NICE $\aleph_1$ GENERATED NON-P-POINTS, I
SH980

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Abstract. We define a family of a (non-principal) ultrafilter on $\mathbb{N}$, i.e. a point which are very far from P-points. We first under reasonable conditions, prove its existence. In a continuation we shall prove that such a point may exist while no P-point exists.

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Anotated Content

§ 0. Introduction

P-point is an important notion in general topology and set theory of the reals. Recall here a P-point is a non-principal ultrafilter on $\mathbb{N}$ for which any countable subset has a lower bound modulo finite in the ultrafilter.

We have some knowledge on preservation of P-points by specific forcing and by say a CS iterated forcing, this is important in many applications; preservation of an ultrafilter means that the ultrafilter in the ground model $\mathbf{V}$ generate an ultrafilter in $\mathbf{V}^{\mathbb{P}_\delta}$, (see [Sh:f, VI]). Of course, a forcing notion $\mathbb{Q}$ preserving P-points (i.e. all $D \in \mathbf{P}_1$, see Definition 0.3 below) preserve every ultrafilter in the closure $\mathbf{P}_2$ of $\mathbf{P}_1$ under sums.

From our point of view the P-points are trackable for independence results because:

$\boxplus_1$ (A) there are quite many forcing notions preserving P-points

(B) a forcing notion $\mathbb{Q}$ which preserving "D being an ultrafilter" preserve its being a P-point (well, when $\mathbb{Q}$ is proper or even less)

(C) the preservation of P-points is preserved in limit for CS-iteration (together this gives a well controlled way to have ultrafilters generated by $\aleph_1 < 2^{\aleph_0}$ sets)

(D) we can destroy a P-point by forcing, i.e. ensure it has no extension to a P-point (so CON(no P-point))

(E) moreover we can "split hairs", i.e. destroy some P-point while preserving others "orthogonal" to it (in the right sense), so can have unique P-point up to isomorphisms

See [Sh:f, Ch.VI,Ch.XVIII,§4] and see the survey Blass [Bla] on the various points, i.e. special ultrafilters on $\mathbb{N}$ and history there.

We may wonder:

Question 0.1. 1) Can we find other ultrafilters preserved by say enough CS- iterations of suitable forcing notions? {z2}
2) In particualr for a limit ordinal δ, having been preserved by $\mathbb{P}_\alpha$ for $\alpha < \delta$ this holds for $\mathbb{P}_\delta$ when $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta \rangle$ a CS-iteration.

We suggest this problem in [Sh:666, 3.13] and speculate about it, even building an ultrafilter on ${}^{\omega >}\omega$ naturally with many quotient. Ultrafilters as in 0.1, for natural CS iterations are naturally generated by $\aleph_1$ sets; moreover CS-iteration is mainly interesting when we start with CH, and "preserve an ultrafilter" is meaningful only when we add reals, naturally $\aleph_2$ ones.

A posteriory this seemed related to the question on the existence of a point of van-Dowen [vD84], see below but at present we do not know neither if they are related nor how to answer it. We have tried to solve the following problems:

$\boxplus_2$ (A) the one of [Sh:666, 3.13], i.e. 0.1; some more specific problems were raised

(B) [Nyikos] is it consistent to have some ultrafilter $D \in \beta^*(N)\backslash\mathbb{N}$ of character $\aleph_1$, but no P-point

(C) [Dow] is it consistent to have $\mathfrak{u} = \aleph_1$, there is a P-point D but no P-point D with $\chi(D) = \aleph_1$,

(D) [van-Dowen] is it consistent that: there is no ultrafilter D on $\mathbb{Q}$ such that every $A \in D$ contains a member of D which is a closed set with no isolated points.

In the series of papers started here the main points are:

$\boxplus_3$ (A) we have an involved family of sets, really well founded trees, appearing in the definition

(B) each ultrafilter has no P-point as a quotient

(C) we have a game characterization

(D) such systems exists; assuming, e.g. $\diamondsuit_{\aleph_1}$

(E) enough relevant forcing notions preserve such systems, in particular, some serving $\boxplus_1(C)$ so answering $\boxplus_2(A)$, question 0.1(1)

(F) can solve Nyikos problem, see $\boxplus_2(B)$

(G) for Dow problem we cannot use shooting a set through an ultrafilter as this adds an unbounded real. Maybe we should try to devise a suitable creature iteration.

(H) we have a preservation theorem for such systems of ultrafilters under, e.g. CS-iteration, see 0.1(2), 0.1(1).

Concerning (H), really presently the condition are probably too strong but holds for many, in particularly those we consider. More specifically, ${}^\omega\omega$-bounding is necessary but we assume COM wins in the bounding games. We intend to deal with it later.

So in $\boxplus_2$ the first two problems (of myself and Nyikos) still be resolved here but not the last two (of Dow and of van-Dowen).

In the first part, i.e. the present work, first we define ultrafilters of the right kinds analgous to P-points but with no P-point as a quotient; this is done in §1,§2. In §3 we deal with necessary properties of forcing notion, intended for use in the independence results, e.g. sufficient conditions for a forcing $\mathbb{Q}$ that: for every old CWT (countable well founded tree) any colouring of the maximal nodes, by 2 colours (in $\mathbf{V}^{\mathbb{Q}}$) contains a monochromatic positive subtree.

In §4 we start to deal with the kind of forcing notion we would like to iterate.

In the second part (presently the first half of [Sh:F1127]) we present those ultrafilters in a more general framework and deal with sufficient conditions for such an ultrafilter to generate an ultrafilter in a suitable generic extension. For the limit case we continue the proof of preservation theorems in [Sh:f], in particular [Sh:f, Ch.VI,1.26,1.27] and Case A with transitivity of [Sh:f, Ch.XVIII,§3]. For the successor case we need that the relevant forcing, $\mathbb{Q}^3_D$, preserve our ultrafilters. We conclude finishing the proof fo CON($\mathfrak{u} = \aleph_1 +$ no P-points).

In the third part (in what should be the second half of [Sh:F1127]) we note that the ultrafilters so far were really analogous to selective (= Ramsey) ultrafilters and give more general framework which really includes P-points.

We thank Alan Dow for asking me about $\boxplus_2(B),(C)$ and for some comments.

{z7}

Remark 0.2. There may be P-point while $\mathfrak{d} > \aleph_1$, see Blass-Shelah [BsSh:242] and references there, but the existence of ultrafilters in the direction here, far from P-point, implies $\mathfrak{d} = \aleph_1$, see the survey of Blass [Bla]. But note that the ultrafilter may be $\aleph_1$-generated in a different sense: union of $\aleph_1$ families of the form $\mathrm{fil}(B) \cap \mathscr{P}(\max(B))$, see 1.2(3E).

Note that it may be harder (than in the P-point case) to build such ultrafilters as here which are μ-generated instead of $\aleph_1$-generated because of the unbounded countable depth involved. We have not looked at this as well as at the natural variants of our definition (not to speak of generalization to reasonable ultrafilters (see [Sh:830] and Roslanowski-Shelah [RoSh:890]).

Originally the idea was to have a system of ultrafilters on $\mathbb{N}$ rather than one. We have nice argument for the naturality of and interest in this approach but eventually we have to discard it, still the system of trees $A^{\mathbf{x}}_\eta, \eta \neq \eta^{\mathbf{x}}_{\mathrm{tr}}$ remains.

Our strategy was to build a system $\langle D_t : t \in T \rangle$ of ultrafilters by a sequence of countable approximations, for each approximation $\mathbf{x}, D^{\mathbf{x}}_t$ look like a member of $\mathbf{P}_2$. We try to use games in which more and more of the ultrafilters are involved, thinking that games will help in the preservation. Another possible way to prove preservation, was using this and nep and faking (see [Sh:630]) we have tried to show that those ultrafilters are preserved by forcing notion which preserve P-points (and are nep enough), i.e. preserve each D_t by faking: the faking is reasonable as our ultrafilters locally (i.e. in some countable N) look like members of $\mathbf{P}_2$. This has not worked out, still we mention those original definitions.

Definition 0.3. 1) Let $\mathbf{P}_1$ be the set of P-points, which are ultrafilters on countable sets, $\mathbf{P}_2$ its closure under sums. {z9}
2) We say D is a Q-point <u>when</u> D is an ultrafilter on a countable set $\mathrm{Dom}(D)$ such that if f is a finite-to-one function with domain $\mathrm{Dom}(D)$, then $f{\restriction}A$ is one-to-one for some $A \in D$.
3) [The Rudin-Keisler order on ultrafilters.] We let D_ℓ be an ultrafilter on $\mathscr{U}_\ell$ for $\ell = 1, 2$. We say $D_1 \leq_{\mathrm{RK}} D_2$ <u>iff</u> some function h witness it which means:

- $\mathrm{Dom}(h) \in D_2$
- $\mathrm{Rang}(h) \in D_1$ and
- $A \in D_1 \Leftrightarrow \{a \in \mathrm{Dom}(h) : h(a) \in A\} \in D_2$ for every $A \subseteq \mathscr{U}_1$.

Definition 0.4. For $\mathscr{X} \subseteq \mathscr{P}(\mathbb{N})$ let $\mathrm{fil}(\mathscr{X})$ be the filter on $\mathbb{N}$ generated by $\mathscr{X}$ and the co-finite subsets of $\mathbb{N}$. {z13}

§ 1. System of filters using well founded trees

{a2}

Notation 1.1. M is a partial order, T is tree here means in the model theoretic sense, i.e. for every $\eta \in T$ the set $\{\nu \in T : \nu <_T \eta\}$ is linearly ordered by $<_T$ but not necessarily well founded. Here B denotes a subset of T or of a partial order M inheriting its order.
1) For $B \subseteq M$ we let $\max_M(B) = \{\eta \in B : B \cap M_{>\eta} = \emptyset\}$.
2) We say Y is a front of $B \subseteq M$ <u>when</u> $Y \subseteq B$ and every branch of B meet Y and the members of Y are pairwise $<_M$-incomparable.
3) $\mathrm{suc}_B(\eta) = \mathrm{suc}(\eta, B) = \mathrm{suc}_T(\eta, B) = \{\nu \in B : \eta <_T \nu$ but for no $\rho \in B$ do we have $\eta <_M \rho <_M \nu\}$; below we may write $\mathrm{suc}_M(\eta, B)$ or $\mathrm{suc}_{\mathbf{x}}(\eta, B)$ when $B \subseteq M$ or $B \subseteq M_{\mathbf{x}}$.
4) $B_{\geq \eta} = \{\nu \in B : \eta \leq \nu\}$ and similarly $B_{>\eta}$ for $\eta \in B$.

{a5}

Definition 1.2. For a partial order M; writing $\mathbf{x}$ instead of M means $M_{\mathbf{x}}$, see Definition 1.5.
1) Let $\mathrm{CWT}(M)$ be the set of $B \subseteq M$ such that (CWT stands for countable well-founded sub-tree):

- (a) B is a countable subset of M
- (b) B has a $<_M$-minimal member called its root, $\mathrm{rt}(B)$
- (c) B, i.e. $(B, <_M \restriction B)$ is a tree with $\leq \omega$ levels (so well ordered!) and no ω-branch
- (d) for each $\nu \in B$ the set $\mathrm{suc}_B(\nu) = \{\rho \in B : \nu <_T \rho$ and $\neg(\exists \rho)(\nu <_T \rho <_T \rho)\}$ is empty or infinite.

2) For $B \in \mathrm{CWT}(M)$ let: $\mathrm{frt}(B) = \mathrm{frt}_M(B) = \mathrm{frt}(B, <_M \restriction B)$ be the set of fronts I of B, which in this case means maximal set of pairwise incomparable members.
2A) For antichains Y_1, Y_2 of M we say Y_2 is above Y_1 when $(\forall \eta \in Y_2)(\exists \nu \in Y_1)[\nu \leq_M \eta]$ and this will be used mainly for $Y_1, Y_2 \in \mathrm{frt}(B), B \in \mathrm{CWT}(M)$.
2B) For Y_1, Y_2 as above let the projection $h_{Y_1,Y_2} : Y_2 \to Y_1$ be the unique function h such that $h(\eta) \leq_M \eta$ for $\eta \in Y_2$.
2C) If Y_1, Y_2 are from $\mathrm{frt}(B)$ <u>then</u> Y_2 is almost above Y_1 <u>when</u> for some $B' \in \mathrm{sb}(B)$, see below, $B' \cap Y_2$ is above $B' \cap Y_1$, still we can define h_{Y_1,Y_2} but its domain is not Y_2 but $\{\eta \in Y_2 : (\exists \nu \in Y_1)(\nu \leq_M \eta)\}$.
3) For $B \in \mathrm{CWT}(M)$ let $\mathrm{sb}_M(B)$ be the set of exhaustive subtrees B' of B where B' is an exhaustive subtree of B <u>when</u>:

- (a) $B' \subseteq B$
- (b) $\mathrm{rt}(B') = \mathrm{rt}(B)$
- (c) if $\nu \in B'$ then $\mathrm{suc}_B(\nu) \backslash \mathrm{suc}_{B'}(\nu)$ is finite.

3A) For $B \in \mathrm{CWT}(M)$ and $Y \in \mathrm{frt}_M(B)$ let $D_{B,Y} = D_{M,B,Y}$ be the filter on Y generated by $E_{B,I} = E_{M,B,I} := \{Y \cap B' : B'$ is an exhaustive subtree of B, i.e. $\in \mathrm{sb}_M(B)\}$.
3B) For B as above and $Y \subseteq B$ let $B[\leq Y] = \{\nu \in B : (\exists \eta)(\nu \leq_T \eta \in Y\}$.
3C) In part (3), we say f witness "$B' \in \mathrm{sb}(B)$" if $f : B' \backslash \max(B) \to [B]^{<\aleph_0}$ satisfies $\nu \in B' \backslash \max(B) \Rightarrow \mathrm{suc}_B(\nu) \backslash \mathrm{suc}_{B'}(\nu) \subseteq f(\nu)$ so only $f \restriction B'$, in fact only $f \restriction B'[\leq Y]$ matters when we are interested in $D_{M,B,Y}$.

3D) For $B \in \mathrm{CWT}(M)$ let $\mathrm{psb}_M(B), p$ for positive, be the set of positive subtrees B' of B which means (a),(b) as in part (3) above and

$(c)'$ if $\nu \in B' \backslash \max(B)$ then $\mathrm{suc}_{B'}(\nu)$ is infinite.

3E) For B as above let $\mathrm{fil}_M(B) = \{X \subseteq B : X$ contains some $B' \in \mathrm{sb}_M(B)\}$.
4) Let $\mathrm{alm}-\mathrm{frt}(B) = \mathrm{alm}-\mathrm{frt}_M(B)$ be the set of almost fronts of B, i.e. set of $Y \subseteq M$ an antichain such that $Y \cap B'$ is a front of B' for some $B' \in \mathrm{sb}(B)$.
4A) For $Y \in \mathrm{alm}-\mathrm{frt}_M(B)$ let $\mathrm{fil}_M(X, B) = \{X \subseteq Y$: for some $B' \in \mathrm{sb}_M(B)$ we have $X \supseteq B' \cap Y\}$; if $M = B$ we may write $\mathrm{fil}(Y, B)$ or $\mathrm{fil}_B(Y)$.
5) Let $\leq^*_M$ be the following two-place relation (actually a partial order) on $\mathrm{CWT}(M)$:
$B_1 \leq^*_M B_2$ iff ($B_1, B_2 \in \mathrm{CWT}(M)$ and) $\mathrm{rt}(B_1) = \mathrm{rt}(B_2)$ and for some $B'_2 \in \mathrm{sb}_M(B_2)$, we have

(a) $B'_2 \cap B_1 \in \mathrm{psb}(B_1)$
(b) every almost front of $B'_2 \cap B_1$ is an almost front of B_2.

5A) The default value of $Y \in \mathrm{frt}(B)$ is $\max(B) = \{\nu \in B : \nu$ is $<_M$-maximal in $B\}$.
5B) For $B \in \mathrm{CTW}(M)$ let $\mathrm{qsb}_M(B) = \{B' \in \mathrm{CWT}(M) : B \leq^*_M B'\}$.
6) $\mathrm{Dp}_M(B)$ is the depth of B, i.e. $\mathrm{Dp}_M(B) = \sup\{\mathrm{Dp}_M(B_{\geq\eta})+1 : \nu \in B\backslash\{\mathrm{rt}_M(B)\}\}$.

{a7}

Remark 1.3. We may use more almost fronts.

{a11}

Observation 1.4. Let M be a partial order.
1) We have $B_1 \leq^*_M B_2$ iff ($B_1, B_2 \in \mathrm{CWT}(M)$ and) every almost front Y_1 of B_1 is an almost front of B_2.
2) For $B \in \mathrm{CWT}(M)$, $\max(B)$ is a front of B and also $\{\mathrm{rt}(B)\}$ is.
3) Every front of $B \in \mathrm{CWT}(M)$ is an almost front of B.
4) If $B \in \mathrm{CWT}(M)$ then $\mathrm{Dp}_M(B)$ is a countable ordinal.

Proof. Straight. $\square_{1.4}$

{7g.1}

Definition 1.5. Let $\mathbf{K}$ be the class of the objects $\mathbf{x}$ which consists of the following objects satisfying the following properties:

(a) $\bar{\mathscr{A}}_{\mathbf{x}} = \bar{\mathscr{A}} = \langle \mathscr{A}_\eta : \eta \in M \rangle = \langle \mathscr{A}^{\mathbf{x}}_\eta : \eta \in M_{\mathbf{x}} \rangle$ and $\mathscr{A}_{\mathbf{x}} = \cup\{\mathscr{A}_\eta : \eta \in M^-_{\mathbf{x}}\}$
(b) $M = M_{\mathbf{x}} = M[\mathbf{x}]$ is a partial order with a root $\mathrm{rt}_{\mathbf{x}} = \mathrm{rt}(\mathbf{x})$ so the partial order is $<_M = <_{M[\mathbf{x}]}$; and let $M^-_{\mathbf{x}} = M_{\mathbf{x}} \backslash \{\mathrm{rt}_{\mathbf{x}}\}$
(c) $\mathscr{A}_\eta \subseteq \mathrm{CWT}(M)$, let $\mathscr{A}^-_\eta = \mathscr{A}_\eta \backslash \{\{\eta\}\}$
(d) $\mathrm{rt}(B) = \eta$ for every $B \in \mathscr{A}_\eta$
(e) $\mathscr{A}_\eta$ is not empty, in fact $\{\eta\} \in \mathscr{A}_\eta$
(f) $\mathscr{B}_{\mathbf{x}} = \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})}$ and $\leq_{\mathbf{x}}$ is a directed partial order on $\mathscr{B}_\eta$ for $\eta = \mathrm{rt}_{\mathbf{x}}$
(g) $B_1 \leq_{\mathbf{x}} B_2$ implies $B_1 \leq^*_T B_2$, see 1.2(5) and, of course, $B_1, B_2 \in \mathscr{A}_{\mathrm{rt}(\mathbf{x})}$
(h) (α) if $\nu \in B \in \mathscr{A}_\eta$ then $B \cap M_{\geq\nu} \in \mathscr{A}_\nu$
(β) if $\nu \in B \in \mathscr{B}_{\mathbf{x}}$ and $\nu \neq \mathrm{rt}_{\mathbf{x}}$ then $B_{\geq\nu} \in \mathscr{A}_\nu$.

{7g.4}

Definition 1.6. Let $\mathbf{x} \in \mathbf{K}$ and $\eta \in M_{\mathbf{x}}$; below we may omit $\mathbf{x}$ when clear from the context.
1) Let $\mathrm{frt}(\eta) = \mathrm{frt}_{\mathbf{x}}(\eta) = \{Y : Y$ is a front of B for some $B \in \mathscr{A}^{\mathbf{x}}_\eta\}$ (and $\mathrm{frt}_{\mathbf{x}}(B)$ similarly). Omitting η means $\eta = \mathrm{rt}_{\mathbf{x}}$.

2) Let $\mathrm{frt}^-(\eta) = \{Y \in \mathrm{frt}(\eta) : Y \neq \{\eta\}\}$. Similarly $\mathrm{alm}-\mathrm{frt}_{\mathbf{x}}(\eta)$, using 1.2(4).
3) Let $\mathrm{Fin}_{\mathbf{x}}(B) = \{f : f$ is a function with domain $B \backslash \max(B)$ such that $f(\nu) \in [\mathrm{suc}_B(\nu)]^{<\aleph_0}\}$ and let $A_f = A_{B,f} = A_{\mathbf{x},B,f} = \{\eta \in B$: if $\rho <_T \varrho \leq_T \eta$ and $\varrho \in \mathrm{suc}_B(\rho)$ so $\rho, \varrho \in B$ then $\varrho \notin f(\rho)\}$. Recall 1.2(3C).
3A) Assume $Y \in \mathrm{frt}_{\mathbf{x}}$ say $Y \in \mathrm{frt}(B_1)$ or just $Y \in \mathrm{alm}-\mathrm{frt}_{\mathbf{x}}(B_1), B_1 \in \mathscr{B}_{\mathbf{x}}$; we let $D_Y = D^{\mathbf{x}}_Y = D_{\mathbf{x}}(Y)$ (see 1.7(2)) be the filter on Y generated by $\{B \cap Y \cap A_{B,f} : B \in \mathscr{B}_{\mathbf{x}}$ is $B_1 \leq_{\mathbf{x}} B$ and $f \in \mathrm{Fin}_{\mathbf{x}}(B)\}$.
3B) Also if $B \in \mathscr{B}_{\mathbf{x}}, Y \subseteq M$ and Y is an almost front of B then $D_Y = D^{\mathbf{x}}_Y$ is $\{Z \subseteq Y$: for some $B \in \mathscr{B}_{\mathbf{x}}$ and $B' \in \mathrm{sb}(B)$ we have $Y \in \mathrm{alm}-\mathrm{frt}$ of B and $B' \cap Y \subseteq Z\}$; we could use $A_{B,f}$'s as in (3A) and vice versa.
3C) Lastly, if $B \in \mathscr{B}_{\mathbf{x}}$ then $D^{\mathbf{x}}_B = D_{\mathbf{x}}(B) = D_{\mathbf{x}}(\max(B))$.
4) If $\eta \in M_{\mathbf{x}}, B \in \mathscr{A}^{\mathbf{x}}_\eta$ let $\mathrm{Dp}_{\mathbf{x}}(B)$ be $\mathrm{Dp}_{T_{\mathbf{x}}}(B)$ as defined in 1.2(6) and let $\mathrm{Dp}_{\mathbf{x}}(\eta) = \sup\{\mathrm{Dp}_{\mathbf{x}}(B) + 1 : B \in \mathscr{A}^{\mathbf{x}}_\eta\}$.

{7g.5g} **Observation 1.7.** Assume $\mathbf{x} \in \mathbf{K}, \eta = \mathrm{rt}_{\mathbf{x}}$ and $Y \in \mathrm{frt}_{\mathbf{x}}$.
1) $\{\eta\} \in \mathrm{frt}_{\mathbf{x}}(\eta)$ and $D^{\mathbf{x}}_{\{\eta\}} = \{\{\eta\}\}$ (and η is uniquely determined).
2) D_Y is a filter on Y.
3) If $B \in \mathscr{B}_{\mathbf{x}}$ and $Y_1, Y_2 \in \mathrm{frt}_{\mathbf{x}}(B)$ and Y_2 is above Y_1 and $h : Y_2 \to Y_1$ is the projection, i.e. $h(\nu_2) = \nu_1 \Leftrightarrow \nu_1 \in Y_1 \wedge \nu_2 \in Y_2 \wedge \nu_1 \leq_T \nu_2$, see 1.2(2B); then $h(D_{Y_2}) = D_{Y_1}$ so h witnesses $D_{Y_1} \leq_{\mathrm{RK}} D_{Y_2}$, i.e. $D_{Y_1} = \{A_1 \subseteq Y_1$: for some $A_2 \in D_2, \{h(\eta) : \eta \in A_2 \cap h^{-1}(Y_1)\} = A_1\}$, in particular $\mathrm{Rang}(h) \in D_{Y_1}$, see part (4).
4) If $B_1 \leq_{\mathbf{x}} B_2$ and $Y_1 \in \mathrm{frt}(B_1)$ and $Y_2 = Y_1 \cap B_2$ hence $\in \mathrm{alm}-\mathrm{frt}_{\mathbf{x}}(B_2)$ <u>then</u> $Y_2 \in D^{\mathbf{x}}_{Y_1}$ and $D^{\mathbf{x}}_{Y_2} = D^{\mathbf{x}}_{Y_1} \restriction Y_2$.
5) If $B_1 \leq_{\mathbf{x}} B_2$ and $Y_\ell = \mathrm{suc}_{\mathbf{x}}(\eta, B_\ell)$ for $\ell = 1, 2$ <u>then</u>:

- (a) Y_ℓ is a front of $(B_\ell)_{\geq \eta}$ and Y_1 almost above Y_2, see 1.2(2C)
- (b) if Y is a front of $(B_\ell)_{\geq \eta}$ and $\neq \{\eta\}$ then Y is above Y_ℓ.

6) $\max(B)$ is the maximal front of B which means that it is above any other.
7) If $\mathbb{Q}$ is an ${}^\omega\omega$-bounding forcing, $\mathbf{x} \in \mathbf{K}, B \in \mathscr{B}_{\mathbf{x}}$ <u>then</u> for any $B' \in \mathrm{sb}_{\mathbf{x}}(B)^{\mathbf{V}[\mathbb{Q}]}$ there is $B'' \in (\mathrm{sb}_{\mathbf{x}}(B))^{\mathbf{V}}$ such that $B'' \subseteq B'$.

Proof. 1) By the definition.
2)-8) Also easy. $\square_{1.7}$

{7g.7} **Definition 1.8.** 1) For an (infinite) cardinal κ let $\mathbf{K}_{<\kappa}$ be the class of $\mathbf{x} \in \mathbf{K}$ such that $\|\mathbf{x}\| := |M_{\mathbf{x}}| + \Sigma\{|\mathscr{A}^{\mathbf{x}}_\eta| : \eta \in M_{\mathbf{x}}\} < \kappa$, similarly $\mathbf{K}_{\leq\kappa}$.
2) $\leq_{\mathbf{K}}$ is the following two-place relation on $\mathbf{K}$; (it is a partial order, see 1.9 below): $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ <u>iff</u>:

- (a) $M_{\mathbf{x}} \subseteq M_{\mathbf{y}}$ (as partial orders)
- (b) $\eta \in M_{\mathbf{x}} \Rightarrow \mathscr{A}^{\mathbf{x}}_\eta \subseteq \mathscr{A}^{\mathbf{y}}_\eta$
- (c) $\mathrm{rt}_{\mathbf{y}} = \mathrm{rt}_{\mathbf{x}}$, really follows by (d)
- (d) $\leq_{\mathbf{x}} = \leq_{\mathbf{y}} \restriction \mathscr{B}_{\mathbf{x}}$.

3) If $\langle \mathbf{x}_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbf{K}}$-increasing sequence we define $\mathbf{x}_\delta = \cup\{\mathbf{x}_\alpha : \alpha < \delta\}$, the union of the sequence by $M_{\mathbf{x}_\delta} = \cup\{M_{\mathbf{x}_\alpha} : \alpha < \delta\}$ as partial orders and $\mathscr{A}^{\mathbf{x}_\delta}_\eta = \cup\{\mathscr{A}^{\mathbf{x}_\alpha}_\eta : \alpha < \delta$ satisfies $\eta \in M_{\mathbf{x}_\alpha}\}$ and $\leq_{\mathbf{x}_\delta} = \cup\{\leq_{\mathbf{x}_\alpha} : \alpha < \delta\}$.
4) We say $\mathbf{x} \in \mathbf{K}$ is principal <u>when</u> there is a B such that $B \in \mathscr{B}_{\mathbf{x}}$ is $\leq_{\mathbf{x}}$-maximal.
5) We say $\mathbf{x} \in \mathbf{K}$ is countable when $\|\mathbf{x}\| \leq \aleph_0$.

6) $\mathbf{K}_{\rm uf}$ is the class of $\mathbf{x} \in \mathbf{K}$ such that $D^{\mathbf{x}}_B$ is an ultrafilter on $\max(B)$ for every $B \in \mathscr{B}_{\mathbf{x}}$.

{7g.9}

Claim 1.9. *1) $\le_{\mathbf{K}}$ is really a partial order.*
2) If $\langle \mathbf{x}_\alpha : \alpha < \delta\rangle$ is $\le_{\mathbf{K}}$-increasing <u>then</u> $\mathbf{x}_\delta$, the union of the sequence is a $\le_{\mathbf{K}}$-lub of the sequence and $\|\mathbf{x}_\delta\| \le \Sigma\|\{\|\mathbf{x}_\alpha\| : \alpha < \delta\}$.

{7g.12}

Remark 1.10. 1) We can use $\le^+_{\mathbf{K}}$: $\mathbf{x} \le^+_{\mathbf{K}} \mathbf{y}$ iff $\mathbf{x} \le_{\mathbf{K}} \mathbf{y}$ and if $B_2 \in \mathscr{B}_{\mathbf{y}} \backslash \mathscr{B}_{\mathbf{x}}$ and $B_1 \in \mathscr{B}_{\mathbf{x}}$ <u>then</u> $B_1 \le^+_{\mathbf{x}} B_2$ where
2) $B_1 \le^+_{\mathbf{x}} B_2$ when for some $B'_1 \in \mathrm{sb}_{\mathbf{x}}(B_1)$ we have $\eta \in B_2 \cap B'_1 \wedge \eta <_T \nu \in B_2 \cap B'_1 \Rightarrow \nu \in B_2 \cap B'_1$.

We can use "fat" $\mathbf{x}$, this is natural for $\mathbf{x} \in \mathbf{K}_{\aleph_1}$ when $\mathfrak{d} = \aleph_1$.

{7g.14}

Definition 1.11. 1) We say $\mathbf{x} \in \mathbf{K}$ is fat <u>when</u>: if $B \in \mathscr{A}^{\mathbf{x}}_\eta$ and $B' \in \mathrm{sb}(B)$ then there is $B'' \in \mathrm{sb}(B')$ which belongs to $\mathscr{A}^{\mathbf{x}}_\eta$ and $\eta = \mathrm{rt}_{\mathbf{x}} \Rightarrow B \le_{\mathbf{x}} B''$.
2) We say that $\mathbf{x} \in \mathbf{K}$ is strongly big when: if $B \in \mathscr{A}^{\mathbf{x}}_\eta$ where $\eta \in M_{\mathbf{x}}$ and $\mathbf{c} : \max(B) \to \{0,1\}$ <u>then</u> for some $B' \in \mathscr{A}^{\mathbf{x}}_\eta$ we have $B' \in \mathrm{psb}(B) \cap \mathscr{A}^{\mathbf{x}}_\eta, \mathbf{c}{\restriction}\max(B')$ is constant and $\eta = \mathrm{rt}_{\mathbf{x}} \Rightarrow B \le_{\mathbf{x}} B'$.
2A) We say $\mathbf{x} \in \mathbf{K}$ is weakly big <u>when</u> in part (2) we restrict ourselves to $\eta \in M^-_{\mathbf{x}}$.
3) We say $\mathbf{x} \in \mathbf{K}$ is strongly large <u>when</u> if $B \in \mathscr{A}^{\mathbf{x}}_\eta$ where $\eta \in M_{\mathbf{x}}$ and $\mathbf{c}$ is a function with domain $\max(B)$ then for some $B' \in \mathrm{psb}(B) \cap \mathscr{A}^{\mathbf{x}}_\eta$ and front Y of B' we have $\mathbf{c}(\eta) = \mathbf{c}(\nu)$ <u>iff</u> for every $\eta, \nu \in \max(B')$ we have $\mathbf{c}(\eta) = \mathbf{c}(\nu) \Leftrightarrow (\exists \rho \in Y)(\rho \le_{M_{\mathbf{x}}} \eta \wedge \rho \le_{M_{\mathbf{x}}} \nu)$.
3A) We say $\mathbf{x} \in \mathbf{K}$ is weakly large <u>when</u> in part (3) we restrict ourselves to $\eta \in M^-_{\mathbf{x}}$.
4) We say $\mathbf{x} \in \mathbf{K}$ is full <u>when</u>: if $\eta \in M^-_{\mathbf{x}}$ and $B \in \mathscr{A}^{\mathbf{x}}_\eta$ then $\mathrm{psb}_{\mathbf{x}}(B) \subseteq \mathscr{A}^{\mathbf{x}}_\eta$.

[repetition: naturally related is (see 3.3).]

{7g.18}

Definition 1.12. 1) We say $\mathscr{P} \subseteq [\mathbb{N}]^{\aleph_0}$ is big <u>when</u> for every $\mathbf{c} : \mathbb{N} \to \{0,1\}$ there is $A \in \mathscr{P}$ such that $\mathbf{c}{\restriction}A$ is constant.
2) For $B \in \mathrm{CTW}({}^{\omega>}\omega, \triangleleft)$ we say $\mathscr{B} \subseteq \mathrm{psb}(B)$ is big (in B) <u>when</u> for every $\mathbf{c} : \max(B) \to \{0,1\}$ there is $B' \in \mathscr{B}$ such that $\mathbf{c}{\restriction}\max(B')$ is constant.
3) For $B \in \mathrm{CTW}({}^{\omega>}\omega, \triangleleft)$ we say $\mathscr{B} \subseteq \mathrm{psb}(B)$ is large (in B) <u>when</u> for every function $\mathbf{c}$ with domain $\max(B)$ there is $B' \in \mathscr{B}$ and front Y of B' such that for every $\eta, \nu \in \max(B')$ we have $\mathbf{c}(\eta) = \mathbf{c}(\nu) \Leftrightarrow (\exists \rho \in Y)(\rho \le_B \nu \wedge \rho \le_B \eta)$.

§ 2. Construction of ultra-systems and games

{8h.3}

Claim 2.1. $\mathbf{K}_{\leq\aleph_0}$ *is non-empty.*

Proof. Let $M_{\mathbf{x}} = \{\eta_*\}, \mathscr{A}^{\mathbf{x}}_{\eta_*} = \{\{\eta_*\}\}$, $\mathrm{rt}_{\mathbf{x}} = \eta_*$.

Now it is easy to check. $\square_{2.1}$

{8h.7}

Claim 2.2. *If $\mathbf{x} \in \mathbf{K}$ and $\eta \in M_{\mathbf{x}}$ satisfies $|\mathscr{A}^{\mathbf{x}}_\eta| = 1$, i.e. $\mathscr{A}^{\mathbf{x}}_\eta = \{\{\eta\}\}$ <u>then</u> for some $\mathbf{y} \in \mathbf{K}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}, |\mathscr{A}^{\mathbf{y}}_\eta| > 1$ and $\|\mathbf{y}\| \leq \|\mathbf{x}\| + \aleph_0$.*

Proof. Let $\langle \eta_n : n < \omega\rangle$ be pairwise distinct and $\notin M_{\mathbf{x}}$. We define $\mathbf{y}$ by:

- (a) $M_{\mathbf{y}}$ has set of elements $M_{\mathbf{x}} \cup \{\eta_n : n < \omega\}$
- (b) $\nu <_{\mathbf{y}} \rho$ <u>when</u> $\nu <_{T_{\mathbf{x}}} \rho$ or $\nu \leq_{\mathbf{x}} \eta \wedge (\exists n)(\rho = \eta_n)$
- (c) $\mathscr{A}^{\mathbf{y}}_\nu$ is:
  - (α) $\mathscr{A}^{\mathbf{x}}_\nu$ when $\nu \in M_{\mathbf{x}}\backslash\{\eta\}$
  - (β) $\{\{\eta\}, \{\eta_n : n < \omega\} \cup \{\eta\}\}$ when $\nu = \eta$
  - (γ) $\{\{\eta_n\}\}$ when $\nu = \eta_n$
- (d) the order $<_{\mathbf{y}}$ is $<_{\mathbf{x}}$ if $\eta \neq \mathrm{rt}_{\mathbf{x}}$ and $\{(\{\eta\}, \{\eta_n : n < \omega\} \cup \{\eta\})\}$ if $\eta = \mathrm{rt}_{\mathbf{x}}$.

Now check. $\square_{2.2}$

{8h.10}

Claim 2.3. *1) If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$ <u>then</u> for some $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and in $\mathscr{B}_{\mathbf{y}}$ there is a $\leq_{\mathbf{y}}$-maximal member.*
2) If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$ and some $B \in \mathscr{B}_{\mathbf{x}}$ is $\leq_{\mathbf{x}}$-maximal <u>then</u> for some $\mathbf{y}$ and B' we have $\mathbf{x} \leq \mathbf{y} \in \mathbf{K}_{\leq\aleph_0}, B' \in \mathscr{B}_{\mathbf{y}}$ and $B <_{\mathbf{y}} B'$.
3) If $\mathbf{x} \in \mathbf{K}_{\aleph_0}, \eta \in M^-_{\mathbf{x}}, B_1 \in \mathscr{A}^{\mathbf{x}}_\eta$ and $B_2 \in \mathrm{psb}_{\mathbf{x}}(B_1)$ <u>then</u> there is $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ such that $x \leq_{\mathbf{K}} y$ and $B_2 \in \mathscr{A}^{\mathbf{y}}_\eta$.

Proof. 1) If in $(\mathscr{B}_{\mathbf{x}}, \leq_{\mathbf{x}})$ there is a maximal member then we let $\mathbf{y} = \mathbf{x}$. Otherwise, as it is directed (see clause (f) of Definition 1.5) and $\|\mathbf{x}\| \leq \aleph_0$ because $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$, there is a $<_{\mathbf{x}}$-increasing cofinal sequence $\langle B_n : n < \omega\rangle$. Let $Y_n = \mathrm{suc}_{\mathbf{x}}(\mathrm{rt}_{\mathbf{x}}, B_n)$.

Note that for each $n, \langle Y_m \cap B_n : m \leq n\rangle$ is a sequence of almost fronts of B_n. So when $m_1 < m_2 \leq n$ we have $Y_{m_1} \cap B_n$ is almost above $Y_{m_2} \cap B_n$, hence for some $B_{m_1,m_2,n} \in \mathrm{sb}_{\mathbf{x}}(B_n)$ we have "$Y_{m_1} \cap B_{m_1,m_2,n}$ is above Y_{m_2}". Let $B^*_n := \cap\{B_{m_1,m_2,n} : m_1 < m_2 \leq n\}$, clearly B^*_n belongs to $\mathrm{sb}_{\mathbf{x}}(B_n)$.

We now choose $\bar\nu_n = \langle \nu_{n,\ell} : \ell \leq n\rangle$ by induction on $n < \omega$ such that

⊛ (a) $\nu_{n,\ell} \in Y_\ell \cap B^*_\ell$
- (b) $\nu_{n,\ell+1} \leq_M \nu_{n,\ell}$
- (c) $\nu_{n+1,m_1}, \nu_{n_1,m_2}$ are $<_M$-incomparable for $n_1 \leq n, m_1 \leq n+1, m_2 \leq n_1$.

This is easy.

Now let $B_* = \{\eta\} \cup \{B_n \cap M_{\geq \nu_{n,n}} : n < \omega\}$.

Lastly, we define $\mathbf{y}$:

⊛ (a) $M_{\mathbf{y}} = M_{\mathbf{x}}$
- (b) $\mathscr{A}^{\mathbf{y}}_\nu = \mathscr{A}^{\mathbf{x}}_\nu$ <u>when</u> $\nu \in M_{\mathbf{x}}\backslash\{\eta\}$
- (c) $\mathscr{A}^{\mathbf{y}}_\eta = \mathscr{A}^{\mathbf{x}}_\eta \cup \{B_*\}$
- (d) $B_1 \leq^{\mathbf{y}}_\eta B_2$ <u>iff</u> $B_1 \leq^{\mathbf{x}}_\eta B_2$ or $B_1 \in A^{\mathbf{y}}_\eta \wedge B_2 = B_*$ or $B_1 = B_2 = B_*$.

Now check.
2) Similarly to part (1) just easier and follows by 2.4 below.
3) Easy. $\square_{2.3}$

{8h.15}

Claim 2.4. *1) If $\mathbf{x} \in \mathbf{K}_{\le\aleph_0}, Y \in \mathrm{alm} - \mathrm{frt}_{\mathbf{x}}$ and $Z \subseteq Y$ <u>then</u> for some $\mathbf{y} \in \mathbf{K}_{\le\aleph_0}$ we have $\mathbf{x} \le_{\mathbf{K}} \mathbf{y}$ and $Z \in D^{\mathbf{y}}_Y \vee (Y\backslash Z) \in D^{\mathbf{y}}_Y$.*
2) Moreover, if h is a function with domain Y <u>then</u> above we can demand that for some $B \in \mathscr{A}_{\mathbf{y}}, Y \cap B$ is a front of B and for some front Y' of B which is below Y and a one-to-one function h' with domain Y' we have $\rho \in Y' \wedge \varrho \in Y \cap B \wedge \rho \le_T \varrho \Rightarrow h(\rho) = h'(\varrho)$, note that possibly $Y' = \{\eta\}$ so $h{\restriction}(Y \cap B)$ is constant.

Proof. 1) By 2.3(1) without loss of generality there is $B \in \mathscr{A}_{\mathbf{x}}$ such that B is $\le_{\mathbf{x}}$-maximal in $\mathscr{A}_{\mathbf{x}}$; clearly $Y \cap B$ is an almost front of B and so without loss of generality $Y \subseteq B$.

We know that $B[\le Y]$ has no ω-branch, so by $<_{T_{\mathbf{x}}}$-downward induction on $\nu \in B[\le Y] = \{\rho \in B : (\exists \nu)[\rho \le_{T_{\mathbf{v}}} \nu \in Y]\}$ we choose $(\mathbf{t}_\nu, Y_\nu)$ such that (where $M = M_{\mathbf{x}}$, of course):

- (a) $\mathbf{t}_\nu \in \{\text{yes, no}\}$ or $\{0,1\}$
- (b) • $Y_\nu \subseteq M_{\ge\nu} \cap Z$ if $\mathbf{t}_\nu =$ yes,
  • $Y_\nu \subseteq M_{\ge\nu} \cap (Y\backslash Z)$ if $\mathbf{t}_\nu =$ no
- (c) $Y_\nu = \max(B'_\nu)$ for some $B'_\nu \in \mathrm{psb}(B_{\ge\nu})$
- (d) if $\nu \in Y$ then $Y_\nu = \{\nu\}$ and $\mathbf{t}_\nu =$ (the truth value of $\nu \in Z$)
- (e) if $\nu \in B[\le Y]\backslash Y$ then
  - (α) $\mathbf{t}_\nu = \min\{\mathbf{t}$: the set $\{\rho \in \mathrm{Suc}_{\mathbf{x}}(\nu, B) : \mathbf{t}_\rho = \mathbf{t}\}$ is infinite$\}$
  - (β) $Y_\nu = \cup\{Y_\rho : \rho \in \mathrm{suc}_B(\nu)$ and $\mathbf{t}_\rho = \mathbf{t}_\nu\}$.

This is easily done and so $\mathbf{t}_\eta$ is well defined. For $\nu \in B[\le Y]$ we let $B^*_\nu = \{\rho \in B_{\ge\nu}$: for some $\varrho \in Y_\nu$ we have $\varrho \le_T \rho \vee \rho \le_{T_{\mathbf{x}}} \varrho\}$. Now define $\mathbf{y}$ by adding B^*_ν to $\mathscr{A}^{\mathbf{x}}_\nu$ for every $\nu \in B[\le Y]$, and check.
2) Similarly noting: if $h : Y \to A, Y \in \mathrm{frt}(B), Z = \{\eta \in B; \mathrm{suc}_B(\eta) \subseteq Y\}$ and $\eta \in Z \Rightarrow h{\restriction}\mathrm{suc}_B(\eta)$ is one-to-one <u>then</u> we can find $B' \in \mathrm{psb}(B)$ such that: $h{\restriction}B' \cap Z$ is one-to-one. $\square_{2.4}$

We can conclude

{8h.17}

Conclusion 2.5. *Assume CH.*
There is a $\mathbf{x} \in \mathbf{K}$ such that:

- (a) (α) *$\mathscr{A}^{\mathbf{x}}_\eta \ne \{\{\eta\}\}$ for $\eta \in M_{\mathbf{x}}$*
  (β) *$\mathscr{B}_{\mathbf{x}} = \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})}$ is $\aleph_1$-directed under $\le_{\mathbf{x}}$*
- (b) *if $B \in \mathscr{B}_{\mathbf{x}}$ and $Y \in \mathrm{frt}_{\mathbf{x}}(B)$ then*
  - (α) *$D^{\mathbf{x}}_Y$ is an ultrafilter on Y*
  - (β) *it is a non-principal ultrafilter iff $Y \ne \{\mathrm{rt}_{\mathbf{x}}\}$*
- (c) *if $B_1 \in \mathscr{B}_{\mathbf{x}}$ <u>then</u> for some $B_2 \in \mathscr{A}_{\mathbf{x}}$ we have $B_1 \le_{\mathbf{x}} B_2$ and $B_1 \cap \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}) = \emptyset$, moreover*[1] $(\forall \varrho \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}))(\exists^\infty \rho \in \mathrm{suc}_{B_1}(\mathrm{rt}_{\mathbf{x}}))[\varrho \le_{M_{\mathbf{x}}} \rho]$.

We may add

[1]Not a serious addition. As always, the number of $\varrho \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}})$ failing this is finite.

(*d*) *if* $\eta \in M_{\mathbf{x}}, B \in \mathscr{A}^{\mathbf{x}}_\eta$ *and* $Y \in \mathrm{frt}_{\mathbf{x}}(\eta)$ *then* $B[\leq Y] \in \mathscr{A}^{\mathbf{x}}_\eta$

(*e*) *if* $\eta \in M_{\mathbf{x}}, B_1 \in \mathscr{A}^{\mathbf{x}}_\eta$ *and* $h : \max(B_1) \to \omega$ *then there is* $B_2 \in A^{\mathbf{x}}_\eta$ *such that* $\max(B_1) \cap B_2$ *is a front of* B_2 *and* $(Y; h')$ *is as in 2.4(2)*

(*f*) *if* $B \in \mathscr{B}_{\mathbf{x}}$ *and* $B_\nu \in \mathscr{A}^{\mathbf{x}}_\nu$ *for every* $\nu \in \max(B)$ *then there is* $B' \in \mathscr{A}^{\mathbf{x}}_\eta$ *such that* $B \leq^*_{\mathbf{x}} B'$ *and* $\nu \in \max(B) \cap B' \Rightarrow B_\nu \in \mathrm{sb}(B'_{\geq \nu})$

(*g*) $D^{\mathbf{x}}_Y$ *is a Q-point, see Definition 0.3(2)*

(*h*) $\mathbf{x}$ *is*

(α) *fat*

(β) *strongly big (follows by (e))*

(γ) *strongly large*

(δ) *full.*

Proof. Straight.

For $(b)(\gamma)$ think of the proof of 2.4(2). $\square_{2.5}$

{8j.20d}

Definition 2.6. 1) Let $\mathbf{K}_{\mathrm{ut}}$ be the class of $\mathbf{x} \in \mathbf{K}$ which are ultra which means $\mathbf{x} \in \mathbf{K}_g \cap \mathbf{K}_u$, see below.
2) Let $\mathbf{K}_r$ be the set of $\mathbf{x} \in \mathbf{K}$ which are reasonable which means (a),(c) of 2.5 holds.
3) Let $\mathbf{K}_g$ be the class of $\mathbf{x} \in \mathbf{K}_2$ which are good, which means: if $\mathscr{A}$ is $\mathbf{x}$-dense, $\mathbf{x}$-open, see below and $B_2 \in \mathscr{B}_{\mathbf{x}}$ then for some $B_2, B_1 \leq_{\mathbf{x}} B_2 \in \mathscr{B}_{\mathbf{x}}$ and for all but finitely many $\eta \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}})$ we have $(B_2)_{\geq \eta} \in \mathscr{I}$.
4) For $\mathbf{x} \in \mathbf{K}$ we say $\mathscr{I}$ is $\mathbf{x}$ densely open when:

(*a*) $\mathscr{I} \subseteq \mathscr{A}^{\mathbf{x}}_{>\mathrm{rt}(\mathbf{x})}$

(*b*) for every $B_1 \in \mathscr{B}_{\mathbf{x}}$ there is B_2 such that $B_1 \leq_{\mathbf{x}} B_2 \in \mathscr{B}_{\mathbf{x}}$ such that for all but finitely many $\eta \in \mathrm{suc}_B(\mathrm{rt}_{\mathbf{x}})$, there is $B_3 \in \mathrm{qsb}_{\mathbf{x}}((B_2)_{\geq\eta})$ such taht $B_3 \in \mathscr{I}$.

5) For $\mathbf{x} \in \mathbf{K}$ we say $\mathscr{I}$ is $\mathbf{x}$ open if clause (a) from part (4) and

(*c*) if $B_1 \in \mathscr{I}$ then $\mathrm{qsb}_{M_{\mathbf{x}}}(B_1) \subseteq \mathscr{I}$.

6) Let $\mathbf{K}_u$ be the set of $\mathbf{x} \in \mathbf{K}_r$ for which clause $(b)(\beta)$ of Definition 2.5 holds.
7) We say $\mathbf{x} \in \mathbf{K}_r$ is large when it satisfies clause (e) of 2.5.
8) Let $\mathbf{K}_{\mathrm{ut}}$ be the set of $\mathbf{x} \in \mathbf{K}$ which are ultra which means $\mathbf{x} \in \mathbf{K}_g \cap \mathbf{K}_u$ and $\mathbf{x}$ is large

- so in particular
for every $Y \in \mathrm{frt}_{\mathbf{x}}(\mathrm{rt}_{\mathbf{x}})$, the filter $D^{\mathbf{x}}_Y$ is an ultrafilter
(equivalently for every $B \in \mathscr{B}_{\mathbf{x}}$), i.e. $\mathbf{x} \in \mathbf{K}_{\mathrm{uf}}$, see Definition 1.8(6).

{8j.28}

Claim 2.7. *Assume* $\diamondsuit_{\aleph_1}$. *In 2.5 we can add:*

(*i*) $\mathbf{x} \in \mathbf{K}_g$.

Proof. Straightforward.

Let $\mathscr{I} = \{B \in \mathscr{A}^{\mathbf{x}}_{>\mathrm{rt}(\mathbf{x})} : Y \in \mathrm{alm}-\mathrm{frt}_{\mathbf{x}}(B)$ and for some $B' \in \mathrm{sb}(B)$, the function $h{\restriction}(Y \cap B')$ is constant$\}$

$(*)_1$ $\mathscr{I}$ is $\mathbf{x}$-dense.

[Why? Let $B_2 \in \mathscr{A}_{\mathbf{x}}$; as $(A_{\mathbf{x}}, \leq_{\mathbf{x}})$ is directed there is $B_2 \in \mathscr{A}_{\mathbf{x}}$ such that

- $B_1 \leq_{\mathbf{x}} B_2$
- $A_n \leq_{\mathbf{x}} B_2$.

So for each n, there is $B'_{2,n} \in \mathrm{sb}(B_2)$ such that $n \notin \mathrm{Rang}(h{\restriction}(Y \cap B'_{2,n})$. $\square$

Claim 2.8. *If $\mathbf{x} \in \mathbf{K}_{\mathrm{ut}}$, i.e. in 2.5 from $(a)(\alpha), (\beta), (b)(\alpha), (\beta), (c), (h), (h)(\gamma)$ we can conclude* {8j.17}

(*f*) *if $B \in \mathscr{B}_{\mathbf{x}}$ and $Y_1, Y_2 \in \mathrm{frt}(B)$ and Y_2 is above Y_1 <u>then</u> $h^{\mathbf{x}}_{Y_2,Y_1}$ exemplify $D^{\mathbf{x}}_{Y_1} \leq_{\mathrm{RK}} D^{\mathbf{x}}_{Y_2}$*

(*g*) *$\{D^{\mathbf{x}}_Y : Y \in \mathrm{frt}^-_{\mathbf{x}}\}$ is $\leq_{\mathrm{RK}}$-directed (even $\aleph_1$-directed)*

(*h*) *below $D^{\mathbf{x}}_Y$ there is no P-point.*

Proof. The main point is:

<u>Clause (*h*)</u>: Let $B_1 \in \mathscr{B}_{\mathbf{x}}$ be such that $B_1 \cap Y$ is an almost front of B; without loss of generality $Y \subseteq B$.

So let $h : Y \to \mathbb{N}$ be such that the set $h^{-1}\{n\}$ is $= \emptyset \mod D^{\mathbf{x}}_Y$ for every n hence there is $A_n \in \mathscr{B}_{\mathbf{x}}$ which witness it and toward contradiction assume that $h(D^{\mathbf{x}}_Y)$ is a P-point; without loss of generality h is onto $\mathbb{N}$. From clause (e) of 2.5 this is immediate but we shall avoid using it. As $\mathscr{A}_{\mathbf{x}}$ is $\aleph_1$-directed by clause (a) of 2.5 without loss of generality $n < \omega \Rightarrow A_n \leq_{\mathbf{x}} B_2$ and $B_1 \leq_{\mathbf{x}} B_2$.

As $\mathbf{x}$ is large, apply the definition 1.11(3) of large to the pair (B_2, h') where $h'(\eta) = h(\nu)$ when $\nu \leq_{M_{\mathbf{x}}} \eta \in \max(B)$ and zero if there is no such nu; as $Y \subseteq B_0$ is an almost front of B_0, h' is well defined. So there is Y_0, a front of B_0 such that for $\eta, \nu \in \max(B)$ we have $h'(\eta) = h'(\nu) \Leftrightarrow (\exists \rho \in B_1)(\rho \leq_{M_{\mathbf{x}}} \eta \wedge \rho \leq_{M_{\mathbf{x}}} \nu)$, clearly Y_0 is below Y, i.e. $(\forall \eta \in Y)(\exists^{\leq} \nu \in Y_0)[\nu \leq_{M_{\mathbf{x}}} \eta]$. Let $Z = \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}})$. So clearly the ultrafilter $D^{\mathbf{x}}_Z$ is $\leq_{\mathrm{RK}} h(D^{\mathbf{x}}_Y)$, hence $D^{\mathbf{x}}$ is a P-point.

By clause (c) of 2.5 there is $B_3 \in \mathscr{B}_{\mathbf{x}}$ such that $B_2 \leq_{\mathbf{x}} B_3$ and $(\forall \varrho \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}))(\exists^\infty \rho \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}))[\varrho \leq_{M_{\mathbf{x}}} \rho]$.

For each $\varrho \in \mathrm{suc}_{B_3}(\mathrm{rt}_{\mathbf{x}})$ let $Z_\varrho = \{\rho \in Z : \varrho \leq_{M_{\mathbf{x}}} \rho\}$, so $\langle Z_\varrho : \rho \in \mathrm{suc}_{B_3}(\mathrm{rt}_{\mathbf{x}})$ is a partition of Z, each Z_ϱ is $= \emptyset \mod D^{\mathbf{x}}_Z$, but by the definitions of "$\mathbf{x} \in K$ and $D^{\mathbf{x}}_Z$" clearly there is no $Z' \in D^{\mathbf{x}}_Z$ such that $Z' \cap Z_\varrho$ is finite for every $\varrho \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}})$, contradiction to "$D^{\mathbf{x}}_Z$" is a P-point. $\square_{2.8}$

§ 3. Toward preserving $D^{\mathbf{x}}_B$

{k2}

Claim 3.1. *If (A) then (B) where:*

- (A) (a) $B \in \mathrm{CWT}(T)$ *for a partial order* T, without loss of generality $T = ({}^{\omega>}\omega, \triangleleft)$
  - (b) $\mathbb{Q}$ *is a forcing notion with the COM player winning the strongly bounding game* $\partial^{\mathrm{sb}}_{\mathbb{Q}}$, *see Definition 3.6 below*
  - $(c)(\alpha)$ *forcing with* $\mathbb{Q}$ *preserving some non-principal ultrafilter on* $\mathbb{N}$ *<u>or just</u>*
    - (β) $([\mathbb{N}]^{\aleph_0})^{\mathbf{V}}$ *is big in* $\mathbf{V}^{\mathbb{Q}}$
  - (d) $p \Vdash$ "$\underset{\sim}{A} \subseteq \max(B)$"
- (B) *there are* $B', p', \mathbf{t}$
  - (a) $\mathbb{Q} \models$ "$p \leq p'$"
  - (b) $B' \in \mathrm{psb}(B)$, *see 1.2(3D)*
  - (c) $\mathbf{t}$ *a truth value*
  - (d) $p' \Vdash$ "$\max(B' \subseteq \underset{\sim}{A}^{[\mathbf{t}]})$".

{k2d}

Remark 3.2. 1) Recall $A^{[1]} = A, A^{[0]} = \mathbb{N} \backslash A$.
2) In 3.1(A)(b) it is enough that the COM player does not lose the game $\partial^{\mathrm{sb}}_{\mathbb{Q}}$, i.e. the INC player has no winning strategy.
3) The following definition put 3.1 in frame.

{k2f}

Definition 3.3. [2011.7.21 redundant by 1.11, 1.12.]
1) A forcing notion $\mathbb{Q}$ is non-tree shattering <u>when</u> if $B \in \mathrm{CWT}(({}^{\omega>}\omega, \triangleleft))$ and $p \in \mathbb{Q}, p \Vdash$ "$\underset{\sim}{\tau} \subseteq \max(B)$" <u>then</u> for some $B' \in \mathrm{psb}(B)$, (from $\mathbf{V}$!) and $q \in \mathbb{Q}$ we have $p \leq q$ and $q \Vdash$ "$B' \subseteq \underset{\sim}{\tau}$" <u>or</u> $q \Vdash$ "$B' \subseteq \max(B) \backslash \underset{\sim}{\tau}$".
2) For $B \in \mathrm{CWT}({}^{\omega>}\omega, \triangleleft)$ and $\mathscr{B} \subseteq \mathrm{psb}(B)$ we say $\mathscr{B}$ is large (in B) <u>when</u> for every function $\mathbf{c} : \max(B) \to \{0,1\}$ there is $B_1 \in \mathscr{B}$ such that $\mathbf{c} \restriction \max(B_1)$ is constant.
3) We say $\mathbf{x} \in \mathbf{K}$ is large <u>when</u> for every $\eta \in T_{\mathbf{x}}$ and $B \in \mathscr{A}_{\mathbf{x},\eta}$ the set $\{B' : B' \in \mathscr{A}_{\mathbf{x},\eta}$ and $B' \cap B \in \mathrm{psb}(B)\}$ is large in B.

An alternative to 3.1 with a similar proof is:

{k2m}

Claim 3.4. *If (A) then(B) where:*

- (A) (a) $B \in \mathrm{CWT}(T)$ *for some* T
  - (b) $\mathbb{Q}$ *is a bounding forcing (i.e. every new* $f : \mathbb{N} \to \mathbb{N}$ *is below some "old" such function)*
  - (c) *forcing with* $\mathbb{Q}$ *preserve some* P*-point*
- (B) *if* $B \in \mathrm{CTW}({}^{\omega>}\omega, \triangleleft)$ *then* $(\mathrm{psb}(B))^{\mathbf{V}}$ *is big in* $\mathbf{V}^{\mathbb{Q}}$*; see Definition.*

Remark 3.5. To use this for iterations we may "change our mind" about which P-point to use.

{k3}

Definition 3.6. For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define $\partial_{\mathrm{sb}} = \partial^{\mathrm{sb}}_p = \partial^{\mathrm{sb}}_{\mathbb{Q},p}$, the strong bounding game between the null player NU and the bounding player BND, omitting p means NU chooses it in his first move; by

- (a) a play last ω moves; a play is between the player COM and INC
- (b) in the n-th move:

(α) the NU player gives a (non-empty) tree $\mathscr{T}_n$ with ω levels and no maximal node and a $\mathbb{Q}$-name $\underset{\sim}{F}_n$ of a function with domain $\mathscr{T}_n$ such that $\eta \in \mathscr{T}_n \Rightarrow p \Vdash_{\mathbb{Q}}$ "$F_n(\eta) \in \mathrm{suc}_{\mathscr{T}_n}(\eta)$"

(β) the BND player chooses $\eta_n \in \mathscr{T}_n$

(c) in the end of the play, the BND player wins the play <u>iff</u> there is $q \in \mathbb{Q}$ above p forcing, for every n, that "$(\exists k < \ell g(\eta_n))(\underset{\sim}{F}_n(\eta_n{\restriction}k) \le_{\mathscr{T}_n} \eta_n \wedge k$ even)" where $\eta_n{\restriction}k$ is the unique $\nu \le_{\mathscr{T}} \eta_n$ of level k.

Proof. <u>Proof of 3.1</u>

We prove this by induction on $\mathrm{rk}_{\mathbf{x}}(B)$, for all such B's. Let $\eta = \mathrm{rt}(B)$.

<u>Case 1</u>: $\mathrm{Dp}_{\mathbf{x}}(B) = 0$

Trivial as then $B = \{\eta\}$, i.e. B is a singleton so $B' = B$ can serve.

<u>Case 2</u>: $\mathrm{Dp}_{\mathbf{x}}(B) = 1$

Without loss of generality $\nu \in B \backslash \{\eta\} \Rightarrow \mathrm{rk}(\nu, B) = 0$. Now $|B\backslash\{\eta\}| = \aleph_0$ and we just need to find $p' \in \mathbb{Q}$ above p such that $\{\nu \in B : \nu \neq \eta$ and p' forces $\nu \in \underset{\sim}{A}$ or forces $\nu \notin \underset{\sim}{A}\}$ is infinite. As $\Vdash_{\mathbb{Q}}$ "no $\underset{\sim}{X} \subseteq \mathbb{N}$ shatters $\mathscr{P}(\mathbb{N})^{\mathbf{V}}$", this is possible.

<u>Case 3</u>: $\alpha = \mathrm{Dp}_{\mathbf{x}}(B) > 1$

Let $Y = \mathrm{suc}_{\mathbf{x}}(\eta, B)$ so for $\nu \in Y$ we have $\mathrm{Dp}_{\mathbf{x}}(B^{[\nu]}) < \alpha$, hence the induction hypothesis applies to $B^{[\nu]}$, let $\langle \nu_n : \nu \in \mathbb{N}\rangle$ list Y.

We simulate a play of $\eth^{\mathrm{sb}}_{\mathbb{Q},p}$ with the BND player using a winning strategy such that in the n-th move the NU player acts such that:

$(*)_1$ (a) $\mathscr{T}_n = \{\langle B_0, \dots, B_{k-1}\rangle : k \in \mathbb{N}, B_\ell \in \mathrm{psb}(B^{[\nu_n]})$ for $\ell < k$ and $B_{\ell+1} \subseteq B_\ell$ if $\ell + 1 < k\}$

(b) $<_{\mathscr{T}_n}$ is being an initial segment

(c) $\underset{\sim}{F}_n(\langle B_0, \dots, B_{k-1}\rangle)$ is $\langle B_0, \dots, B_{k-1}, B'\rangle$ for some member B' of $\mathrm{psb}(B_k)$ from $\mathbf{V}$ such that $\max(B') \subseteq \underset{\sim}{A} \vee \max(B') \cap \underset{\sim}{A} = \emptyset$.

There is such a function $\underset{\sim}{F}_n$ because of the induction hypothesis.

Clearly we can do this. As the player BND has used a winning strategy, COM has won the play so there is q such that

$(*)_2$ (a) $q \in \mathbb{Q}$

(b) $\mathbb{Q} \models$ "$p \le q$"

(c) $q \Vdash$ "for every n for some even $k <$ $\mathrm{level}_{\mathscr{T}_n}(\eta_n)$ we have $\underset{\sim}{F}_n(\eta_n{\restriction}k) \le_{\mathscr{T}_n} \eta_n$".

Hence by the choice of $(\mathscr{T}_n, \underset{\sim}{F}_m)$, letting $\eta_n = \langle B_{n,0}, \dots, B_{n,k(n)}\rangle$

$(*)_3$ for some $\langle \underset{\sim}{\mathbf{t}}_n : n \in \mathbb{N}\rangle$

(α) $B_{n,k(n)} \in \mathrm{psb}(B_{[\ge \nu_n]})$

(β) $\underset{\sim}{\mathbf{t}}_n$ is a $\mathbb{Q}$-name of the truth value

(γ) $q \Vdash$ "if $\underset{\sim}{\mathbf{t}}_n = 1$ then $\max(B_{n,k(n)}) \subseteq \underset{\sim}{A}$ and if $\underset{\sim}{\mathbf{t}}_n = 0$ then $\max(B_{n,k(n)}), \underset{\sim}{A} = \emptyset$".

Now by clause $(c)(\beta)$ of the claim assumption

$(*)$ there is an infinite $\mathscr{U} \subseteq \mathbb{N}$, truth value $\mathbf{t}$ and r such that $q \le_{\mathbb{Q}} r$ and $r \Vdash$ "$\underset{\sim}{\mathbf{t}}_n = \mathbf{t}$ for $n \in \mathscr{U}$".

Lastly, let $B_* = \cup\{B_{n,k(n)} : n \in \mathscr{U}\} \cup \{\eta\}$ and clearly B_*, r are as required. $\square_{3.1}$

{k11}

Definition 3.7. 1) For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define a game $\partial_{\rm bd} = \partial^{\rm bd}_p = \partial^{\rm bd}_{\mathbb{Q},p}$; omitting p means that the player NU chooses it in his first move; by

- (a) a play last ω-moves
- (b) in the n-th move
  - (α) the NU player gives a $\mathbb{Q}$-name $\underset{\sim}{\tau}_n$ of a member of $\mathbf{V}$ and then
  - (β) the BND player gives a finite set $w_n \subseteq \mathbf{V}$
- (c) in the end of the play the COM player wins the play <u>iff</u> there is $q \in \mathbb{Q}$ above p forcing "$\underset{\sim}{\tau}_n \in w_n$" for every n.

2) The game $\partial^{\rm bd}_{\mathbb{Q},p,f}$ where $\mathbb{Q}$ is a forcing notion and $p \in \mathbb{Q}$ and $f : \mathbb{N} \to \mathbb{N}\backslash\{0\}$ going to infinity, is defined similarly, but we demand $|w_n| \le f(n)$.
3) We say the forcing notion $\mathbb{Q}$ is (f,g)-bounding <u>when</u> $f, g \in {}^{\mathbb{N}}(\mathbb{N}\backslash\{0\}), g \le f$ and for every $\eta \in (\prod\limits_n (f(n))^{\mathbf{V}[\mathbb{Q}]}$ there is $\bar{w} \in (\prod\limits_n [f(n)]^{g(n)})^{\mathbf{V}}$ such that $(\forall n)(\eta(n) \in w_n)$.

Recalling Definition 1.11, 1.12

{k12}

Claim 3.8. *1) If (A) below and $B \in \mathrm{CTW}(M)^{\mathbf{V}}$ <u>then</u> $(\mathrm{psb}(B))^{\mathbf{V}}$ is large in $\mathbf{V}^{\mathbb{Q}}$.
2) If (A) + (B) below <u>then</u> in $\mathbf{V}^{\mathbb{Q}}, \mathbf{x}$ is large; <u>where</u>*

- (A)
  - (a) *$\mathbb{Q}$ is a proper forcing notion*
  - (b) *D_* is a Ramsey ultrafilter in $\mathbf{V}$*
  - (c) *$\Vdash_{\mathbb{Q}}$ "$\mathrm{fil}(D_*)$ is a Ramsey ultrafilter*
  - (d) *$\Vdash_{\mathbb{Q}}$ "$D^{\mathbf{x}}_Y$ is an ultrafilter, equivalently $(D^{\mathbf{x}}_Y)^{\mathbf{V}}$ generates one"*
- (B)
  - (a) *$\mathbf{x}$ is as in 4.1, i.e. $\mathbf{x} \in \mathbf{K}_g$ (see Definition 2.6)*
  - (b) *$\mathbf{x}$ is full, i.e. if $B_1 \in \mathscr{A}^{\mathbf{x}}_\eta, \eta \ne \mathrm{tr}_{\mathbf{x}}$ and $B_2 \in \mathrm{psb}(B_1)$ then $B_2 \in \mathscr{A}^{\mathbf{x}}_\eta$.*

Proof. 1) We prove this by induction on $\mathrm{Dp}(B)$, let $\mathbf{c} : \max(B) \to \mathbb{N}$ be from $\mathbf{V}^{\mathbb{Q}}$ and we should find (B^1, Y) as promised. If $\mathrm{Dp}(B) = 0$, i.e. $|B| = 1$ this is trivial.

If $\mathrm{Dp}(B) = 1$ let $\langle \eta_n : \eta \in \mathbb{N}\rangle \in \mathbf{V}$ list $\mathrm{suc}_B(\mathrm{rt}_B)$: by (A),(C) in $\mathbf{V}^{\mathbb{Q}}$, for some $A \in \mathrm{fil}(D_*)$ the sequence $\langle \mathbf{c}(\eta_n) : n \in A\rangle$ is constant or without repetitions, so by (A)(c), without loss of generality $A \in D_* \subseteq \mathbf{V}$ and $\{\mathrm{rt}_B\} \cup \{\eta_n : n \in A\}$ is as required.

So assume $\mathrm{Dp}(B) > 1$. Without loss of generality $0 \notin \mathrm{Rang}(\mathbf{c})$. For $\nu \in B\backslash \max(B)$ let $\langle \eta_{\nu,n} : n \in \mathbb{N}\rangle$ list $\mathrm{suc}_B(\mathrm{rt}_{\mathbf{x}})$ and without loss of generality the function $(\nu, n) \mapsto \eta_{\nu,n}$ belongs to $\mathbf{V}$. By downward induction on $\nu \in B$ we choose $(k_\nu, A_\nu) = (k(\nu), A_\nu)$

- $(*)$
  - (a) $k_\nu \in \mathbb{N}$
  - (b) $A_\nu \in D_*$
  - (c) if $\nu \in \max(B)$ then $k_n = \mathbf{c}(\nu)$ so > 0
  - (d) if $\nu \notin \max(B)$ then $(\alpha)_\nu$ or $(\beta)_\nu$ where
    - $(\alpha)_\nu$ $k_\nu = 0$ and $\langle k(\eta_{\nu,n}) : n \in A_\nu\rangle$ is with no repetitions, all non-zero
    - $(\beta)_\nu$ $\langle k(\eta_{\nu,n}) : n \in A_\nu\rangle$ is constantly k_ν.

[Why we can? This is possible by (A)(c).]

(∗) for $\nu, \rho \in B \setminus \max(B)$ choose $A_{\nu,\rho} \in D_*$ such that either $n \in A_{\nu,\rho} \Rightarrow k(\eta_{\rho,n}) = k(\eta_{\nu,n})$ or $\{k(\eta_{\rho,n}) : n \in A_{\nu,\ell}\}$ is disjoint to $\{k(\eta_{\nu,n}) : n \in A_{\nu,\rho}\}$ needed?

Now by (A)(c) there is A_* such that

(∗) (a) $A_* \in \mathrm{fil}(D_*)$

(b) $\nu \in B \setminus \max(B) \Rightarrow A_* \subseteq^* A_\nu$

(c) if $\nu, \rho \in B \setminus \max(B) \Rightarrow$ then $A_* \subseteq^* A_{\nu,\rho} \vee A_* \subseteq^* (\mathbb{N} \setminus A_{\nu,\rho})$

(d) without loss of generality $A_* \in D_*$.

Let $\langle \nu_n : n \in \mathbb{N} \rangle$ list $B \setminus \max(B)$ and let f_1 be the function with domain $B \setminus \max(B)$ such that $f_1(\nu) = \{\eta_{\nu,n} : n \in A_\nu \setminus A_*\} \in [\mathrm{suc}_B(\nu)]^{<\aleph_0}$.

As the forcing $\mathbb{Q}$ is bounding, there is a function $f_2 \in \mathbf{V}$ with domain $B \setminus \max(B)$ such that $\nu \in f_1(\nu) \subseteq f_2(\nu) \in [\mathrm{suc}_B(\nu)]^{<\aleph_0}$. Clearly

(∗) $B_1 \in \mathrm{psb}(B)^{\mathbf{V}}$ where $B_1 = A_{B,f} = \{\nu \in B$: if $\rho \in B$ satisfies $\mathrm{rt}_{\mathbf{x}} \leq_B \rho <_B \nu$ and n is such that $\eta_{\rho,n} \leq_B \nu$ then $n \in A_*$ but $\eta_{\rho,n} \notin f_2(\nu)\}$.

Also, let

(∗) $Y = \{\nu \in B_1 : k_\nu \neq 0$ and $\rho <_B \nu \Rightarrow k_\rho = 0\}$.

Now

(∗) (a) Y is a front of B_1

(b) if $\nu \in Y$ then $\mathbf{c}\restriction(B_1)_{\geq \nu}$ is constantly k_ν

(∗) there is $B_2 \in \mathrm{psb}(B_1)^{\mathbf{V}}$ such that: if $\nu \in B_2 \setminus \max(B_2)$ and $\mathrm{suc}_{B_2}(\nu)$ is not disjoint to Y then $\mathrm{suc}_{B_1}(\nu) \subseteq Y$.

[Why? Similar to the above proof. In fact

- if $B' \in \mathrm{CTW}(M)^{\mathbf{V}}, \mathbf{d}$ is a function with domain B' then for some $B'' \in \mathrm{psb}(B')^{\mathbf{V}}$, for every $\eta \in B'' \setminus \max(B''), \mathbf{d}\restriction\mathrm{suc}(\eta, B'')$ is constant or one-to-one.

If $Y = \{\mathrm{rt}_{\mathbf{x}}\}$ we are done, so assume not and let

(∗) $Z = \{\eta \in B_2 : \eta \notin \max(B_2)$ and $\mathrm{suc}_{B_2}(\eta) \subseteq Y\}$.

So

(∗) Z is a front of B_2.

Also if $Z = \{\mathrm{rt}_{B_2}\}$ we are done so assume not and let $\langle \nu_n : n \in \mathbb{N} \rangle$ list Z. As $\mathrm{fil}(D_*)$ is a Ramsey ultrafilter we can find $\bar{n}$ such that

(∗) (a) $\bar{n} = \langle n_i = n(i) : i \in \mathbb{N} \rangle$ is increasing

(b) $\bar{n}$ list a member of D_* hence $\in \mathbf{V}$

(c) if $i \geq \ell$ then $\eta_{\nu_\ell, n_i} \in B_2$

(d) if $i_1 < i_2$ then $\{k(\eta_{\nu_i, n(i_1)}) : \ell \leq \ell_1\}$ and $\{k(\eta_{\nu_\ell, n(i_2)}) : \ell \leq i_2\}$ are disjoint.

Lastly, as $n(i) \in \mathbf{V}$ we can find in $\mathbf{V}$ a partition $\langle C_\ell : \ell \in \mathbb{N}\rangle$ of $\mathbb{N}$ to infinite sets and let $B_3 = \{\varrho$: if $\nu_\ell <_{B_2} \varrho$ and for some $i \in C_\ell, i > \ell$ and $\eta_{\nu_\ell, n(i)} \le_{B_2} \varrho\}$.

Easily $B_3 \in \mathbf{V}, B_3 \in \mathrm{psb}(B_2)$ and is as required.]

2) FILL.

3) We use "for $\mathscr{I}$ dense open...". $\square_{3.8}$

{k13}

Definition 3.9. 1) For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define a game $\Game_{\rm ufbd} = \Game^{\rm ufbd}_p = \Game^{\rm ufbd}_{\mathbb{Q},p}$; omitting p means that the player NU chooses it in his first move; by

- (a) a play last ω-moves
- (b) in the n-th move
  - (α) the NU player chooses an ultrafilter E_n on some set I_n from $\mathbf{V}$ and a $\mathbb{Q}$-name $\underset{\sim}{E}^+_n$ of an ultrafilter on I_n extending E_n and a $\mathbb{Q}$-name $\underset{\sim}{X}_n$ of a member of $\underset{\sim}{E}^+_n$
  - (β) the BND player chooses $t_n \in I_n$
- (c) in the end of the play the BND player wins the play <u>iff</u> there is $q \in \mathbb{Q}$ above p forcing "$t_n \in \underset{\sim}{X}_n$" for every n.

2) For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define a game $\Game_{\rm vfbd} = \Game^{\rm vfbd}_p = \Game^{\rm vfbd}_{\mathbb{Q},p}$ as in (1) but $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{X}_n \in E_n$ or just include a member of E_n" so $\underset{\sim}{E}^+_n$ is redundant.

{k14}

Observation 3.10. Let $\mathbb{Q}$ be a forcing notion.

1) If BND wins in $\Game^{\rm sb}_{\mathbb{Q},p}$ <u>then</u> BND wins in $\Game^{\rm bd}_{\mathbb{Q},p}$ which implies that $\mathbb{Q}$ is a bounding forcing.

2) The player BND wins in $\Game^{\rm sb}_{\mathbb{Q},p}$ <u>iff</u> BND wins in $\Game^{\rm vfbd}_{\mathbb{Q},p}$.

3) If the player BND wins in $\Game^{\rm ufbd}_{\mathbb{Q}.p}$ <u>then</u> BND wins in $\Game^{\rm vfbd}_{\mathbb{Q},p}$.

4) We can replace in 1) - 3) above "wins" by "do not lose".

Proof. 1) The second implication is obvious, so we concentrate on the first. For every $\underset{\sim}{\tau}$, a $\mathbb{Q}$-name of an ordinal we define a pair $(T_{\underset{\sim}{\tau}}, \underset{\sim}{F}_{\underset{\sim}{\tau}})$ as follows:

$(*)_1$ (a) let $u = \{\alpha : \nVdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \neq \alpha$"$\}$ it is a set of $|\mathbb{Q}|$ ordinals, non-empty

(b) $T_{\underset{\sim}{\tau}}$ is the tree $\{\eta : \eta \in {}^{\omega>}u\}$, i.e. order by $\triangleleft$ being an initial segment

(c) $\underset{\sim}{F}_{\underset{\sim}{\tau}}(\eta) = \eta\hat{\ }\langle \underset{\sim}{\tau}\rangle$ for $\eta \in T_{\underset{\sim}{\tau}}$.

Clearly

$(*)_2$ (a) $T_{\underset{\sim}{\tau}}$ is in $\mathbf{V}$, a tree with ω levels

(b) $\underset{\sim}{F}_{\underset{\sim}{\tau}}$ is a $\mathbb{Q}$-name of a function with domain $\underset{\sim}{F}$ such that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{F}_{\underset{\sim}{\tau}}(\eta) \in \mathrm{suc}_{T_{\underset{\sim}{\tau}}}(\eta)$".

[Why? Read the definitions.]

$(*)_3$ if $q \in \mathbb{Q}$ and $\eta \in T_{\underset{\sim}{\tau}}$ (so $\mathrm{Rang}(\eta)$ is a finite subset of u) then the following are equivalent:

(a) $q \Vdash$ "$\underset{\sim}{\tau} \in \mathrm{Rang}(\eta)$"

(b) $q \Vdash$ "for some $\nu \triangleleft \eta$" we have $q \Vdash$ "$\nu\hat{\ }\langle \underset{\sim}{F}_{\underset{\sim}{\tau}}(\nu)\rangle \trianglelefteq \eta$".

[Why? Read the definitions.]

So clearly playing the game $\partial^{\rm bd}_{\mathbb{Q},p}$ we can "translate" it to a play of $\partial^{\rm sb}_{\mathbb{Q},p}$ replacing the NU choice of $\underset{\sim}{\tau}_n$ to the choice of $(T_{\underset{\sim}{\tau}}, \underset{\sim}{F}_{\underset{\sim}{\tau}})$. So for every strategy $\mathbf{st}_1$ of COM in $\partial^{\rm sb}_{\mathbb{Q},p}$ we can translate it to a strategy $\mathbf{st}_2$ of the player BND in $\partial^{\rm bd}_{\mathbb{Q},p}$, and finish easily.

2) We now need two translations.

<u>Translating $\mathbb{Q}^{\rm vfbd}_{\mathbb{Q},p}$ to $\partial^{\rm bd}_{\mathbb{Q},p}$:</u>

So we are given a choice $y = (I, E, \underset{\sim}{X}) \in \mathbf{Y}_{{\rm vfbd},\mathbb{Q}}$, of INC, as in 3.9(1)($\alpha$), i.e.

- $\bullet_1$ $I \in \mathbf{V}$
- $\bullet_2$ E an ultrafilter on I, in $\mathbf{V}$
- $\bullet_3$ $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{X} \in E$ or just include a member $\underset{\sim}{X}'$ of E".

Now

($*$) if $q \Vdash$ "$\underset{\sim}{X}' \in \mathscr{W}$" where $\mathscr{W} \subseteq E$ is finite ($\mathscr{W}$ an object in $\mathbf{V}$ not a name), <u>then</u> $\cap\{A : A \in \mathscr{W}\}$ is non-empty and $t \in \cap\{A : A \in cW\} \Rightarrow q \Vdash$ "$t \in \underset{\sim}{X}' \subseteq \underset{\sim}{X}$".

<u>Translating $\partial^{\rm bd}_{\mathbb{Q},p}$ to $\partial^{\rm vfbd}_{\mathbb{Q},p}$:</u>

Given $y = (I_1, \underset{\sim}{\tau}), \underset{\sim}{\tau}$ a $\mathbb{Q}$-name of a member I_1 of $\mathbf{V}$ we define $I_y = [I_1]^{<\aleph_0} \in \mathbf{V}$ and choose $E_y \in \mathbf{V}$ an ultrafilter on I_y such that $u_* \in [I_1]^{<\aleph_0} \Rightarrow \{u \in [I_1]^{<\aleph_0} : u_* \subseteq u\} \in E$; lastly we choose

$$\underset{\sim}{X}_y = \{u \in [I_1]^{<\aleph_0} : \underset{\sim}{\tau} \in u\}.$$

So $(I_y, E_y, \underset{\sim}{X}_y)$ is a legal move in $\partial^{\rm vfbd}_{\mathbb{Q},p}$ and

($*$) if $q \Vdash$ "$t \in X_{\mathbf{y}}$" then $q \Vdash$ "$\underset{\sim}{\tau} \in t$", t a finite subset of I_1".

3) Obvious.

4) The same proof. $\square_{3.10}$

{k16}

Claim 3.11. *Assume $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*)\rangle$ is a CS-iteration of proper forcing.*

If $\underset{\sim}{\mathbb{Q}}_\beta$ is strongly bounding (in $\mathbf{V}^{\mathbb{P}_\beta}$ for $\beta < \alpha(*)$) <u>then</u> $P_{\alpha(*)}$ is strongly bounding (hence $\mathbb{P}_\beta/\mathbb{P}_\alpha$ is strongly bounding in $\mathbf{V}^{\mathbb{P}_\alpha}$ for every $\alpha \leq \beta \leq \alpha(*)$).

Proof. Straight, using the characterization by the game $\partial^{\rm vfbd}_{\mathbb{Q},p}$, see Definition 3.9(2) and Observation 3.10(2). $\square_{3.11}$

§ 4. Specific Forcing Notions

§ 4(A). On $\mathbb{Q}^2_D$.

See [Sh:F1091], [Sh:594]. Note that the forcing of [Sh:F1091, §1] is not O.K. as it makes every old ultrafilter on $\mathbb{N}$ not generate an ultrafilter in the extension.

{m2}

Definition 4.1. For D a non-principal ultrafilter on $\mathbb{N}$ let $\mathbb{Q} = \mathbb{Q}^2_D$ be the following forcing notion:

- (A) $p \in \mathbb{Q}$ <u>iff</u> p consists of
  - (a) $\mathscr{U} = \mathscr{U}_p = \emptyset \text{ mod } D$, i.e. $\mathbb{N}\backslash\mathscr{U} \in D$
  - (b) $f = f_p$, a function from $\mathbb{N}$ to $\{-1, 1\}$
  - (c) $E = E_p$, an equivalence relation on $\mathbb{N}\backslash\mathscr{U}$
  - (d) if $n \in \mathbb{N}\backslash\mathscr{U}$ then $(n/E_p) = \emptyset \mod D$
- (B) $\mathbb{Q} \models$ "$p \le q$" <u>iff</u>
  - (a) $p, q \in \mathbb{Q}$
  - (b) $\mathscr{U}_p \subseteq \mathscr{U}_q$
  - (c) $\mathscr{U}_q\backslash\mathscr{U}_p$ is E_p-closed, i.e. $n_1 E_p n_2 \Rightarrow (n_1 \in \mathscr{U}_q \equiv n_2 \in \mathscr{U}_q)$
  - (d) $E_p{\restriction}\mathscr{U}_q$ refine E_q
  - (e) if $n \in \mathbb{N}\backslash\mathscr{U}_p$ <u>then</u> $f_q{\restriction}(n/E_p)$ is $\pm f_p{\restriction}(n/E_p)$.

{m4}

Definition 4.2. For $\mathbb{Q} = \mathbb{Q}^2_D$:

1)

- (a) $\text{First}(p) = \{n : n \notin \mathscr{U}_p \text{ and } n = \min(n/E_p\}$,
- (b) $\text{first}_n(p) =$ the n-th member of $\text{First}(p)$,
- (c) $\text{first}_{<n}(p) = \{\text{first}_k(p) : k < n\}$.

2) $p \le_{\text{pr},n} q$ means

- (a) $p \le q$
- (b) $\mathscr{U}_p = \mathscr{U}_q$
- (c) if $k \in \text{first}_n(p)$ then $k/E_q = k/E_k$ hence $\text{first}_{<n}(p) = \text{first}_{<n}(q)$.

3) For $p \in \mathbb{Q}, n < \omega, \eta \in {}^n\{1, -1\}$ let $q = p^{[\eta]}$ be defined by

- (a) $\mathscr{U}_q = \mathscr{U}_p \cup \{\bigcup\{m/E_p : m \in \text{ first}_{<n}(p)\}\}$
- (b) $E_q = E_p{\restriction}(\mathbb{N}\backslash\mathscr{U}_p)$
- (c) $f_q{\restriction}\mathscr{U}_p = f_p{\restriction}\mathscr{U}_p$
- (d) $f_q{\restriction}(\mathbb{N}\backslash\mathscr{U}_q) = f_p{\restriction}(\mathbb{N}\backslash\mathscr{U}_q)$ or just $n \in \mathbb{N}\backslash\mathscr{U}_q \Rightarrow f_q{\restriction}(n/E_q) = \pm(f_p{\restriction}(n/E_p))$
- (e) if $m \in \text{first}_{<n}(p)$ then $f_q{\restriction}(m/E_p) = \eta(m) \times f_p{\restriction}(m/E_p)$.

4) $p \le_{\text{apr},n} q$ <u>iff</u> $q \in \{p^{[\eta]} : \eta \in {}^n\{1, -1\}\}$.

{m6}

Remark 4.3. Do we in the Definition preserve convexity? No, we allow infinite parts but the partition is discussed below.

{m9}

Claim 4.4. *For D a P-point or just a non-principal ultrafilter on $\mathbb{N}, \mathbb{Q} = \mathbb{Q}^2_D$ satisfies*
1) $\mathbb{Q}$ is a proper forcing of cardinality $2^{\aleph_0}$.
2) COM *wins in the game $\eth^{\rm bd}_{\mathbb{Q},p}$ and even in $\eth^{\rm bd}_{\mathbb{Q},p,f}$ for $p \in \mathbb{Q}$ and $f \in {}^{\mathbb{N}}(\mathbb{N})\backslash\{0\}$ which goes to infinity.*
3) $\mathbb{Q}^2_D$ has the (f,g)-bounding property <u>*when*</u> *$f, g \in {}^{\mathbb{N}}(\mathbb{N}\backslash\{0\})$ are increasing, $g \leq f$ and $0 = \lim\{g(n)/f(n) : n < \omega\}$.*
4) $\mathbb{Q}$ has the PP-property, see [Sh:f, Ch.VI].

Remark 4.5. 1) Now then in [Sh:f, Ch.VI,§4] the forcing does not have "COM win the play $\eth^{\rm bd}_{\mathbb{Q},p,f}$".
2) But in [Sh:594] it does.

Proof. <u>Proof of 4.4</u>
1) Easy, using the properties of $\leq_{{\rm pr},n}$.
2),3),4) Follows. $\square_{4.4}$

The following is closer to [Sh:f, Ch.VI,§4] and can be used similarly but later we use another way.

Definition 4.6. Let $\underset{\sim}{A}$ be the generic of $\underset{\sim}{\mathbb{Q}}^2_D$, i.e. $\cup\{f_p{\restriction}\mathscr{U}_p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_\alpha}\}$. {m11}
2) Let $\mathbb{Q}^{2,*}_D = (\mathbb{Q}^2_D)^{\mathbb{N}}$, the product of $\aleph_0$ many copies of $\mathbb{Q}^2_D$.

Claim 4.7. *1) If D is a non-principal ultrafilter on $\mathbb{N}$ then $\Vdash_{\underset{\sim}{\mathbb{Q}}^2_D}$ "$\underset{\sim}{A} \subseteq \mathbb{N}$ is a new subset of $\mathbb{N}, \underset{\sim}{A} \neq \emptyset$ mod D".* {m13}
2) For D a non-principal ultrafilter (or P-point) forcing with $\mathbb{Q}^{2,*}_D$ has the properties of $\mathbb{Q}^2_D$ from 4.4.

Remark 4.8. Compare with way B below.

Proof. As in [Sh:f, Ch.VI,§4]. $\square_{4.7}$

Remark 4.9. To prove the consistency of "$\mathfrak{u} = \aleph_2 +$ no P-point" iterating $\mathbb{Q}^2_D$ is not enough, we need a relative $\mathbb{Q}^3_D$, which presently is in [Sh:F1127, §(5A)].

§ 5. Private Appendix
No P-point below

Old proof of 2.8, moved 7/2011, pgs.12-13:

$A_n \le_{\mathbf{x}} B$. So for each n there is $A'_n \in \mathrm{sb}(B)$ such that $n \notin \mathrm{Rang}(h{\restriction}(Y \cap A'_n))$ and let $f_n : B \backslash \max(B) \to [B]^{\le \aleph_0}$ witness $A'_n \in \mathrm{sb}(B)$it, see 1.2(3c).

By clause (c) of 2.5 without loss of generality $\mathrm{suc}_{\mathbf{x}}(\mathrm{rt}_{\mathbf{x}}, B)$ is disjoint to Y. Let $\langle \rho_n : n < \omega \rangle$ list $\mathrm{suc}_B(\mathrm{rt}_{\mathbf{x}})$ and define $f : B \backslash \max(B) \to [B]^{<\aleph_0}$ by $\rho_n \trianglelefteq \nu \in B \backslash \max(B) \Rightarrow f(\nu) = \cup\{f_m(\nu) : m \le n\}$ and $f(\mathrm{rt}_{\mathbf{x}}) = \emptyset$. Let $B'_1 \in \mathrm{sb}_{\mathbf{x}}(B_1)$ be witnessed by f, so $Y \cap B_1 \in D^{\mathbf{x}}_Y$ and let $Y_* = Y \cap B'_1$

(∗) if $\rho_n \le_{T[\mathbf{x}]} \eta \in Y$ and $h(\eta) = k$ <u>then</u> $k \ge n$.

[Why? By the choice of B'_1 and of $\langle A_\ell : \ell \le n \rangle$.]

Let $D_1 = h(D_{\mathbf{x},Y})$ and $D_2 = D_{\mathbf{x},Y}$, so $D_1 \le_{\mathrm{RK}} D_2$ as witnessed by h.

By clause (c) of 2.5 there is $B_2 \in \mathscr{A}_{\mathbf{x}}$ such that $B_2 <_{\mathbf{x}} B_1$ and $\mathrm{suc}(\mathrm{tr}_{\mathbf{x}}, B_2)$ is disjoint to B_1; moreover $(\forall \varrho \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}))(\exists^\infty \rho \in \mathrm{suc}_{B_1}(\mathrm{rt}_{\mathbf{x}})))[\varrho <_{T_{\mathbf{x}}} \rho]$.

Let $\langle \varrho_k : k < \omega \rangle$ list $\mathrm{suc}(\mathrm{tr}_{\mathbf{x}}, B_2)$, and let $u_k = \{n : \varrho_k <_{\mathbf{x}} \rho_n\}$, so $\langle u_k : k < \omega \rangle$ is a partition of ω to infinite sets. Let $Z_n = \{h(\eta) : \eta \in Y_*$ and $\rho_k <_{\mathbf{x}} \eta\}$, so $\langle Z_k : k < \omega \rangle$ is a sequence of subsets of $\mathbb{N}, Z = \cup\{Z_k : k < \omega\} \in D_1$ but $Z_k \notin D_1$ for $k < \omega$.

Hence there is $X \in D_1$ such that $k < \omega \Rightarrow Z_k \cap X$ is finite. Hence $h^{-1}(X) = \{\eta \in Y_* : h(\eta) \in X\}$ belongs to D_2, but $D_2 = D^{\mathbf{x}}_Y$ hence we can find (B_3, f) such that:

⊡ (a) $B_3 \in \mathscr{A}_{\mathbf{x}}$
  (b) $B_2 \le_{\mathbf{x}} B_3$
  (c) $f : (B_3 \backslash \max(B_3))) \to [B_3]^{<\aleph_0}$
  (d) $A_{\mathbf{x},B_3,f} \cap Y \subseteq h^{-1}(X)$.

So for each k, the set $Y_{3,k} := \{\eta \in A_{\mathbf{x},B_3,f} \cap Y_* : h(\eta) \in Z_k\}$ is included in $h^{-1}(Z_k \cap X)$. Hence $\{n < \omega : \rho_n \triangleleft \eta \in Y_{3,k}\}$ is included in $\{n < \omega : (\exists \eta)(\rho_n \le_{\mathbf{x}} \eta \in Y \wedge h(\eta) \in Z_k \backslash X)\}$ but $Z_k \cap X \subseteq \mathbb{N}$ is finite so by (∗) the latter is $\subseteq [0, n_k)$ for some n_k so $\rho_n \triangleleft \eta \in Y_{3,k} \Rightarrow n \le n_k$.

However necessarily for some $k, \varrho_k \in A_{\mathbf{x},B_3,f}$ hence $(\exists^{\aleph_0} n)(\varrho_k \le_{T_{\mathbf{x}}} \rho_n \in B_3)$ hence $(\exists^{\aleph_0} n)(\exists \eta)(\varrho_k <_{T_{\mathbf{x}}} \rho_n \le_{T_{\mathbf{x}}} \eta \in B_3)$, contradiction.

Moved 2011.7.21; was in §3 after 3.7, pg.16:

{k12f} *Remark* 5.1. 1) Considering $\mathbf{x} \in \mathbf{K}$, note that $\le_{\mathbf{x}}$ is directed so if PO do not know which $B \in \mathscr{A}^{\mathbf{x}}_\eta$ (or $B \in \underset{\sim}{A}^{\mathbf{x}}_\nu$) choose, but we have finitely many possibilities, PO can choose an upper bound and pretend this was NU's choice.

2) We should still sort out which version of the property is most convenient. We need more than $\mathbf{x} \in \mathbf{K}^{\mathrm{uf}}_{\aleph_1} \cap \mathbf{K}_{\mathrm{nt}}$, for which we may add §5.

Part II

§ 6. The Games[2]

It seems helpful to use

{8j.19}

Definition 6.1. Let $\mathbf{x} \in \mathbf{K}$.
1) We say E is a Y-filter for $\mathbf{x}$ when: $Y \in \mathrm{frt}_{\mathbf{x}}$ and E is a filter on some $I = I_E \subseteq I_{\mathbf{x},Y} = \cup\{\mathrm{psb}_{\mathbf{x}}(B_{\geq\nu}) : B \in \mathscr{A}_{\mathbf{x}}, \nu \in B\backslash\{\mathrm{rt}_{\mathbf{x}}\}$ and Y is an almost front of B and of $B_\nu\}$.
1A) If below we omit I we mean $I_{\mathbf{x},Y}$.
1B) Writing B instead of Y means $\max(B)$.
2) We say E is a standard Y-filter for $\mathbf{x}$ on I when in addition any set $\mathscr{Y}$ of the following form belongs to E:

- $\odot$ for some finite $Z \subseteq T_{>\mathrm{rt}(\mathbf{x})}$ and $\mathbf{B} \subseteq \{B \in I_E : \mathrm{rt}_{\mathbf{x}}(B)$ is $\leq_{T_{\mathbf{x}}}$-incomparable with every member of $Z\}$ we have
  - (a) $\mathscr{Y} = \cup\{\mathrm{psb}_{\mathbf{x}}(B) : B \in \mathbf{B}\} \cap I_E$
  - (b) if $B \in I_E$ and $B_1 \in \mathrm{psb}_{\mathbf{x}}(B) \cap I_E$ as above then there is $B_2 \in \mathbf{B} \cap \mathrm{psb}_{\mathbf{x}}(B_1)$.

2A) Let $E_{Y,I} = E^{\mathbf{x}}_{Y,I} = E_{Y,I}[\mathbf{x}]$ be the minimal standard Y-filter on $I = I_{\mathbf{x},Y}$, easily exists.
3) If E is a Y-filter for $\mathbf{x}$ in the universe $\mathbf{V}_1, \mathbb{P}$ a forcing notion and $\mathbf{V}_2 = \mathbf{V}_1^{\mathbb{P}}$, then E, in $\mathbf{V}_2$, means the minimal Y-filter for $\mathbf{x}$ on I_E which includes E in $\mathbf{V}_2$. Similarly if $E \subseteq \mathscr{P}(I_E)$.
4) For E as above and $\iota = 1$, let $\Game^\iota_E = \Game^\iota_{\mathbf{x},E} = \Game^\iota_{\mathbf{x},Y,E}$ be the following game:

- (A) A play last ω moves
- (B) in the n-th move
  - (a) the player NU chooses $\mathscr{Y}_n \in E$
  - (b) the player PO chooses $B_n \in \mathscr{Y}_n$
- (C) in the end of the play, the player PO wins the play when $\cup\{B_n : n < \omega\} \cup \{\eta_{\mathbf{x}}\}$ belongs to $\mathscr{A}_{\mathbf{x}}$.

5) For $\iota = 2$ we define $\Game^\iota_E$ as in part (4) except that in $(B)(a)$, $\mathscr{Y}_n \in E^+$.

{8j.20}

Definition 6.2. Assume

$\boxplus_{\mathbf{R}}$ $\mathbf{R}$ is a Borel relation on ${}^{\mathbb{N}}\mathbb{N} \times [\mathbb{N}]^{\aleph_0}$ written $f\mathbf{R}A$, such that $(\forall f \in {}^{\mathbb{N}}\mathbb{N})(\exists A \in [\mathbb{N}]^{\aleph_0})(f\mathbf{R}A)$.

1) As in Definition 6.1(4),(5), for $\iota = 3, 4$, instead $\iota = 1, 2$ we define the game $\Game^\iota_{\mathbf{x},Y,E,\mathbf{R}}$, but

- (α) during a play, in the n-th move the NU player chooses also $k_n \in \mathbb{N}$
- (β) in the end of the play, the player PO win the play iff there is $u \in [\mathbb{N}]^{\aleph_0}$ such that $\langle k_n : n < \omega\rangle\mathbf{R}u$ and $\cup\{B_n : n \in u\} \in D^{\mathbf{x}}_Y$

2) Replacing $\mathbf{R}$ by a family $\mathscr{R}$ of such Borel relations means "for every $\mathbf{R} \in \mathscr{R}$". If we write B instead of Y we mean $Y = \max(B)$.

[2] was the second half of §2 till 2011.6

{8j.20n}

Definition 6.3. In Definition 6.1(4),(5) and 6.2.
1) Replacing "game" by "N-game", where $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is countable and $\mathbf{x} \in N$, means that both sides are allowed to choose only objects from N.
2) Replacing "game" by "$\mathscr{S}$-game", where $\mathscr{S} \subseteq [\lambda]^{\le \aleph_0}$ is stationary, means that (for χ large enough) the player NU in the start chooses $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ such that N is countable and $\{\mathbf{x}, \lambda, \mathscr{S}\} \in A$ and $N \cap \lambda \in \mathscr{S}$ and then we continue as in part (1).

Remark 6.4. Mostly it does not matter whether we use the $\mathscr{S}$-game or the game, but proving $\mathbf{x} \in \mathbf{K}_{\rm ut} \cap \mathbf{K}^\iota_{\rm uf}$ is preserved in the limit of CS iterations, it seems that yes, see 9.2.

{8h.20}

Definition 6.5. 1) For $\iota = 1, 2$

- (A) $\mathbf{K}^\iota_{\min} = \mathbf{K}^{\iota,\min}$ but we may omit the min, is the class of $\mathbf{x} \in \mathbf{K}$ such that for every $B \in \mathscr{A}_\mathbf{x}$ for some standard max(B)-filter E, (if $\iota = 1$ equivalently for the minimal one), the player PO does not lose in the game $\partial^\iota_{\mathbf{x},B,E}$ (and $\mathbf{K}^\iota_{<\kappa} = \{\mathbf{x} \in \mathbf{K}_\iota : \|x\| < \kappa\}$, etc)
- (B) let $\mathbf{K}^\iota_{\rm uf} = \mathbf{K}^{\iota,{\rm uf}}$ be the class of $\mathbf{x} \in \mathbf{K}$ such that for every $B \in \mathscr{A}_\mathbf{x}$ for some standard $\max_\mathbf{x}(B)$-ultrafilter E, the player PO does not lose the game $\partial^\iota_{\mathbf{x},B,E}$
- (C) $K^\iota_E = \mathbf{K}^{\iota,E}$ means as in (A) for a fix filter E
- (D) $\mathbf{K}^{\iota,{\rm alt}}_E$ means for every standard ultrafilter extending E
- (E) $\mathbf{K}^{\iota,\min}_E$ means $\mathbf{K}^\iota_{c\ell_\mathbf{x}(E)}$ where $c\ell(E)$ is the minimal standard Y-filter extending E (assuing Y is determined by E and $\emptyset \notin c\ell(E)$.

Recall $T_\mathbf{x}$ a partial order not a tree.
2) For $\iota = 3, 4$ and $\mathbf{R}$ as in Definition 6.2

- (A) let $\mathbf{K}^{\iota,\min}_\mathbf{R}$ be as in (1)(A) using the game $\partial^\iota_{\mathbf{x},B,R}$
- (B) let $\mathbf{K}^{\iota,{\rm uf}}_\mathbf{R}$ be as in (1)(B) using the game $\partial^\iota_{\mathbf{x},B,\mathbf{R}}$
- (C) $\mathbf{K}^{\iota,E}_\mathbf{R} = K^\iota_{E,\mathbf{R}}$ means as in (1)(C) using the game $\partial^\iota_{\mathbf{x},B,E,\mathbf{R}}$
- (D) omitting $\mathbf{R}$ (in (2)(A),(B),(C)) means "for every $\mathbf{R}$ from $\mathbf{L}$ (separately).

3) For uniformizing notation, if $\iota = 1, 2$ then $\mathbf{K}^{\iota,\min}_\mathbf{R}, K^{\iota,{\rm uf}}_\mathbf{R}, K^{\iota,E}_\mathbf{R}$ means the same as in part (1), ignoring the $\mathbf{R}$; (this helps), similarly for the game.
4) We may above replace E and/or $\mathbf{R}$ by a set of such objects.
5) $\mathbf{K}^\iota_{\rm ut}$ is the set of ultra $\mathbf{x} \in \mathbf{K}_\iota$, see 2.6 below; we define $\mathbf{K}^{\iota,{\rm ut}}_E$ for $\iota = 1, 2$ when we restrict ourselves to the game $\partial^\iota_{\mathbf{x},{\rm fil}(E)}$, fil$(E)$ the filter generated by E (on the relevant set) and similarly $\mathbf{K}^{\iota,{\rm ut}}_{E,\mathbf{R}}$ for $\iota = 3, 4$.
6) We may omit the "min".
7) Adding a stationary $\mathscr{S} \subseteq [\lambda]^{\aleph_0}$ as a parameter means that we use the $\mathscr{S}$-game.

{8h.20b}

Remark 6.6. Presently we shall concentrate on $\mathbf{K}_\iota$ for $\iota = 1, 2$. The others are O.K., too.

{8j.21}

Claim 6.7. *1)* $\mathbf{K}_{\iota_1} \subseteq \mathbf{K}_{\iota_2}$ *when* $(\iota_1, \iota_2) = (1, 2), (2, 4), (1, 3), (2, 4)$.
2) Assume $\mathbf{x} \in \mathbf{K}$ *and* $E_1 \subseteq E_2$ *are standard* Y*-filters for* $\mathbf{x}$ *and* $\iota \in \{1, 3\}$.

The player PO not losing in $\partial^{\iota+1}_{\mathbf{x},E_1}$ *implies the player PO not losing in* $\partial^{\iota+1}_{\mathbf{x},E_2}$ *implies the player PO not losing* $\partial^{\iota}_{\mathbf{x},E_2}$ *implies the player PO not losing in* $\partial^{\iota}_{\mathbf{x},E_1}$.
3) Results for B, *i.e.* $\max(B)$ *implies ones for any* $Y \in \mathrm{frt}_{\mathbf{x}}(B)$.
4) If E *is a standard* Y*-ultra filter,* *<u>then</u>* $\mathbf{K}^{\iota}_{E,\mathbf{R}} = K^{\iota+1}_{E,\mathbf{R}}$ *for* $\iota = 1, 3$, *similarly in the other versions.*

Our aim now is to prove $\mathbf{K}^{\iota}_{\mathrm{ut}} \neq \emptyset$. The following three claims do the main work.

Claim 6.8. *1) Assume* N *is countable, transitive model of* ZFC^-, $N \models$ "$\mathbf{x} \in \mathbf{K}$ *and* $Y \in \mathrm{frt}_{\mathbf{x}}$ *and* E *is a* Y*-filter for* $\mathbf{x}$ *on* I_E, *see Definition 6.1" so necessarily* $\|\mathbf{x}\| \le \aleph_0$, $\mathbf{st}$ *a strategy of the player NU in the* N*-game* $\partial^{\iota}_{\mathbf{x},Y}$, *and* $\iota = 1$. *<u>Then</u> there is* $\mathbf{y} \in \mathbf{K}$ *such that* $\mathbf{x} \le_{\mathbf{K}} \mathbf{y}$ *and some* $\langle \mathscr{Y}_n, B_n : n < \omega \rangle$ *such that* {8j.23}

- (a) *this is a play of the* N*-game* $\partial^{\iota}_{\mathbf{x},Y,E}$
- (b) *every finite initial segment of the play belongs to* N *and*
- (c) *the player NU uses the strategy* $\mathbf{st}$, *well defined by (b)*
- (d) $\cup\{B_n : n < \omega\} \cup \{\eta_{\mathbf{x}}\} \in \mathscr{A}_{\mathbf{y}}$.

2) If $N \models$ "$\mathbf{st}$ *is a strategy of the player NU in* $\partial^{\iota}_{\mathbf{x},y}$ *and* $\iota = 1$" *<u>then</u> we have a similar conclusion.*

Proof. Easy. $\square_{6.8}$

Claim 6.9. *In 6.8, change the assumptions by* $\iota = 3$ *and* $\mathbf{R}$ *is as in 6.2, that is,* $\boxplus_{\mathbf{R}}$ *of 6.2 = 2.9 holds.* {8j.25}
<u>Then</u> we get a similar conclusion only

- $(a)'$ *this is a play of* $\partial^{\iota}_{\mathbf{x},Y,E,\mathbf{R}}$
- $(d)'$ *for some* $\mathscr{U}$ *we have* $\langle k_n : n < \omega \rangle \mathbf{R} B$ *and* $\cup\{B_n : n \in \mathscr{U}\} \cup \{\eta_{\mathbf{x}}\} \in \mathscr{A}_{\mathbf{y}}$.

Proof. Straight (see more in 6.17 = 2.19 proof). $\square$

Conclusion 6.10. *Assume* $\diamondsuit_{\aleph_1}$. *There is* $\mathbf{x} \in \mathbf{K}^{\iota}_{ut}$ *satisfying (a)-(c) of 2.5 for* $\iota = 1$. {8h.26}

Proof. Straightforward: let $\langle B^*_n : n < \omega \rangle$ be $\le_{\mathbf{x}}$-increasing cofinal in $(\mathscr{A}_{\mathbf{x}}, \le_{\mathbf{x}})$ as in the proof of 2.3 = 2.3. Choose a play $\langle \mathscr{Y}_n, B_n, k_n : n < \omega \rangle$ of the game. Any initial segment belongs to N such that for $B_n \in \cup\{\mathrm{sb}(B^*_{\ge \eta}) : \eta \in \mathrm{suc}_{\mathrm{rt}(\mathbf{x})}(B^*_{\eta})$. By $\boxplus_{\mathbf{R}}$ of Definition 6.2=2.3 choose $u \in [\mathbb{N}]^{\aleph_0}$ such that $\langle k_n : n < \omega \rangle \mathbf{R} u$, <u>then</u> we define $B_* = \cup\{B_n : n \in u\} \cup \{\mathrm{rt}_{\mathbf{x}}\}$ and define $\mathbf{y} \in \mathbf{K}_{\aleph_0}$ such that $\mathbf{x} \le \mathbf{y}, \mathscr{A}_{\mathbf{y}} = \mathscr{A}_{\mathbf{x}} \cup \{B_*\}$, etc. $\square_{6.9}$

Remark 6.11. In 6.8, 6.9 we can deal also with $\iota = 3, 4$, but this holds by monotonicity by 6.7, (2),(4). {8h.27}

Discussion 6.12. Note that if $\mathbf{x} \in \mathbf{K}_{\mathrm{ut}}, Y \in \mathrm{alm} - \mathrm{frt}(\mathbf{x})$ it does not follow that the ultrafilter $D^{\mathbf{x}}_Y$ is generated by $\le \|\mathbf{x}\|$ sets. It follows that it is equal to $\cup\{Y \cap \mathrm{sb}(B) : B \in \mathscr{A}_{\mathbf{x}}\}$; still... {8h.30}

Claim 6.13. *For any* $\mathbf{x} \in \mathbf{K}$ *and* $Y \in \mathrm{alm} - \mathrm{frt}(\mathbf{x})$, *the filter* $D^{\mathbf{x}}_Y$ *is generated by* $\le \|\mathbf{x}\| + \mathfrak{d}$ *sets.* {8h.33}

Claim 6.14. *Define* $\mathbf{R} = \{(\eta, u) : \eta \in {}^{\mathbb{N}}\mathbb{N}$ *and* $u = \mathbb{N}\}$. {8h.34}
1) $\mathbf{R}$ *satisfies* $\boxplus_{\mathbf{R}}$ *from 6.2.*
2) For $\iota = 1, 2$ *the games* $\Game^{\iota}_{\mathbf{x},E}$ *is equivalent to the game* $\Game^{\iota+2}_{\mathbf{x},E,\mathbf{R}}$ *so in particular the player PO has a winning strategy (do not lose in one game)* <u>*iff*</u> *this holds in the other game; similarly for the player NU.*
3) If we replace in $\mathbf{x}, T_{\mathbf{x}}$ *by* $\{B[< Y] : B \in T_{\mathbf{x}}, Y \in \mathrm{frt}_{\mathbf{x}}(B)\}$ *nothing changes.*

Proof. Obvious. $\square$

Discussion 6.15. It may be sometimes more transparent to use the following variant of $\mathbf{K}$ from Definition 1.5. It makes no real difference.

Definition 6.16. We have $\mathbf{x}$ has a roof $Y = Y_{\mathbf{x}} = Y[\mathbf{x}]$ where

- (a) $Y = \{\eta \in T_{\mathbf{x}} : \eta \neq \mathrm{rt}_{\mathbf{x}}$ and η is $\leq_{T_{\mathbf{x}}}$-maximal
- (b) if $B \in \mathscr{A}_{\mathbf{x},\eta}, \eta \in T_{\mathbf{x}}$ then $\max(B) \subseteq Y$.

{8h.26} **Conclusion 6.17.** *Assume* $\diamondsuit_{\aleph_1}$. *There is* $\mathbf{x} \in \mathbf{K}^{\iota}_{\mathrm{ut}}$ *satisfying (a)-(e) of 2.5 for* $\iota = 1$.

Proof. Straight. $\square_{6.17}$

§ 7. When $\mathbb{Q}$ preserves $\mathbf{x}$

Note: was middle part of §3

Claim 7.1. *Let $\iota \in \{1,3\}$ and $x \in \{\min, \text{uf}\}$, see Definitions 6.3, 6.5. Forcing with $\mathbb{Q}$ preserves "$\mathbf{x} \in \mathbf{K}^{\iota,x}_{\text{ut}}$" $\underline{\text{when}}$:* {k9}

(a) $\mathbf{x} \in \mathbf{K}^{\iota,x}_{\text{ut}}$

(b) *forcing with $\mathbb{Q}$ preserves*[3] $\mathbf{x} \in \mathbf{K}_{\text{uf}}$, *(i.e."$D^{\mathbf{x}}_B$ is an ultrafilter for $B \in \mathscr{A}_{\mathbf{x}}$)*

(c) (α) *if $x = \min$ then in $\partial^{\text{sb}}_{\mathbb{Q}}$, the ${}^{\omega}\omega$-bounding game for $\mathbb{Q}$, the* COM *player wins (see Definition 3.7 below)*

(β) *if $x = \text{uf}$, $\underline{\text{then}}$ in the game $\partial^{\text{ufbd}}_{\mathbb{Q}}$, the COM player wins, (see Definition 3.9 below).*

Claim 7.2. *1) Like 7.1 for $\iota = 3$ $\underline{\text{but}}$ with $\mathbf{R}$.* {k10}
2) Like 7.1 for $\iota = 1$, fixing E the minimal $\underline{\text{or}}$ $\iota = 3$ $\underline{\text{but}}$ we fix E and $\mathbf{R}$.

Remark 7.3. 0) As $\iota = 1$ is a special case of $\iota = 3, \mathbf{R} =$ trivial we can concentrate on $\iota = 3$. {b10d}
1) Presently the main case is $x = \min$ so having the minimal standard Y-filter E for some $Y \in \text{alm} - \text{frt}(\mathbf{x})$.
2) The case we fix E and use $\mathbf{K}^{\iota,\min}_E$ have not sort out.
3) The ultrafilter case is done but have not sort out usefulness.

Remark 7.4. Claims 7.1, 7.2 should be a parallel to the theorem: if D is a P-point and $\mathbb{Q}$ is a proper forcing and in $\mathbf{V}^{\mathbb{Q}}, D$ is an ultrafilter $\underline{\text{then}}$ in $\mathbf{V}^{\mathbb{Q}}, D$ is a P-point.

Proof. $\underline{\text{Proof of 7.1, 7.2}}$:
By 6.13, without loss of generality $\iota = 3$. Clearly it suffices to prove 7.2(3) and recalling Definition 6.5(3) we are dealing with the game $\partial^{\iota}_{E,\mathbf{R}}$.

Note that fixing E also fixes Y and for notational simplicity (justify by 6.13) there is $B \in \mathscr{A}_{\mathbf{x}}$ and $Y = \max(B)$.

First we assume

$\boxplus_1$ (a) E is a standard Y-filter for $\mathbf{x}$ in $\mathbf{V}$ and $\boxplus_{\mathbf{R}}$ of 6.2 holds

(b) the player NU has no winning strategy in the game $\partial^{\iota}_{\mathbf{x},E,\mathbf{R}}$ in $\mathbf{V}$

(c) $\iota = 3$.

Let

$\boxplus_2$ (a) $\underset{\sim}{E}_1$ is the $\mathbb{Q}$-name of the standard Y-filter generated by E in $\mathbf{V}^{\mathbb{Q}}$ suffice in the $x = \min$ case

(b) $\underset{\sim}{E}_2$ is a $\mathbb{Q}$-name of a Y-ultrafilter extending $\underset{\sim}{E}_1$ (used only in the $x = \text{uf}$ case

(c) let $\underset{\sim}{E}_x$ be $\underset{\sim}{E}_1$ if $x = \min$ and $\underset{\sim}{E}_2$ if $x = \text{uf}$.

We shall prove

$\boxplus_3$ the player NU has no winning strategy in the game $\partial^{\iota,x}_{\mathbf{x},\underset{\sim}{E}_x,\mathbf{R}}$ in $\mathbf{V}^{\mathbb{Q}}$.

[3]this implies the version with "$Y \subseteq B$ is an almost front of B'

Toward contradiction assume that $p_* \in \mathbb{Q}$ and $p_* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbf{s}}t_1$ is a winning strategy for the player NU in $\partial^{\iota,x}_{\mathbf{x},\underset{\sim}{E}_x,\mathbf{R}}$".

Let $\mathbf{st}_2$ be a winning strategy for the COM player in the game $\partial^{\mathrm{vfbd}}_{\mathbb{Q},p_*}$ if $x =$ min, $\partial^{ufbd}_{\mathbb{Q},p_x}$ if $x = $ uf.

We now define, in $\mathbf{V}$, a strategy $\mathbf{st}_3$ for the player NU in the game $\partial^{\iota,x}_{\mathbf{x},\underset{\sim}{E}_x,\mathbf{R}}$. On the side NU simulates plays of the game $\partial^{x}_{\mathbb{Q},p}$ while playing the game $\partial^{\iota,x}_{\mathbf{x},\underset{\sim}{E}_1,\mathbf{R}}$.

<u>Case 1</u>: $x = \mathrm{uf}, \iota = 1$

In the n-th move we do the following.

First, let $\underset{\sim}{Y}_n$ be the $\mathbb{Q}$-name of NU's n-th move by $\underset{\sim}{\mathbf{s}}t_1$ and $I_n = I_E$.

Second, we consider the following set of possible moves of the player INC in the game $\partial^{\mathrm{ufbd}}_{\mathbb{Q},p}, \Lambda_n = \{(I_n, E, \underset{\sim}{E}_2, \underset{\sim}{Y}_n \cap Z) : Z \in E\}$.

So the strategy $\mathbf{st}_2$ of the COM player in $\partial^{\mathrm{ufbd}}_{\mathbb{Q},p}$ gives us a function F_n with domain Λ_n such that $F_n((I, E, \underset{\sim}{E}_n, \underset{\sim}{Y}_n \cap \underset{\sim}{Y})) \in I_n$.

Third, let him make his move in $\partial^{\iota,x}_{\mathbf{x},E,\mathbf{R}}$: Rang(F_n) so a subset of I_n

$(*)_1$ Range(F_n) is a legal move in $\partial^{\iota,x}_{\mathbf{x},E,\mathbf{R}}$.

[Why? It is a subset $I_E = I_n$ and if it does not belong to E then we get a contradiction to $F_n(I_n, E, \underset{\sim}{E}_2, \underset{\sim}{Y}_n \cap 2)$ being well defined.]

Fourth, let PO make his move $\partial^{\iota,x}_{\mathbf{x},E,R}$: it is $(B_n, k_n) \in \mathrm{Rang}(F_n) \times \mathbb{N}$.

Fifth, (on the side) we choose Z_n such that $F_n(I_n, E, \underset{\sim}{E}_2, \underset{\sim}{Y}_n, Z) = (B_n, k_n)$.

Clearly $\mathbf{st}_3$ is well defined (strategy for the player NU in the game $\partial^{\iota,x}_{\mathbf{x},E,\mathbf{R}}$ in $\mathbf{V}$) hence, in $\mathbf{V}$, for some play of $\partial^{\iota,x}_{\mathbf{x},E,\mathbf{R}}$ in which NU uses the strategy $\mathbf{st}_3$, the player PO, wins the play.

This play defines a sequence $\langle(\underset{\sim}{Y}_n, F_n, B_n, k_n, Z_n) : n < \omega\rangle$ as above has been defined but in this play PO wins so

$(*)_2$ there is an infinite $u \subseteq \mathbb{N}$ such that $\cup\{B_n : n \in u\} \cup \{\mathrm{rt}_{\mathbf{x}}\} \in \mathscr{A}_{\mathbf{x}}$ and $\langle k_n : n < \omega\rangle \mathbf{R} U$.

Now

$(*)_3$ $\langle(I_n, E, \underset{\sim}{E}_2, \underset{\sim}{Y}_n \cap Z_n), B_n : n < \omega\rangle$ is a play of the game $\partial^{ufbd}_{\mathbb{Q},p}$ in which COM uses the strategy $\mathbf{st}_2$.

[Why? Read our choices.]

By the choice of $\mathbf{st}_2$ and

$(*)_4$ the COM player wins this play, hence there are q, u such that

$(*)_5$ (a) $\mathbb{Q} \models$ "$p_* \leq q$"

(b) $q \Vdash_{\mathbb{Q}}$ "$B_n \in \underset{\sim}{Y}_n$".

Together

$(*)_6$ (a) $\mathbb{Q} \models$ "$p_* \leq q$"

(b) $u \in [\mathbb{N}]^{\aleph_0}$

(c) $\langle k_n : n < \omega\rangle \mathbf{R} u$

(d) $\cup\{B_n : n \in u\} \cup \{\mathrm{rt}_{\mathbf{x}}\} \in \mathscr{A}_{\mathbf{x}}$

(e) $q \Vdash$ "$B_n \in \underset{\sim}{Y}_n$ for $n < \omega$".

Also

$(*)_7$ (a) $p_* \Vdash$ "$\langle \underset{\sim}{Y}_n, (B_n, k_n) : n < \omega\rangle$ is a play of $\partial^{\iota,x}_{\mathbf{x},\underset{\sim}{E}_x,\mathbf{R}}$ in which the player NU uses the strategy $\mathbf{st}_1$ and (by $(*)_5$), PO wins the play.

[Why? Read.]

This contradicts the choice of $(p_*, \underset{\sim}{\mathbf{s}}t_1)$, so we are done.

Case 2: $x = \min$

In the n-th move let NU's move as dictated by $\underset{\sim}{\mathbf{s}}t_1$ be $\underset{\sim}{Y}_n$ which is a $\mathbb{Q}$-name of a member of $\underset{\sim}{E}_1$.

So without loss of generality

$(*)_1$ (a) $\underset{\sim}{Y}_n = \underset{\sim}{Y}^1_n \cup \underset{\sim}{Y}^2_2$

(b) $\underset{\sim}{Y}^1_n$ is a $\mathbb{Q}$-name of a member of E (so of an old set, one frmo $\mathbf{V}$)

(c) $\underset{\sim}{Y}^2_n$ is a $\mathbb{Q}$-name of a member of the minimal standard Y-filter, generated in $\mathbf{V}^Q$ by E, i.e. a set of the form from $\odot$ of Definition 6.1(2).

So there are $\mathbb{Q}$-names $\underset{\sim}{Z}_n, \underset{\sim}{\mathbf{B}}_n$ such that

(α) $\underset{\sim}{Y}^2_n = \cup\{\text{psb}(B) : B \in \mathbf{B}_n\} \cap I_E$

(β) $\underset{\sim}{Z}_n \subseteq T_{\mathbf{x}} \backslash \{\text{rt}_{\mathbf{x}}\}$ finite (i.e. this is forced)

(γ) $\underset{\sim}{\mathbf{B}}_n \subseteq \underset{\sim}{\mathbf{B}}^0_n := \{B \in I_T : \text{rt}_{B_{\mathbf{x}}}(B)$ is $\leq_{T_{\mathbf{x}}}$-incomparable, with every member of $\underset{\sim}{Z}_n\}$

(δ) $\underset{\sim}{\mathbf{B}}_n$ is dense: if $B' \in \mathbf{B}^0_n$ then there is $B'' \in \underset{\sim}{\mathbf{B}}^0_n, B'' \in \text{psb}(B')$ (this corrects 6.1).

Delayed

NOTE: If we use $E =$ the minimal standard Y-filter then we can omit $\underset{\sim}{Y}^1_n$.

Case 2A: $x = \min, E$ minimal

[We reinterpret $\partial^{\text{sb}}_{Q,p}$: and $\mathbf{st}_2$: in the n-th move first INC gives a name $\underset{\sim}{\alpha}$ of an ordinal (or just member of $\mathbf{V}$), second COM gives $w_n \subseteq$ Ord finite, third INC givews $\mathscr{T}_n, \underset{\sim}{F}_n$ as before, fourth COM gives η_n, in the end we demand $q \Vdash \underset{\sim}{\alpha}_n \in w_n, (\exists \nu <_{\mathscr{T}_n} \eta_n)(\underset{\sim}{F}_n \backslash \nu) \leq_{\mathscr{T}_n} \eta_n)$.]

Second, we on the side give $\partial^{\text{sb}}_{\mathbb{Q},p}$ the first choice for INC in the first half of the n-th move as $\underset{\sim}{\alpha}_n$ he chooses $\underset{\sim}{Z}_n$ (a $\mathbb{Q}$-name of a member of $\mathbf{V}$) let COM by $\mathbf{st}_2$ choose Z^*_n a finite subset of $\mathbf{V}$, so $\in \mathbf{V}$ and without loss of generality $Z^*_n \subseteq \mathscr{T}_{\mathbf{x}} \backslash \{\text{rt}_{\mathbf{x}}\}$ (reclal $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{Z}_n \subseteq T_{\mathbf{x}} \backslash \{\text{rt}_{\mathbf{x}}\}$".

Third, we make on the side the second choice for INC in the second half of the n-th move: let

$$\mathscr{T}_n = \{\eta : \quad \eta \in {}^{\omega>}(I_E) \text{ such that}$$

- $\ell + 1 < \ell g(\eta) \Rightarrow \eta(\ell + 1) \in \text{psb}(\eta(\ell))$
- if $\ell g(\eta) \neq \emptyset$ then $\text{rt}_{\mathbf{x}}(\eta(0))$

is $\leq_{\mathscr{T}_{\mathbf{x}}}$ -incomparable with every member of $Z_n\}$

choose

$$\underset{\sim}{F}_n(\eta) = \eta\hat{\ }\langle B\rangle \in \mathscr{T}_n \text{ for some } B \in \mathbf{B}.$$

The rest should be clear. $\square$

§ 8. Private Appendix
Part III

Discussion 8.1. Should [Sh:F1127, §5] appear here? What about the claim using [Sh:F1127, §(1B)]? Is it written in a way compatible with $\mathbf{x}$? [Sh:480]?

Anotated Content of Part II

* * *

Moved 2011.6.20 from §0, pg.4:

This work hopefully will be continued in [Sh:F1112]. The ultrafilters introduced here are in a sense parallel to Ramsey ultrafilters, that is the basic building blocks are the filter of cofinite sets (i.e. tree sums). We shall there start with a suitable ommitory creature forcing.

Moved 2011.6.26 from before 3.1 and from part (4) of 3.2, pg. 13:

§3 was revised (2011.5.13) thinking on the case $\iota = 3, \mathbf{x} \in \mathbf{K}^\iota_{\rm min}$. But now (2011.5.19) it seems only an initial segment: claims 3.1, 3.10, Definition 3.6 and to some extent 3.9 are useful.

As for Claims 7.1, 7.2 not so clear; the use of $\eth^{\rm uf,sb}$ seems O.K. but not sure it will help; and 9.2 on where not checked. Another avenue is with details on the iteration is in the works. Replace the games from the end of §2 by relations.

4) In [Sh:F1127, §(13B)] we will intend to give a better and more general result. Note: here we concentrate on the case $\mathfrak{d} = \aleph_1$, but $\mathbf{x} \in \mathbf{K}_{\rm ut} \cap \mathbf{K}^{\rm uf}_{\aleph_1}$ with $\|X\| = \aleph_2$ can exist even if $\mathfrak{d} > \aleph_1$.

Moved 2011.6.26 from the proof of 3.10, pg.17:
4) Translating from $\partial^{\rm ufbd}_{\mathbb{Q},p}$ to $\partial^{\rm sb}_{\mathbb{Q},p}$:
We are given $y = (T, \underset{\sim}{F})$, a possible choice of INC in $\partial^{\rm sb}_{Q,p}$.
We define

$(*)_1$ (a) $I_y = \mathscr{T}$
(b) $E_y \in \mathbf{V}$ is any ultrafilter on $\mathscr{T}$ such that:
$F \in \mathscr{F}_{\mathscr{T}} = \{F : {\rm Dom}(F) = \mathscr{T}, F(\eta) \in {\rm suc}_{\mathscr{T}}(\eta) \text{ for } \eta \in T\}$
then $A_F = \{\eta \in \mathscr{T} : (\exists \nu, \varrho)(\nu <_I \varrho \leq_I \eta_1), \varrho \in {\rm suc}_{\mathscr{T}}(\nu)\hat{\ }\rho = F(\nu)\}$
(c) $\underset{\sim}{X}_y = \{\eta \in T$: there is $\rho <_{\mathscr{T}} \nu \leq_{\mathscr{T}} \eta, \nu = \underset{\sim}{F}(\rho)\}$
(d) $\underset{\sim}{E}^+_y$ is a Q-name of an ultrafilter on $\mathscr{T}$ extending E_y such that
$V^Q \Vdash_Q$ "if $F \in \mathscr{F}_F$ then $A_{\mathscr{F}} \in \underset{\sim}{E}^+_y$".

[Why exists? I_y obvious, E_y easy, $\underset{\sim}{E}^+_y$ have to prove

- if $p \in \mathbb{Q}, p \Vdash$ "$\underset{\sim}{F}_0, \dots, \underset{\sim}{F}_n \in \mathscr{F}_{\mathscr{T}}$" and $X \in E$ then for some (q, η) we have $p \leq_{\mathbb{Q}} q, \eta \in X_0 \subseteq \mathscr{T}, q \Vdash$ "$\eta \in A_{F_\ell}$" for $\ell < n$.

Let $\mathscr{T}_k = \{\eta \in T : \eta \text{ of level } k\}$.
By induction on n we choose $\langle F'_i{\restriction}T_k : i \leq n\rangle$ such that $F'_\ell \in \eta \in {\rm suc}_{\mathscr{T}}(\eta)$ and if $\eta \in T_k$ then for some $q \in Q$:
$\mathbb{Q} \models$ "$p \leq q$"
$q \Vdash$ "$F'_i(\rho) = \underset{\sim}{F}_i(\rho)$" if $\rho \trianglelefteq \eta$ is of level i.
So

$(*)$ $(I_y, E_y, \underset{\sim}{E}^+_y, \underset{\sim}{X}_y)$ is a possible choice for NU in $\partial^{\rm ufnb}_{Q,p}$.

Also if $q \Vdash$ "$\eta \in \underset{\sim}{X}_y$" then q forces η is as required for y in $\partial^{\rm sb}_{\mathbb{Q},p}$.
5) The same proof.
Translating from $\partial^{\rm sb}_{Q,p}$ to $\partial^{\rm ufbd}_{\mathbb{Q},p}$:
So we are given $y = (I, E, \underset{\sim}{E}^+, \underset{\sim}{X})$ such that

$(*)$ (a) E an ultrafilter on I, in $\mathbf{V}$
(b) $\underset{\sim}{E}^+ \supseteq E$ an ultrafilter on I, a $\mathbb{Q}$-name
(c) $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{X} \in \underset{\sim}{E}^+$".

Let $\mathscr{T}_y = {}^{\omega >}I$
$\underset{\sim}{F}_y : \mathscr{T}_y \to \mathscr{T}_y$ is $\underset{\sim}{F}_y$
NOT CLEAR.

§ 9. When $\mathbb{Q}$ preserves $D^{\mathbf{x}}_Y$ ultrafilter

Note: Was the second half of §3 till 2011.6

Notation 9.1. All $Y \subseteq I_E$ should be $\mathscr{Y}$.

{k19} **Claim 9.2.** *Major Claim Assume* $\mathbf{x} \in \mathbf{K}_{\rm ut} \cap \mathbf{K}^\iota_{\rm uf}$ *for* $\iota = 1, 3$.
1) The forcing $\mathbb{Q} = \mathbb{Q}^2_D$ *preserves* $\mathbf{x} \in \mathbf{K}_{\rm uf}$ *when* D *is a P-point (and* $\mathbb{Q}^2_D$ *defined in 4.1).*
2) Instead "D is a P-point", it suffices D is a non-principal ultrafilter on $\mathbb{N}$ *such that* $D \perp D^{\mathbf{x}}_B$ *for* $B \in \mathscr{A}_{\mathbf{x}}$.

Remark 9.3. 1) The $\iota = 3$ case will follow.
2) Not for part (3) as the relevant place.
3) Recall $D_1 \perp D_2$ where D_1, D_2 are non-principal ultrafilters on $\mathbb{N}$ when they do not have a common $\leq_{\rm RK}$-lower bound.
4) Similarly for $\mathbb{Q}^{2,*}_{2,D}$.

Proof. See 4.4 = 4.4.

Stage A: Fix

$\boxplus_1$ (a) $B_* \in \mathscr{A}_{\mathbf{x}}$
(b) $Y_* = \max(B_*)$
(c) $p_* \in \mathbb{Q}$ and
(d) $p_* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A} \subseteq Y_*$".

We should prove that some $p' \geq p_*$ forces that $\underset{\sim}{A}$ or $Y \backslash \underset{\sim}{A}$ include some member of $(D^{\mathbf{x}}_{\eta,B})^{\mathbf{V}[\mathbb{Q}]}$.

Case 1: $\iota = 1$.
Let E be a standard ultrafilter on $I_{\mathbf{x}}$, see Definition.
Let $\mathbf{X} = \{(p, n_*) : p_* \leq p \in \mathbb{Q}$ and $n_* \in \mathbb{N}\}$

$\boxplus$ for any pair $(p, \mathbf{t}), \mathbf{t}$ a truth value, $p \geq p_*$ we define a strategy $\mathbf{St} = \mathbf{St}_{p,n_*}/\mathbf{St}_{p,\mathbf{t}}$ for the player NU for the game $\eth^\iota_{\mathbf{x},E}$ so in the n-the move a pair (Y_n, B_n) is chosen.

On the side NU also chooses p_n (the following leaves some free choice) such that $p_{-1} = p$

$\oplus$ (a) $p_{n+1} \leq_{{\rm pr},n} p_n$
(b) [old for some truth value $\mathbf{t}$] for every $\eta \in {}^{n(*)}2, p^{[\eta\hat{}<1>]}_n \Vdash$ "$B_n \cap Y \subseteq \underset{\sim}{A}^{[\mathbf{t}]}$"
(c) p_n is chosen by the player NU in the n-th move after choosing Y_n, B_n
(d) $Y_n = Y[q^{n,n(*)}, p_{n-1}]$ where $Y[n, n(*), \mathbf{t}, p] = \{B \in I_{\mathbf{x}}$: there is a q/a pair$(q, \mathbf{t})$ such that $p \leq_{{\rm pr},n} q$ and for every $\nu \in {}^{[n(*),n]}2/{}^n2$ [old for some $\eta \in {}^{n(*)}2$] we have $q^{[\eta\hat{}(\nu^{-1}<1>]} \Vdash$ "$B \subseteq \underset{\sim}{A}^{[\mathbf{t}]}\}$.

We claim

$\boxplus$ for some pair$(p, n_*, \mathbf{t})/(p, \mathbf{t}) \in \mathbf{X}$ the strategy $\mathbf{st}_{p,\mathbf{t}}/\mathbf{st}_{p,n_*}$ is a well defined strategy, i.e. always give a legal move, i.e. $Y_n \in E$.

[Why? Assume not; first by $(p_*, 0)$ so by the assumption there are $n_1 > 0, p'$ such that $p \le_{\mathbb{Q}} p_1$ and $Y[0, n_1, p'] = \emptyset \bmod E$, i.e. p' is p_{n-1}, n_1 is n when we are stuck in some play of the game $\partial^\iota_{\mathbf{x},E}$ in which the player NU uses the strategy $\mathbf{st}_{p_*,0}$. Second, try the pair(p', n_1) and find p'' and $n_2 > n_1, p' \le_{\mathbb{Q}} p''$ and $Y[n_2, n_1, p'] = \emptyset \bmod E$. Now for every $B \in I_{\mathbf{x}}$ $\quad p_{n+1} \le_{\mathrm{pr},n} p_n$ there are q_B and sequence $\langle \mathbf{t}_{(B,\eta)} : \eta \in {}^{n_2}2 \rangle$ of truth value and C_β such that

- (a) $p'' \le q_B$ in $\mathbb{Q}$
- (b) $C_\beta \in \mathrm{psb}_{\mathbf{x}}(B)$
- (c) if $\eta \in {}^{n_2}2$ then $q_B^{[\eta]} \Vdash$ "$C_B \subseteq \underset{\sim}{A}^{[\mathbf{t}(B,\eta)]}$" hence this holds for every $C \in \mathrm{psb}_{\mathbf{x}}(C_B)$.

As E is standard we can deduce

- $(*)$ the set of $B \in I_{\mathbf{x}}$ such that there are $\langle q_\beta, \mathbf{t}(B, \eta) : \eta \in {}^{n_2}2 \rangle$ as above belongs to E.

As E is an ultrafilter there is a sequence $\bar{\mathbf{t}} = \langle \mathbf{t}(\eta) : \eta \in {}^{(n_2)}2 \rangle$ such that

- $(*)$ $\mathscr{Y}_* \in E$ where $\mathscr{Y}_* = \{B \in I_{\mathbf{x}}$: for some $q \ge p'', q \Vdash C \subseteq \underset{\sim}{A}^{[\mathbf{t}(\eta)]}$ for $\eta \in {}^{(n_2)}2$".

By the choice of (p'', n_2) we have

- $(*)$ for $\mathbf{t} \in \{0, 1\}$ for some $\eta \in {}^{[n_1,n_2]}2$, in the sequence $\langle t(\nu\hat{}\eta) : \nu \in {}^{(n_1)}2 \rangle$, the value $\mathbf{t}$ does not appear, so it is constantly $1 - \mathbf{t}$.

But this contradicts the choice of (p', n_1). So indeed the strategy is well defined, i.e. $\boxplus$ holds.

$\boxplus$ for some (q_*, n_*) and play $\langle (Y_n, B_n) : n < \omega \rangle$ of $\partial^\iota_{\mathbf{x},B_*}$ we have
- (a) the player NU uses the strategy $\mathbf{st}_{q_*,n_*}$
- (b) let the play be $\langle (Y_n, B_n) : n < \omega \rangle$ and $\langle p_n : n < \omega \rangle$ played on the side
- (c) the player PO wins, i.e. $(\cup\{B_n : n < \omega\}) \cap Y \in D^{\mathbf{x}}_B$
- (d) let $\mathbf{t}_n \in \{0, 1\}$ be such that $(\forall \eta_n \in {}^{[n_*,n)}2))(\exists \nu \in {}^{n_*}2)(p_{n+1}^{[\eta\hat{}<1>]} \Vdash$ "$B_n \subseteq \underset{\sim}{A}^{[\mathbf{t}]}$".

[Why? By the $\boxplus$ above recalling $\mathbf{x} \in \mathbf{K}^\iota_{\mathrm{ul}}$ so NU cannot have a winning strategy.]

$\boxplus$ (a) let $\mathbf{t}_*$ be such that $\cup\{B_n \cap Y : \mathbf{t}_n = \mathbf{t}_*\} \in D^{\mathbf{x}}_{B_*}$.

[Why exist? As $D^{\mathbf{x}}_{B_*}$ is an ultrafilter $\mathbf{t}_n \in \{0, 1\}$ for $n < \omega$ adn $\cup\{B_n \cap Y : n < \omega\} \in D^{\mathbf{x}}_{B_*}$.]

$\boxplus$ (a) $p_\omega = \lim\langle p_n : n < \omega \rangle$ belongs to $\mathbb{Q}$ and is a $\le_{\mathbb{Q}}$-lub of $\{p_n : n < \omega\}$
(b) $\mathrm{first}_n(p_\omega/E_{p_\omega} = \mathrm{first}_n(p_{n+1})/E^{p_{n+1}}$.

[Why?

$\boxplus$ there is $X \subseteq \{n : \mathbf{t}_n = t_*\}$ such that
- (a) $\cup\{B_n : n \in X\} \in D^{\mathbf{x}}_{B_*}$ but
- (b) $\cup\{\mathrm{first}_n(p_\omega)/E_{p_\omega} : n \in X\} \in D$.

[Why? First the quotient $D_1 := D/E^{p_\omega}$, i.e. $\{X \subseteq \mathbb{N} : \cup\{\mathrm{first}_n(p_\omega)/E_{p_\omega} : n \in X\}$ belongs to $D\}$ is a non-principal ultrafilter on $\mathbb{N}$ which is $\leq_{\mathrm{RK}} D$ hence is a P-point.

Second, $D_2 = \{X \subseteq \mathbb{N} : \cup\{B_n \cap Y : n \subseteq X\} \in D^{\mathbf{x}}_{B_*}\}$ a non-principal ultrafilter on $\mathbb{N}$ which is a quotient of $D^{\mathbf{x}}_{B_*}$; (why non-principal? as $\mathrm{rt}(B_n) \neq \mathrm{rt}_{\mathbf{x}}$ hence $B_n \cap Y = \emptyset \bmod D^{\mathbf{x}}_{B_*}$). But $\leq_{\mathrm{RK}}$-below $D^{\mathbf{x}}_{B_*}$ there is no P-point, see **??** and prove. But is there is no such X, <u>then</u> $D_1 = D_2$ clear contradiction. So we are done.]

Choosing X as above define $q = (f_q, E_q)$ by: $f_q = f_p, E_q = E_p {\restriction} \cup\{\mathrm{first}_n(p^\omega)/E_{p_\omega} : n \in X\}$. Now q is as required.

* * *

<u>Case 2</u>: $\iota = 3$

We try to use the partition theorem of Goldstern-Shelah [GoSh:388] in the middle not in the end. So fixing $(p_*, \underset{\sim}{A})$ we let

$\boxplus$ for $p \geq p_*$ and n let
- (a) $\Lambda_n = \{\bar{\mathbf{t}} : \bar{\mathbf{t}} = \langle \mathbf{t}_\eta : \eta \in {}^n\{-1,1\}\rangle$ where $t_\eta \in \{0,1\}\}$
- (b) for $\bar{\mathbf{t}} \in \Lambda_n$ let $I_{p,n,\bar{\mathbf{t}}} = \{B \in I_{\mathbf{x}}$: there is $q \geq p$ such that $\eta \in {}^n\{-1,1\} \Rightarrow q \Vdash_{\mathbb{Q}}$ "$B \subseteq \underset{\sim}{A}^{[\mathbf{t}_\eta]}$"
- (c) $\Lambda_{p,n} = \{\bar{\mathbf{t}} \in \Lambda_n : I_{p,n,\bar{\mathbf{t}}} \in E\}$.

Now

$\boxplus$ for $p \geq p_*$ and n:
- (a) $\cup\{I_{p,n,\bar{\mathbf{t}}} : \bar{\mathbf{t}} \in \Lambda_n\} \in E_n$
- (b) $\Lambda_{p,n} \neq \emptyset$
- (c) if $p_n \leq p_1 \leq_{m,m} p_2$ then $\Lambda_{p_2,n} \in \Lambda_{p_1,n}$
- (d) for some $p_1, p \leq_{\mathrm{pr},n} p_1$ and $p_1 \leq_{\mathrm{pr}} p_2 \Rightarrow \Lambda_{p_2,n} = \Lambda_{p_1,n}$

$\boxplus$ there is $\bar{p} = \langle p_n : n \leq \omega\rangle$ such that
- (a) $p_0 = p_*$
- (b) $p_n \leq_{\mathrm{pr},n} p_{n+1}$
- (c) if $p_{n+1} \leq_{\mathrm{pr},m} q$ then $\Lambda_{q,n} = \Lambda_{p_{n+1},n}$
- (d) p_ω is $\lim\langle p_n : n < \omega\rangle$.

Note

$\boxplus$ if $m < n, \bar{t} \in \Lambda_{p_{n+1},n}, \mathbf{t}' \in \Lambda$ and $(\forall \eta \in {}^m\{-1,1\})(\exists \nu \in {}^n\{-1,1\})(\eta \triangleleft \nu \wedge \mathbf{t}'_\eta = \mathbf{t}_\nu)$ <u>then</u> $\bar{\mathbf{t}}' \in \Lambda_{p_{m+1},m}$.

Hence

$\boxplus$ there is a sequence $\langle \bar{\mathbf{t}}_n : n < \omega\rangle$ such that $\mathbf{t}_n \in \Lambda_{p_{n+1},n}$ for $n < \omega$.

By the partition theorem of Goldstern-Shelah [GoSh:388]

$\boxplus$ there are increasing $\langle n_i : i < \omega\rangle$ and $\eta_i \in {}^{[n_{i_1},n_i)}\{-1,1\}$ stipulation $n_{-1} = 0$ and $\mathbf{t}$ such that if $\rho \in {}^i\{-1,1\}, \eta = \eta_0 \cup \rho(0) \cup \eta_1 \cup \rho(1) \cup \eta_2 \cup \ldots \cup \rho(i-1) = \eta_{i-1} \cup \eta_i$ then $\mathbf{t}_{p_{n_i},\eta} = \mathbf{t}$.

Note that any subsequence works, too.

Also we have much choice in the choice of the p_n's and the $\bar{\mathbf{t}}_n$ depend on $\bar{p}$.

Now as we use $\iota = 3, 4$ that is we choose $\mathbf{R}$ such that letting $f : \mathbb{N} \to \mathbb{N}$ be interpreted as $f(n) \in \{\langle \mathbf{t}_\eta : \eta \in {}^n\{-1,1\}\rangle : \mathbf{t}_\eta \in \{0,1\}\}$ so $f((n)(n) \in \{0,1\}$. Let $f\mathbf{R}, \mathscr{U}$ <u>iff</u> letting $f(n) = \langle \mathbf{t}_\eta : \eta \in {}^n\{-1,1\}$ and $\langle \eta_i : i < \omega\rangle$ listing u in increasing order the demand there holds.

We define a strategy for $\mathbf{st}$ for the player NU in the game $\partial^\iota_{\mathbf{x},E}$ as follows. The player NU onthe side chooses $p_n, \bar{\mathbf{t}}_n$ ($\mathbf{t}_n$ is coded by a finite number k_n) if

- $(*)$ (a) $\langle p_m : m \leq n\rangle$ as above, p_m chosen in the n-th step
  - (b) $\mathbf{t}_n \in \Lambda_{p_m}$
  - (c) $\mathscr{Y}_n = \{B \in I_{\mathbf{a}}$: there is $q \in \mathbb{Q}$ such that $p_n \leq_n q$ and $q \Vdash_{\mathbb{Q}}$ "$B \subseteq \underset{\sim}{A}^{[\mathbf{t}_{n,\eta}]}$ for $\eta \in {}^n\{-1,1\}$.

As $\mathbf{x} \in \mathbf{K}^\iota_{\mathrm{ut}}$, there is $\langle(\mathscr{Y}_n, B_n) : n < \omega\rangle$ be a player such that the player NU uses the strategy NU as witnessed by $\langle p_n, \mathbf{t}_n : n < \omega\rangle$ and the player PO wins. Let $u = \{n_i : i < \omega\}$ and $\langle \eta_i : i < \omega\rangle$ witness this. Let q_1 be the limit of the p_n's

- $\boxplus$ there is $C \subseteq \mathbb{N}$ such that:
  - (a) $\cup\{\mathrm{first}_{q_i}(b)$: for some $i, \eta \in [n_i, n_{i+1})$ and $i \in C\} = \emptyset \mod D$
  - (b) $\cup\{B_{n_i} : i \in C\} \in D^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})}$.

If so, we define $q_2 \in \mathbb{Q}$:

- $\mathscr{U}_{q_2} = \mathscr{U}_{q_1} \cup \bigcup\{\mathrm{first}_{q_1}(\ell)/E_{q_1} : \ell < n_0$ or $\ell = n_i\} \cup \mathscr{U}_q$
- $f_{q_2}{\restriction}\mathscr{U}_{q_1} = f_{q_1}{\restriction}\mathscr{U}_{q_1}$
- $f_{q_2}{\restriction}(\mathrm{first}_{q_1}(\ell)/E_{q_1}) \subseteq \eta_0(\ell) \cdot f_{q_1}$ for $\ell < n_0$ or $\ell = n_i \wedge i \in C$
- the E_{q_2}-equivalence classes are $\{m$: for some $\ell \in [n_i, n_{i+1})$, we have mE_{q_1} $\mathrm{first}_{q_1}(\ell)\}$ for $i \in \mathbb{N}\backslash C$
- if $\ell \in [n_i, n_{i+1})$ then $f_{q_2}{\restriction}(\mathrm{first}_{q_1}(\ell)/E_{q_1}) \subseteq \eta_i(\ell) \cdot f_{q_1}$.

Now check. $\square_{9.2}$

Second part of old §4: the Theorem

Claim 9.4. **??**_noindent If_ $\underset{\sim}{\mathbb{Q}}$ _is a_ $\mathbb{Q}^3_D$_-name of a_ ${}^\omega\omega$_-bounding proper forcing with the PP property then_ $\Vdash_{\mathbb{Q}^3_D * \underset{\sim}{\mathbb{Q}}}$ _"D cannot be extended to a P-point"._

{m17} **Theorem 9.5.** _Assume_ CH $+2^{\aleph_1} = \aleph_2 + \diamondsuit_{S^2_1}$ _and_ $\mathbf{x} \in \mathbf{K}^\iota_{\rm ut}$ _with_ $\iota = 3$.
Then there is $\mathbb{P}$ _such that_

- (a) $\mathbb{P}$ _is a proper_ $\aleph_2$_-c.c. forcing notion_
- (b) $\mathbb{P}$ _of cardinality_ $\aleph_2$
- (c) $\mathbf{V}^{\mathbb{P}}$ _there is no_ P_-point_
- (d) _in_ $\mathbf{V}^{\mathbb{P}}$ _still_ $\mathbf{x} \in \mathbf{K}^\iota_{\rm ut}$ _hence_ $\mathfrak{u} = \aleph_1$.

Proof. Stage A: The forcing
We can find $\mathbb{P}_\alpha, \mathbb{Q}_\beta, \underset{\sim}{D}_\beta (\alpha \le \aleph_2, \beta < \aleph_2)$ such that

$\boxplus$ (a) $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \le \aleph_2, \beta < \aleph_2 \rangle$ is CS iteration with limit $\mathbb{P} = \mathbb{P}_{\aleph_2}$

(b) if $\alpha < \aleph_2$ then $\mathbb{P}_\alpha$ has density $\aleph_1$ hence $\Vdash_{\mathbb{P}_\alpha}$ "$2^{\aleph_0} = \aleph_1$" of cardinality $\aleph_0$

(c) $\underset{\sim}{D}_\alpha$ is a $\mathbb{P}_\alpha$-name of an ultrafilter on $\mathbb{N}$

(d) if $\underset{\sim}{D}_\alpha$ is a P-point then $\underset{\sim}{\mathbb{Q}}_\alpha$ is the $\mathbb{P}_\alpha$-name of $\mathbb{Q}^2_{\underset{\sim}{D}_\alpha}$, see Definition 4.1

(e) if $\underset{\sim}{D}$ is a $\mathbb{P}$-name of an ultrafilter on $\mathbb{N}$ then $\{\delta \in S^2_1 :\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{D}_\delta \subseteq \underset{\sim}{D}$"$\}$ is a stationary subset of $\aleph_2$.

Now

- $(*)_{1.1}$ every $\mathbb{P}_\alpha$ (and $\mathbb{P}_\alpha/\mathbb{P}_\beta$ for $\beta < \alpha$) is a proper forcing
- $(*)_{1.2}$ $\mathbb{P} = \mathbb{P}_{\aleph_2}$ satisfies the $\aleph_2$-c.c.
- $(*)_{1.3}$ $\mathbb{P}_\alpha$ and $\mathbb{P}_\alpha/\mathbb{P}_\beta$ are proper ${}^\omega\omega$-bounding and has the PP-property.

[Why? As each $\mathbb{Q}_\alpha$ has those properties by 4.4(1),(2),(4) and those properties are preserved by CS iteration, see [Sh:f, Ch.VI,§4].]

- $(*)_{1.4}$ $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} \le \aleph_2 = 2^{\aleph_1}$.

[Why? See [Sh:f].]

- $(*)_{1.5}$ $\underset{\sim}{A}_{\mathbb{Q}_\alpha}$ is the generic of $\underset{\sim}{\mathbb{Q}}_\alpha$, so $\Vdash$ "$\underset{\sim}{A}_\alpha \notin \mathscr{P}(\mathbb{N})^{\mathbf{V}[\mathbb{P}_\alpha]}$" hence $\langle \underset{\sim}{A}_\alpha : \alpha < \aleph_2 \rangle$ witness $2^{\aleph_0} \ge \aleph_2$.

Stage B: In $\mathbf{V}^{\mathbb{P}}$ there is no P-point.
Why? Otherwise there is $\underset{\sim}{D}, f_*$ such that

- (a) $\underset{\sim}{D}$ is a $\mathbb{P}$-name of a non-principal ultrafilter on $\mathbb{N}$
- (b) $p_* \in \mathbb{P}$ and $p_* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{D}$ is a P-point".

Clearly

- (c) $S_{\underset{\sim}{D}} := \{\delta \in S^2_1$: we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{D}_\delta \subseteq \underset{\sim}{D}\}$ is a stationary subset of $\aleph_2$
- (d) $S'_{\underset{\sim}{D}} := \{\delta \in S_{\underset{\sim}{D}} : p_* \in \mathbb{P}_\delta$ and $p_* \Vdash_{\mathbb{P}_\delta}$ "$\underset{\sim}{D}_\delta$ is a P-point$\}$
- (e) $\underset{\sim}{S}'_{\underset{\sim}{D}}$ is stationary in $\aleph_2$ and even $S_{\underset{\sim}{D}} \backslash S'_{\underset{\sim}{D}}$ is not stationary.

Way A:

So choose $\delta(*) \in S'_{\underset{\sim}{D}}$ and use 4.7, using the "$\mathbb{P}_{\aleph_2}/\mathbb{P}_{\delta(*)+1}$" is ${}^\omega\omega$-bounding and has the PP-property by $(*)_{1.3}$.

Way B: For every $\delta \in S'_{\mathscr{D}}$ choose $p_\delta \in \mathbb{P}_{\aleph_2}$ and $\mathbf{t}_\delta \in \{0,1\}$ such that $p_\delta \Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_\delta^{[\mathbf{t}_\delta]} \in \underset{\sim}{D}$" and $\delta \in \mathrm{Dom}(p_\delta)$.

For some stationary $S''_{\underset{\sim}{D}} \subseteq S'_{\underset{\sim}{D}}$ we have

(∗) if $\delta \in S''_{\underset{\sim}{D}}$ then
- $p_\delta{\restriction}\delta = p_{**}$
- $p_\delta(\delta)$ is the same
- $\mathbf{t}_\delta$ is the same.

Now choose an increasing sequence $\bar{\delta} = \langle \delta_n : n \in \mathbb{N}\rangle$ such that $\delta_n \in S''_{\underset{\sim}{D}_n}$ and $p_{\delta_n} \in \mathbb{P}_{\delta_{n+1}}$.

Now continue as in [Sh:f, Ch.VI,§4] but now all the $p_{\delta_n}(\delta_n)$ are equal so the proof is easier.

Stage C: In $\mathbf{V}^{\mathbb{P}}$ still $\mathbf{x} \in \mathbf{K}^\iota_{\mathrm{ut}}$.

There are two ways.

We prove that this holds in $\mathbf{V}^{\mathbb{P}_\alpha}$ by induction on $\alpha \le \aleph_2$. For $\alpha = 0$ this is assumed, for $\alpha = \beta + 1$ we use Claim 9.2 to show $\Vdash_{\mathbb{P}_\alpha}$ "$\mathbf{x} \in \mathbf{K}_{\mathrm{uf}}$" and this by Claim 7.1 deduction $\Vdash_{\mathbb{P}_{\alpha+1}}$ "$\mathbf{x} \in \mathbf{K}^\iota_{\mathrm{ut}}$".

Lastly, for limit α, work more in 9.2 to deal with iteration of such forcing notions. $\square_{9.5}$

§ 10. Private Appendix

Moved from Remark 9.2, pg.20:
3) For way 2 of the proof of 9.5, an iterated version of [?]. Maybe get a tree + return to [Sh:64].
4) Alternative: faking for $\mathbf{P}_2$.

§10 parts moved in 2011.1.30

From Content, pg.2:
§5 Preservation in limit using the bounding game, pg.

From §2: was clause (d) of 2.5(e)$\bullet_1, \bullet_2, \bullet_2$) becomes (d),(e),(f), pg.10:

(d) if $\eta \in T_{\mathbf{x}}, B_1 \in \mathscr{A}^{\mathbf{x}}_\eta$ and $B_2 \in \mathrm{sb}_{\mathbf{x}}(B)$ then for some $B_3 \in \mathscr{A}^{\mathbf{x}}_\eta$ we have $B_1 \cap B_3 \subseteq B_2$

From 6.1 on we change 9,10,11,12 to 1,2,3,4 respectively.

From §3, pgs. 22-23:

Discussion 10.1. 1) We have to strengthen the definition of $\mathbf{K}_\iota$, maybe adding (d) form the conclusion, and then have to strengthen the proof of "if forcing with $\mathbb{Q}$ preserves $\mathbf{x} \in \mathbf{K}_{\mathrm{uf}}$ then it preserves $\mathbf{x} \in \mathbf{K}^\iota_{\mathrm{uf}}$" when COM wins the $\partial^{\mathrm{bd}}_{\mathbb{Q}}$-game.
2) Possibly for preservation under CS iteration, the limit case.
Maybe better:

(a) $E \subseteq \mathscr{P}(I_{\mathbf{a}})$

(b) if $X \in E$ and $\langle I_\ell : \ell < k\rangle$ is a partition of I_a then there is $X' \in E$ such that $X' \subseteq X$ and $(\exists \ell < k)(A' \subseteq I_\ell)$

(c) in the game in the n-th move:
- NU chooses $X_N \in E$
- PO chooses k and a partition $\langle I_{n,\ell} : \ell < k\rangle$ of $I_{\mathbf{a}}$
- NU chooses $\mathscr{Y}'_n \in E, Y'_n \subseteq I_{n,\ell} \cap \mathscr{Y}_n$ for some n
- PO chooses $B_n \in \mathscr{Y}'_n$.

3) We can strengthen the statement of the theorem: for any ultrafilter $\underset{\sim}{E}' \supseteq E$. But simpler use $\iota = 10$ by $Y_{n+1} \subseteq Y_n$.
4) Moreover in the claim, PU can win playing ω plays, if

$\boxplus$ $\mathbb{B}$ is a Boolean algebra of subsets of $I_{\mathbf{x}}$ and E an ultrafilter of $\mathbb{B}$.

So we have $\langle (N_\delta, \bar{E}_\delta, \bar{B}_\delta) : \delta \in S\rangle$ such that

- $S \subseteq \omega_1$ stationary
- N_δ countable hereditary $N_\delta \models$ "$\aleph_1 = \delta$"
- $\mathbf{x}_\delta = \mathbf{x}{\restriction}\mathscr{H}(\aleph_1)^{N_\delta} \in N_\delta$
- $\bar{E}_\delta = \langle E_{\delta,B} : B \in \mathscr{A}^{\mathbf{x}_\delta}_{u(\mathbf{x}_\delta)}\rangle$
- $N_\delta \models$ "$\bar{E}_\delta$ is an ultrafilter standard Y-filter for $\mathbf{x}_\delta$ and an ultrafilter
- $\bar{B}_\delta = \langle B_{\delta,Y}, \delta \in \mathbf{Y}_{\mathbf{x}}\rangle, \bar{B}_{\delta,Y} = \langle B_{\delta,Y,n} : n \in \mathbb{N}\rangle, \mathbf{Y}_{\mathbf{x}} = \{\max(B) : B \in \mathscr{A}_{\mathbf{x}}\}$

- the guessing property (with $(\mathbb{B}, E)$ "upstairs"? originally with diamond yes, later not.

5) This seems to demand "E is an ultrafilter" but not so only some countable version reasonable as any forcing preserving some ultrafilter on $\mathbb{N}$ does not add a shattering real.
6) Instead proving the preservation by CS iteration, we can point out the properties of it, while weaker than the one we use for Q^2_D, are enough. Also we have to prove (see 4.2) that $(\mathbb{Q}^2_D)^{\mathbb{N}}$ "kill" the P-point D and/or the parallel of [Sh:F1089] - no P-point and large continuation does ?

Discussion 10.2. The analysis seems to lead to the question

Question 10.3. Assume

(a) $\mathbf{x} \in \mathbf{K}_{\mathrm{ul}}, \eta \in T_{\mathbf{x}}, B \in \mathscr{A}^{\mathbf{x}}_\eta$ and $h : \mathbb{N} \to B$ be a one-to-one onto
(b) $\mathbf{c}$ is a function from ${}^{\omega>}\{1, -1\}$ to $\{0, 1\}$.

Can we find monochromatic $(p, \mathfrak{c}_0, \mathfrak{c}_1, \ldots)$ as in [GoSh:388] such that $\{m^{\mathrm{up}}_{\mathfrak{c}_n} : m < \omega\} \in h^{-1}(D^{\mathbf{x}}_{\eta,B})$?

We may split the question: first, does it hold in Conclusion **??**, i.e. when we construct $\mathbf{x}$ assuming CH? Probably (e) of **??** does it. Second, is this preserved under suitable forcing (best)?

Need less - see above: leave free $(0, n_*)$ and now for every $\eta \in {}^{[n_*,n)}\{-1, 1\}$ we can get $\mathbf{t}$ varying $\nu \in {}^{[0,n_*)}\{-1, 1\}$.

Discussion 10.4 (2010.9.26)**.** Concerning the games we have to make a choice

(a) The old way: $\mathscr{A}^{\mathbf{x}}_\eta$ is directed and $B \in \mathscr{A}^{\mathbf{x}}_\eta \wedge \nu \in B \Rightarrow B^{[\nu]} \in \mathscr{A}^{\mathbf{x}}_\nu$. The old games fit. But seemingly there are problems in building $\mathbf{x} \in \mathbf{K}_\iota$
(b) The new way should be that on $\mathscr{A}^{\mathbf{x}}_\eta$ there is an equivalence relation $E^{\mathbf{x}}_\eta$ each $E^{\mathbf{x}}_\eta$-equivalence class is directed by $\leq^{\mathbf{x}}_\eta$. The filter corresponds to pairs $(\eta, e), e = B/E^x_\eta$ for some $B \in \mathscr{A}^{\mathbf{x}}_\eta$.

§ 11. Discussion

{p2}
Discussion 11.1. (2010.10.18) 1) We can replace the co-finite filter with other definable P-filters, e.g. $\{\mathscr{U} \subseteq \mathbb{N} : |[0,n) \backslash \mathscr{U}|/f(n)$ goes to $0\}$ for any increasing $f : \mathbb{N} \to \mathbb{N} \backslash \{0\}$.
2) We can prove preservation using faking - as possibly in many countable $N \subseteq \mathscr{H}, \mathbf{y} = \mathbf{x} \restriction N$ is well defined, has $<_{\mathbf{y}}$-maximal B and is $N \models$ "$D_{\mathbf{y},B} \in \mathbf{P}_2$", see 0.3. This gives way to prove preservation of $\mathbf{x} \in \mathbf{K}_{\rm uf}$ hence $\mathbf{x} \in \mathbf{K}_{\rm ut}$.
3) Can we not use approximations from $\mathbf{P}_2$; see later in xxx.

{p4}
Definition 11.2. We say $\bar{N}$ represents $\mathbf{x} \in \mathbf{K}$ <u>when</u>

$\boxplus$ (a) $\bar{N} = \langle N_\zeta : \zeta < \aleph_1 \rangle$ is $\subseteq$-increasing continuous
(b) $N_\zeta \subseteq (\mathscr{H}(\aleph_1), \in)$ is transitive
(c) $\mathbf{x}_\zeta = \mathbf{x}$ is well defined
- $N^+_\zeta = (N_\zeta, \mathbf{x}_\zeta) \models \mathrm{ZFC}^-$, so $\mathbf{x}_\zeta$ is a class of N_ζ
- $N_\zeta \models$ "$\mathbf{x}_\zeta \in \mathbf{K}$"
- $\mathbf{x}_\zeta \le \mathbf{x}_\xi \le \mathbf{x}$ for $\varepsilon < \xi < \aleph_1$.

{p6}
Claim 11.3. *A sufficient condition for "PO does not lose in $\partial_{\mathbf{x},\min}$" is:*

$\boxplus$ (a) *$\bar{N}$ represents $\mathbf{x}$*
(b) *$\bigcup\limits_\zeta N_\zeta \cap {}^{\mathbb{N}}\mathbb{N}$ is cofinal in $({}^{\mathbb{N}}\mathbb{N}, <^*)$ so $\mathfrak{d} = \aleph_0$*
(c) *the set $S_1[\bar{N}]$ is stationary in $\aleph_1$ where $\delta \in S_1[\bar{N}]$ iff*
(α) *$N^+_{\delta+1} \models$ "...?*

Remark 11.4. See later

{p8}
Discussion 11.5. (2010.10.19) 1) In a sense our ultrafilter is build by approximations which are trees of depth some countable ordinal; but in each node we use the filter of co-finite sets. So in the games the PO player chooses $B'_n \in [B_n]^{[\nu_n]}$. Can we do otherwise? Certainly, yes.
2) We can repalce B by $(B, \bar{p}), \bar{p} = \langle p_\eta : \eta \in B \backslash \max(B) \rangle, p_\eta$ belongs to some creature forcing $\mathbb{Q}_\eta$ omittory with the ultrafilter property, we order then by $\le^*_{\mathbb{Q}_\eta}$, i.e. mod finite. We can use one such $\mathbb{Q}$ or a family.
3) More abstractly we can use ideals $\mathscr{I}$ on $\mathbb{N}$ such that:

(a) "$A \in \mathscr{I}$" is Π^1_2
(b) in the game $\partial_{\mathscr{I}}$ the NU player has no winning strategy
  (α) A play last ω moves
  (β) before the n-th move we have a pair $(w_n, A_n), A_n \in \mathscr{I}^+, w_n \subseteq A_n$ finite such that $A_{n+1} \subseteq A_n, w_n \subseteq w_{n+1} \subseteq w_n \cup A_n$, we stipulate $w_{-1} = \emptyset, A_{-1} = \mathbb{N}$
  (γ) in the n-th move
  (δ) NU chooses A_n under the above restriction then
  (η) PO chooses w_n under the above restriction
(c) in the end of the game, PO wins if $\cup\{w_n : n \in \mathbb{N}\} \in \mathscr{I}^+$.

4) In both cases not much is changed.
5) But can we avoid the explicit mentioning of the trees.

{p10}
Discussion 11.6. (2010.10.19) Maybe better in games to have PO choose from the set of trees already appearing in $I_{\mathbf{x}} = \cup\{\mathscr{A}^{\mathbf{x}}_\eta : \eta \in T_{\mathbf{x}}\backslash\{\mathrm{rt}_{\mathbf{x}}\}\}$ but demand that $\mathscr{B}_{\mathbf{x}}$ has some density property like

(*) the ultrafilter property: if $B \in \mathscr{A}^{\mathbf{x}}_\eta$ and $A \subseteq \max_{\mathbf{x}}(B)$ then for some $B' \in \mathscr{A}^{\mathbf{x}}_\eta$ we have $A \subseteq \max(B')$ or $A \cap \max(B') = \emptyset$.

Discussion 11.7. (2010.10.19) 1) Can we use the forcing notion from [Sh:f, Ch.VI] "against P-points"? Yes. {p13}
2) Can we use [?]? Yes.
3) (Here?) Can we use faking? Yes.

Definition 11.8. 1) For a witness/representation $(\bar{N}, \bar{\mathbf{x}})$ of $\mathbf{x}$, we say that $\mathbf{x}$ is simple in α when $N^+_{\alpha+1} \models$ "$A_{\mathbf{x},\alpha+1}$ has a $<_{\mathbf{x}_{\alpha+1}}$-maximal member, $\mathbf{x}_{\alpha+1} \in \mathbf{K}_{\mathrm{uf}}$ so is $\aleph_1$-directed. {p25}
2) Letting $B_{\mathbf{x},\alpha}$ be this maximal member for every $\eta \in B_{\mathbf{x},\alpha}\backslash \max(B_{\mathbf{x},\alpha}), Y_{\mathbf{x},\alpha,\eta} = \mathrm{suc}_{\mathbf{x}}(\eta, B_{\mathbf{x},\alpha}), N_{\alpha+1} \models$ "$D_{\mathbf{x}_{\alpha+1},\eta_1}$ is an ultrafilter on $Y_{\mathbf{x},\eta,\alpha}$ defined naturally.
3) We say is $\mathbb{Q}$-simple in α where $\mathbb{Q}$ is omittory, with ultrafilter property when $N_{\alpha+1} \models$ "$D_{\mathbf{x}_{\alpha+1},\eta}$ is a $\mathbb{Q}$-ultrafilter". Let $D_{\mathbf{x},\alpha+1,\eta} = -D_{\mathbf{x}_{\alpha+1},\eta}$.
4) So for $\mathbb{Q}$ = Mathias forcing we get $N_{\alpha+1} \models$ "each $D_{\mathbf{x},\alpha+1,\eta}$ is a selective (= Ramsey) ultrafilter".
5) Recheck - is $\mathfrak{d} = \aleph_1$ necessarily? By faking as iteration as in [BsSh:242] preserve members of $\mathbb{P}_2$, so faking indicates this is not so. Sort out.

Definition 11.9. 1) Let $\mathbf{I}$ be the set of definitions $\mathscr{I}$ such that {p17}

(a) $\mathscr{I}$ (define) anideal on $\mathbb{N}$

(a) on the game of NU choosing $A_n \in I^+_i$, PO chooses $w_n \subseteq A_n$, PO winning if $\cup\{w_n : n \in \mathbb{N}\} \in \mathscr{I}^+_r$, NU does not win.

Definition 11.10. We define $K_{\mathscr{I}}$ as in §2, but for {p19}

(b) $\mathscr{A}_\eta = \mathscr{A}^\eta_{\mathbf{x}}$ is a set of pairs $\mathbf{b} = (B, \bar{I})$, where
  (α) B as before
  (β) $\bar{\mathscr{I}} = \langle I_\eta : \eta \in B\backslash \max(B)\rangle$
  (γ) $\mathscr{I}_\eta$ is an ideal on $\mathrm{suc}_B(\eta)$ which up to isomorphism is from $\mathbf{I}$.

Definition 11.11. We say $\mathbf{w}$ is a witness for $\mathbf{x}$ when $\mathbf{x} \in \mathbf{K}_{\mathrm{ut}}$ and $\mathbf{w}$ consists of {p27}

(α) N_ζ for $\zeta < \aleph_1$

(β) $N_\zeta \subseteq (\mathscr{H}(\aleph_1, \in)$ is a countable and transitive model of ZFC$^-$

(γ) N_ζ is $\subseteq$-increasing continuous

(δ) $\mathbf{x}_\zeta$ is a class of N_ζ, i.e. $(N_\zeta, \mathbf{x}_\zeta) \models$ ZFC$^-$, not necessarily definable in N_ζ.

Definition 11.12. For $\mathbf{w}$ witness for $\mathbf{x}$ and $\mathbf{q}$ a parameter family of ideals {p29}

(a) $\mathscr{S}_{\mathbf{q}}(\mathbf{w}) = \{\delta < \aleph_2 : N_{\delta+1} \models \mathbf{x}_{\delta+1}$ is $\mathbf{q}$-simple$\}$.

Remark 11.13. 1) Try faking.
2) Check: does in [BsSh:242] kill those ultrafilters?

Definition 11.14. Let $\mathfrak{K}$ be the class of form $\mathbf{q}$ consisting of:

(a) $\ell g(\mathbf{q}) = \alpha_q$ an ordinal

(b) $\mathbb{P}_\alpha$ is a forcing notion $\lessdot$-increasing with $\alpha \leq \ell g(\mathbf{q})$

(c) $\underset{\sim}{\mathbb{Q}}_\alpha$, a $\mathbb{P}_\alpha$-name of a forcing notion which is $\subseteq \mathbb{Q}_\alpha, \mathbb{Q}_\alpha$ a forcing notion from $\in \mathbf{V}$, so every element of $\underset{\sim}{\mathbb{Q}}_\alpha$ as well as the order are old but the set of elements is in general new

(d) the members of $\mathbb{P}_\alpha$ are functions with domain, a countable subset of α

(e) if $\alpha < \beta \leq \alpha_{\mathbf{q}}$ and $p \in \mathbb{P}_\beta$ then
  - (α) $p \restriction \alpha \in \mathbb{P}_\alpha$
  - (β) $(p \restriction \alpha) \leq_{\mathbb{P}_\delta} p$
  - (γ) if $p \restriction \alpha \leq_{\mathbb{P}_\alpha} q$ then $r = q \cup (p \restriction [\alpha, \beta]) \in \mathbb{P}_\beta$ is a common upper bound of p, q and $q \restriction \alpha = q$

(f) if $p \in \mathbb{P}_\alpha, \beta \in \mathrm{dom}(p) \Rightarrow$ then
  - $p(\beta) \in \mathbb{Q}_\beta$
  - $p \restriction \beta \models p(\beta) \in \underset{\sim}{\mathbb{Q}}_\beta$.

Remark 11.15. 1) This work for $\mathbb{Q}^2_D$ from 3.1, but seems to fail for [Sh:f, Ch.VI].
2) We can think of having trees as in [Sh:64] and so any condition is a base of pre-conditions some "false" the set of values for $\mathrm{Dom}(\eta) = \mathrm{Dom}(p) \cap \alpha, \alpha \in \mathrm{Dom}$ is I_α the set of true one or more elaborately as in 2-completeness systems.

§ 12. Private Appendix

<u>Moved 2010.10.18; Debts: 1) moved from debts to §0, pg.2</u>:

[Debt - Dow question, note that we can use [Sh:407] only, not Q^2_D! against P-points generated by $\aleph_1$.]

<u>Moved from end of the introduction, pg.4</u>:

Frelim asked whether consistently there are exactly 2 (or k) P-points, recalling that by [Sh:f, Ch.VI], consistently there is exactly one (modulo permuting $\mathbb{N}$). The same proof gives then (DETAILS)

<u>Moved from after 1.5, pg.6</u>:

{7g.3}

Definition 12.1. Let T be a tree (not necessarily well founded). For $B_1, B_2 \in$ CWT(M) let: $B_2 \le_{\rm side} B_1$ in words B_1 side extend B_2 (or B_2 is a side-subset of B_1) <u>when</u> $B_1 \supseteq B_2$, they have the same root and $\nu \in B_2 \Rightarrow \text{suc}_{B_2}(\nu) \subseteq \text{suc}_{B_1}(\nu)$.

<u>Moved from pg.10</u>:

Remark 12.2 (omit?). We can demand in addition that the following set belongs to $A^{\mathbf{y}}_\eta$:

$$B^* := \{\rho \in T_{\mathbf{x}} : \text{ for some } \nu \in Y \text{ we have } \rho \in f(\nu) \vee (\rho \in B \wedge \rho \le_T \nu)\}$$

but the proof is somewhat cumbersome. [09.9.27 - have not checked]

Proof. By 2.3 there is $\mathbf{z} \in \mathbf{K}$ such that $\mathbf{x} \le_{\mathbf{K}} \mathbf{z}$ and $\mathscr{A}_\nu$ has a $\le^{\mathbf{z}}_\nu$-maximal member B_ν for every $\nu \in T_{\mathbf{z}}$. Clearly $Y \cap B_\eta \in \text{alm-frt}_{\mathbf{x}}(B_\eta)$ so let $B_1 \in \text{sb}_{\mathbf{z}}(B_\eta)$ be such that $Y_1 := Y \cap B_1$ is a front of B_1.

Let $B^* = \{\rho \in T_{\mathbf{y}}$: for some $\nu \in Y_1$ we have $\rho \in B_\nu \vee (\rho \in B_1 \wedge \rho \le_T \nu)\}$ we define $\mathbf{y} \in \mathbf{K}$ as follows:

$(*)$ (a) $T_{\mathbf{y}} = T_{\mathbf{z}}$

(b) $\mathscr{A}^{\mathbf{y}}_\nu$ is: $\mathscr{A}^{\mathbf{z}}_\nu \cup \{B^* \cap T_{\ge \nu}\}$ <u>when</u> $\nu \in B^{**}$
$\mathscr{A}^{\mathbf{z}}_\nu$ <u>when</u> otherwise

(c) $\le^{\mathbf{y}}_\nu$ is defined naturally.

Now check. $\square_{16.9}$

<u>Moved from pgs.11,12</u>:

Below we concentrate on the case of ultra $\mathbf{x}$, see below.

{8j.18}

Definition 12.3. (2010.10.10 HERE?) 1) Let $\iota = 6$. For $\mathbf{x} \in \mathbf{K}$ and $Y \in$ alm-frt$_{\mathbf{x}}(\text{rt}_{\mathbf{x}})$ we define a game $\Game^\iota = \Game^\iota_Y = \Game^{\mathbf{x},\iota}_Y = \Game^\iota_{\mathbf{x},Y}$ where $\iota = 6$, (writing $\Game^\iota_{\mathbf{x},B}$ where $B \in \mathscr{A}^{\mathbf{x}}_{\text{rt}(\mathbf{x})}$ means $\Game^\iota_{x,\max(B)}$

(A) a play last ω moves and is between the player PO (positive) and NU (null).

(B) in the n-th move

(a) first, NU chooses $B_{n,1} \in \mathscr{A}^{\mathbf{x}}_{\text{rt}(\mathbf{x})}$ and finite $Z_n \subseteq \text{suc}(\text{rt}_{\mathbf{x}}, B_{n,1})$

(b) second, PO chooses $B_{n,2} \subseteq \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})}$ such that $B_{n,1} \le_{\mathbf{x}} B_{n,2}$ and $\nu_n \in \mathrm{suc}(\mathrm{rt}_{\mathbf{x}}, B_{n,2})$ not $\le_{\mathbf{x}}$-below Z_n and $B_{n,3} \in \mathrm{psb}_{\mathbf{x}}(B^{[\nu_n]}_{n,2})$ such that $B_{n,3} \cap Y \subseteq B_{n,1} \cap Y$ is an almost front of $B_{n,3}$ and a filter D_n on $B^{[\nu_n]}_{n,3} \cap Y$ extending $\mathrm{sb}_{\mathbf{x}}(B^{[\nu_n]}_{n,3})$

(c) third, NU chooses $B_{n,4} \in \mathrm{psb}_{\mathbf{x}}(B^{[\nu]}_{n,3})$ such that $B_{n,3} \cap Y \in D_n$.

In the end of the play PO wins the play when $\cup\{B_{n,4} \cap Y : n < \omega\} \in D^{\mathbf{x}}_Y$, (why $B_{n,4}$? for 3.9).
2) If $\iota = 8$ we allow in the end we demand $\cup\{B_{n,2} \cap Y : n < \omega\} \in (D^{\mathbf{x}}_Y)^+$. If $\iota = 7$ we also allow NU to choose also $A_n \in (D^{\mathbf{x}}_{\eta,Y})^+$ and demand $B_{n,2} \cap Y \subseteq A_n$.

Moved from pgs.13,14,15:

{8j.22yajan} **Claim 12.4.** *Assume $\mathbf{x} \in \mathbf{K}$ in $\mathbf{V}$ and E is a standard B-filter for $\mathbf{x}$ and $\mathbb{Q}$ a forcing notion as in 3.1.*
2) If E is an ultrafilter then we can find in $\mathbf{V}^{\mathbb{Q}}$ a standard B-ultrafilter E_1 for $\mathbf{x}$ extending E.
3) There is a minimal extension of E to a standard B-filter for $\mathbf{x}$ in $\mathbb{Q}$.
4) Continuing in limit of iterations.

{8h.23} **Claim 12.5.** *Assume $N \subseteq (\mathscr{H}(\aleph_1), \in)$ is countable and transitive and $N \models$ "$\mathbf{x} \in \mathbf{K}_{\le\aleph_1}, Y \in \mathrm{alm}-\mathrm{frt}_{\mathbf{x}}$ and $\mathbf{st}$ is a strategy of the NU player in the game $\Game^{\iota}_{\mathbf{x},\eta,Y}$", so $\mathbf{x} \in \mathbf{K}_{\le\aleph_0}, Y \in \mathrm{alm}-\mathrm{frt}_{\mathbf{x}}$. Then there is $\mathbf{y}$ satisfying $\mathbf{x} \le_{\mathbf{K}} \mathbf{y} \in \mathbf{K}_{\aleph_0}$ and a sequence $\bar{b}$ such that*

(a) $\bar{b} = \langle (B_{n,1}, B_{n,2}, B_{n,3}, B_{n,4}, \nu_n) : n < \omega\rangle$

(b) every finite initial segment of $\bar{b}$ is an initial segment of a play of the game $\Game^{\mathbf{x},\iota}_{\eta,Y}$ in which NU uses the strategy $\mathbf{st}$

(c) the player PO wins the play for $\mathbf{y}$ hence for $\mathbf{y}'$ too when $\mathbf{y} \le_{\mathbf{K}} \mathbf{y}'$. [check with D_n?]

2) Moreover, if $\mathbf{x} \le_{\mathbf{K}} \mathbf{y}_0 \in \mathbf{K}_{\aleph_0}$ we can demand $\mathbf{y}_0 \le \mathbf{y}$. [check]

Proof. Case 1: Let $\iota = 6$ (or 7 or 8, see 16.12)

Let $\langle B^*_k : k < \omega\rangle$ be $\le^{\mathbf{x}}_{\eta}$-increasing cofinal in $\mathscr{A}^{\mathbf{x}}_{\eta}$. We simulate a play and in the n-th move the player NU uses $\mathbf{st}$ to choose $B_{n,1}, \mathbb{Z}_n$ and the player PO chooses $B_{n,2} = B^*_{k(n)}$ with $k(n)$ large enough and then $\nu_n \in \mathrm{Suc}(\eta, B_{n,2})$ such that ν_n is not below any $\nu \in Z_n \cup \{\nu_m : m < n\}$ and $B^{[\nu_n]}_{n,2} \cap Y \subseteq B_{n,1}$ and $B_{n,3} \in \mathrm{sp}(B^{[\nu_n]}_{k(n)})$.

Having finished the play note $\{B_{n,3} : n < \omega\}$ are pairwise disjoint and define $\mathbf{y}$ by

$\boxplus$ (a) $T_{\mathbf{y}} = T_{\mathbf{x}}$

(b) $\mathscr{A}^{\mathbf{y}}_{\eta} = A^{\mathbf{x}}_{\eta} \cup \{\cup\{B_{n,2} : n < \omega\} \cup \{\eta\}\}$

(c) if $\rho \in T_{\mathbf{x}} \backslash \{\eta\} \backslash \cup \{B_{n,3} : n < \omega\}$ then $\mathscr{A}^{\mathbf{y}}_{\rho} = \mathscr{A}^{\mathbf{x}}_{\rho}$

(d) if $\rho \in B_{n,3}, n < \omega$ then $\mathscr{A}^{\mathbf{y}}_{\rho} = \mathscr{A}^{\mathbf{x}}_{\rho} \cup \{\mathrm{frt}(B^{[\rho]}_{n,3}, B) : B \in \mathscr{A}^{\mathbf{x}}_{\rho}$ satisfies $B_{n,2} \le^{\mathbf{x}}_{\eta} B\}$ where

($*$) if $\rho \in T_{\mathbf{x}}, B_1, B'' \in \mathscr{A}^{\mathbf{x}}_\eta, B_1 \le^{\mathbf{x}}_\eta B''$ and $B' \in \mathrm{sp}_{\mathbf{x}}(B_1)$ then $\mathrm{fus}_{\mathbf{x}}(B', B'') = \{\nu \in B''$: there is no $\varrho \in B - 1 \backslash B'$ such that $\varrho \triangleleft \nu\}$.

Case 2: Similarly. $\square_{12.5}$

Or older proof of **??** for $\iota < 6$ - check:

Problem (?): Maybe $\eta_n \in t_n, \eta_{n+1} \in A^n_{\eta_n}$, rk $A^n_{t_n}(t_n) \ge \omega$ (nec?) result not well founded.

Answer: Not really as in (C) of **??**

1) For notational simplicity assume no $(\mathscr{A}^{\mathbf{x}}_\rho, \le^{\mathbf{x}}_\rho)$ has a last member. For $\nu \in T_{\mathbf{x}}$ let $\langle B^*_{\nu,n} : n < \omega\rangle$ be $<_\nu$-increasing cofinal in $(\mathscr{A}^{\mathbf{x}}_\nu, \le^{\mathbf{x}}_\nu)$; without loss of generality Y is an almost front of $B^*_{\eta,0}$.

Now we choose $9t_n, \bar{s}_n, \bar{A}_n, \bar{B}_n)$ by induction on $n < \omega$ such that:

(a) $\langle (t_m, \bar{A}_m) : m \le n\rangle$ is an initial segment of the play

(b) in this initial segment the NU player uses the strategy **st**

(c)
- $\bar{A}_m = \langle A^m_\rho : \rho \in t_m\rangle$
- $\bar{B}_m = \langle B^m_\rho : \rho \in t_m\rangle$
- $B^n_\rho \in \mathscr{A}^{\mathbf{x}}_\rho$
- $A^m_\rho \in \mathrm{sb}(B^m_\rho)$

(d) $\bar{s}_n = \langle s_{n,\nu} : \nu \in t_n\rangle$ such that $s_{n,\nu} \in (t_n) > \nu$

(e) if $n = m+1$ then $\nu \in t_m \Rightarrow |s_{n,\nu} \backslash s_{m,\nu}| = 1$ and $\nu \in t_n \backslash t_m \Rightarrow s_{n,\nu} = \emptyset$

(f) for $n < \omega$ and $\nu \in t_n$
- let $k_{n,\nu}, k(n,\nu) = \min\{k < \omega : k \ge n$ and $(\forall ln)(m \le n \wedge \nu \in t_m \to B^m_\nu \in \{B^*_{\nu,\ell} : \ell \le k\}$
- there are $\eta_{\nu,k(n,\nu)} <_{T_{\mathbf{x}}} \eta_{\nu,k(n,\nu)-1} <_{T_{\mathbf{x}}} \ldots < \eta_{\nu,0}$ such that $\eta_{nu,\ell} \in B^*_{nu,\ell}$, moreover $\eta_{\nu,\ell} \in \mathrm{suc}_{B^*_{\nu,\ell}}(\nu)$ and $s_{n+1,\nu} \backslash s_{n,\nu} = \{\eta_{\nu,\ell}\}$

(g) if $\nu \in t_n, \nu <_{T_{\mathbf{x}}} \varrho \in t_{n+1}, k < k(n,\nu)$ and $\varrho \in B^*_{\nu,k}$ then $\{\rho \in B^*_{\nu,k} : \rho \le_{T_{\mathbf{x}}} \varrho\} \subseteq t_{n+1}$.

There are no problems to carry the definition and let $t_\omega = \cup\{t_n : n < \omega\}$. Now we define $\mathbf{y}$ as follows:

(α) $T_{\mathbf{y}} = T_{\mathbf{x}}$

(β) if $\nu \notin t_\omega$ then $\mathscr{A}^{\mathbf{y}}_\nu = A^{\mathbf{x}}_\nu$

(γ) if $\nu \in t_\omega$ then $\mathscr{A}^{\mathbf{y}}_\nu$ is the union
- $A^{\mathbf{x}}_\nu$
- $\{\{\rho \in B_{\nu,k} : \rho \in t_\omega$ or $(\exists \varrho \in t_\omega \cap Y)(\varrho <_{T_{\mathbf{x}}} \rho\} : k < \omega\}$ if $\nu \in t_\omega$
- $\{\{\nu\} \cup \bigcup\{B^*_{\rho,k} \cap t_\omega$: for some $n, \{\rho\} = s_{n+1,\rho} \backslash s_{n,\rho}$ and $k = k(n,\rho)$.

Now check.
2) (09.10.25) - rethink.

Moved from 3.1's proof, pgs.19,20:

We can use in (A)(b) there and in proof the game $\Game^3_{\mathbb{Q},p}$. So in Case 3 third line:

Now we simulate a play of $\Game^3_{\mathbb{Q},p}$ with COM using a winning strategy such that in the n-move

- $(*)$ in the subgame in the ℓ-th move, also $B_{n,\ell}, t_{\mathbf{n},\ell}$ for $\ell < \ell_n \times r_n$ are chosen such that
  - $B_{n,\ell} \in$ psb$(B^{[\nu_n]})$
  - $B_{n,\ell} \subseteq B$
  - $B_{n,\ell} \subseteq B_{n,m}$ if $m < \ell$
  - $\mathbf{t}_{n,\ell} \in \{0,1\}$
  - $q_{n,\ell} \Vdash B_{n,\ell} \subseteq \underset{\sim}{A}$ if $\mathbf{t}_{n,\ell} = 1, B_{n,\ell} \cap \underset{\sim}{A} = \emptyset$ if $\mathbf{t}_{n,\ell} = 1$.

There is no problem to carry this. As the player COM was a winning strategy, CON wins the play, so there is p' such that

- $(*)$ (a) $p \le p'$ in $\mathbb{Q}$
  - (b) $\{p_n, \ell_n, r_n, i : i < \ell_n\}$ is predense above p'.

Hence for every $n, B'_n := B_{n,\ell_n * \ell_n} \subseteq \mathbb{N}$ is infinite and for every $n \in \mathbb{N}$ we have $p' \Vdash$ "$\max(B'_n) \subseteq \underset{\sim}{A}$ or $\max(B'_n) \cap \underset{\sim}{A} = \emptyset$".

Let $\underset{\sim}{\mathbf{t}}_n$ be the truth value of $B_n \subseteq \underset{\sim}{A}$. So by the induction hypothesis for some p'' above p' and $\mathbf{t}$ and infinite $\mathscr{U}_1 \subseteq \mathbb{N}$ we have $n \in \mathscr{U}_1 = p'' \Vdash$ "$\underset{\sim}{\mathbf{t}}_n = \mathbf{t}$". So $B'' = \cup\{B'_n : n \in \mathscr{U}_1\} \cup \{\eta\} \in$ psb$_{\mathbf{x}}(B)$ and $B'', p'', \mathbf{t}$ are as required.

{k4} **Discussion 12.6.** [here?] 1) We seem quite close to our goal but is being in $\mathbf{K}_{\rm uf}$ preserved by forcing notions preserving its being ultra? The parallel holds for P-point and takes care of the successor case of the iteration. The limit case is also done in [Sh:f, Ch.VI]. Do we have the parallel here?

As long as we do not know this, we may try to strengthen $\Pr^3$ in 3.1. We need to strengthen

- (a) starting with countable $N \prec (\mathscr{H}(\chi), \in)$ such that $\mathbb{Q} \in N, p \in \mathbb{Q} \cap N$ the choice of the infinite set $\mathscr{U}$ does not depend on $\mathbb{Q}$ just on $(N, p, \mathbb{Q})/\cong$.

This seems to succeed for each forcing but fails for the iteration in [Sh:f, Ch.VI], but holds in the original proof [**?**]?, but not the second demand (still holds for the individual forcing)

- (b) INC choosing $q_{n,c}$ chooses also $\mathbf{t}_{n,c} \in \{0,1\}$ and demand thre N in the end there are $(q, \mathscr{U}, \mathbf{t})$ such that (as before and)
  - $\oplus$ $\{q_{n,\ell} : \ell < \ell(n), \mathbf{t}_{n,\ell} = \mathbf{t}\}$ is predense above q for every $n \in \mathscr{U}$.

Relevant is the partition theorem of [**?**] for this combines [Sh:F1091] and [**?**].
2) NO! This works for product of $\aleph_1$ not of $\aleph_2$.
3) But

- (A) if for $\mathbb{Q}$, COM wins the ${}^\omega\omega$-bounding games and forcing with $\mathbb{Q}$ preserves $\mathbf{x}$ is ultra then forcing with $\mathbb{Q}$ preserves $\mathbf{x} \in \mathbf{K}_0$

(B) using one $\mathbb{Q}_D$ maybe we should immitate $\mathbb{Q}_D$ preserve the P-point $D' \perp D$

(C) lately the tree involves try for finite sequence of $\bar{p}$ in the n-th move (so has a finite subsidiary game).

If we cannot in it step to $\mathbf{t} = 1$, multiply the board! Think of $(p_*, \underset{\sim}{A}), p_* \Vdash$ "$\underset{\sim}{A} \subseteq \mathbb{N}$".

That is we fix $p, \mathbf{t}, n_0$ large enough (if we fail, increase it); we by induction $p_i, \nu_n, B_{n,3}$

$\boxplus$ (α) n_i increases
(β) $p_* \le p_i \le_{n_i} p_{i+1}$
(γ) for every $g \in {}^{[i]}2, g \cup [p_{i+1} \restriction [i, \omega)$ force $A, B_{3,n} \subseteq \underset{\sim}{A}^{[\mathbf{t}_{i,g}]} \in \{\underset{\sim}{A}, \mathbb{N} \backslash \underset{\sim}{A}\}$.

Etc.

(D) if we succeed we still have the limit case.

Moved from after 7.1, pgs.22,23:

Discussion 12.7. 1) What about $\iota = 10, 12$? {k17.ya}

May consider

$\boxplus_3$ $(a), (b)$ as in $\boxplus_1$
(c) $\iota = 10$.

The proof should be as above but $\underset{\sim}{Y}'$ should be $\underset{\sim}{Y}'_{\ge p_n} = \{B' \in A^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})} : Y'$ is an almost front of B' and...

But this needs using more carefully the forcing games and the game here. Hence we avoid it.

2) We may choose "for every standard Y-filter for $\mathbf{x}$" which is an ultrafilter. In this case for $\partial'(Q, p)$ this works. Should consider.

3) Maybe add "if $B \in \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})}$ and $B' \in \mathrm{sb}(B)$ then for some $B'' \in \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})}$ we have $B'' \subseteq B' \wedge B'' <^* B'$ (check).

Normality (end of §1) solve this. Add it to Definition.

* * *

Way B:

$\boxplus_3$ $\iota = 10$

We are given also $\mathbf{R}$ (but in $\mathbf{V}$! not new, check maybe in $\mathbf{L}$ - FILL!)

$\boxplus_4$ $\iota = 11$

Now the bounding game simulated by the player NU is $\partial^1(\mathbb{Q}, p_*)$. So first E is a standard Y-filter for $\mathbf{x}$ which is an ultrafilter given. We choose $\underset{\sim}{E}_1$ a Q-name of an ultrafilter extending it in. Why NU chooses $\underset{\sim}{Y}_n$ by $\underset{\sim}{\mathbf{st}}_1$ let $\mathbf{st}_2$ still COM chooses $\langle p_\ell : \ell < \ell_n \rangle$ from $\mathbb{Q}$.

Let $Y_{n,\ell} = \{B \in \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})} : p_\ell \nVdash_{\mathbb{Q}}$ "$B \cap Y \notin \underset{\sim}{Y}_n$"$\}$. So $\ell < \ell_n \Rightarrow Y_{n,\ell} \in E$ hence $Y_n := \cap\{Y_{n,\ell} : \ell < \ell_n\} \in E$ and $\underset{\sim}{\mathbf{st}}_3$ till NU to choose Y_n.

Note that in 3.1 we prove more.

{k3.yajan}

Claim 12.8. *1) Let $\iota = 9, 11$; fixing E, i.e. $\mathrm{fil}(E)$, really standard?*
2) Let $\iota = 10, 12$ for every E there is E; check.

§ 13

Question 13.1. Can we replace "wins $\partial^{\rm bd}_{\mathbb{Q}}$" by proper + ${}^\omega\omega$-bounding?

Claim 13.2. *Assume* $\mathbf{x} \in \mathbf{K}_{\rm ut}$. *Forcing* $\mathbb{P}_\delta$ *preserves* $\mathbf{x} \in \mathbf{K}_{uf}$ *when:* {k19.yaj}

- (a) $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \alpha\rangle$ *is a CS iteration*
- (b) $\mathbf{x} \in \mathbf{K}^\iota_{\rm uf}$ *and* $\iota \in ...9$ *or* $\mathbf{x} \in \mathbf{K}^{\iota,{\rm uf}}_{\mathbf{R}'}, \iota = 11$
- (c) $\underset{\sim}{\mathbb{Q}}_\beta$ *is game bounding, i.e.* $\Vdash_{\mathbb{P}_\beta}$ "COM *wins in* $\partial^{\rm bd}_{\mathbb{Q}_\beta}$" *for* $\beta < \delta$ *(hence also* $\mathbb{P}_\alpha/\mathbb{P}_\beta$ *for* $\alpha \in [\beta, \delta]$*).*

Remark 13.3. 1) Still we like to have also preservation.
2) Before this claim we should prove by induction on $\varepsilon < \aleph_1$, that if ${\rm Dp}(B) = \varepsilon, B \in \mathbf{V}, p \Vdash_{\mathbb{P}_\delta}$ "if $X \subseteq \max(B), X \neq \emptyset \bmod D_B$" (e.g. $X = B' \cap \max(B), B' \in {\rm psb}(A)$ then there is $B'' \in V$ and $q \supseteq p, q \Vdash B'' \cap \max Y \subseteq X^{[\mathbf{t}]}$ for some $\mathbf{t}$. See 3.1.

Discussion 13.4. 1) We need in §3 to write the proof for $\mathbb{Q}^{2,*}_D$.
2) Define and prove preservation for $\mathbf{x} \in \mathbf{K}_{\rm uf}$ as in [Sh:f, Ch.XVIII,§3].
3) For Dow question, we have at least the following ways:

- (A) toward simple $\aleph_2$-points eliminate almost all P-point as usual, but for one we in each case add $\mathscr{A}_\alpha \in P$ such that $\langle \mathscr{A}_\beta : \beta \leq \alpha\rangle$ is $\subseteq^*$-decreasing. Maybe $\underset{\sim}{\mathbb{Q}}_\alpha$ does two jobs fighting $\underset{\sim}{D}_\alpha$ as in $\underset{\sim}{\mathbb{Q}}^{2,*}_D$ and adding $\mathscr{A}_\beta$ or in $\alpha \in S^2$, as for no P-points, $\alpha \in \aleph_2 \backslash S^2$ we add $\underset{\sim}{A}_\alpha$. But what about the problem which Dow found in [BsSh:242]?
  Sort out: may the proof in [BsSh:242] work for this purpose (2010.10.11 worse this rotate $\mathfrak{d} = \aleph_0$) ?
- (B) iterate a forcing notion making $\underset{\sim}{D}_\alpha$ not (generate an) ultrafilter. Maybe just iterate $\mathbb{Q}^2_{\underset{\sim}{D}}$; (2010.10.11 or simpler as in [Sh:407])
- (C) let $\langle u_{\alpha,\ell} : \ell < C_\alpha\rangle$ is $\subseteq$-increasing union α, quite agree ($u_{\alpha,i} \cap \beta = u_{\beta,i}$ if $\beta \in u_{\beta,i}, C_\alpha \subseteq^* C_\beta$ for $\beta < \alpha$) add $\langle \underset{\sim}{A}_{\alpha,i} : \alpha \in \ell \in C_\alpha\rangle$?

4) The forcing $\mathbb{Q}^2_D, \mathbb{Q}^{2,*}_D$ preserves the P-point E, if $E \perp D$ (i.e. no common $\leq_{\rm RK}$-lower bound.
5) The forcing $\mathbb{Q}^2_D$ similar to [40],[5xx] Haifa but the case "against P-points" is similar to [Sh:f, Ch.VI,§x].
6) Use oracles.

§ 14. Preservation under iteration of a system

{u.3}
Definition 14.1. For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define a game $\Game = \Game_p = \Game_{\mathbb{Q},p}$: a play last ω moves, in the n-th move sequences $\bar{p}_n = \langle p_\rho : \rho \in u_n \rangle, \bar{q}_n = \langle q_\rho : \rho \in u_n \rangle$ are chosen such that $u_n \subseteq {}^n\omega$, $\bigcup\limits_{m \le n} u_m$ is a subtree of ${}^{\omega >}\omega$ with u_n its set of maximal nodes, $p_\rho \le_{\mathbb{Q}} q_\rho$ and $\rho_1 \trianglelefteq \rho_2 \in u_n \Rightarrow q_{\rho_1} \le_{\mathbb{Q}} p_{\rho_2}, u_0 = \{<>\}, p_{<>} = p$.

The player COM chooses $\bar{p}_n$ (in particular u_n) and the player INC chooses $\bar{q}_n$.

In the end COM wins the play if for some $r \in \mathbb{Q}$ we have $r \Vdash$ "$\bigwedge\limits_n \bigvee\limits_{\rho \in u_n} q_\rho \in \mathop{G}\limits_{\sim}{}_{\mathbb{Q}}$".

{u.5}
Claim 14.2. *Let $\iota = 2$. The forcing notion $\mathbb{Q}$ preserves "$\mathbf{x} \in \mathbf{K}^{st}_\iota$" when: for stationarily many $N \prec (\mathscr{H}(\chi), \in)$ we have $\boxplus_{\mathbf{x},N,\mathbb{Q}} \Rightarrow \oplus_{\mathbf{x},N,\mathbb{Q}}$ where*

$\boxplus_{\mathbf{x},N,\mathbb{Q}}$ *assume*

- (a) *$\bar{N} = \langle N_n : n < \omega \rangle$ is an increasing sequence of finite subsets of N with union N*
- (b) *if a is definable by a formula $\in N_n$ with parameters from N_n and is finite, then $a \subseteq N_{n+1}$,*
- (c) *$\mathbb{Q} \in N_0$,*
- (d) *$p \in \mathbb{Q} \cap N_0$,*
- (e) *$\mathbf{st} \in N_0$, a strategy of INC in the game $\Game^\iota_{\mathbf{x},\eta,Y}$ from **??**,*

$\oplus_{\mathbf{x},N,\mathbb{Q}}$ *there is $\mathscr{U} \in [\omega]^{\aleph_0}$ not containing any two successive integers such that letting $\langle n_\ell : \ell < \omega \rangle$ list $\mathscr{U}$ in increasing order, play of the game defined in **??** above with INC using $\mathbf{st}$, and COM making his ℓ-th move from N_{n_ℓ} such that COM wins.*

Discussion 14.3. We deal with $\eta \in T_{\mathbf{x}}, B \in \mathscr{A}_\eta \backslash \{\{\eta\}\}, Y = \max(B)$ and $D = D_{\mathbf{x},Y}$, so assume $\Vdash$ "$\mathop{A}\limits_{\sim} \subseteq {}^{\omega >}\omega, \mathop{A}\limits_{\sim} \neq \emptyset \bmod \mathrm{fil}(D)$".

For every $p \in \mathbb{Q}$ we can find $A_{[p]} \subseteq \omega$ such that

$\circledast_1$ $(\forall \text{ finite } u \subseteq Y)(\exists q)(p \le_{\mathbb{Q}} q \wedge q \Vdash$ "$\mathop{A}\limits_{\sim} \cap u = A_{[p]} \cap u$").

As $A_{[r]}$ can serve for $A_{[q]}$ if $q \le_{\mathbb{Q}} r$, without loss of generality $(\forall p)(A_{[p]} \in D)$ or $(\forall p)({}^\omega Y \backslash A_{[p]} \in D)$. By symmetry without loss of generality.

$\circledast_2$ $A_{[p]} \in D$ for every $p \in \mathbb{Q}$.

But we have to do the same for any $\nu \in t_n$ during a play, i.e. during a play of $\Game^\iota_{\mathbf{x},\eta,Y}$ we consider additional filters, so this is not good enough.

{u.6}
Before we prove 14.2 sd Fix $\eta \in T_{\mathbf{x}}, B \in \mathscr{A}^{\mathbf{x}}_\eta$ and $Y \in$ front(B).

For $p \in \mathbb{Q}, \mathop{A}\limits_{\sim}$ a $\mathbb{Q}$-name of a subset of Y and $N \prec (\mathscr{H}(\chi), \in)$ and $\{\mathbb{Q}, p, \mathop{A}\limits_{\sim}\} \in N$, we define a game $\Game_{\mathbb{Q},p,\mathop{A}\limits_{\sim},N}$ as follows: (if $N = (\mathscr{A}(\chi), \in)$ we may omit N)

it is between FO and AF (forcer, anti-forcer)

a play last ω-moves, during a play before the n-th move a finite subset t_{n-1} of $C_{\mathbf{x},\eta,Y}$ is chosen with $t_{-1} = \{\eta\}$ and a condition p_0; all from N

$\boxplus$ in the n-th move
- $\circledast_1$ the AF player chooses $q_n, p_n \le_{\mathbb{Q}} q_n$ (but $q_0 = p$)
- $\circledast_2$ the FO player chooses (t_{n+1}, p_{n+1}) such that
  - (a) $t_n \subseteq t_{n+1} \subseteq C_{\mathbf{x},\eta,Y}$ are finite

(*b*) $q_n \le_{\mathbb{Q}} p_{n+1}$

(*c*) $p_{n+1} \Vdash t_{n+1} \subseteq \underaccent{\tilde}{A}$ or $C_{\mathbf{x},\eta,Y}$.

In the end of the play FO wins if, letting $t_\omega = \cup\{t_n : n < \omega\}$, FO wins the play when $\nu \in t_\omega \Rightarrow t_\omega \cap Y \cap T_{\ge\nu} \in D^{\mathbf{x}}_\nu$.

Now {u.7}

Subclaim 14.4.

$\boxplus$ *if* $\underaccent{\tilde}{A}$ *and* $p \in \mathbb{Q}$ *letting* $\underaccent{\tilde}{A}_0 = Y\backslash\underaccent{\tilde}{A}, \underaccent{\tilde}{A}_1 = \underaccent{\tilde}{A}$ *<u>then</u> for some* $\underaccent{\tilde}{A}'$ *satisfying* $\underaccent{\tilde}{A}' \in \{\underaccent{\tilde}{A}_0, \underaccent{\tilde}{A}_1\}$ *for some* $q, p \le_{\mathbb{Q}} q$*, and in the game* $\partial_{\mathbb{Q},q,\underaccent{\tilde}{A}'_\ell,N}$*, FO does not lose.*

Proof. Why? Otherwise there are winning strategies $\mathrm{st}_0, \mathrm{st}_1$ for the player AF in the game $\partial_0 = \partial_{\mathbb{Q},p,\underaccent{\tilde}{A}_0}, \partial_1 = \partial_{Q,p,\underaccent{\tilde}{A}_1}$, respectively.

$\boxtimes$ let $\mathbf{S}$ be the set of pairs (u, q) such that $q \in \mathbb{Q}$ (or $q \in \mathbb{Q} \cap N$), u is a non-empty finite subset of $\mathbf{U}$ where

$\mathbf{U} = \{\mathbf{p} \in N :$ $\mathbf{p}$ is an initial segment of a play of $\partial_\iota, \iota = \iota(\mathbf{p})$ (so $\mathbf{p}$ contains the information $\iota(\mathbf{p}) \in \{0,1\}$) it is the turn of AF and his last choice $r_{\mathbf{p}}$ is $\le q\}$

and either FO is <u>not</u> in a losing role <u>or</u> if AF chooses q (or above) as his next move then it is a winning position for AF.

Notation: for $\mathbf{p} \in \mathbf{U}$ we let $r_{\mathbf{p}}$ be the current forcing condition.

Now

$(*)_1$ $\mathbf{U} \ne \emptyset$

[just the initial position]

$(*)_2$ if $(u_1, q_1), \mathbf{S}, \mathbf{p}_1 \in u_1, \nu \in \mathbf{T_x} \cap N$ and in $\mathbf{p}_1$ we let AF choose q_1 and t_n is such that $(\exists r)[(t, r)$ is a legal move for FO in this situation], <u>then</u> we can find $\mathbf{p}^* \in \mathbf{U}, r_*$ such that $\mathbf{p}^*$ is gotten from $\mathbf{p}$ by letting AF choose q_1 and then FO chooses (t, r_*) such that $u' := (u \cup \mathbf{p}^*), r \in \mathbf{S}$.

[Why? Obvious.]

$(*)_3$ moreover, in the case of a win for AF we use the strategy ∂_ι!

[in fact by monotonicity this always occurs but we do not use this]

$(*)_4$ there is $\bar{U} = \langle (u_n, q_n) : n < \omega\rangle$ such that $u_n \in \mathbf{S}, q_n \le_{\mathbb{Q}} q_{n+1}, u_n \subseteq u_{n+1}$ and if $\nu \in Y$ then $(\exists n)(q_n) \vdash \nu \in \underaccent{\tilde}{A}]$ or $(\exists n)[q_n \Vdash \nu \notin \underaccent{\tilde}{A}]$.

Let $A^* = \{\nu \in Y : (\exists n)(q_n \Vdash_{\mathbb{Q}} \nu \in \underaccent{\tilde}{A})$.

So $A^*_q = A^*, A^*_0 = Y\backslash A^*$, so for some $\iota \in \{0,1\}, A^*_\iota \in D^*_Y$, hence there is a witness $B \in T_{\mathbf{x}}$, i.e. $B \in \mathscr{A}^{\mathbf{x}}_\eta$, without loss of generality for simplicity $\max(B) = Y$ and $\nu \in B \Rightarrow B \cap T_{\ge\nu} \cap Y \in D_{\nu,Y} = D_{\nu,Y} \cap t_{\ge\nu}$ in particular $B \cap Y \in D^{\mathbf{x}}_{\eta,Y}$ and $B \cap Y \subseteq A^*_{\iota(*)}, \iota(*) \in \{0,1\}$.

Now we can choose $\mathbf{p}_n \in u_n$ by induction on $n < \omega$ such that $\iota(\mathbf{p}_n) = \iota(*), \mathbf{p}_n$ an initial segment of $\mathbf{p}_{n+1}$ and $t[\mathbf{p}_n] \subseteq B$ and if $t \in B$ then for some $n, \nu \in t[\mathbf{p}_n]$.

So $\langle \mathbf{p}_n : n < \omega \rangle$ is a play of $\partial_{\iota(*)}$, we start with a losing postion so AF uses his strategy hence it wins, but $t_\omega = \cup\{t_n : n < \omega\} = B$ so FO wins, contradiction. $\square$

{27.2b}

Subconclusion 14.5. If (q,t) is a non-losing state for FO (i.e. there is $\mathbf{p} \in \mathbf{U}$, etc., or define it naturally) then there is $B \in \mathscr{A}_\mathbf{x}$ such that $t \subseteq B, B$ canonical (like t_ω) and for every finite $g', t \subseteq t' \subseteq B$ for some $q', q \leq_\mathbb{Q} q'$ $(\in N?)$ (q',t') is a non-losing state for FO.

Proof. Play. Proof of 14.2:
Let $\bar{N}$ be such that

⊛ (a) $\bar{N} = \langle N_n : n < \omega \rangle$
(b) $N_n \subseteq N_{n+1}$ are finite
(c) N_n is fnite and $\mathbf{x} \in N_0$
(d) $N_n \cap \omega = k_n, N_n \cap {}^{\omega >}\omega = s_n$, a subtree
(e) as in the assumption of the claim.

Also $\langle N'_{10n} : n < \omega \rangle$ is O.K. and let $\mathscr{U}$ be as in the Claim 14.4 and let $\langle n_k : k < \omega \rangle$ list $\mathscr{U}$ and $\mathbf{st}$ be the stratey fo COM in the revised game - see the claim. We play a play with COM using $\mathbf{st}_{\mathrm{COM}}$ simulating a play of the relevant game with PO using $\mathbf{st}^*$.

We describe a play: in the k-move, gives $\mathbf{s} = \mathbf{s}_n = \langle t_\varrho, \bar{B}_\varrho : \varrho \in u^1_k \rangle$ and $\langle p_\rho, q_\rho : \rho \in u^2_n \rangle$, e.g. in $N_{10 n_{k-1}+5}$ and on the side we have t_n.

We let COM choose $\langle p_\rho : \rho \in u^2_k \rangle$ in $N_{10 n_k}$. Now to decide on INC player answer (i.e. we are describing a strategy for INC).

We preserve:

⊞ (q_ρ, t_ϱ) is as in the subclaim: a non-losing positin for FO for every $\rho \in U^r_k, \varrho \in u^1_k$.

The rest should be clear. $\square$

{u9}

Claim 14.6. *The $\mathbb{Q}$ from Definition 15.1, page 24 satisfies the demand in claim 14.2.*

Remark 14.7. Which forcing?

Proof. We just need $\mathbf{U}$ and $\langle n_k : k < \omega \rangle$ to satisfy

$$\cup\{(N_{n_k+1} \backslash N_{n_k-1}) \cap \omega : k < \omega\} = \emptyset \bmod E.$$

$\square$

{u11}

Conclusion 14.8. *If $\mathbf{V} \models 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$, then for some proper $\aleph_2$-c.c. forcing notion $\mathbb{P}$*
$\Vdash_\mathbb{P}$ "$2^{\aleph_0} = \aleph_2$, no P-point and $\mathbf{x} \in K^\mathbf{V}_2$ is preserved, i.e. the $D^\mathbf{x}_{\eta,Y}$ continue to be (pedantically generates) as ultrafilter".

Proof. We iterate as in [Sh:f, VI] for this purpose. So $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha; \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ is CS iteration, $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{E}_\alpha$ a P-point", $\underset{\sim}{\mathbb{Q}}_\alpha = Q_{\underset{\sim}{E}_\alpha}$ as in 15.1. Use the theorem there. Each $\underset{\sim}{\mathbb{Q}}_\alpha$ preserves any such $\mathbf{x}$ by **??**,14.6.

By §24 the iteration does. $\square$

Remark 14.9. We can use my first proof of CON no P-point as in Wimer []?, have to deal also with the condition from **??** which is straight.

§ 15. Against one P-point

For proving CON($\exists!P$-point) we had defined:

{3c.1}

Definition 15.1. For a P-point or just a P-filter E we define $\mathbb{Q} = \mathbb{Q}_E$ as in [Sh:f, VI,xxx]:

(A) $p \in \mathbb{Q}$ iff
  (α) p is a perfect subtree of $\prod\limits_{n<\omega} {}^n 2$
  (β) the following set $\in E : \{n : (\forall \eta \in p)(\ell g(\eta) = n \wedge \rho \in {}^n 2 \Rightarrow \eta\hat{\ }\langle\rho\rangle \in p)\}$ (not $\eta\hat{\ }\rho$!)

(B) $p \le q$ iff $p \supseteq q$.

{k2.yajan}

Claim 15.2. *1) $\mathbb{Q}_E$ is proper ${}^\omega\omega$-bounding.*
2) For E a P-point and $p \in \mathbb{Q}_E$ in the game $\Game_{\mathbb{Q}_t,p}$ the Player COM *do not lose (see works with Roslanowski* [?] *- ask him).*

{k2.yaj}

Definition 15.3. 1) For $p \in \mathbb{P}$ we define $\Game^x_p = \Game^x_p$ as follows:

(a) a play last ω moves
(b) in the n-moves, INC (the incompletness player) chooses $k_n < \omega$ and pairwise incompatible $p_{n,\ell} \ge p$ for $\ell < k_n$ then COM chooses $q_{n,\ell} \ge p_{n,\ell}$ for $\ell < k_n$
(c) in the end COM wins if there is $q \ge p$ such that the following set is ω if $x = a$, is infinite if $x = b, \{n < \omega : q \Vdash$ "$q_{n,\ell} \in \underset{\sim}{G}$ for some $\ell < k_n$"$\}$; notation?

2) We define $\Game_p = \Game^c_{p,D}$ similarly where $D \subseteq [\omega]^{\aleph_0}$ but:

$(b)_1$ in the n-th move INC also chooses $A_{2n} \in D, A_{2n} \subset \bigcap\limits_{\ell<n} A_\ell$ but $A_0 = \omega$ (?)
  CON also chooses $A_{2n+1} \subset A_{2n}, A_{2n+1} \in D$
(c) in the end if $X = \emptyset$ mod fil$(D), Y = \cup\{A_{2n} \backslash A_{2n+2} : n \in X\}$ (or so) then for some $q \ge p$ we have $q \Vdash$ "if $n \in X$ then $q_{n,\ell} \in \underset{\sim}{G}$ for some $\ell < k_n$".

3) We[4] define $\Game^d_{p,\infty}$ similarly but:

(d) in the end for every infinite $X \subseteq \omega$ there are infinite $Y \subseteq X$ and q such that $p \le_{\mathbb{P}} q$ and $q \vdash \bigwedge\limits_{n\in Y} \bigvee\limits_{\ell<k_n} q_{n,c} \in \underset{\sim}{G}_{\mathbb{Q}}$.

4) $\Game^\forall_{\mathbb{Q}}$ - similarly $X = \omega$ forgetting D.

Proof. 1) See there, any follows by (2).
2) Easy (see quotations).
3) Also straight. $\square$

{3c.5}

Claim 15.4. *Assume*

(A) $\bar{D} \in \mathbf{K}_{2,one}$ *(i.e. $\in \mathbf{K}$, so D_η ultrafilter ... and in $\Game^{one/one}_{\bar{D}^{[\eta]}}$ PO does not lose for $\eta \in {}^{\omega>}\omega$; this covers Sachs forcing.*

Then :

[4] we can restrict ourselves $N, \{(H(\chi)$ continue for stationary many N

(1) *if $\mathbb{Q}$ is a forcing notion as in Definition* **??***(1) for $x = a$ then $\mathbb{Q}$ is $\bar{D}$-preserving*

(1A) *also for $x = c$ and*

(2) *E a P-filter and no extension of it is a quotient of any D_η (I think this can replace "$\mathscr{P}(\omega)/E$ is c.c.c.) forcing by $\mathbb{Q}_T$ preserves the "PO does not lose is $\Game^{fin}_{\bar{D}}$".*

Proof. 1) By symmetry we deal with $D_{<>}$. Let $p \Vdash$ "$\underset{\sim}{A}, \underset{\sim}{T} \subseteq \omega$" without loss of generality $p \Vdash$ "$\underset{\sim}{A} \in \text{fil}(D_{<>})^+$" and $\underset{\sim}{T}$ is a strategy for NU is the game $\Game_{\bar{D}}$ (any variant)". Now NU will simulate on the side finitely many of the game.
2) FILL (the main case is D a P-point). $\square$

Intended Theorem 15.5. *$CON(\exists \bar{D} \in \mathbf{K}_2)$ such that PO does not lose and any P-point is above some $D_\eta/E^*_\eta, \eta \in {}^{\omega>}\omega$.*

Proof. Immitating [Sh:f, VI,xxx]. $\square$

§ 16. Alan Dow question

We try to prove the consistency of $\mathfrak{u} = \aleph_1 +$ there is a P-point but no P-point of character $\aleph_1$.

We may like

Context 16.1. $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$ and $\diamondsuit_S, S \subseteq \{\delta < \aleph_2 \colon \mathrm{cf}(\delta) = \aleph_2\}$.

Program 16.2. We shall use a proper forcing and $\bar{D}$:

(a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \omega_2 \rangle$
(b) $\bar{Q}$ is a CS iteration with limit $\mathbb{P} = \lim(\bar{\mathbb{Q}})$
(c) each $\underset{\sim}{\mathbb{Q}}_\alpha$ is proper
(d) $\underset{\sim}{D}_\alpha$ is a P_α-name of a P-point
(e) $\underset{\sim}{D}_\alpha$ increases with α
(f) for $\delta \in S, \diamondsuit_S$ gives us $\underset{\sim}{E}_\delta$, a $\mathbb{P}_\delta$-name of a P-point such that
 (α) $\underset{\sim}{D}_\alpha, \underset{\sim}{E}_\delta$ are $\leq_{\mathrm{RK}}$-incomparable
 (β) $\mathbb{Q}_\delta = \mathbb{Q}_E$
 (γ) for unboundedly many $\alpha < \delta, \mathbb{Q}_\alpha$ is $\mathbb{Q}[\underset{\sim}{D}_\alpha]$ probably the forcing from [Sh:407]
 (δ) let $\bar{D}_\alpha = \underset{\sim}{D}_\beta : \beta = 0$ or β successor, $\beta < \alpha\}$.

To carry this we need

$\boxplus_1$ a preservation theorem in limit for ordinals δ of cofinality $\aleph_0$ such that D^-_δ can be complete to a P-point, e.g.

$\boxplus_2$ if $\mathrm{cf}(\delta) = \aleph_0, \Vdash_{\mathbb{P}}$ "if $A_n \subseteq \omega, A_n \neq \emptyset \mod \underset{\sim}{D}^-_\delta$ and $\underset{\sim}{A}_n \geq A_{n^+}$ then for some $B \neq \emptyset \mod \underset{\sim}{D}^-_\delta$ we have $B \subseteq^* \underset{\sim}{A}_n$ for every n".

This is similar to preservation theorem but does not seem so. This depends on the forcing notions used earlier.

Saharon Continue?

Moved 2010.10.10, pg.4:

Definition 16.3. Let $\mathbf{K}$ be the class of the objects $\mathbf{x}$ which consists of the following objects satisfying the following properties: {z17}

(a) $\bar{\mathscr{A}}_{\mathbf{x}} = \bar{\mathscr{A}} = \langle \mathscr{A}_\eta : \eta \in T \rangle = \langle \mathscr{A}^{\mathbf{x}}_\eta : \eta \in T_{\mathbf{x}} \rangle$
(b) $T = T_{\mathbf{x}}$ is a partial order which is a tree (but maybe $(\exists \eta, \nu \in T)\neg(\exists \rho \in T)[\rho \leq_T \eta_1, \rho \leq_T \nu]$
(c) $\mathscr{A}_\eta = \mathscr{A}^{\mathbf{x}}_\eta \subseteq \mathrm{CWT}(T)$, let $\mathscr{A}^-_\eta = \mathscr{A}_\eta \backslash \{\{\eta\}\}$
(d) $\mathrm{rt}(B) = \eta$ for every $B \in \mathscr{A}_\eta$
(e) $\mathscr{A}_\eta$ is not empty, $\{\eta\} \in \mathscr{A}_\eta$
(f) $\mathscr{A}_\eta$ is closed under finite intersection
(g) $\leq_\eta = \leq^{\mathbf{x}}_\eta$ is a partial order on $\mathscr{A}_\eta$ such that $(\mathscr{A}_\eta, \leq_\eta)$ is directed
(h) $B_1 \leq_\eta B_2$ implies $B_1 \leq^*_T B_2$ and, of course, $B_1, B_2 \in \mathscr{A}_\eta$
(i) if $\nu \in B \in \mathscr{A}_\eta$ then $B \cap T_{\geq \nu} \in \mathscr{A}_\nu$

(j) (used?) if $\mathscr{A}_\eta \neq \{\{\eta\}\}$ and $\eta <_M \nu$ then for some $B \in \mathscr{A}_\eta$ we have $\{\rho \in \operatorname{suc}_B(\eta) : \nu \leq_T \rho\}$ is finite.

{z19}

Definition 16.4. For $\mathbf{x} \in \mathbf{K}$ and $\eta \in T_{\mathbf{x}}$ and $Y \in \text{alm-frt}_{\mathbf{x}}(\eta)$ we define a game $\eth^\iota = \eth^\iota_Y = \eth^{\mathbf{x},\iota}_{\eta,Y} = \eth^\iota_{\mathbf{x},\eta,Y}$ where $\iota < 6$, the default value is 0.

(A) a play last ω moves and is between the player PO (positive) and NU (null).

(B) in the n-th move, PO chooses t_n and NU chooses $\bar{A}_n$ such that:

(a) t_n is a finite subset of $t_{\geq \eta}$
(b) $\bar{A}_n = \langle A^n_\nu : \nu \in t_n \rangle$

such that

(c) $t_0 = \{\eta\}$
(d) $t_m \subseteq t_n$ for $m < n$
(e) $A^n_\nu \in D^+_{\nu,Y}$ and $\in D_{\nu,Y}$ if $\iota < 3$ and $A^n_\nu \cap t_n = \emptyset$ moreover $\rho \in t_n \wedge A^n_\nu \cap T_{\geq \rho} \neq \emptyset \Rightarrow \rho \leq_T \nu$
(f) if $n = m+1$, then $t_n \backslash t_m \subseteq \cup\{A^m_\nu : \nu \in t_m\}$
(g) also if $\iota = 0, 1 \bmod 3$, NU also chooses $\eta_n \in t_n$ in the n-the move and in the next move PO has to satisfy $t_{n+1} \backslash t_n \subseteq T_{>\eta_n}$ and if $\iota = 0 \bmod 3$ then $|t_{n+1} \backslash t_n| \leq 1$.

(C) in the end of the play PO wins when letting $t_\omega := \cup\{t_n : n < \omega\}$ we have $\iota \geq 3 \rightarrow t_\omega \cap Y_{\geq \nu} \in D^+_{\nu,Y}$ and $\iota < 3 \Rightarrow t_\omega \cap Y_{\geq \nu} \in D_{\nu,Y}$ whenever $\nu \in \{t_n : n < \omega\}$; and $\iota = 0, 1 \bmod 3 \Rightarrow (\exists^\infty n)(\eta_n = \nu)$.

Discussion 16.5. A middle ground between 16.3 and **??** is: to add in 16.3 to clause (f):

(f) $(\alpha) quad \leq_\eta = \leq^{\mathbf{x}}_\eta$ is a partial order on $\mathscr{A}_\eta$
(β) $E_\eta = E^{\mathbf{x}}_\eta$ is an equivalence relation on $A^{\mathbf{x}}_\eta \backslash \{\eta\}$
(γ) if $e \in \mathscr{A}^{\mathbf{x}}_\eta / E_\eta$ then $(e, <^{\mathbf{x}}_\eta)$ is directed
[check:make a tree]

Discussion 16.6. Adopting 16.3 we have:
(A) in 1.6(3)(B)
3A) Assume $Y \in \operatorname{front}_{\mathbf{x}}$ say $Y \in \operatorname{front}(B_1), B_1 \in \mathscr{A}_{\eta(\mathrm{rt})}, \eta = \eta_{\mathbf{x}}$ we let $D_Y = D^{\mathbf{x}}_Y = D_{\eta,Y} = D^{\mathbf{x}}_{\eta,Y}$ (see 1.7(2)) be the filter on Y generated by $\{B \cap Y \cap A_f : B \in \mathscr{A}_\eta$ is $B_1 \leq^{\mathbf{x}}_\eta B$ and $f \in \operatorname{Fin}_{\mathbf{x}}(B)\}$; note that η is determined by Y.
3B) Also if $B \in \mathscr{A}^{\mathbf{x}}_\nu, Y \subseteq T$ and $Y \cap B$ is an almost front of B then $D_{\eta,Y} = D^{\mathbf{x}}_{\eta,Y}$ is $D_{\eta,Y \cap T_{\geq \nu}}$.
3C) Lastly, if $B \in \mathscr{A}^{\mathbf{x}}_\eta$ then $D^{\mathbf{x}}_B = D^{\mathbf{x}}_{\eta,\max(B)}$.

{7j.2} *Remark* 16.7. (B) In Definition 1.7 add:
A middle way is to add in clause 4:
1A) η is uniquely determined by Y.
2A) If $B \in \mathscr{A}_\eta, B \neq \{\eta\}$ and $A \in D_B$ then $\{\nu \in A : A \cap T_{\geq \nu} \in D_{(B_{\geq \nu})}\} \in D_B$.
8) If $\mathbb{Q}$ is an ${}^\omega\omega$-bounding forcing, $\mathbf{x} \in \mathbf{K}, \eta \in \mathscr{T}_{\mathbf{x}}, B \in T^{\mathbf{x}}_\eta$ then for any $B' \in \operatorname{sb}_{\mathbf{x}}(B)^{\mathbf{V}[\mathbf{Q}]}$ there is $B'' \in \operatorname{sb}_{\mathbf{x}}(B)^{\mathbf{V}}$ such that $B'' \subseteq B'$.

(C) In Definition 1.8 add:
4) We say $\mathbf{x} \in \mathbf{K}$ is principal <u>when</u> there is a sequence $\bar{B} = \langle B_\eta : \eta \in T_{\mathbf{x}} \rangle$ such that $B_\eta \in \mathscr{A}^{\mathbf{x}}_\eta$ is $<_\eta$-maximal for each $\eta \in T_{\mathbf{x}}$.
4A) We say $\mathbf{x} \in \mathbf{K}$ is principal in η <u>when</u> $\eta \in T_{\mathbf{x}}$ and there is $B_\eta \in \mathscr{A}^{\mathbf{x}}_\eta$.
8) $\mathbf{K}_{\rm uf}$ is the class of $\mathbf{x} \in \mathbf{K}$ such that $D^{\mathbf{x}}_B$ is an ultrafitler on $\max(B)$ for every $B \in \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})}$.

(D) Replace 2.3 by:

Claim 16.8. *1) If $\mathbf{x} \in \mathbf{K}_{\le\aleph_0}$ and $\eta \in \mathbf{X}$ <u>then</u> for some $\mathbf{y} \in \mathbf{K}_{\le\aleph_0}$ we have $\mathbf{x} \le \mathbf{y}$ and in $\mathscr{A}^{\mathbf{y}}_\eta$ there is a $<^{\mathbf{y}}_\eta$-maximal member (and if $\nu \in T_{\mathbf{x}} \backslash \{\eta\}$ and $\le^{\mathbf{x}}_\nu$ has a maximal member then so does $\le^{\mathbf{y}}_\nu$) and $T_{\mathbf{y}} = T_{\mathbf{x}}$.* {8h.10yaj}
2) If $\mathbf{x} \in \mathbf{K}_{\le\aleph_0}$ <u>then</u> for some $\mathbf{y} \in \mathbf{K}_{\le\aleph_0}, \mathbf{x} \le_{\mathbf{K}} \mathbf{y}$ and in $\mathscr{A}^{\mathbf{y}}_\eta$ there is a $\le^{\mathbf{y}}_\eta$-maximal member for every $\eta \in T_{\mathbf{y}}$, that is $\mathbf{y}$ is principal. [check]
3) If $\mathbf{x} \in \mathbf{K}_{\le\aleph_0}$ and $\eta \in T_{\mathbf{x}}$ and $B \in \mathscr{A}^{\mathbf{x}}_\eta$ is $\le^{\mathbf{x}}_\eta$-maximal and $B' \in \mathrm{sb}_{T_{\mathbf{x}}}(B)$ <u>then</u> there is $\mathbf{y}$ such that $\mathbf{x} \le \mathbf{y} \in \mathbf{K}_{<\aleph_0}$ and $\mathscr{A}^{\mathbf{y}} = \mathscr{A}^{\mathbf{x}}_\eta \cup \{B'\}$ in B' is $\le^{\mathbf{y}}_\eta$-maximal. [check]

Similarly

Claim 16.9. *Assume $\mathbf{x} \in \mathbf{K}, \eta \in T_{\mathbf{x}}, Y \in \text{alm-frt}_{\mathbf{x}}(B), B \in \mathscr{A}^{\mathbf{x}}_\eta$ and f is a function with domain Y such that $\nu \in Y \Rightarrow f(\nu) \in \mathscr{A}^{\mathbf{x}}_\nu$.* {8h.13}

<u>Then</u> for some $\mathbf{y} \in \mathbf{K}$ we have $\mathbf{x} \le_{\mathbf{K}} \mathbf{y}$ and for some $B' \in \mathscr{A}^{\mathbf{y}}_\eta$ we have $B \le^{\mathbf{y}}_\eta B' \wedge (\forall \nu \in Y)(\nu \in B' \to f(\nu) \le^{\mathbf{y}}_\nu B' \cap (T_{\mathbf{y}})_{\ge\eta})$.

(E) In 2.4, $\eta \in T_{\mathbf{x}}$ instead essentially only $\eta = \mathrm{it}_{\mathbf{x}}$.

<u>Moved 2010.10.09 from §1, pg.7:</u>
<u>An alternative to 1.5 is:</u>

Definition 16.10. Let $\mathbf{K}$ be the set of objects $\mathbf{x}$ consisting of {7j.1}

- (a) $\bar{\mathscr{A}}_{\mathbf{x}} = \bar{\mathscr{A}} = \langle \mathscr{A}_\eta : \eta \in T \rangle = \langle \mathscr{A}^{\mathbf{x}}_\eta : \eta \in T_{\mathbf{x}} \rangle$; let $T[\mathbf{x}] = T_{[\mathbf{x}]}$
- (b) T is a partial order which is a tree with root $\eta_{\mathbf{x}} = \mathrm{rt}_{\mathbf{x}} = \mathrm{rt}(\mathbf{x})$, so the order is $<_T = <_{T[\mathbf{x}]}$, but we may write $<_T$ or $<_{\mathbf{x}}$; tree (in the model theoretic sense, i.e. for every $\eta \in T$ the set $\{\nu \in T : \nu <_T\}$ is linearly ordered by $<_T$; but not necessarily well founded, moreover maybe $(\exists \eta, \nu \in T)\neg(\exists \rho \in T)[\rho \le_T \eta_1 \wedge \rho \le_T \nu]$
- (c) $\mathscr{A}_\eta \subseteq \mathrm{CWT}(T)$, let $\mathscr{A}^-_\eta = \mathscr{A}_\eta \backslash \{\{\eta\}\}$
- (d) $\mathrm{rt}(B) = \eta$ for every $B \in \mathscr{A}_\eta$
- (e) $\mathscr{A}_\eta$ is not empty, in fact $\{\eta\} \in \mathscr{A}_\eta$
- (f) $<_{\mathbf{x}} = \le^{\mathbf{x}}_{\eta_{\mathbf{x}}}$ a partial order of $\mathscr{A}_{\eta_{\mathbf{x}}}$ directed.

<u>Moved from Claim 2.3, pg.10</u>
(and if $\nu \in T_{\mathbf{x}} \backslash \{\eta\}$ and $\le^{\mathbf{x}}_\nu$ has a maximal member then so does $\le^{\mathbf{y}}_\nu$) and $T_{\mathbf{y}} = T_{\mathbf{x}}$.
2) If $\mathbf{x} \in \mathbf{K}_{\le\aleph_0}$ <u>then</u> for some $\mathbf{y} \in \mathbf{K}_{\le\aleph_0}, \mathbf{x} \le_{\mathbf{K}} \mathbf{y}$ and in $\mathscr{A}^{\mathbf{y}}_\eta$ there is a $\le^{\mathbf{y}}_\eta$-maximal member for every $\eta \in T_{\mathbf{y}}$, that is $\mathbf{y}$ is principal. [check]
3) If $\mathbf{x} \in \mathbf{K}_{\le\aleph_0}$ and $\eta \in T_{\mathbf{x}}$ and $B \in \mathscr{A}^{\mathbf{x}}_\eta$ is $\le^{\mathbf{x}}_\eta$-maximal and $B' \in \mathrm{sb}_{T_{\mathbf{x}}}(B)$ <u>then</u> there is $\mathbf{y}$ such that $\mathbf{x} \le \mathbf{y} \in \mathbf{K}_{<\aleph_0}$ and $\mathscr{A}^{\mathbf{y}} = \mathscr{A}^{\mathbf{x}}_\eta \cup \{B'\}$ in B' is $\le^{\mathbf{y}}_\eta$-maximal. [check]

<u>Moved from after 12.3, pg.13</u>

Definition 16.11. 1) For $\mathbf{x} \in \mathbf{K}$ and $\eta \in T_{\mathbf{x}}, \eta = \text{ rt}_{\mathbf{x}}$ and $Y \in \text{ alm-frt}_{\mathbf{x}}(\eta)$ we define a game $\eth^\iota = \eth^\iota_Y = \eth^{\mathbf{x},\iota}_Y = \eth^\iota_{\mathbf{x},Y}$ where $\iota < 6$, (writing $\eth^\iota_{\mathbf{x},B}$ where $B \in \mathscr{A}^{\mathbf{x}}_\eta$ means $\eth^\iota_{x,\max(B)}$ {8j.19yaj}

(A) a play last ω moves and is between the player PO (positive) and NU (null).

(B) in the n-th move PO chooses t_n then DU chooses $\bar{B}_n$ such that lastly PO chooses $\bar{D}^n$:

- (a) t_n is a fintie subset of $\nu \in 1 : \nu$ is below some $\rho \in Y\}$
- (b) $\bar{B}_n = \langle B^n_\nu : \nu \in t_n \rangle$ and $\bar{D}^n = \langle D^n_\nu : \nu \in t_n \backslash \{\text{rt}\}\rangle$ such that
- (c) $t_0 = \{\eta\}$
- (d) for $m < n$
  - (α) $t_m \subseteq t_m$
  - (β) if $\eta \in t_m, \rho \in \text{ suc}(\eta, t_m)$ then $\rho \in \text{ suc}(\eta, t_n)$ moreover $\rho \in \text{ suc}(\eta, A^m_\eta)$
- (e) $B^n_\eta \in A^{\mathbf{x}}_\eta$ for $\eta \in t_n$ and $Y \cap B^n_\eta$ is an almost front of B^n_η (e.g. is $\{\eta\} \subseteq Y$)
- (f) if $n = m+1, \eta \in t_m$ <u>then</u>
  - (α) $B^n_\eta \in \text{ psb}_{\mathbf{x}}(B^m_\eta)$
  - (β) $B^n_\rho \in \text{ psb}_{\mathbf{x}}((B^m_\eta)^{[\rho]})$ if $\rho \in \text{ suc}(\eta, t_n) \backslash t_m$
- (g) D^n_ν is an ultrafilter on B^n_ν which extends $\text{sb}_{\mathbf{x}}(B^n_\nu)$
- (h) also if $\iota = 0, 1 \bmod 3$, NU also chooses $\eta_n \in t_n$ in the n-the move and in the next move PO has to satisfy $t_{n+1} \backslash t_n \subseteq T_{>\eta_n}$ and if $\iota = 0 \bmod 3$ then $|t_{n+1} \backslash t_n| \leq 1$.

(C) in the end of the play PO wins when letting $t_\omega := \cup\{t_n : n < \omega\}$ we have $\iota \geq 3 \to t_\omega \cap Y \in D^+$ and $\iota < 3 \Rightarrow t_\omega \cap Y \in D_Y$ whenever $\nu \in \{t_n : n < \omega\}$; and $\iota = 0, 1 \bmod 3 \Rightarrow (\exists^\infty n)(\eta_n = \nu)$.

<u>Moved from after 6.5, pg.14</u>

{8h.21} **Claim 16.12.** *1)* $\mathbf{K}_{\iota_1} \subseteq \mathbf{K}_{\iota_2}$ *when* $(\iota_1, \iota_2) = (0,1), (0,2)$.
2) For ultra $\mathbf{x}, x \in \mathbf{K}_{\rm o} \Leftrightarrow \mathbf{x} \in \mathbf{K}_\iota$ *for* $\iota = 0, 1, 2$.

<u>Moved 2010.10.10, pg.23</u>

<u>Case 1</u>: $\iota = 6$

<u>Stage B</u>:

$\boxplus_2$ there are ... ?

Why? Let $p \geq p_n$ we define a strategy **St** for the player NU in the game $\eth = \eth_{\mathbf{x},\eta,B}$, so

$(*)_{2.1}$ in the i-th move $(B_{1,n}, B_{2,n}, B_{3,n}, \nu_n, B_{n,4})$ are chosen.

The strategy is to choose, on the side, also p such that

$(*)_{2.2}$ (a) $p_0 = p_*$

(b) $p_n \le_{\mathrm{pr},n} p_{n+1}$ in $\mathbb{Q}$

(c) $p_{n+1}, B_{n,4}$ are chosen in the third part of the n-th move

(d) if $g \in {}^n 2$ then for some $\mathbf{t} = \mathbf{t}_{n,g}$ we have $p^{[g]}_{n+1} \Vdash$ "$B_{n,4} \subseteq \underset{\sim}{A}^{[\mathbf{t}]}$".

So really this is a partial strategy. By 3.1 we are not stuck hence, as "$\mathbf{x} \in \mathbf{K}_{\mathrm{ul}}$":

$(*)_{2.3}$ there is a play $\langle B_{n,1}, B_{n,2}, \nu_n, B_{n,4} : n < \omega\rangle$ in which NU uses the strategy **st** as witnessed by $p_n, \{\mathbf{t}_{n,g}(g \in {}^n 2)$ and $B_u = \cup\{B_{n,4} : n < \omega\} \in D^{\mathbf{x}}_{\eta,B}$.

Note that $p_\omega = \lim\langle p_n : n < \omega\rangle$ is well defined and $\mathbb{Q}^2_D$, no need of some diagonal union using D is a P-point.

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