MONOTONE HULLS FOR $\mathcal{N}\cap\mathcal{M}$

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Dedicated to László Fuchs for his ninetieth birthday

Abstract. Using the method of decisive creatures (see Kellner and Shelah [8]) we show the consistency of "there is no increasing ω_2–chain of Borel sets and $\mathrm{non}(\mathcal{N})=\mathrm{non}(\mathcal{M})=\mathrm{non}(\mathcal{N}\cap\mathcal{M})=\omega_2=2^{\omega}$". Hence, consistently, there are no monotone Borel hulls for the ideal $\mathcal{M}\cap\mathcal{N}$. This answers Balcerzak and Filipczak [1, Questions 23, 24]. Next we use finite support iteration of ccc forcing notions to show that there may be monotone Borel hulls for the ideals $\mathcal{M},\mathcal{N}$ even if they are not generated by towers.

0. Introduction

Brendle and Fuchino [4, Section 3] considered the following spectrum of cardinal numbers

$$\begin{aligned}\mathfrak{DO}=\big\{\mathrm{cf}(\mathrm{otp}(\langle X,R{\restriction}X\rangle)):\ & R\subseteq{}^{\omega}2\times{}^{\omega}2 \text{ is a projective binary relation,}\\ & X\subseteq{}^{\omega}2 \text{ and } R\cap X^2 \text{ is a well ordering of } X\big\}\end{aligned}$$

and they introduced a cardinal invariant $\mathfrak{do}=\sup\mathfrak{DO}$. The invariant $\mathfrak{do}$ satisfies $\min\{\mathrm{non}(\mathcal{I}),\mathrm{cov}(\mathcal{I})\}\leq\mathfrak{do}$ for every ideal $\mathcal{I}$ on $\mathbb{R}$ with Borel basis (see [4, Lemma 3.6]). The proof of Kunen [9, Theorem 12.7] essentially shows that adding any number of Cohen (or random) reals to a model of CH results in a model in which $\mathfrak{do}=\aleph_1$. Thus both

$$\begin{aligned}&\mathrm{non}(\mathcal{N})=\mathrm{cov}(\mathcal{M})=\aleph_2+\mathrm{non}(\mathcal{M})=\mathrm{cov}(\mathcal{N})=\mathfrak{do}=\aleph_1, \text{ and}\\ &\mathrm{non}(\mathcal{M})=\mathrm{cov}(\mathcal{N})=\aleph_2+\mathrm{non}(\mathcal{N})=\mathrm{cov}(\mathcal{M})=\mathfrak{do}=\aleph_1\end{aligned}$$

are consistent (where $\mathcal{M},\mathcal{N}$ stand for the ideals of meager and null sets, respectively). This naturally leads to the question if

$$(\circledast)\ \ \mathrm{non}(\mathcal{M})=\mathrm{non}(\mathcal{N})=\mathrm{non}(\mathcal{N}\cap\mathcal{M})=\aleph_2+\mathfrak{do}=\aleph_1=\mathrm{cov}(\mathcal{N})=\mathrm{cov}(\mathcal{M})$$

is consistent. In this note we show the consistency of $(\circledast)$ using the method of *decisive creatures* developed in Kellner and Shelah [8], and this method is in turn a special case of the method of *norms on possibilities* of Rosłanowski and Shelah [11].

Note that if there is a $\subset$–increasing κ–chain of Borel subsets of ${}^{\omega}2$, then $\mathrm{cf}(\kappa)\in\mathfrak{DO}$. (Just consider a relation R on ${}^{\omega}2\simeq{}^{\omega}2\times{}^{\omega}2$ given by: $(x,y)\ R\ (x',y')$ if and only if " y,y' are Borel codes and x belongs to the set coded by y' "; cf. Elekes and Kunen [6, Lemma 2.4].) Thus if we set

$$\mathfrak{d}_{\mathcal{B}}=\sup\big\{\mathrm{cf}(\gamma):\ \text{there is a }\subset\text{–increasing chain of Borel subset of }\mathbb{R}\text{ of length }\gamma\ \big\}$$

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then $\mathfrak{d}_{\mathcal{B}} \leq \mathfrak{d}\mathfrak{o}$. If $\mathfrak{d}_{\mathcal{B}}$ is smaller than the cofinality of the uniformity number $\mathrm{non}(\mathcal{I})$ of a Borel ideal $\mathcal{I}$, then there is no monotone Borel hull operation on $\mathcal{I}$ (see Elekes and Máthé [7, Theorem 2.1], Balcerzak and Filipczak [1, Theorem 5]). Thus

($\otimes$) if $\mathcal{I}$ is an ideal with Borel basis on $\mathbb{R}$, $\mathfrak{d}_{\mathcal{B}} < \mathrm{non}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I})$ is a regular cardinal, then there is no $\subset$–monotone mapping $\psi : \mathcal{I} \longrightarrow \mathrm{Borel}(\mathbb{R}) \cap \mathcal{I}$.

Therefore in our model for ($\circledast$) we will have (Corollary 3.2)

"there are no monotone Borel hull operations on the ideals $\mathcal{M}, \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N}$". This answers Balcerzak and Filipczak [1, Question 23].

We also obtain a positive result providing a new situation in which monotone hulls exist. Consistently, the ideals $\mathcal{M}, \mathcal{N}$ do not possess tower–basis but they do admit monotone Borel hulls (Corollary 3.9). This model is obtained by finite support iterations of partial Amoeba for Category and Amoeba for Measure $\mathbb{A}$ forcing notions.

Notation Most of our notation is standard and compatible with that of classical textbooks (like Bartoszyński and Judah [2]). However, in forcing we keep the older convention that *a stronger condition is the larger one.*

• For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\ell g(\eta)$. A *tree* is a family T of finite sequences closed under initial segments. For a tree T, the family of all ω–branches through T is denoted by $[T]$.

• The Cantor space ${}^{\omega}2$ is the space of all functions from ω to 2, equipped with the product topology generated by sets of the form $[\nu] = \{\eta \in {}^{\omega}2 : \nu \vartriangleleft \eta\}$ for $\nu \in {}^{\omega>}2$. This space is also equipped with the standard product measure μ.

• For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g. $\underset{\sim}{A}$, $\underset{\sim}{\eta}$). The canonical name for a $\mathbb{P}$–generic filter over $\mathbf{V}$ is denoted $\underset{\sim}{G}_{\mathbb{P}}$. Our notation and terminology concerning creatures and forcing with creatures will be compatible with that in [8] (except of the reversed orders). While this is a slight departure from the original terminology established for creature forcing in [11], the reader may find it more convenient when verifying the results on decisive creatures that are quoted in the next section.

1. Background on decisive creatures

As declared in the introduction, we will follow the notation and the context of [8] (which slightly differs from that of [11]). For reader's convenience we will recall here all relevant definitions and results from that paper.

Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\aleph_0)$ (where $\mathcal{H}(\aleph_0)$ is the family of all hereditarily finite sets). A *creating pair* for $\mathbf{H}$ is a pair $(\mathbf{K}, \mathbf{\Sigma})$, where

- $\mathbf{K} = \bigcup\limits_{n<\omega} \mathbf{K}(n)$, where each $\mathbf{K}(n)$ is a finite set; elements of $\mathbf{K}$ are called *creatures*, each creature $\mathfrak{c} \in \mathbf{K}(n)$ has some norm $\mathrm{nor}(\mathfrak{c})$ (a non-negative real number) and a non-empty set of possible values $\mathrm{val}(\mathfrak{c}) \subseteq \mathbf{H}(n)$,
- if $\mathfrak{c} \in \mathbf{K}(n)$, $\mathrm{nor}(\mathfrak{c}) > 0$, then $|\mathrm{val}(\mathfrak{c})| > 1$
- $\mathbf{\Sigma} : \mathbf{K} \longrightarrow \mathcal{P}(\mathbf{K})$ is such that if $\mathfrak{c} \in \mathbf{K}(n)$ and $\mathfrak{c}' \in \mathbf{\Sigma}(\mathfrak{c})$, then $\mathfrak{c}' \in \mathbf{K}(n)$,
- $\mathfrak{c} \in \mathbf{\Sigma}(\mathfrak{c})$ and $\mathfrak{c}' \in \mathbf{\Sigma}(\mathfrak{c})$ implies $\mathbf{\Sigma}(\mathfrak{c}') \subseteq \mathbf{\Sigma}(\mathfrak{c})$,
- if $\mathfrak{c}' \in \mathbf{\Sigma}(\mathfrak{c})$, then $\mathrm{nor}(\mathfrak{c}') \leq \mathrm{nor}(\mathfrak{c})$ and $\mathrm{val}(\mathfrak{c}') \subseteq \mathrm{val}(\mathfrak{c})$.

If $\mathfrak{c} \in \mathbf{K}$ and $x \in \mathbf{H}(n)$, then we write $x \in \mathbf{\Sigma}(\mathfrak{c})$ if and only if $x \in \mathrm{val}(\mathfrak{c})$. For $x \in \mathbf{H}(n)$ we also set $\mathbf{\Sigma}(x) = \mathrm{val}(x) = \{x\}$.

Definition 1.1 (See [8, Definitions 3.1, 4.1])**.** Let $0 < r \leq 1$, B, K, m be positive integers and $(\mathbf{K},\mathbf{\Sigma})$ be a creating pair for $\mathbf{H}$.

(1) A creature $\mathfrak{c}$ is r–halving if there is a $\mathrm{half}(\mathfrak{c}) \in \mathbf{\Sigma}(\mathfrak{c})$ such that
- $\mathrm{nor}(\mathrm{half}(\mathfrak{c})) \geq \mathrm{nor}(\mathfrak{c}) - r$, and
- if $\mathfrak{d} \in \mathbf{\Sigma}(\mathrm{half}(\mathfrak{c}))$ and $\mathrm{nor}(\mathfrak{d}) > 0$, then there is a $\mathfrak{d}' \in \mathbf{\Sigma}(\mathfrak{c})$ such that

$$\mathrm{nor}(\mathfrak{d}') \geq \mathrm{nor}(\mathfrak{c}) - r \quad \text{and} \quad \mathrm{val}(\mathfrak{d}') \subseteq \mathrm{val}(\mathfrak{d}).$$

$\mathbf{K}(n)$ is r–halving, if all $\mathfrak{c} \in \mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1$ are r–halving.

(2) A creature $\mathfrak{c}$ is (B, r)–big if for every function $F : \mathrm{val}(\mathfrak{c}) \longrightarrow B$ there is a $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - r$ and the restriction $F{\restriction}\mathrm{val}(\mathfrak{d})$ is constant. We say that $\mathfrak{c}$ is hereditarily (B, r)-big, if every $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ with $\mathrm{nor}(\mathfrak{d}) > 1$ is (B, r)-big. Also, $\mathbf{K}(n)$ is (B, r)–big if every $\mathfrak{c} \in \mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1$ is (B, r)–big.

(3) We say that $\mathfrak{c}$ is (K, m, r)–decisive, if for some $\mathfrak{d}^-, \mathfrak{d}^+ \in \mathbf{\Sigma}(\mathfrak{c})$ we have: $\mathfrak{d}^+$ is hereditarily $(2^{K^m}, r)$–big, and $|\mathrm{val}(\mathfrak{d}^-)| \leq K$ and $\mathrm{nor}(\mathfrak{d}^-), \mathrm{nor}(\mathfrak{d}^+) \geq \mathrm{nor}(\mathfrak{c}) - r$. The creature $\mathfrak{c}$ is (m, r)–decisive if $\mathfrak{c}$ is (K', m, r)–decisive for some K'.

(4) $\mathbf{K}(n)$ is (m, r)–decisive if every $\mathfrak{c} \in \mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1$ is (m, r)–decisive.

Lemma 1.2 (See [8, Lemma 4.3])**.** *Assume that $(\mathbf{K},\mathbf{\Sigma})$ is a creating pair for $\mathbf{H}$, $k, m, t \geq 1$, $0 < r \leq 1$. Suppose that $\mathbf{K}(n)$ is (k, r)–decisive and $\mathfrak{c}_0, \ldots, \mathfrak{c}_{k-1} \in \mathbf{K}(n)$ are hereditarily $(2^{m^t}, r)$–big with $\mathrm{nor}(\mathfrak{c}_i) > 1 + r \cdot (k-1)$ (for each $i < k$). Let $F : \prod\limits_{i<k} \mathrm{val}(\mathfrak{c}_i) \longrightarrow 2^{m^t}$. Then there are $\mathfrak{d}_0, \ldots, \mathfrak{d}_{k-1} \in \mathbf{K}(n)$ such that:*

$$\mathfrak{d}_i \in \mathbf{\Sigma}(\mathfrak{c}_i), \quad \mathrm{nor}(\mathfrak{d}_i) \geq \mathrm{nor}(\mathfrak{c}_i) - r \cdot k, \quad \textit{and } F{\restriction}\prod_{i\in k} \mathrm{val}(\mathfrak{d}_i) \textit{ is constant.}$$

A creating pair $(\mathbf{K},\mathbf{\Sigma})$ determines the forcing notion $\mathbb{Q}^*_\infty(\mathbf{K},\mathbf{\Sigma})$ and its special product $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ as described by the following definition. (The forcing notion $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ is a relative of the CS product of $\mathbb{Q}^*_\infty(\mathbf{K},\mathbf{\Sigma})$ indexed by the set I.)

Definition 1.3 (See [8, Definitions 2.1, 5.2, 5.3])**.** (1) A condition in the forcing $\mathbb{Q}^*_\infty(\mathbf{K},\mathbf{\Sigma})$ is an ω–sequence $p = \langle p(i) : i < \omega\rangle$ such that for some $n < \omega$ (called the trunk-length of p) we have $p(i) \in \mathbf{H}(i)$ if $i < n$, $p(i) \in \mathbf{K}(i)$ and $\mathrm{nor}(p(i)) > 0$ if $i \geq n$, and $\lim\limits_{i\to\infty}(\mathrm{nor}(p(i))) = \infty$.

The order on $\mathbb{Q}^*_\infty(\mathbf{K},\mathbf{\Sigma})$ is defined by $q \geq p$ if and only if (both belong to $\mathbb{Q}^*_\infty(\mathbf{K},\mathbf{\Sigma})$ and) $q(i) \in \mathbf{\Sigma}(p(i))$ for all i.[1]

(2) Let I be a non-empty (index) set. A condition p in $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ consists of a countable subset $\mathrm{dom}(p)$ of I, of objects $p(\alpha, n)$ for $\alpha \in \mathrm{dom}(p)$, $n \in \omega$, and of a function $\mathrm{trunklg}(p, \cdot) : \mathrm{dom}(p) \longrightarrow \omega$ satisfying the following demands for all $\alpha \in \mathrm{dom}(p)$:

(α) If $n < \mathrm{trunklg}(p, \alpha)$, then $p(\alpha, n) \in \mathbf{H}(n)$.

(β) If $n \geq \mathrm{trunklg}(p, \alpha)$, then $p(\alpha, n) \in \mathbf{K}(n)$ and $\mathrm{nor}(p(\alpha, n)) > 0$.

(γ) Setting $\mathrm{supp}(p, n) = \{\alpha \in \mathrm{dom}(p) : \mathrm{trunklg}(p, \alpha) \leq n\}$, we have $|\mathrm{supp}(p, n)| < n$ for all $n > 0$ and $\lim\limits_{n\to\infty}(|\mathrm{supp}(p, n)|/n) = 0$.

(δ) $\lim\limits_{n\to\infty}(\min(\{\mathrm{nor}(p(\alpha, n)) : \alpha \in \mathrm{supp}(p, n)\})) = \infty$.

The order on $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ is defined by $q \geq p$ if and only if (both belong to $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ and) $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and

[1] Remember our convention that for $x, y \in \mathbf{H}(i)$ and $\mathfrak{c} \in \mathbf{K}(i)$ we write $x \in \mathbf{\Sigma}(\mathfrak{c})$ iff $x \in \mathrm{val}(\mathfrak{c})$, and $x \in \mathbf{\Sigma}(y)$ iff $x = y$.

(ε) if $\alpha \in \mathrm{dom}(p)$ and $n \in \omega$, then $q(\alpha, n) \in \mathbf{\Sigma}(p(\alpha, n))$,
(ζ) the set $\{\alpha \in \mathrm{dom}(p) : \mathrm{trunklg}(q, \alpha) \neq \mathrm{trunklg}(p, \alpha)\}$ is finite.

Note that for $\alpha \in \mathrm{dom}(p)$ the sequence $\langle p(\alpha, n) : n \in \omega\rangle$ is in $\mathbb{Q}^*_\infty(\mathbf{K}, \mathbf{\Sigma})$. However, $\mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$ is not a subforcing of the CS product of I copies of $\mathbb{Q}^*_\infty(\mathbf{K}, \mathbf{\Sigma})$ because of a slight difference in the definition of the order relation.

Proposition 1.4 (See [8, Lemmas 5.4, 5.5])**.** (1) *If* $J \subseteq I$*, then* $\mathbb{P}_J(K, \Sigma) = \{p \in \mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma}) : \mathrm{dom}(p) \subseteq J\}$ *is a complete subforcing of* $\mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$*.*

(2) *Assume* CH*. Then* $\mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$ *satisfies the* $\aleph_2$*–chain condition.*

Definition 1.5 (See [8, Definition 5.6])**.** (1) For a condition $p \in \mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$ we define[2]

$$\mathrm{val}^{\Pi}(p, {<}n) = \prod_{\alpha \in \mathrm{dom}(p)} \prod_{m<n} \mathrm{val}(p(\alpha, m)).$$

(2) If $w \subseteq \mathrm{dom}(p)$ and $t \in \prod\limits_{\alpha \in w} \prod\limits_{m<n} \mathbf{H}(m)$, then $p \wedge t$ is defined by

$$\mathrm{trunklg}(p \wedge t, \alpha) = \begin{cases} \max(\mathrm{trunklg}(p, \alpha), n) & \text{if } \alpha \in w, \\ \mathrm{trunklg}(p, \alpha) & \text{otherwise} \end{cases}$$

and

$$(p \wedge t)(\alpha, m) = \begin{cases} t(\alpha, m) & \text{if } m < n \text{ and } \alpha \in w, \\ p(\alpha, m) & \text{otherwise.} \end{cases}$$

(3) If $\underset{\sim}{\tau}$ is a name of an ordinal, then we say that p ${<}n$–decides $\underset{\sim}{\tau}$, if for every $t \in \mathrm{val}^{\Pi}(p, {<}n)$ the condition $p \wedge t$ forces a value to $\underset{\sim}{\tau}$. The condition p essentially decides $\underset{\sim}{\tau}$, if p ${<}n$-decides $\underset{\sim}{\tau}$ for some n.

Proposition 1.6. (1) $p \wedge t \in \mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$*, and if* $t \in \mathrm{val}^{\Pi}(p, {<}n)$*, then* $p \wedge t \geq p$*.*

(2) $\mathrm{val}^{\Pi}(p, {<}n) \leq \prod\limits_{m<n} |\mathbf{H}(m)|^m$*.*

(3) $\{p \wedge t : t \in \mathrm{val}^{\Pi}(p, {<}n)\}$ *is predense above* p

Theorem 1.7 (See [8, Theorems 5.8, 5.9])**.** *Let* $\varphi({<}n) = \prod\limits_{m<n} |\mathbf{H}(m)|^m$ *and* $0 < r(n) \leq 1/(n^2\varphi({<}n))$*. Assume that each* $\mathbf{K}(n)$ *is* $(n, r(n))$*-decisive and* $r(n)$*–halving (for* $n \in \omega$*).*

(1) *The forcing notion* $\mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$ *is proper and* ${}^\omega\omega$*-bounding. If* $|I| \geq 2$ *and* $\lambda = |I|^{\aleph_0}$*, then* $\mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$ *forces* $|I| \leq 2^{\aleph_0} \leq \lambda$*.*

(2) *Moreover, if* $\underset{\sim}{\tau}(n)$ *is a* $\mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$*–name for an ordinal (for* $n < \omega$*) and* $p \in \mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$*, then there is a condition* $q \geq p$ *which essentially decides all the names* $\underset{\sim}{\tau}(n)$*.*

(3) *Assume, additionally, that each* $\mathbf{K}(n)$ *is* $(g(n), r(n))$*– big, where* $g \in {}^\omega\omega$ *is strictly increasing. Suppose that* $\underset{\sim}{\nu}(n)$ *is a* $\mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$*–name and* $p \in \mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$ *forces that* $\underset{\sim}{\nu}(n) < 2^{g(n)}$ *for all* $n < \omega$*. Then there is a* $q \geq p$ *which* ${<}n$*–decides* $\underset{\sim}{\nu}(n)$ *for all* n*.*

The next theorem is a consequence of (the proof of) [4, Corollaries 4.8(e), 3.9(b)]. However, the results in [4] are stated for products, while $\mathbb{P}_I(\mathbf{K}, \mathbf{\Sigma})$ is not exactly a product (though it does have all the required features). Therefore we will present the relatively simple proof of this result fully.

[2] Remember our convention that, for $x \in \mathbf{H}(i)$, $\mathrm{val}(x) = \{x\}$.

Theorem 1.8. *Assume* CH. *Let* $r, \varphi, \mathbf{K}$ *and* $\mathbf{\Sigma}$ *be as in the assumptions of Theorem 1.7. Then* $\Vdash_{\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})} \mathfrak{do} = \mathfrak{d}_{\mathcal{B}} = \aleph_1$.

Proof. If $|I| \leq \aleph_1$, then $\Vdash_{\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})}$ CH, so let us assume $|I| \geq \aleph_2$.

Every bijection $\pi : I \xrightarrow{\text{onto}} I$ determines an automorphism $\tilde{\pi}$ of the forcing $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ in a natural way. Then, for $J \subseteq I$, $\tilde{\pi}{\restriction}\mathbb{P}_J(\mathbf{K},\mathbf{\Sigma})$ is an isomorphism from $\mathbb{P}_J(\mathbf{K},\mathbf{\Sigma})$ onto $\mathbb{P}_{\pi[J]}(\mathbf{K},\mathbf{\Sigma})$. Also, π gives rise to a natural bijection from $\mathrm{val}^{\Pi}(p,{<}n)$ onto $\mathrm{val}^{\Pi}(\tilde{\pi}(p),{<}n)$; we will denote this mapping by $\tilde{\pi}$ as well.

Suppose that $\varphi(x,y,\underset{\sim}{\tau})$ is a projective definition of a binary relation on ${}^{\omega}2$, where $\underset{\sim}{\tau}$ is a $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$–name for a real parameter. Assume towards contradiction that there are $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$–names $\underset{\sim}{\eta}_\alpha$ (for $\alpha < \omega_2$) and a condition $p \in \mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ such that

(i) $p \Vdash_{\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})}$ " $\big(\forall \alpha,\beta<\omega_2\big)\big(\varphi(\underset{\sim}{\eta}_\alpha,\underset{\sim}{\eta}_\beta,\underset{\sim}{\tau}) \ \Leftrightarrow\ \alpha<\beta\big)$ ".

For each $\alpha<\omega_2$ choose a condition $p_\alpha \geq p$ which essentially decides all $\underset{\sim}{\eta}_\alpha(n)$ (for $n<\omega$). Then we may also pick an increasing sequence $\bar{N}^\alpha = \langle N^\alpha_n : n<\omega\rangle \subseteq \omega$ and a mapping $f_\alpha : \bigcup\limits_{n<\omega} \mathrm{val}^{\Pi}(p_\alpha,{<}N^\alpha_n) \longrightarrow 2$ such that for each $t \in \mathrm{val}^{\Pi}(p_\alpha,{<}N^\alpha_n)$ we have $(p_\alpha \wedge t) \Vdash \underset{\sim}{\eta}_\alpha(n) = f_\alpha(t)$.

By CH, we may use a standard Δ–system argument and the fact that $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ satisfies the $\aleph_2$–cc (see 1.4) to choose $J \in [I]^{\aleph_1}$, $X \in [\omega_2]^{\aleph_2}$ and bijections $\pi_{\alpha,\beta} : \mathrm{dom}(p_\alpha) \xrightarrow{\text{onto}} \mathrm{dom}(p_\beta)$ such that

(ii) $\mathrm{dom}(p) \subseteq J$ and $\underset{\sim}{\tau}$ is a $\mathbb{P}_J(\mathbf{K},\mathbf{\Sigma})$–name, and for distinct $\alpha,\beta \in X$:
(iii) $\mathrm{dom}(p_\alpha)\cap\mathrm{dom}(q_\beta) = \mathrm{dom}(p_\alpha)\cap J$ and $\pi_{\alpha,\beta}{\restriction}(\mathrm{dom}(p_\alpha)\cap J)$ is the identity,
(iv) $\tilde{\pi}_{\alpha,\beta}(p_\alpha) = p_\beta$, $\bar{N}^\alpha = \bar{N}^\beta$, and $f_\alpha = f_\beta \circ \tilde{\pi}_{\alpha,\beta}$.

Pick $\alpha<\beta$ from X. Let π be a bijection from I onto I such that $\pi_{\alpha,\beta} \subseteq \pi$, $(\pi_{\alpha,\beta})^{-1} \subseteq \pi$ and $\pi{\restriction}J$ is the identity. Then

(v) $\tilde{\pi}(p_\alpha) = p_\beta$, $\tilde{\pi}(p_\beta) = p_\alpha$ and $\tilde{\pi}(\underset{\sim}{\tau}) = \underset{\sim}{\tau}$.

Note that $p_\alpha \cup p_\beta$ does not have to be a condition in $\mathbb{P}_I(\mathbf{K},\mathbf{\Sigma})$ as the demand 1.3(2)(γ) may fail. But extending finitely many trunks will easily resolve this problem and we get a condition q stronger than both p_α and p_β. We may even do this in such a manner that the condition q satisfies $\tilde{\pi}(q) = q$. Since $q \geq p_\alpha, p_\beta$, clause (iv) implies

(vi) $q \Vdash$ " $\tilde{\pi}(\underset{\sim}{\eta}_\alpha) = \underset{\sim}{\eta}_\beta$ & $\tilde{\pi}(\underset{\sim}{\eta}_\beta) = \underset{\sim}{\eta}_\alpha$ ".

Since $q \geq p$ and $\alpha<\beta$ we have $q \Vdash \varphi(\underset{\sim}{\eta}_\alpha,\underset{\sim}{\eta}_\beta,\underset{\sim}{\tau})$. Applying the automorphism $\tilde{\pi}$ and remembering (vi) we conclude that then also $\tilde{\pi}(q) = q \Vdash \varphi(\underset{\sim}{\eta}_\beta,\underset{\sim}{\eta}_\alpha,\underset{\sim}{\tau})$, contradicting (i). $\square$

2. Consistency of $\mathfrak{do} < \mathrm{non}(\mathcal{M}\cap\mathcal{N})$

Definition 2.1. Let $n<\omega$.

(1) A *basic n–block* is a finite non-empty set B of functions from some non-empty $v \in [\omega]^{<\omega}$ to 2 (i.e., $B \subseteq {}^{v}2$) such that $|B|/2^{|v|} < 2^{-n}$. If $\eta \in {}^{\omega>}2 \cup {}^{\omega}2$ and $B \subseteq {}^{v}2$ is a basic block, then we write $\eta \prec B$ whenever $\eta{\restriction}v \in B$. For an n–block $B \subseteq {}^{v}2$ we set $v(B) = v$.
(2) Let H_n be the family of all pairs $(b,\mathcal{B})$ such that b is a positive integer and $\mathcal{B}$ is a non-empty finite set of basic n–blocks.

(3) We define a function pnor : $H_n \longrightarrow \omega$ by declaring inductively when $\mathrm{pnor}(b,\mathcal{B}) \geq k$. We set $\mathrm{pnor}(b,\mathcal{B}) \geq 0$ always, and then
- $\mathrm{pnor}(b,\mathcal{B}) \geq 1$ if and only if $(\forall F \in [{}^{\omega}2]^{b})(\exists B \in \mathcal{B})(\forall \eta \in F)(\eta \prec B)$,
- $\mathrm{pnor}(b,\mathcal{B}) \geq k+1$ if and only if there are positive integers $b_0,\ldots,b_{M-1}$ and disjoint sets $\mathcal{B}_0,\ldots,\mathcal{B}_{M-1} \subseteq \mathcal{B}$ such that
  (α) $M > b^{k+1}$, $b_0 \geq b$ and
  (β) $\mathrm{pnor}(b_i,\mathcal{B}_i) \geq k$ and $(b_i)^2 \cdot 2^{|\mathcal{B}_i|^n} < b_{i+1}$ for all $i < M$.

Proposition 2.2. *Let $n < \omega$, $(b,\mathcal{B}),(b',\mathcal{B}') \in H_n$.*

(1) $\mathrm{pnor}(b,\mathcal{B}) \in \omega$ *is well defined and* $2^{\mathrm{pnor}(b,\mathcal{B})} \leq |\mathcal{B}|$.
(2) *If* $\mathcal{B} \subseteq \mathcal{B}'$ *and* $b' \leq b$, *then* $\mathrm{pnor}(b,\mathcal{B}) \leq \mathrm{pnor}(b',\mathcal{B}')$.
(3) *For each N there is $(b^*,\mathcal{B}^*) \in H_n$ such that*

$$b^* \geq N \text{ and } \mathrm{pnor}(b^*,\mathcal{B}^*) \geq N \text{ and } \min(v(B)) > N \text{ for all } B \in \mathcal{B}^*.$$

(4) *If* $\mathrm{pnor}(b,\mathcal{B}) \geq k+1 \geq 2$ *and* $c : \mathcal{B} \longrightarrow \{0,\ldots,b-1\}$, *then for some* $\ell < b$ *we have* $\mathrm{pnor}(b, c^{-1}[\{\ell\}]) \geq k$.

Proof. (1,2) Easy induction on $\mathrm{pnor}(b,\mathcal{B})$.

(3) Note that if $w \in [\omega]^{<\omega}$, $2^n \cdot N < 2^{|w|}$ and $\mathcal{B}_w$ consists of all basic n–blocks B with $v(B) = w$, then $\mathrm{pnor}(N,\mathcal{B}_w) \geq 1$. Now proceed inductively.

(4) Induction on $k \geq 1$. Assume $\mathrm{pnor}(b,\mathcal{B}) \geq 2$ and $c : \mathcal{B} \longrightarrow b$. We claim that for some $\ell < b$ we have $\mathrm{pnor}(b, c^{-1}[\{\ell\}]) \geq 1$. If not, then for each $\ell < b$ we may choose $F_\ell \in [{}^{\omega}2]^{b}$ such that

$$\big(\forall B \in \mathcal{B}\big)\big(\exists \eta \in F_\ell\big)\big(c(B) = \ell \ \Rightarrow\ \eta \not\prec B\big).$$

Set $F = \bigcup\limits_{\ell<b} F_\ell$. Let $b_0,\ldots,b_{M-1},\mathcal{B}_0,\ldots,\mathcal{B}_{M-1}$ witness $\mathrm{pnor}(b,\mathcal{B}) \geq 2$, in particular, $b_1 > b^2$ and $\mathrm{pnor}(b_1,\mathcal{B}_1) \geq 1$. Since $|F| \leq b^2$ we conclude that there is $B \in \mathcal{B}_1$ such that $(\forall \eta \in F)(\eta \prec B)$. Then B contradicts the choice of $F_{c(B)}$.
Now, for the inductive step, assume our statement holds for k. Let $\mathrm{pnor}(b,\mathcal{B}) \geq k+2$ and $c : \mathcal{B} \longrightarrow \{0,\ldots,b-1\}$. Let $\{(b_i,\mathcal{B}_i) : i < M\}$ witness $\mathrm{pnor}(b,\mathcal{B}) \geq (k+1)+1$, so $M > b^{k+2}$ and $\mathrm{pnor}(b_i,\mathcal{B}_i) \geq k+1$ and $b_i \geq b$. For each $i < M$ apply the inductive hypothesis to choose $\ell_i < b$ such that $\mathrm{pnor}(b_i, \mathcal{B}_i \cap c^{-1}[\{\ell_i\}]) \geq k$. Choose $\ell^* < b$ such that $|\{i < M : \ell^* = \ell_i\}| > b^{k+1}$. Then $\{(b_i, \mathcal{B}_i \cap c^{-1}[\{\ell_i\}]) : \ell_i = \ell^*\}$ witnesses that $\mathrm{pnor}(b, c^{-1}[\{\ell^*\}]) \geq k+1$. □

Now, by induction on $n < \omega$ we define the following objects

$(\oplus)_n$ $\varphi_{\mathbf{H}^*}({<}n), r_{\mathbf{H}^*}(n), a(n), N_n, g(n), \mathbf{H}^*(n), \mathbf{K}^*(n), \mathbf{\Sigma}^*{\restriction}\mathbf{K}^*(n), \varphi_{\mathbf{H}^*}({=}\,n)$.

We start with stipulating $N_0 = 0$, $\varphi_{\mathbf{H}^*}(<0) = 1$.
Assume we have defined objects listed in $(\oplus)_k$ for $k < n$, and that we also have defined integers $N_n, \varphi_{\mathbf{H}^*}({<}n)$. We set

(i) $g(n) = 2^{N_n} + \varphi_{\mathbf{H}^*}({<}n)$, $r_{\mathbf{H}^*}(n) = \frac{1}{(n+2)^2 \varphi_{\mathbf{H}^*}(<n)}$ and $a(n) = 2^{1/r_{\mathbf{H}^*}(n)}$.

Choose $(b^*,\mathcal{B}^*) \in H_n$ such that

(ii) $b^* > g(n)$, $\min(v(B)) > N_n$ for all $B \in \mathcal{B}^*$ and $\mathrm{pnor}(b^*,\mathcal{B}^*) > a(n)^{n+972}$

(possible by 2.2(3)). Set

(iii) $N_{n+1} = \max\big(\bigcup\{v(B) : B \in \mathcal{B}^*\}\big) + 1$.

We let $\mathbf{H}^*(n)$ be the set of all basic n–blocks B such that $v(B)\subseteq[N_n,N_{n+1})$, and $\mathbf{K}^*(n)$ consist of all triples $\mathfrak{c}=(k^{\mathfrak{c}},b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})$ such that

$$(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})\in H_n,\quad \mathcal{B}^{\mathfrak{c}}\subseteq\mathbf{H}^*(n),\quad b^{\mathfrak{c}}>g(n),\quad\text{and}\quad k^{\mathfrak{c}}\in\omega,\quad k^{\mathfrak{c}}<\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-1.$$

For $\mathfrak{c}\in\mathbf{K}^*(n)$ we set

(iv) $\mathrm{nor}(\mathfrak{c})=\log_{a(n)}\big(\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}}\big)$, $\mathrm{val}(\mathfrak{c})=\mathcal{B}^{\mathfrak{c}}$ and $\mathbf{\Sigma}^*(\mathfrak{c})=\{\mathfrak{d}\in\mathbf{K}^*(n): k^{\mathfrak{c}}\le k^{\mathfrak{d}},\ b^{\mathfrak{c}}\le b^{\mathfrak{d}},\ \mathcal{B}^{\mathfrak{d}}\subseteq\mathcal{B}^{\mathfrak{c}}\}$.

Finally, we put $\varphi_{\mathbf{H}^*}(=n)=|\mathbf{H}^*(n)|^n$ and $\varphi_{\mathbf{H}^*}(<n+1)=\varphi_{\mathbf{H}^*}(<n)\cdot\varphi_{\mathbf{H}^*}(=n)$. This completes our inductive definition.

Proposition 2.3. $(\mathbf{K}^*,\mathbf{\Sigma}^*)$ *is a creating pair for* $\mathbf{H}^*$ *such that, for each* $n<\omega$, $\mathbf{K}^*(n)$ *is* $(n,r_{\mathbf{H}^*}(n))$*–decisive,* $r_{\mathbf{H}^*}(n)$*–halving and* $(g(n),r_{\mathbf{H}^*}(n))$*–big.*

Proof. To verify halving, for each $\mathfrak{c}\in\mathbf{K}^*(n)$ with $\mathrm{nor}(\mathfrak{c})>1$ set

$$\mathrm{half}(\mathfrak{c})=(k^{\mathfrak{c}}+\lfloor\frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}})\rfloor,b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}}).$$

Note that $\mathrm{nor}(\mathfrak{c})>1$ implies $\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}}>2$ and hence

$$k^{\mathfrak{c}}+\lfloor\frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}})\rfloor<\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-1.$$

Therefore, $\mathrm{half}(\mathfrak{c})\in\mathbf{\Sigma}^*(\mathfrak{c})$ and $\mathrm{nor}(\mathrm{half}(\mathfrak{c}))\ge\mathrm{nor}(\mathfrak{c})-r_{\mathbf{H}^*}(n)$. Now suppose $\mathfrak{d}\in\mathbf{\Sigma}^*(\mathrm{half}(\mathfrak{c}))$, so $k^{\mathfrak{c}}+\lfloor\frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}})\rfloor\le k^{\mathfrak{d}}$, $b^{\mathfrak{c}}\le b^{\mathfrak{d}}$ and $\mathcal{B}^{\mathfrak{d}}\subseteq\mathcal{B}^{\mathfrak{c}}$. Also, $k^{\mathfrak{d}}<\mathrm{pnor}(b^{\mathfrak{d}},\mathcal{B}^{\mathfrak{d}})-1$, so $\mathrm{pnor}(b^{\mathfrak{d}},\mathcal{B}^{\mathfrak{d}})>k^{\mathfrak{c}}+\lfloor\frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}})\rfloor+1$. Consider $\mathfrak{d}'=(k^{\mathfrak{c}},b^{\mathfrak{d}},\mathcal{B}^{\mathfrak{d}})$. Plainly $\mathfrak{d}'\in\mathbf{\Sigma}^*(\mathfrak{c})$, $\mathrm{val}(\mathfrak{d}')\subseteq\mathrm{val}(\mathfrak{d})$ and

$$\mathrm{nor}(\mathfrak{d}')\ge\log_{a(n)}\big(\lfloor\tfrac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}})\rfloor+1\big)\ge\log_{a(n)}\big(\tfrac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}})\big)=\mathrm{nor}(\mathfrak{c})-r_{\mathbf{H}^*}(n).$$

It follows from 2.2(4) that

$(*)$ if $\mathfrak{c}\in\mathbf{K}^*(n)$, $\mathrm{nor}(\mathfrak{c})>r_{\mathbf{H}^*}(n)$, then $\mathfrak{c}$ is $(b^{\mathfrak{c}},r_{\mathbf{H}^*}(n))$–big.

Hence $\mathbf{K}^*(n)$ is $(g(n),r_{\mathbf{H}^*}(n))$–big (remember the definition of $\mathbf{K}^*(n)$).

Now suppose $\mathfrak{c}\in\mathbf{K}^*(n)$, $\mathrm{nor}(\mathfrak{c})>1$. Then $\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-k^{\mathfrak{c}}>2$, so by the definition of pnor (see 2.1(3)) we may find $b^{\mathfrak{c}}\le b_0<b_1<\ldots<b_{M-1}$ and disjoint $\mathcal{B}_0,\ldots,\mathcal{B}_{M-1}\subseteq\mathcal{B}^{\mathfrak{c}}$ such that $\mathrm{pnor}(b_i,\mathcal{B}_i)\ge\mathrm{pnor}(b^{\mathfrak{c}},\mathcal{B}^{\mathfrak{c}})-1$ and $(b_i)^2\cdot 2^{|\mathcal{B}_i|^n}<b_{i+1}$. Set

$$\mathfrak{d}^-=(k^{\mathfrak{c}},b_0,\mathcal{B}_0),\quad \mathfrak{d}^+=(k^{\mathfrak{c}},b_1,\mathcal{B}_1),\quad\text{and } K=|\mathcal{B}_0|.$$

Plainly, $\mathfrak{d}^-,\mathfrak{d}^+\in\mathbf{\Sigma}(\mathfrak{c})$, $\min\{\mathrm{nor}(\mathfrak{d}^-),\mathrm{nor}(\mathfrak{d}^+)\}\ge\mathrm{nor}(\mathfrak{c})-r_{\mathbf{H}^*}(n)>r_{\mathbf{H}^*}(n)$ and $|\mathrm{val}(\mathfrak{d}^-)|=K$. Also $\mathfrak{d}^+$ is hereditarily $(2^{K^n},r_{\mathbf{H}^*}(n))$–big (remember $b_1>2^{K^n}$, use $(*)$). Thus $\mathfrak{d}^-,\mathfrak{d}^+$ witness that $\mathfrak{c}$ is $(K,n,r_{\mathbf{H}^*}(n))$–decisive. □

Definition 2.4. (1) For a cardinal λ we consider the forcing notion $\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)$ determined by the creating pair $(\mathbf{K}^*,\mathbf{\Sigma}^*)$ as in 1.3(2). For $\alpha<\lambda$, a $\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)$–name $\underset{\sim}{\rho}_\alpha$ is defined by

$$\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)}\underset{\sim}{\rho}_\alpha=\bigcup\big\{p(\alpha,n):\alpha\in\mathrm{dom}(p)\ \&\ n<\mathrm{trunklg}(p,\alpha)\ \&\ p\in\underset{\sim}{G}_{\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)}\big\}.$$

(2) For $\rho\in\prod\limits_{n<\omega}\mathbf{H}^*(n)$ we set $F(\rho)=\big\{\eta\in{}^\omega 2:(\forall^\infty n<\omega)(\eta\prec\rho(n)\big\}$.

Plainly, for each $\alpha<\lambda$, $\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)}\underset{\sim}{\rho}_\alpha\in\prod\limits_{n<\omega}\mathbf{H}^*(n)$. Also, for $\rho\in\prod\limits_{n<\omega}\mathbf{H}^*(n)$, the set $F(\rho)$ is a meager and null $\mathbf{\Sigma}^0_2$–subset of ${}^\omega 2$.

Theorem 2.5. *Assume CH. Let λ be an uncountable cardinal, $\lambda=\lambda^{\aleph_0}$.*

(1) *Forcing with $\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)$ preserves cardinalities and cofinalities and $\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)}$ " $2^{\aleph_0}=\lambda$ ".*

(2) *If $\beta<\lambda$ and $\underset{\sim}{\nu}$ is a $\mathbb{P}_{\lambda\setminus\{\beta\}}(\mathbf{K}^*,\mathbf{\Sigma}^*)$–name for a member of ${}^\omega 2$, then $\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)}$ " $\underset{\sim}{\nu}\in F(\underset{\sim}{\rho}_\beta)$ ".*

(3) *Consequently, $\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)}$ " $\mathrm{non}(\mathcal{N})=\mathrm{non}(\mathcal{M})=\lambda$ ".*

Proof. (1) It follows from 2.3+1.4(2)+1.7.

(2) The proof is parallel to that of [8, Lemma 9.1]. Assume $p\in\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)$. Remembering 1.4(1) we may use 1.7(3) to find a condition $q\geq p$ such that

$(*)_1$ the condition $q{\restriction}(\lambda\setminus\{\beta\})$ $<n$–decides the value of $\underset{\sim}{\nu}{\restriction}N_n$ (for each n), and

$(*)_2$ $\mathrm{trunklg}(q,\alpha)\geq 972$ for all $\alpha\in\mathrm{dom}(q)$ and $\mathrm{nor}(q(\alpha,m))\geq 972$ whenever $\alpha\in\mathrm{supp}(q,m)$, and

$(*)_3$ $\beta\in\mathrm{dom}(q)$ and if $\mathrm{supp}(q,m)\neq\emptyset$, then $|\mathrm{supp}(q,m)|\geq 972$.

Thus, for each n, we have a mapping $E_n:\mathrm{val}^\Pi(q{\restriction}(\lambda\setminus\{\beta\}),{<}n)\longrightarrow {}^{N_n}2$ such that

$$\big(q{\restriction}(\lambda\setminus\{\beta\})\big)\wedge t\Vdash_{\mathbb{P}_{\lambda\setminus\{\beta\}}(\mathbf{K}^*,\mathbf{\Sigma}^*)} \text{“ } \underset{\sim}{\nu}{\restriction}N_n=E_n(t)\text{ ”}.$$

We will further strengthen q to a condition q^* such that $\mathrm{dom}(q^*)=\mathrm{dom}(q)$ and

$(*)^{\rm goal}$ for all $n\geq\mathrm{trunklg}(q^*,\beta)$ and $t\in\mathrm{val}^\Pi\big(q^*{\restriction}(\lambda\setminus\{\beta\}),{<}(n+1)\big)$ we have

$$\big(\forall B\in q^*(\beta,n)\big)\big(E_{n+1}(t)\prec B\big).$$

Then clearly we will have $q^*\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)}$ " $\underset{\sim}{\nu}\in F(\underset{\sim}{\rho}_\beta)$ " and the proof of 2.5(2) will follow by the standard density argument.

To construct the condition q^* we set $\mathrm{dom}(q^*)=\mathrm{dom}(q)$, $\mathrm{trunklg}(q^*,\alpha)=\mathrm{trunklg}(q,\alpha)$, and we define $q^*(\alpha,m)$ by induction on m so that:

$q^*(\alpha,m)=q(\alpha,m)$ whenever $\alpha\notin\mathrm{supp}(q,m)$ or $\beta\notin\mathrm{supp}(q,m)$, and

$q^*(\alpha,m)\in\mathbf{\Sigma}^*(q(\alpha,m))$, $\mathrm{nor}(q^*(\alpha,m))\geq\mathrm{nor}(q(\alpha,m))-2$ for $\alpha\in\mathrm{supp}(q,m)$.

These demands guarantee that q^* is a condition in $\mathbb{P}_\lambda(\mathbf{K}^*,\mathbf{\Sigma}^*)$ stronger than q.

Fix an $n\geq\mathrm{trunklg}(q,\beta)$. Put $A=\mathrm{supp}(q,n)$ and note that $\beta\in A$, A has at least 972 elements (remember $(*)_3$), and $|A|<n$ (by 1.3(2)(γ)).

Set $\mathfrak{c}^0_\alpha=q(\alpha,n)$ for $\alpha\in A$.

We choose inductively an enumeration $\{\alpha_0,\ldots,\alpha_{|A|-1}\}$ of A and creatures $\mathfrak{c}^\ell_{\alpha_k}$ (for $\ell\leq k<|A|$) and $\mathfrak{d}_{\alpha_k}$ from $\mathbf{\Sigma}^*(\mathfrak{c}^0_{\alpha_k})$. So assume that for some $\ell\geq 0$ we already have defined a list $\{\alpha_k:k<\ell\}$ of distinct elements of A and creatures $\mathfrak{c}^\ell_\alpha$ for $\alpha\in A\setminus\{\alpha_0,\ldots,\alpha_{\ell-1}\}$. Each $\mathfrak{c}^\ell_\alpha$ is $(K^\ell_\alpha,n,r_{\mathbf{H}^*}(n))$–decisive for some K^ℓ_α. Put $K_\ell=\min(\{K^\ell_\alpha:\alpha\in A\setminus\{\alpha_0,\ldots,\alpha_{\ell-1}\}\})$, and choose α_ℓ such that $K^\ell_{\alpha_\ell}=K_\ell$. Let $\mathfrak{d}_{\alpha_\ell}\in\mathbf{\Sigma}^*(\mathfrak{c}^\ell_{\alpha_\ell})$ be such that $|\mathrm{val}(\mathfrak{d}_{\alpha_\ell})|\leq K_\ell$ and $\mathrm{nor}(\mathfrak{d}_{\alpha_\ell})\geq\mathrm{nor}(\mathfrak{c}^\ell_{\alpha_\ell})-r_{\mathbf{H}^*}(n)$. For $\alpha\in A\setminus\{\alpha_0,\ldots,\alpha_\ell\}$, let $\mathfrak{c}^{\ell+1}_\alpha\in\mathbf{\Sigma}^*(\mathfrak{c}^\ell_\alpha)$ be hereditarily $(2^{(K_\ell)^n},r_{\mathbf{H}^*}(n))$–big and such that $\mathrm{nor}(\mathfrak{c}^{\ell+1}_\alpha)\geq\mathrm{nor}(\mathfrak{c}^\ell_{\alpha_\ell})-r_{\mathbf{H}^*}(n)$. Iterate this procedure $|A|-1$ times. At the end, there remains one α that has not been listed as an α_ℓ, so we set $\alpha_{|A|-1}=\alpha$ and $\mathfrak{d}_{\alpha_{|A|-1}}=\mathfrak{c}^{|A|-1}_\alpha$.

Since $\mathfrak{c}^{\ell+1}_{\alpha_{\ell+1}}$ is hereditarily $(2^{(K_\ell)^n},r_{\mathbf{H}^*}(n))$–big, we see that $2^{(K_\ell)^n}<K_{\ell+1}$. Let m be such that $\beta=\alpha_m$, and put

$$K=K_m,\quad S=\{\alpha_\ell:\ell<m\},\quad L=\{\alpha_\ell:\ell>m\}.$$

It is possible that (at most) one of the sets S,L is empty. By our choices,

$(*)_4$ (a) $\mathfrak{d}_\alpha\in\mathbf{\Sigma}^*(q(\alpha,n))$, $\mathrm{nor}(\mathfrak{d}_\alpha)\geq\mathrm{nor}(q(\alpha,n))-(n-1)\cdot r_{\mathbf{H}^*}(n)>900$, and

(b) if $S \neq \emptyset$ then $\mathfrak{d}_\beta$ is $(2^{(K_{m-1})^n}, r_{\mathbf{H}^*}(n))$–big and hence in particular $(K_{m-1})^{n-2} < K$; if $S = \emptyset$ then $K = K_0$,

(c) $\prod\limits_{\alpha \in S} |\mathrm{val}(\mathfrak{d}_\alpha)| \le (K_{m-1})^{n-2} < K$ and $|\mathrm{val}(\mathfrak{d}_\beta)| \le K$,

(d) $\varphi_{\mathbf{H}^*}(<n) < K_0 \le K$ (remember that $\mathbf{K}(n)$ is $(g(n), r_{\mathbf{H}^*}(n))$–big and $g(n) > \varphi_{\mathbf{H}^*}(<n)$),

(e) if $\alpha \in L$, then $\mathfrak{d}_\alpha$ is $(2^{K^n}, r_{\mathbf{H}^*}(n))$–big.

Let $Z = \{t \in \mathrm{val}^\Pi(q{\restriction}(\lambda \setminus \{\beta\}), <(n+1)) : t(\alpha, n) \in \mathrm{val}(\mathfrak{d}_\alpha)$ for $\alpha \in A \setminus \{\beta\}\}$ and for $s \in \prod\limits_{\alpha \in L} \mathrm{val}(\mathfrak{d}_\alpha)$ let $Z_s = \{t \in Z : t(\alpha, n) = s(\alpha)$ for $\alpha \in L\}$. Next, for $t \in Z$ put $\mathcal{C}_t = \{B \in \mathcal{B}^{\mathfrak{d}_\beta} : E_{n+1}(t) \not\prec B\}$.

If $S = \emptyset$, then in what follows ignore $\prod\limits_{\alpha \in S} \mathrm{val}(\mathfrak{d}_\alpha)$ and set $K_{m-1} = 1$. Assume L is non-empty (otherwise move to $(*)_6$). For each $s \in \prod\limits_{\alpha \in L} \mathrm{val}(\mathfrak{d}_\alpha)$ consider a function

$$c(s) : \mathrm{val}^\Pi(q{\restriction}(\lambda \setminus \{\beta\}), <n) \times \prod_{\alpha \in S} \mathrm{val}(\mathfrak{d}_\alpha) \longrightarrow \mathcal{P}(\mathrm{val}(\mathfrak{d}_\beta))$$

such that $c(s)(t_0, t_1) = \mathcal{C}_{t_0 \frown t_1 \frown s}$, where $t_0 {}^\frown t_1 {}^\frown s \in Z_s$ is obtained by natural concatenation. This determines a coloring c on $\prod\limits_{\alpha \in L} \mathrm{val}(\mathfrak{d}_\alpha)$ with the range of size at most

$$\big(2^K\big)^{\varphi_{\mathbf{H}^*}(<n) \cdot (K_{m-1})^{n-2}} \le \big(2^K\big)^{K \cdot K} = 2^{K^3} < 2^{K^n}.$$

Since $\mathbf{K}^*(n)$ is $(n, r_{\mathbf{H}^*}(n))$–decisive, and each $\mathfrak{d}_\alpha$ is hereditarily $(2^{K^n}, r_{\mathbf{H}^*}(n))$–big (for $\alpha \in L$), $\mathrm{nor}(\mathfrak{d}_\alpha) > 900$ and $|L| \le n - 2$, therefore we may use Lemma 1.2 to find $q^*(\alpha, n) \in \mathbf{\Sigma}^*(\mathfrak{d}_\alpha)$ for $\alpha \in L$ such that

$(*)_5$ (a) $\mathrm{nor}(q^*(\alpha, n)) \ge \mathrm{nor}(\mathfrak{d}_\alpha) - r_{\mathbf{H}^*}(n) \cdot n \ge \mathrm{nor}(q(\alpha, n)) - 2$, and

(b) $c{\restriction} \prod\limits_{\alpha \in L} \mathrm{val}(q^*(\alpha, n))$ is constant.

If $L = \emptyset$ then the procedure described above is not needed. In any case, letting

$$X = \mathrm{val}^\Pi(q{\restriction}(\lambda \setminus \{\beta\}), <n) \times \prod_{\alpha \in S} \mathrm{val}(\mathfrak{d}_\alpha),$$

we have a mapping $d : X \longrightarrow \mathcal{P}(\mathrm{val}(\mathfrak{d}_\beta))$ and $q^*(\alpha, n)$ for $\alpha \in L$ such that

$(*)_6$ if $t \in Z$ and $t(\alpha, n) \in \mathrm{val}(q^*(\alpha, n))$ for $\alpha \in L$, then $\mathcal{C}_t = d(t_0, t_1)$, where $t_0 = t{\restriction}\big((\mathrm{dom}(q) \setminus \{\beta\}) \times n\big) \in \mathrm{val}^\Pi(q{\restriction}(\lambda \setminus \{\beta\}), <n)$ and $t_1 = t{\restriction}(S \times \{n\}) \in \prod\limits_{\alpha \in S} \mathrm{val}(\mathfrak{d}_\alpha)$.

For each $(t_0, t_1) \in X$ fix one $t = t[t_0, t_1] \in Z$ such that $t(\alpha, n) \in \mathrm{val}(q^*(\alpha, n))$ for $\alpha \in L$, $t_0 = t{\restriction}\big((\mathrm{dom}(q) \setminus \{\beta\}) \times n\big)$ and $t_1 = t{\restriction}(S \times \{n\})$. Now, for $B \in \mathrm{val}(\mathfrak{d}_\beta)$ we (try to) choose $(t_0^B, t_1^B) \in X$ such that $B \in \mathcal{C}_{t[t_0^B, t_1^B]}$, if possible. Consider a coloring $e : \mathrm{val}(\mathfrak{d}_\beta) \longrightarrow {}^{N_{n+1}}2 \cup \{*\}$ defined by

$$e(B) = \begin{cases} E_{n+1}(t[t_0^B, t_1^B]) & \text{if } (t_0^B, t_1^B) \in X \text{ is defined,} \\ * & \text{otherwise.} \end{cases}$$

Since $|X| \le \varphi_{\mathbf{H}^*}(<n) \cdot (K_{m-1})^{n-2} \le \max\{(K_{m-1})^{n-1}, \varphi_{\mathbf{H}^*}(<n)\}$, we know that the range of the coloring e has at most $\max\{(K_{m-1})^{n-1}, \varphi_{\mathbf{H}^*}(<n)\} + 1$ members. Thus $\mathfrak{d}_\beta$ is $(|\mathrm{rng}(e)|, r_{\mathbf{H}^*}(n))$–big and we may choose $q^*(\beta, n) \in \mathbf{\Sigma}^*(\mathfrak{d}_\beta)$ such that $\mathrm{nor}(q^*(\beta, n)) \ge \mathrm{nor}(\mathfrak{d}_\beta) - r_{\mathbf{H}^*}(n) \ge \mathrm{nor}(q(\alpha, n)) - 2 > 900$ and $e{\restriction}\mathrm{val}(q^*(\alpha, n))$ is constant. If the constant value were $\eta \in {}^{N_{n+1}}2$, then we would have $\eta \not\prec B$ for all $B \in \mathrm{val}(q^*(\alpha,n))$, contradicting $\mathrm{nor}(q^*(\beta,n)) > 0$. Therefore,

$(*)_7$ (t_0^B, t_1^B) is defined for no $B \in \mathrm{val}(q^*(\beta,n))$ and hence

$$\mathrm{val}(q^*(\beta,n)) \cap \bigcup\{\mathcal{C}_{t[t_0,t_1]} : (t_0,t_1) \in X\} = \emptyset.$$

For $\alpha \in S$ we set $q^*(\alpha,n) = \mathfrak{d}_\alpha$. Now note that

$(*)_8$ if $t \in Z$ is such that $t(\alpha,n) \in q^*(\alpha,n)$ for $\alpha \in S \cup L$ and $B \in \mathrm{val}(q^*(\beta,n))$, then $E_{n+1}(t) \prec B$.

Why? Assume towards contradiction that $E_{n+1}(t) \not\prec B$, i.e., $B \in \mathcal{C}_t$. Represent t as $t = t_0{}^\frown t_1{}^\frown s$ where $(t_0,t_1) \in X$. Then $\mathcal{C}_t = \mathcal{C}_{t[t_0,t_1]}$ (by $(*)_6$) and therefore $B \in \mathcal{C}_{t[t_0,t_1]}$, contradicting $(*)_7$.

This completes the definition of q^*. It follows from $(*)_8$ (for $n \geq \mathrm{trunklg}(q^*,\beta)$) that $(*)^{\rm goal}$ is satisfied.

(3) Follows from (2) and the fact that $F(\rho) \in \mathcal{N} \cap \mathcal{M}$ for $\rho \in \prod\limits_{n<\omega} \mathbf{H}(m)$. □

Corollary 2.6. *It is consistent that*

$$\mathrm{non}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N} \cap \mathcal{M}) = \aleph_2 = 2^{\aleph_0} \textit{ and } \mathfrak{do} = \aleph_1.$$

Proof. Start with a model of CH and force with $\mathbb{P}_{\aleph_2}(\mathbf{K}^*, \mathbf{\Sigma}^*)$. It follows from 2.5 and 1.8 that the resulting model is as required. □

In models for the statement in Corollary 2.6 necessarily $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$. However, it is not clear if we could not get a parallel result for $\mathfrak{d}_{\mathcal{B}}$ and cov.

Problem 2.7. Is it consistent that

$$\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_2 = 2^{\aleph_0} \text{ and } \mathfrak{d}_{\mathcal{B}} = \aleph_1 \ ?$$

In particular, is it consistent that $\mathfrak{do} > \mathfrak{d}_{\mathcal{B}}$?

Directly from 2.6 we also obtain

Corollary 2.8. *It is consistent that* $\mathrm{non}(\mathcal{N} \cap \mathcal{M}) = \aleph_2$ *and there is no* $\subset$*–increasing chain of Borel subset of* ${}^\omega 2$ *of length* ω_2.

3. Monotone hulls

The interest in Corollary 2.8 came from the questions concerning Borel hulls.

Definition 3.1. Let $\mathrm{Borel}({}^\omega 2)$ be the family of all Borel subsets of ${}^\omega 2$, $\mathcal{I}$ be a σ–ideal on ${}^\omega 2$ with Borel basis and $\mathcal{S}_{\mathcal{I}}$ be the σ–algebra of subsets of ${}^\omega 2$ generated by $\mathrm{Borel}({}^\omega 2) \cup \mathcal{I}$. Let $\mathcal{F} \subseteq \mathcal{S}_{\mathcal{I}}$. A *monotone Borel hull* on $\mathcal{F}$ with respect to $\mathcal{I}$ is a mapping $\psi : \mathcal{F} \longrightarrow \mathrm{Borel}({}^\omega 2)$ such that

- $A \subseteq \psi(A)$ and $\psi(A) \setminus A \in \mathcal{I}$ for all $A \in \mathcal{F}$, and
- if $A_1 \subseteq A_2$ are from $\mathcal{F}$, then $\psi(A_1) \subseteq \psi(A_2)$.

If the range of ψ consists of sets of some Borel class $\mathcal{K}$, then we say that ψ is a monotone $\mathcal{K}$ hull operation.

As discussed in Balcerzak and Filipczak [1, Question 24], 2.8 implies the following.

Corollary 3.2. *It is consistent that*

- *there are no monotone Borel hulls on* $\mathcal{M}$ *with respect to* $\mathcal{M}$*, and*
- *there are no monotone Borel hulls on* $\mathcal{N}$ *with respect to* $\mathcal{N}$*, and*

- *there are no monotone Borel hulls on $\mathcal{M}\cap\mathcal{N}$ with respect to $\mathcal{M}\cap\mathcal{N}$.*

The non-existence of monotone Borel hulls on $\mathcal{I}$ implies non-existence of such hulls on $\mathcal{S}_{\mathcal{I}}$. While some partial results were presented in [7] and [1], not much is known about the converse implication.

Problem 3.3 (Cf. Balcerzak and Filipczak [1, Question 26])**.** Let $\mathcal{I}\in\{\mathcal{M},\mathcal{N}\}$. Is it consistent that there exists a monotone Borel hull on $\mathcal{I}$ (with respect to $\mathcal{I}$) but there is no such hull on $\mathcal{S}_{\mathcal{I}}$? In particular, is it consistent that $\mathrm{add}(\mathcal{I})=\mathrm{cof}(\mathcal{I})$ but there is no monotone Borel hull on $\mathcal{S}_{\mathcal{I}}$?

It was noted in [1, Proposition 7] (see also Elekes and Máthé [7, Theorem 2.4]) that $\mathrm{add}(\mathcal{I})=\mathrm{cof}(\mathcal{I})$ implies that there exists a monotone Borel hull on $\mathcal{I}$ (with respect to $\mathcal{I}$). It appears that was the only situation in which the positive result of this kind was known. Using a finite support iteration of ccc forcing notions we will show in this section that, consistently, we may have $\mathrm{add}(\mathcal{I})<\mathrm{cof}(\mathcal{I})$ (for $\mathcal{I}\in\{\mathcal{N},\mathcal{M}\}$) and yet there are monotone hulls for $\mathcal{I}$.

Definition 3.4. Let $\mathcal{I}$ be an ideal of subsets of ${}^{\omega}2$.

(1) We say that a family $\mathcal{B}\subseteq\mathrm{Borel}({}^{\omega}2)\cap\mathcal{I}$ is *an mhg–base for $\mathcal{I}$* if[3]
  (a) $\mathcal{B}$ is a basis for $\mathcal{I}$, i.e., $(\forall A\in\mathcal{I})(\exists B\in\mathcal{B})(A\subseteq B)$, and
  (b) if $\langle B_i: i<\omega_1\rangle$ is a sequence of elements of $\mathcal{B}$, then for some $i<j<\omega_1$ we have $B_i\subseteq B_j$.

(2) Let α^*,β^* be limit ordinals. *An $\alpha^*\times\beta^*$–base for $\mathcal{I}$* is a sequence $\langle B_{\alpha,\beta}: \alpha<\alpha^*\ \&\ \beta<\beta^*\rangle$ of Borel sets from $\mathcal{I}$ such that it forms a basis for $\mathcal{I}$ (i.e., (a) above holds) and
  (c) for each $\alpha_0,\alpha_1<\alpha^*$, $\beta_0,\beta_1<\beta^*$ we have
$$B_{\alpha_0,\beta_0}\subseteq B_{\alpha_1,\beta_1}\quad\Leftrightarrow\quad \alpha_0\le\alpha_1\ \&\ \beta_0\le\beta_1.$$

Proposition 3.5. *Assume that $\langle B_{\alpha,\beta}:\alpha<\alpha^*\ \&\ \beta<\beta^*\rangle$ is an $\alpha^*\times\beta^*$–base for $\mathcal{I}$. Then:*

(i) $B_{\alpha,\beta}\neq B_{\alpha',\beta'}$ *whenever* $(\alpha,\beta)\neq(\alpha',\beta')$, $\alpha,\alpha'<\alpha^*$, $\beta,\beta'<\beta^*$.
(ii) $\{B_{\alpha,\beta}:\alpha<\alpha^*\ \&\ \beta<\beta^*\}$ *is an mhg–base for $\mathcal{I}$.*
(iii) $\mathrm{add}(\mathcal{I})=\min\{\mathrm{cf}(\alpha^*),\mathrm{cf}(\beta^*)\}$ *and* $\mathrm{cof}(\mathcal{I})=\max\{\mathrm{cf}(\alpha^*),\mathrm{cf}(\beta^*)\}$.

Proof. Straightforward. □

Proposition 3.6. *Suppose that an ideal $\mathcal{I}$ on ${}^{\omega}2$ has an mhg–base $\mathcal{B}\subseteq\mathrm{Borel}({}^{\omega}2)\cap\mathcal{I}$. Then there exists a monotone hull operation $\psi:\mathcal{I}\longrightarrow\mathrm{Borel}({}^{\omega}2)\cap\mathcal{I}$ on $\mathcal{I}$. If, additionally, $\mathcal{B}\subseteq\Pi^0_\xi$, $\xi<\omega_1$, then ψ can be taken to have values in Π^0_ξ.*

Proof. For a set $A\in\mathcal{I}$ let $\mathcal{S}_A$ be the family of all sequences $\bar{B}=\langle B_i: i<\gamma\rangle\subseteq\mathcal{B}$ satisfying

$(*)_1$ $(\forall i<\gamma)(A\subseteq B_i)$ and
$(*)_2$ $(\forall i<j<\gamma)(B_i\not\subseteq B_j)$.

Note that for each $\bar{B}\in\mathcal{S}_A$ we have $\ell g(\bar{B})<\omega_1$ (by 3.4(1)(b) and $(*)_2$). Clearly, every $\trianglelefteq$–increasing chain of elements of $\mathcal{S}_A$ has a $\trianglelefteq$–upper bound in $\mathcal{S}_A$, so we may choose $\bar{B}_A=\langle B^A_i: i<\gamma_A\rangle\in\mathcal{S}_A$ which has no proper extension in $\mathcal{S}_A$. Put $\psi(A)=\bigcap\limits_{i<\gamma_A}B^A_i$. Plainly, $A\subseteq\psi(A)\in\mathcal{I}$ and $\psi(A)$ is a Borel set, and if $\mathcal{B}\subseteq\Pi^0_\xi$ then also $\psi(A)\in\Pi^0_\xi$.

[3] "mhg" stands for "monotone hull generating"

Claim 3.6.1. $\psi(A) = \bigcap\{B \in \mathcal{B} : A \subseteq B\}$

Proof of the Claim. By $(*)_1$ we see that $\psi(A) \supseteq \bigcap\{B \in \mathcal{B} : A \subseteq B\}$. To show the converse inclusion suppose $B \in \mathcal{B}$, $A \subseteq B$. By the choice of $\bar{B}_A$ we know that $\bar{B}_A{}^\frown\langle B\rangle \notin \mathcal{S}_A$ and hence $B^A_i \subseteq B$ for some $i < \gamma_A$. Consequently $\psi(A) \subseteq B$. □

It follows from the above claim that $A_1 \subseteq A_2 \in \mathcal{I}$ implies $\psi(A_1) \subseteq \psi(A_2)$. □

Bartoszyński and Kada [3] showed that for any σ–directed partial order Q there is a ccc forcing notion $\mathbb{P}$ such that

$\Vdash_{\mathbb{P}}$ " $\mathcal{M}$ has a basis order isomorphic to Q with respect to set–inclusion ".

A parallel result for $\mathcal{N}$ was given by Burke and Kada [5]. These theorems imply that for uncountable cardinals κ and λ we may force that $\mathcal{M}$ has a $\kappa\times\lambda$–basis, and we may also force that $\mathcal{N}$ has a $\kappa\times\lambda$–basis. The corresponding forcing notions (for both cases) were essentially versions of "FS iterations with partial memories" used in Shelah [13, 14, 15], Mildenberger and Shelah [10] and Shelah and Thomas [16]. We will use explicitly the method of "FS iterations with partial memories" to construct a model in which *both* ideals have $\kappa\times\lambda$–bases.

Theorem 3.7. *Let κ,λ be cardinals of uncountable cofinality, $\kappa\leq\lambda$. There is a ccc forcing notion $\mathbb{Q}^{\kappa,\lambda}$ of size $\lambda^{\aleph_0}$ such that*

$\Vdash_{\mathbb{Q}^{\kappa,\lambda}}$ *" the meager ideal $\mathcal{M}$ has a $\kappa\times\lambda$–basis consisting of Σ^0_2 sets, and the null ideal $\mathcal{N}$ has a $\kappa\times\lambda$–basis consisting of Π^0_2 sets ".*

Proof. The forcing notion $\mathbb{Q}^{\kappa,\lambda}$ will be obtained by means of finite support iteration of ccc forcing notions. The iterands will be products of the Amoeba for Category $\mathbb{B}$ and Amoeba for Measure $\mathbb{A}$ but *considered over partial sub-universes only.*

We will use the notation and some basic facts stated in the third section of [16].

Let us recall the forcings $\mathbb{A}$ and $\mathbb{B}$ used as iterands.

- A condition in $\mathbb{A}$ is a tree $T\subseteq {}^{\omega>}2$ such that $\mu([T])>\frac{1}{2}$ and $\mu([t]\cap[T])>0$ for all $t\in T$. The order $\leq_{\mathbb{A}}$ of $\mathbb{A}$ is the reverse inclusion.
- A condition in $\mathbb{B}$ is a pair (n,T) such that $n\in\omega$, $T\subseteq{}^{\omega>}2$ is a tree with no maximal nodes and $[T]$ is a nowhere dense subset of ${}^{\omega}2$. The order $\leq_{\mathbb{B}}$ of $\mathbb{B}$ is given by:
  $(n,T)\leq_{\mathbb{B}}(n',T')$ if and only if $n\leq n'$, $T\subseteq T'$ and $T\cap{}^{n}2=T'\cap{}^{n}2$.

Both $\mathbb{A}$ and $\mathbb{B}$ are (nice definitions of) ccc forcing notions, $\mathbb{B}$ is σ–centered and if $\mathbf{V}'\subseteq\mathbf{V}''$ are universes of set theory then $\mathbb{A}^{\mathbf{V}'}$ is still ccc in $\mathbf{V}''$. We will use the following immediate properties of these forcing notions.

$(\circledast)_1$ If $G\subseteq\mathbb{A}$ is generic over $\mathbf{V}$, $F=\bigcap\{[T]:T\in G\}$, then F is a closed subset of ${}^{\omega}2$, $\mu(F)=\frac{1}{2}$ and F is disjoint from every Borel null set coded in $\mathbf{V}$. Hence the set $F^*=\{x\in{}^{\omega}2:(\forall y\in F)(\exists^\infty n)(x(n)\neq y(n))\}$ is a null Π^0_2 set and it includes all Borel null sets coded in $\mathbf{V}$.
Let $\underaccent{\tilde}{F}_{\mathbb{A}},\underaccent{\tilde}{F}^*_{\mathbb{A}}$ be $\mathbb{A}$–names for the sets F,F^*, respectively.

$(\circledast)_2$ If $G\subseteq\mathbb{B}$ is generic over $\mathbf{V}$, $F=\bigcup\{[T]:(\exists n)((n,T)\in G)\}$, then F is a closed nowhere dense subset of ${}^{\omega}2$. Letting $F^*=\{x\in{}^{\omega}2:(\exists y\in F)(\forall^\infty n)(x(n)=y(n))\}$ we get a meager Σ^0_2 set including all Borel meager sets coded in $\mathbf{V}$.
Let $\underaccent{\tilde}{F}_{\mathbb{B}},\underaccent{\tilde}{F}^*_{\mathbb{B}}$ be $\mathbb{B}$–names for the sets F,F^*, respectively.

$(\circledast)^a_3$ If $T\in\mathbb{A}$, $t\in T$, then there is $T'\geq_{\mathbb{A}}T$ such that $T'\Vdash_{\mathbb{A}}[t]\cap\underaccent{\tilde}{F}_{\mathbb{A}}\neq\emptyset$.

$(\circledast)^b_3$ If $T\in\mathbb{A}$, $n\in\omega$, then there is $N>n$ such that for each $\nu\in {}^{[n,N)}2$ there is $T'\geq_{\mathbb{A}} T$ with $T'\Vdash_{\mathbb{A}} (\forall y\in \underset{\sim}{F}_{\mathbb{A}})(y{\restriction}[n,N)\neq\nu)$.

$(\circledast)^a_4$ If $(n,T)\in\mathbb{B}$, $t\in T$, $\ell g(t)\leq n$, $m_1>m_0\geq n$ and $\nu\in {}^{[m_0,m_1)}2$, then there are $(n',T')\geq_{\mathbb{B}}(n,T)$ and $s\in T'$ such that $t\lhd s$ and $s{\restriction}[m_0,m_1)=\nu$ (and $(n',T')\Vdash_{\mathbb{B}} [s]\cap \underset{\sim}{F}_{\mathbb{B}}\neq\emptyset$).

$(\circledast)^b_4$ If $(n,T)\in\mathbb{B}$, $m_0<\omega$, then there are $m_1>m_0$ and $\nu\in {}^{[m_0,m_1)}2$ and $(n',T')\geq_{\mathbb{B}}(n,T)$ such that $(n',T')\Vdash_{\mathbb{B}} (\forall y\in \underset{\sim}{F}_{\mathbb{B}})(y{\restriction}[m_0,m_1)\neq\nu)$.

Fix an ordinal γ and a bijection $\pi:\kappa\times\lambda\xrightarrow{\text{onto}}\gamma$ such that

$$\alpha_0\leq\alpha_1<\kappa\ \&\ \beta_0\leq\beta_1<\lambda\quad\Rightarrow\quad \pi(\alpha_0,\beta_0)\leq\pi(\alpha_1,\beta_1).$$

For $i=\pi(\alpha_1,\beta_1)$ let $a_i=\{\pi(\alpha_0,\beta_0):\alpha_0\leq\alpha_1\ \&\ \beta_0\leq\beta_1\}\setminus\{i\}$. We say that a set $b\subseteq\gamma$ is *closed* if $a_i\subseteq b$ for all $i\in b$. It follows from our choice of π that for each $i<\gamma$ we have

$(\circledast)_5$ $a_i\subseteq i$ and the sets $a_i, i, a_i\cup\{i\}$ are closed.

Now, by induction we define $\langle\mathbb{P}_i,\underset{\sim}{\mathbb{Q}}_i,\underset{\sim}{F}^0_i,\underset{\sim}{F}^1_i,\underset{\sim}{F}^{\mathbb{A}}_i,\underset{\sim}{F}^{\mathbb{B}}_i:i<\gamma\rangle$ and $\mathbb{P}^*_b$ for closed $b\subseteq\gamma$ simultaneously proving the correctness of the definition and the desired properties listed below.[4]

$(\circledast)_6$ $\langle\mathbb{P}_j,\underset{\sim}{\mathbb{Q}}_i:j\leq\gamma,i<\gamma\rangle$ is a finite support iteration of ccc forcing notions.

$(\circledast)_7$ $\mathbb{P}^*_b=\big\{p\in\mathbb{P}_\gamma:\mathrm{supp}(p)\subseteq b\ \&\ p(i)$ is a $\mathbb{P}^*_{a_i}$–name (for a member of $\underset{\sim}{\mathbb{Q}}_i$) for each $i\in\mathrm{supp}(p)\big\}$.

$(\circledast)_8$ $\mathbb{P}^*_b$ is a complete suborder of $\mathbb{P}_\gamma$, $\mathbb{P}^*_{a_i\cup\{i\}}$ is isomorphic with the composition $\mathbb{P}^*_{a_i}*\underset{\sim}{\mathbb{Q}}_i$.

$(\circledast)_9$ $\underset{\sim}{\mathbb{Q}}_i$ is a $\mathbb{P}^*_{a_i}$–name for the product[5] $\mathbb{A}\times\mathbb{B}$.

$(\circledast)_{10}$ $\underset{\sim}{F}^0_i,\underset{\sim}{F}^1_i,\underset{\sim}{F}^{\mathbb{A}}_i,\underset{\sim}{F}^{\mathbb{B}}_i$ are $\mathbb{P}^*_{a_i\cup\{i\}}$–names for the sets $\underset{\sim}{F}_{\mathbb{A}},\underset{\sim}{F}_{\mathbb{B}},\underset{\sim}{F}^*_{\mathbb{A}},\underset{\sim}{F}^*_{\mathbb{B}}$ added by the forcings at the last coordinate of $\mathbb{P}^*_{a_i\cup\{i\}}\simeq\mathbb{P}^*_{a_i}*(\mathbb{A}\times\mathbb{B})$.

$(\circledast)_{11}$ (a) $\mathbb{P}^*_i$ is a dense subset of $\mathbb{P}_i$ (for $i\leq\gamma$).
(b) If $q\in\mathbb{P}^*_\gamma$, then $q{\restriction}b\in\mathbb{P}^*_b$.
(c) If $p,q\in\mathbb{P}^*_\gamma$, $p\leq q$ and $i\in\mathrm{supp}(q)$ then $p{\restriction}a_i\leq_{\mathbb{P}^*_{a_i}} q{\restriction}a_i$ and $q{\restriction}a_i\Vdash_{\mathbb{P}^*_{a_i}} p(i)\leq q(i)$.
(d) If $q\in\mathbb{P}^*_\gamma$, $p\in\mathbb{P}^*_b$ and $p\leq q$, then $p\leq_{\mathbb{P}^*_b} q{\restriction}b$.
(e) If $q\in\mathbb{P}^*_b$, $p\in\mathbb{P}^*_\gamma$, $p{\restriction}b\leq_{\mathbb{P}^*_b} q$ and r is defined by

$$r(\xi)=\begin{cases} q(\xi) & \text{if } \xi\in b,\\ p(\xi) & \text{otherwise}\end{cases}\qquad\text{for } \xi<\gamma$$

then $r\in\mathbb{P}^*_\gamma$ and $r\geq q$, $r\geq p$.

Also,

$(\circledast)_{12}$ if $\underset{\sim}{\tau}$ is a canonical[6] $\mathbb{P}^*_\gamma$–name for a member of ${}^\omega 2$, then $\underset{\sim}{\tau}$ is a $\mathbb{P}^*_{a_i}$–name for some $i<\gamma$.

[Why? Note that if $(\alpha_n,\beta_n)\in\kappa\times\lambda$, $n<\omega$, then there is $(\alpha^*,\beta^*)\in\kappa\times\lambda$ such that $\alpha_n\leq\alpha^*$, $\beta_n\leq\beta^*$ for all $n<\omega$.]

The main technical point of our argument is given in the following observation.

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[4]See [16, 3.1–3.7] for the order in which these should be shown.

[5]Since $\mathbb{B}^{\mathbf{V}^{\mathbb{P}^*_{a_i}}}$ is σ–centered we know that the product is ccc.

[6]i.e., determined in a standard way by a sequence of maximal antichains

$(\circledast)_{13}$ Suppose $i,j<\gamma$, $i\notin a_j$, $j\notin a_i$, $i\neq j$, $\ell\in\{0,1\}$. Assume that $p\in\mathbb{P}^*_\gamma$, $\eta\in{}^n 2$, $n<\omega$ and $p\Vdash_{\mathbb{P}^*_\gamma}[\eta]\cap\underset{\sim}{F}{}^\ell_i\neq\emptyset$. Then there are $\nu\in{}^{[n,N)}2$, $n<N<\omega$ and $q\geq_{\mathbb{P}^*_\gamma}p$ such that

$$q\Vdash_{\mathbb{P}^*_\gamma}\text{“ }[\eta^\frown\nu]\cap\underset{\sim}{F}{}^\ell_i\neq\emptyset\text{ and }\big(\forall y\in\underset{\sim}{F}{}^\ell_j\big)\big(y{\restriction}[n,N)\neq\nu\big)\text{ ”}.$$

[Why? Let us provide detailed arguments for $\ell=0$. By $(\circledast)^b_3+(\circledast)_9+(\circledast)_{11}$ we may find $N>n$ and a condition $p_0'\in\mathbb{P}^*_{a_j}$ such that $p_0'\geq p{\restriction}a_j$ and

$$p_0'\Vdash_{\mathbb{P}^*_{a_j}}\text{“ for each }\nu\in{}^{[n,N)}2\text{ there is }p_j\geq_{\underset{\sim}{\mathbb{Q}}_j}p(j)\text{ such that }p_j\Vdash_{\underset{\sim}{\mathbb{Q}}_j}(\forall y\in\underset{\sim}{F}_{\mathbb{A}})(y{\restriction}[n,N)\neq\nu)\text{ ”}.$$

Let $p_0\in\mathbb{P}^*_\gamma$ be such that $p_0(\xi)=p_0'(\xi)$ for $\xi\in a_j$ and $p_0(\xi)=p(\xi)$ otherwise (see $(\circledast)_{11}(e)$; so p_0 is a common extension of p_0' and p). Note that $p_0(j)=p(j)$. Use $(\circledast)^a_3$ to choose $\nu\in{}^{[n,N)}2$ and a condition $p_1'\in\mathbb{P}^*_{a_i\cup\{i\}}$ such that $p_1'\geq p_0{\restriction}(a_i\cup\{i\})$ and $p_1'\Vdash_{\mathbb{P}^*_{a_i\cup\{i\}}}[\eta^\frown\nu]\cap\underset{\sim}{F}{}^0_i\neq\emptyset$. Let $p_1\in\mathbb{P}^*_\gamma$ be such that $p_1(\xi)=p_1'(\xi)$ if $\xi\in a_i\cup\{i\}$ and $p_1(\xi)=p_0(\xi)$ otherwise. Then p_1 is stronger than both p_1' and p_0, and $p_1(j)=p_0(j)=p(j)$. Hence

$$p_1{\restriction}a_j\Vdash_{\mathbb{P}^*_{a_j}}\text{“ there is }p_j\geq_{\underset{\sim}{\mathbb{Q}}_j}p_1(j)\text{ such that }p_j\Vdash_{\underset{\sim}{\mathbb{Q}}_j}(\forall y\in\underset{\sim}{F}_{\mathbb{A}})(y{\restriction}[n,N)\neq\nu)\text{ ”}.$$

Let $q(j)$ be a $\mathbb{P}^*_{a_j}$–name for a p_j as above and let $q(\xi)=p_1(\xi)$ for $\xi\neq j$. Clearly $q\in\mathbb{P}^*_\gamma$ and $q{\restriction}(a_j\cup\{j\})\Vdash_{\mathbb{P}^*_{a_j\cup\{j\}}}(\forall y\in\underset{\sim}{F}{}^0_j)(y{\restriction}[n,N)\neq\nu)$, and (as $q{\restriction}(a_i\cup\{i\})=p_1{\restriction}(a_i\cup\{i\})=p_1'$) $q{\restriction}(a_i\cup\{i\})\Vdash_{\mathbb{P}^*_{a_i\cup\{i\}}}[\eta^\frown\nu]\cap\underset{\sim}{F}{}^0_i\neq\emptyset$. Using $(\circledast)_8+(\circledast)_{10}+(\circledast)_{11}$ we get that the condition q is as required. If $\ell=1$ then the arguments are similar, but instead of $(\circledast)^a_3,(\circledast)^b_3$ we use $(\circledast)^a_4,(\circledast)^b_4$.]

For $\alpha<\kappa$, $\beta<\lambda$ let $\underset{\sim}{B}{}^{\mathbb{A}}_{\alpha,\beta}=\underset{\sim}{F}{}^{\mathbb{A}}_{\pi(\alpha,\beta)}$ and $\underset{\sim}{B}{}^{\mathbb{B}}_{\alpha,\beta}=\underset{\sim}{F}{}^{\mathbb{B}}_{\pi(\alpha,\beta)}$. Immediately from $(\circledast)_{12}+(\circledast)_1+(\circledast)_2$ we conclude that

$(\circledast)_{14}$ $\Vdash_{\mathbb{P}^*_\gamma}$ “ $\{\underset{\sim}{B}{}^{\mathbb{A}}_{\alpha,\beta}:\alpha<\kappa\ \&\ \beta<\lambda\}$ is a basis for $\mathcal{N}$ and $\{\underset{\sim}{B}{}^{\mathbb{B}}_{\alpha,\beta}:\alpha<\kappa\ \&\ \beta<\lambda\}$ is a basis for $\mathcal{M}$ ”

and

$(\circledast)_{15}$ if $\alpha_0\leq\alpha_1<\kappa$, $\beta_0\leq\beta_1<\lambda$, $(\alpha_0,\beta_0)\neq(\alpha_1,\beta_1)$, then

$$\Vdash_{\mathbb{P}^*_\gamma}\text{“ }\underset{\sim}{B}{}^{\mathbb{A}}_{\alpha_0,\beta_0}\subsetneq\underset{\sim}{B}{}^{\mathbb{A}}_{\alpha_1,\beta_1}\ \&\ \underset{\sim}{B}{}^{\mathbb{B}}_{\alpha_0,\beta_0}\subsetneq\underset{\sim}{B}{}^{\mathbb{B}}_{\alpha_1,\beta_1}\text{ ”}.$$

Also

$(\circledast)_{16}$ if $\alpha_0,\alpha_1<\kappa$, $\beta_0,\beta_1<\lambda$ and $\neg\big(\alpha_0\leq\alpha_1\ \&\ \beta_0\leq\beta_1\big)$ then

$$\Vdash_{\mathbb{P}^*_\gamma}\text{“ }\underset{\sim}{B}{}^{\mathbb{A}}_{\alpha_0,\beta_0}\not\subseteq\underset{\sim}{B}{}^{\mathbb{A}}_{\alpha_1,\beta_1}\ \&\ \underset{\sim}{B}{}^{\mathbb{B}}_{\alpha_0,\beta_0}\not\subseteq\underset{\sim}{B}{}^{\mathbb{B}}_{\alpha_1,\beta_1}\text{ ”}.$$

[Why? If $\alpha_1\leq\alpha_0$ and $\beta_1\leq\beta_0$, then $(\circledast)_{15}$ applies, so we may assume additionally $\neg\big(\alpha_1\leq\alpha_0\ \&\ \beta_1\leq\beta_0\big)$. Then our assumptions on $\alpha_0,\alpha_1,\beta_0,\beta_1$ mean that, letting $j=\pi(\alpha_0,\beta_0)$ and $i=\pi(\alpha_1,\beta_1)$, we have $i\notin a_j$, $j\notin a_i$, $i\neq j$. So using $(\circledast)_{13}$ for $\ell=0$ we easily build a $\mathbb{P}^*_\gamma$–name $\underset{\sim}{\eta}$ for a member of ${}^\omega 2$ such that

$$\Vdash_{\mathbb{P}^*_\gamma}\text{“ }\underset{\sim}{\eta}\in[\underset{\sim}{F}{}^0_i]\subseteq{}^\omega 2\setminus\underset{\sim}{F}{}^{\mathbb{A}}_i={}^\omega 2\setminus\underset{\sim}{B}{}^{\mathbb{A}}_{\alpha_1,\beta_1}\ \&\ \underset{\sim}{\eta}\in\underset{\sim}{F}{}^{\mathbb{A}}_j=\underset{\sim}{B}{}^{\mathbb{A}}_{\alpha_0,\beta_0}\text{ ”}.$$

Similarly, using $(\circledast)_{13}$ for $\ell=1$ and interchanging the role of i and j we may construct a $\mathbb{P}^*_\gamma$–name $\underset{\sim}{\eta}'$ such that $\Vdash_{\mathbb{P}^*_\gamma}$ “ $\underset{\sim}{\eta}'\in\underset{\sim}{B}{}^{\mathbb{B}}_{\alpha_0,\beta_0}\setminus\underset{\sim}{B}{}^{\mathbb{B}}_{\alpha_1,\beta_1}$ ”.]

Finally we note that $\mathbb{P}^*_\gamma$ has a dense subset of size $\lambda^{\aleph_0}$, so we may choose it as our desired forcing $\mathbb{Q}^{\kappa,\lambda}$. $\square$

Remark 3.8. In a manner similar to our proof of $(\circledast)_{13}$ above one may argue for the following stronger property.

$(\circledast)^2_{13}$ Suppose $i,j<\gamma$, $i\notin a_j$, $j\notin a_i$, $i\neq j$, $\ell\in\{0,1\}$. Assume that $p\in\mathbb{P}^*_\gamma$, $\eta\in {}^n2$, $n<\omega$ and $p\Vdash_{\mathbb{P}^*_\gamma}[\eta]\cap \underset{\sim}{F}^\ell_i\neq\emptyset$. Then there are $\nu_0,\nu_1\in {}^{[n,N)}2$, $n<N<\omega$ and $q\geq_{\mathbb{P}^*_\gamma} p$ such that $\nu_0\neq\nu_1$ and

$$q\Vdash_{\mathbb{P}^*_\gamma} \text{“ } [\eta^\frown\nu_0]\cap \underset{\sim}{F}^\ell_i\neq\emptyset\neq[\eta^\frown\nu_1]\cap\underset{\sim}{F}^\ell_i \text{ and } \big(\forall y\in \underset{\sim}{F}^\ell_j\big)\big(y{\restriction}[n,N)\notin\{\nu_0,\nu_1\}\big) \text{ ”.}$$

Then, if $i=\pi(\alpha_0,\beta_0)$, $j=\pi(\alpha_1,\beta_1)$, $i\notin a_j$, $j\notin a_i$ and $i\neq j$, we may use this property to construct $\mathbb{P}^*_\gamma$–names $\underset{\sim}{T}^{\mathbb{A}}$ and $\underset{\sim}{T}^{\mathbb{B}}$ for perfect subtrees of ${}^{\omega>}2$ such that

$$\Vdash_{\mathbb{P}^*_\gamma} \text{“ } [\underset{\sim}{T}^{\mathbb{A}}]\subseteq \underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0}\setminus \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1} \text{ and } [\underset{\sim}{T}^{\mathbb{B}}]\subseteq \underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0}\setminus \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1} \text{ ”.}$$

Also $(\circledast)_{15}$ can easily strengthen to

$(\circledast)^+_{15}$ if $\alpha_0\leq\alpha_1<\kappa$, $\beta_0\leq\beta_1<\lambda$, $(\alpha_0,\beta_0)\neq(\alpha_1,\beta_1)$, then

$$\Vdash_{\mathbb{P}^*_\gamma} \text{“ both } \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1}\setminus \underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0} \text{ and } \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1}\setminus \underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0} \text{ are uncountable ”.}$$

Consequently, in $\mathbf{V}^{\mathbb{P}^*_\gamma}$, the $\kappa\times\lambda$–bases $\{\underset{\sim}{B}^{\mathbb{A}}_{\alpha,\beta}:\alpha<\kappa,\ \beta<\lambda\}$ and $\{\underset{\sim}{B}^{\mathbb{B}}_{\alpha,\beta}:\alpha<\kappa,\ \beta<\lambda\}$ have the additional property that

$$\Vdash_{\mathbb{P}^*_\gamma} \text{“ } \alpha_0>\alpha_1\ \vee\ \beta_0>\beta_1 \quad\Rightarrow\quad \big|\underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0}\setminus \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1}\big| = \big|\underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0}\setminus \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1}\big| = 2^{\aleph_0} \text{ ”.}$$

This is used in Rosłanowski and Shelah [12].

Corollary 3.9. *It is consistent that*

- $\mathrm{add}(\mathcal{N})=\mathrm{add}(\mathcal{M})<\mathrm{cof}(\mathcal{N})=\mathrm{cof}(\mathcal{M})=2^\omega$ *(and hence the ideals $\mathcal{M},\mathcal{N}$ do not poses tower bases) , and*
- *there is a monotone Π^0_3 hull operation on $\mathcal{M}$ with respect to $\mathcal{M}$, and*
- *there is a monotone Π^0_2 hull operation on $\mathcal{N}$ with respect to $\mathcal{N}$, and*
- *there is a monotone Π^0_3 hull operation on $\mathcal{M}\cap\mathcal{N}$ with respect to $\mathcal{M}\cap\mathcal{N}$.*

Proof. Start with a universe satisfying CH and use the forcing given by Theorem 3.7 for $\kappa=\aleph_1$ and $\lambda=\aleph_2$. Propositions 3.6 and 3.5 imply that the resulting model is as required. □

Remark 3.10. In Theorem 3.7 we obtained a universe of set theory in which both $\mathcal{N}$ and $\mathcal{M}$ have bases that are (with respect to the inclusion) order isomorphic to $\kappa\times\lambda$. We may consider any partial order $(S,\sqsubseteq)$ such that

(a) $|S|=\lambda$ and $(S,\sqsubseteq)$ is well founded, and
(b) every countable subset of S has a common $\sqsubseteq$–upper bound.

Then by a very similar construction we get a forcing extension in which both $\mathcal{N}$ and $\mathcal{M}$ have bases order isomorphic to $(S,\sqsubseteq)$. If additionally

(c) for every sequence $\langle s_i:i<\omega_1\rangle\subseteq S$ there are $i<j<\omega_1$ such that $s_i\sqsubseteq s_j$,

then those bases will be mhg. (Note that forcings with the Knaster property preserve the demand described in (c).)

References

[1] Marek Balcerzak and Tomasz Filipczak. On monotone hull operations. *Mathematical Logic Quarterly*, 57:186–193, 2011.

[2] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line.* A K Peters, Wellesley, Massachusetts, 1995.

[3] Tomek Bartoszyński and Masaru Kada. Hechler's theorem for the meager ideal. *Topology Appl.*, 146/147:429–435, 2005.
[4] Jörg Brendle and Sakaé Fuchino. Coloring ordinals by reals. *Fundamenta Mathematicae*, 196:151–195, 2007.
[5] Maxim R. Burke and Masaru Kada. Hechler's theorem for the null ideal. *Arch. Math. Logic*, 43:703–722, 2004.
[6] Márton Elekes and Kenneth Kunen. Transfinite sequences of continuous and Baire class 1 functions. *Proc. Amer. Math. Soc.*, 131:2453–2457, 2003.
[7] Márton Elekes and András Máthé. Can we assign the Borel hulls in a monotone way? *Fundamenta Mathematicae*, 205:105–115, 2009.
[8] Jakob Kellner and Saharon Shelah. Decisive creatures and large continuum. *Journal of Symbolic Logic*, 74:73–104, 2009. math.LO/0601083.
[9] Kenneth Kunen. *Inaccessibility properties of cardinals*. PhD thesis, Stanford University, 1968.
[10] Heike Mildenberger and Saharon Shelah. Changing cardinal characteristics without changing ω–sequences or cofinalities. *Annals of Pure and Applied Logic*, 106:207–261, 2000. math.LO/9901096.
[11] Andrzej Roslanowski and Saharon Shelah. Norms on possibilities I: forcing with trees and creatures. *Memoirs of the American Mathematical Society*, 141(671):xii + 167, 1999. math.LO/9807172.
[12] Andrzej Roslanowski and Saharon Shelah. Around `cofin`. *Colloquium Mathematicum*, 134:211–225, 2014. 1304.5683.
[13] Saharon Shelah. Covering of the null ideal may have countable cofinality. *Fundamenta Mathematicae*, 166:109–136, 2000. math.LO/9810181.
[14] Saharon Shelah. Was Sierpiński right? IV. *Journal of Symbolic Logic*, 65:1031–1054, 2000. math.LO/9712282.
[15] Saharon Shelah. The null ideal restricted to some non-null set may be $\aleph_1$-saturated. *Fundamenta Mathematicae*, 179:97–129, 2003. math.LO/9705213.
[16] Saharon Shelah and Simon Thomas. The Cofinality Spectrum of The Infinite Symmetric Group. *Journal of Symbolic Logic*, 62:902–916, 1997. math.LO/9412230.

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