BOREL CONJECTURE AND DUAL BOREL CONJECTURE

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Abstract. We show that it is consistent that the Borel Conjecture and the dual Borel Conjecture hold simultaneously.

Introduction

History. A set X of reals[1] is called "strong measure zero" (smz), if for all functions $f : \omega \to \omega$ there are intervals I_n of measure $\leq 1/f(n)$ covering X. Obviously, a smz set is a null set (i.e., has Lebesgue measure zero), and it is easy to see that the family of smz sets forms a σ-ideal and that perfect sets (and therefore uncountable Borel or analytic sets) are not smz.

At the beginning of the 20th century, Borel [Bor19, p. 123] conjectured:

> Every smz set is countable.

This statement is known as the "Borel Conjecture" (BC). In the 1970s it was proved that BC is *independent*, i.e., neither provable nor refutable.

Let us very briefly comment on the notion of independence: A sentence φ is called independent of a set T of axioms, if neither φ nor $\neg\varphi$ follows from T. (As a trivial example, $(\forall x)(\forall y) x \cdot y = y \cdot x$ is independent from the group axioms.) The set theoretic (first order) axiom system ZFC (Zermelo Fraenkel with the axiom of choice) is considered to be the standard axiomatization of all of mathematics: A mathematical proof is generally accepted as valid iff it can be formalized in ZFC. Therefore we just say "φ is independent" if φ is independent of ZFC. Several mathematical statements are independent, the earliest and most prominent example is Hilbert's first problem, the Continuum Hypothesis (CH).

BC is independent as well: Sierpiński [Sie28] showed that CH implies ¬BC (and, since Gödel showed the consistency of CH, this gives us the consistency of ¬BC). Using the method of forcing, Laver [Lav76] showed that BC is consistent.

Galvin, Mycielski and Solovay [GMS73] proved the following conjecture of Prikry:

> $X \subseteq 2^\omega$ is smz if and only if every comeager (dense G_δ) set contains a translate of X.

Prikry also defined the following dual notion:

> $X \subseteq 2^\omega$ is called "strongly meager" (sm) if every set of Lebesgue measure 1 contains a translate of X.

The dual Borel Conjecture (dBC) states:

> Every sm set is countable.

Prikry noted that CH implies ¬dBC and conjectured dBC to be consistent (and therefore independent), which was later proved by Carlson [Car93].

Numerous additional results regarding BC and dBC have been proved: The consistency of variants of BC or of dBC, the consistency of BC or dBC together with certain assumptions on cardinal characteristics,

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[1] In this paper, we use 2^ω as the set of reals. ($\omega = \{0, 1, 2, \ldots\}$.) By well-known results both the definition and the theorem also work for the unit interval $[0, 1]$ or the torus $\mathbb{R}/\mathbb{Z}$. Occasionally we also write "x is a real" for "$x \in \omega^\omega$".

etc. See [BJ95, Ch. 8] for several of these results. In this paper, we prove the consistency (and therefore independence) of BC+dBC (i.e., consistently BC and dBC hold simultaneously).

The problem. The obvious first attempt to force BC+dBC is to somehow combine Laver's and Carlson's constructions. However, there are strong obstacles:

Laver's construction is a countable support iteration of Laver forcing. The crucial points are:

- Adding a Laver real makes every old uncountable set X non-smz.
- And this set X remains non-smz after another forcing P, provided that P has the "Laver property".

So we can start with CH and use a countable support iteration of Laver forcing of length ω_2. In the final model, every set X of reals of size $\aleph_1$ already appeared at some stage $\alpha < \omega_2$ of the iteration; the next Laver real makes X non-smz, and the rest of the iteration (as it is a countable support iteration of proper forcings with the Laver property) has the Laver property, and therefore X is still non-smz in the final model.

Carlson's construction on the other hand adds ω_2 many Cohen reals in a finite support iteration (or equivalently: finite support product). The crucial points are:

- A Cohen real makes every old uncountable set X non-sm.
- And this set X remains non-sm after another forcing P, provided that P has precaliber $\aleph_1$.

So we can start with CH, and use more or less the same argument as above: Assume that X appears at $\alpha < \omega_2$. Then the next Cohen makes X non-sm. It is enough to show that X remains non-sm at all subsequent stages $\beta < \omega_2$. This is guaranteed by the fact that a finite support iteration of Cohen reals of length $< \omega_2$ has precaliber $\aleph_1$.

So it is unclear how to combine the two proofs: A Cohen real makes all old sets smz, and it is easy to see that whenever we add Cohen reals cofinally often in an iteration of length, say, ω_2, all sets of any intermediate extension will be smz, thus violating BC. So we have to avoid Cohen reals,[2] which also implies that we cannot use finite support limits in our iterations. So we have a problem even if we find a replacement for Cohen forcing in Carlson's proof that makes all old uncountable sets X non-sm and that does not add Cohen reals: Since we cannot use finite support, it seems hopeless to get precaliber $\aleph_1$, an essential requirement to keep X non-sm.

Note that it is the *proofs* of BC and dBC that are seemingly irreconcilable; this is not clear for the models. Of course Carlson's model, i.e., the Cohen model, cannot satisfy BC, but it is not clear whether maybe already the Laver model could satisfy dBC. (It is even still open whether a single Laver forcing makes every old uncountable set non-sm.) Actually, Bartoszyński and Shelah [BS03] proved that the Laver model does satisfy the following weaker variant of dBC (note that the continuum has size $\aleph_2$ in the Laver model):

Every sm set has size less than the continuum.

In any case, it turns out that one *can* reconcile Laver's and Carlson's proof, by "mixing" them "generically", resulting in the following theorem:

Theorem. *If ZFC is consistent, then ZFC+BC+dBC is consistent.*

Prerequisites. To understand anything of this paper, the reader

- should have some experience with finite and countable support iteration, proper forcing, $\aleph_2$-cc, σ-closed, etc.,
- should know what a quotient forcing is,
- should have seen some preservation theorem for proper countable support iteration,
- should have seen some tree forcings (such as Laver forcing).

To understand everything, additionally the following is required:

- The "case A" preservation theorem from [She98], more specifically we build on the proof of [Gol93] (or [GK06]).
- In particular, some familiarity with the property "preservation of randoms" is recommended. We will use the fact that random and Laver forcing have this property.

[2] An iteration that forces dBC without adding Cohen reals was given in [BS10], using non-Cohen oracle-cc.

- We make some claims about (a rather special case of) ord-transitive models in Section 3.A. The readers can either believe these claims, or check them themselves (by some rather straightforward proofs), or look up the proofs (of more general settings) in [She04] or [Kel].

From the theory of strong measure zero and strongly meager, we only need the following two results (which are essential for our proofs of BC and dBC, respectively):

- Pawlikowski's result from [Paw96a] (which we quote as Theorem 0.2 below), and
- Theorem 8 of Bartoszyński and Shelah's [BS10] (which we quote as Lemma 2.1).

We do not need any other results of Bartoszyński and Shelah's paper [BS10]; in particular we do not use the notion of non-Cohen oracle-cc (introduced in [She06]); and the reader does not have to know the original proofs of Con(BC) and Con(dBC), by Laver and Carlson, respectively.

The third author claims that our construction is more or less the same as a non-Cohen oracle-cc construction, and that the extended version presented in [She10] is even closer to our preparatory forcing.

Notation and some basic facts on forcing, strongly meager (sm) and strong measure zero (smz) sets. We call a lemma "Fact" if we think that no proof is necessary — either because it is trivial, or because it is well known (even without a reference), or because we give an explicit reference to the literature.

Stronger conditions in forcing notions are smaller, i.e., $q \leq p$ means that q is stronger than p.

Let $P \subseteq Q$ be forcing notions. (As usual, we abuse notation by not distinguishing between the underlying set and the quasiorder on it.)

- For $p_1, p_2 \in P$ we write $p_1 \perp_P p_2$ for "p_1 and p_2 are incompatible". Otherwise we write $p_1 \parallel_P p_2$. (We may just write $\perp$ or $\parallel$ if P is understood.)
- $q \leq^* p$ (or: $q \leq^*_P p$) means that q forces that p is in the generic filter, or equivalently that every $q' \leq q$ is compatible with p. And $q =^* p$ means $q \leq^* p \ \wedge \ p \leq^* q$.
- P is separative, if $\leq$ is the same as $\leq^*$, or equivalently, if for all $q \leq p$ with $q \neq p$ there is an $r \leq p$ incompatible with q. Given any P, we can define its "separative quotient" Q by first replacing (in P) $\leq$ by $\leq^*$ and then identifying elements p, q whenever $p =^* q$. Then Q is separative and forcing equivalent to P.
- "*P is a subforcing of Q*" means that the relation $\leq_P$ is the restriction of $\leq_Q$ to P.
- "*P is an incompatibility-preserving subforcing of Q*" means that P is a subforcing of Q and that $p_1 \perp_P p_2$ iff $p_1 \perp_Q p_2$ for all $p_1, p_2 \in P$.

Let additionally M be a countable transitive[3] model (of a sufficiently large subset of ZFC) containing P.

- "P is an M-complete subforcing of Q" (or: $P \lessdot_M Q$) means that P is a subforcing of Q and: if $A \subseteq P$ is in M a maximal antichain, then it is a maximal antichain of Q as well. (Or equivalently: P is an incompatibility-preserving subforcing of Q and every predense subset of P in M is predense in Q.) Note that this means that every Q-generic filter G over V induces a P-generic filter over M, namely $G^M := G \cap P$ (i.e., every maximal antichain of P in M meets $G \cap P$ in exactly one point). In particular, we can interpret a P-name τ in M as a Q-name. More exactly, there is a Q-name τ' such that $\tau'[G] = \tau[G^M]$ for all Q-generic filters G. We will usually just identify τ and τ'.
- Analogously, if $P \in M$ and $i : P \to Q$ is a function, then i is called an M-complete embedding if it preserves $\leq$ (or at least $\leq^*$) and $\perp$ and moreover: If $A \in M$ is predense in P, then $i[A]$ is predense in Q.

There are several possible characterizations of sm ("strongly meager") and smz ("strong measure zero") sets; we will use the following as definitions:

A set X is not sm if there is a measure 1 set into which X cannot be translated; i.e., if there is a null set Z such that $(X + t) \cap Z \neq \emptyset$ for all reals t, or, in other words, $Z + X = 2^\omega$. To summarize:

(0.1) $\quad X$ is *not* sm iff there is a Lebesgue null set Z such that $Z + X = 2^\omega$.

We will call such a Z a "witness" for the fact that X is not sm (or say that Z witnesses that X is not sm).

The following theorem of Pawlikowski [Paw96a] is central for our proof[4] that BC holds in our model:

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[3] We will also use so-called ord-transitive models, as defined in Section 3.A.

[4] We thank Tomek Bartoszyński for pointing out Pawlikowski's result to us, and for suggesting that it might be useful for our proof.

Theorem 0.2. *$X \subseteq 2^\omega$ is smz iff $X + F$ is null for every closed null set F.*
Moreover, for every dense G_δ set H we can construct *(in an absolute way) a closed null set F such that for every $X \subseteq 2^\omega$ with $X + F$ null there is $t \in 2^\omega$ with $t + X \subseteq H$.*

In particular, we get:

(0.3) X is *not* smz iff there is a closed null set F such that $X+F$ has positive outer Lebesgue measure.

Again, we will say that the closed null set F "witnesses" that X is not smz (or call F a witness for this fact).

Annotated contents.

An informal overview of the proof, including two illustrations, can be found at `http://arxiv.org/abs/1112.4424/`.

1. Ultralaver forcing

In this section, we define the family of *ultralaver forcings* $\mathbb{L}_{\bar{D}}$, variants of Laver forcing which depend on a system $\bar{D}$ of ultrafilters.

In the rest of the paper, we will use the following properties of $\mathbb{L}_{\bar{D}}$. (And we will use *only* these properties. So readers who are willing to take these properties for granted could skip to Section 2.)

(1) $\mathbb{L}_{\bar{D}}$ is σ-centered, hence ccc.
(This is Lemma 1.2.)
(2) $\mathbb{L}_{\bar{D}}$ is separative.
(This is Lemma 1.3.)
(3) *Ultralaver kills smz:* There is a canonical $\mathbb{L}_{\bar{D}}$-name $\underaccent{\tilde}{\bar{\ell}}$ for a fast growing real in ω^ω called the ultralaver real. From this real, we can define (in an absolute way) a closed null set F such that $X+F$ is positive for all uncountable X in V (and therefore F witnesses that X is not smz, according to Theorem 0.2).
(This is Corollary 1.21.)
(4) Whenever X is uncountable, then $\mathbb{L}_{\bar{D}}$ forces that X is not "thin".
(This is Corollary 1.24.)
(5) If $(M,\in)$ is a countable model of ZFC* and if $\mathbb{L}_{\bar{D}^M}$ is an ultralaver forcing in M, then for any ultrafilter system $\bar{D}$ extending $\bar{D}^M$, $\mathbb{L}_{\bar{D}^M}$ is an M-complete subforcing of the ultralaver forcing $\mathbb{L}_{\bar{D}}$.
(This is Lemma 1.5.)
Moreover, the real $\underaccent{\tilde}{\bar{\ell}}$ of item (3) is so "canonical" that we get: If (in M) $\underaccent{\tilde}{\bar{\ell}}^M$ is the $\mathbb{L}_{\bar{D}^M}$-name for the $\mathbb{L}_{\bar{D}^M}$-generic real, and if (in V) $\underaccent{\tilde}{\bar{\ell}}$ is the $\mathbb{L}_{\bar{D}}$-name for the $\mathbb{L}_{\bar{D}}$-generic real, and if H is $\mathbb{L}_{\bar{D}}$-generic over V and thus $H^M := H \cap \mathbb{L}_{\bar{D}^M}$ is the induced $\mathbb{L}_{\bar{D}^M}$-generic filter over M, then $\underaccent{\tilde}{\bar{\ell}}[H]$ is equal to $\underaccent{\tilde}{\bar{\ell}}^M[H^M]$.
Since the closed null set F is constructed from $\underaccent{\tilde}{\bar{\ell}}$ in an absolute way, the same holds for F, i.e., the Borel codes $F[H]$ and $F[H^M]$ are the same.
(6) Moreover, given M and $\mathbb{L}_{\bar{D}^M}$ as above, and a random real r over M, we can choose $\bar{D}$ extending $\bar{D}^M$ such that $\mathbb{L}_{\bar{D}}$ forces that randomness of r is preserved (in a strong way that can be preserved in a countable support iteration).
(This is Lemma 1.30.)

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1.A. Definition of ultralaver.

Notation. We use the following fairly standard notation:

A *tree* is a nonempty set $p \subseteq \omega^{<\omega}$ which is closed under initial segments and has no maximal elements.[5] The elements ("nodes") of a tree are partially ordered by $\subseteq$.

For each sequence $s \in \omega^{<\omega}$ we write $\mathrm{lh}(s)$ for the length of s.

For any tree $p \subseteq \omega^{<\omega}$ and any $s \in p$ we write $\mathrm{succ}_p(s)$ for one of the following two sets:

$$\{k \in \omega : s^\frown k \in p\} \quad \text{or} \quad \{t \in p : (\exists k \in \omega)\ t = s^\frown k\}$$

and we rely on the context to help the reader decide which set we mean.

A *branch* of p is either of the following:

- A function $f : \omega \to \omega$ with $f \restriction n \in p$ for all $n \in \omega$.
- A maximal chain in the partial order $(p, \subseteq)$. (As our trees do not have maximal elements, each such chain C determines a branch $\bigcup C$ in the first sense, and conversely.)

We write $[p]$ for the set of all branches of p.

For any tree $p \subseteq \omega^{<\omega}$ and any $s \in p$ we write $p^{[s]}$ for the set $\{t \in p : t \supseteq s \text{ or } t \subseteq s\}$, and we write $[s]$ for either of the following sets:

$$\{t \in p : s \subseteq t\} \quad \text{or} \quad \{x \in [p] : s \subseteq x\}.$$

The stem of a tree p is the shortest $s \in p$ with $|\mathrm{succ}_p(s)| > 1$. (The trees we consider will never be branches, i.e., will always have finite stems.)

Definition 1.1.
- For trees q, p we write $q \le p$ if $q \subseteq p$ ("q is stronger than p"), and we say that "*q is a pure extension of p*" ($q \le_0 p$) if $q \le p$ and $\mathrm{stem}(q) = \mathrm{stem}(p)$.
- A filter system $\bar{D}$ is a family $(D_s)_{s\in\omega^{<\omega}}$ of filters on ω. (All our filters will contain the Fréchet filter of cofinite sets.) We write D_s^+ for the collection of D_s-positive sets (i.e., sets whose complement is not in D_s).
- We define $\mathbb{L}_{\bar{D}}$ to be the set of all trees p such that $\mathrm{succ}_p(t) \in D_t^+$ for all $t \in p$ above the stem.
- The generic filter is determined by the generic branch $\bar{\ell} = (\ell_i)_{i\in\omega} \in \omega^\omega$, called the *generic real*: $\{\bar{\ell}\} = \bigcap_{p\in G}[p]$ or equivalently, $\bar{\ell} = \bigcup_{p\in G} \mathrm{stem}(p)$.
- An ultrafilter system is a filter system consisting of ultrafilters. (Since all our filters contain the Fréchet filter, we only consider nonprincipal ultrafilters.)
- An *ultralaver forcing* is a forcing $\mathbb{L}_{\bar{D}}$ defined from an ultrafilter system. The generic real for an ultralaver forcing is also called the *ultralaver real*.

Recall that a forcing notion $(P, \le)$ is *σ-centered* if $P = \bigcup_n P_n$, where for all $n, k \in \omega$ and for all $p_1, \dots, p_k \in P_n$ there is $q \le p_1, \dots, p_k$.

Lemma 1.2. *All ultralaver forcings $\mathbb{L}_{\bar{D}}$ are σ-centered (hence ccc).*

Proof. Every finite set of conditions sharing the same stem has a common lower bound. □

Lemma 1.3. *$\mathbb{L}_{\bar{D}}$ is separative.*[6]

Proof. If $q \le p$, and $q \ne p$, then there is $s \in p \setminus q$. Now $p^{[s]} \perp q$. □

If each D_s is the Fréchet filter, then $\mathbb{L}_{\bar{D}}$ is Laver forcing (often just written $\mathbb{L}$).

1.B. *M*-complete embeddings.

Note that for all ultrafilter systems $\bar{D}$ we have:

(1.4) Two conditions in $\mathbb{L}_{\bar{D}}$ are compatible if and only if their stems are comparable and moreover, the longer stem is an element of the condition with the shorter stem.

Lemma 1.5. *Let M be countable.*[7] *In M, let $\mathbb{L}_{\bar{D}^M}$ be an ultralaver forcing. Let $\bar{D}$ be (in V) a filter system extending*[8] *$\bar{D}^M$. Then $\mathbb{L}_{\bar{D}^M}$ is an M-complete subforcing of $\mathbb{L}_{\bar{D}}$.*

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[5] Except for the proof of Lemma 1.5, where we also allow trees with maximal elements, and even empty trees.

[6] See page 3 for the definition.

[7] Here, we can assume that M is a countable transitive model of a sufficiently large finite subset ZFC* of ZFC. Later, we will also use ord-transitive models instead of transitive ones, which does not make any difference as far as properties of $\mathbb{L}_{\bar{D}}$ are concerned, as our arguments take place in transitive parts of such models.

[8] I.e., $D_s^M \subseteq D_s$ for all $s \in \omega^{<\omega}$.

Proof. For any tree[9] T, any filter system $\bar{E} = (E_s)_{s\in\omega^{<\omega}}$, and any $s_0 \in T$ we define a sequence $(T^\alpha_{\bar{E},s_0})_{\alpha\in\omega_1}$ of "derivatives" (where we may abbreviate $T^\alpha_{\bar{E},s_0}$ to T^α) as follows:

- $T^0 := T^{[s_0]}$.
- Given T^α, we let $T^{\alpha+1} := T^\alpha \setminus \bigcup\{[s] : s \in T^\alpha, s_0 \subseteq s, \mathrm{succ}_{T^\alpha}(s) \notin E_s^+\}$, where $[s] := \{t : s \subseteq t\}$.
- For limit ordinals $\delta > 0$ we let $T^\delta := \bigcap_{\alpha<\delta} T^\alpha$.

Then we have

(a) Each T^α is closed under initial segments. Also: $\alpha < \beta$ implies $T^\alpha \supseteq T^\beta$.
(b) There is an $\alpha_0 < \omega_1$ such that $T^{\alpha_0} = T^{\alpha_0+1} = T^\beta$ for all $\beta > \alpha_0$. We write T^∞ or $T^\infty_{\bar{E},s_0}$ for T^{α_0}.
(c) If $s_0 \in T^\infty_{\bar{E},s_0}$, then $T^\infty_{\bar{E},s_0} \in \mathbb{L}_{\bar{E}}$ with stem s_0.
Conversely, if $\mathrm{stem}(T) = s_0$, and $T \in \mathbb{L}_{\bar{E}}$, then $T^\infty = T$.
(d) If T contains a tree $q \in \mathbb{L}_{\bar{E}}$ with $\mathrm{stem}(q) = s_0$, then T^∞ contains $q^\infty = q$, so in particular $s_0 \in T^\infty$.
(e) Thus: T contains a condition in $\mathbb{L}_{\bar{E}}$ with stem s_0 iff $s_0 \in T^\infty_{\bar{E},s_0}$.
(f) The computation of T^∞ is absolute between any two models containing T and $\bar{E}$. (In particular, any transitive ZFC*-model containing T and $\bar{E}$ will also contain α_0.)
(g) Moreover: Let $T \in M$, $\bar{E} \in M$, and let $\bar{E}'$ be a filter system extending $\bar{E}$ such that for all s_0 and all $A \in \mathscr{P}(\omega) \cap M$ we have: $A \in (E_{s_0})^+$ iff $A \in (E'_{s_0})^+$. (In particular, this will be true for any $\bar{E}'$ extending $\bar{E}$, provided that each E_{s_0} is an M-ultrafilter.)
Then for each $\alpha \in M$ we have $T^\alpha_{\bar{E},s_0} = T^\alpha_{\bar{E}',s_0}$ (and hence $T^\alpha_{\bar{E}',s_0} \in M$). (Proved by induction on α.)

Now let $A = (p_i : i \in I) \in M$ be a maximal antichain in $\mathbb{L}_{\bar{D}^M}$, and assume (in V) that $q \in \mathbb{L}_{\bar{D}}$. Let $s_0 := \mathrm{stem}(q)$.

We will show that q is compatible with some p_i (in $\mathbb{L}_{\bar{D}}$). This is clear if there is some i with $s_0 \in p_i$ and $\mathrm{stem}(p_i) \subseteq s_0$, by (1.4). (In this case, $p_i \cap q$ is a condition in $\mathbb{L}_{\bar{D}}$ with stem s_0.)

So for the rest of the proof we assume that this is not the case, i.e.:

(1.6) There is no i with $s_0 \in p_i$ and $\mathrm{stem}(p_i) \subseteq s_0$.

Let $J := \{i \in I : s_0 \subseteq \mathrm{stem}(p_i)\}$. We claim that there is $j \in J$ with $\mathrm{stem}(p_j) \in q$ (which as above implies that q and p_j are compatible).

Assume towards a contradiction that this is not the case. Then q is contained in the following tree T:

$$T := (\omega^{<\omega})^{[s_0]} \setminus \bigcup_{j\in J} [\mathrm{stem}(p_j)]. \tag{1.7}$$

Note that $T \in M$. In V we have:

(1.8) The tree T contains a condition q with stem s_0.

So by (e) (applied in V), followed by (g), and again by (e) (now in M) we get:

(1.9) The tree T also contains a condition $p \in M$ with stem s_0.

Now p has to be compatible with some p_i. The sequences $s_0 = \mathrm{stem}(p)$ and $\mathrm{stem}(p_i)$ have to be comparable, so by (1.4) there are two possibilities:

(1) $\mathrm{stem}(p_i) \subseteq \mathrm{stem}(p) = s_0 \in p_i$. We have excluded this case in our assumption (1.6).
(2) $s_0 = \mathrm{stem}(p) \subseteq \mathrm{stem}(p_i) \in p$. So $i \in J$. By construction of T (see (1.7)), we conclude $\mathrm{stem}(p_i) \notin T$, contradicting $\mathrm{stem}(p_i) \in p \subseteq T$ (see 1.9). □

1.C. **Ultralaver kills strong measure zero.** The following lemma appears already in [Bla88, Theorem 9]. We will give a proof below in Lemma 1.35.

Lemma 1.10. *If A is a finite set, $\underset{\sim}{\alpha}$ an $\mathbb{L}_{\bar{D}}$-name, $p \in \mathbb{L}_{\bar{D}}$, and $p \Vdash \underset{\sim}{\alpha} \in A$, then there is $\beta \in A$ and a pure extension $q \leq_0 p$ such that $q \Vdash \underset{\sim}{\alpha} = \beta$.*

Definition 1.11. Let $\bar{\ell}$ be an increasing sequence of natural numbers. We say that $X \subseteq 2^\omega$ is *smz with respect to* $\bar{\ell}$, if there exists a sequence $(I_k)_{k\in\omega}$ of basic intervals of 2^ω of measure $\leq 2^{-\ell_k}$ (i.e., each I_k is of the form $[s_k]$ for some $s_k \in 2^{\ell_k}$) such that $X \subseteq \bigcap_{m\in\omega} \bigcup_{k\geq m} I_k$.

Remark 1.12. It is well known and easy to see that the properties

[9] Here we also allow empty trees, and trees with maximal nodes.

- For all $\bar{\ell}$ there exists exists a sequence $(I_k)_{k\in\omega}$ of basic intervals of 2^ω of measure $\le 2^{-\ell_k}$ such that $X \subseteq \bigcup_{k\in\omega} I_k$.
- For all ℓ there exists exists a sequence $(I_k)_{k\in\omega}$ of basic intervals of 2^ω of measure $\le 2^{-\ell_k}$ such that $X \subseteq \bigcap_{m\in\omega}\bigcup_{k\ge m} I_k$.

are equivalent. Hence, a set X is smz iff X is smz with respect to all $\bar{\ell} \in \omega^\omega$.

The following lemma is a variant of the corresponding lemma (and proof) for Laver forcing (see for example [Jec03, Lemma 28.20]): Ultralaver makes old uncountable sets non-smz.

Lemma 1.13. *Let $\bar{D}$ be a system of ultrafilters, and let $\bar{\underset{\sim}{\ell}}$ be the $\mathbb{L}_{\bar{D}}$-name for the ultralaver real. Then each uncountable set $X \in V$ is forced to be non-smz (witnessed by the ultralaver real $\bar{\underset{\sim}{\ell}}$).*

More precisely, the following holds:

$$\Vdash_{\mathbb{L}_{\bar{D}}} \forall X \in V \cap [2^\omega]^{\aleph_1} \quad \forall (x_k)_{k\in\omega} \subseteq 2^\omega \quad X \not\subseteq \bigcap_{m\in\omega}\bigcup_{k\ge m}[x_k \restriction \underset{\sim}{\ell}_k]. \tag{1.14}$$

We first give two technical lemmas:

Lemma 1.15. *Let $p \in \mathbb{L}_{\bar{D}}$ with stem $s \in \omega^{<\omega}$, and let $\underset{\sim}{x}$ be a $\mathbb{L}_{\bar{D}}$-name for a real in 2^ω. Then there exists a pure extension $q \le_0 p$ and a real $\tau \in 2^\omega$ such that for every $n \in \omega$,*

$$\{i \in \mathrm{succ}_q(s) :\ q^{[s^\frown i]} \Vdash \underset{\sim}{x}\restriction n = \tau\restriction n\} \in D_s. \tag{1.16}$$

Proof. For each $i \in \mathrm{succ}_p(s)$, let $q_i \le_0 p^{[s^\frown i]}$ be such that q_i decides $\underset{\sim}{x}\restriction i$, i.e., there is a t_i of length i such that $q_i \Vdash \underset{\sim}{x}\restriction i = t_i$ (this is possible by Lemma 1.10).

Now we define the real $\tau \in 2^\omega$ as the D_s-limit of the t_i's. In more detail: For each $n \in \omega$ there is a (unique) $\tau_n \in 2^n$ such that $\{i :\ t_i\restriction n = \tau_n\} \in D_s$; since D_s is a filter, there is a real $\tau \in 2^\omega$ with $\tau\restriction n = \tau_n$ for each n. Finally, let $q := \bigcup_i q_i$. □

Lemma 1.17. *Let $p \in \mathbb{L}_{\bar{D}}$ with stem s, and let $(\underset{\sim}{x}_k)_{k\in\omega}$ be a sequence of $\mathbb{L}_{\bar{D}}$-names for reals in 2^ω. Then there exists a pure extension $q \le_0 p$ and a family of reals $(\tau_\eta)_{\eta\in q,\,\eta\supseteq s} \subseteq 2^\omega$ such that for each $\eta \in q$ above s, and every $n \in \omega$,*

$$\{i \in \mathrm{succ}_q(\eta) :\ q^{[\eta^\frown i]} \Vdash \underset{\sim}{x}_{|\eta|}\restriction n = \tau_\eta\restriction n\} \in D_\eta. \tag{1.18}$$

Proof. We apply Lemma 1.15 to each node η in p above s (and to $\underset{\sim}{x}_{|\eta|}$) separately: We first get a $p_1 \le_0 p$ and a $\tau_s \in 2^\omega$; for every immediate successor $\eta \in \mathrm{succ}_{p_1}(s)$, we get $q_\eta \le_0 p_1^{[\eta]}$ and a $\tau_\eta \in 2^\omega$, and let $p_2 := \bigcup_\eta q_\eta$; in this way, we get a (fusion) sequence $(p, p_1, p_2, \ldots)$, and let $q := \bigcap_k p_k$. □

Proof of Lemma 1.13. We want to prove (1.14). Assume towards a contradiction that X is an uncountable set in V, and that $(\underset{\sim}{x}_k)_{k\in\omega}$ is a sequence of names for reals in 2^ω and $p \in \mathbb{L}_{\bar{D}}$ such that

$$p \Vdash X \subseteq \bigcap_{m\in\omega}\bigcup_{k\ge m}[\underset{\sim}{x}_k\restriction \underset{\sim}{\ell}_k]. \tag{1.19}$$

Let $s \in \omega^{<\omega}$ be the stem of p.

By Lemma 1.17, we can fix a pure extension $q \le_0 p$ and a family $(\tau_\eta)_{\eta\in q,\,\eta\supseteq s} \subseteq 2^\omega$ such that for each $\eta \in q$ above the stem s and every $n \in \omega$, condition (1.18) holds.

Since X is (in V and) uncountable, we can find a real $x^* \in X$ which is different from each real in the countable family $(\tau_\eta)_{\eta\in q,\,\eta\supseteq s}$; more specifically, we can pick a family of natural numbers $(n_\eta)_{\eta\in q,\,\eta\supseteq s}$ such that $x^*\restriction n_\eta \neq \tau_\eta\restriction n_\eta$ for any η.

We can now find $r \le_0 q$ such that:

- For all $\eta \in r$ above s and all $i \in \mathrm{succ}_r(\eta)$ we have $i > n_\eta$.
- For all $\eta \in r$ above s and all $i \in \mathrm{succ}_r(\eta)$ we have $r^{[\eta^\frown i]} \Vdash \underset{\sim}{x}_{|\eta|}\restriction n_\eta = \tau_\eta\restriction n_\eta \neq x^*\restriction n_\eta$.

So for all $\eta \in r$ above s we have, writing k for $|\eta|$, that $r^{[\eta^\frown i]}$ forces $x^* \notin [\underset{\sim}{x}_k\restriction n_\eta] \supseteq [\underset{\sim}{x}_k\restriction \ell_k]$. We conclude that r forces $x^* \notin \bigcup_{k\ge|s|}[\underset{\sim}{x}_k\restriction \ell_k]$, contradicting (1.19). □

Corollary 1.20. *Let $(t_k)_{k\in\omega}$ be a dense subset of 2^ω.*

Let $\bar D$ be a system of ultrafilters, and let $\underset{\sim}{\bar\ell}$ be the $\mathbb{L}_{\bar D}$-name for the ultralaver real. Then the set

$$\underset{\sim}{H} := \bigcap_{m\in\omega} \bigcup_{k\geq m} [t_k \restriction \underset{\sim}{\ell}_k]$$

is forced to be a comeager set with the property that $\underset{\sim}{H}$ does not contain any translate of any old uncountable set.

Pawlikowski's theorem 0.2 gives us:

Corollary 1.21. *There is a canonical name F for a closed null set such that $X + F$ is positive for all uncountable X in V.*

In particular, no uncountable ground model set is smz in the ultralaver extension.

1.D. **Thin sets and strong measure zero.** For the notion of "(very) thin" set, we use an increasing function $B^*(k)$ (the function we use will be described in Corollary 2.2). We will assume that $\bar\ell^* = (\ell^*_k)_{k\in\omega}$ is an increasing sequence of natural numbers with $\ell^*_{k+1} \gg B^*(k)$. (We will later use a subsequence of the ultralaver real $\bar\ell$ as $\bar\ell^*$, see Lemma 1.23).

Definition 1.22. For $X \subseteq 2^\omega$ and $k \in \omega$ we write $X\restriction[\ell^*_k, \ell^*_{k+1})$ for the set $\{x\restriction[\ell^*_k, \ell^*_{k+1}) : x \in X\}$. We say that

- $X \subseteq 2^\omega$ is "*very thin with respect to $\bar\ell^*$ and B^**", if there are infinitely many k with $|X\restriction[\ell^*_k, \ell^*_{k+1})| \leq B^*(k)$.
- $X \subseteq 2^\omega$ is "*thin with respect to $\bar\ell^*$ and B^**", if X is the union of countably many very thin sets.

Note that the family of thin sets is a σ-ideal, while the family of very thin sets is not even an ideal. Also, every very thin set is covered by a closed very thin (in particular nowhere dense) set. In particular, every thin set is meager and the ideal of thin sets is a proper ideal.

Lemma 1.23. *Let B^* be an increasing function. Let $\bar\ell$ be an increasing sequence of natural numbers. We define a subsequence $\bar\ell^*$ of $\bar\ell$ in the following way: $\ell^*_k = \ell_{n_k}$ where $n_{k+1} - n_k = B^*(k) \cdot 2^{\ell^*_k}$.*
Then we get: If X is thin with respect to $\bar\ell^*$ and B^*, then X is smz with respect to $\bar\ell$.

Proof. Assume that $X = \bigcup_{i\in\omega} Y_i$, each Y_i very thin with respect to $\bar\ell^*$ and B^*. Let $(X_j)_{j\in\omega}$ be an enumeration of $\{Y_i : i \in \omega\}$ where each Y_i appears infinitely often. So $X \subseteq \bigcap_{m\in\omega} \bigcup_{j\geq m} X_j$.

By induction on $j \in \omega$, we find for all $j > 0$ some $k_j > k_{j-1}$ such that

$$|X_j\restriction[\ell^*_{k_j}, \ell^*_{k_j+1})| \leq B^*(k_j) \quad \text{hence} \quad |X_j\restriction[0, \ell^*_{k_j+1})| \leq B^*(k_j) \cdot 2^{\ell^*_{k_j}} = n_{k_j+1} - n_{k_j}.$$

So we can enumerate $X_j\restriction[0, \ell^*_{k_j+1})$ as $(s_i)_{n_{k_j}\leq i<n_{k_j+1}}$. Hence X_j is a subset of $\bigcup_{n_{k_j}\leq i<n_{k_j+1}}[s_i]$; and each s_i has length $\ell^*_{k_j+1} \geq \ell_i$, since $\ell^*_{k_j+1} = \ell_{n_{k_j+1}}$ and $i < n_{k_j+1}$. This implies

$$X \subseteq \bigcap_{m\in\omega} \bigcup_{j\geq m} X_j \subseteq \bigcap_{m\in\omega} \bigcup_{i\geq m} [s_i].$$

Hence X is smz with respect to $\bar\ell$. □

Lemma 1.13 and Lemma 1.23 yield:

Corollary 1.24. *Let B^* be an increasing function. Let $\bar D$ be a system of ultrafilters, and $\underset{\sim}{\bar\ell}$ the name for the ultralaver real. Let $\underset{\sim}{\bar\ell}^*$ be constructed from B^* and $\underset{\sim}{\bar\ell}$ as in Lemma 1.23.*
Then $\mathbb{L}_{\bar D}$ forces that for every uncountable $X \subseteq 2^\omega$:

- *X is not smz with respect to $\underset{\sim}{\bar\ell}$.*
- *X is not thin with respect to $\underset{\sim}{\bar\ell}^*$ and B^*.*

1.E. **Ultralaver and preservation of Lebesgue positivity.** It is well known that both Laver forcing and random forcing preserve Lebesgue positivity; in fact they satisfy a stronger property that is preserved under countable support iterations. (So in particular, a countable support iteration of Laver and random also preserves positivity.)

Ultralaver forcing $\mathbb{L}_{\bar D}$ will in general not preserve positivity. Indeed, if all ultrafilters D_s are equal to the same ultrafilter D^*, then the range $L := \{\ell_0, \ell_1, \ldots\} \subseteq \omega$ of the ultralaver real $\bar\ell$ will diagonalize D^*, so every ground model real $x \in 2^\omega$ (viewed as a subset of ω) will either almost contain L or be almost disjoint to L, which implies that the set $2^\omega \cap V$ of old reals is covered by a null set in the extension. However, later in this paper it will become clear that if we choose the ultrafilters D_s in a sufficiently generic way, then many old positive sets will stay positive. More specifically, in this section we will show (Lemma 1.30): If $\bar{D}^M$ is an ultrafilter system in a countable model M and r a random real over M, then we can find an extension $\bar{D}$ such that $\mathbb{L}_{\bar{D}}$ forces that r remains random over $M[H^M]$ (where H^M denotes the $\mathbb{L}_{\bar{D}}$-name for the restriction of the $\mathbb{L}_{\bar{D}}$-generic filter H to $\mathbb{L}_{\bar{D}^M} \cap M$). Additionally, some "side conditions" are met, which are necessary to preserve the property in forcing iterations.

In Section 3.D we will see how to use this property to preserve randoms in limits.

The setup we use for preservation of randomness is basically the notation of "Case A" preservation introduced in [She98, Ch.XVIII], see also [Gol93, GK06] or the textbook [BJ95, 6.1.B]:

Definition 1.25. We write CLOPEN for the collection of clopen sets on 2^ω. We say that the function $Z : \omega \to$ CLOPEN is a code for a null set, if the measure of $Z(n)$ is at most 2^{-n} for each $n \in \omega$.

For such a code Z, the set nullset(Z) coded by Z is

$$\text{nullset}(Z) := \bigcap_n \bigcup_{k \geq n} Z(k).$$

The set nullset(Z) obviously is a null set, and it is well known that every null set is contained in such a set nullset(Z).

Definition 1.26. For a real r and any code Z, we define $Z \sqsubset_n r$ by:

$$(\forall k \geq n)\ r \notin Z(k).$$

We write $Z \sqsubset r$ if $Z \sqsubset_n r$ holds for some n; i.e., if $r \notin \text{nullset}(Z)$.

For later reference, we record the following trivial fact:

(1.27) $p \Vdash \utilde{Z} \sqsubset r$ iff there is a name $\utilde{n}$ for an element of ω such that $p \Vdash \utilde{Z} \sqsubset_{\utilde{n}} r$.

Let P be a forcing notion, and $\utilde{Z}$ a P-name of a code for a null set. An interpretation of $\utilde{Z}$ below p is some code Z^* such that there is a sequence $p = p_0 \geq p_1 \geq p_2 \geq \dots$ such that p_m forces $\utilde{Z}\restriction m = Z^* \restriction m$. Usually we demand (which allows a simpler proof of the preservation theorem at limit stages) that the sequence $(p_0, p_1, \dots)$ is inconsistent, i.e., p forces that there is an m such that $p_m \notin G$. Note that whenever P adds a new ω-sequence of ordinals, we can find such an interpretation for any $\utilde{Z}$.

If $\bar{\utilde{Z}} = (\utilde{Z}_1, \dots, \utilde{Z}_m)$ is a tuple of names of codes for null sets, then an interpretation of $\bar{\utilde{Z}}$ below p is some tuple $(Z_1^*, \dots, Z_m^*)$ such that there is a single sequence $p = p_0 \geq p_1 \geq p_2 \geq \dots$ interpreting each $\utilde{Z}_i$ as Z_i^*.

We now turn to preservation of Lebesgue positivity:

Definition 1.28. (1) A forcing notion P *preserves Borel outer measure*, if P forces $\text{Leb}^*(A^V) = \text{Leb}(A^{V[G_P]})$ for every code A for a Borel set. (Leb^* denotes the outer Lebesgue measure, and for a Borel code A and a set-theoretic universe V, A^V denotes the Borel set coded by A in V.)

(2) *P strongly preserves randoms*, if the following holds: Let $N \prec H(\chi^*)$ be countable for a sufficiently large regular cardinal χ^*, let $P, p, \bar{\utilde{Z}} = (\utilde{Z}_1, \dots, \utilde{Z}_m) \in N$, let $p \in P$ and let r be random over N. Assume that in N, $\bar{Z}^*$ is an interpretation of $\bar{\utilde{Z}}$, and assume $Z_i^* \sqsubset_{k_i} r$ for each i. Then there is an N-generic $q \leq p$ forcing that r is still random over $N[G]$ and moreover, $\utilde{Z}_i \sqsubset_{k_i} r$ for each i. (In particular, P has to be proper.)

(3) Assume that P is absolutely definable. *P strongly preserves randoms over countable models* if (2) holds for all countable (transitive[10]) models N of ZFC*.

It is easy to see that these properties are increasing in strength. (Of course (3)⇒(2) works only if ZFC* is satisfied in $H(\chi^*)$.)

In [KS05] it is shown that (1) implies (3), provided that P is nep ("non-elementary proper", i.e., nicely definable and proper with respect to countable models). In particular, every Suslin ccc forcing notion such as random forcing, and also many tree forcing notions including Laver forcing, are nep. However $\mathbb{L}_{\bar{D}}$ is not nicely definable in this sense, as its definition uses ultrafilters as parameters.

[10] Later we will introduce ord-transitive models, and it is easy to see that it does not make any difference whether we demand transitive or not; this can be seen using a transitive collapse.

Lemma 1.29. *Both Laver forcing and random forcing strongly preserve randoms over countable models.*

Proof. For random forcing, this is easy and well known (see, e.g., [BJ95, 6.3.12]).

For Laver forcing: By the above, it is enough to show (1). This was done by Woodin (unpublished) and Judah-Shelah [JS90]. A nicer proof (including a variant of (2)) is given by Pawlikowski [Paw96b]. □

Ultralaver will generally not preserve Lebesgue positivity, let alone randomness. However, we get the following "local" variant of strong preservation of randoms (which will be used in the preservation theorem 3.33). The rest of this section will be devoted to the proof of the following lemma.

Lemma 1.30. *Assume that M is a countable model, $\bar{D}^M$ an ultrafilter system in M and r a random real over M. Then there is (in V) an ultrafilter system $\bar{D}$ extending*[11] *$\bar{D}^M$, such that the following holds:*
If

- $p \in \mathbb{L}_{\bar{D}^M}$,
- *in M, $\underaccent{\tilde}{\bar{Z}} = (\underaccent{\tilde}{Z}_1, \dots, \underaccent{\tilde}{Z}_m)$ is a sequence of $\mathbb{L}_{\bar{D}^M}$-names for codes for null sets,*[12] *and $Z_1^*, \dots, Z_m^*$ are interpretations under p, witnessed by a sequence $(p_n)_{n\in\omega}$ with strictly increasing stems,*
- $Z_i^* \sqsubset_{k_i} r$ *for* $i = 1, \dots, m$,

then *there is a $q \leq p$ in $\mathbb{L}_{\bar{D}}$ forcing that*

- *r is random over $M[G^M]$,*
- $\underaccent{\tilde}{Z}_i \sqsubset_{k_i} r$ *for* $i = 1, \dots, m$.

For the proof of this lemma, we will use the following concepts:

Definition 1.31. Let $p \subseteq \omega^{<\omega}$ be a tree. A "*front name below p*" is a function[13] $h : F \to \textsc{clopen}$, where $F \subseteq p$ is a front (a set that meets every branch of p in a unique point). (For notational simplicity we also allow h to be defined on elements $\notin p$; this way, every front name below p is also a front name below q whenever $q \leq p$.)

If h is a front name and $\bar{D}$ is any filter system with $p \in \mathbb{L}_{\bar{D}}$, we define the corresponding $\mathbb{L}_{\bar{D}}$-name (in the sense of forcing) $\underaccent{\tilde}{z}^h$ by

$$\underaccent{\tilde}{z}^h := \{(\check{y}, p^{[s]}) : s \in F,\ y \in h(s)\}. \tag{1.32}$$

(This does not depend on the $\bar{D}$ we use, since we set $\check{y} := \{(\check{x}, \omega^{<\omega}) : x \in y\}$.)

Up to forced equality, the name $\underaccent{\tilde}{z}^h$ is characterized by the fact that $p^{[s]}$ forces (in any $\mathbb{L}_{\bar{D}}$) that $\underaccent{\tilde}{z}^h = h(s)$, for every s in the domain of h.

Note that the same object h can be viewed as a front name below p with respect to different forcings $\mathbb{L}_{\bar{D}_1}$, $\mathbb{L}_{\bar{D}_2}$, as long as $p \in \mathbb{L}_{\bar{D}_1} \cap \mathbb{L}_{\bar{D}_2}$.

Definition 1.33. Let $p \subseteq \omega^{<\omega}$ be a tree. A "*continuous name below p*" is either of the following:

- An ω-sequence of front names below p.
- A $\subseteq$-increasing function $g : p \to \textsc{clopen}^{<\omega}$ such that $\lim_{n\to\infty} \mathrm{lh}(g(c{\restriction}n)) = \infty$ for every branch $c \in [p]$.

For each n, the set of minimal elements in $\{s \in p : \mathrm{lh}(g(s)) > n\}$ is a front, so each continuous name in the second sense naturally defines a name in the first sense, and conversely. Being a continuous name below p does not involve the notion of $\Vdash$ nor does it depend on the filter system $\bar{D}$.

If g is a continuous name and $\bar{D}$ is any filter system, we can again define the corresponding $\mathbb{L}_{\bar{D}}$-name $\underaccent{\tilde}{Z}^g$ (in the sense of forcing); we leave a formal definition of $\underaccent{\tilde}{Z}^g$ to the reader and content ourselves with this characterization:

$$(\forall s \in p) : p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}}} g(s) \subseteq \underaccent{\tilde}{Z}^g. \tag{1.34}$$

Note that a continuous name below p naturally corresponds to a continuous function $F : [p] \to \textsc{clopen}^{\omega}$, and $\underaccent{\tilde}{Z}^g$ is forced (by p) to be the value of F at the generic real $\underaccent{\tilde}{\ell}$.

Lemma 1.35. $\mathbb{L}_{\bar{D}}$ *has the following "pure decision properties":*

[11]This implies, by Lemma 1.5, that the $\mathbb{L}_{\bar{D}}$-generic filter G induces an $\mathbb{L}_{\bar{D}^M}$-generic filter over M, which we call G^M.

[12]Recall that $\mathrm{nullset}(\underaccent{\tilde}{Z}) = \bigcap_n \bigcup_{k\geq n} \underaccent{\tilde}{Z}(k)$ is a null set in the extension.

[13]Instead of CLOPEN we may also consider other ranges of front names, such as the class of all ordinals, or the set ω.

(1) *Whenever $\underset{\sim}{y}$ is a name for an element of* CLOPEN, $p \in \mathbb{L}_{\bar{D}}$, *then there is a pure extension $p_1 \leq_0 p$ such that $\underset{\sim}{y} = \underset{\sim}{z}^h$ (is forced) for a front name h below p_1.*
(2) *Whenever $\underset{\sim}{Y}$ is a name for a sequence of elements of* CLOPEN, $p \in \mathbb{L}_{\bar{D}}$, *then there is a pure extension $q \leq_0 p$ such that $\underset{\sim}{Y} = \underset{\sim}{Z}^g$ (is forced) for some continuous name g below q.*
(3) *(This is Lemma 1.10.) If A is a finite set, $\underset{\sim}{\alpha}$ a name, $p \in \mathbb{L}_{\bar{D}}$, and p forces $\underset{\sim}{\alpha} \in A$, then there is $\beta \in A$ and a pure extension $q \leq_0 p$ such that $q \Vdash \underset{\sim}{\alpha} = \beta$.*

Proof. Let $p \in \mathbb{L}_{\bar{D}}$, $s_0 := \mathrm{stem}(p)$, $\underset{\sim}{y}$ a name for an element of CLOPEN.

We call $t \in p$ a "good node in p" if $\underset{\sim}{y}$ is a front name below $p^{[t]}$ (more formally: forced to be equal to $\underset{\sim}{z}^h$ for a front name h). We can find $p_1 \leq_0 p$ such that for all $t \in p_1$ above s_0: If there is $q \leq_0 p_1^{[t]}$ such that t is good in q, then t is already good in p_1.

We claim that s_0 is now good (in p_1). Note that for any bad node s the set $\{ t \in \mathrm{succ}_{p_1}(s) : \ t \text{ bad} \}$ is in D_s^+. Hence, if s_0 is bad, we can inductively construct $p_2 \leq_0 p_1$ such that all nodes of p_2 are bad nodes in p_1. Now let $q \leq p_2$ decide $\underset{\sim}{y}$, $s := \mathrm{stem}(q)$. Then $q \leq_0 p_1^{[s]}$, so s is good in p_1, contradiction. This finishes the proof of (1).

To prove (2), we first construct p_1 as in (1) with respect to $\underset{\sim}{y}_0$. This gives a front $F_1 \subseteq p_1$ deciding $\underset{\sim}{y}_0$. Above each node in F_1 we now repeat the construction from (1) with respect to $\underset{\sim}{y}_1$, yielding p_2, etc. Finally, $q := \bigcap_n p_n$.

To prove (3): Similar to (1), we can find $p_1 \leq_0 p$ such that for each $t \in p_1$: If there is a pure extension of $p_1^{[t]}$ deciding $\underset{\sim}{\alpha}$, then $p_1^{[t]}$ decides $\underset{\sim}{\alpha}$; in this case we again call t good. Since there are only finitely many possibilities for the value of $\underset{\sim}{\alpha}$, any bad node t has D_t^+ many bad successors. So if the stem of p_1 is bad, we can again reach a contradiction as in (1). □

Corollary 1.36. *Let $\bar{D}$ be a filter system, and let $G \subseteq \mathbb{L}_{\bar{D}}$ be generic. Then every $Y \in$ CLOPEN$^\omega$ in $V[G]$ is the evaluation of a continuous name $\underset{\sim}{Z}^g$ by G.*

Proof. In V, fix a $p \in \mathbb{L}_{\bar{D}}$ and a name $\underset{\sim}{Y}$ for an element of CLOPEN$^\omega$. We can find $q \leq_0 p$ and a continuous name g below q such that $q \Vdash \underset{\sim}{Y} = \underset{\sim}{Z}^g$. □

We will need the following modification of the concept of "continuous names".

Definition 1.37. Let $p \subseteq \omega^{<\omega}$ be a tree, $b \in [p]$ a branch. An "*almost continuous name below p (with respect to b)*" is a $\subseteq$-increasing function $g : p \to$ CLOPEN$^{<\omega}$ such that $\lim_{n\to\infty} \mathrm{lh}(g(c \restriction n)) = \infty$ for every branch $c \in [p]$, except possibly for $c = b$.

Note that "except possibly for $c = b$" is the only difference between this definition and the definition of a continuous name.

Since for any $\bar{D}$ it is forced[14] that the generic real (for $\mathbb{L}_{\bar{D}}$) is not equal to the exceptional branch b, we again get a name $\underset{\sim}{Z}^g$ of a function in CLOPEN$^\omega$ satisfying:

$$(\forall s \in p) : p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}}} g(s) \subseteq \underset{\sim}{Z}^g.$$

An almost continuous name naturally corresponds to a continuous function F from $[p] \setminus \{b\}$ into CLOPEN$^\omega$.

Note that being an almost continuous name is a very simple combinatorial property of g which does not depend on $\bar{D}$, nor does it involve the notion $\Vdash$. Thus, the same function g can be viewed as an almost continuous name for two different forcing notions $\mathbb{L}_{\bar{D}_1}$, $\mathbb{L}_{\bar{D}_2}$ simultaneously.

Lemma 1.38. *Let $\bar{D}$ be a system of filters (not necessarily ultrafilters).*

Assume that $\bar{p} = (p_n)_{n\in\omega}$ witnesses that Y^* is an interpretation of $\underset{\sim}{Y}$, and that the lengths of the stems of the p_n are strictly increasing.[15] *Then there exists a sequence $\bar{q} = (q_n)_{n\in\omega}$ such that*

(1) $q_0 \geq q_1 \geq \cdots$.
(2) $q_n \leq p_n$ *for all n.*
(3) *$\bar{q}$ also interprets $\underset{\sim}{Y}$ as Y^*. (This follows from the previous two statements.)*
(4) *$\underset{\sim}{Y}$ is almost continuous below q_0, i.e., there is an almost continuous name g such that q_0 forces $\underset{\sim}{Y} = \underset{\sim}{Z}^g$.)*

[14] This follows from our assumption that all our filters contain the Fréchet filter.

[15] It is easy to see that for every $\mathbb{L}_{\bar{D}}$-name $\underset{\sim}{Y}$ we can find such $\bar{p}$ and Y^*: First find $\bar{p}$ which interprets both $\underset{\sim}{Y}$ and $\bar{\underset{\sim}{\ell}}$, and then thin out to get a strictly increasing sequence of stems.

(5) $\underset{\sim}{Y}$ is almost continuous below q_n, for all n. (This follows from the previous statement.)

Proof. Let b be the branch described by the stems of the conditions p_n:

$$b := \{s : (\exists n)\ s \subseteq \text{stem}(p_n)\}.$$

We now construct a condition q_0. For every $s \in b$ satisfying $\text{stem}(p_n) \subseteq s \subsetneq \text{stem}(p_{n+1})$ we set $\text{succ}_{q_0}(s) = \text{succ}_{p_n}(s)$, and for all $t \in \text{succ}_{q_0}(s)$ except for the one in b we let $q_0^{[t]} \leq_0 p_n^{[t]}$ be such that $\underset{\sim}{Y}$ is continuous below $q_0^{[t]}$. We can do this by Lemma 1.35(2).

Now we set

$$q_n := p_n \cap q_0 = q_0^{[\text{stem}(p_n)]} \leq p_n.$$

This takes care of (1) and (2). Now we show (4): Any branch c of q_0 not equal to b must contain a node $s^\frown k \notin b$ with $s \in b$, so c is a branch in $q_0^{[s^\frown k]}$, below which $\underset{\sim}{Y}$ was continuous. □

The following lemmas and corollaries are the motivation for considering continuous and almost continuous names.

Lemma 1.39. *Let $\bar{D}$ be a system of filters (not necessarily ultrafilters). Let $p \in \mathbb{L}_{\bar{D}}$, let b be a branch, and let $g : p \to \text{CLOPEN}^{<\omega}$ be an almost continuous name below p with respect to b; write $\underset{\sim}{Z}^g$ for the associated $\mathbb{L}_{\bar{D}}$-name.*

Let $r \in 2^\omega$ be a real, $n_0 \in \omega$. Then the following are equivalent:

(1) $p \Vdash_{\mathbb{L}_{\bar{D}}} r \notin \bigcup_{n \geq n_0} \underset{\sim}{Z}^g(n)$, i.e., $\underset{\sim}{Z}^g \sqsubset_{n_0} r$.
(2) For all $n \geq n_0$ and for all $s \in p$ for which $g(s)$ has length $> n$ we have $r \notin g(s)(n)$.

Note that (2) does not mention the notion $\Vdash$ and does not depend on $\bar{D}$.

Proof. $\neg(2) \Rightarrow \neg(1)$: Assume that there is $s \in p$ for which $g(s) = (C_0, \ldots, C_n, \ldots, C_k)$ and $r \in C_n$. Then $p^{[s]}$ forces that the generic sequence $\underset{\sim}{Z}^g = (\underset{\sim}{Z}(0), \underset{\sim}{Z}(1), \ldots)$ starts with $C_0, \ldots, C_n$, so $p^{[s]}$ forces $r \in \underset{\sim}{Z}^g(n)$.

$\neg(1) \Rightarrow \neg(2)$: Assume that p does not force $r \notin \bigcup_{n \geq n_0} \underset{\sim}{Z}^g(n)$. So there is a condition $q \leq p$ and some $n \geq n_0$ such that $q \Vdash r \in \underset{\sim}{Z}^g(n)$. By increasing the stem of q, if necessary, we may assume that $s := \text{stem}(q)$ is not on b (the "exceptional" branch), and that $g(s)$ has already length $> n$. Let $C_n := g(s)(n)$ be the n-th entry of $g(s)$. So $p^{[s]}$ already forces $\underset{\sim}{Z}^g(n) = C_n$; now $q^{[s]} \leq p^{[s]}$, and $q^{[s]}$ forces the following statements: $r \in \underset{\sim}{Z}^g(n)$, $\underset{\sim}{Z}^g(n) = C_n$. Hence $r \in C_n$, so (2) fails. □

Corollary 1.40. *Let $\bar{D}_1$ and $\bar{D}_2$ be systems of filters, and assume that p is in $\mathbb{L}_{\bar{D}_1} \cap \mathbb{L}_{\bar{D}_2}$. Let $g : p \to \text{CLOPEN}^{<\omega}$ be an almost continuous name of a sequence of clopen sets, and let $\underset{\sim}{Z}_1^g$ and $\underset{\sim}{Z}_2^g$ be the associated $\mathbb{L}_{\bar{D}_1}$-name and $\mathbb{L}_{\bar{D}_2}$-name, respectively.*

Then for any real r and $n \in \omega$ we have

$$p \Vdash_{\mathbb{L}_{\bar{D}_1}} \underset{\sim}{Z}_1^g \sqsubset_n r \quad \Leftrightarrow \quad p \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_n r.$$

(We will use this corollary for the special case that $\mathbb{L}_{\bar{D}_1}$ is an ultralaver forcing, and $\mathbb{L}_{\bar{D}_2}$ is Laver forcing.)

Lemma 1.41. *Let $\bar{D}_1$ and $\bar{D}_2$ be systems of filters, and assume that p is in $\mathbb{L}_{\bar{D}_1} \cap \mathbb{L}_{\bar{D}_2}$. Let $g : p \to \text{CLOPEN}^{<\omega}$ be a continuous name of a sequence of clopen sets, let $F \subseteq p$ be a front and let $h : F \to \omega$ be a front name. Again we will write $\underset{\sim}{Z}_1^g, \underset{\sim}{Z}_2^g$ for the associated names of codes for null sets, and we will write $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ for the associated $\mathbb{L}_{\bar{D}_1}$- and $\mathbb{L}_{\bar{D}_2}$-names, respectively, of natural numbers.*

Then for any real r we have:

$$p \Vdash_{\mathbb{L}_{\bar{D}_1}} \underset{\sim}{Z}_1^g \sqsubset_{\underset{\sim}{n}_1} r \quad \Leftrightarrow \quad p \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_{\underset{\sim}{n}_2} r.$$

Proof. Assume $p \Vdash_{\mathbb{L}_{\bar{D}_1}} \underset{\sim}{Z}_1^g \sqsubset_{\underset{\sim}{n}_1} r$. So for each $s \in F$ we have: $p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}_1}} \underset{\sim}{Z}_1^g \sqsubset_{h(s)} r$. By Corollary 1.40, we also have $p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_{h(s)} r$. So also $p^{[s]} \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_{\underset{\sim}{n}_2} r$ for each $s \in F$. Hence $p \Vdash_{\mathbb{L}_{\bar{D}_2}} \underset{\sim}{Z}_2^g \sqsubset_{\underset{\sim}{n}_2} r$. □

Corollary 1.42. *Assume $q \in \mathbb{L}$ forces in Laver forcing that $\underset{\sim}{Z}^{g_k} \sqsubset r$ for $k = 1, 2, \ldots$, where each g_k is a continuous name of a code for a null set. Then there is a Laver condition $q' \leq_0 q$ such that for all ultrafilter systems $\bar{D}$ we have:*

If $q' \in \mathbb{L}_{\bar{D}}$, then q' forces (in ultralaver forcing $\mathbb{L}_{\bar{D}}$) that $\underset{\sim}{Z}^{g_k} \sqsubset r$ for all k.

Proof. By (1.27) we can find a sequence $(\underset{\sim}{n}_k)_{k=1}^{\infty}$ of $\mathbb{L}$-names such that $q \Vdash \underset{\sim}{Z}^{g_k} \sqsubset_{\underset{\sim}{n}_k} r$ for each k. By Lemma 1.35(2) we can find $q' \leq_0 q$ be such that this sequence is continuous below q'. Since each $\underset{\sim}{n}_k$ is now a front name below q', we can apply the previous lemma. □

Lemma 1.43. *Let M be a countable model, $r \in 2^\omega$, $\bar{D}^M \in M$ an ultrafilter system, $\bar{D}$ a filter system extending $\bar{D}^M$, $q \in \mathbb{L}_{\bar{D}}$. For any V-generic filter $G \subseteq \mathbb{L}_{\bar{D}}$ we write G^M for the (M-generic, by Lemma 1.5) filter on $\mathbb{L}_{\bar{D}^M}$.*

The following are equivalent:

(1) $q \Vdash_{\mathbb{L}_{\bar{D}}} r$ is random over $M[G^M]$.
(2) For all names $\underset{\sim}{Z} \in M$ of codes for null sets: $q \Vdash_{\mathbb{L}_{\bar{D}}} \underset{\sim}{Z} \sqsubset r$.
(3) For all continuous names $g \in M$: $q \Vdash_{\mathbb{L}_{\bar{D}}} \underset{\sim}{Z}^g \sqsubset r$.

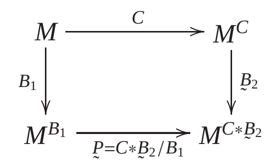
Proof. (1)⇔(2) holds because every null set is contained in a set of the form nullset(Z), for some code Z.

(2)⇔(3): Every code for a null set in $M[G^M]$ is equal to $\underset{\sim}{Z}^g[G^M]$, for some $g \in M$, by Corollary 1.36. □

The following lemma may be folklore. Nevertheless, we prove it for the convenience of the reader.

Lemma 1.44. *Let r be random over a countable model M and $A \in M$. Then there is a countable model $M' \supseteq M$ such that A is countable in M', but r is still random over M'.*

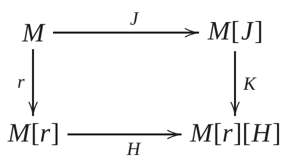
Proof. We will need the following forcing notions, all defined in M:



- Let C be the forcing that collapses the cardinality of A to ω with finite conditions.
- Let B_1 be random forcing (trees $T \subseteq 2^{<\omega}$ of positive measure).
- Let $\underset{\sim}{B}_2$ be the C-name of random forcing.
- Let $i : B_1 \to C * \underset{\sim}{B}_2$ be the natural complete embedding $T \mapsto (1_C, T)$.
- Let $\underset{\sim}{P}$ be a B_1-name for the forcing $C * \underset{\sim}{B}_2 / i[G_{B_1}]$, the quotient of $C * \underset{\sim}{B}_2$ by the complete subforcing $i[B_1]$.

The random real r is B_1-generic over M. In $M[r]$ we let $P := \underset{\sim}{P}[r]$. Now let $H \subseteq P$ be generic over $M[r]$. Then $r * H \subseteq B_1 * \underset{\sim}{P} \simeq C * \underset{\sim}{B}_2$ induces an M-generic filter $J \subseteq C$ and an $M[J]$-generic filter $K \subseteq \underset{\sim}{B}_2[J]$; it is easy to check that K interprets the $\underset{\sim}{B}_2$-name of the canonical random real as the given random real r.

Hence r is random over the countable model $M' := M[J]$, and A is countable in M'.



□

Proof of Lemma 1.30. We will first describe a construction that deals with a single triple $(\bar{p}, \bar{\underset{\sim}{Z}}, \bar{Z}^*)$ (where $\bar{p}$ is a sequence of conditions with strictly increasing stems which interprets $\bar{\underset{\sim}{Z}}$ as $\bar{Z}^*$); this construction will yield a condition $q' = q'(\bar{p}, \bar{\underset{\sim}{Z}}, \bar{Z}^*)$. We will then show how to deal with all possible triples.

So let p be a condition, and let $\bar{p} = (p_k)_{k\in\omega}$ be a sequence interpreting $\bar{\underset{\sim}{Z}}$ as $\bar{Z}^*$, where the lengths of the stems of p_n are strictly increasing and $p_0 = p$. It is easy to see that it is enough to deal with a single null set, i.e., $m = 1$, and with $k_1 = 0$. We write $\underset{\sim}{Z}$ and Z^* instead of $\underset{\sim}{Z}_1$ and Z_1^*.

Using Lemma 1.38 we may (strengthening the conditions in our interpretation) assume (in M) that the sequence $(\underset{\sim}{Z}(k))_{k\in\omega}$ is almost continuous, witnessed by $g : p \to \text{CLOPEN}^{<\omega}$. By Lemma 1.44, we can find a model $M' \supseteq M$ such that $(2^\omega)^M$ is countable in M', but r is still random over M'.

We now work in M'. Note that g still defines an almost continuous name, which we again call $\underset{\sim}{Z}$.

Each filter in D^M_s is now countably generated; let A_s be a pseudo-intersection of D^M_s which additionally satisfies $A_s \subseteq \mathrm{succ}_p(s)$ for all $s \in p$ above the stem. Let D'_s be the Fréchet filter on A_s. Let $p' \in \mathbb{L}_{\bar{D}'}$ be the tree with the same stem as p which satisfies $\mathrm{succ}_{p'}(s) = A_s$ for all $s \in p'$ above the stem.

By Lemma 1.5, we know that $\mathbb{L}_{\bar{D}^M}$ is an M-complete subforcing of $\mathbb{L}_{\bar{D}'}$ (in M' as well as in V). We write G^M for the induced filter on $\mathbb{L}_{\bar{D}^M}$.

We now work in V. Note that below the condition p', the forcing $\mathbb{L}_{\bar{D}'}$ is just Laver forcing $\mathbb{L}$, and that $p' \leq_{\mathbb{L}} p$. Using Lemma 1.29 we can find a condition $q \leq p'$ (in Laver forcing $\mathbb{L}$) such that:

(1.45) q is M'-generic.

(1.46) $q \Vdash_{\mathbb{L}} r$ is random over $M'[G_{\mathbb{L}}]$ (hence also over $M[G^M]$).

(1.47) Moreover, $q \Vdash_{\mathbb{L}} \underaccent{\sim}{Z} \sqsubset_0 r$.

Enumerate all continuous $\mathbb{L}_{\bar{D}^M}$-names of codes for null sets from M as $\underaccent{\sim}{Z}^{g_1}, \underaccent{\sim}{Z}^{g_2}, \dots$ Applying Corollary 1.42 yields a condition $q' \leq q$ such that for all filter systems $\bar{E}$ satisfying $q' \in \mathbb{L}_{\bar{E}}$, we have $q' \Vdash_{\mathbb{L}_{\bar{E}}} \underaccent{\sim}{Z}^{g_i} \sqsubset r$ for all i. Corollary 1.40 and Lemma 1.43 now imply:

(1.48) For every filter system $\bar{E}$ satisfying $q' \in \mathbb{L}_{\bar{E}}$, q' forces in $\mathbb{L}_{\bar{E}}$ that r is random over $M[G^M]$ and that $\underaccent{\sim}{Z} \sqsubset_0 r$.

By thinning out q' we may assume that

(1.49) For each $\nu \in \omega^\omega \cap M$ there is k such that $\nu \restriction k \notin q'$.

We have now described a construction of $q' = q'(\bar{p}, \underaccent{\sim}{Z}, Z^*)$.

Let $(\bar{p}^n, \underaccent{\sim}{Z}^n, Z^{*n})$ enumerate all triples $(\bar{p}, \underaccent{\sim}{Z}, Z^*) \in M$ where $\bar{p}$ interprets $\underaccent{\sim}{Z}$ as Z^* (and consists of conditions with strictly increasing stems). For each n write ν^n for $\bigcup_k \mathrm{stem}(p^n_k)$, the branch determined by the stems of the sequence $\bar{p}^n$. We now define by induction a sequence q^n of conditions:

- $q^0 := q'(\bar{p}^0, \underaccent{\sim}{Z}^0, Z^{*0})$.
- Given q^{n-1} and $(\bar{p}^n, \underaccent{\sim}{Z}^n, Z^{*n})$, we find k_0 such that $\nu^n \restriction k_0 \notin q^0 \cup \dots \cup q^{n-1}$ (using (1.49)). Let k_1 be such that $\mathrm{stem}(p^n_{k_1})$ has length $> k_0$. We replace $\bar{p}^n$ by $\bar{p}' := (p^n_k)_{k \geq k_1}$. (Obviously, $\bar{p}'$ still interprets $\underaccent{\sim}{Z}^n$ as Z^{*n}.) Now let $q^n := q'(\bar{p}', \underaccent{\sim}{Z}^n, Z^{*n})$.

Note that the stem of q^n is at least as long as the stem of $p^n_{k_1}$, and is therefore not in $q^0 \cup \dots \cup q^{n-1}$, so $\mathrm{stem}(q^i)$ and $\mathrm{stem}(q^j)$ are incompatible for all $i \neq j$. Therefore we can choose for each s an ultrafilter D_s extending D^M_s such that $\mathrm{stem}(q^i) \subseteq s$ implies $\mathrm{succ}_{q^i}(s) \in D_s$.

Note that all q^i are in $\mathbb{L}_{\bar{D}}$. Therefore, we can use (1.48). Also, $q^i \leq p^i_0$. □

Below, in Lemma 3.33, we will prove a preservation theorem using the following "local" variant of "random preservation":

Definition 1.50. Fix a countable model M, a real $r \in 2^\omega$ and a forcing notion $Q^M \in M$. Let Q^M be an M-complete subforcing of Q. We say that "*Q locally preserves randomness of r over M*", if there is in M a sequence $(D^{Q^M}_n)_{n\in\omega}$ of open dense subsets of Q^M such that the following holds:

Assume that

- M thinks that $\bar{p} := (p^n)_{n\in\omega}$ interprets $(\underaccent{\sim}{Z}_1, \dots, \underaccent{\sim}{Z}_m)$ as $(Z^*_1, \dots, Z^*_m)$ (so each $\underaccent{\sim}{Z}_i$ is a Q^M-name of a code for a null set and each Z^*_i is a code for a null set, both in M);
- moreover, each p^n is in $D^{Q^M}_n$ (we call such a sequence $(p^n)_{n\in\omega}$, or the according interpretation, "*quick*";
- r is random over M;
- $Z^*_i \sqsubset_{k_i} r$ for $i = 1, \dots, m$.

Then there is a $q \leq_Q p^0$ forcing that

- r is random over $M[G^M]$;
- $\underaccent{\sim}{Z}_i \sqsubset_{k_i} r$ for $i = 1, \dots, m$.

Note that this is trivially satisfied if r is not random over M.

For a variant of this definition, see Section 6.

Setting $D^{Q^M}_n$ to be the set of conditions with stem of length at least n, Lemma 1.30 gives us:

Corollary 1.51. *If Q^M is an ultralaver forcing in M and r a real, then there is an ultralaver forcing Q over*[16] *Q^M locally preserving randomness of r over M.*

2. Janus forcing

In this section, we define a family of forcing notions that has two faces (hence the name "*Janus forcing*"): Elements of this family may be countable (and therefore equivalent to Cohen), and they may also be essentially random.

In the rest of the paper, we will use the following properties of Janus forcing notions $\mathbb{J}$. (And we will use *only* these properties. So readers who are willing to take these properties for granted could skip to Section 3.)

Throughout the whole paper we fix a function $B^* : \omega \to \omega$ given by Corollary 2.2. The Janus forcings will depend on a real parameter $\bar\ell^* = (\ell^*_m)_{m\in\omega} \in \omega^\omega$ which grows fast with respect to B^*. (In our application, $\bar\ell^*$ will be given by a subsequence of an ultralaver real.)

The sequence $\bar\ell^*$ and the function B^* together define a notion of a "thin set" (see Definition 1.22).

(1) There is a canonical $\mathbb{J}$-name for a (code for a) null set $\underset{\sim}{Z}_\nabla$.
Whenever $X \subseteq 2^\omega$ is not thin, and $\mathbb{J}$ is countable, then $\mathbb{J}$ forces that X is not strongly meager, witnessed[17] by nullset($\underset{\sim}{Z}_\nabla$) (the set we get when we evaluate the code $\underset{\sim}{Z}_\nabla$). Moreover, for any $\mathbb{J}$-name $\underset{\sim}{Q}$ of a σ-centered forcing, also $\mathbb{J} * \underset{\sim}{Q}$ forces that X is not strongly meager, again witnessed by nullset($\underset{\sim}{Z}_\nabla$).
(This is Lemma 2.9; "thin" is defined in Definition 1.22.)

(2) Let M be a countable transitive model and $\mathbb{J}^M$ a Janus forcing in M. Then $\mathbb{J}^M$ is a Janus forcing in V as well (and of course countable in V). (Also note that trivially the forcing $\mathbb{J}^M$ is an M-complete subforcing of itself.)
(This is Fact 2.8.)

(3) Whenever M is a countable transitive model and $\mathbb{J}^M$ is a Janus forcing in M, then there is a Janus forcing $\mathbb{J}$ such that
   - $\mathbb{J}^M$ is an M-complete subforcing of $\mathbb{J}$.
   - $\mathbb{J}$ is (in V) equivalent to random forcing (actually we just need that $\mathbb{J}$ preserves Lebesgue positivity in a strong and iterable way).

(This is Lemma 2.16 and Lemma 2.20.)

(4) Moreover, the name $\underset{\sim}{Z}_\nabla$ referred to in (1) is so "canonical" that it evaluates to the same code in the $\mathbb{J}$-generic extension over V as in the $\mathbb{J}^M$-generic extension over M.
(This is Fact 2.7.)

2.A. **Definition of Janus.** A Janus forcing $\mathbb{J}$ will consist of:[18]

- A countable "core" (or: backbone) ∇ which is defined in a combinatorial way from a parameter $\bar\ell^*$. (In our application, we will use a Janus forcing immediately after an ultralaver forcing, and $\bar\ell^*$ will be a subsequence of the ultralaver real.) This core is of course equivalent to Cohen forcing.
- Some additional "stuffing" $\mathbb{J} \setminus \nabla$ (countable[19] or uncountable). We allow great freedom for this, we just require that the core ∇ is a "sufficiently" complete subforcing (in a specific combinatorial sense, see Definition 2.5(3)).

We will use the following combinatorial theorem from [BS10]:

Lemma 2.1 ([BS10, Theorem 8][20]). *For every $\varepsilon, \delta > 0$ there exists $N_{\varepsilon,\delta} \in \omega$ such that for all sufficiently large finite sets $I \subseteq \omega$ there is a nonempty family $\mathcal{A}_I$ consisting of sets $A \subseteq 2^I$ with $\frac{|A|}{2^{|I|}} \leq \varepsilon$ such that if*

[16]"Q over Q^M" just means that Q^M is an M-complete subforcing of Q.

[17]in the sense of (0.1)

[18]We thank Andreas Blass and Jindřich Zapletal for their comments that led to an improved presentation of Janus forcing.

[19]Also the trivial case $\mathbb{J} = \nabla$ is allowed.

[20]The theorem in [BS10] actually says "for a sufficiently large I", but the proof shows that this should be read as "for all sufficiently large I".

$X \subseteq 2^I$, $|X| \geq N_{\varepsilon,\delta}$ then

$$\frac{|\{A \in \mathcal{A}_I : X + A = 2^I\}|}{|\mathcal{A}_I|} \geq 1 - \delta.$$

(Recall that $X + A \coloneqq \{x + a : x \in X, a \in A\}$.)

Rephrasing and specializing to $\delta = \frac{1}{4}$ and $\varepsilon = \frac{1}{2^i}$ we get:

Corollary 2.2. *For every $i \in \omega$ there exists $B^*(i)$ such that for all finite sets I with $|I| \geq B^*(i)$ there is a nonempty family $\mathcal{A}_I$ satisfying the following:*

- *$\mathcal{A}_I$ consists of sets $A \subseteq 2^I$ with $\frac{|A|}{2^{|I|}} \leq \frac{1}{2^i}$.*
- *For every $X \subseteq 2^I$ satisfying $|X| \geq B^*(i)$, the set $\{A \in \mathcal{A}_I : X + A = 2^I\}$ has at least $\frac{3}{4}|\mathcal{A}_I|$ elements.*

Assumption 2.3. We fix a sufficiently fast increasing sequence $\bar{\ell}^* = (\ell^*_i)_{i\in\omega}$ of natural numbers; more precisely, the sequence $\bar{\ell}^*$ will be a subsequence of an ultralaver real $\bar{\ell}$, defined as in Lemma 1.23 using the function B^* from Corollary 2.2. Note that in this case $\ell^*_{i+1} - \ell^*_i \geq B^*(i)$; so we can fix for each i a family $\mathcal{A}_i \subseteq \mathscr{P}(2^{L_i})$ on the interval $L_i \coloneqq [\ell^*_i, \ell^*_{i+1})$ according to Corollary 2.2.

Definition 2.4. First we define the "core" $\nabla = \nabla_{\bar{\ell}^*}$ of our forcing:

$$\nabla = \bigcup_{i\in\omega} \prod_{j<i} \mathcal{A}_j.$$

In other words, $\sigma \in \nabla$ iff $\sigma = (A_0, \ldots, A_{i-1})$ for some $i \in \omega$, $A_0 \in \mathcal{A}_0, \ldots, A_{i-1} \in \mathcal{A}_{i-1}$. We will denote the number i by height(σ).

The forcing notion ∇ is ordered by reverse inclusion (i.e., end extension): $\tau \leq \sigma$ if $\tau \supseteq \sigma$.

Definition 2.5. Let $\bar{\ell}^* = (\ell^*_i)_{i\in\omega}$ be as in the assumption above. We say that $\mathbb{J}$ is a Janus forcing based on $\bar{\ell}^*$ if:

(1) $(\nabla, \supseteq)$ is an incompatibility-preserving subforcing of $\mathbb{J}$.
(2) For each $i \in \omega$ the set $\{\sigma \in \nabla : \text{height}(\sigma) = i\}$ is predense in $\mathbb{J}$. So in particular, $\mathbb{J}$ adds a branch through ∇. The union of this branch is called $\underset{\sim}{C}^\nabla = (\underset{\sim}{C}^\nabla_0, \underset{\sim}{C}^\nabla_1, \underset{\sim}{C}^\nabla_2, \ldots)$, where $\underset{\sim}{C}^\nabla_i \subseteq 2^{L_i}$ with $\underset{\sim}{C}^\nabla_i \in \mathcal{A}_i$.
(3) "Fatness":[21] For all $p \in \mathbb{J}$ and all real numbers $\varepsilon > 0$ there are arbitrarily large $i \in \omega$ such that there is a core condition $\sigma = (A_0, \ldots, A_{i-1}) \in \nabla$ (of length i) with

$$\frac{|\{A \in \mathcal{A}_i : \sigma^\frown A \parallel_{\mathbb{J}} p\}|}{|\mathcal{A}_i|} \geq 1 - \varepsilon.$$

(Recall that $p \parallel_{\mathbb{J}} q$ means that p and q are compatible in $\mathbb{J}$.)
(4) $\mathbb{J}$ is ccc.
(5) $\mathbb{J}$ is separative.[22]
(6) (To simplify some technicalities:) $\mathbb{J} \subseteq H(\aleph_1)$.

We now define $\underset{\sim}{Z}_\nabla$, which will be a canonical $\mathbb{J}$-name of (a code for) a null set. We will use the sequence $\underset{\sim}{C}^\nabla$ added by $\mathbb{J}$ (see Definition 2.5(2)).

Definition 2.6. Each $\underset{\sim}{C}^\nabla_i$ defines a clopen set $\underset{\sim}{Z}^\nabla_i = \{x \in 2^\omega : x\restriction L_i \in \underset{\sim}{C}^\nabla_i\}$ of measure at most $\frac{1}{2^i}$. The sequence $\underset{\sim}{Z}_\nabla = (\underset{\sim}{Z}^\nabla_0, \underset{\sim}{Z}^\nabla_1, \underset{\sim}{Z}^\nabla_2, \ldots)$ is (a name for) a code for the null set

$$\text{nullset}(\underset{\sim}{Z}_\nabla) = \bigcap_{n<\omega} \bigcup_{i\geq n} \underset{\sim}{Z}^\nabla_i.$$

Since $\underset{\sim}{C}^\nabla$ is defined "canonically" (see in particular Definition 2.5(1),(2)), and $\underset{\sim}{Z}^\nabla$ is constructed in an absolute way from $\underset{\sim}{C}^\nabla$, we get:

[21] This is the crucial combinatorial property of Janus forcing. Actually, (3) implies (2).
[22] Separative is defined on page 3.

Fact 2.7. If $\mathbb{J}$ is a Janus forcing, M a countable model and $\mathbb{J}^M$ a Janus forcing in M which is an M-complete subset of $\mathbb{J}$, if H is $\mathbb{J}$-generic over V and H^M the induced $\mathbb{J}^M$-generic filter over M, then $\underset{\sim}{C}^\nabla$ evaluates to the same real in $M[H^M]$ as in $V[H]$, and therefore $\underset{\sim}{Z}^\nabla$ evaluates to the same code (but of course not to the same set of reals).

For later reference, we record the following trivial fact:

Fact 2.8. Being a Janus forcing is absolute. In particular, if $V \subseteq W$ are set theoretical universes and $\mathbb{J}$ is a Janus forcing in V, then $\mathbb{J}$ is a Janus forcing in W. In particular, if M is a countable model in V and $\mathbb{J} \in M$ a Janus forcing in M, then $\mathbb{J}$ is also a Janus forcing in V.
Let $(M^n)_{n\in\omega}$ be an increasing sequence of countable models, and let $\mathbb{J}^n \in M^n$ be Janus forcings. Assume that $\mathbb{J}^n$ is M^n-complete in $\mathbb{J}^{n+1}$. Then $\bigcup_n \mathbb{J}^n$ is a Janus forcing, and an M^n-complete extension of $\mathbb{J}^n$ for all n.

2.B. **Janus and strongly meager.** Carlson [Car93] showed that Cohen reals make every uncountable set X of the ground model not strongly meager in the extension (and that not being strongly meager is preserved in a subsequent forcing with precaliber $\aleph_1$). We show that a *countable* Janus forcing $\mathbb{J}$ does the same (for a subsequent forcing that is even σ-centered, not just precaliber $\aleph_1$). This sounds trivial, since any (nontrivial) countable forcing is equivalent to Cohen forcing anyway. However, we show (and will later use) that the canonical null set $\underset{\sim}{Z}_\nabla$ defined above witnesses that X is not strongly meager (and not just some null set that we get out of the isomorphism between $\mathbb{J}$ and Cohen forcing). The point is that while ∇ is not a complete subforcing of $\mathbb{J}$, the condition (3) of the Definition 2.5 guarantees that Carlson's argument still works, if we assume that X is non-thin (not just uncountable). This is enough for us, since by Corollary 1.24 ultralaver forcing makes any uncountable set non-thin.

Recall that we fixed the increasing sequence $\bar{\ell}^* = (\ell^*_i)_{i\in\omega}$ and B^*. In the following, whenever we say "(very) thin" we mean "(very) thin with respect to $\bar{\ell}^*$ and B^*" (see Definition 1.22).

Lemma 2.9. *If X is not thin, $\mathbb{J}$ is a countable Janus forcing based on $\bar{\ell}^*$, and $\underset{\sim}{R}$ is a $\mathbb{J}$-name for a σ-centered forcing notion, then $\mathbb{J} * \underset{\sim}{R}$ forces that X is not strongly meager witnessed by the null set $\underset{\sim}{Z}_\nabla$.*

Proof. Let $\underset{\sim}{c}$ be a $\mathbb{J}$-name for a function $\underset{\sim}{c} : \underset{\sim}{R} \to \omega$ witnessing that $\underset{\sim}{R}$ is σ-centered.

Recall that "$\underset{\sim}{Z}_\nabla$ witnesses that X is not strongly meager" means that $X + \underset{\sim}{Z}_\nabla = 2^\omega$. Assume towards a contradiction that $(p, r) \in \mathbb{J} * \underset{\sim}{R}$ forces that $X + \underset{\sim}{Z}_\nabla \neq 2^\omega$. Then we can fix a $(\mathbb{J} * \underset{\sim}{R})$-name $\underset{\sim}{\xi}$ such that $(p, r) \Vdash \underset{\sim}{\xi} \notin X + \underset{\sim}{Z}_\nabla$, i.e., $(p, r) \Vdash (\forall x \in X)\ \underset{\sim}{\xi} \notin x + \underset{\sim}{Z}_\nabla$. By definition of $\underset{\sim}{Z}_\nabla$, we get

$$(p, r) \Vdash (\forall x \in X)\, (\exists n \in \omega)\, (\forall i \geq n)\ \underset{\sim}{\xi}{\restriction}L_i \notin x{\restriction}L_i + \underset{\sim}{C}^\nabla_i.$$

For each $x \in X$ we can find $(p_x, r_x) \leq (p, r)$ and natural numbers $n_x \in \omega$ and $m_x \in \omega$ such that p_x forces that $\underset{\sim}{c}(r_x) = m_x$ and

$$(p_x, r_x) \Vdash (\forall i \geq n_x)\ \underset{\sim}{\xi}{\restriction}L_i \notin x{\restriction}L_i + \underset{\sim}{C}^\nabla_i.$$

So $X = \bigcup_{p\in\mathbb{J},m\in\omega,n\in\omega} X_{p,m,n}$, where $X_{p,m,n}$ is the set of all x with $p_x = p$, $m_x = m$, $n_x = n$. (Note that $\mathbb{J}$ is countable, so the union is countable.) As X is not thin, there is some p^*, m^*, n^* such that $X^* := X_{p^*,m^*,n^*}$ is not very thin. So we get for all $x \in X^*$:

$$(p^*, r_x) \Vdash (\forall i \geq n^*)\ \underset{\sim}{\xi}{\restriction}L_i \notin x{\restriction}L_i + \underset{\sim}{C}^\nabla_i. \tag{2.10}$$

Since X^* is not very thin, there is some $i_0 \in \omega$ such that for all $i \geq i_0$

$$\text{the (finite) set } X^*{\restriction}L_i \text{ has more than } B^*(i) \text{ elements.} \tag{2.11}$$

Due to the fact that $\mathbb{J}$ is a Janus forcing (see Definition 2.5 (3)), there are arbitrarily large $i \in \omega$ such that there is a core condition $\sigma = (A_0, \ldots, A_{i-1}) \in \nabla$ with

$$\frac{|\{A \in \mathcal{A}_i :\ \sigma^\frown A \parallel_{\mathbb{J}} p^*\}|}{|\mathcal{A}_i|} \geq \frac{2}{3}. \tag{2.12}$$

Fix such an i larger than both i_0 and n^*, and fix a condition σ satisfying (2.12).

We now consider the following two subsets of $\mathcal{A}_i$:

$$\{A \in \mathcal{A}_i :\ \sigma^\frown A \parallel_{\mathbb{J}} p^*\} \ \text{ and } \ \{A \in \mathcal{A}_i :\ X^*{\restriction}L_i + A = 2^{L_i}\}. \tag{2.13}$$

By (2.12), the relative measure (in $\mathcal{A}_i$) of the left one is at least $\frac{2}{3}$; due to (2.11) and the definition of $\mathcal{A}_i$ according to Corollary 2.2, the relative measure of the right one is at least $\frac{3}{4}$; so the two sets in (2.13) are not disjoint, and we can pick an A belonging to both.

Clearly, $\sigma^\frown A$ forces (in $\mathbb{J}$) that $\underset{\sim}{C}_i^\nabla$ is equal to A. Fix $q \in \mathbb{J}$ witnessing $\sigma^\frown A \parallel_{\mathbb{J}} p^*$. Then

$$q \Vdash_{\mathbb{J}} X^* \restriction L_i + \underset{\sim}{C}_i^\nabla = X^* \restriction L_i + A = 2^{L_i}. \tag{2.14}$$

Since p^* forces that for each $x \in X^*$ the color $\underset{\sim}{c}(r_x) = m^*$, we can find an r^* which is (forced by $q \leq p^*$ to be) a lower bound of the *finite* set $\{r_x : x \in X^{**}\}$, where $X^{**} \subseteq X^*$ is any finite set with $X^{**} \restriction L_i = X^* \restriction L_i$.

By (2.10),

$$(q, r^*) \Vdash \underset{\sim}{\xi} \restriction L_i \notin X^{**} \restriction L_i + \underset{\sim}{C}_i^\nabla = X^* \restriction L_i + \underset{\sim}{C}_i^\nabla,$$

contradicting (2.14). □

Corollary 2.15. *Let X be uncountable. If $\mathbb{L}_{\bar{D}}$ is any ultralaver forcing adding an ultralaver real $\bar{\ell}$, and $\bar{\ell}^*$ is defined from $\bar{\ell}$ as in Lemma 1.23, and if $\underset{\sim}{\mathbb{J}}$ is a countable Janus forcing based on $\bar{\ell}^*$, $\underset{\sim}{Q}$ is any σ-centered forcing, then $\mathbb{L}_{\bar{D}} * \underset{\sim}{\mathbb{J}} * \underset{\sim}{Q}$ forces that X is not strongly meager.*

2.C. **Janus forcing and preservation of Lebesgue positivity.** We show that every Janus forcing in a countable model M can be extended to locally preserve a given random real over M. (We showed the same for ultralaver forcing in Section 1.E.)

We start by proving that every countable Janus forcing can be embedded into a Janus forcing which is equivalent to random forcing, preserving the maximality of countably many maximal antichains. (In the following lemma, the letter M is just a label to distinguish $\mathbb{J}^M$ from $\mathbb{J}$, and does not necessarily refer to a model.)

Lemma 2.16. *Let $\mathbb{J}^M$ be a countable Janus forcing (based on $\bar{\ell}^*$) and let $\{D_k : k \in \omega\}$ be a countable family of open dense subsets of $\mathbb{J}^M$. Then there is a Janus forcing $\mathbb{J}$ (based on the same $\bar{\ell}^*$) such that*

- *$\mathbb{J}^M$ is an incompatibility-preserving subforcing of $\mathbb{J}$.*
- *Each D_k is still predense in $\mathbb{J}$.*
- *$\mathbb{J}$ is forcing equivalent to random forcing.*

Proof. Recall that $\nabla = \nabla^{\mathbb{J}^M}$ was defined in Definition 2.4. Note that for each j the set $\{\sigma \in \nabla : \text{height}(\sigma) = j\}$ is predense in $\mathbb{J}^M$, so the set

$$E_j := \{p \in \mathbb{J}^M : \exists \sigma \in \nabla : \text{height}(\sigma) = j,\ p \leq \sigma\} \tag{2.17}$$

is dense open in $\mathbb{J}^M$; hence without loss of generality each E_j appears in our list of D_k's.

Let $\{r^n : n \in \omega\}$ be an enumeration of $\mathbb{J}^M$.

We now fix n for a while (up to (2.19)). We will construct a finitely splitting tree $S^n \subseteq \omega^{<\omega}$ and a family $(\sigma_s^n, p_s^n, \tau_s^{*n})_{s \in S^n}$ satisfying the following (suppressing the superscript n):

(a) $\sigma_s \in \nabla$, $\sigma_{\langle\rangle} = \langle\rangle$, $s \subseteq t$ implies $\sigma_s \subseteq \sigma_t$, and $s \perp_{S^n} t$ implies $\sigma_s \perp_\nabla \sigma_t$.
 (So in particular the set $\{\sigma_t : t \in \text{succ}_{S^n}(s)\}$ is a (finite) antichain above σ_s in ∇.)
(b) $p_s \in \mathbb{J}^M$, $p_{\langle\rangle} = r^n$; if $s \subseteq t$ then $p_t \leq_{\mathbb{J}^M} p_s$ (hence $p_t \leq r^n$); $s \perp_{S^n} t$ implies $p_s \perp_{\mathbb{J}^M} p_t$.
(c) $p_s \leq_{\mathbb{J}^M} \sigma_s$.
(d) $\sigma_s \subseteq \tau_s^* \in \nabla$, and $\{\sigma_t : t \in \text{succ}_{S^n}(s)\}$ is the set of all $\tau \in \text{succ}_\nabla(\tau_s^*)$ which are compatible with p_s.
(e) The set $\{\sigma_t : t \in \text{succ}_{S^n}(s)\}$ is a subset of $\text{succ}_\nabla(\tau_s^*)$ of relative size at least $1 - \frac{1}{\text{lh}(s)+10}$.
(f) Each $s \in S^n$ has at least 2 successors (in S^n).
(g) If $k = \text{lh}(s)$, then $p_s \in D_k$ (and therefore also in all D_l for $l < k$).

Set $\sigma_{\langle\rangle} = \langle\rangle$ and $p_{\langle\rangle} = r^n$. Given s, σ_s and p_s, we construct $\text{succ}_{S^n}(s)$ and $(\sigma_t, p_t)_{t \in \text{succ}_{S^n}(s)}$: We apply fatness 2.5(3) to p_s with $\varepsilon = \frac{1}{\text{lh}(s)+10}$. So we get some $\tau_s^* \in \nabla$ of height bigger than the height of σ_s such that the set B of elements of $\text{succ}_\nabla(\tau_s^*)$ which are compatible with p_s has relative size at least $1 - \varepsilon$. Since $p_s \leq_{\mathbb{J}^M} \sigma_s$ we get that τ_s^* is compatible with (and therefore stronger than) σ_s. Enumerate B as $\{\tau_0, \ldots, \tau_{l-1}\}$. Set $\text{succ}_{S^n}(s) = \{s^\frown i : i < l\}$ and $\sigma_{s^\frown i} = \tau_i$. For $t \in \text{succ}_{S^n}(s)$, choose $p_t \in \mathbb{J}^M$ stronger than both σ_t and p_s (which is obviously possible since σ_t and p_s are compatible), and moreover $p_t \in D_{\text{lh}(t)}$. This concludes the construction of the family $(\sigma_s^n, p_s^n, \tau_s^{*n})_{s \in S^n}$.

So $(S^n, \subseteq)$ is a finitely splitting nonempty tree of height ω with no maximal nodes and no isolated branches. $[S^n]$ is the (compact) set of branches of S^n. The closed subsets of $[S^n]$ are exactly the sets of the form $[T]$, where $T \subseteq S^n$ is a subtree of S^n with no maximal nodes. $[S^n]$ carries a natural ("uniform") probability measure μ_n, which is characterized by

$$\mu_n((S^n)^{[t]}) = \frac{1}{|\mathrm{succ}_{S^n}(s)|} \cdot \mu_n((S^n)^{[s]})$$

for all $s \in S^n$ and all $t \in \mathrm{succ}_{S^n}(s)$. (We just write $\mu_n(T)$ instead of $\mu_n([T])$ to increase readability.)

We call $T \subseteq S^n$ positive if $\mu_n(T) > 0$, and we call T pruned if $\mu_n(T^{[s]}) > 0$ for all $s \in T$. (Clearly every tree T contains a pruned tree T' of the same measure, which can be obtained from T by removing all nodes s with $\mu_n(T^{[s]}) = 0$.)

Let $T \subseteq S^n$ be a positive, pruned tree and $\varepsilon > 0$. Then on all but finitely many levels k there is an $s \in T$ such that

(2.18) $$\mathrm{succ}_T(s) \subseteq \mathrm{succ}_{S^n}(s) \text{ has relative size } \geq 1 - \varepsilon.$$

(This follows from Lebesgue's density theorem, or can easily be seen directly: Set $C_m = \bigcup_{t \in T,\, \mathrm{lh}(t)=m} (S^n)^{[t]}$. Then C_m is a decreasing sequence of closed sets, each containing $[T]$. If the claim fails, then $\mu_n(C_{m+1})) \leq \mu_n(C_m) \cdot (1-\varepsilon)$ infinitely often; so $\mu_n(T) \leq \mu_n(\bigcap_m C_m) = 0$.)

It is well known that the set of positive, pruned subtrees of S^n, ordered by inclusion, is forcing equivalent to random forcing (which can be defined as the set of positive, pruned subtrees of $2^{<\omega}$).

We have now constructed S^n for all n. Define

(2.19) $$\mathbb{J} = \mathbb{J}^M \cup \bigcup_n \{ (n, T) : \ T \subseteq S^n \text{ is a positive pruned tree} \}$$

with the following partial order:

- The order on $\mathbb{J}$ extends the order on $\mathbb{J}^M$.
- $(n', T') \leq (n, T)$ if $n = n'$ and $T' \subseteq T$.
- For $p \in \mathbb{J}^M$: $(n, T) \leq p$ if there is a k such that $p^n_t \leq p$ for all $t \in T$ of length k. (Note that this will then be true for all bigger k as well.)
- $p \leq (n, T)$ never holds (for $p \in \mathbb{J}^M$).

The lemma now easily follows from the following properties:

(1) The order on $\mathbb{J}$ is transitive.
(2) $\mathbb{J}^M$ is an incompatibility-preserving subforcing of $\mathbb{J}$.
In particular, $\mathbb{J}$ satisfies item (1) of Definition 2.5 of Janus forcing.
(3) For all k: the set $\{(n, T^{[t]}) : \ t \in T, \ \mathrm{lh}(t) = k\}$ is a (finite) predense antichain below (n, T).
(4) $(n, T^{[t]})$ is stronger than p^n_t for each $t \in T$ (witnessed, e.g., by $k = \mathrm{lh}(t)$). Of course, $(n, T^{[t]})$ is stronger than (n, T) as well.
(5) Since $p^n_t \in D_k$ for $k = \mathrm{lh}(t)$, this implies that each D_k is predense below each (n, S^n) and therefore in $\mathbb{J}$.
Also, since each set E_j appeared in our list of open dense subsets (see (2.17)), the set $\{\sigma \in \nabla : \mathrm{height}(\sigma) = j\}$ is still predense in $\mathbb{J}$, i.e., item (2) of the Definition 2.5 of Janus forcing is satisfied.
(6) The condition (n, S^n) is stronger than r^n, so $\{(n, S^n) : n \in \omega\}$ is predense in $\mathbb{J}$ and $\mathbb{J} \setminus \mathbb{J}^M$ is dense in $\mathbb{J}$.
Below each (n, S^n), the forcing $\mathbb{J}$ is isomorphic to random forcing.
Therefore, $\mathbb{J}$ itself is forcing equivalent to random forcing. (In fact, the complete Boolean algebra generated by $\mathbb{J}$ is isomorphic to the standard random algebra, Borel sets modulo null sets.) This proves in particular that $\mathbb{J}$ is ccc, i.e., satisfies property 2.5(4).
(7) It is easy (but not even necessary) to check that $\mathbb{J}$ is separative, i.e., property 2.5(5). In any case, we could replace $\leq_{\mathbb{J}}$ by $\leq^*_{\mathbb{J}}$, thus making $\mathbb{J}$ separative without changing $\leq_{\mathbb{J}^M}$, since $\mathbb{J}^M$ was already separative.
(8) Property 2.5(6), i.e., $\mathbb{J} \in H(\aleph_1)$, is obvious.
(9) The remaining item of the definition of Janus forcing, fatness 2.5(3), is satisfied.
I.e., given $(n, T) \in \mathbb{J}$ and $\varepsilon > 0$ there is an arbitrarily high $\tau^* \in \nabla$ such that the relative size of the set $\{\tau \in \mathrm{succ}_\nabla(\tau^*) : \ \tau \parallel (n, T)\}$ is at least $1 - \varepsilon$. (We will show $\geq (1-\varepsilon)^2$ instead, to simplify the notation.)

We show (9): Given $(n, T) \in \mathbb{J}$ and $\varepsilon > 0$, we use (2.18) to get an arbitrarily high $s \in T$ such that $\mathrm{succ}_T(s)$ is of relative size $\geq 1 - \varepsilon$ in $\mathrm{succ}_{S^n}(s)$. We may choose s of length $> \frac{1}{\varepsilon}$. We claim that τ^*_s is as required:

- Let $B := \{\sigma_t : t \in \text{succ}_{S^n}(s)\}$. Note that $B = \{\tau \in \text{succ}_\nabla(\tau^*_s) : \tau \parallel p_s\}$.
  B has relative size $\geq 1 - \frac{1}{\text{lh}(s)} \geq 1 - \varepsilon$ in $\text{succ}_\nabla(\tau^*_s)$ (according to property (e) of S^n).
- $C := \{\sigma_t : t \in \text{succ}_T(s)\}$ is a subset of B of relative size $\geq 1 - \varepsilon$ according to our choice of s.
- So C is of relative size $(1 - \varepsilon)^2$ in $\text{succ}_\nabla(\tau^*_s)$.
- Each $\sigma_t \in C$ is compatible with (n, T), as $(n, T^{[t]}) \leq p_t \leq \sigma_t$ (see (4)). □

So in particular if $\mathbb{J}^M$ is a Janus forcing in a countable model M, then we can extend it to a Janus forcing $\mathbb{J}$ which is in fact random forcing. Since random forcing strongly preserves randoms over countable models (see Lemma 1.29), it is not surprising that we get local preservation of randoms for Janus forcing, i.e., the analoga of Lemma 1.30 and Corollary 1.51. (Still, some additional argument is needed, since the fact that $\mathbb{J}$ (which is now random forcing) "strongly preserves randoms" just means that a random real r over M is preserved with respect to random forcing in M, not with respect to $\mathbb{J}^M$.)

Lemma 2.20. *If $\mathbb{J}^M$ is a Janus forcing in a countable model M and r a random real over M, then there is a Janus forcing $\mathbb{J}$ such that $\mathbb{J}^M$ is an M-complete subforcing of $\mathbb{J}$ and the following holds:*
If

- *$p \in \mathbb{J}^M$,*
- *in M, $\bar{\underset{\sim}{Z}} = (\underset{\sim}{Z}_1, \dots, \underset{\sim}{Z}_m)$ is a sequence of $\mathbb{J}^M$-names for codes for null sets, and $Z^*_1, \dots, Z^*_m$ are interpretations under p, witnessed by a sequence $(p_n)_{n\in\omega}$,*
- *$Z^*_i \sqsubset_{k_i} r$ for $i = 1, \dots, m$,*

***then** there is a $q \leq p$ in $\mathbb{J}$ forcing that*

- *r is random over $M[G^M]$,*
- *$\underset{\sim}{Z}_i \sqsubset_{k_i} r$ for $i = 1, \dots, m$.*

Remark 2.21. In the version for ultralaver forcings, i.e., Lemma 1.30, we had to assume that the stems of the witnessing sequence are strictly increasing. In the Janus version, we do not have any requirement of that kind.

Proof. Let $\mathcal{D}$ be the set of dense subset of $\mathbb{J}^M$ in M. According to Lemma 1.44, we can first find some countable M' such that r is still random over M' and such that in M' both $\mathbb{J}^M$ and $\mathcal{D}$ are countable. According to Fact 2.8, $\mathbb{J}^M$ is a (countable) Janus forcing in M', so we can apply Lemma 2.16 to the set $\mathcal{D}$ to construct a Janus forcing $\mathbb{J}^{M'}$ which is equivalent to random forcing such that (from the point of V) $\mathbb{J}^M \lessdot_M \mathbb{J}^{M'}$. In V, let $\mathbb{J}$ be random forcing; since this is a Suslin ccc forcing we know that $\mathbb{J}^{M'}$ is an M'-complete subforcing of $\mathbb{J}$ and therefore that $\mathbb{J}^M \lessdot_M \mathbb{J}$. Moreover, as was noted in Lemma 1.29, we even know that random forcing strongly preserves randoms over M' (see Definition 1.50).

So assume that (in M) the sequence $(p_n)_{n\in\omega}$ of $\mathbb{J}^M$-conditions interprets $\bar{\underset{\sim}{Z}}$ as $\bar{Z}^*$. In M', $\mathbb{J}^M$-names can be reinterpreted as $\mathbb{J}^{M'}$-names, and the $\mathbb{J}^{M'}$-name $\bar{\underset{\sim}{Z}}$ is interpreted as $\bar{Z}^*$ by the same sequence $(p_n)_{n\in\omega}$. Let $k_1, \dots, k_m$ be such that $Z^*_i \sqsubset_{k_i} r$ for $i = 1, \dots, m$. So by strong preservation of randoms, we can in V find some $q \leq p_0$ forcing that r is random over $M'[H^{M'}]$ (and therefore also over the subset $M[H^M]$), and that $\underset{\sim}{Z}_i \sqsubset_{k_i} r$ (where $\underset{\sim}{Z}_i$ can be evaluated in $M'[H^{M'}]$ or equivalently in $M[H^M]$). □

So Janus forcing is locally preserving randoms (just as ultralaver forcing):

Corollary 2.22. *If Q^M is a Janus forcing in M and r a real, then there is a Janus forcing Q over Q^M (which is in fact equivalent to random forcing) locally preserving randomness of r over M.*

Proof. In this case, the notion of "quick" interpretations is trivial, i.e., $D^{Q^M}_k = Q^M$ for all k, and the claim follows from the previous lemma. □

3. Almost finite and almost countable support iterations

A main tool to construct the forcing for BC+dBC will be "partial countable support iterations", more particularly "almost finite support" and "almost countable support" iterations. A partial countable support iteration is a forcing iteration $(P_\alpha, Q_\alpha)_{\alpha<\omega_2}$ such that for each limit ordinal δ the forcing notion P_δ is a subset of the countable support limit of $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ which satisfies some natural properties (see Definition 3.6).

Instead of transitive models, we will use ord-transitive models (which are transitive when ordinals are considered as urelements). Why do we do that? We want to "approximate" the generic iteration $\bar{\mathbf{P}}$ of length

ω_2 with countable models; this can be done more naturally with ord-transitive models (since obviously countable transitive models only see countable ordinals). We call such an ord-transitive model a "candidate" (provided it satisfies some nice properties, see Definition 3.1). A basic point is that forcing extensions work naturally with candidates.

In the following, $x = (M^x, \bar{P}^x)$ will denote a pair such that M^x is a candidate and $\bar{P}^x$ is (in M^x) a partial countable support iteration; similarly we write, e.g., $y = (M^y, \bar{P}^y)$ or $x_n = (M^{x_n}, \bar{P}^{x_n})$.

We will need the following results to prove BC+dBC. (However, as opposed to the case of the ultralaver and Janus section, the reader will probably have to read this section to understand the construction in the next section, and not just the following list of properties.)

Given $x = (M^x, \bar{P}^x)$, we can construct by induction on α a partial countable support iteration $\bar{P} = (P_\alpha, Q_\alpha)_{\alpha<\omega_2}$ satisfying:

There is a canonical M^x-complete embedding from $\bar{P}^x$ to $\bar{P}$.

In this construction, we can use at each stage β any desired Q_β, as long as P_β forces that Q^x_β is (evaluated as) an $M^x[H^x_\beta]$-complete subforcing of Q_β (where $H^x_\beta \subseteq P^x_\beta$ is the M^x-generic filter induced by the generic filter $H_\beta \subseteq P_\beta$).
Moreover, we can demand either of the following two additional properties[23] of the limit of this iteration $\bar{P}$:

(1) If all Q_β are forced to be σ-centered, and Q_β is trivial for all $\beta \notin M^x$, then P_{ω_2} is σ-centered.
(2) If r is random over M^x, and all Q_β locally preserve randomness of r over $M^x[H^x_\beta]$ (see Definition 1.50), then also P_{ω_2} locally preserves the randomness of r.

Actually, we need the following variant: Assume that we already have P_{α_0} for some $\alpha_0 \in M^x$, and that $P^x_{\alpha_0}$ canonically embeds into P_{α_0}, and that the respective assumption on Q_β holds for all $\beta \geq \alpha_0$. Then we get that P_{α_0} forces that the quotient $P_{\omega_2}/P_{\alpha_0}$ satisfies the respective conclusion.

We also need:[24]

(3) If instead of a single x we have a sequence x_n such that each P^{x_n} canonically (and M^{x_n}-completely) embeds into $P^{x_{n+1}}$, then we can find a partial countable support iteration $\bar{P}$ into which all P^{x_n} embed canonically (and we can again use any desired Q_β, assuming that $Q^{x_n}_\beta$ is an $M^{x_n}[H^{x_n}_\beta]$-complete subforcing of Q_β for all $n \in \omega$).
(4) (A fact that is easy to prove but awkward to formulate.) If a Δ-system argument produces two x_1, x_2 as in Lemma 4.7(3), then we can find a partial countable support iteration $\bar{P}$ such that $\bar{P}^{x_i}$ canonically (and M^{x_i}-completely) embeds into $\bar{P}$ for $i = 1, 2$.

3.A. **Ord-transitive models.** We will use "ord-transitive" models, as introduced in [She04] (see also the presentation in [Kel]). We briefly summarize the basic definitions and properties (restricted to the rather simple case needed in this paper):

Definition 3.1. Fix a suitable finite subset ZFC* of ZFC (that is satisfied by $H(\chi^*)$ for sufficiently large regular χ^*).

(1) A set M is called a *candidate*, if
  - M is countable,
  - $(M, \in)$ is a model of ZFC*,
  - M is ord-absolute: $M \models \alpha \in \mathrm{Ord}$ iff $\alpha \in \mathrm{Ord}$, for all $\alpha \in M$,
  - M is ord-transitive: if $x \in M \setminus \mathrm{Ord}$, then $x \subseteq M$,
  - $\omega + 1 \subseteq M$.
  - "α is a limit ordinal" and "$\alpha = \beta + 1$" are both absolute between M and V.
(2) A candidate M is called *nice*, if "α has countable cofinality" and "the countable set A is cofinal in α" both are absolute between M and V. (So if $\alpha \in M$ has countable cofinality, then $\alpha \cap M$ is cofinal in α.) Moreover, we assume $\omega_1 \in M$ (which implies $\omega_1{}^M = \omega_1$) and $\omega_2 \in M$ (but we do not require $\omega_2{}^M = \omega_2$).
(3) Let P^M be a forcing notion in a candidate M. (To simplify notation, we can assume without loss of generality that $P^M \cap \mathrm{Ord} = \emptyset$ (or at least $\subseteq \omega$) and that therefore $P^M \subseteq M$ and also $A \subseteq M$

[23] The σ-centered version is central for the proof of dBC; the random preserving version for BC.
[24] This will give σ-closure and $\aleph_2$-cc for the preparatory forcing $\mathbb{R}$.

whenever M thinks that A is a subset of P^M.) Recall that a subset H^M of P^M is M-generic (or: P^M-generic over M), if $|A \cap H^M| = 1$ for all maximal antichains A in M.

(4) Let H^M be P^M-generic over M and $\underset{\sim}{\tau}$ a P^M-name in M. We define the evaluation $\underset{\sim}{\tau}[H^M]^M$ to be x if M thinks that $p \Vdash_{P^M} \underset{\sim}{\tau} = \check{x}$ for some $p \in H^M$ and $x \in M$ (or equivalently just for $x \in M \cap \mathrm{Ord}$), and $\{\underset{\sim}{\sigma}[H^M]^M : (\underset{\sim}{\sigma}, p) \in \underset{\sim}{\tau},\ p \in H^M\}$ otherwise. Abusing notation we write $\underset{\sim}{\tau}[H^M]$ instead of $\underset{\sim}{\tau}[H^M]^M$, and we write $M[H^M]$ for $\{\underset{\sim}{\tau}[H^M] : \underset{\sim}{\tau}$ is a P^M-name in $M\}$.

(5) The ord-collapse k (or k^M) is a recursively defined function with domain M: $k(x) = x$ if $x \in \mathrm{Ord}$, and $k(x) = \{k(y) : y \in x \cap M\}$ otherwise.

(6) The ord-transitive closure of a set x is defined inductively on the rank:

$$\mathrm{ordclos}(x) = x \cup \bigcup\{\mathrm{ordclos}(y) : y \in x \setminus \mathrm{Ord}\}.$$

So ordclos(x) is the smallest ord-transitive set containing x as a subset. HCON is the collection of all sets x such that the ord-transitive closure of x is countable. x is in HCON iff x is element of some candidate. In particular, all reals and all ordinals are HCON.

We write HCON_α for the family of all sets x in HCON whose ord-transitive closure (or, in this case equivalently, transitive closure) only contains ordinals $< \alpha$.

The following facts can be found in [She04] or [Kel] (they can be proven by rather straightforward, if tedious, inductions on the ranks of the according objects).

Fact 3.2. (1) The ord-collapse of a countable elementary submodel of $H(\chi^*)$ is a nice candidate.

(2) Unions, intersections etc. are generally not absolute for candidates. For example, let $x \in M \setminus \mathrm{Ord}$. In M we can construct a set y such that $M \models y = \omega_1 \cup \{x\}$. Then y is not an ordinal and therefore a subset of M, and in particular y is countable and $y \neq \omega_1 \cup \{x\}$.

(3) Let $j : M \to M'$ be the transitive collapse of a candidate M, and $f : \omega_1 \cap M' \to \mathrm{Ord}$ the inverse (restricted to the ordinals). Obviously M' is a countable transitive model of ZFC*; moreover M is characterized by the pair (M', f) (we call such a pair a "labeled transitive model"). Note that f satisfies $f(\alpha + 1) = f(\alpha) + 1$, $f(\alpha) = \alpha$ for $\alpha \in \omega \cup \{\omega\}$. $M \models (\alpha$ is a limit) iff $f(\alpha)$ is a limit. $M \models \mathrm{cf}(\alpha) = \omega$ iff $\mathrm{cf}(f(\alpha)) = \omega$, and in that case $f[\alpha]$ is cofinal in α. On the other hand, given a transitive countable model M' of ZFC* and an f as above, then we can construct a (unique) candidate M corresponding to (M', f).

(4) All candidates M with $M \cap \mathrm{Ord} \subseteq \omega_1$ are hereditarily countable, so their number is at most $2^{\aleph_0}$. Similarly, the cardinality of HCON_α is at most continuum whenever $\alpha < \omega_2$.

(5) If M is a candidate, and if H^M is P^M-generic over M, then $M[H^M]$ is a candidate as well and an end-extension of M such that $M \cap \mathrm{Ord} = M[H^M] \cap \mathrm{Ord}$. If M is nice and (M thinks that) P^M is proper, then $M[H^M]$ is nice as well.

(6) Forcing extensions commute with the transitive collapse j: If M corresponds to (M', f), then $H^M \subseteq P^M$ is P^M-generic over M iff $H' \coloneqq j[H^M]$ is $P' \coloneqq j(P^M)$-generic over M', and in that case $M[H^M]$ corresponds to $(M'[H'], f)$. In particular, the forcing extension of $M[H^M]$ of M satisfies the forcing theorem (everything that is forced is true, and everything true is forced).

(7) For elementary submodels, forcing extensions commute with ord-collapses: Let N be a countable elementary submodel of $H(\chi^*)$, $P \in N$, $k : N \to M$ the ord-collapse (so M is a candidate), and let H be P-generic over V. Then H is P-generic over N iff $H^M \coloneqq k[H]$ is $P^M \coloneqq k(P)$-generic over M; and in that case the ord-collapse of $N[H]$ is $M[H^M]$.

Assume that a nice candidate M thinks that $(\bar{P}^M, \bar{Q}^M)$ is a forcing iteration of length ${\omega_2}^V$ (we will usually write ω_2 for the length of the iteration, by this we will always mean ${\omega_2}^V$ and not the possibly different ${\omega_2}^M$). In this section, we will construct an iteration $(\bar{P}, \bar{Q})$ in V, also of length ω_2, such that each P^M_α canonically and M-completely embeds into P_α for all $\alpha \in \omega_2 \cap M$. Once we know (by induction) that P^M_α M-completely embeds into P_α, we know that a P_α-generic filter H_α induces a P^M_α-generic (over M) filter which we call H^M_α. Then $M[H^M_\alpha]$ is a candidate, but nice only if P^M_α is proper. We will not need that $M[H^M_\alpha]$ is nice, actually we will only investigate set of reals (or elements of $H(\aleph_1)$) in $M[H^M_\alpha]$, so it does not make any difference whether we use $M[H^M_\alpha]$ or its transitive collapse.

Remark 3.3. In the discussion so far we omitted some details regarding the theory ZFC* (that a candidate has to satisfy). The following "fine print" hopefully absolves us from any liability. (It is entirely irrelevant for the understanding of the paper.)

We have to guarantee that each $M[H^M_\alpha]$ that we consider satisfies enough of ZFC to make our arguments work (for example, the definitions and basic properties of ultralaver and Janus forcings should work). This turns out to be easy, since (as usual) we do not need the full power set axiom for these arguments (just the existence of, say, $\beth_5$). So it is enough that each $M[H^M_\alpha]$ satisfies some fixed finite subset of ZFC minus power set, which we call ZFC*.

Of course we can also find a bigger (still finite) set ZFC** that implies: $\beth_{10}$ exists, and each forcing extension of the universe with a forcing of size $\le \beth_4$ satisfies ZFC*. And it is provable (in ZFC) that each $H(\chi)$ satisfies ZFC** for sufficiently large regular χ.

We define candidate using the weaker theory ZFC*, and require that nice candidates satisfies the stronger theory ZFC**. This guarantees that all forcing extensions (by small forcings) of nice candidates will be candidates (in particular, satisfy enough of ZFC such that our arguments about Janus or ultralaver forcings work). Also, every ord-collapse of a countable elementary submodel N of $H(\chi)$ will be a nice candidate.

3.B. **Partial countable support iterations.** We introduce the notion of "partial countable support limit": a subset of the countable support (CS) limit containing the union (i.e., the direct limit) and satisfying some natural requirements.

Let us first describe what we mean by "forcing iteration". They have to satisfy the following requirements:

- A "*topless forcing iteration*" $(P_\alpha, Q_\alpha)_{\alpha<\varepsilon}$ is a sequence of forcing notions P_α and P_α-names Q_α of quasiorders with a weakest element 1_{Q_α}. A "*topped iteration*" additionally has a final limit P_ε. Each P_α is a set of partial functions on α (as, e.g., in [Gol93]). More specifically, if $\alpha < \beta \le \varepsilon$ and $p \in P_\beta$, then $p{\restriction}\alpha \in P_\alpha$. Also, $p{\restriction}\beta \Vdash_{P_\beta} p(\beta) \in Q_\beta$ for all $\beta \in \mathrm{dom}(p)$. The order on P_β will always be the "natural" one: $q \le p$ iff $q{\restriction}\alpha$ forces (in P_α) that $q^{\mathrm{tot}}(\alpha) \le p^{\mathrm{tot}}(\alpha)$ for all $\alpha < \beta$, where $r^{\mathrm{tot}}(\alpha) = r(\alpha)$ for all $\alpha \in \mathrm{dom}(r)$ and 1_{Q_α} otherwise. $P_{\alpha+1}$ consists of *all* p with $p{\restriction}\alpha \in P_\alpha$ and $p{\restriction}\alpha \Vdash p^{\mathrm{tot}}(\alpha) \in Q_\alpha$, so it is forcing equivalent to $P_\alpha * Q_\alpha$.
- $P_\alpha \subseteq P_\beta$ whenever $\alpha < \beta \le \varepsilon$. (In particular, the empty condition is an element of each P_β.)
- For any $p \in P_\varepsilon$ and any $q \in P_\alpha$ $(\alpha < \varepsilon)$ with $q \le p{\restriction}\alpha$, the partial function $q \wedge p := q \cup p{\restriction}[\alpha, \varepsilon)$ is a condition in P_ε as well (so in particular, $p{\restriction}\alpha$ is a reduction of p, hence P_α is a complete subforcing of P_ε; and $q \wedge p$ is the weakest condition in P_ε stronger than both q and p).
- Abusing notation, we usually just write $\bar{P}$ for an iteration (be it topless or topped).
- We usually write H_β for the generic filter on P_β (which induces P_α-generic filters called H_α for $\alpha \le \beta$). For topped iterations we call the filter on the final limit sometimes just H instead of H_ε.

We use the following notation for quotients of iterations:

- For $\alpha < \beta$, in the P_α-extension $V[H_\alpha]$, we let P_β/H_α be the set of all $p \in P_\beta$ with $p{\restriction}\alpha \in H_\alpha$ (ordered as in P_β). We may occasionally write P_β/P_α for the P_α-name of P_β/H_α.
- Since P_α is a complete subforcing of P_β, this is a quotient with the usual properties, in particular P_β is equivalent to $P_\alpha * (P_\beta/H_\alpha)$.

Remark 3.4. It is well known that quotients of proper countable support iterations are naturally equivalent to (names of) countable support iterations. In this paper, we can restrict our attention to proper forcings, but we do not really have countable support iterations. It turns out that it is not necessary to investigate whether our quotients can naturally be seen as iterations of any kind, so to avoid the subtle problems involved we will not consider the quotient as an iteration by itself.

Definition 3.5. Let $\bar{P}$ be a (topless) iteration of limit length ε. We define three limits of $\bar{P}$:

- The "*direct limit*" is the union of the P_α (for $\alpha < \varepsilon$). So this is the smallest possible limit of the iteration.
- The "*inverse limit*" consists of *all* partial functions p with domain $\subseteq \varepsilon$ such that $p{\restriction}\alpha \in P_\alpha$ for all $\alpha < \varepsilon$. This is the largest possible limit of the iteration.
- The "*full countable support limit* $P^{\mathrm{CS}}_\varepsilon$" of $\bar{P}$ is the inverse limit if $\mathrm{cf}(\varepsilon) = \omega$ and the direct limit otherwise.

We say that P_ε is a "*partial CS limit*", if P_ε is a subset of the full CS limit and the sequence $(P_\alpha)_{\alpha\le\varepsilon}$ is a topped iteration. In particular, this means that P_ε contains the direct limit, and satisfies the following for each $\alpha<\varepsilon$: P_ε is closed under $p\mapsto p\restriction\alpha$, and whenever $p\in P_\varepsilon$, $q\in P_\alpha$, $q\le p\restriction\alpha$, then also the partial function $q\wedge p$ is in P_ε.

So for a given topless $\bar P$ there is a well-defined inverse, direct and full CS limit. If $\mathrm{cf}(\varepsilon)>\omega$, then they all coincide. If $\mathrm{cf}(\varepsilon)=\omega$, then the direct limit and the full CS limit (=inverse limit) differ. Both of them are partial CS limits, but there are many more possibilities for partial CS limits. By definition, all of them will yield iterations.

Note that the name "CS limit" is slightly inappropriate, as the size of supports of conditions is not part of the definition. To give a more specific example: Consider a topped iteration $\bar P$ of length $\omega+\omega$ where P_ω is the direct limit and $P_{\omega+\omega}$ is the full CS limit. Let p be any element of the full CS limit of $\bar P\restriction\omega$ which is not in P_ω; then p is not in $P_{\omega+\omega}$ either. So not every countable subset of $\omega+\omega$ can appear as the support of a condition.

Definition 3.6. A forcing iteration $\bar P$ is called a "*partial CS iteration*", if

- every limit is a partial CS limit, and
- every Q_α is (forced to be) separative.[25]

The following fact can easily be proved by transfinite induction:

Fact 3.7. Let $\bar P$ be a partial CS iteration. Then for all α the forcing notion P_α is separative.

From now on, all iterations we consider will be partial CS iterations. In this paper, we will only be interested in proper partial CS iterations, but properness is not part of the definition of partial CS iteration. (The reader may safely assume that all iterations are proper.)

Note that separativity of the Q_α implies that all partial CS iterations satisfy the following (trivially equivalent) properties:

Fact 3.8. Let $\bar P$ be a topped partial CS iteration of length ε. Then:

(1) Let H be P_ε-generic. Then $p\in H$ iff $p\restriction\alpha\in H_\alpha$ for all $\alpha<\varepsilon$.
(2) For all $q,p\in P_\varepsilon$: If $q\restriction\alpha\le^* p\restriction\alpha$ for each $\alpha<\varepsilon$, then $q\le^* p$.
(3) For all $q,p\in P_\varepsilon$: If $q\restriction\alpha\le^* p\restriction\alpha$ for each $\alpha<\varepsilon$, then $q\parallel p$.

We will be concerned with the following situation:

Assume that M is a nice candidate, $\bar P^M$ is (in M) a topped partial CS iteration of length ε (a limit ordinal in M), and $\bar P$ is (in V) a topless partial CS iteration of length $\varepsilon':=\sup(\varepsilon\cap M)$. (Recall that "$\mathrm{cf}(\varepsilon)=\omega$" is absolute between M and V, and that $\mathrm{cf}(\varepsilon)=\omega$ implies $\varepsilon'=\varepsilon$.) Moreover, assume that we already have a system of M-complete coherent[26] embeddings $i_\beta: P^M_\beta\to P_\beta$ for $\beta\in\varepsilon'\cap M=\varepsilon\cap M$. (Recall that any potential partial CS limit of $\bar P$ is a subforcing of the full CS limit $P^{\mathrm{CS}}_{\varepsilon'}$.) It is easy to see that there is only one possibility for an embedding $j: P^M_\varepsilon\to P^{\mathrm{CS}}_{\varepsilon'}$ (in fact, into any potential partial CS limit of $\bar P$) that extends the i_β's naturally:

Definition 3.9. For a topped partial CS iteration $\bar P^M$ in M of length ε and a topless one $\bar P$ in V of length $\varepsilon':=\sup(\varepsilon\cap M)$ together with coherent embeddings i_β, we define $j: P^M_\varepsilon\to P^{\mathrm{CS}}_{\varepsilon'}$, the "*canonical extension*", in the obvious way: Given $p\in P^M_\varepsilon$, take the sequence of restrictions to M-ordinals, apply the functions i_β, and let $j(p)$ be the union of the resulting coherent sequence.

We do not claim that $j: P^M_\varepsilon\to P^{\mathrm{CS}}_{\varepsilon'}$ is M-complete.[27] In the following, we will construct partial CS limits $P_{\varepsilon'}$ such that $j: P^M_\varepsilon\to P_{\varepsilon'}$ is M-complete. (Obviously, one requirement for such a limit is that

[25]The reason for this requirement is briefly discussed in Section 6. Separativity, as well as the relations $\le^*$ and $=^*$, are defined on page 3.

[26]I.e., they commute with the restriction maps: $i_\alpha(p\restriction\alpha)=i_\beta(p)\restriction\alpha$ for $\alpha<\beta$ and $p\in P^M_\beta$.

[27] For example, if $\varepsilon=\varepsilon'=\omega$ and if P^M_ω is the finite support limit of a nontrivial iteration, then $j: P^M_\omega\to P^{\mathrm{CS}}_\omega$ is not complete: For notational simplicity, assume that all Q^M_n are (forced to be) Boolean algebras. In M, let c_n be (a P^M_n-name for) a nontrivial element of Q^M_n (so $\neg c_n$, the Boolean complement, is also nontrivial). Let p_n be the P^M_n-condition $(c_0,\dots,c_{n-1})$, i.e., the truth value of "$c_m\in H(m)$ for all $m<n$". Let q_n be the P^M_{n+1}-condition $(c_0,\dots,c_{n-1},\neg c_n)$, i.e., the truth value of "n is minimal with $c_n\notin H(n)$". In M, the set $A=\{q_n: n\in\omega\}$ is a maximal antichain in P^M_ω. Moreover, the sequence $(p_n)_{n\in\omega}$ is a decreasing coherent sequence, therefore $i_n(p_n)$ defines an element p_ω in P^{CS}_ω, which is clearly incompatible with all $j(q_n)$, hence $j[A]$ is not maximal.

$j[P^M_\varepsilon] \subseteq P_{\varepsilon'}$.) We will actually define two versions: The almost FS ("almost finite support") and the almost CS ("almost countable support") limit.

Note that there is only one effect that the "top" of $\bar{P}^M$ (i.e., the forcing P^M_ε) has on the canonical extension j: It determines the domain of j. In particular it will generally depend on P^M_ε whether j is complete or not. Apart from that, the value of any given $j(p)$ does not depend on P^M_ε.

Instead of arbitrary systems of embeddings i_α, we will only be interested in "canonical" ones. We assume for notational convenience that Q^M_α is a subset of Q_α (this will naturally be the case in our application anyway).

Definition 3.10 (The canonical embedding)**.** Let $\bar{P}$ be a partial CS iteration in V and $\bar{P}^M$ a partial CS iteration in M, both topped and of length $\varepsilon \in M$. We construct by induction on $\alpha \in (\varepsilon + 1) \cap M$ the canonical M-complete embeddings $i_\alpha : P^M_\alpha \to P_\alpha$. More precisely: We try to construct them, but it is possible that the construction fails. If the construction succeeds, then we say that "$\bar{P}^M$ *(canonically) embeds into* $\bar{P}$", or "*the canonical embeddings work*", or just: "$\bar{P}$ *is over* $\bar{P}^M$", or "*over* P^M_ε".

- Let $\alpha = \beta + 1$. By induction hypothesis, i_β is M-complete, so a V-generic filter $H_\beta \subseteq P_\beta$ induces an M-generic filter $H^M_\beta := i^{-1}_\beta[H_\beta] \subseteq P^M_\beta$. We require that (in the H_β extension) the set $Q^M_\beta[H^M_\beta]$ is an $M[H^M_\beta]$-complete subforcing of $Q_\beta[H_\beta]$. In this case, we define i_α in the obvious way.
- For α limit, let i_α be the canonical extension of the family $(i_\beta)_{\beta\in\alpha\cap M}$. We require that P_α contains the range of i_α, and that i_α is M-complete; otherwise the construction fails. (If $\alpha' := \sup(\alpha \cap M) < \alpha$, then i_α will actually be an M-complete map into $P_{\alpha'}$, assuming that the requirement is fulfilled.)

In this section we try to construct a partial CS iteration $\bar{P}$ (over a given $\bar{P}^M$) satisfying additional properties.

Remark 3.11. What is the role of $\varepsilon' := \sup(\varepsilon \cap M)$? When our inductive construction of $\bar{P}$ arrives at P_ε where $\varepsilon' < \varepsilon$, it would be too late[28] to take care of M-completeness of i_ε at this stage, even if all i_α work nicely for $\alpha \in \varepsilon \cap M$. Note that $\varepsilon' < \varepsilon$ implies that ε is uncountable in M, and that therefore $P^M_\varepsilon = \bigcup_{\alpha\in\varepsilon\cap M} P^M_\alpha$. So the natural extension j of the embeddings $(i_\alpha)_{\alpha\in\varepsilon\cap M}$ has range in $P_{\varepsilon'}$, which will be a complete subforcing of P_ε. So we have to ensure M-completeness already in the construction of $P_{\varepsilon'}$.

For now we just record:

Lemma 3.12. *Assume that we have topped iterations $\bar{P}^M$ (in M) of length ε and $\bar{P}$ (in V) of length $\varepsilon' :=$ $\sup(\varepsilon \cap M)$, and that for all $\alpha \in \varepsilon \cap M$ the canonical embedding $i_\alpha : P^M_\alpha \to P_\alpha$ works. Let $i_\varepsilon : P^M_\varepsilon \to P^{\mathrm{CS}}_{\varepsilon'}$ be the canonical extension.*

(1) If P^M_ε is (in M) a direct limit (which is always the case if ε has uncountable cofinality) then i_ε (might not work, but at least) has range in $P_{\varepsilon'}$ and preserves incompatibility.
(2) If i_ε has a range contained in $P_{\varepsilon'}$ and maps predense sets $D \subseteq P^M_\varepsilon$ in M to predense sets $i_\varepsilon[D] \subseteq P_{\varepsilon'}$, then i_ε preserves incompatibility (and therefore works).

Proof. (1) Since P^M_ε is a direct limit, the canonical extension i_ε has range in $\bigcup_{\alpha<\varepsilon'} P_\alpha$, which is subset of any partial CS limit $P_{\varepsilon'}$. Incompatibility in P^M_ε is the same as incompatibility in P^M_α for sufficiently large $\alpha \in \varepsilon \cap M$, so it by assumption it is preserved by i_α and hence also by i_ε.

(2) Fix $p_1, p_2 \in P^M_\varepsilon$, and assume that their images are compatible in $P_{\varepsilon'}$; we have to show that they are compatible in P^M_ε. So fix a generic filter $H \subseteq P_{\varepsilon'}$ containing $i_\varepsilon(p_1)$ and $i_\varepsilon(p_2)$.

In M, we define the following set D:

$$D := \{q \in P^M_\varepsilon : (q \leq p_1 \wedge q \leq p_2) \text{ or } (\exists\alpha < \varepsilon : q{\restriction}\alpha \perp_{P^M_\alpha} p_1{\restriction}\alpha) \text{ or } (\exists\alpha < \varepsilon : q{\restriction}\alpha \perp_{P^M_\alpha} p_2{\restriction}\alpha)\}.$$

Using Fact 3.8(3) it is easy to check that D is dense. Since i_ε preserves predensity, there is $q \in D$ such that $i_\varepsilon(q) \in H$. We claim that q is stronger than p_1 and p_2. Otherwise we would have without loss

[28] For example: Let $\varepsilon = \omega_1$ and $\varepsilon' = \omega_1 \cap M$. Assume that $P^M_{\omega_1}$ is (in M) a (or: the unique) partial CS limit of a nontrivial iteration. Assume that we have a topless iteration $\bar{P}$ of length ε' in V such that the canonical embeddings work for all $\alpha \in \omega_1 \cap M$. If we set $P_{\varepsilon'}$ to be the full CS limit, then we cannot further extend it to any iteration of length ω_1 such that the canonical embedding i_{ω_1} works: Let p_α and q_α be as in footnote 27. In M, the set $A = \{q_\alpha : \alpha \in \omega_1\}$ is a maximal antichain, and the sequence $(p_\alpha)_{\alpha\in\omega_1}$ is a decreasing coherent sequence. But in V there is an element $p_{\varepsilon'} \in P^{\mathrm{CS}}_{\varepsilon'}$ with $p_{\varepsilon'}{\restriction}\alpha = p_\alpha$ for all $\alpha \in \varepsilon \cap M$. This condition $p_{\varepsilon'}$ is clearly incompatible with all elements of $j[A] = \{j(p_\alpha) : \alpha \in \varepsilon \cap M\}$. Hence $j[A]$ is not maximal.

of generality $q\restriction\alpha \perp_{P^M_\alpha} p_1\restriction\alpha$ for some $\alpha < \varepsilon$. But the filter $H\restriction\alpha$ contains both $i_\alpha(q\restriction\alpha)$ and $i_\alpha(p_1\restriction\alpha)$, contradicting the assumption that i_α preserves incompatibility. □

3.C. **Almost finite support iterations.** Recall Definition 3.9 (of the canonical extension) and the setup that was described there: We have to find a subset $P_{\varepsilon'}$ of $P^{\mathrm{CS}}_{\varepsilon'}$ such that the canonical extension $j : P^M_\varepsilon \to P_{\varepsilon'}$ is M-complete.

We now define the almost finite support limit. (The direct limit will in general not do, as it may not contain the range $j[P^M_\varepsilon]$. The almost finite support limit is the obvious modification of the direct limit, and it is the smallest partial CS limit $P_{\varepsilon'}$ such that $j[P^M_\varepsilon] \subseteq P_{\varepsilon'}$, and it indeed turns out to be M-complete as well.)

Definition 3.13. Let ε be a limit ordinal in M, and let $\varepsilon' := \sup(\varepsilon \cap M)$. Let $\bar P^M$ be a topped iteration in M of length ε, and let $\bar P$ be a topless iteration in V of length ε'. Assume that the canonical embeddings i_α work for all $\alpha \in \varepsilon \cap M = \varepsilon' \cap M$. Let i_ε be the canonical extension. We define the *almost finite support limit of $\bar P$ over $\bar P^M$* (or: almost FS limit) as the following subforcing $P_{\varepsilon'}$ of $P^{\mathrm{CS}}_{\varepsilon'}$:

$$P_{\varepsilon'} := \{\, q \wedge i_\varepsilon(p) \in P^{\mathrm{CS}}_{\varepsilon'} :\ p \in P^M_\varepsilon \text{ and } q \in P_\alpha \text{ for some } \alpha \in \varepsilon \cap M \text{ such that } q \leq_{P_\alpha} i_\alpha(p\restriction\alpha)\,\}.$$

Note that for $\mathrm{cf}(\varepsilon) > \omega$, the almost FS limit is equal to the direct limit, as each $p \in P^M_\varepsilon$ is in fact in P^M_α for some $\alpha \in \varepsilon \cap M$, so $i_\varepsilon(p) = i_\alpha(p) \in P_\alpha$.

Lemma 3.14. *Assume that $\bar P$ and $\bar P^M$ are as above and let $P_{\varepsilon'}$ be the almost FS limit. Then $\bar P^\frown P_{\varepsilon'}$ is a partial CS iteration, and i_ε works, i.e., i_ε is an M-complete embedding from P^M_ε to $P_{\varepsilon'}$. (As $P_{\varepsilon'}$ is a complete subforcing of P_ε, this also implies that i_ε is M-complete from P^M_ε to P_ε.)*

Proof. It is easy to see that $P_{\varepsilon'}$ is a partial CS limit and contains the range $i_\varepsilon[P^M_\varepsilon]$. We now show preservation of predensity; this implies M-completeness by Lemma 3.12.

Let $(p_j)_{j\in J} \in M$ be a maximal antichain in P^M_ε. (Since P^M_ε does not have to be ccc in M, J can have any cardinality in M.) Let $q \wedge i_\varepsilon(p)$ be a condition in $P_{\varepsilon'}$. (If $\varepsilon' < \varepsilon$, i.e., if $\mathrm{cf}(\varepsilon) > \omega$, then we can choose p to be the empty condition.) Fix $\alpha \in \varepsilon \cap M$ be such that $q \in P_\alpha$. Let H_α be P_α-generic and contain q, so $p\restriction\alpha$ is in H^M_α. Now in $M[H^M_\alpha]$ the set $\{p_j : j \in J, p_j \in P^M_\varepsilon/H^M_\alpha\}$ is predense in $P^M_\varepsilon/H^M_\alpha$ (since this is forced by the empty condition in P^M_α). In particular, p is compatible with some p_j, witnessed by $p' \leq p, p_j$ in $P^M_\varepsilon/H^M_\alpha$.

We can find $q' \leq_{P_\alpha} q$ deciding j and p'; since certainly $q' \leq^* i_\alpha(p'\restriction\alpha)$, we may assume even $\leq$ without loss of generality. Now $q' \wedge i_\varepsilon(p') \leq q \wedge i_\varepsilon(p)$ (since $q' \leq q$ and $p' \leq p$), and $q' \wedge i_\varepsilon(p') \leq i_\varepsilon(p_j)$ (since $p' \leq p_j$). □

Definition and Claim 3.15. Let $\bar P^M$ be a topped partial CS iteration in M of length ε. We can construct by induction on $\beta \in \varepsilon + 1$ an *almost finite support iteration $\bar P$ over $\bar P^M$* (or: almost FS iteration) as follows:

(1) As induction hypothesis we assume that the canonical embedding i_α works for all $\alpha \in \beta \cap M$. (So the notation $M[H^M_\alpha]$ makes sense.)
(2) Let $\beta = \alpha + 1$. If $\alpha \in M$, then we can use any Q_α provided that (it is forced that) Q^M_α is an $M[H^M_\alpha]$-complete subforcing of Q_α. (If $\alpha \notin M$, then there is no restriction on Q_α.)
(3) Let $\beta \in M$ and $\mathrm{cf}(\beta) = \omega$. Then P_β is the almost FS limit of $(P_\alpha, Q_\alpha)_{\alpha<\beta}$ over P^M_β.
(4) Let $\beta \in M$ and $\mathrm{cf}(\beta) > \omega$. Then P_β is again the almost FS limit of $(P_\alpha, Q_\alpha)_{\alpha<\beta}$ over P^M_β (which also happens to be the direct limit).
(5) For limit ordinals not in M, P_β is the direct limit.

So the claim includes that the resulting $\bar P$ is a (topped) partial CS iteration of length ε over $\bar P^M$ (i.e., the canonical embeddings i_α work for all $\alpha \in (\varepsilon + 1) \cap M$), where we only assume that the Q_α satisfy the obvious requirement given in (2). (Note that we can always find some suitable Q_α for $\alpha \in M$, for example we can just take Q^M_α itself.)

Proof. We have to show (by induction) that the resulting sequence $\bar P$ is a partial CS iteration, and that $\bar P^M$ embeds into $\bar P$. For successor cases, there is nothing to do. So assume that α is a limit. If P_α is a direct limit, it is trivially a partial CS limit; if P_α is an almost FS limit, then the easy part of Lemma 3.14 shows that it is a partial CS limit.

So it remains to show that for a limit $\alpha \in M$, the (naturally defined) embedding $i_\alpha : P^M_\alpha \to P_\alpha$ is M-complete. This was the main claim in Lemma 3.14. □

The following lemma is natural and easy.

Lemma 3.16. *Assume that we construct an almost FS iteration $\bar{P}$ over $\bar{P}^M$ where each Q_α is (forced to be) ccc. Then P_ε is ccc (and in particular proper).*

Proof. We show that P_α is ccc by induction on $\alpha \leq \varepsilon$. For successors, we use that Q_α is ccc. For α of uncountable cofinality, we know that we took the direct limit coboundedly often (and all P_β are ccc for $\beta < \alpha$), so by a result of Solovay P_α is again ccc. For α a limit of countable cofinality not in M, just use that all P_β are ccc for $\beta < \alpha$, and the fact that P_α is the direct limit. This leaves the case that $\alpha \in M$ has countable cofinality, i.e., the P_α is the almost FS limit. Let $A \subseteq P_\alpha$ be uncountable. Each $a \in A$ has the form $q \wedge i_\alpha(p)$ for $p \in P^M_\alpha$ and $q \in \bigcup_{\gamma<\alpha} P_\gamma$. We can thin out the set A such that p are the same and all q are in the same P_γ. So there have to be compatible elements in A. □

All almost FS iterations that we consider in this paper will satisfy the countable chain condition (and hence in particular be proper).

We will need a variant of this lemma for σ-centered forcing notions.

Lemma 3.17. *Assume that we construct an almost FS iteration $\bar{P}$ over $\bar{P}^M$ where only countably many Q_α are nontrivial (e.g., only those with $\alpha \in M$) and where each Q_α is (forced to be) σ-centered. Then P_ε is σ-centered as well.*

Proof. By induction: The direct limit of countably many σ-centered forcings is σ-centered, as is the almost FS limit of σ-centered forcings (to color $q \wedge i_\alpha(p)$, use p itself together with the color of q). □

We will actually need two variants of the almost FS construction: Countably many models M^n; and starting the almost FS iteration with some α_0.

Firstly, we can construct an almost FS iteration not just over one iteration $\bar{P}^M$, but over an increasing chain of iterations. Analogously to Definition 3.13 and Lemma 3.14, we can show:

Lemma 3.18. *For each $n \in \omega$, let M^n be a nice candidate, and let $\bar{P}^n$ be a topped partial CS iteration in M^n of length[29] $\varepsilon \in M^0$ of countable cofinality, such that $M^m \in M^n$ and M^n thinks that $\bar{P}^m$ canonically embeds into $\bar{P}^n$, for all $m < n$. Let $\bar{P}$ be a topless iteration of length ε into which all $\bar{P}^n$ canonically embed.*

Then we can define the almost FS limit P_ε over $(\bar{P}^n)_{n\in\omega}$ as follows: Conditions in P_ε are of the form $q \wedge i^n_\varepsilon(p)$ where $n \in \omega$, $p \in P^n_\varepsilon$, and $q \in P_\alpha$ for some $\alpha \in M^n \cap \varepsilon$ with $q \leq i^n_\alpha(p{\restriction}\alpha)$. Then P_ε is a partial CS limit over each $\bar{P}^n$.

As before, we get the following corollary:

Corollary 3.19. *Given M^n and $\bar{P}^n$ as above, we can construct a topped partial CS iteration $\bar{P}$ such that each $\bar{P}^n$ embeds M^n-completely into it; we can choose Q_α as we wish (subject to the obvious restriction that each Q^n_α is an $M^n[H^n_\alpha]$-complete subforcing). If we always choose Q_α to be ccc, then $\bar{P}$ is ccc; this is the case if we set Q_α to be the union of the (countable) sets Q^n_α.*

Proof. We can define P_α by induction. If $\alpha \in \bigcup_{n\in\omega} M^n$ has countable cofinality, then we use the almost FS limit as in Lemma 3.18. Otherwise we use the direct limit. If $\alpha \in M^n$ has uncountable cofinality, then $\alpha' \coloneqq \sup(\alpha \cap M)$ is an element of M^{n+1}. In our induction we have already considered α' and have defined $P_{\alpha'}$ by Lemma 3.18 (applied to the sequence $(\bar{P}^{n+1}, \bar{P}^{n+2}, \ldots)$). This is sufficient to show that $i^n_\alpha : P^n_\alpha \to P_{\alpha'} \lessdot P_\alpha$ is M^n-complete. □

Secondly, we can start the almost FS iteration after some α_0 (i.e., $\bar{P}$ is already given up to α_0, and we can continue it as an almost FS iteration up to ε), and get the same properties that we previously showed for the almost FS iteration, but this time for the quotient $P_\varepsilon/P_{\alpha_0}$. In more detail:

Lemma 3.20. *Assume that $\bar{P}^M$ is in M a (topped) partial CS iteration of length ε, and that $\bar{P}$ is in V a topped partial CS iteration of length α_0 over $\bar{P}^M{\restriction}\alpha_0$ for some $\alpha_0 \in \varepsilon \cap M$. Then we can extend $\bar{P}$ to a (topped) partial CS iteration of length ε over $\bar{P}^M$, as in the almost FS iteration (i.e., using the almost FS limit at limit points $\beta > \alpha_0$ with $\beta \in M$ of countable cofinality; and the direct limit everywhere else). We can use any Q_α for $\alpha \geq \alpha_0$ (provided Q^M_α is an $M[H^M_\alpha]$-complete subforcing of Q_α). If all Q_α are ccc, then P_{α_0} forces that $P_\varepsilon/H_{\alpha_0}$ is ccc (in particular proper); if moreover all Q_α are σ-centered and only countably many are nontrivial, then P_{α_0} forces that $P_\varepsilon/H_{\alpha_0}$ is σ-centered.*

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[29] Or only: $\varepsilon \in M^{n_0}$ for some n_0.

3.D. **Almost countable support iterations.** "Almost countable support iterations $\bar{P}$" (over a given iteration $\bar{P}^M$ in a candidate M) will have the following two crucial properties: There is a canonical M-complete embedding of $\bar{P}^M$ into $\bar{P}$, and $\bar{P}$ preserves a given random real (similar to the usual countable support iterations).

Definition and Claim 3.21. Let $\bar{P}^M$ be a topped partial CS iteration in M of length ε. We can construct by induction on $\beta \in \varepsilon + 1$ the *almost countable support iteration $\bar{P}$ over $\bar{P}^M$* (or: almost CS iteration):

(1) As induction hypothesis, we assume that the canonical embedding i_α works for every $\alpha \in \beta \cap M$. We set[30]

$$\delta := \min(M \setminus \beta), \quad \delta' := \sup(\alpha + 1 : \alpha \in \delta \cap M). \tag{3.22}$$

Note that $\delta' \leq \beta \leq \delta$.

(2) Let $\beta = \alpha + 1$. We can choose any desired forcing Q_α; if $\beta \in M$ we of course require that

$$Q^M_\alpha \text{ is an } M[H^M_\alpha]\text{-complete subforcing of } Q_\alpha. \tag{3.23}$$

This defines P_β.

(3) Let $\mathrm{cf}(\beta) > \omega$. Then P_β is the direct limit.

(4) Let $\mathrm{cf}(\beta) = \omega$ and assume that $\beta \in M$ (so $M \cap \beta$ is cofinal in β and $\delta' = \beta = \delta$). We define $P_\beta = P_\delta$ as the union of the following two sets:
- The almost FS limit of $(P_\alpha, Q_\alpha)_{\alpha<\delta}$, see Definition 3.13.
- The set P^{gen}_δ of M-generic conditions $q \in P^{\mathrm{CS}}_\delta$, i.e., those which satisfy

$$q \Vdash_{P^{\mathrm{CS}}_\delta} i^{-1}_\delta[H_{P^{\mathrm{CS}}_\delta}] \subseteq P^M_\delta \text{ is } M\text{-generic.}$$

(5) Let $\mathrm{cf}(\beta) = \omega$ and assume that $\beta \notin M$ but $M \cap \beta$ is cofinal in β, so $\delta' = \beta < \delta$. We define $P_\beta = P_{\delta'}$ as the union of the following two sets:
- The direct limit of $(P_\alpha, Q_\alpha)_{\alpha<\delta'}$.
- The set $P^{\mathrm{gen}}_{\delta'}$ of M-generic conditions $q \in P^{\mathrm{CS}}_{\delta'}$, i.e., those which satisfy

$$q \Vdash_{P^{\mathrm{CS}}_{\delta'}} i^{-1}_\delta[H_{P^{\mathrm{CS}}_{\delta'}}] \subseteq P^M_\delta \text{ is } M\text{-generic.}$$

(Note that the M-generic conditions form an open subset of $P^{\mathrm{CS}}_\beta = P^{\mathrm{CS}}_{\delta'}$.)

(6) Let $\mathrm{cf}(\beta) = \omega$ and $M \cap \beta$ not cofinal in β (so $\beta \notin M$). Then P_β is the full CS limit of $(P_\alpha, Q_\alpha)_{\alpha<\beta}$ (see Definition 3.5).

So the claim is that for every choice of Q_α (with the obvious restriction (3.23)), this construction always results in a partial CS iteration $\bar{P}$ over $\bar{P}^M$. The proof is a bit cumbersome; it is a variant of the usual proof that properness is preserved in countable support iterations (see e.g. [Gol93]).

We will use the following fact in M (for the iteration $\bar{P}^M$):

(3.24) Let $\bar{P}$ be a topped iteration of length ε. Let $\alpha_1 \leq \alpha_2 \leq \beta \leq \varepsilon$. Let p_1 be a P_{α_1}-name for a condition in P_ε, and let D be an open dense set of P_β. Then there is a P_{α_2}-name p_2 for a condition in D such that the empty condition of P_{α_2} forces: $p_2 \leq p_1 \restriction \beta$ and: if p_1 is in $P_\varepsilon/H_{\alpha_2}$, then the condition p_2 is as well.

(Proof: Work in the P_{α_2}-extension. We know that $p' := p_1 \restriction \beta$ is a P_β-condition. We now define p_2 as follows: If $p' \notin P_\beta/H_{\alpha_2}$ (which is equivalent to $p_1 \notin P_\epsilon/H_{\alpha_2}$), then we choose any $p_2 \leq p'$ in D (which is dense in P_β). Otherwise (using that $D \cap P_\beta/H_{\alpha_2}$ is dense in P_β/H_{α_2}) we can choose $p_2 \leq p'$ in $D \cap P_\beta/H_{\alpha_2}$.)

The following easy fact will also be useful:

(3.25) Let P be a subforcing of Q. We define $P \restriction p := \{r \in P : r \leq p\}$. Assume that $p \in P$ and $P \restriction p = Q \restriction p$.
Then for any P-name $\underset{\sim}{x}$ and any formula $\varphi(x)$ we have: $p \Vdash_P \varphi(\underset{\sim}{x})$ iff $p \Vdash_Q \varphi(\underset{\sim}{x})$.

We now prove by induction on $\beta \leq \varepsilon$ the following statement (which includes that the Definition and Claim 3.21 works up to β). Let δ, δ' be as in (3.22).

Lemma 3.26. *(a) The topped iteration $\bar{P}$ of length β is a partial CS iteration.*
(b) The canonical embedding $i_\delta : P^M_\delta \to P_{\delta'}$ works, hence also $i_\delta : P^M_\delta \to P_\delta$ works.

[30] So for successors $\beta \in M$, we have $\delta' = \beta = \delta$. For $\beta \in M$ limit, $\beta = \delta$ and δ' is as in Definition 3.9.

(c) Moreover, assume that
- *$\alpha \in M \cap \delta$,*
- *$\underaccent{\tilde}{p} \in M$ is a P^M_α-name of a P^M_δ-condition,*
- *$q \in P_\alpha$ forces (in P_α) that $\underaccent{\tilde}{p}\restriction\alpha[H^M_\alpha]$ is in H^M_α.*

Then there is a $q^+ \in P_{\delta'}$ (and therefore in P_β) extending q and forcing that $\underaccent{\tilde}{p}[H^M_\alpha]$ is in H^M_δ.

Proof. First let us deal with the trivial cases. It is clear that we always get a partial CS iteration.
- Assume that $\beta = \beta_0 + 1 \in M$, i.e., $\delta = \delta' = \beta$. It is clear that i_β works. To get q^+, first extend q to some $q' \in P_{\beta_0}$ (by induction hypothesis), then define q^+ extending q' by $q^+(\beta_0) := \underaccent{\tilde}{p}(\beta_0)$.
- If $\beta = \beta_0 + 1 \notin M$, there is nothing to do.
- Assume that $\mathrm{cf}(\beta) > \omega$ (whether $\beta \in M$ or not). Then $\delta' < \beta$. So $i_\delta : P^M_\delta \to P_{\delta'}$ works by induction, and similarly (c) follows from the inductive assumption. (Use the inductive assumption for $\beta = \delta'$; the δ that we got at that stage is the same as the current δ, and the q^+ we obtained at that stage will still satisfy all requirements at the current stage.)
- Assume that $\mathrm{cf}(\beta) = \omega$ and that $M \cap \beta$ is bounded in β. Then the proof is the same as in the previous case.

We are left with the cases corresponding to (4) and (5) of Definition 3.21: $\mathrm{cf}(\beta) = \omega$ and $M \cap \beta$ is cofinal in β. So either $\beta \in M$, then $\delta' = \beta = \delta$, or $\beta \notin M$, then $\delta' = \beta < \delta$ and $\mathrm{cf}(\delta) > \omega$.

We leave it to the reader to check that P_β is indeed a partial CS limit. The main point is to see that for all $p, q \in P_\beta$ the condition $q \wedge p$ is in P_β as well, provided $q \in P_\alpha$ and $q \leq p\restriction\alpha$ for some $\alpha < \beta$. If $p \in P^{\mathrm{gen}}_\beta$, then this follows because P^{gen}_β is open in P^{CS}_β; the other cases are immediate from the definition (by induction).

We now turn to claim (c). Assume $q \in P_\alpha$ and $\underaccent{\tilde}{p} \in M$ are given, $\alpha \in M \cap \delta$.

Let $(D_n)_{n\in\omega}$ enumerate all dense sets of P^M_δ which lie in M, and let $(\alpha_n)_{n\in\omega}$ be a sequence of ordinals in M which is cofinal in β, where $\alpha_0 = \alpha$.

Using (3.24) in M, we can find a sequence $(\underaccent{\tilde}{p}_n)_{n\in\omega}$ satisfying the following in M, for all $n > 0$:
- $\underaccent{\tilde}{p}_0 = \underaccent{\tilde}{p}$.
- $\underaccent{\tilde}{p}_n \in M$ is a $P^M_{\alpha_n}$-name of a P^M_δ-condition in D_n.
- $\Vdash_{P^M_{\alpha_n}} \underaccent{\tilde}{p}_n \leq_{P^M_\delta} \underaccent{\tilde}{p}_{n-1}$.
- $\Vdash_{P^M_{\alpha_n}}$ If $\underaccent{\tilde}{p}_{n-1}\restriction\alpha_n \in H^M_{\alpha_n}$, then $\underaccent{\tilde}{p}_n\restriction\alpha_n \in H^M_{\alpha_n}$ as well.

Using the inductive assumption for the α_n's, we can now find a sequence $(q_n)_{n\in\omega}$ of conditions satisfying the following:
- $q_0 = q$, $q_n \in P_{\alpha_n}$.
- $q_n\restriction\alpha_{n-1} = q_{n-1}$.
- $q_n \Vdash_{P_{\alpha_n}} \underaccent{\tilde}{p}_{n-1}\restriction\alpha_n \in H^M_{\alpha_n}$, so also $\underaccent{\tilde}{p}_n\restriction\alpha_n \in H^M_{\alpha_n}$.

Let $q^+ \in P^{\mathrm{CS}}_\beta$ be the union of the q_n. Then for all n:

(1) $q_n \Vdash_{P^{\mathrm{CS}}_\beta} \underaccent{\tilde}{p}_n\restriction\alpha_n \in H^M_{\alpha_n}$, so also q^+ forces this.
(Using induction on n.)

(2) For all n and all $m \geq n$: $q^+ \Vdash_{P^{\mathrm{CS}}_\beta} \underaccent{\tilde}{p}_m\restriction\alpha_m \in H^M_{\alpha_m}$, so also $\underaccent{\tilde}{p}_n\restriction\alpha_m \in H^M_{\alpha_m}$.
(As $\underaccent{\tilde}{p}_m \leq \underaccent{\tilde}{p}_n$.)

(3) $q^+ \Vdash_{P^{\mathrm{CS}}_\beta} \underaccent{\tilde}{p}_n \in H^M_\delta$.
(Recall that P^{CS}_β is separative, see Fact 3.7. So $i_\delta(\underaccent{\tilde}{p}_n) \in H_\delta$ iff $i_{\alpha_n}(\underaccent{\tilde}{p}\restriction\alpha_m) \in H_{\alpha_m}$ for all large m.)

As $q^+ \Vdash_{P^{\mathrm{CS}}_\beta} \underaccent{\tilde}{p}_n \in D_n \cap H^M_\delta$, we conclude that $q^+ \in P^{\mathrm{gen}}_\beta$ (using 3.12, applied to P^{CS}_β). In particular, P^{gen}_β is dense in P_β: Let $q \wedge i_\delta(p)$ be an element of the almost FS limit; so $q \in P_\alpha$ for some $\alpha < \beta$. Now find a generic q^+ extending q and stronger than $i_\delta(p)$, then $q^+ \leq q \wedge i_\delta(p)$.

It remains to show that i_δ is M-complete. Let $A \in M$ be a maximal antichain of P^M_δ, and $p \in P_\beta$. Assume towards a contradiction that p forces in P_β that $i^{-1}_\delta[H_\beta]$ does not intersect A in exactly one point.

Since P^{gen}_β is dense in P_β, we can find some $q \leq p$ in P^{gen}_β. Let

$$P' := \{r \in P^{\mathrm{CS}}_\beta : r \leq q\} = \{r \in P_\beta : r \leq q\},$$

where the equality holds because P^{gen}_β is open in P^{CS}_β.

Let Γ be the canonical name for a P'-generic filter, i.e.: $\Gamma := \{(\check{r}, r) : r \in P'\}$. Let R be either P^{CS}_β or P_β. We write $\langle\Gamma\rangle_R$ for the filter generated by Γ in R, i.e., $\langle\Gamma\rangle_R := \{r \in R : (\exists r' \in \Gamma)\ r' \leq r\}$. So

$$q \Vdash_R H_R = \langle\Gamma\rangle_R. \tag{3.27}$$

We now see that the following hold:

- $q \Vdash_{P_\beta} i^{-1}_\delta[H_{P_\beta}]$ does not intersect A in exactly one point. (By assumption.)
- $q \Vdash_{P_\beta} i^{-1}_\delta[\langle\Gamma\rangle_{P_\beta}]$ does not intersect A in exactly one point. (By (3.27).)
- $q \Vdash_{P^{\mathrm{CS}}_\beta} i^{-1}_\delta[\langle\Gamma\rangle_{P_\beta}]$ does not intersect A in exactly one point. (By (3.25).)
- $q \Vdash_{P^{\mathrm{CS}}_\beta} i^{-1}_\delta[\langle\Gamma\rangle_{P^{\mathrm{CS}}_\beta}]$ does not intersect A in exactly one point. (Because i_δ maps A into $P_\beta \subseteq P^{\mathrm{CS}}_\beta$, so $A \cap i^{-1}_\delta[\langle Y\rangle_{P_\beta}] = A \cap i^{-1}_\delta[\langle Y\rangle_{P^{\mathrm{CS}}_\beta}]$ for all Y.)
- $q \Vdash_{P^{\mathrm{CS}}_\beta} i^{-1}_\delta[H_{P^{\mathrm{CS}}_\beta}]$ does not intersect A in exactly one point. (Again by (3.27).)

But this, according to the definition of P^{gen}_β, implies $q \notin P^{\mathrm{gen}}_\beta$, a contradiction. □

We can also show that the almost CS iteration of proper forcings Q_α is proper. (We do not really need this fact, as we could allow non-proper iterations in our preparatory forcing, see Section 6.A(4). In some sense, M-completeness replaces properness, so the proof of M-completeness was similar to the "usual" proof of properness.)

Lemma 3.28. *Assume that in Definition 3.21, every Q_α is (forced to be) proper. Then also each P_δ is proper.*

Proof. By induction on $\delta \leq \varepsilon$ we prove that for all $\alpha < \delta$ the quotient P_δ/H_α is (forced to be) proper. We use the following facts about properness:

(3.29) If P is proper and P forces that Q is proper, then $P * Q$ is proper.

(3.30) If $\bar{P}$ is an iteration of length ω and if each Q_n is forced to be proper, then the inverse limit P_ω is proper, as are all quotients P_ω/H_n.

(3.31) If $\bar{P}$ is an iteration of length δ with $\mathrm{cf}(\delta) > \omega$, and if all quotients P_β/H_α (for $\alpha < \beta < \delta$) are forced to be proper, then the direct limit P_δ is proper, as are all quotients P_δ/H_α.

If δ is a successor, then our inductive claim easily follows from the inductive assumption together with (3.29).

Let δ be a limit of countable cofinality, say $\delta = \sup_n \delta_n$. Define an iteration $\bar{P}'$ of length ω with $Q'_n := P_{\delta_{n+1}}/H_{\delta_n}$. (Each Q'_n is proper, by inductive assumption.) There is a natural forcing equivalence between P^{CS}_δ and P'^{CS}_ω, the full CS limit of $\bar{P}'$.

Let $N \prec H(\chi^*)$ contain $\bar{P}, P_\delta, \bar{P}', M, \bar{P}^M$. Let $p \in P_\delta \cap N$. Without loss of generality $p \in P^{\mathrm{gen}}_\delta$. So below p we can identify P_δ with P^{CS}_δ and hence with P'^{CS}_ω; now apply (3.30).

The case of uncountable cofinality is similar, using (3.31) instead. □

Recall the definition of $\sqsubset_n$ and $\sqsubset$ from Definition 1.26, the notion of (quick) interpretation Z^* (of a name $\underset{\sim}{Z}$ of a code for a null set) and the definition of local preservation of randoms from Definition 1.50. Recall that we have seen in Corollaries 1.51 and 2.22:

Lemma 3.32.
- *If Q^M is an ultralaver forcing in M and r a real, then there is an ultralaver forcing Q over Q^M locally preserving randomness of r over M.*
- *If Q^M is a Janus forcing in M and r a real, then there is a Janus forcing Q over Q^M locally preserving randomness of r over M.*

We will prove the following preservation theorem:

Lemma 3.33. *Let $\bar{P}$ be an almost CS iteration (of length ε) over $\bar{P}^M$, r random over M, and $p \in P^M_\varepsilon$. Assume that each P_α forces that Q_α locally preserves randomness of r over $M[H^M_\alpha]$. Then there is some $q \leq p$ in P_ε forcing that r is random over $M[H^M_\varepsilon]$.*

What we will actually need is the following variant:

Lemma 3.34. *Assume that $\bar{P}^M$ is in M a topped almost CS iteration of length ε, and we already have some topped partial CS iteration $\bar{P}$ over $\bar{P}^M\restriction\alpha_0$ of length $\alpha_0 \in M\cap\varepsilon$. Let $\underset{\sim}{r}$ be a P_{α_0}-name of a random real over $M[H^M_{\alpha_0}]$. Assume that we extend $\bar{P}$ to length ε as an almost CS iteration[31] using forcings Q_α which locally preserve the randomness of $\underset{\sim}{r}$ over M, witnessed by a sequence $(D_k^{Q^M_\alpha})_{k\in\omega}$. Let $p\in P^M_\varepsilon$. Then we can find a $q\le p$ in P_ε forcing that $\underset{\sim}{r}$ is random over $M[H^M_\varepsilon]$.*

Actually, we will only prove the two previous lemmas under the following additional assumption (which is enough for our application, and saves some unpleasant work). This additional assumption is not really necessary; without it, we could use the method of [GK06] for the proof.

Assumption 3.35.

- For each $\alpha\in M\cap\varepsilon$, ($P^M_\alpha$ forces that) Q^M_α is either trivial[32] or adds a new ω-sequence of ordinals. Note that in the latter case we can assume without loss of generality that $\bigcap_{n\in\omega} D_n^{Q^M_\alpha}=\emptyset$ (and, of course, that the $D_n^{Q^M_\alpha}$ are decreasing).
- Moreover, we assume that already in M there is a set $T\subseteq\varepsilon$ such that P_α forces: Q_α is trivial iff $\alpha\in T$. (So whether Q_α is trivial or not does not depend on the generic filter below α, it is already decided in the ground model.)

The result will follow as a special case of the following lemma, which we prove by induction on β. (Note that this is a refined version of the proof of Lemma 3.26 and similar to the proof of the preservation theorem in [Gol93, 5.13].)

Definition 3.36. Under the assumptions of Lemma 3.34 and Assumption 3.35, let $\underset{\sim}{Z}$ be a P_δ-name, $\alpha_0\le\alpha<\delta$, and let $\bar{p}=(p^k)_{k\in\omega}$ be a sequence of P_α-names of conditions in P_δ/H_α. Let Z^* be a P_α-name.

We say that $(\bar{p},Z^*)$ is a *quick* interpretation of $\underset{\sim}{Z}$ if $\bar{p}$ interprets $\underset{\sim}{Z}$ as Z^* (i.e., P_α forces that p^k forces $\underset{\sim}{Z}\restriction k = Z^*\restriction k$ for all k), and moreover:

> Letting $\beta\ge\alpha$ be minimal with Q^M_β nontrivial (if such β exists): P_β forces that the sequence $(p^k(\beta))_{k\in\omega}$ is quick in Q^M_β, i.e., $p^k(\beta)\in D_k^{Q^M_\beta}$ for all k.

It is easy to see that:

(3.37) For every name $\underset{\sim}{Z}$ there is a quick interpretation $(\bar{p},Z^*)$.

Lemma 3.38. *Under the same assumptions as above, let β,δ,δ' be as in* (3.22) *(so in particular we have $\delta'\le\beta\le\delta\le\varepsilon$).*
Assume that

- *$\alpha\in M\cap\delta$ ($=M\cap\beta$) and $\alpha\ge\alpha_0$ (so $\alpha<\delta'$),*
- *$p\in M$ is a P^M_α-name of a P^M_δ-condition,*
- *$\underset{\sim}{Z}\in M$ is a P^M_δ-name of a code for null set,*
- *$Z^*\in M$ is a P^M_α-name of a code for a null set,*
- *P^M_α forces: $\bar{p}=(p^k)_{k\in\omega}\in M$ is a quick sequence in P^M_δ/H^M_α interpreting $\underset{\sim}{Z}$ as Z^* (as in Definition 3.36),*
- *P^M_α forces: if $p\restriction\alpha\in H^M_\alpha$, then $p^0\le p$,*
- *$q\in P_\alpha$ forces $p\restriction\alpha\in H^M_\alpha$,*
- *q forces that r is random over $M[H^M_\alpha]$, so in particular there is (in V) a P_α-name $\underset{\sim}{c}_0$ below q for the minimal c with $Z^*\sqsubset_c r$.*

Then *there is a condition $q^+\in P_{\delta'}$, extending q, and forcing the following:*

- *$p\in H^M_\delta$,*
- *r is random over $M[H^M_\delta]$,*
- *$\underset{\sim}{Z}\sqsubset_{\underset{\sim}{c}_0} r$.*

We actually claim a slightly stronger version, where instead of Z^* and $\underset{\sim}{Z}$ we have finitely many codes for null sets and names of codes for null sets, respectively. We will use this stronger claim as inductive assumption, but for notational simplicity we only prove the weaker version; it is easy to see that the weaker version implies the stronger version.

[31] Of course our official definition of almost CS iteration assumes that we start the construction at 0, so we modify this definition in the obvious way.

[32] More specifically, $Q^M_\alpha=\{\emptyset\}$.

Proof. ***The nontrivial successor case:*** $\beta = \gamma + 1 \in M$.

If Q^M_γ is trivial, there is nothing to do.

Now let $\gamma_0 \geq \alpha$ be minimal with $Q^M_{\gamma_0}$ nontrivial. We will distinguish two cases: $\gamma = \gamma_0$ and $\gamma > \gamma_0$.

Consider first the case that $\gamma = \gamma_0$. Work in $V[H_\gamma]$ where $q \in H_\gamma$. Note that $M[H^M_\gamma] = M[H^M_\alpha]$. So r is random over $M[H^M_\gamma]$, and $(p^k(\gamma))_{k\in\omega}$ quickly interprets $\underaccent{\sim}{Z}$ as Z^* in Q^M_γ. Now let $q^+ \restriction \gamma = q$, and use the fact that Q_γ locally preserves randomness to find $q^+(\gamma) \leq p^0(\gamma)$.

Next consider the case that Q^M_γ is nontrivial and $\gamma \geq \gamma_0+1$. Again work in $V[H_\gamma]$. Let k^* be maximal with $p^{k^*} \restriction \gamma \in H^M_\gamma$. (This k^* exists, since the sequence $(p^k)_{k\in\omega}$ was quick, so there is even a k with $p^k \restriction (\gamma_0 + 1) \notin H^M_{\gamma_0+1}$.) Consider $\underaccent{\sim}{Z}$ as a Q^M_γ-name, and (using (3.37)) find a quick interpretation Z' of $\underaccent{\sim}{Z}$ witnessed by a sequence starting with $p^{k^*}(\gamma)$. In $M[H^M_\alpha]$, Z' is now a P^M_γ/H^M_α-name. Clearly, the sequence $(p^k \restriction \gamma)_{k\in\omega}$ is a quick sequence interpreting Z' as Z^*. (Use the fact that $p^k \restriction \gamma$ forces $k^* \geq k$.)
Using the induction hypothesis, we can first extend q to a condition $q' \in P_\gamma$ and then (again by our assumption that Q_γ locally preserves randomness) to a condition $q^+ \in P_{\gamma+1}$.

The nontrivial limit case: $M \cap \beta$ unbounded in β, i.e., $\delta' = \beta$. (This deals with cases (4) and (5) in Definition 3.21. In case (4) we have $\beta \in M$, i.e., $\beta = \delta$; in case (5) we have $\beta \notin M$ and $\beta < \delta$.)

Let $\alpha = \delta_0 < \delta_1 < \cdots$ be a sequence of M-ordinals cofinal in $M \cap \delta' = M \cap \delta$. We may assume[33] that each $Q^M_{\delta_n}$ is nontrivial.

Let $(\underaccent{\sim}{Z}_n)_{n\in\omega}$ be a list of all P^M_δ-names in M of codes for null sets (starting with our given null set $\underaccent{\sim}{Z} = \underaccent{\sim}{Z}_0$). Let $(E_n)_{n\in\omega}$ enumerate all open dense sets of P^M_δ from M, without loss of generality[34] we can assume that:

(3.39) $\qquad E_n$ decides $\underaccent{\sim}{Z}_0 \restriction n, \ldots, \underaccent{\sim}{Z}_n \restriction n$.

We write p^k_0 for p^k, and $Z_{0,0}$ for Z^*; as mentioned above, $\underaccent{\sim}{Z} = \underaccent{\sim}{Z}_0$.

By induction on n we can now find a sequence $\bar{p}_n = (p^k_n)_{k\in\omega}$ and $P^M_{\delta_n}$-names $Z_{i,n}$ for $i \in \{0, \ldots, n\}$ satisfying the following:

1. $P^M_{\delta_n}$ forces that $p^0_n \leq p^k_{n-1}$ whenever $p^k_{n-1} \in P^M_\delta/H^M_{\delta_n}$.
2. P_{δ_n} forces that $p^0_n \in E_n$. (Clearly $E_n \cap P^M_\delta/H^M_{\delta_n}$ is a dense set.)
3. $\bar{p}_n \in M$ is a $P^M_{\delta_n}$-name for a quick sequence interpreting $(\underaccent{\sim}{Z}_0, \ldots, \underaccent{\sim}{Z}_n)$ as $(Z_{0,n}, \ldots, Z_{n,n})$ (in $P^M_\delta/H^M_{\delta_n}$), so $Z_{i,n}$ is a $P^M_{\delta_n}$-name of a code for a null set, for $0 \leq i \leq n$.

Note that this implies that the sequence $(p^k_{n-1} \restriction \delta_n)$ is (forced to be) a quick sequence interpreting $(Z_{0,n}, \ldots, Z_{n-1,n})$ as $(Z_{0,n-1}, \ldots, Z_{n-1,n-1})$.

Using the induction hypothesis, we now define a sequence $(q_n)_{n\in\omega}$ of conditions $q_n \in P_{\delta_n}$ and a sequence $(c_n)_{n\in\omega}$ (where c_n is a P_{δ_n}-name) such that (for $n > 0$) q_n extends q_{n-1} and forces the following:

- $p^0_{n-1} \restriction \delta_n \in H^M_{\delta_n}$.
- Therefore, $p^0_n \leq p^0_{n-1}$.
- r is random over $M[H^M_{\delta_n}]$.
- Let c_n be the least c such that $Z_{n,n} \sqsubset_c r$.
- $Z_{i,n} \sqsubset_{c_i} r$ for $i = 0, \ldots, n-1$.

Now let $q = \bigcup_n q_n \in P^{\text{CS}}_{\delta'}$. As in Lemma 3.26 it is easy to see that $q \in P^{\text{gen}}_{\delta'} \subseteq P_{\delta'}$. Moreover, by (3.39) we get that q forces that $\underaccent{\sim}{Z}_i = \lim_n Z_{i,n}$. Since each set $C_{c,r} \coloneqq \{x : x \sqsubset_c r\}$ is closed, this implies that q forces $\underaccent{\sim}{Z}_i \sqsubset_{c_i} r$, in particular $\underaccent{\sim}{Z} = \underaccent{\sim}{Z}_0 \sqsubset_{c_0} r$.

The trivial cases: In all other cases, $M \cap \beta$ is bounded in β, so we already dealt with everything at stage $\beta_0 \coloneqq \sup(\beta \cap M)$. Note that δ'_0 and δ_0 used at stage β_0 are the same as the current δ' and δ. $\square$

4. The forcing construction

In this section we describe a σ-closed "preparatory" forcing notion $\mathbb{R}$; the generic filter will define a "generic" forcing iteration $\bar{\mathbf{P}}$, so elements of $\mathbb{R}$ will be approximations to such an iteration. In Section 5 we will show that the forcing $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces BC and dBC.

From now on, we assume CH in the ground model.

[33]If from some γ on all Q^M_ζ are trivial, then $P^M_\delta = P^M_\gamma$, so by induction there is nothing to do. If Q^M_α itself is trivial, then we let $\delta_0 \coloneqq \min\{\zeta : Q^M_\zeta \text{ nontrivial}\}$ instead.

[34]well, if we just enumerate a basis of the open sets instead of all of them. . .

4.A. **Alternating iterations, canonical embeddings and the preparatory forcing $\mathbb{R}$.** The preparatory forcing $\mathbb{R}$ will consist of pairs $(M, \bar{P})$, where M is a countable model and $\bar{P} \in M$ is an iteration of ultralaver and Janus forcings.

Definition 4.1. An alternating iteration[35] is a topped partial CS iteration $\bar{P}$ of length ω_2 satisfying the following:

- Each P_α is proper.[36]
- For α even, either both Q_α and $Q_{\alpha+1}$ are (forced by the empty condition to be) trivial,[37] or P_α forces that Q_α is an ultralaver forcing adding the generic real $\bar{\ell}_\alpha$, and $P_{\alpha+1}$ forces that $Q_{\alpha+1}$ is a Janus forcing based on $\bar{\ell}^*_\alpha$ (where $\bar{\ell}^*$ is defined from $\bar{\ell}$ as in Lemma 1.23).

We will call an even index an "ultralaver position" and an odd one a "Janus position".

As in any partial CS iteration, each P_δ for $\mathrm{cf}(\delta) > \omega$ (and in particular P_{ω_2}) is a direct limit.

Recall that in Definition 3.10 we have defined the notion "$\bar{P}^M$ canonically embeds into $\bar{P}$" for nice candidates M and iterations $\bar{P} \in V$ and $\bar{P}^M \in M$. Since our iterations now have length ω_2, this means that the canonical embedding works up to and including[38] ω_2.

In the following, we will use pairs $x = (M^x, \bar{P}^x)$ as conditions in a forcing, where $\bar{P}^x$ is an alternating iteration in the nice candidate M^x. We will adapt our notation accordingly: Instead of writing M, $\bar{P}^M$, P^M_α H^M_α (the induced filter), Q^M_α, etc., we will write M^x, $\bar{P}^x$, P^x_α, H^x_α, $\underset{\sim}{Q}^x_\alpha$, etc. Instead of "$\bar{P}^x$ canonically embeds into $\bar{P}$" we will say[39] "x canonically embeds into $\bar{P}$" or "$(M^x, \bar{P}^x)$ canonically embeds into $\bar{P}$" (which is a more exact notation anyway, since the test whether the embedding is M^x-complete uses both M^x and $\bar{P}^x$, not just $\bar{P}^x$).

The following rephrases Definition 3.10 of a canonical embedding in our new notation, taking into account that:

$$\mathbb{L}_{\bar{D}^x} \text{ is an } M^x\text{-complete subforcing of } \mathbb{L}_{\bar{D}} \quad \text{iff} \quad \bar{D} \text{ extends } \bar{D}^x$$

(see Lemma 1.5).

Fact 4.2. $x = (M^x, \bar{P}^x)$ canonically embeds into $\bar{P}$, if (inductively) for all $\beta \in \omega_2 \cap M^x \cup \{\omega_2\}$ the following holds:

- Let $\beta = \alpha+1$ for α even (i.e., an ultralaver position). Then either Q^x_α is trivial (and Q_α can be trivial or not), or we require that (P_α forces that) the $V[H_\alpha]$-ultrafilter system $\bar{D}$ used for Q_α extends the $M^x[H^x_\alpha]$-ultrafilter system $\bar{D}^x$ used for Q^x_α.
- Let $\beta = \alpha + 1$ for α odd (i.e., a Janus position). Then either Q^x_α is trivial, or we require that (P_α forces that) the Janus forcing Q^x_α is an $M^x[H^x_\alpha]$-complete subforcing of the Janus forcing Q_α.
- Let β be a limit. Then the canonical extension $i_\beta : P^x_\beta \to P_\beta$ is M^x-complete. (The canonical extension was defined in Definition 3.9.)

Fix a sufficiently large regular cardinal χ^* (see Remark 3.3).

Definition 4.3. The "*preparatory forcing*" $\mathbb{R}$ consists of pairs $x = (M^x, \bar{P}^x)$ such that $M^x \in H(\chi^*)$ is a nice candidate (containing ω_2), and $\bar{P}^x$ is in M^x an alternating iteration (in particular topped and of length ω_2). We define y to be stronger than x (in symbols: $y \leq_{\mathbb{R}} x$), if the following holds: either $x = y$, or:

- $M^x \in M^y$ and M^x is countable in M^y.
- M^y thinks that $(M^x, \bar{P}^x)$ canonically embeds into $\bar{P}^y$.

Note that this order on $\mathbb{R}$ is transitive.

We will sometimes write $i_{x,y}$ for the canonical embedding (in M^y) from $P^x_{\omega_2}$ to $P^y_{\omega_2}$.

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[35]See Section 6 for possible variants of this definition.

[36]This does not seem to be necessary, see Section 6, but it is easy to ensure and might be comforting to some of the readers and/or authors.

[37]For definiteness, let us agree that the trivial forcing is the singleton $\{\emptyset\}$.

[38]This is stronger than to require that the canonical embedding works for every $\alpha \in \omega_2 \cap M$, even though both P_{ω_2} and $P^M_{\omega_2}$ are just direct limits; see footnote 28.

[39]Note the linguistic asymmetry here: A symmetric and more verbose variant would say "$x = (M^x, \bar{P}^x)$ canonically embeds into $(V, \bar{P})$".

There are several variants of this definition which result in equivalent forcing notions. We will briefly come back to this in Section 6.

The following is trivial by elementarity:

Fact 4.4. Assume that $\bar{P}$ is an alternating iteration (in V), that $x = (M^x, \bar{P}^x) \in \mathbb{R}$ canonically embeds into $\bar{P}$, and that $N \prec H(\chi^*)$ contains x and $\bar{P}$. Let $y = (M^y, \bar{P}^y)$ be the ord-collapse of $(N, \bar{P})$. Then $y \in \mathbb{R}$ and $y \leq x$.

This fact will be used, for example, to get from the following Lemma 4.5 to Corollary 4.6.

Lemma 4.5. *Given $x \in \mathbb{R}$, there is an alternating iteration $\bar{P}$ such that x canonically embeds into $\bar{P}$.*

Proof. For the proof, we use either of the partial CS constructions introduced in the previous chapter (i.e., an almost CS iteration or an almost FS iteration over $\bar{P}^x$). The only thing we have to check is that we can indeed choose Q_α that satisfy the definition of an alternating iteration (i.e., as ultralaver or Janus forcings) and such that Q^x_α is M^x-complete in Q_α.

In the ultralaver case we arbitrarily extend $\bar{D}^x$ to an ultrafilter system $\bar{D}$, which is justified by Lemma 1.5.

In the Janus case, we take $Q_\alpha := Q^x_\alpha$ (this works by Fact 2.8). Alternatively, we could extend Q^x_α to a random forcing (using Lemma 2.20). □

Corollary 4.6. *Given $x \in \mathbb{R}$ and an HCON object $b \in H(\chi^*)$ (e.g., a real or an ordinal), there is a $y \leq x$ such that $b \in M^y$.*

What we will actually need are the following three variants:

Lemma 4.7. *(1) Given $x \in \mathbb{R}$ there is a σ-centered alternating iteration $\bar{P}$ above x.*

(2) Given a decreasing sequence $\bar{x} = (x_n)_{n\in\omega}$ in $\mathbb{R}$, there is an alternating iteration $\bar{P}$ such that each x_n embeds into $\bar{P}$. Moreover, we can assume that for all Janus positions β, the Janus [40] forcing Q_β is (forced to be) the union of the $Q^{x_n}_\beta$, and that for all limits α, the forcing P_α is the almost FS limit over $(x_n)_{n\in\omega}$ (as in Corollary 3.19).

(3) Let $x, y \in \mathbb{R}$. Let j^x be the transitive collapse of M^x, and define j^y analogously. Assume that $j^x[M^x] = j^y[M^y]$, that $j^x(\bar{P}^x) = j^y(\bar{P}^y)$ and that there are $\alpha_0 \leq \alpha_1 < \omega_2$ such that:
- *$M^x \cap \alpha_0 = M^y \cap \alpha_0$ (and thus $j^x \restriction \alpha_0 = j^y \restriction \alpha_0$).*
- *$M^x \cap [\alpha_0, \omega_2) \subseteq [\alpha_0, \alpha_1)$.*
- *$M^y \cap [\alpha_0, \omega_2) \subseteq [\alpha_1, \omega_2)$.*

Then there is an alternating iteration $\bar{P}$ such that both x and y canonically embed into it.

Proof. For (1), use an almost FS iteration. We only use the coordinates in M^x, and use the (countable!) Janus forcings $Q_\alpha := Q^x_\alpha$ for all Janus positions $\alpha \in M^x$ (see Fact 2.8). Ultralaver forcings are σ-centered anyway, so P_ε will be σ-centered, by Lemma 3.17.

For (2), use the almost FS iteration over the sequence $(x_n)_{n\in\omega}$ as in Corollary 3.19, and at Janus positions α set Q_α to be the union of the $Q^{x_n}_\alpha$. (By Fact 2.8, $Q^{x_n}_\alpha$ is M^{x_n}-complete in Q_α, so Corollary 3.19 can be applied here.)

For (3), we again use an almost FS construction. This time we start with an almost FS construction over x up to α_1, and then continue with an almost FS construction over y. □

As above, Fact 4.4 gives us the following consequences:

Corollary 4.8. *(1) $\mathbb{R}$ is σ-closed. Hence $\mathbb{R}$ does not add new HCON objects (and in particular: no new reals).*

(2) $\mathbb{R}$ forces that the generic filter $G \subseteq \mathbb{R}$ is σ-directed, i.e., for every countable subset B of G there is a $y \in G$ stronger than each element of B.

(3) $\mathbb{R}$ forces CH. (Since we assume CH in V.)

(4) Given a decreasing sequence $\bar{x} = (x_n)_{n\in\omega}$ in $\mathbb{R}$ and any HCON object $b \in H(\chi^*)$, there is a $y \in \mathbb{R}$ such that
- *$y \leq x_n$ for all n,*
- *M^y contains b and the sequence $\bar{x}$,*

[40] If all $Q^{x_n}_\beta$ are trivial, then we may also set Q_β to be the trivial forcing, which is formally not a Janus forcing.

- *for all Janus positions β, M^y thinks that the Janus forcing Q^y_β is (forced to be) the union of the $Q^{x_n}_\beta$,*
- *for all limits α, M^y thinks that P^y_α is the almost FS limit*[41] *over $(x_n)_{n\in\omega}$ (of $(P^y_\beta)_{\beta<\alpha}$).*

Proof. Item (4) directly follows from Lemma 4.7(2) and Fact 4.4. Item (1) is a special case of (4), and (2) and (3) are trivial consequences of (1). □

Another consequence of Lemma 4.7 is:

Lemma 4.9. *The forcing notion $\mathbb{R}$ is $\aleph_2$-cc.*

Proof. Recall that we assume that V (and hence $V[G]$) satisfies CH.

Assume towards a contradiction that $(x_i : i < \omega_2)$ is an antichain. Using CH we may without loss of generality assume that for each $i \in \omega_2$ the transitive collapse of $(M^{x_i}, \bar{P}^{x_i})$ is the same. Set $L_i := M^{x_i} \cap \omega_2$. Using the Δ-lemma we find some uncountable $I \subseteq \omega_2$ such that the L_i for $i \in I$ form a Δ-system with root L. Set $\alpha_0 = \sup(L) + 3$. Moreover, we may assume $\sup(L_i) < \min(L_j \setminus \alpha_0)$ for all $i < j$.

Now take any $i, j \in I$, set $x := x_i$ and $y := x_j$, and use Lemma 4.7(3). Finally, use Fact 4.4 to find $z \leq x_i, x_j$. □

4.B. **The generic forcing $\mathbf{P}'$.** Let G be $\mathbb{R}$-generic. Obviously G is a $\leq_{\mathbb{R}}$-directed system. Using the canonical embeddings, we can construct in $V[G]$ a direct limit $\mathbf{P}'_{\omega_2}$ of the directed system G: Formally, we set

$$\mathbf{P}'_{\omega_2} := \{(x, p) : x \in G \text{ and } p \in P^x_{\omega_2}\},$$

and we set $(y, q) \leq (x, p)$ if $y \leq_{\mathbb{R}} x$ and q is (in y) stronger than $i_{x,y}(p)$ (where $i_{x,y} : P^x_{\omega_2} \to P^y_{\omega_2}$ is the canonical embedding). Similarly, we define for each α

$$\mathbf{P}'_\alpha := \{(x, p) : x \in G, \alpha \in M^x \text{ and } p \in P^x_\alpha\}$$

with the same order.

To summarize:

Definition 4.10. For $\alpha \leq \omega_2$, the direct limit of the P^x_α with $x \in G$ is called $\mathbf{P}'_\alpha$.

Formally, elements of $\mathbf{P}'_{\omega_2}$ are defined as pairs (x, p). However, the x does not really contribute any information. In particular:

Fact 4.11. (1) Assume that (x, p^x) and (y, p^y) are in $\mathbf{P}'_{\omega_2}$, that $y \leq x$, and that the canonical embedding $i_{x,y}$ witnessing $y \leq x$ maps p^x to p^y. Then $(x, p^x) =^* (y, p^y)$.

(2) (y, q) is in $\mathbf{P}'_{\omega_2}$ stronger than (x, p) iff for some (or equivalently: for any) $z \leq x, y$ in G the canonically embedded q is in $P^z_{\omega_2}$ stronger than the canonically embedded p. The same holds if "stronger than" is replaced by "compatible with" or by "incompatible with".

(3) If $(x, p) \in \mathbf{P}'_\alpha$, and if y is such that $M^y = M^x$ and $\bar{P}^y \restriction \alpha = \bar{P}^x \restriction \alpha$, then $(y, p) =^* (x, p)$.

In the following, we will therefore often abuse notation and just write p instead of (x, p) for an element of $\mathbf{P}'_\alpha$.

We can define a natural restriction map from $\mathbf{P}'_{\omega_2}$ to $\mathbf{P}'_\alpha$, by mapping (x, p) to $(x, p\restriction\alpha)$. Note that by the fact above, we can assume without loss of generality that $\alpha \in M^x$. More exactly: There is a $y \leq x$ in G such that $\alpha \in M^y$ (according to Corollary 4.6). Then in $\mathbf{P}'_{\omega_2}$ we have $(x, p) =^* (y, p)$.

Fact 4.12. The following is forced by $\mathbb{R}$:

- $\mathbf{P}'_\beta$ is completely embedded into $\mathbf{P}'_\alpha$ for $\beta < \alpha \leq \omega_2$ (witnessed by the natural restriction map).
- If $x \in G$, then P^x_α is M^x-completely embedded into $\mathbf{P}'_\alpha$ for $\alpha \leq \omega_2$ (by the identity map $p \mapsto (x, p)$).
- If $\mathrm{cf}(\alpha) > \omega$, then $\mathbf{P}'_\alpha$ is the union of the $\mathbf{P}'_\beta$ for $\beta < \alpha$.
- By definition, $\mathbf{P}'_{\omega_2}$ is a subset of V.

G will always denote an $\mathbb{R}$-generic filter, while the $\mathbf{P}'_{\omega_2}$-generic filter over $V[G]$ will be denoted by H'_{ω_2} (and the induced $\mathbf{P}'_\alpha$-generic by H'_α). Recall that for each $x \in G$, the map $p \mapsto (x, p)$ is an M^x-complete embedding of $P^x_{\omega_2}$ into $\mathbf{P}'_{\omega_2}$ (and of P^x_α into $\mathbf{P}'_\alpha$). This way $H'_\alpha \subseteq \mathbf{P}'_\alpha$ induces an M^x-generic filter $H^x_\alpha \subseteq P^x_\alpha$.

So $x \in \mathbb{R}$ forces that $\mathbf{P}'_\alpha$ is approximated by P^x_α. In particular we get:

[41] constructed in Lemma 3.18

Lemma 4.13. *Assume that $x \in \mathbb{R}$, that $\alpha \leq \omega_2$ in M^x, that $p \in P^x_\alpha$, that $\varphi(x)$ is a first order formula of the language $\{\in\}$ with one free variable x and that $\dot\tau$ is a P^x_α-name in M^x. Then $M^x \models p \Vdash_{P^x_\alpha} \varphi(\dot\tau)$ iff $x \Vdash_{\mathbb{R}} (x, p) \Vdash_{\mathbf{P}'_\alpha} M^x[H^x_\alpha] \models \varphi(\dot\tau[H^x_\alpha])$.*

Proof. "$\Rightarrow$" is clear. So assume that $\varphi(\dot\tau)$ is not forced in M^x. Then some $q \leq_{P^x_\alpha} p$ forces the negation. Now x forces that $(x, q) \leq (x, p)$ in $\mathbf{P}'_\alpha$; but the conditions (x, p) and (x, q) force contradictory statements. □

4.C. **The inductive proof of ccc.** We will now prove by induction on α that $\mathbf{P}'_\alpha$ is (forced to be) ccc and (equivalent to) an alternating iteration. Once we know this, we can prove Lemma 4.28, which easily implies all the lemmas in this section. So in particular these lemmas will only be needed to prove ccc and not for anything else (and they will probably not aid the understanding of the construction).

In this section, we try to stick to the following notation: $\mathbb{R}$-names are denoted with a tilde underneath (e.g., $\underset{\sim}{\tau}$), while P^x_α-names or $\mathbf{P}'_\alpha$-names (for any $\alpha \leq \omega_2$) are denoted with a dot accent (e.g., $\dot\tau$). We use both accents when we deal with $\mathbb{R}$-names for $\mathbf{P}'_\alpha$-names (e.g., $\underset{\sim}{\dot\tau}$).

We first prove a few lemmas that are easy generalizations of the following straightforward observation:

Assume that $x \Vdash_{\mathbb{R}} (\underset{\sim}{z}, \underset{\sim}{p}) \in \mathbf{P}'_\alpha$. In particular, $x \Vdash \underset{\sim}{z} \in G$. We first strengthen x to some x_1 that decides $\underset{\sim}{z}$ and $\underset{\sim}{p}$ to be z^* and p^*. Then $x_1 \leq^* z^*$ (the order $\leq^*$ is defined on page 3), so we can further strengthen x_1 to some $y \leq z^*$. By definition, this means that z^* is canonically embedded into $\bar{P}^y$; so (by Fact 4.11) the $P^{z^*}_\alpha$-condition p^* can be interpreted as a P^y_α-condition as well. So we end up with some $y \leq x$ and a P^y_α-condition p^* such that $y \Vdash_{\mathbb{R}} (\underset{\sim}{z}, \underset{\sim}{p}) =^* (y, p^*)$.

Since $\mathbb{R}$ is σ-closed, we can immediately generalize this to countably many ($\mathbb{R}$-names for) $\mathbf{P}'_\alpha$-conditions:

Fact 4.14. Assume that $x \Vdash_{\mathbb{R}} \underset{\sim}{p}_n \in \mathbf{P}'_\alpha$ for all $n \in \omega$. Then there is a $y \leq x$ and there are $p^*_n \in P^y_\alpha$ such that $y \Vdash_{\mathbb{R}} \underset{\sim}{p}_n =^* p^*_n$ for all $n \in \omega$.

Recall that more formally we should write: $x \Vdash_{\mathbb{R}} (\underset{\sim}{z}_n, \underset{\sim}{p}_n) \in \mathbf{P}'_\alpha$; and $y \Vdash_{\mathbb{R}} (\underset{\sim}{z}_n, \underset{\sim}{p}_n) =^* (y, p^*_n)$.

We will need a variant of the previous fact:

Lemma 4.15. *Assume that $\mathbf{P}'_\beta$ is forced to be ccc, and assume that x forces (in $\mathbb{R}$) that $\underset{\sim}{\dot r}_n$ is a $\mathbf{P}'_\beta$-name for a real (or an HCON object) for every $n \in \omega$. Then there is a $y \leq x$ and there are P^y_β-names $\dot r^*_n$ in M^y such that $y \Vdash_{\mathbb{R}} (\Vdash_{\mathbf{P}'_\beta} \underset{\sim}{\dot r}_n = \dot r^*_n)$ for all n.*

(Of course, we mean: $\underset{\sim}{\dot r}_n$ is evaluated by $G * H'_\beta$, while $\dot r^*_n$ is evaluated by H^y_β.)

Proof. The proof is an obvious consequence of the previous fact, since names of reals in a ccc forcing can be viewed as a countable sequence of conditions.

In more detail: For notational simplicity assume all $\underset{\sim}{\dot r}_n$ are names for elements of 2^ω. Working in V, we can find for each $n, m \in \omega$ names for a maximal antichain $\underset{\sim}{A}_{n,m}$ and for a function $\underset{\sim}{f}_{n,m} : \underset{\sim}{A}_{n,m} \to 2$ such that x forces that ($\mathbf{P}'_\beta$ forces that) $\underset{\sim}{\dot r}_n(m) = \underset{\sim}{f}_{n,m}(a)$ for the unique $a \in \underset{\sim}{A}_{n,m} \cap H'_\beta$. Since $\mathbf{P}'_\beta$ is ccc, each $\underset{\sim}{A}_{n,m}$ is countable, and since $\mathbb{R}$ is σ-closed, it is forced that the sequence $\underset{\sim}{\Xi} = (\underset{\sim}{A}_{n,m}, \underset{\sim}{f}_{n,m})_{n,m\in\omega}$ is in V.

In V, we strengthen x to x_1 to decide $\underset{\sim}{\Xi}$ to be some Ξ^*. We can also assume that $\Xi^* \in M^{x_1}$ (see Corollary 4.6). Each $A^*_{n,m}$ consists of countably many a such that x_1 forces $a \in \mathbf{P}'_\beta$. Using Fact 4.14 iteratively (and again the fact that $\mathbb{R}$ is σ-closed) we get some $y \leq x_1$ such that each such a is actually an element of P^y_β. So in M^y, we can use $(A^*_{n,m}, f^*_{n,m})_{n,m\in\omega}$ to construct P^y_β-names $\dot r^*_n$ in the obvious way.

Now assume that $y \in G$ and that H'_β is $\mathbf{P}'_\beta$-generic over $V[G]$. Fix any $a \in A^*_{n,m} = \underset{\sim}{A}_{n,m}$. Since $a \in P^y_\beta$, we get $a \in H^y_\beta$ iff $a \in H'_\beta$. So there is a unique element a of $A^*_{n,m} \cap H^y_\beta$, and $\dot r^*_n(m) = f^*_{n,m}(a) = \underset{\sim}{f}_{n,m}(a) = \underset{\sim}{\dot r}_n(m)$. □

We will also need the following modification:

Lemma 4.16. *(Same assumptions as in the previous lemma.) In $V[G][H'_\beta]$, let $\mathbf{Q}_\beta$ be the union of $Q^z_\beta[H^z_\beta]$ for all $z \in G$. In V, assume that x forces that each $\underset{\sim}{\dot r}_n$ is a name for an element of $\mathbf{Q}_\beta$. Then there is a $y \leq x$ and there is in M^y a sequence $(\dot r^*_n)_{n\in\omega}$ of P^y_β-names for elements of Q^y_β such that y forces $\underset{\sim}{\dot r}_n = \dot r^*_n$ for all n.*

So the difference to the previous lemma is: We additionally assume that $\underset{\sim}{\dot r}_n$ is in $\bigcup_{z\in G} Q^z_\beta$, and we additionally get that $\dot r^*_n$ is a name for an element of Q^y_β.

Proof. Assume $x \in G$ and work in $V[G]$. Fix n. $\mathbf{P}'_\beta$ forces that there is some $y_n \in G$ and some $P^{y_n}_\beta$-name $\tau_n \in M^{y_n}$ of an element of $Q^{y_n}_\beta$ such that $\underset{\sim}{\dot r}_n$ (evaluated by H'_β) is the same as τ_n (evaluated by $H^{y_n}_\beta$). Since we assume that $\mathbf{P}'_\beta$ is ccc, we can find a countable set $Y_n \subseteq G$ of the possible y_n, i.e., the empty condition of $\mathbf{P}'_\beta$ forces $y_n \in Y_n$. (As $\mathbb{R}$ is σ-closed and $Y_n \subseteq \mathbb{R} \subseteq V$, we must have $Y_n \in V$.)

So in V, there is (for each n) an $\mathbb{R}$-name $\underset{\sim}{Y}_n$ for this countable set. Since $\mathbb{R}$ is σ-closed, we can find some $z_0 \leq x$ deciding each $\underset{\sim}{Y}_n$ to be some countable set $Y^*_n \subseteq \mathbb{R}$. In particular, for each $y \in Y^*_n$ we know that $z_0 \Vdash_{\mathbb{R}} y \in G$, i.e., $z_0 \leq^* y$; so using once again that $\mathbb{R}$ is σ-closed we can find some z stronger than z_0 and all the $y \in \bigcup_{n\in\omega} Y^*_n$. Let X contain all $\tau \in M^y$ such that for some $y \in \bigcup_{n\in\omega} Y^*_n$, τ is a P^y_β-name for a Q^y_β-element. Since $z \leq y$, each $\tau \in X$ is actually[42] a P^z_β-name for an element of Q^z_β.

So X is a set of P^z_β-names for Q^z_β-elements; we can assume that $X \in M^z$. Also, z forces that $\dot{\underset{\sim}{r}}_n \in X$ for all n. Using Lemma 4.15, we can additionally assume that there are names P^z_β-name $\dot{r}^*_n$ in M^z such that z forces that $\dot{\underset{\sim}{r}}_n = \dot{r}^*_n$ is forced for each n. By Lemma 4.13, we know that M^z thinks that P^z_β forces that $\dot{r}^*_n \in X$. Therefore $\dot{r}^*_n$ is a P^z_β-name for a Q^z_β-element. □

We now prove by induction on α that $\mathbf{P}'_\alpha$ is equivalent to a ccc alternating iteration:

Lemma 4.17. *The following holds in $V[G]$ for $\alpha < \omega_2$:*

(1) $\mathbf{P}'_\alpha$ is equivalent to an alternating iteration. More formally: There is an iteration $(\mathbf{P}_\beta, \mathbf{Q}_\beta)_{\beta<\alpha}$ with limit $\mathbf{P}_\alpha$ that satisfies the definition of alternating iteration (up to α), and there is a naturally defined dense embedding $j_\alpha : \mathbf{P}'_\alpha \to \mathbf{P}_\alpha$, such that for $\beta < \alpha$ we have $j_\beta \subseteq j_\alpha$, and the embeddings commute with the restrictions.[43] Each $\mathbf{Q}_\alpha$ is the union of all Q^x_α with $x \in G$. For $x \in G$ with $\alpha \in M^x$, the function $i_{x,\alpha} : P^x_\alpha \to \mathbf{P}_\alpha$ that maps p to $j_\alpha(x, p)$ is the canonical M^x-complete embedding.
(2) In particular, a $\mathbf{P}'_\alpha$-generic filter H'_α can be translated into a $\mathbf{P}_\alpha$-generic filter which we call H_α (and vice versa).
(3) $\mathbf{P}_\alpha$ has a dense subset of size $\aleph_1$.
(4) $\mathbf{P}_\alpha$ is ccc.
(5) $\mathbf{P}_\alpha$ forces CH.

Proof. $\alpha = 0$ is trivial (since $\mathbf{P}_0$ and $\mathbf{P}'_0$ both are trivial: $\mathbf{P}_0$ is a singleton, and $\mathbf{P}'_0$ consists of pairwise compatible elements).

So assume that all items hold for all $\beta < \alpha$.

Proof of (1).

Ultralaver successor case: Let $\alpha = \beta + 1$ with β an ultralaver position. Let H_β be $\mathbf{P}_\beta$-generic over $V[G]$. Work in $V[G][H_\beta]$. By induction, for every $x \in G$ the canonical embedding j_β defines a P^x_β-generic filter over M^x called H^x_β.

Definition of $\mathbf{Q}_\beta$ (and thus of $\mathbf{P}_\alpha$): In $M^x[H^x_\beta]$, the forcing notion Q^x_β is defined as $\mathbb{L}_{\bar{D}^x}$ for some system of ultrafilters $\bar{D}^x$ in $M^x[H^x_\beta]$. Fix some $s \in \omega^{<\omega}$. If $y \leq x$ in G, then D^y_s extends D^x_s. Let D_s be the union of all D^x_s with $x \in G$. So D_s is a proper filter. It is even an ultrafilter: Let r be a $\mathbf{P}_\beta$-name for a real. Using Lemma 4.15, we know that there is some $y \in G$ and some P^y_β-name $\underset{\sim}{r}^y \in M^y$ such that (in $V[G][H_\beta]$) we have $\underset{\sim}{r}^y[H^y_\beta] = r$. So $r \in M^y[H^y_\beta]$, hence either r or its complement is in D^y_s and therefore in D_s. So all filters in the family $\bar{D} = (D_s)_{s\in\omega^{<\omega}}$ are ultrafilters.

Now work again in $V[G]$. We set $\mathbf{Q}_\beta$ to be the $\mathbf{P}_\beta$-name for $\mathbb{L}_{\bar{D}}$. (Note that $\mathbf{P}_\beta$ forces that $\mathbf{Q}_\beta$ literally is the union of the $Q^x_\beta[H^x_\beta]$ for $x \in G$, again by Lemma 4.15.)

Definition of j_α: Let (x, p) be in $\mathbf{P}'_\alpha$. If $p \in P^x_\beta$, then we set $j_\alpha(x, p) = j_\beta(x, p)$, i.e., j_α will extend j_β. If $p = (p\restriction\beta, p(\beta))$ is in P^x_α but not in P^x_β, we set $j_\alpha(x, p) = (r, s) \in \mathbf{P}_\beta * \mathbf{Q}_\beta$ where $r = j_\beta(x, p\restriction\beta)$ and s is the ($\mathbf{P}_\alpha$-name for) $p(\beta)$ as evaluated in $M^x[H^x_\beta]$. From $\mathbf{Q}_\beta = \bigcup_{x\in G} Q^x_\beta[H^x_\beta]$ we conclude that this embedding is dense.

The canonical embedding: By induction we know that $i_{x,\beta}$ which maps $p \in P^x_\beta$ to $j_\beta(x, p)$ is (the restriction to P^x_β of) the canonical embedding of x into $\mathbf{P}_{\omega_2}$. So we have to extend the canonical embedding to $i_{x,\alpha} : P^x_\alpha \to \mathbf{P}_\alpha$. By definition of "canonical embedding", $i_{x,\alpha}$ maps $p \in P^x_\alpha$ to the pair $(i_{x,\beta}(p\restriction\beta), p(\beta))$. This is the same as $j_\alpha(x, p)$. We already know that D^x_s is (forced to be) an $M^x[H^x_\beta]$-ultrafilter that is extended by D_s.

[42]Here we use two consequences of $z \leq y$: Every P^y_β-name in M^y can be canonically interpreted as a P^z_β-name in M^z, and Q^y_β is (forced to be) a subset of Q^z_β.

[43]I.e., $j_\beta(x, p\restriction\beta) = j_\alpha(x, p\restriction\beta) = j_\alpha(x, p)\restriction\beta$.

Janus successor case: This is similar, but simpler than the previous case: Here, $\mathbf{Q}_\beta$ is just defined as the union of all $Q^x_\beta[H^x_\beta]$ for $x \in G$. We will show below that this union satisfies the ccc; just as in Fact 2.8, it is then easy to see that this union is again a Janus forcing.

In particular, $\mathbf{Q}_\beta$ consists of hereditarily countable objects (since it is the union of Janus forcings, which by definition consist of hereditarily countable objects). So since $\mathbf{P}_\beta$ forces CH, $\mathbf{Q}_\beta$ is forced to have size $\aleph_1$. Also note that since all Janus forcings involved are separative, the union (which is a limit of an incompatibility-preserving directed system) is trivially separative as well.

Limit case: Let α be a limit ordinal.

Definition of $\mathbf{P}_\alpha$ *and* j_α*:* First we define $j_\alpha : \mathbf{P}'_\alpha \to \mathbf{P}^{\mathrm{CS}}_\alpha$: For each $(x, p) \in \mathbf{P}'_\alpha$, let $j_\alpha(x, p) \in \mathbf{P}^{\mathrm{CS}}_\alpha$ be the union of all $j_\beta(x, p{\restriction}\beta)$ (for $\beta \in \alpha \cap M^x$). (Note that $\beta_1 < \beta_2$ implies that $j_{\beta_1}(x, p{\restriction}\beta_1)$ is a restriction of $j_{\beta_2}(x, p{\restriction}\beta_2)$, so this union is indeed an element of $\mathbf{P}^{\mathrm{CS}}_\alpha$.)

$\mathbf{P}_\alpha$ is the set of all $q \wedge p$, where $p \in j_\alpha[\mathbf{P}'_\alpha]$, $q \in \mathbf{P}_\beta$ for some $\beta < \alpha$, and $q \leq p{\restriction}\beta$.

It is easy to check that $\mathbf{P}_\alpha$ actually is a partial countable support limit, and that j_α is dense. We will show below that $\mathbf{P}_\alpha$ satisfies the ccc, so in particular it is proper.

The canonical embedding: To see that $i_{x,\alpha}$ is the (restriction of the) canonical embedding, we just have to check that $i_{x,\alpha}$ is M^x-complete. This is the case since $\mathbf{P}'_\alpha$ is the direct limit of all P^y_α for $y \in G$ (without loss of generality $y \leq x$), and each $i_{x,y}$ is M^x-complete (see Fact 4.12).

Proof of (3).

Recall that we assume CH in the ground model.

The successor case, $\alpha = \beta + 1$, follows easily from (3)–(5) for $\mathbf{P}_\beta$ (since $\mathbf{P}_\beta$ forces that $\mathbf{Q}_\beta$ has size $2^{\aleph_0} = \aleph_1 = \aleph_1^V$).

If $\mathrm{cf}(\alpha) > \omega$, then $\mathbf{P}_\alpha = \bigcup_{\beta<\alpha} \mathbf{P}_\beta$, so the proof is easy.

So let $\mathrm{cf}(\alpha) = \omega$. The following straightforward argument works for any ccc partial CS iteration where all iterands $\mathbf{Q}_\beta$ are of size $\leq \aleph_1$.

For notational simplicity we assume $\Vdash_{\mathbf{P}_\beta} \mathbf{Q}_\beta \subseteq \omega_1$ for all $\beta < \alpha$ (this is justified by inductive assumption (5)). By induction, we can assume that for all $\beta < \alpha$ there is a dense $\mathbf{P}^*_\beta \subseteq \mathbf{P}_\beta$ of size $\aleph_1$ and that every $\mathbf{P}^*_\beta$ is ccc. For each $p \in \mathbf{P}_\alpha$ and all $\beta \in \mathrm{dom}(p)$ we can find a maximal antichain $A^p_\beta \subseteq \mathbf{P}^*_\beta$ such that each element $a \in A^p_\beta$ decides the value of $p(\beta)$, say $a \Vdash_{\mathbf{P}_\beta} p(\beta) = \gamma^p_\beta(a)$. Writing[44] $p \sim q$ if $p \leq q$ and $q \leq p$, the map $p \mapsto (A^p_\beta, \gamma^p_\beta)_{\beta\in\mathrm{dom}(p)}$ is 1-1 modulo $\sim$. Since each A^p_β is countable, there are only $\aleph_1$ many possible values, therefore there are only $\aleph_1$ many $\sim$-equivalence classes. Any set of representatives will be dense.

Alternatively, we can prove (3) directly for $\mathbf{P}'_\alpha$. I.e., we can find a $\leq^*$-dense subset $\mathbf{P}'' \subseteq \mathbf{P}'_\alpha$ of cardinality $\aleph_1$. Note that a conditions $(x, p) \in \mathbf{P}'_\alpha$ essentially depends only on p (cf. Fact 4.11). More specifically, given (x, p) we can “transitively[45] collapse x above α”, resulting in a $=^*$-equivalent condition (x', p'). Since $|\alpha| = \aleph_1$, there are only ${\aleph_1}^{\aleph_0} = 2^{\aleph_0}$ many such candidates x' and since each x' is countable and $p' \in x'$, there are only $2^{\aleph_0}$ many pairs (x', p').

Proof of (4).

Ultralaver successor case: Let $\alpha = \beta + 1$ with β an ultralaver position. We already know that $\mathbf{P}_\alpha = \mathbf{P}_\beta * \mathbf{Q}_\beta$ where $\mathbf{Q}_\beta$ is an ultralaver forcing, which in particular is ccc, so by induction $\mathbf{P}_\alpha$ is ccc.

Janus successor case: As above it suffices to show that $\mathbf{Q}_\beta$, the union of the Janus forcings $Q^x_\beta[H^x_\beta]$ for $x \in G$, is (forced to be) ccc.

Assume towards a contradiction that this is not the case, i.e., that we have an uncountable antichain in $\mathbf{Q}_\beta$. We already know that $\mathbf{Q}_\beta$ has size $\aleph_1$ and therefore the uncountable antichain has size $\aleph_1$. So, working in V, we assume towards a contradiction that

(4.18) $$x_0 \Vdash_{\mathbb{R}} p_0 \Vdash_{\mathbf{P}_\beta} \{\dot{q}_i : i \in \omega_1\} \text{ is a maximal (uncountable) antichain in } \mathbf{Q}_\beta.$$

[44] Since $\leq$ is separative, $p \sim q$ iff $p =^* q$, but this fact is not used here.

[45] In more detail: We define a function $f : M^x \to V$ by induction as follows: If $\beta \in M^x \cap \alpha + 1$ or if $\beta = \omega_2$, then $f(\beta) = \beta$. Otherwise, if $\beta \in M^x \cap \mathrm{Ord}$, then $f(\beta)$ is the smallest ordinal above $f[\beta]$. If $a \in M^x \setminus \mathrm{Ord}$, then $f(a) = \{f(b) : b \in a \cap M^x\}$. It is easy to see that f is an isomorphism from M^x to $M^{x'} := f[M^x]$ and that $M^{x'}$ is a candidate. Moreover, the ordinals that occur in $M^{x'}$ are subsets of $\alpha + \omega_1$ together with the interval $[\omega_2, \omega_2 + \omega_1]$; i.e., there are $\aleph_1$ many ordinals that can possibly occur in $M^{x'}$, and therefore there are $2^{\aleph_0}$ many possible such candidates. Moreover, setting $p' := f(p)$, it is easy to check that $(x, p) =^* (x, p)$ (similarly to Fact 4.11).

We construct by induction on $n \in \omega$ a decreasing sequence of conditions such that x_{n+1} satisfies the following:

(i) For all $i \in \omega_1 \cap M^{x_n}$ there is (in $M^{x_{n+1}}$) a $P^{x_{n+1}}_\beta$-name $\dot{a}^*_i$ for a $Q^{x_{n+1}}_\beta$-condition such that

$$x_{n+1} \Vdash_{\mathbb{R}} p_0 \Vdash_{\mathbf{P}_\beta} \dot{\underset{\sim}{q}}_i = \dot{a}^*_i.$$

Why can we get that? Just use Lemma 4.16.

(ii) If τ is in M^{x_n} a $P^{x_n}_\beta$-name for an element of $Q^{x_n}_\beta$, then there is $k^*(\tau) \in \omega_1$ such that

$$x_{n+1} \Vdash_{\mathbb{R}} p_0 \Vdash_{\mathbf{P}_\beta} (\exists i < k^*(\tau))\, \dot{\underset{\sim}{q}}_i \parallel_{\mathbf{P}_\beta} \tau.$$

Also, all these $k^*(\tau)$ are in $M^{x_{n+1}}$.

Why can we get that? First note that $x_n \Vdash p_0 \Vdash (\exists i \in \omega_1)\, \dot{\underset{\sim}{q}}_i \parallel \tau$. Since $\mathbf{P}_\beta$ is ccc, x_n forces that there is some bound $\underset{\sim}{k}(\tau)$ for i. So it suffices that x_{n+1} determines $\underset{\sim}{k}(\tau)$ to be $k^*(\tau)$ (for all the countably many τ).

Set $\delta^* := \omega_1 \cap \bigcup_{n\in\omega} M^{x_n}$. By Corollary 4.8(4), there is some y such that

- $y \leq x_n$ for all $n \in \omega$,
- $(x_n)_{n\in\omega}$ and $(\dot{a}^*_i)_{i\in\delta^*}$ are in M^y,
- (M^y thinks that) P^y_β forces that Q^y_β is the union of $Q^{x_n}_\beta$, i.e., as a formula: $M^y \models P^y_\beta \Vdash Q^y_\beta = \bigcup_{n\in\omega} Q^{x_n}_\beta$.

Let G be $\mathbb{R}$-generic (over V) containing y, and let H_β be $\mathbf{P}_\beta$-generic (over $V[G]$) containing p_0.

Set $A^* := \{\dot{a}^*_i[H^y_\beta] : i < \delta^*\}$. Note that A^* is in $M^y[H^y_\beta]$. We claim

$$A^* \subseteq Q^y_\beta[H^y_\beta] \text{ is predense.} \tag{4.19}$$

Pick any $q_0 \in Q^y_\beta$. So there is some $n \in \omega$ and some τ which is in M^{x_n} a $P^{x_n}_\beta$-name of a $Q^{x_n}_\beta$-condition, such that $q_0 = \tau[H^{x_n}_\beta]$. By (ii) above, x_{n+1} and therefore y forces (in $\mathbb{R}$) that for some $i < k^*(\tau)$ (and therefore some $i < \delta^*$) the condition p_0 forces the following (in $\mathbf{P}_\beta$):

> The conditions $\dot{\underset{\sim}{q}}_i$ and τ are compatible in $\mathbf{Q}_\beta$. Also, $\dot{\underset{\sim}{q}}_i = \dot{a}^*_i$ and τ both are in Q^y_β, and Q^y_β is an incompatibility-preserving subforcing of $\mathbf{Q}_\beta$. Therefore $M^y[H^y_\beta]$ thinks that $\dot{a}^*_i$ and τ are compatible.

This proves (4.19).

Since $Q^y_\beta[H^y_\beta]$ is $M^y[H^y_\beta]$-complete in $\mathbf{Q}_\beta[H_\beta]$, and since $A^* \in M^y[H^y]$, this implies (as $\dot{a}^*_i[H^y_\beta] = \dot{\underset{\sim}{q}}_i[G * H_\beta]$ for all $i < \delta^*$) that $\{\dot{\underset{\sim}{q}}_i[G * H_\beta] : i < \delta^*\}$ already is predense, a contradiction to (4.18).

Limit case: We work with $\mathbf{P}'_\alpha$, which by definition only contains HCON objects.

Assume towards a contradiction that $\mathbf{P}'_\alpha$ has an uncountable antichain. We already know that $\mathbf{P}'_\alpha$ has a dense subset of size $\aleph_1$ (modulo $=^*$), so the antichain has size $\aleph_1$.

Again, work in V. We assume towards a contradiction that

$$x_0 \Vdash_{\mathbb{R}} \{\underset{\sim}{q}_i : i \in \omega_1\} \text{ is a maximal (uncountable) antichain in } \mathbf{P}'_\alpha. \tag{4.20}$$

So each $\underset{\sim}{q}_i$ is an $\mathbb{R}$-name for an HCON object (x, p) in V.

To lighten the notation we will abbreviate elements $(x, p) \in \mathbf{P}'_\alpha$ by p; this is justified by Fact 4.11.

Fix any HCON object p and $\beta < \alpha$. We will now define the $(\mathbb{R} * \mathbf{P}'_\beta)$-names $\underset{\sim}{i}(\beta, p)$ and $\underset{\sim}{r}(\beta, p)$: Let G be $\mathbb{R}$-generic and containing x_0, and H'_β be $\mathbf{P}'_\beta$-generic. Let R be the quotient $\mathbf{P}'_\alpha / H'_\beta$. If p is not in R, set $\underset{\sim}{i}(\beta, p) = \underset{\sim}{r}(\beta, p) = 0$. Otherwise, let $\underset{\sim}{i}(\beta, p)$ be the minimal i such that $\underset{\sim}{q}_i \in R$ and $\underset{\sim}{q}_i$ and p are compatible (in R), and set $\underset{\sim}{r}(\beta, p) \in R$ to be a witness of this compatibility. Since $\mathbf{P}'_\beta$ is (forced to be) ccc, we can find (in $V[G]$) a countable set $\underset{\sim}{X}^\iota(\beta, p) \subseteq \omega_1$ containing all possibilities for $\underset{\sim}{i}(\beta, p)$ and similarly $\underset{\sim}{X}^r(\beta, p)$ consisting of HCON objects for $\underset{\sim}{r}(\beta, p)$.

To summarize: For every $\beta < \alpha$ and every HCON object p, we can define (in V) the $\mathbb{R}$-names $\underset{\sim}{X}^\iota(\beta, p)$ and $\underset{\sim}{X}^r(\beta, p)$ such that

$$x_0 \Vdash_{\mathbb{R}} \Vdash_{\mathbf{P}'_\beta} \Big(p \in \mathbf{P}'_\alpha / H'_\beta \ \rightarrow\ (\exists i \in \underset{\sim}{X}^\iota(\beta, p))\, (\exists r \in \underset{\sim}{X}^r(\beta, p))\, r \leq_{\mathbf{P}'_\alpha / H'_\beta} p, \underset{\sim}{q}_i\Big). \tag{4.21}$$

Similarly to the Janus successor case, we define by induction on $n \in \omega$ a decreasing sequence of conditions such that x_{n+1} satisfies the following: For all $\beta \in \alpha \cap M^{x_n}$ and $p \in P^{x_n}_\alpha$, x_{n+1} decides $\underset{\sim}{X}^\iota(\beta, p)$ and $\underset{\sim}{X}^r(\beta, p)$ to be some $X^{\iota *}(\beta, p)$ and $X^{r*}(\beta, p)$. For all $i \in \omega_1 \cap M^{x_n}$, x_{n+1} decides $\underset{\sim}{q}_i$ to be some $a^*_i \in P^{x_{n+1}}_\alpha$.

Moreover, each such $X^{\iota *}$ and X^{r*} is in $M^{x_{n+1}}$, and every $r \in X^{r*}(\beta, p)$ is in $P_\alpha^{x_{n+1}}$. (For this, we just use Fact 4.14 and Lemma 4.15.)

Set $\delta^* := \omega_1 \cap \bigcup_{n\in\omega} M^{x_n}$, and set $A^* := \{a_i^* : i \in \delta^*\}$. By Corollary 4.8(4), there is some y such that

$$y \leq x_n \text{ for all } n \in \omega, \tag{4.22}$$

$$\bar{x} := (x_n)_{n\in\omega} \text{ and } A^* \text{ are in } M^y, \tag{4.23}$$

$$(M^y \text{ thinks that) } P_\alpha^y \text{ is defined as the almost FS limit over } \bar{x}. \tag{4.24}$$

We claim that y forces

$$A^* \text{ is predense in } P_\alpha^y. \tag{4.25}$$

Then P_α^y is M^y-completely embedded into $\mathbf{P}'_\alpha$, and since $A^* \in M^y$ (and since $\underset{\sim}{a}_i = a_i^*$ for all $i \in \delta^*$) we get that $\{\underset{\sim}{a}_i : i \in \delta^*\}$ is predense, a contradiction to (4.20).

So it remains to show (4.25). Let G be $\mathbb{R}$-generic containing y. Let r be a condition in P_α^y; we will find $i < \delta^*$ such that r is compatible with a_i^*. Since P_α^y is the almost FS limit over $\bar{x}$, there is some $n \in \omega$ and $\beta \in \alpha \cap M^{x_n}$ such that r has the form $q \wedge p$ with p in $P_\alpha^{x_n}$, $q \in P_\beta^y$ and $q \leq p{\restriction}\beta$.

Now let H'_β be $\mathbf{P}'_\beta$-generic containing q. Work in $V[G][H'_\beta]$. Since $q \leq p{\restriction}\beta$, we get $p \in \mathbf{P}'_\alpha/H'_\beta$. Let ι^* be the evaluation by $G * H'_\beta$ of $\underset{\sim}{i}(\beta, p)$, and let r^* be the evaluation of $\underset{\sim}{\dot r}(\beta, p)$. Note that $\iota^* < \delta^*$ and $r^* \in P_\alpha^y$. So we know that $a_{\iota^*}^*$ and p are compatible in $\mathbf{P}'_\alpha/H'_\beta$ witnessed by r^*. Find $q' \in H'_\beta$ forcing $r^* \leq_{\mathbf{P}'_\alpha/H'_\beta} p, a_{\iota^*}^*$. We may find $q' \leq q$. Now $q' \wedge r^*$ witnesses that $q \wedge p$ and $a_{\iota^*}^*$ are compatible in P_α^y.

To summarize: The crucial point in proving the ccc is that "densely" we choose (a variant of) a finite support iteration, see (4.24). Still, it is a bit surprising that we get the ccc, since we can also argue that densely we use (a variant of) a countable support iteration. But this does not prevent the ccc, it only prevents the generic iteration from having direct limits in stages of countable cofinality.[46]

Proof of (5).

This follows from (3) and (4). □

4.D. **The generic alternating iteration $\bar{\mathbf{P}}$.** In Lemma 4.17 we have seen:

Corollary 4.26. *Let G be $\mathbb{R}$-generic. Then we can construct*[47] *(in $V[G]$) an alternating iteration $\bar{\mathbf{P}}$ such that the following holds:*

- *$\bar{\mathbf{P}}$ is ccc.*
- *If $x \in G$, then x canonically embeds into $\bar{\mathbf{P}}$. (In particular, a $\mathbf{P}_{\omega_2}$-generic filter H_{ω_2} induces a $P_{\omega_2}^x$-generic filter over M^x, called $H_{\omega_2}^x$.)*
- *Each $\mathbf{Q}_\alpha$ is the union of all $Q_\alpha^x[H_\alpha^x]$ with $x \in G$.*
- *$\mathbf{P}_{\omega_2}$ is equivalent to the direct limit $\mathbf{P}'_{\omega_2}$ of G: There is a dense embedding $j : \mathbf{P}'_{\omega_2} \to \mathbf{P}_{\omega_2}$, and for each $x \in G$ the function $p \mapsto j(x, p)$ is the canonical embedding.*

Lemma 4.27. *Let $x \in \mathbb{R}$. Then $\mathbb{R}$ forces the following: $x \in G$ iff x canonically embeds into $\bar{\mathbf{P}}$.*

Proof. If $x \in G$, then we already know that x canonically embeds into $\bar{\mathbf{P}}$.

So assume (towards a contradiction) that y forces that x embeds, but $y \Vdash x \notin G$. Work in $V[G]$ where $y \in G$. Both x (by assumption) and $y \in G$ canonically embed into $\bar{\mathbf{P}}$. Let N be an elementary submodel of $H^{V[G]}(\chi^*)$ containing $x, y, \bar{\mathbf{P}}$; let $z = (M^z, \bar{P}^z)$ be the ord-collapse of $(N, \bar{\mathbf{P}})$. Then $z \in V$ (as $\mathbb{R}$ is σ-closed) and $z \in \mathbb{R}$, and (by elementarity) $z \leq x, y$. This shows that $x \parallel_{\mathbb{R}} y$, i.e., y cannot force $x \notin G$, a contradiction. □

Using ccc, we can now prove a lemma that is in fact stronger than the lemmas in the previous section 4.C:

Lemma 4.28. *The following is forced by $\mathbb{R}$: Let $N \prec H^{V[G]}(\chi^*)$ be countable, and let y be the ord-collapse of $(N, \bar{\mathbf{P}})$. Then $y \in G$. Moreover, if $x \in G \cap N$, then $y \leq x$.*

[46] Assume that x forces that $\mathbf{P}'_\alpha$ is the union of the $\mathbf{P}'_\beta$ for $\beta < \alpha$; then we can find a stronger y that uses an almost CS iteration over x. This almost CS iteration contains a condition p with unbounded support. (Take any condition in the generic part of the almost CS limit; if this condition has bounded domain, we can extend it to have unbounded domain, see Definition **??**.) Now p will be in $\mathbf{P}'_\alpha$ and have unbounded domain.

[47] in an "absolute way": Given G, we first define $\mathbf{P}'_{\omega_2}$ to be the direct limit of G, and then inductively construct the $\mathbf{P}_\alpha$'s from $\mathbf{P}'_{\omega_2}$.

Proof. Work in $V[G]$ with $x \in G$. Pick an elementary submodel N containing x and $\bar{\mathbf{P}}$. Let y be the ord-collapse of $(N, \bar{\mathbf{P}})$ via a collapsing map k. As above, it is clear that $y \in \mathbb{R}$ and $y \leq x$. To show $y \in G$, it is (by the previous lemma) enough to show that y canonically embeds. We claim that k^{-1} is the canonical embedding of y into $\bar{\mathbf{P}}$. The crucial point is to show M^y-completeness. Let $B \in M^y$ be a maximal antichain of $P^y_{\omega_2}$, say $B = k(A)$ where $A \in N$ is a maximal antichain of $\mathbf{P}_{\omega_2}$. So (by ccc) A is countable, hence $A \subseteq N$. So not only $A = k^{-1}(B)$ but even $A = k^{-1}[B]$. Hence k^{-1} is an M^y-complete embedding. □

Remark 4.29. We used the ccc of $\mathbf{P}_{\omega_2}$ to prove Lemma 4.28; this use was essential in the sense that we can in turn easily prove the ccc of $\mathbf{P}_{\omega_2}$ if we assume that Lemma 4.28 holds. In fact Lemma 4.28 easily implies all other lemmas in section 4.C as well.

5. The proof of BC+dBC

We first[48] prove that no uncountable X in V will be smz or sm in the final extension $V[G * H]$. Then we show how to modify the argument to work for all uncountable sets in $V[G * H]$.

5.A. BC+dBC for ground model sets.

Lemma 5.1. *Let $X \in V$ be an uncountable set of reals. Then $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that X is not smz.*

Proof.

(1) Fix any even $\alpha < \omega_2$ (i.e., an ultralaver position) in our iteration. The ultralaver forcing $\bar{\mathbf{Q}}_\alpha$ adds a (canonically defined code for a) closed null set $\dot{F}$ constructed from the ultralaver real $\bar{\ell}_\alpha$. (Recall Corollary 1.21.) In the following, when we consider various ultralaver forcings $\mathbf{Q}_\alpha$, Q_α, Q^x_α, we treat $\dot{F}$ not as an actual name, but rather as a definition which depends on the forcing used.

(2) According to Theorem 0.2, it is enough to show that $X + \dot{F}$ is non-null in the $\mathbb{R} * \mathbf{P}_{\omega_2}$-extension, or equivalently, in every $\mathbb{R} * \mathbf{P}_\beta$-extension ($\alpha < \beta < \omega_2$). So assume towards a contradiction that there is a $\beta > \alpha$ and an $\mathbb{R} * \mathbf{P}_\beta$-name $\dot{\underset{\sim}{Z}}$ of a (code for a) Borel null set such that some $(x, p) \in \mathbb{R} * \mathbf{P}_{\omega_2}$ forces that $X + \dot{F} \subseteq \dot{\underset{\sim}{Z}}$.

(3) Using the dense embedding $j_{\omega_2} : \mathbf{P}'_{\omega_2} \to \mathbf{P}_{\omega_2}$, we may replace (x, p) by a condition $(x, p') \in \mathbb{R} * \mathbf{P}'_{\omega_2}$. According to Fact 4.14 (recall that we now know that $\mathbf{P}_{\omega_2}$ satisfies ccc) and Lemma 4.15 we can assume that p' is already a P^x_β-condition p^x and that $\dot{\underset{\sim}{Z}}$ is (forced by x to be the same as) a P^x_β-name $\dot{Z}^x$ in M^x.

(4) We construct (in V) an iteration $\bar{P}$ in the following way:
   (a) Up to α, we take an arbitrary alternating iteration into which x embeds. In particular, P_α will be proper and hence force that X is still uncountable.
   (b) Let Q_α be any ultralaver forcing (over Q^x_α in case $\alpha \in M^x$). So according to Corollary 1.21, we know that Q_α forces that $X + \dot{F}$ is not null.
   Therefore we can pick (in $V[H_{\alpha+1}]$) some $\dot{r}$ in $X + \dot{F}$ which is random over (the countable model) $M^x[H^x_{\alpha+1}]$, where $H^x_{\alpha+1}$ is induced by $H_{\alpha+1}$.
   (c) In the rest of the construction, we preserve randomness of $\dot{r}$ over $M^x[H^x_\zeta]$ for each $\zeta \leq \omega_2$. We can do this using an almost CS iteration over x where at each Janus position we use a random version of Janus forcing and at each ultralaver position we use a suitable ultralaver forcing; this is possible by Lemma 3.32. By Lemma 3.34, this iteration will preserve the randomness of $\dot{r}$.
   (d) So we get $\bar{P}$ over x (with canonical embedding i_x) and $q \leq_{P_{\omega_2}} i_x(p^x)$ such that $q \restriction \beta$ forces (in P_β) that $\dot{r}$ is random over $M^x[H^x_\beta]$, in particular that $\dot{r} \notin \dot{Z}^x$.

   We now pick a countable $N \prec H(\chi^*)$ containing everything and ord-collapse $(N, \bar{P})$ to $y \leq x$. (See Fact 4.4.) Set $X^y := X \cap M^y$ (the image of X under the collapse). By elementarity, M^y thinks that (a)–(d) above holds for $\bar{P}^y$ and that X^y is uncountable. Note that $X^y \subseteq X$.

(5) This gives a contradiction in the obvious way: Let G be $\mathbb{R}$-generic over V and contain y, and let H_β be $\mathbf{P}_\beta$-generic over $V[G]$ and contain $q \restriction \beta$. So $M^y[H^y_\beta]$ thinks that $r \notin \dot{Z}^x$ (which is absolute) and that $r = x + f$ for some $x \in X^y \subseteq X$ and $f \in F$ (actually even in F as evaluated in $M^y[H^y_{\alpha+1}]$). So

[48]Note that for this weak version, it would be enough to produce a generic iteration of length 2 only, i.e., $\mathbf{Q}_0 * \mathbf{Q}_1$, where $\mathbf{Q}_0$ is an ultralaver forcing and $\mathbf{Q}_1$ a corresponding Janus forcing.

in $V[G][H_\beta]$, r is the sum of an element of X and an element of F. So $(y,q) \le (x,p')$ forces that $r \in X + \dot F \setminus \dot Z$, a contradiction to (2). $\square$

Of course, we need this result not just for ground model sets X, but for $\mathbb{R} * \mathbf{P}_{\omega_2}$-names $\dot{\underset{\sim}{X}} = (\dot{\underset{\sim}{x}}_i : i \in \omega_1)$ of uncountable sets. It is easy to see that it is enough to deal with $\mathbb{R} * \mathbf{P}_\beta$-names for (all) $\beta < \omega_2$. So given $\dot{\underset{\sim}{X}}$, we can (in the proof) pick α such that $\dot{\underset{\sim}{X}}$ is actually an $\mathbb{R} * \mathbf{P}_\alpha$-name. We can try to repeat the same proof; however, the problem is the following: When constructing $\bar P$ in (4), it is not clear how to simultaneously make all the uncountably many names $(\dot{\underset{\sim}{x}}_i)$ into $\bar P$-names in a sufficiently "absolute" way. In other words: It is not clear how to end up with some M^y and $\dot X^y$ uncountable in M^y such that it is guaranteed that $\dot X^y$ (evaluated in $M^y[H^y_\alpha]$) will be a subset of $\dot{\underset{\sim}{X}}$ (evaluated in $V[G][H_\alpha]$). We will solve this problem in the next section by factoring $\mathbb{R}$.

Let us now give the proof of the corresponding weak version of dBC:

Lemma 5.2. *Let $X \in V$ be an uncountable set of reals. Then $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that X is not strongly meager.*

Proof. The proof is parallel to the previous one:

(1) Fix any even $\alpha < \omega_2$ (i.e., an ultralaver position) in our iteration. The Janus forcing $\mathbf{Q}_{\alpha+1}$ adds a (canonically defined code for a) null set $\dot Z_\nabla$. (See Definition 2.6 and Fact 2.7.)
(2) According to (0.1), it is enough to show that $X + \dot Z_\nabla = 2^\omega$ in the $\mathbb{R} * \mathbf{P}_{\omega_2}$-extension, or equivalently, in every $\mathbb{R} * \mathbf{P}_\beta$-extension ($\alpha < \beta < \omega_2$). (For every real r, the statement $r \in X + \dot Z_\nabla$, i.e., $(\exists x \in X)\ x + r \in \dot Z_\nabla$, is absolute.) So assume towards a contradiction that there is a $\beta > \alpha$ and an $\mathbb{R} * \mathbf{P}_\beta$-name $\dot{\underset{\sim}{r}}$ of a real such that some $(x,p) \in \mathbb{R} * \mathbf{P}_{\omega_2}$ forces that $\dot{\underset{\sim}{r}} \notin X + \dot Z_\nabla$.
(3) Again, we can assume that $\dot{\underset{\sim}{r}}$ is a P^x_β-name $\dot r^x$ in M^x.
(4) We construct (in V) an iteration $\bar P$ in the following way:
  (a) Up to α, we take an arbitrary alternating iteration into which x embeds. In particular, P_α again forces that X is still uncountable.
  (b1) Let Q_α be any ultralaver forcing (over Q^x_α). Then Q_α forces that X is not thin (see Corollary 1.24).
  (b2) Let $Q_{\alpha+1}$ be a countable Janus forcing. So $Q_{\alpha+1}$ forces $X + \dot Z_\nabla = 2^\omega$. (See Lemma 2.9.)
  (c) We continue the iteration in a σ-centered way. I.e., we use an almost FS iteration over x of ultralaver forcings and countable Janus forcings, using trivial Q_ζ for all $\zeta \notin M^x$; see Lemma 3.17.
  (d) So P_β still forces that $X + \dot Z_\nabla = 2^\omega$, and in particular that $\dot r^x \in X + \dot Z_\nabla$. (Again by Lemma 2.9.)

  Again, by collapsing some N as in the previous proof, we get $y \le x$ and $X^y \subseteq X$.
(5) This again gives the obvious contradiction: Let G be $\mathbb{R}$-generic over V and contain y, and let H_β be $\mathbf{P}_\beta$-generic over $V[G]$ and contain p. So $M^y[H^y_\beta]$ thinks that $r = x + z$ for some $x \in X^y \subseteq X$ and $z \in Z_\nabla$ (this time, $\dot Z_\nabla$ is evaluated in $M^y[H^y_\beta]$), contradicting (2). $\square$

5.B. **A factor lemma.** We can restrict $\mathbb{R}$ to any $\alpha^* < \omega_2$ in the obvious way: Conditions are pairs $x = (M^x, \bar P^x)$ of nice candidates M^x (containing α^*) and alternating iterations $\bar P^x$, but now M^x thinks that $\bar P^x$ has length α^* (and not ω_2). We call this variant $\mathbb{R}\restriction\alpha^*$.

Note that all results of Section 4 about $\mathbb{R}$ are still true for $\mathbb{R}\restriction\alpha^*$. In particular, whenever $G \subseteq \mathbb{R}\restriction\alpha^*$ is generic, it will define a direct limit (which we call $\mathbf{P}'^*$), and an alternating iteration of length α^* (called $\mathbf{P}^*$); again we will have that $x \in G$ iff x canonically embeds into $\bar{\mathbf{P}}^*$.

There is a natural projection map from $\mathbb{R}$ (more exactly: from the dense subset of those x which satisfy $\alpha^* \in M^x$) into $\mathbb{R}\restriction\alpha^*$, mapping $x = (M^x, \bar P^x)$ to $x\restriction\alpha^* := (M^x, \bar P^x\restriction\alpha^*)$. (It is obvious that this projection is dense and preserves $\le$.)

There is also a natural embedding φ from $\mathbb{R}\restriction\alpha^*$ to $\mathbb{R}$: We can just continue an alternating iteration of length α^* by appending trivial forcings.

φ is complete: It preserves $\le$ and $\perp$. (Assume that $z \le \varphi(x), \varphi(y)$. Then $z\restriction\alpha^* \le x, y$.) Also, the projection is a reduction: If $y \le x\restriction\alpha^*$ in $\mathbb{R}\restriction\alpha^*$, then let M^z be a model containg both x and y. In M^z, we can first construct an alternating iteration of length α^* over y (using almost FS over y, or almost CS — this does not matter here). We then continue this iteration $\bar P^z$ using almost FS or almost CS over x. So x and y both embed into $\bar P^z$, hence $z = (M^z, \bar P^z) \le x, y$.

So according to the general factor lemma of forcing theory, we know that $\mathbb{R}$ is forcing equivalent to $\mathbb{R}\restriction\alpha^* * (\mathbb{R}/\mathbb{R}\restriction\alpha^*)$, where $\mathbb{R}/\mathbb{R}\restriction\alpha^*$ is the quotient of $\mathbb{R}$ and $\mathbb{R}\restriction\alpha^*$, i.e., the ($\mathbb{R}\restriction\alpha^*$-name for the) set of $x \in \mathbb{R}$ which are compatible (in $\mathbb{R}$) with all $\varphi(y)$ for $y \in G\restriction\alpha^*$ (the generic filter for $\mathbb{R}\restriction\alpha^*$), or equivalently, the set of $x \in \mathbb{R}$ such that $x\restriction\alpha^* \in G\restriction\alpha^*$. So Lemma 4.27 (relativized to $\mathbb{R}\restriction\alpha^*$) implies:

(5.3) $\mathbb{R}/\mathbb{R}\restriction\alpha^*$ is the set of $x \in \mathbb{R}$ that canonically embed (up to α^*) into $\mathbf{P}_{\alpha^*}$.

Setup. *Fix some $\alpha^* < \omega_2$ of uncountable cofinality.*[49] *Let $G\restriction\alpha^*$ be $\mathbb{R}\restriction\alpha^*$-generic over V and work in $V^* := V[G\restriction\alpha^*]$. Set $\bar{\mathbf{P}}^* = (\mathbf{P}_\beta)_{\beta<\alpha^*}$, the generic alternating iteration added by $\mathbb{R}\restriction\alpha^*$. Let $\mathbb{R}^*$ be the quotient $\mathbb{R}/\mathbb{R}\restriction\alpha^*$.*

We claim that $\mathbb{R}^*$ satisfies (in V^*) all the properties that we proved in Section 4 for $\mathbb{R}$ (in V), with the obvious modifications. In particular:

(A)$_{\alpha^*}$ $\mathbb{R}^*$ is $\aleph_2$-cc, since it is the quotient of an $\aleph_2$-cc forcing.

(B)$_{\alpha^*}$ $\mathbb{R}^*$ does not add new reals (and more generally, no new HCON objects), since it is the quotient of a σ-closed forcing.[50]

(C)$_{\alpha^*}$ Let G^* be $\mathbb{R}^*$-generic over V^*. Then G^* is $\mathbb{R}$-generic over V, and therefore Corollary 4.26 holds for G^*. (Note that $\mathbf{P}'_{\omega_2}$ and then $\mathbf{P}_{\omega_2}$ is constructed from G^*.) Moreover, it is easy to see[51] that $\bar{\mathbf{P}}$ starts with $\bar{\mathbf{P}}^*$.

(D)$_{\alpha^*}$ In particular, we get a variant of Lemma 4.28: The following is forced by $\mathbb{R}^*$: Let $N \prec H^{V[G^*]}(\chi^*)$ be countable, and let y be the ord-collapse of $(N, \bar{\mathbf{P}})$. Then $y \in G^*$. Moreover: If $x \in G^* \cap N$, then $y \leq x$.

We can use the last item to prove the $\mathbb{R}^*$-version of Fact 4.14:

Corollary 5.4. *In V^*, the following holds:*

(1) Assume that $x \in \mathbb{R}^*$ forces that $p \in \mathbf{P}_{\omega_2}$. Then there is a $y \leq x$ and a $p^y \in P^y_{\omega_2}$ such that y forces $p^y =^* p$.

(2) Assume that $x \in \mathbb{R}^*$ forces that $\dot{\underset{\sim}{r}}$ is a $\mathbf{P}_{\omega_2}$-name of a real. Then there is a $y \leq x$ and a $P^y_{\omega_2}$-name $\dot{r}^y$ such that y forces that $\dot{r}^y$ and $\dot{\underset{\sim}{r}}$ are equivalent as $\mathbf{P}_{\omega_2}$-names.

Proof. We only prove (1), the proof of (2) is similar.

Let G^* contain x. In $V[G^*]$, pick an elementary submodel N containing $x, p, \bar{\mathbf{P}}$ and let $(M^z, \bar{P}^z, p^z)$ be the ord-collapse of $(N, \bar{\mathbf{P}}, p)$. Then $z \in G^*$. This whole situation is forced by some $y \leq z \leq x \in G^*$. So y and p^y is as required, where $p^y \in P^y_{\omega_2}$ is the canonical image of p^z. □

We also get the following analogue of Fact 4.4:

(5.5) In V^* we have: Let $x \in \mathbb{R}^*$. Assume that $\bar{P}$ is an alternating iteration that extends $\bar{\mathbf{P}}\restriction\alpha^*$ and that $x = (M^x, \bar{P}^x) \in \mathbb{R}$ canonically embeds into $\bar{P}$, and that $N \prec H(\chi^*)$ contains x and $\bar{P}$. Let $y = (M^y, \bar{P}^y)$ be the ord-collapse of $(N, \bar{P})$. Then $y \in \mathbb{R}^*$ and $y \leq x$.

We now claim that $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces BC+dBC. We know that $\mathbb{R}$ is forcing equivalent to $\mathbb{R}\restriction\alpha^* * \mathbb{R}^*$. Obviously we have

$$\mathbb{R} * \mathbf{P}_{\omega_2} = \mathbb{R}\restriction\alpha^* * \mathbb{R}^* * \mathbf{P}_{\alpha^*} * \mathbf{P}_{\alpha^*,\omega_2}$$

(where $\mathbf{P}_{\alpha^*,\omega_2}$ is the quotient of $\mathbf{P}_{\omega_2}$ and $\mathbf{P}_{\alpha^*}$). Note that $\mathbf{P}_{\alpha^*}$ is already determined by $\mathbb{R}\restriction\alpha^*$, so $\mathbb{R}^* * \mathbf{P}_{\alpha^*}$ is (forced by $\mathbb{R}\restriction\alpha^*$ to be) a product $\mathbb{R}^* \times \mathbf{P}_{\alpha^*} = \mathbf{P}_{\alpha^*} \times \mathbb{R}^*$.

But note that this is not the same as $\mathbf{P}_{\alpha^*} * \mathbb{R}^*$, where we evaluate the definition of $\mathbb{R}^*$ in the $\mathbf{P}_{\alpha^*}$-extension of $V[G\restriction\alpha^*]$: We would get new candidates and therefore new conditions in $\mathbb{R}^*$ after forcing with $\mathbf{P}_{\alpha^*}$. In other words, we can *not* just argue as follows:

Wrong argument. $\mathbb{R} * \mathbf{P}_{\omega_2}$ is the same as $(\mathbb{R}\restriction\alpha^* * \mathbf{P}_{\alpha^*}) * (\mathbb{R}^* * \mathbf{P}_{\alpha^*,\omega_2})$; so given an $\mathbb{R} * \mathbf{P}_{\omega_2}$-name X of a set of reals of size $\aleph_1$, we can choose α^* large enough so that X is an $(\mathbb{R}\restriction\alpha^* * \mathbf{P}_{\alpha^*})$-name. Then, working in the $(\mathbb{R}\restriction\alpha^* * \mathbf{P}_{\alpha^*})$-extension, we just apply Lemmas 5.1 and 5.2.

[49] Probably the cofinality is completely irrelevant, but the picture is clearer this way.

[50] It is easy to see that $\mathbb{R}^*$ is even σ-closed, by "relativizing" the proof for $\mathbb{R}$, but we will not need this.

[51] Let $\mathbf{P}'^*_\beta$ be the direct limit of $G\restriction\alpha^*$ (for $\beta \leq \alpha^*$), and $\mathbf{P}'_\beta$ the direct limit of G^*. The function $k_\beta : \mathbf{P}'^*_\beta \to \mathbf{P}'_\beta$ that maps (x, p) to $(\varphi(x), p)$ preserves $\leq$ and $\perp$ and is surjective modulo $=^*$, see Fact 4.11(3). So it is clear that defining $\bar{\mathbf{P}}^*$ by induction from $\mathbf{P}'^*_{\omega_2}$ yields the same result as defining $\bar{\mathbf{P}}$ from $\mathbf{P}'_{\omega_2}$.

So what do we do instead? Assume that $\dot{\underaccent{\tilde}{X}} = \{\dot{\underaccent{\tilde}{\xi}}_i : i \in \omega_1\}$ is an $\mathbb{R} * \mathbf{P}_{\omega_2}$-name for a set of reals of size $\aleph_1$. So there is a $\beta < \omega_2$ such that $\dot{\underaccent{\tilde}{X}}$ is added by $\mathbb{R} * \mathbf{P}_\beta$ (using $\aleph_2$-cc of $\mathbb{R}$). In the $\mathbb{R}$-extension, $\mathbf{P}_\beta$ is ccc, therefore we can assume that each $\dot{\underaccent{\tilde}{\xi}}_i$ is a system of countably many countable antichains $\underaccent{\tilde}{A}_i^m$ of $\mathbf{P}_\beta$, together with functions $\underaccent{\tilde}{f}_i^m : \underaccent{\tilde}{A}_i^m \to \{0,1\}$. For the following argument, we prefer to work with the equivalent $\mathbf{P}'_\beta$ instead of $\mathbf{P}_\beta$. We can assume that each of the sequences $B_i := (\underaccent{\tilde}{A}_i^m, \underaccent{\tilde}{f}_i^m)_{m\in\omega}$ is an element of V (since $\mathbf{P}'_\beta$ is a subset of V and since $\mathbb{R}$ is σ-closed). So each B_i is decided by a maximal antichain Z_i of $\mathbb{R}$. Since $\mathbb{R}$ is $\aleph_2$-cc, these $\aleph_1$ many antichains all are contained in some $\mathbb{R}\restriction\alpha^*$ with $\alpha^* \geq \beta$.

So in the $\mathbb{R}\restriction\alpha^*$-extension V^* we have the following situation: Each ξ_i is a very "absolute[52]" $\mathbb{R}^* * \mathbf{P}_{\alpha^*}$-name (or equivalently, $\mathbb{R}^* \times \mathbf{P}_{\alpha^*}$-name), in fact they are already determined by antichains that are in $\mathbf{P}_{\alpha^*}$ and do not depend on $\mathbb{R}^*$. So we can interpret them as $\mathbf{P}_{\alpha^*}$-names.

Note that:

(5.6) The ξ_i are forced (by $\mathbb{R}^* * \mathbf{P}_{\alpha^*}$) to be pairwise different, and therefore already by $\mathbf{P}_{\alpha^*}$.

Now we are finally ready to prove that $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that every uncountable X is neither smz nor sm. It is enough to show that for every name $\dot{\underaccent{\tilde}{X}}$ of an uncountable set of reals of size $\aleph_1$ the forcing $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that $\dot{\underaccent{\tilde}{X}}$ is neither smz nor sm. For the rest of the proof we fix such a name $\dot{\underaccent{\tilde}{X}}$, the corresponding $\dot{\underaccent{\tilde}{\xi}}_i$'s, $i \in \omega_1$, and the appropriate α^* as above. From now on, we work in the $\mathbb{R}\restriction\alpha^*$ extension V^*.

So we have to show that $\mathbb{R}^* * \mathbf{P}_{\omega_2}$ forces that $\dot{\underaccent{\tilde}{X}}$ is neither smz nor sm.

After all our preparations, we can just repeat the proofs of BC (Lemma 5.1) and dBC (Lemma 5.2) of Section 5.A, with the following modifications. The modifications are the same for both proofs; for better readability we concentrate on the proof of dBC.

(1) Change: Instead of an arbitrary ultralaver position $\alpha < \omega_2$, we obviously have to choose $\alpha \geq \alpha^*$.
For the dBC: we choose $\alpha > \alpha^*$ an arbitrary Laver position. The Janus forcing $\mathbf{Q}_{\alpha+1}$ adds a (canonically defined code for a) null set $\dot{Z}_\nabla$.

(2) Change: No change here. (Of course we now have an $\mathbb{R}^* * \mathbf{P}_{\alpha^*}$-name $\dot{\underaccent{\tilde}{X}}$ instead of a ground model set.)
For the dBC: It is enough to show that $\dot{\underaccent{\tilde}{X}} + \dot{Z}_\nabla = 2^\omega$ in the $\mathbb{R}^* * \mathbf{P}_{\omega_2}$-extension of V^*, or equivalently, in every $\mathbb{R}^* * \mathbf{P}_\beta$-extension ($\alpha < \beta < \omega_2$). So assume towards a contradiction that there is a $\beta > \alpha$ and an $\mathbb{R}^* * \mathbf{P}_\beta$-name $\dot{\underaccent{\tilde}{r}}$ of a real such that some $(x, p) \in \mathbb{R}^* * \mathbf{P}_{\omega_2}$ forces that $\dot{\underaccent{\tilde}{r}} \notin \dot{\underaccent{\tilde}{X}} + \dot{Z}_\nabla$.

(3) Change: no change. (But we use Corollary 5.4 instead of Lemma 4.15.)
For dBC: Using Corollary 5.4(2), without loss of generality x forces $p^x =^* p$ and there is a $\mathbb{R}^* * P^x_\beta$-name $\dot{\underaccent{\tilde}{r}}^x$ in M^x such that $\dot{r}^x = \dot{\underaccent{\tilde}{r}}$ is forced.

(4) Change: The iteration obviously has to start with the $\mathbb{R}\restriction\alpha^*$-generic iteration which we call $\bar{\mathbf{P}}^*$ (which is ccc), the rest is the same.
For dBC: In V^* we construct an iteration $\bar{P}$ in the following way:
- (a1) Up to α^*, we use the iteration $\bar{\mathbf{P}}^*$ (which already lives in our current universe V^*). As explained above in the paragraph preceding (5.6), $\dot{\underaccent{\tilde}{X}}$ can be interpreted as a $\mathbf{P}_{\alpha^*}$-name $\dot{X}$, and by (5.6), $\dot{X}$ is forced to be uncountable.
- (a2) We continue the iteration from α^* to α in way that embeds x and such that P_α is proper. So P_α will force that $\dot{X}$ is still uncountable.
- (b1) Let Q_α be any ultralaver forcing (over Q^x_α). Then Q_α forces that $\dot{X}$ is not thin.
- (b2) Let $Q_{\alpha+1}$ be a countable Janus forcing. So $Q_{\alpha+1}$ forces $\dot{X} + \dot{Z}_\nabla = 2^\omega$.
- (c) We continue the iteration in a σ-centered way. I.e., we use an almost FS iteration over x of ultralaver forcings and countable Janus forcings, using trivial Q_ζ for all $\zeta \notin M^x$.
- (d) So P_β still forces that $\dot{X} + \dot{Z}_\nabla = 2^\omega$, and in particular that $\dot{r}^x \in \dot{X} + \dot{Z}_\nabla$.

We now pick (in V^*) a countable $N \prec H(\chi^*)$ containing everything and ord-collapse $(N, \bar{P})$ to $y \leq x$, by (5.5). The HCON object y is of course in V (and even in $\mathbb{R}$), but we can say more: Since the iteration $\bar{P}$ starts with the $(\mathbb{R}\restriction\alpha^*)$-generic iteration $\bar{\mathbf{P}}^*$, the condition y will be in the quotient forcing $\mathbb{R}^*$.
Set $\dot{X}^y := \dot{X} \cap M^y$ (which is the image of $\dot{X}$ under the collapse, since we view $\dot{X}$ as a set of HCON-names). By elementarity, M^y thinks that (a)–(d) above holds for $\bar{P}^y$ and that $\dot{X}^y$ is forced to be uncountable by P^y. Note that $\dot{X}^y \subseteq \dot{X}$ in the following sense: Whenever $G^* * H$ is $\mathbb{R}^* * \mathbf{P}_{\omega_2}$-generic

[52] or: "nice" in the sense of [Kun80, 5.11]

over V^*, and $y \in G^*$, then the evaluation of $\dot{X}^y$ in $M^y[H^y]$ is a subset of the evaluation of $\dot{X}$ in $V^*[G^* * H]$.

(5) Change: No change here.
For dBC: We get our desired contradiction as follows:
Let G^* be $\mathbb{R}^*$-generic over V^* and contain y. Let H_β be $\mathbf{P}^*_\beta$-generic over $V^*[G^*]$ and contain p. So $M^y[H^y_\beta]$ thinks that $\dot{r} = x + z$ for some $x \in \dot{X}^y \subseteq \dot{X}$ and [53] $z \in \dot{Z}_\nabla$, contradicting (2).

6. A word on variants of the definitions

The following is not needed for understanding the paper, we just briefly comment on alternative ways some notions could be defined.

6.A. **Regarding "*alternating iterations*".** We call the set of $\alpha \in \omega_2$ such that Q_α is (forced to be) nontrivial the "*true domain*" of $\bar{P}$ (we use this notation in this remark only). Obviously $\bar{P}$ is naturally isomorphic to an iteration whose length is the order type of its true domain. In Definitions 4.1 and 4.3, we could have imposed the following additional requirements. All these variants lead to equivalent forcing notions.

(1) M^x is (an ord-collapse of) an *elementary* submodel of $H(\chi^*)$.
This is equivalent, as conditions coming from elementary submodels are dense in our $\mathbb{R}$, by Fact 4.4.
While this definition looks much simpler and therefore nicer (we could replace ord-transitive models by the better understood elementary models), it would not make things easier and just "hides" the point of the construction: For example, we use models M^x that are (an ord-collapse of) an elementary submodel of $H^{V'}(\chi^*)$ for some forcing extension V' of V.

(2) Require that (M^x thinks that) the true domain of $\bar{P}^x$ is ω_2.
This is equivalent for the same reason as (1) (and this requirement is compatible with (1)).
This definition would allow to drop the "trivial" option from the definition. The whole proof would still work with minor modifications — in particular, because of the following fact: [54]

(6.1) The finite support iteration of σ-centered forcing notions of length $< (2^{\aleph_0})^+$ is again σ-centered.

We chose our version for two reasons: first, it seems more flexible, and second, we were initially not aware of (6.1).

(3) Alternatively, require that (M^x thinks that) the true domain of $\bar{P}^x$ is countable.
Again, equivalence can be seen as in (1), again (3) is compatible with (1) but obviously not with (2).
This requirement would not make the definition easier, so there is no reason to adopt it. It would have the slight inconvenience that instead of using ord-collapses as in Fact 4.4, we would have to put another model on top to make the iteration countable. Also, it would have the (purely aesthetic) disadvantage that the generic iteration itself does not satisfy this requirement.

(4) Also, we could have dropped the requirement that the iteration is proper. It is never directly used, and "densely" $\bar{P}$ is proper anyway. (E.g., in Lemma 5.1(4)(a), we would just construct $\bar{P}$ up to α to be proper or even ccc, so that X remains uncountable.)

6.B. **Regarding "*almost CS iterations and separative iterands*".** Recall that in Definition 3.6 we required that each iterand Q_α in a partial CS iteration is separative. This implies the property (actually: the three equivalent properties) from Fact 3.8. Let us call this property "*suitability*" for now. Suitability is a property of the limit P_ε of $\bar{P}$. Suitability always holds for finite support iterations and for countable support iterations. However, if we do not assume that each Q_α is separative, then suitability may fail for partial CS iterations. We could drop the separativity assumption, and instead add suitability as an additional natural requirement to the definition of partial CS limit.

The disadvantage of this approach is that we would have to check in all constructions of partial CS iterations that suitability is indeed satisfied (which we found to be straightforward but rather cumbersome, in particular in the case of the almost CS iteration).

[53] Note that we get the same Borel code, whether we evaluate $\dot{Z}_\nabla$ in $M^y[H^y_\beta]$ or in $V^*[G^* * H_\beta]$. Accordingly, the actual Borel set of reals coded by Z_∇ in the smaller universe is a subset of the corresponding Borel set in the larger universe.

[54] We are grateful to Stefan Geschke and Andreas Blass for pointing out this fact. The only reference we are aware of is [Bla11].

In contrast, the disadvantage of assuming that Q_α is separative is minimal and purely cosmetic: It is well known that every quasiorder Q can be made into a separative one which is forcing equivalent to the original Q (e.g., by just redefining the order to be $\leq^*_Q$).

6.C. **Regarding "*preservation of random and quick sequences*".** Recall Definition 1.50 of local preservation of random reals and Lemma 3.32.

In some respect the dense sets D_n are unnecessary. For ultralaver forcing $\mathbb{L}_{\bar{D}}$, the notion of a "quick" sequence refers to the sets D_n of conditions with stem of length at least n.

We could define a new partial order on $\mathbb{L}_{\bar{D}}$ as follows:

$$q \leq' p \;\Leftrightarrow\; (q = p) \text{ or } (q \leq p \text{ and the stem of } q \text{ is strictly longer than the stem of } p).$$

Then $(\mathbb{L}_{\bar{D}}, \leq)$ and $(\mathbb{L}_{\bar{D}}, \leq')$ are forcing equivalent, and any $\leq'$-interpretation of a new real will automatically be quick.

Note however that $(\mathbb{L}_{\bar{D}}, \leq')$ is now not separative any more. Therefore we chose not to take this approach, since losing separativity causes technical inconvenience, as described in 6.B.

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