On the p-rank of $\mathrm{Ext}(A, B)$ for countable abelian groups A and B

Lutz Strüngmann and Saharon Shelah

Abstract. In this note we show that the p-rank of $\mathrm{Ext}(A, B)$ for countable torsion-free abelian groups A and B is either countable or the size of the continuum.

1. Introduction

The structure of $\mathrm{Ext}(A, B)$ for torsion-free abelian groups A has received much attention in the literature. In particular in the case of $B = \mathbb{Z}$ complete characterizations are available in various models of ZFC (see [**EkMe**, Sections on the structure of Ext], [**ShSt1**], and [**ShSt2**] for references). It is easy to see that $\mathrm{Ext}(A, B)$ is always a divisible group for any torsion-free group A. Hence it is of the form

$$\mathrm{Ext}(A, B) = \bigoplus_{p \in \Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for some cardinals ν_p, ν_0 $(p \in \Pi)$ which are uniquely determined. Here $\mathbb{Z}(p^\infty)$ is the p-Prüfer group and $\mathbb{Q}$ is the group of rational numbers. Thus, the obvious question that arises is which sequences $(\nu_0, \nu_p : p \in \Pi)$ can appear as the cardinal invariants of $\mathrm{Ext}(A, B)$ for some (which) torsion-free abelian group A and arbitrary abelian group B? As mentioned above for $B = \mathbb{Z}$ the answer is pretty much known but for general B there is little known so far. Some results were obtained in [**Fr**] and [**FrSt**] for countable abelian groups A and B. However, one essential question was left open, namely if the situation is similar to the case $B = \mathbb{Z}$ when it comes to p-ranks. It was conjectured that the p-rank of $\mathrm{Ext}(A, B)$ can only be either countable or the size of the continuum whenever A and B are countable. Here we prove

This is number 968 in the first author's list of publication. The authors would like to thank the German Israeli Foundation for their support project No. I-963-98.6/2007 ; the second author was also supported by project STR 627/1-6 of the German Research Foundation DFG

2000 Mathematics Subject Classification. Primary 20K15, 20K20, 20K35, 20K40; Secondary 18E99, 20J05.

that the conjecture is true.

Our notation is standard and we write maps from the left. If H is a pure subgroup of the abelian group G, then we will write $H \subseteq_* G$. The set of natural primes is denoted by Π. For further details on abelian groups and set-thoretic methods we refer to [**Fu**] and [**EkMe**].

2. Proof of the conjecture on the p-rank of Ext

It is well-known that for torsion-free abelian groups A the group of extensions $\mathrm{Ext}(A,B)$ is divisible for any abelian group B. Hence it is of the form

$$\mathrm{Ext}(A,B) = \bigoplus_{p\in\Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for some cardinals ν_p, ν_0 $(p \in \Pi)$ which are uniquely determined.
The invariant $r_p(\mathrm{Ext}(A,B)) := \nu_p$ is called the *p-rank* of $\mathrm{Ext}(A,B)$ while $r_0(\mathrm{Ext}(A,B)) := \nu_0$ is called the *torsion-free rank* of $\mathrm{Ext}(A,B)$. The following was shown in [**FrSt**] and gives an almost complete description of the structure of $\mathrm{Ext}(A,B)$ for countable torsion-free A and B. Recall that the *nucleus* G_0 of a torsion-free abelian group G is the largest subring R of $\mathbb{Q}$ such that G is a module over R.

Proposition 2.1. *Let A and B be torsion-free groups with A countable and $|B| < 2^{\aleph_0}$. Then either*

i) *$r_0(\mathrm{Ext}(A,B)) = 0$ and $A \otimes B_0$ is a free B_0-module <u>or</u>*
ii) *$r_0(\mathrm{Ext}(A,B)) = 2^{\aleph_0}$.*

Recall that for a torsion-free abelian group G the *p-rank* $r_p(G)$ is defined to be the $\mathbb{Z}/p\mathbb{Z}$-dimension of the vectorspace G/pG.

Proposition 2.2. *The following holds true:*

i) *If A and B are countable torsion-free abelian groups and A has finite rank then $r_p(\mathrm{Ext}(A,B)) \leq \aleph_0$. Moreover, all cardinals $\leq \aleph_0$ can appear as the p-rank of some $\mathrm{Ext}(A,B)$.*
ii) *If A and B are countable torsion-free abelian groups with $\mathrm{Hom}(A,B) = 0$ then*

$$r_p(\mathrm{Ext}(A,B)) = \begin{cases} 0 & \text{if } r_p(A) = 0 \text{ or } r_p(B) = 0 \\ r_p(A)\cdot r_p(B) & \text{if } 0 < r_p(A), r_p(B) < \aleph_0 \\ \aleph_0 & \text{if } 0 < r_p(A) < \aleph_0 \text{ and } r_p(B) = \aleph_0 \\ 2^{\aleph_0} & \text{if } r_p(A) = \aleph_0 \text{ and } 0 < r_p(B) \leq \aleph_0 \end{cases}$$

.

Recall the following way of calculating the p-rank of $\mathrm{Ext}(A,B)$ for torsion-free groups A and B (see [**EkMe**, page 389]).

Lemma 2.3. *For a torsion-free abelian group A and an arbitrary abelian group B let φ^p be the map that sends $\psi \in \mathrm{Hom}(A,B)$ to $\pi \circ \psi$ with π :*

$B \longrightarrow B/pB$ *the canonical epimorphism. Then*

$$r_p(\mathrm{Ext}(A,B)) = dim_{\mathbb{Z}/p\mathbb{Z}}(\mathrm{Hom}(A,B/pB)/\varphi^p(\mathrm{Hom}(A,B))).$$

We now need some preparation from descriptive set-theory in order to prove our main result. Recall that a subset of a topological space is called *perfect* if it is closed and contains no isolated points. Moreover, a subset X of a Polish space V is *analytic* if there is a Polish space Y and a Borel (or closed) set $B \subseteq V \times Y$ such that X is the projection of B; that is,

$$X = \{v \in V | (\exists y \in Y)\langle v, y\rangle \in B\}.$$

PROPOSITION 2.4. *If E is an analytic equivalence relation on $\Gamma = \{X : X \subseteq \omega\}$ that satisfies*

(†) *if $X, Y \subseteq \omega, n \notin Y, X = Y \cup \{n\}$ then X and Y are not E-equivalent*

then there is a perfect subset T of Γ of pairwise nonequivalent $X \subseteq \omega$.

PROOF. See [**Sh**, Lemma 13]. □

We are now in the position to prove our main result.

THEOREM 2.5. *Let A be a countable torsion-free abelian group and B an arbitrary countable abelian group. Then either*

- $r_p(\mathrm{Ext}(A,B)) \leq \aleph_0$ <u>*or*</u>
- $r_p(\mathrm{Ext}(A,B)) = 2^{\aleph_0}$.

PROOF. The proof uses descriptive set theory, relies on [**Sh**, Lemma 13] and is inspired by [**HaSh**, Lemma 2.2]. By the above Lemma 2.3 the p-rank of $\mathrm{Ext}(A,B)$ is the dimension κ of the $\mathbb{Z}/p\mathbb{Z}$ vector space $L := \mathrm{Hom}(A,B/pB)/\varphi^p(\mathrm{Hom}(A,B))$ where φ^p is the natural map. We choose a basis $\{[\varphi_\alpha] \mid \alpha < \kappa\}$ of L with $\varphi_\alpha \in \mathrm{Hom}(A,B/pB)$ and assume that $\aleph_0 < \kappa$. Note that $[\varphi_\alpha] \neq 0$ means, that $\varphi_\alpha : A \to B/pB$ has no lifting to an element $\psi \in \mathrm{Hom}(A,B)$ such that $\varphi_\alpha = \pi \circ \psi$.
Now let $A = \bigcup_{i<\omega} A_i$ with $rk(A_i)$ finite. By a pigeonhole-principle there are $\alpha_1 \neq \beta_1$ such that $\varphi_{\alpha_1} \restriction A_1 = \varphi_{\beta_1} \restriction A_1$. We define $\psi_1 := \varphi_{\alpha_1} - \varphi_{\beta_1}$ and obtain $A_1 \subseteq Ker(\psi_1)$. Obviously, ψ_1 has no lifting because $\{[\varphi_\alpha] \mid \alpha < \kappa\}$ is a basis of L, hence $[\psi_1] \neq 0$. Since $\alpha_1 \neq \beta_1$ there exists $x_1 \in A$ satisfying $\varphi_{\alpha_1}(x_1) \neq \varphi_{\beta_1}(x_1)$. Let n be minimal with $x_1 \in A_n \setminus A_{n-1}$. Because A_n has finite rank we similarly get $\alpha_2 \neq \beta_2$ such that $\varphi_{\alpha_2} \restriction A_n = \varphi_{\beta_2} \restriction A_n$ and define $\psi_2 := \varphi_{\alpha_2} - \varphi_{\beta_2}$ with $A_n \subseteq Ker(\psi_2)$. Clearly we have $\psi_1 \neq \psi_2$ since $\psi_1(x_1) \neq 0 = \psi_2(x_1)$ and $[\psi_2] \neq 0$. Continuing this construction we get $\aleph_0$ pairwise different $\psi_n \in \mathrm{Hom}(A,B/pB)$ with $[\psi_n] \neq 0$.
Now let $\eta \in {}^{\omega}2$, which means that η is a countable $\{0,1\}$-sequence and choose $\psi_\eta := \sum_{n \in supp(\eta)} \psi_n$ where $supp(\eta) := \{n \in \omega \mid \eta(n) = 1\}$. This is well-defined since for any $a \in A$, the sum $\psi_\eta(a)$ consists of only finitely many summands because there is $n \in \mathbb{N}$ such that $a \in A_n$ and A_n is in the kernel of ψ_m for sufficiently large m. Note that also $\psi_\eta \neq \psi_{\eta'}$ for all $\eta \neq \eta'$.

It remains to prove that the size of $\{[\psi_\eta] : \eta \in {}^\omega 2\}$ is $2^{\aleph_0}$ which then implies that $\kappa = 2^{\aleph_0}$.
We now define an equivalence relation on $\Gamma = \{X \subseteq \omega\}$ in the following way: $X \sim X'$ if and only if $[\psi_{\eta_X}] = [\psi_{\eta_{X'}}]$, where $\eta_X \in {}^\omega 2$ is the characteristic function of X. We claim that

(‡) for $X, X' \subseteq \omega$ and $n \notin X$ such that $X' = X \cup \{n\}$ we have $X \not\sim X'$.

But this is obvious since in this case $\psi_{\eta_X} - \psi_{\eta_{X'}} = -\psi_n$ and $[-\psi_n] \neq 0$, so $[\psi_{\eta_X}] \neq [\psi_{\eta_{X'}}]$. We claim that the above equivalence relation is analytic. It then follows from Proposition 2.4 that there is a perfect subset of Γ of pairwise nonequivalent $X \subseteq \omega$ and hence there are $2^{\aleph_0}$ distinct equivalence classes for $\sim$. Thus the p-rank of $\mathrm{Ext}(A, B)$ must be the size of the continuum as claimed.
In order to see that $\sim$ is analytic recall that both, the cantor space ${}^\omega 2$ (and hence Γ) and the Baire space ${}^\omega\omega$ are Polish spaces with the natural topologies. For $\sim$ to be analytic we therefore have to show that $\Delta = \{(X, X') : X, X' \in \Gamma \textit{ and } X \sim X'\}$ is an analytic subset of the product space $\Gamma \times \Gamma$. We enumerate $A = \{a_n : n \in \omega\}$ and $B = \{b_n : n \in \omega\}$ and let $C_1 \subseteq {}^\omega\omega$ be the set of all $\rho \in {}^\omega\omega$ such that ρ induces a homomorphism $f_\rho \in \mathrm{Hom}(A, B/pB)$, i.e. the map $f_\rho : A \to B/pB$ sending a_n onto $b_{\rho(n)} + pB$ is a homomorphism. C_2 is defined similarly replacing $\mathrm{Hom}(A, B/pB)$ by $\mathrm{Hom}(A, B)$, i.e. $\nu \in C_2$ if the map $g_\nu : A \to B$ with $a_n \mapsto b_{\nu(n)}$ is a homomorphism. Clearly C_1 and C_2 are closed subsets of ${}^\omega\omega$. Put $P = \{(\rho_1, \rho_2, \nu) : \rho_1, \rho_2 \in C_1; \nu \in C_2 \text{ and } f_{\rho_1} - f_{\rho_2} = \varphi^p(g_\nu)\}$. Then P is a closed subset of ${}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$. Now the construction above gives a continuous function Θ from ${}^\omega 2$ to ${}^\omega\omega$ by sending η to $\rho = \Theta(\eta)$ where ρ is induced by the homomorphism ψ_η. Then $X \sim X'$ if and only if there is $\nu \in C_2$ such that $(\Theta(\eta_X), \Theta(\eta_{X'}), \nu) \in P$ and thus $\sim$ is an analytic equivalence relation. □

References

[EkMe] **P.C. Eklof and A. Mekler**, *Almost Free Modules, Set- Theoretic Methods (revised edition)*, Amsterdam, New York, North-Holland, Math. Library (2002).

[Fr] **S. Friedenberg**, *Extensions of Abelian Groups*, Südwestdeutscher Verlag für Hochschulschrift (2010).

[FrSt] **S. Friedenberg and L. Strüngmann**, *The structure of* $\mathrm{Ext}(A, B)$ *for countable abelian groups* A *and* B, submitted (2010).

[Fu] **L. Fuchs**, *Infinite Abelian Groups, Vol. I and II*, Academic Press (1970 and 1973).

[HaSh] **L. Harrington and S. Shelah**, *Counting equivalence classes for co-κ-Souslin relation*, Proc. Conf. Prague, 1980, Logic Colloq. North-Holland, 1982, 147-152.

[Sh] **S. Shelah**, *Can the fundamental (homotopy) group of a space be the rationals?*, Proc. AMS **103** (1988), 627–632.

[ShSt1] **S. Shelah and L. Strüngmann**, *A characterization of* $\mathrm{Ext}(G, \mathbb{Z})$ *assuming (V=L)*, Fundamenta Math. **193** (2007), 141-151.

[ShSt2] **S. Shelah and L. Strüngmann**, *On the p-rank of* $\mathrm{Ext}(G, \mathbb{Z})$ *in certain models of ZFC*, Algebra and Logic **46** (2007), 200-215.

Faculty of Mathematics, University of Duisburg–Essen, 45117 Essen, Germany
E-mail address: lutz.struengmann@uni-due.de

Department of Mathematics, The Hebrew University of Jerusalem, Israel and Rutgers University, New Brunswick, NJ U.S.A.
E-mail address: shelah@math.huji.ac.il