^+_\alpha$
- (d) if $p \in \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ and $\alpha < \lambda$ <u>then</u> there is $q \in \mathscr{I}_\alpha$ such that $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]} \models$ "$p \le q$".

Also if $\alpha \in \lambda \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$ then $\mathscr{I}_\alpha \in N_{\underset{\sim}{\eta}[\mathbf{G}]}$ and the set $\{p \in \mathscr{T} \cap N_{\underset{\sim}{\eta}[\mathbf{G}]} : p \le_{\mathscr{T}} p_*\}$ is not empty, let p^*_α be in it and let its level in $\mathscr{T}$ be γ^*_α.

Now

$(*)_2$ if $\alpha \in \mathfrak{h} \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$ then $\gamma^*_\alpha \in N_{\underset{\sim}{\eta}[\mathbf{G}]} \cap \mathfrak{h} = N_{<>} \cap \mathfrak{h}$ hence

$(*)_3$ in $\mathbf{V}[\mathbf{G}]$ the following function h_* is well defined
- (a) $\operatorname{Dom}(h_*) = N_{<>} \cap \mathfrak{h}$
- (b) $h_*(\gamma)$ is the unique $p \in N_{\underset{\sim}{\eta}[\mathbf{G}]} \cap \mathscr{T}$ of level γ which is $\le_{\mathscr{T}} p_*$.

also by the choice of $\bar{h}$ (and genericity) clearly

$(*)_4$ $\operatorname{Rang}(h_*)$ is equal to $u := (2^{\aleph_0}) \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$.

Lastly,

$(*)_5$ $h_* \in \mathbf{V}$.

[Why? As its domain, $N_{<>} \cap \mathfrak{h}$ belongs to $\mathbf{V}$ and $h_*(\gamma)$ is defined from $\langle \mathscr{T}, \gamma, p_* \rangle \in \mathbf{V}$ and $\mathscr{T}$ is a tree.]

$(*)_6$ (a) from $u := \lambda \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$ we can define $\underset{\sim}{\eta}[\mathbf{G}]$
- (b) $u = \cup\{N_{\underset{\sim}{\eta}\restriction n[\mathbf{G}]} \cap \lambda : n < \omega\}$.

[Why? By the choice of $\bar{N}$.]

Together we get that $\underset{\sim}{\eta}[\mathbf{G}] \in \mathbf{V}$, contradiction. $\square_{1.4}$

{2b.23} **Claim 1.7.** *We have* $\neg \mathrm{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$ <u>*when*</u>

- (a) $2^{\aleph_0} \ge \lambda = \operatorname{cf}(\lambda) > \kappa = \mathfrak{h}$
- (b) $\alpha < \lambda \Rightarrow \operatorname{cf}([\alpha]^{\le\kappa}, \subseteq) \le \kappa) < \lambda$
- (c) $\mathbb{Q}$ *is* (λ, κ)*-newly proper.*

{2b.10} *Proof.* Similar to 1.4. $\square_{1.7}$

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{2b.26} **Conclusion 1.8.** *If* $\mathfrak{h} < 2^{\aleph_0}$ *and* $\mathbb{Q}$ *is a* $(\mathfrak{h}^+, \mathfrak{h})$*-newly proper* *__then__* $\neg \mathrm{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$.

modified:2013-10-06

(960) revision:2013-10-05

§ 2. General sufficient conditions

{general}{3c.1}

Claim 2.1. *Assume CH, i.e.* $\mathbf{V} \models CH$.

If $\mathbb{Q}$ *is c.c.c.* <u>then</u> $\mathrm{Pr}_2(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$.

Remark 2.2. 1) This works replacing $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ by any $\aleph_1$-complete $\mathbb{P}$ and strengthening the conclusions to Pr_1, see 2.3. {3c.3}

{1a.7} 2) See Definition 0.3(1).

Proof. Let $\mathbb{P} = \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$. The point is

(∗) if $r \in \mathbb{P}$ and $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathscr{I}}$ is a dense open subset of $\mathbb{P}$" <u>then</u> there is r' such that:

(a) $r \le_{\mathbb{P}} r'$

(b) $\Vdash_{\mathbb{Q}}$ "$r' \in \underset{\sim}{\mathscr{I}} \subseteq \mathbb{P}$".

Why (∗) holds? We try (all in $\mathbf{V}$) to choose (r_α, q_α) by induction on $\alpha < \omega_1$ but choosing q_α togegther with $r_{\alpha+1}$ such that:

⊛ (a) $r_0 = r$

(b) $r_\alpha \in \mathbb{P}$ is $\le_{\mathbb{P}}$-increasing

(c) $q_\alpha \in \mathbb{Q}$

(d) q_α, q_β are incompatible in $\mathbb{Q}$ for $\beta < \alpha$

(e) $q_\alpha \Vdash_{\mathbb{Q}}$ "$r_{\alpha+1} \in \underset{\sim}{\mathscr{I}}$".

We cannot succeed because $\mathbb{Q} \models$ c.c.c.

For $\alpha = 0$ no problem as only clause (a) is relevant.

For α limit - easy as $\mathbb{P}$ is $\aleph_1$-complete (and the only relevant clause is (b)).

For $\alpha = \beta + 1$, we first ask:

<u>Question</u>: Is $\langle q_\gamma : \gamma < \beta \rangle$ a maximal antichain of $\mathbb{Q}$?

<u>If yes</u>, then r_β is as required: if $\mathbf{G}_{\mathbb{Q}} \subseteq \mathbb{Q}$ is generic over $\mathbf{V}$ <u>then</u> for some $\gamma < \beta, q_\gamma \in \mathbf{G}_{\mathbb{Q}}$ hence $r_{\gamma+1} \in \underset{\sim}{\mathscr{I}}[\mathbf{G}_Q]$ but $\underset{\sim}{\mathscr{I}}[\mathbf{G}_{\mathbb{Q}}]$ is a dense subset of $\mathbb{P}$ and is <u>open</u> and $r_{\gamma+1} \le_{\mathbb{R}} r_\beta$ so $r_\beta \in \underset{\sim}{\mathscr{I}}[\mathbf{G}_{\mathbb{Q}}]$.

<u>If no</u>, let $q^\beta \in \mathbb{Q}$ be incompatible with q_γ for every $\gamma < \beta$. Recalling $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathscr{I}}$ is <u>dense</u> and open" the set $X_\beta = \{r \in \mathbb{P}$: for some $q, q^\beta \le_{\mathbb{Q}} q$ and $q \Vdash$ "$r \in \underset{\sim}{\mathscr{I}}$"$\}$ is a dense subset of $\mathbb{P}$ hence there is a member of X_β above r_β, let r_α be such member. By $r_\alpha \in X_\beta$, there is $q, q^\beta \le q$ such that $q \Vdash$ "$r_\alpha \in \underset{\sim}{\mathscr{I}}$". So we choose q_β as such q, so we can carry the induction step.

As said above we cannot carry the induction for all $\alpha < \omega_1$ because then $\{q_\alpha : \alpha < \omega_1\}$ contradicts "$\mathbb{Q}$ satisfies the c.c.c." So for some α we cannot continue, α is neither 0 no limit hence for some $\beta, \alpha = \beta + 1$. So the answer to the question is yes, hence we get the desired conclusion of (∗).

So (∗) indeed holds and this is clearly enough. $\square_{2.1}$

We can weaken the demand on the second forcing (above $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$).

{3c.3}

Claim 2.3. *If (A) then (B) where:*

(A) (a) $\mathbb{P}, \mathbb{Q}$ *are forcing notions*

(b) $\mathbb{Q}$ *is c.c.c. moreover* $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ *is c.c.c.*"

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(c) *forcing with $\mathbb{P}$ and no new ω-sequences,*[2] *from* λ

(d) $\mathbb{Q}$ *has cardinality* $\leq \lambda$

(B) (a) *if* $\mathbb{P}$ *is proper in* $\mathbf{V}$ *then* $\Pr_2(\mathbb{Q}, \mathbb{P})$

(b) *for every* $\mathbb{Q}$*-name* $\underset{\sim}{\mathscr{I}}$ *of a dense open subset of* $\mathbb{P}$*, the set* $\mathscr{J} = \{r \in \mathbb{P} : \Vdash_{\mathbb{Q}} \text{“}r \in \underset{\sim}{\mathscr{I}}\text{”}\}$ *is dense and open.*

Proof. First we prove clause (b); so fix $\mathscr{I}$ and $\mathscr{J}$ as there. Let $\langle q_\varepsilon : \varepsilon < \kappa := |\mathbb{Q}|\rangle$ list $\mathbb{Q}$.

For every $r \in \mathbb{P}$ we define a sequence η_r of ordinals $< \lambda$ as follows:

$\circledast_1$ $\eta_r(\alpha)$ is the minimal ordinal $\varepsilon < \kappa$ such that (so $\ell g(\eta_r) = \alpha$ when there is no such ε)

(a) $q_\varepsilon \Vdash$ “$r \in \underset{\sim}{\mathscr{I}}$”

(b) if $\beta < \alpha$ then $q_\varepsilon, q_{\eta_r(\beta)}$ are incompatible in $\mathbb{Q}$”.

Now

$\circledast_2$ (a) η_r is well defined

(b) $\ell g(\eta_r) < \omega_1$.

[Why? As $\mathbb{Q} \models$ c.c.c.]

Note

$\circledast_3$ if $r_1 \leq_{\mathbb{P}} r_2$ then <u>either</u> $\eta_{r_1} \trianglelefteq \eta_{r_2}$ <u>or</u> for some $\alpha < \ell g(\eta_{r_1})$ we have

$$\eta_{r_1} \restriction \alpha = \eta_{r_2} \restriction \alpha$$

$$\eta_{r_1}(\alpha) > \eta_{r_2}(\alpha).$$

[Why? Think about the definition.]

For $s \in \mathbb{P}$ let η'_s be $\cap\{\eta_{s_1} : s \leq_{\mathbb{P}} s_1\}$, i.e. the longest common initial segment of $\{\eta_{s_1} : s \leq_{\mathbb{P}} s_1\}$; clearly $s_1 \leq_R s_2 \Rightarrow \eta'_{s_1} \trianglelefteq \eta'_{s_2}$. So

$\circledast_4$ $\underset{\sim}{\eta}^* = \cup\{\eta'_s : s \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}\}$ is an $\mathbb{P}$-name of a sequence of pairwise incompatible members of $\mathbb{Q}$.

But by clause (A)(b) of the claim, forcing with $\mathbb{P}$ preserve “$\mathbb{Q} \models$ c.c.c.”, so $\ell g(\underset{\sim}{\eta}^*)$ is countable in $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$. But by clause (A)(c) of the claim, forcing by $\mathbb{P}$ adds no new ω-sequences to $\kappa = |\mathbb{Q}|$ (and $\mathbb{Q}$ is infinite) and $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$ has the same $\aleph_1$ as $\mathbf{V}$, so

$\circledast_5$ $\underset{\sim}{\eta}^*$ is a sequence of countable length of ordinals $< \kappa$ so is old.

Hence

$\circledast_6$ the following set is dense open in $\mathbb{P}$

$$\mathscr{J} = \{r \in \mathbb{P} : r \text{ forces } (\Vdash_{\mathbb{P}}) \text{ that } \underset{\sim}{\eta}^* = \eta^*_r \text{ for some } \eta^*_r \in \mathbf{V}\}$$

Let us define:

$\circledast_7$ $\mathscr{J}_* = \{s \in \mathscr{J} : \eta_s = \eta^*_s \text{ and moreover } s \leq_{\mathbb{P}} t \Rightarrow \eta_t = \eta^*_t = \eta_s\}$.

Next

[2] if you assume $\mathbb{Q}, \mathbb{P}$ are proper, $\lambda = \aleph_0$ the proof may be easier to read

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$\circledast_8$ $\mathscr{J}_*$ is dense open.

Clearly $\mathscr{J}_*$ is open, but why is it dense? Let $p \in \mathbb{P}$ and we shall find $t \in \mathscr{J}_*$ such that $p \le_{\mathbb{P}} t$; so we can replace p by any $p' \in \mathbb{P}$ above p. By $\circledast_6$ there is $p_1 \in \mathscr{J}$ above p so $\eta^*_{p_1}$ is well defined, so $p_1 \Vdash$ "$\underset{\sim}{\eta}^* = \eta^*_{p_1}$". Let $\eta_* = \eta^*_{p_1}$, so by the definition of $\underset{\sim}{\eta}^*$ in $\circledast_4$ we have $\eta'_{p_1} \trianglelefteq \eta_*$ and let

$\circledast_{8.1}$ $\eta'_{p_1} = \eta_*$.

[Why? If not then necessarily $\eta'_{p_1} \triangleleft \eta_*$, hence letting $\xi = \ell g(\eta'_{p_1})$, by the definition of η'_{p_1} there is p_2 such that $p_1 \le_{\mathbb{P}} p_2$ and $\eta_{p_2} \restriction (\xi + 1) = \eta_* \restriction (\xi + 1)$, but necessarily $\eta'_{p_1} \triangleleft \eta_{p_2}$. Hence the condition $p_2, q_{\eta_*(\xi)}$ are compatible; let $p_3 \in \mathbb{P}$ be a common upper bound of them so $(\forall t)(p_3 \le_{\mathbb{P}} t \Rightarrow q_{\eta_*(\xi)} \le_{\mathbb{P}} t \Rightarrow \eta_* \restriction (\xi+1) \ntrianglelefteq \eta_t)$ hence $p_3 \models$ "$\underset{\sim}{\eta}^* \ne \eta_*$", contradiction.]

We can conclude

$\circledast_{8.2}$ if $p_1 \le t$ then $\eta_* \trianglelefteq \eta_t$.

Let $u = \{\eta_t(\ell g(\eta_*)) : t \in \mathbb{P}$ is above $\mathbb{P}\}$. If u is empty then $p_1 \in \mathscr{I}_*$ by the definition of $\mathscr{I}_*$, so we are done proving $\circledast_8$. If u is not empty, let $p_2 \in \mathbb{P}$ be such that $\ell g(\eta_{p_2}) > \ell g(\eta_*)) \wedge \eta_{p_2}(\ell g(\eta_*)) = \min(u)$. By the definition of $\underset{\sim}{\eta}^*$ and of $\langle \eta_t : t \in \mathbb{P} \rangle$ there is $p_3 \in \mathbb{P}$ above p_2 such that $\ell g(\eta_{p_3}) > \ell g(\eta_*)) \wedge \eta_{p_3}(\ell g(\eta_*)) < \min(u)$, contradiction. So $u = \emptyset$ and p_1 is as required.

$\circledast_9$ if $r \in \mathscr{J}$ <u>then</u> $\langle q_{\eta^*_r(\varepsilon)} : \varepsilon < \ell g(\eta^*_r) \rangle$ is a maximal antichain of $\mathbb{Q}$.

{3c.1} [Why? As in the proof of 2.1.]

Fixing $r_* \in \mathscr{J}_* \subseteq \mathbb{P}$ and $\alpha < \ell g(\eta^*_{r_*})$ let

$(*)_0$ $\mathscr{J}_{r_*,\alpha} = \{r \in \mathbb{P} : r_* \le_{\mathbb{P}} r$ and $q_{\eta^*_{r_*}(\alpha)}$ forces (for $\Vdash_{\mathbb{Q}}$) that $r \in \underset{\sim}{\mathscr{I}}\}$

$(*)_1$ $\mathscr{J}_{r_*,\alpha}$ is dense in $\mathbb{P}$ above r_*.

[Why? Assume $\mathbb{P} \models$ "$r_* \le r_1$" so $r_1 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta}^*(\alpha) = \eta^*_{r_*}(\alpha)$" hence for some r_2 we have $\mathbb{P} \models$ "$r_1 \le r_2$" and $\eta^* \restriction (\alpha+1) \trianglelefteq \underset{\sim}{\eta}'_{r_2}$, so by clause (a) of $\circledast_1$ we have $q_{\eta^*_{r_*}(\varepsilon)} \Vdash_{\mathbb{Q}}$ "$r_2 \in \underset{\sim}{\mathscr{I}}$" hence $r_2 \in \mathscr{J}_{r_*,\alpha}$ as required.]

So

$(*)_2$ $\mathscr{J}_{r_*,\alpha}$ is a (dense and) open subset of $\mathbb{P}_{\ge r_*}$ (i.e. above r_*).

[Why? As $\Vdash_{\mathbb{Q}}$ "$\mathscr{I}$ is an open subset", for density use $(*)_1$.]

As forcing with $\mathbb{P}$ add no new ω-sequence of ordinals $< \lambda$ (by clause (A)(c) of the claim)

$(*)_3$ $\mathscr{J}^+_{r_*} := \cap\{\mathscr{J}_{r_*,\alpha} : \alpha < \ell g(\eta^*_{r_*})\}$ is dense open in $\mathbb{P}$ above r_*.

[Why? Let $\mathscr{J}^*_{r_*,\alpha}$ be a maximal antichain $\subseteq \mathscr{J}_{r_*,\alpha}$ for $\alpha < \ell g(\eta^*_{r_*})$ let $\underset{\sim}{f}$ be the $\mathbb{P}$-name of $\{(\alpha, q) : \alpha < \ell g(\eta^*_{r_*})$ and $q \in \mathscr{J}_{r_*,\alpha} \cap \underset{\sim}{G}_{\mathbb{P}}\}$ so $r_* \Vdash$ "f a function from $\ell g(\eta^*_{r_*})$ to $\mathbb{P}$" hence $r_* \Vdash$ "$\underset{\sim}{f} \in \mathbf{V}$".]

{3c.1} Clearly by the definition (recalling $\circledast_7$, as in the proof of 2.1)

$(*)_4$ if $r \in \mathscr{J}^+_{r_*}$ then $\Vdash_{\mathbb{Q}}$ "$r \in \underset{\sim}{\mathscr{I}}$".

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As $\cap\{\mathscr{J}^+_{r_*} : r_* \in \mathscr{J}\}$ is dense open in $\mathbb{P}$ we are done proving clause (b) of the claim.

{1a.3} As for clause (a), let χ, N, q_1, r_1 be as in the assumption of $(*)_1$ of 0.2, so $\mathbb{P}, \mathbb{Q} \in N$. We have to find q_2, r_2.

Let $q_2 = q_1$ and let $r_2 \in \mathbb{P}$ be $(N, \mathbb{P})$-generic and above r_1, exists as $\mathbb{P}$ is a proper forcing in $\mathbf{V}$. So let $\mathbf{G} \subseteq \mathbb{Q}$ be a subset of $\mathbb{Q}$ generic over $\mathbf{V}$ such that $q_2 = q_1 \in \mathbf{G}$. Now if $N[\mathbf{G}] \models$ "$\mathscr{I}$ is a dense open subset of $\mathbb{P}$", then by the definition of $N[\mathbf{G}]$ for some $\mathbb{Q}$-name $\underset{\sim}{\mathscr{I}}$ from N of a dense open subset of $\mathbb{P}$ we have $\underset{\sim}{\mathscr{I}}[\mathbf{G}] = \mathscr{I}$. By clause (β) the set $\mathscr{J} = \{r \in \mathbb{P} : \Vdash_{\mathbb{Q}} \text{"} r \in \underset{\sim}{\mathscr{I}}\text{"}\}$ is dense and open in $\mathbb{P}$ and clearly $\in N$ hence $r_2 \Vdash$ "$\mathscr{J} \cap \underset{\sim}{\mathbf{G}}_{\mathbb{P}} \neq \emptyset$".

So we are done. $\square_{2.3}$

Remark 2.4. In 2.1, 2.3 we can replace "c.c.c." by "strongly proper". {3c.5} {3c.3}

<u>But</u> such $\mathbb{Q}$ preserves "$({}^\omega 2)^{\mathbf{V}}$-non-meagre". {3c.7}

Claim 2.5. *1) There is a proper forcing $\mathbb{Q}$ which forces "$\mathbb{P}_{\mathscr{A}_*}[\mathbf{V}]$ as a forcing notion is not proper", (i.e. $\neg\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P})$).*
2) Even (A) of 0.4(3) fail, i.e. $\neg\mathrm{Pr}_5(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*}[\mathbf{V}])$. {1a.10}

Proof. We use the proof of [Sh:f, Ch.17,Sec.2] and see references there. We repeat in short.

We use a finite iteration so let $\mathbb{P}_0$ be the trivial forcing notion, $\mathbb{P}_{k+1} = \mathbb{P}_k * \underset{\sim}{\mathbb{Q}}_k$ for $k \leq 3$ and the $\mathbb{P}_k$-name $\underset{\sim}{\mathbb{Q}}_k$ is defined below.

<u>Step A</u>: $\mathbb{Q}_0 =$ Levy$(\aleph_1, 2^{\aleph_0})$ so $\Vdash_{\mathbb{Q}_0}$ "CH".

<u>Step B</u>: $\mathbb{Q}_1$ is Cohen forcing.

<u>Step C</u>: In $\mathbf{V}^{\mathbb{P}_2}, \mathbb{Q}_2$ in the Levy collapse of $2^{2^{\aleph_0}}$ to $\aleph_1$, i.e. $\mathbb{Q}_2 =$ Levy$(\aleph_1, \beth_2)^{\mathbf{V}[\mathbb{P}_2]}$.

<u>Step D</u>: Let $\mathscr{T} = ({}^{(\omega_1>)}\omega_1)^{\mathbf{V}[\mathbb{P}_1]} = ({}^{(\omega_1>)}\omega_1)^{\mathbf{V}[\mathbb{P}_0]}$ a tree, so we know that $\lim_{\omega_1}(\mathscr{T})^{\mathbf{V}[\mathbb{P}_1]} = \lim_{\omega_1}(\mathscr{T})^{\mathbf{V}[\mathbb{P}_2]} = \lim_{\omega_1}(\mathscr{T})^{\mathbf{V}[\mathbb{P}_3]}$ and

$(*)_1$ in $\mathbf{V}^{\mathbb{P}_1}, \mathscr{T}$ is isomorphic to a dense subset of $\mathbb{P}_{\mathscr{A}_*[\mathbb{P}_1]} = \mathbb{P}_{\mathscr{A}_*[\mathbb{P}_0]}$.

So in $\mathbf{V}^{\mathbb{P}_3}$ there is a list $\langle \eta^*_\varepsilon : \varepsilon < \omega_1\rangle$ of $\lim_{\omega_1}(\mathscr{T})^{\mathbf{V}[\mathbb{P}_1]}$. Let $\langle \eta^*_\varepsilon \restriction [\gamma_\varepsilon, \omega_1) : \varepsilon < \omega_1\rangle$ be pairwise disjoint end segments.

<u>Step E</u>: In $\mathbf{V}^{\mathbb{P}_3}$ there is $\mathbb{Q}_3$, a c.c.c. forcing notion specializing $\mathscr{T}$ in the sense of [Sh:74], i.e. there is $h_* \in \mathbf{V}^{\mathbb{P}_4}$ such that $h_* : \mathscr{T} \to \omega, h$ is increasing in $\mathscr{T}$ except on the end segments $\eta^*_\varepsilon \restriction [\gamma_\varepsilon, \omega_1)$ for $\varepsilon < \omega_1$, i.e. $\rho <_{\mathscr{T}} \nu \wedge h_*(\rho) = h(\nu) \Rightarrow (\exists \varepsilon)[\rho, \nu \in \{\eta^*_\varepsilon \restriction \gamma : \gamma \in [\gamma_\varepsilon, \omega_1)\}$.

Now

$\boxtimes$ after forcing with $\mathbb{P}_4 = \mathbb{Q}_0 * \mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}}_2 * \underset{\sim}{\mathbb{Q}}_3$, i.e. in $\mathbf{V}^{\mathbb{P}_4}$ the forcing notion $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ is not proper, in fact it collapses $\aleph_1$.

Why? Recall $(*)_1$ and note

$(*)_2$ $\mathscr{I}_n := \{\rho \in \mathscr{T} : (\forall \nu)(\rho \leq_{\mathscr{T}} \nu \to h_*(\nu) \neq n\}$ is dense open in $\mathscr{T}$

and trivially

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$(*)_3$ $\bigcap\limits_n \mathscr{I}_n = \emptyset$; in fact if $\mathbf{G} \subseteq \mathscr{T}$ is generic, then

- (A) $\mathbf{G}$ is a branch of $\mathscr{T}$ of order type $\omega_1^{\mathbf{V}}$ let its name be $\langle \underset{\sim}{\rho}_\gamma : \gamma < \omega_1 \rangle$
- (B) letting $\underset{\sim}{\gamma}_n = \text{ Min}\{\gamma < \omega_2 : \underset{\sim}{\rho}_\gamma \in \mathscr{I}_n\}$ we have $\Vdash_{\mathscr{T}}$ "$\{\underset{\sim}{\gamma}_n : n < \omega\}$ is unbounded in ω_1".

$\square_{2.5}$

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§ 3. Private Appendix
§3

Discussion 3.1. 1) With CH, we still do not know what occurs for $\mathbb{Q}$ proper not c.c.c.

A case which looks to point out where is the problem.

Question 3.2. Assume CH, and $\prod\limits_{\alpha<\omega_2} P_\alpha$, CS product, each $\mathbb{P}_\alpha$ adds a dense subset of $\mathscr{T}_* = {}^{\omega>}(\omega_1)$ or $F : \mathscr{T}_* \to \mathscr{T}_*, \eta <_{\mathscr{T}_*} F(\eta)$ (so the generic $\eta \in \mathscr{T}'_\delta$ will satisfy $(\exists^\infty \alpha < \delta)[F(\eta){\restriction}\alpha) \triangleleft \eta]$.
2) Without CH, we know that every c.c.c. adding a real and may proper one destroy properness. We know that such forcing does not exist when $\mathbf{V} = \mathbf{L}$ by xxx Boban Velichovic xxx Macia Groszek xxx.

Generally there are, but quotients but does this give examples here?
3) In 2.1, for $\mathrm{Pr}_2(\mathbb{Q}, \mathbb{P})$, it seemed that CH is not necessary. {3c.1}
4) Question: in 2.3, can we weaken "$\mathbb{Q}$ is c.c.c." to "$\mathbb{Q}$ is ω-proper"? (or less games...) {3c.3}
5) If $\mathbb{Q}$ is nep, $\mathbb{R}$ is $\aleph_1$-complete then $\mathrm{Pr}_5(\mathbb{Q}, \mathbb{P})$.
6) In 2.3, "Q not adding a new $\eta \in {}^\omega|Q|$", not adding new $\eta \in {}^\omega\mathrm{Ord}$ are equivalent..Check use. {3c.3}

Moved 09.11.24: try to get "$\mathrm{Pr}(Q, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$ fail" from newly $(\lambda, *)$-proper when $\lambda \le \mathfrak{h}$ (another direction: weaken the demand on λ)

Case 2: $\mathfrak{h} \ge \lambda$.

For $\alpha < 2^{\aleph_0}$ we define $\underset{\sim}{T}_\alpha$, a $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$-name, equivalently a $\mathscr{T}$-name by: $\underset{\sim}{\tau}_\alpha[\mathbf{G}] = p$ if for some $p_0 \in \mathscr{T}, p \in \mathrm{suc}_{\mathscr{T}}(p_0)$ we have $h_{p_0}(p) = \alpha$.

Let $\underset{\sim}{h}(\alpha) = \underset{\sim}{\tau}_\alpha$ for $\alpha < 2^{\aleph_0}$.

So

$$q_* \Vdash \text{“}\underset{\sim}{h}{\restriction}(2^{\aleph_0} \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}) \text{ is a function from } 2^{\aleph_0} \cap N_{\underset{\sim}{\eta}[\mathbf{G}]} \text{ onto } 2^{\aleph_0} \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}\text{”}.$$

Let h_{p_*} be the function $p_* \in \mathscr{T}$ induce so h_{p_*} maps $2^{\aleph_0} \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$ onto itself.

Fixing p_* in $\mathbf{V}$ we define $\mathrm{rk}_{p_*} : {}^{\omega>}\lambda \to \mathrm{Ord} \cup \{\infty\}$ by:
$\mathrm{rk}_{p_*}(\eta) \ge \varepsilon$ iff for every $\alpha \in N_\eta \cap {}^{\aleph_0}2$ and $\zeta < \varepsilon$ there is ν such that

- $\eta \triangleleft \nu \in {}^{\omega>}\lambda$
- $\mathrm{rk}_{p_*}(\nu) \ge \zeta$
- $h_{p_*}(\alpha) \in N_\nu$.

Clearly

$(*)$ $\mathrm{rk}_{p_*}(\underset{\sim}{\eta}[\mathbf{G}]{\restriction}n) = \infty$ for every λ
$(*)$ let $\mathscr{S}_1 = \{\nu \in {}^{\omega>}\lambda : \mathrm{rk}_{p_*}(\nu) = \infty\}$.

So clearly

$(*)$ (a) $\mathscr{S}_1$ is a perfect subtree of ${}^{\omega>}\lambda$ in $\mathbf{V}$
(b) $\underset{\sim}{\eta}[\mathbf{G}]$ is a branch of $\mathscr{S}_\eta$ [in $\mathbf{V}[\mathbf{G}]$]

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- (c) there is a perfect subtree $\mathscr{S}_2$ of $\mathscr{S}_1$ such that: if $\eta \in \lim_\omega(\mathscr{S}_1)$ then h_{p_*} maps $N_\eta \cap {}^{\aleph_0}2$ onto itself [in $\mathbf{V}$]
- (d) we can demand moreover that p_* is $(N_\eta, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$ generic absolutely in particular even for branches of $\mathscr{S}_2$ not (in $\mathbf{V}[\mathbf{G}]$) (recall $|\mathscr{A}_*| = |\mathscr{T}| = 2^{\aleph_0}$), [used?]

$(*)$ without loss of generality for some limit $\delta_* < \lambda$, $\mathrm{cf}(\delta_*) = \aleph_0$ and for every $\eta \in \mathscr{S}_2$, $\sup\{\mathrm{Rang}(\nu_{\eta\restriction n}) : n < \omega\} = \delta_*$

$(*)$ without loss of generality in $\boxdot$ we can add
$(i)'_2$ we have $\langle \beta_\eta : \eta \in {}^{\omega>}\lambda\rangle$ such that $\beta_\eta \in \mathfrak{h} \cap N_\eta$ but $\ell g(\eta) = n+1 \Rightarrow \beta_\eta > \sup\{N_\rho \cap \mathfrak{h} : \rho \in {}^{\omega>}(\sup \mathrm{Rang}(\nu_{\eta\restriction n}))\}$

$(*)$ let
- (a) $\beta_\varepsilon = \sup\{\beta_\eta : \eta \in {}^{\omega>}\varepsilon\}$ for $\varepsilon < \lambda$ so $\langle \beta_\varepsilon : \varepsilon < \lambda\rangle$ is increasing
- (b) $\beta_* = \sup\{\beta_\varepsilon : \varepsilon < \lambda\}$
- (c) $\gamma_\rho = \sup(N_\eta \cap \beta_*)$

$(*)$ if $\eta_1 \neq \eta_2 \in \lim(\mathscr{S}_2)$, so $\eta_1 \cap \eta_2 = \eta \in {}^{\omega>}\lambda$, then p_* is $(N_\eta, \mathscr{T})$-generic

$(*)$ for $\rho \triangleleft \eta \in \lim(\mathscr{S}_2)$ let $w_{\eta,\rho} = \{p\restriction\gamma_\rho : p \in N_\eta \cap \mathscr{T}, \mathrm{level}_{\mathscr{T}}(p) > \gamma_\rho\}$

$(*)$ we can choose $\bar N$ such that: if η, η_1, η_2 as above and $p \in w_{\eta_1,\eta} \cap w_{\eta_2,\eta}$ is contained in w^*_η (choose $N^-_\eta \prec N^+_\eta$, etc.)

<u>Toward the case $\mathfrak{h} \geq \lambda$ is similar:</u>

Let $\bar M^* = \langle M^*_\gamma : \gamma < \lambda\rangle$ is $\prec$-increasing continuous, $\|M^*_\gamma\| < \mathfrak{h}, \bar M^*\restriction(\gamma+1) \in M^*_{\gamma+1} \prec (\mathscr{H}(\chi), \in)$. Use $\langle C_\delta : \delta \in \mathfrak{S}^\lambda_{\aleph_0}\rangle$ guess clubs.

Choose $\delta_* \in S^\lambda_{\aleph_0}$. Choose countable $N^*_1 \prec (\mathscr{H}(\chi), \in), \{\bar M^*, \delta_*\} \in N_0$. Choose similarly N^*_2 such that $N^*_1 \subset N^*_2$. Now by the game produce $\langle (N^\ell_\eta, \nu_\eta) : \eta \in {}^{\omega>}\lambda\rangle$.

<u>Old Proof</u>:

$\circledast$ it suffices to prove that for $(q, \mathscr{T}')$ as above
$\boxtimes$ $q \Vdash$ "for some $\eta \in \lim_\omega(\mathscr{T}')$, the tree $\mathscr{T}_\eta := \mathscr{T} \cap N_\eta$ which has a set of levels $u_\eta = N_\eta \cap \mathfrak{h}$, has no u_η-branch with an upper bound in $\mathscr{T}$".

However

$(*)$ without loss of generality for every $\eta \in \lim_\omega(\mathscr{T}')$, even new one, $\gamma_\eta = \sup(N_\eta \cap \mathfrak{h})$ is the same γ_* (as we can replace); moreover in $\boxdot$ we can add
$\boxdot^+$
- (b) if $\mathfrak{h} < \lambda$ and $\eta \in {}^\omega\lambda$ and $\alpha_1 < \alpha_2 < \lambda$ and $t_1 \in N_{\eta\hat{}<\alpha_1>} \cap \mathscr{T}\backslash N_\eta, t_2 \in N_{\eta\hat{}<\alpha_2>\rangle} \cap \mathscr{T}\backslash N_\eta$ then in $\mathscr{T}, t_1, t_2$ are $<_I$-incomparable
- $(b)'$ if $\mathfrak{h} \geq \lambda$, if $(\alpha_1, \alpha_2, t_1, t_2)$ fails the above then t_1, t_2 has a common $<_{\mathscr{T}}$-upper bound in $N_\eta \cap \mathscr{T}$.

<u>Case 1</u>: $\mathfrak{h} < \lambda$

So $\boxtimes$ holds, just use any <u>new</u> $\eta \in \lim_\omega(\delta)$.

<u>Case 2</u>: Not Case 1

We just weaken $\boxtimes$: add "except branches which are included in a branch of $\mathscr{T}$ which belongs to N_η, i.e. $N_{\eta\restriction n}$ for some $n < \omega$".

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Discussion 3.3. Second, how strong is that assumption $\otimes$? Well, e.g. Cohen $*$ Levy$(\aleph_1, 2^{\aleph_0})$ fail it but

$\boxplus$ any c.c.c. forcing is O.K.

Also we can weaken it

$\otimes$ if $\underset{\sim}{\eta}$ in $N_{<>}$ is a $\mathbb{Q}$-name of a new real, <u>then</u> for some perfect embedding h of ${}^{\omega>}2$ into ${}^{\omega>}(\omega_2)$ (so $h(\eta) \triangleleft h(\eta\char`\^\langle\ell\rangle)$ for $\ell = 1,2, h(\eta\char`\^\langle 0\rangle), h(\eta^1\langle 1\rangle)$ are $\triangleleft$-incomparable let $\underset{\sim}{\eta}' = h(\underset{\sim}{\eta})$, i.e. $\cup\{h(\underset{\sim}{\eta}\restriction n) : n < \omega\}$ so $\Vdash \underset{\sim}{\eta}' \in \lim(\mathscr{T}')$ where $\mathscr{T}' = \{\nu : \nu \triangleleft h(\rho)$ so $h \in \mathbf{V}$ for some $\rho \in {}^{\omega>}2\}$ and we demand: <u>there is</u> $q \in \mathbb{Q}$ such that q is "$(N_{\eta'}, \mathbb{Q})$-generic".

The proof is as before.

<u>Moved 09.11.25 from the proof of 2.3 trying to prove $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{R})$:</u>
(Problem: why N_2^* old - rethink)
So without loss of generality $\mathbb{R} \subseteq \mathscr{H}_{<\aleph_1}(\lambda)$ and $\lambda \geq \aleph_0$. So toward contradiction assume $q_* \in \mathbb{Q}$ and

$(*)_1$ $q_* \Vdash_{\mathbb{Q}}$ "$\mathbb{R}$ is not proper".

So letting χ be large enough and

$(*)_2$ (a) $q_* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{N}_1 \prec (\mathscr{H}(\chi)^{\mathbf{V}[\underset{\sim}{G}]}, \mathscr{H}(\chi)^{\mathbf{V}}, \in)$ is countable hence
(b) $\underset{\sim}{N}_0 = \underset{\sim}{N}_1 \restriction \mathscr{H}(\chi)^{\mathbf{V}}$ is a countable elementary submodel of $(\mathscr{H}(\chi), \in)$
(c) let $\underset{\sim}{N}_2 = \underset{\sim}{N}_0 \restriction \mathscr{H}_{<\aleph_1}(\lambda)$ even expanding by $\mathbb{R}$, note $\mathbb{R} \cap \underset{\sim}{N}_0 = \mathbb{R} \cap \underset{\sim}{N}_2$ <u>and</u> $\underset{\sim}{r}_* \in \underset{\sim}{N}_1 \cap \mathbb{R}$ and no r' satisfying $r' \leq_{\mathbb{R}} r_*$ is $(\underset{\sim}{N}_1, \mathbb{R})$-generic".

Possibly increasing q_* without loss of generality

$(*)_3$ $\underset{\sim}{r}_* = r_*$ a member of $\mathbb{R}$ as a forcing.

Similarly without loss of generality

$(*)_4$?? $\underset{\sim}{N}_2 = N_2^*$, i.e. N_2^* is an object from $\mathbf{V}$ and $(N_2, \mathbb{R}) \prec (\mathscr{H}_{<\aleph_1}(\lambda), \in, \mathbb{R})$.

So as $\mathbb{R}$ is $(< \aleph_1)$-complete in $\mathbf{V}$, letting Gen $= \{\mathbf{g} \subseteq N_2^* \cap \mathbb{R} : g$ is directed and $r_* \in \mathbf{g}\}$ in $\mathbf{V}$

$(*)_5$ if $r \in \mathbb{R}$ is $\leq_{\mathbb{R}}$-above r_* then $\{r' \in \mathbb{R} \cap N_0^* : r' \leq_{\mathbb{R}} r\} \in$ Gen.

Hence

$(*)_6$ $q_* \Vdash_{\mathbb{Q}}$ "for no $\mathbf{g} \in$ Gen for every $\mathscr{I} \in \underset{\sim}{N}_0$ which is a dense open subset of $\mathbb{R}$ do we have $\mathbf{g} \cap \mathscr{I} \neq \emptyset$".

By clause (β) of the claim which we have proved

$(*)_7$ $q_* \Vdash_{\mathbb{Q}}$ "for no $\mathbf{g} \in$ Gen for every $\mathscr{I} \in \underset{\sim}{N}_1 \cap \mathbf{V}$ which is a dense open subset of $\mathbb{R}$ do we have $\mathbf{g} \cap \mathscr{I} \neq \emptyset$".

As $\mathbb{Q}$ is c.c.c. (or as it is proper and we can increase q_*) there is N_3 such that

$(*)_8$ $N_3 \in \mathbf{V}, N_3 \prec (\mathscr{H}(\chi), \in)$ is countable and $q_* \Vdash$ "$\underset{\sim}{N}_1 \subseteq N_3$".

But obviously

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$(*)_9$ if $\mathscr{I} \in \mathbf{V}$ and $q_* \nVdash$ "$\mathscr{I} \notin \underset{\sim}{N}_2$" is a dense open subset of $\mathbb{R}$ then

(a) $\mathscr{I} \in N_3$

(b) $\mathscr{I}$ is a dense open subset of $\mathbb{R}$

(c) $\mathscr{I} \cap N_2^*$ is a dense open subset of $\mathbb{R} \cap N_2^*$.

We can finish as

$(*)_{10}$ there is $\mathbf{g} \in$ Gen such that: if $\mathscr{I}$ satisfies (a),(b),(c) of $(*)_3$ then $\mathbf{g} \cap \mathscr{I} \neq \emptyset$.

[Why? As N_3 is countable.]

We get contradiction so we are done.

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modified:2013-10-06

(960) revision:2013-10-05