A Hanf number for saturation and omission: the superstable case

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Abstract

Suppose $\boldsymbol{t} = (T, T_1, p)$ is a triple of two theories in vocabularies $\tau \subset \tau_1$ with cardinality λ, $T \subseteq T_1$ and a τ_1-type p over the empty set that is consistent with T_1. We show the Hanf number for the property: 'There is a model M_1 of T_1 which omits p, but $M_1 \restriction \tau$ is saturated' is less than $\beth_{(2^{(2^\lambda)^+})^+}$ if T is superstable.

We showed in [BS11] that with no stability restriction the Hanf number is essentially equal to the Löwenheim number of second order logic.

Hanf observed [Han60] that if one asks for each $\boldsymbol{K}$ in a set of classes of structures, 'Does $\boldsymbol{K}$ have arbitrarily large members?', there is a cardinal κ (the sup of the maxima of the bounded $\boldsymbol{K}$) such that any class with a member at least of cardinality κ has arbitrarily large models. In many cases this bound κ can be calculated (For a countable first order theory, it is $\aleph_0$.) In this paper we call a Hanf number for a family $\mathcal{K}$ of classes *calculable* if it is bounded by a function that can be computed by an arithmetic function in ZFC (See Definition 0.1.) and if not it is *incalculable*.

The following definition is more abstract than needed for this paper but we include it for comparison with other works where other Hanf functions are shown to be not calculable.

Definition 0.1 *1. A function f (a class-function from cardinals to cardinals) is* strongly calculable *if f can (provably in ZFC) be defined in terms of cardinal addition, multiplication, exponentiation, and iteration of the $\beth$ function.*

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2. *A function f is* calculable *if it is (provably in ZFC) eventually dominated by a strongly calculable function. If not, it is* incalculable.

We extend our work on Newelski's [New12] question about calculating the Hanf number of the following property:

Definition 0.2 1. *Let $\boldsymbol{t} = (T, T_1, p) = (T_{\boldsymbol{t}}, T_{1,\boldsymbol{t}}, p_{\boldsymbol{t}})$ be a triple of two theories T, T_1 in vocabularies $\tau \subset \tau_1$, respectively, such that $|\tau_1| \leq \lambda$, $T \subseteq T_1$ and p is a τ_1-type over the empty set consistent with T_1. We say $M_1 \models \boldsymbol{t}$ if M_1 is a model of T_1 which omits p, but $M_1 \restriction \tau$ is saturated.*

2. *Let $\boldsymbol{N}_\lambda$ denote[1] the set of $\boldsymbol{t}$ with $|\tau_1| = \lambda$. Then $H(\boldsymbol{N}_\lambda)$ denotes the Hanf number of $\boldsymbol{N}_\lambda$. That is, $H(\boldsymbol{N}_\lambda)$ is the least cardinal so that if $\boldsymbol{t} \in \boldsymbol{N}_\lambda$ has a model of cardinality $H(\boldsymbol{N}_\lambda)$ it has arbitrarily large models.*

3. *The Hanf number of a logic $\mathcal{L}$ (e.g. $L_{\kappa^+,\kappa}$) is the least cardinal μ such that if an $\mathcal{L}$-sentence has a model in cardinal μ, then it has arbitrarily large models.*

Under mild set theoretic hypotheses, we showed in [BS11] that $H(\boldsymbol{N}_\lambda)$ essentially equals the Löwenheim number of second order logic, which is incalculable. In Section 1 we restrict the question by requiring that the theory T be superstable; the number is then easily calculable in terms of Beth numbers.

The phenomena that stability considerations can greatly lower Hanf number estimates was earlier explored in [HS91]. Work in preparation extends the current context to strictly stable theories. References of the form X.x.y are to [She91].

Much of this paper depends on a standard way of translating between sentences in languages of the form $L_{\lambda,\omega}(\tau)$ and first order theories in an expanded vocabulary τ that omit a family of types. This translation dates back to [Cha68]; a short explanation of the process appears in Chapter 6.1 of [Bal09]. Chapter VII.5 of [She91] is an essential reference for this paper. In those references, these (equivalent) Hanf numbers of sentences and associated pair of a family of types and theory are calculated using the 'well-ordering number of a class'. We begin with a slight rewording of Definition VII.5.1 of [She91], using language from [Cha68].

Definition 0.3 1. *The Morley number $\mu(\lambda, \kappa)$ is the least cardinal μ such that if a first order theory T in a vocabulary of cardinality λ has a model in cardinality μ which omits a family of κ types over the empty set, it has arbitrarily large such models.*

2. *The well-ordering number $\delta(\lambda, \kappa)$ is the least ordinal α such that if a first order theory T in a vocabulary τ of cardinality λ, which includes a symbol $<$ has a model which omits a family κ types over the empty set and $<$ is well ordering of type α, then there is such a model where $<$ is not a well-order.*

The connection between these two notions is in section VII of [She91].

[1] Thus, 'there is an M with cardinality κ such that $M \models \boldsymbol{t}$ and $\boldsymbol{t} \in \boldsymbol{N}_\lambda$' replaces the notation in [BS11], '$P^\lambda_N(\boldsymbol{K}_{\boldsymbol{t}}, \kappa)$ holds'.

Fact 0.4 *1. If $\kappa > 0$, $\mu(\lambda, \kappa) = \beth_{\delta(\lambda,\kappa)}$.*

2. For every infinite cardinal θ, $H(L_{\theta^+,\omega}) \leq \mu(\theta, 1) < \beth_{(2^\theta)^+}$.

Proof. Item 1 is VII.5.4 of [She91]. Recall that Lopez-Escobar and Chang (e.g. [Cha68]) showed how to code sentences of $L_{\lambda^+,\omega}$ as first order theories omitting types. More strongly (as in proof of VII.5.1.4) one can code by omitting a single type. $H(L_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+}$ is now clear from Theorems VII.5.4 and VII.5.5.7 of [She91].

1 Computing $H(N_\lambda^{ss})$

1.1 Introduction

We study the following notions in this section.

Definition 1.1 *Let $\boldsymbol{N}_\lambda^{ss}$ denote the set*[2] *of $\boldsymbol{t}$ with $|\tau_1| = \lambda$ with the additional requirement that $T_{\boldsymbol{t}}$ is a superstable theory. Now we have the natural notion of the Hanf number, $H(\boldsymbol{N}_\lambda^{ss})$ for this set: If $\boldsymbol{t} \in \boldsymbol{N}_\lambda^{ss}$ has a model of cardinality $\geq H(\boldsymbol{N}_\lambda^{ss})$, it has arbitrarily large models.*

We will prove the following theorem:

Theorem 1.2

$$H(L_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+} < H(\boldsymbol{N}_\lambda^{ss}) < H(L_{(2^\lambda)^+,\omega}) < \beth_{(2^{(2^\lambda)^+})^+}.$$

The first and fourth of these inequalities are immediate from Fact 0.4.2 taking θ first as λ and then as 2^λ.

In Subsection 1.2, we give a rather involved proof that $\beth_{(2^\lambda)^+}$ is strictly less than $H(\boldsymbol{N}_\lambda^{ss})$; together with the first inequality, this implies immediately that $H(L_{\lambda^+,\omega}) < H(\boldsymbol{N}_\lambda^{ss})$. Note that less than or equal, $H(L_{\lambda^+,\omega}) \leq H(\boldsymbol{N}_\lambda^{ss})$ is straightforward. Just set $\boldsymbol{t}$ as (T_0, T_1, p) where T_0 is pure equality and (T_1, p) encode a given sentence $\psi \in L_{\lambda^+,\omega}$. Then T_0 is superstable and every model is saturated, so we have the desired interpretation.

The second and third inequalities are in Subsections 1.2 and 1.3, respectively.

1.2 The Second Inequality

As noted the first inequality in Theorem 1.2.1 is standard. Thus by showing in Theorem 1.4, the second inequality appearing in Theorem 1.2 we will have $H(L_{\lambda^+,\omega}) < H(\boldsymbol{N}_\lambda^{ss})$ and in fact

Theorem 1.3 $H(L_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+} < H(\boldsymbol{N}_\lambda^{ss})$.

[2] Technically, this is not a set since a vocabulary is a sequence of relation symbols and we could use different names for the symbols; this pedantry can be avoided in at least two ways: restrict the symbols to come from a specified set; return to Tarski's convention of discussing not vocabularies but similarity types, the equivalence classes of enumerated vocabularies such that the ith symbol has aritity n_i.

The proof of the second inequality requires the construction of two triples $\boldsymbol{t}_1, \boldsymbol{t}_2$. The first characterizes 2^λ; the second $\beth_{(2^\lambda)^+}$.

Fix $\mathbf{L}_\lambda$ as the set of constructible sets of hereditary cardinality less than λ. In fact, any transitive model of a very weak set theory of cardinality λ would suffice.

Theorem 1.4 *There is a $\boldsymbol{t}_1 = (T, T_1, p) \in \boldsymbol{N}_\lambda^{ss}$ with $|\tau(T_1)| \leq \lambda$ such that:*

1. *There is an $M \models \boldsymbol{t}_2$ with cardinality $\beth_{(2^\lambda)^+}$*
2. *but there is no $M \models \boldsymbol{t}_2$ with cardinality greater than $\beth_{(2^\lambda)^+}$.*

Proof. We first introduce $\boldsymbol{t}_1 \in \boldsymbol{N}_\lambda^{ss}$ and prove several properties of it.

In the first stage, we first define T to be the prototypic superstable theory with λ-independent unary predicates, $P_{1,t}$ (i.e. in the vocabulary τ). The set P_2 will have cardinality λ when the type p is omitted. E_3 will be an extensional relation on $P_2 \times P_3$ so that P_3 has cardinality at most 2^λ. The function F maps the universe into P_3 while respecting the $P_{1,t}$ (and $\neg P_{1,t}$) and so that for any d, $F(d)$ codes via E_3 the τ-type of d. Thus saturation with respect to the $P_{1,t}$ guarantees that P_3 has cardinality exactly 2^λ.

Now we begin the formal development: τ contains unary predicates $P_{1,t}$ for $t \in \mathbf{L}_\lambda$; let T assert any Boolean combination of the $P_{1,t}$ is consistent. Let $\tau_1 = \tau \cup \{c_t : t \in \mathbf{L}_\lambda\} \cup \{P_2, P_3, E_2, E_3, F, F_1\}$. Here the P_i and E_i are unary and binary relations while F is a unary function and F_1 is a binary function. The various names of the symbols are chosen to keep the names different and not for any evocative purpose.

In the following definition, clauses 1) through 4) set the scene; clauses 5) through 7) are the crux of proving Lemmas 1.7 and 1.8; the other clauses are preparation for the proof of Lemma 1.11.

Definition 1.5 *Let T_1 be the τ_1-theory such that for any τ_1-structure M, $M \models T_1$ iff:*

1. *$M \restriction \tau \models T$;*
2. *$\langle c_t^M : t \in \mathbf{L}_\lambda \rangle$ are pairwise distinct elements of M.*
3. *$c_t^M \in P_2^M$ for $t \in \mathbf{L}_\lambda$.*
4. *$E_2^M \subseteq P_2^M \times P_2^M$ and (P_2^M, E_2^M) is a model of $Th(\mathbf{L}_\lambda, \epsilon)$;*
5. *F^M is a function from M onto P_3^M such that $M \models (\forall x)[P_{1,t}(x) \leftrightarrow P_{1,t}(F(x))]$ for every $t \in \mathbf{L}_\lambda$.*
6. *(extensionality) $E_3^M \subseteq P_2^M \times P_3^M$ satisfies*

$$(\forall x_1)(\forall x_2)[P_3(x_1) \wedge P_3(x_2) \rightarrow (\exists y)[P_2(y) \wedge (yE_3x_1 \leftrightarrow \neg yE_3x_2)]).$$

   So, letting $A^2_{M,b}$ denote $\{a \in P_2^M : aE_3^M b\}$ we know that $b_1 \neq b_2$ implies $A^2_{M,b_1} \neq A^2_{M,b_2}$.

7. *For every* $d \in M$*:*
$$d \in P_{1,t}^M \leftrightarrow c_t^M \in A^2_{M,F^M(d)}.$$
That is, $c_t^M E_3 F^M(d)$.

8. *If* $t_1, t_2 \in \mathbf{L}_\lambda$*, then* $M \models c_{t_1} E_2 c_{t_2}$ *if and only if* $\mathbf{L}_\lambda \models t_1 \in t_2$*.*

9. *For every* $d \in P_3^M$*,* $\langle F_1(e,d) : e \in M\rangle$ *is a* 1-1 *function from* M *into* $\{f \in M : F^M(f) = d\}$*.*

The required τ_1*-type* p *to complete the definition of* $\boldsymbol{t}$ *is*

$$p(x) = \{P_2(x)\} \cup \{x \neq c_t : t \in \mathbf{L}_\lambda\}.$$

Definition 1.6 *We call a model* M *of* T_1 standard *when*

1. $c_t^M = t$ *for* $t \in \mathbf{L}_\lambda$*.*
2. $P_2^M = \{t : t \in \mathbf{L}_\lambda\}$
3. *For every* $X \subseteq \mathbf{L}_\lambda$*, there is* $b \in P_3^M$ *such that* $A^2_{M,b} = X$*. Note* b *is unique by Definition 1.5.6.*

Lemma 1.7 *If* M *is a standard model of* T_1 *then* M *omits* p *and* $M \restriction \tau$ *is saturated. That is,* $M \models \boldsymbol{t}$*.*

Proof. Condition 2) asserts p is omitted. A saturated model M of T is one where for each $X \subseteq \mathbf{L}_\lambda$, $q_X(x) = \bigwedge_{t \in \mathbf{L}_\lambda} P_{1,t}(x)^{t \in X}$ is realized $|M|$ times. Clauses 1) and 3) of Definition 1.6 guarantee there is a $b_X \in P_3^M$ such that $X = A^2_{M,b_X}$. By clause 7) of Definition 1.5 any element of $(F^M)^{-1}(b_X)$ realizes q_X. Finally, Condition 8) of Definition 1.5 implies $|(F^M)^{-1}(b_X)| = |M|$. $\square_{1.7}$

Lemma 1.8 *If* $M \models \boldsymbol{t}_1$ *(*$\boldsymbol{t}_1 = (T, T_1, p)$*) then* M *is (isomorphic to) a standard model of* T_1*.*

Proof. Since M omits p, $P_2^M = \{c_t^M : t \in \mathbf{L}\}$. The map $g : P_2^M \to \mathbf{L}_\lambda$ is a well-defined isomorphism from (P_2^M, E_2^M) by condition 9 of Definition 1.5.

Finally, condition 3) of Definition 1.6 holds because the saturation provides a realization d_X of $q_X(x) = \bigwedge_{t \in \mathbf{L}_\lambda} P_{1,t}(x)^{t \in X}$. But then by condition 7) of Definition 1.5, $A^2_{M,F^M(d_X)} = \{c_t^M \in P_2^M : c_t^M E_3^M F^M(d_X)\} = X$ as required.

$\square_{1.8}$

Now we introduce a second triple, $\boldsymbol{t}_2 = (T, T_2, p) \in \boldsymbol{N}_\lambda^{ss}$, which will have models up to but no larger than $\beth_{(2^\lambda)^+}$.

We add a new predicate P_4 which will be linearly (indeed well) ordered by $<_4$ (say as $\{a_\alpha : \alpha < \beta\}$). The well-ordering is obtained by first requiring that every non-empty definable subset of P_4 has a least element and then showing every countable subset is definable (using a set theoretic structure imposed on P_2). A function G_4 projects the universe onto P_4. The predicate R will code the subsets of $(G_4)^{-1}(a_\alpha)$ by elements of $(G_4)^{-1}(a_{\alpha+1})$. An induction then bounds the cardinality of any model of $\boldsymbol{t}_2$.

Definition 1.9 *τ_2 expands τ_1 by adding $<_4, P_4, R, G_4, G_5$. G_4 unary, G_5 binary.*
Let T_2 be the τ_2-theory such that for any τ_1-structure M, $M \models T_2$ iff:

1. *$M \restriction \tau_1 \models T_1$.*
2. *$<_4$ is a linear order of P_4^M satisfying the first order theory of well-orderings.*
3. *G_4 is a function from M onto P_4^M.*
4. *If $cR^M d$ then $G_4(c) <_4^M G_4(d)$.*
5. *If $d_1 \neq d_2$ then for some $d \in M$, $dR^M d_1 \equiv \neg dR^M d_2$.*
6. *G_5^M is a partial function from $P_4^M \times P_4^M$ to P_3^M. If $d \in P_M^4$ then and $d_1 <_4^M d_2 <_4^M d$ then $G_5(d_1, d) \neq G_5(d_2, d)$.*

   So every proper initial segment of P_4^M has cardinality $\leq |P_3^M|$.
7. *For any $\phi(x, \mathbf{y}) \in L(\tau_2)$ and $\mathbf{d}$ in M with the same length as $\mathbf{y}$, $\{a \in P_4^M : M \models \phi(a, \mathbf{d})\}$ is either empty or has a first element.*

Observe that $\boldsymbol{t}_2 = (T, T_2, p) \in \boldsymbol{N}_\lambda^{ss}$.

Lemma 1.10 *There is an $M \models \boldsymbol{t}_2$ of cardinality $\beth_{(2^\lambda)^+}$.*

Proof. We define a τ_2-model as follows.

1. The universe of M is $V_{(2^\lambda)^+}$ where V_α is the α'th stage in the cumulative hierarchy.
2. $c_t^M = t$ for $t \in \mathbf{L}_\lambda$.
3. $P_2^M = \mathbf{L}_\lambda = \{c_t^M : t \in \mathbf{L}_\lambda\}$.
4. $E_2^M = \epsilon \restriction P_2^M$.
5. $P_M^3 = \mathcal{P}(\mathbf{L}_\lambda)$.
6. $E_3^M = \{(t, s) : t \in \mathbf{L}_\lambda, s \in \mathcal{P}(\mathbf{L}_\lambda), t \in s\}$.
7. Let $\langle Y_s : s \in \mathcal{P}(\lambda)\rangle$ be a partition of $|M| = V_{(2^\lambda)^+}$ such that each Y_s has cardinality $||M||$ and $s \in Y_s$ (This implies $Y_s \cap \mathcal{P}(\mathbf{L}_\lambda) = \{s\}$).
8. $P_{1,t}^M = \bigcup\{Y_s : s \in \mathcal{P}(\mathbf{L}_\lambda) \wedge t \in s\}$.
9. F^M maps M to $\mathcal{P}(\mathbf{L}_\lambda) = P_M^3$ by for $d \in Y_s$, $F(d) = s$.
10. Choose $F_1^M : M \times M \to M$ as 1-1 as function that maps $M \times Y_s$ into Y_s.
11. $P_4^M = (2^\lambda)^+$.
12. Let $<_4^M$ be the natural order $\epsilon \restriction (2^\lambda)^+$ on P_4^M.

13. G_5^M is any binary function from P_4^M into P_3^M such that if $d_1 <_4^M d_2 <_4^M d$ then $G_5(d_1, d) \neq G_5(d_2, d)$.

14. G_4^M maps M to P_4^M by $G_4^M(a)$ is the least α such that $a \in V_{\alpha+1}$.

15. $R^M = \epsilon \restriction V_{(2^\lambda)^+}$.

We have defined M to satisfy T_2; it omits p by clause 2) and 3). And conditions 5) and 6) show $M\restriction\tau_1$ is a standard model of T_1. So by Lemma 1.7, $M\restriction\tau$ is saturated and $M \models \boldsymbol{t}_1$.

$\square_{1.10}$

Now we show $\boldsymbol{t}_2$ has no model of cardinality greater than $\beth_{(2^\lambda)^+}$.

Lemma 1.11 *If $M \models \boldsymbol{t}_2$, $|M| \leq \beth_{(2^\lambda)^+}$.*

Proof. Since $M \models \boldsymbol{t}_2$, $M \models \boldsymbol{t}_1$ so by Lemma 1.8, without loss of generality, $M\restriction\tau_1$ is standard. The proof of Theorem 1.11 is easy from the next two claims.

Claim 1.12 *If $d_n <_M^4 d_{n-1} <_M^4 d$ for $n < \omega$, there is $\phi(x, \mathbf{y}) \in L(\tau_2)$ and $\boldsymbol{a} \in M$ with same length as $\mathbf{y}$ such that $\{b : M \models \phi(b, \boldsymbol{a})\} = \{d_n : n < \omega\}$.*

Proof. Let $D = \{d_n : n < \omega\}$ and writing $G_5^M(d, d_n)$ as b_n, let $B = \{b_n : n < \omega\}$. Our goal is to show that D is τ_2-definable. Letting $g(x)$ denote the function $G_5^M(d, x)$, $D = g^{-1}(B)$ and g is τ_2-definable. So it suffices to show B is definable.

Let X_n denote $A^2_{M,b_n} = \{a \in M : P_2^M(a) \wedge aE_2^M b_n\}$ and set $X = \bigcup_{n<\omega}\{n\} \times X_n$. Now X_n is a τ_2-definable subset of P_2^M, so X is definable in (P_2^M, E_2^M) using the set theoretic operations. And $b \in B$ if and only $(n, b)) \in X$ so B is τ_2-definable. $\square_{1.12}$

Claim 1.13 *$(P_4^M, <_4^M)$ is well-ordered of order type at most $(2^\lambda)^+$.*

Proof. By Lemma 1.12 the range of any infinite descending sequence is τ_2-definable. But then by clause 7 of Definition 1.9, it has a least element.

Since $M\restriction\tau_1$ is standard, $|P_3^M| = 2^\lambda$. Then condition 6) of Definition 1.9 implies the order type of $(P_4^M, <_4^M)$ is at most $(2^\lambda)^+$. So we can write $(P_4^M, <_4^M)$ as $\langle a_\alpha : \alpha < \beta\rangle$ for some $\beta \leq (2^\lambda)^+$. $\square_{1.13}$.

To complete the proof, we can show by induction that $|\{a \in M : G_4^M(a) < a_\alpha\}| \leq \beth_\alpha(\lambda)$. Condition 4) of Definition 1.9 shows that for any $d \in M$ with $G_4(d) = a_\alpha$, if bRd, then $G_4^M(b) < a_\alpha$. So with respect to R, d codes a subset of $(G_4^M)^{-1}(a_\alpha)$. Since R is extensional by Condition 4) of Definition 1.9, the $|(G_4^M)^{-1}(a_\alpha| \leq \beth_{\alpha+1}(\lambda)$ and we finish by induction. $\square_{HN}$

Theorem 1.4 is immediate from Lemmas 1.10 and 1.11. $\square_{1.4}$.

The Hanf number $L_{\lambda^+,\omega}$ can consistently be less than $\beth_{(2^\lambda)^+}$. See [SVPV05] and chapter VII.5 of [She91].

1.3 The third inequality

The previous section completed the proof of the second inequality in Theorem 1.2; we pass to the third.

Lemma 1.14 $H(\boldsymbol{N}^{ss}_\lambda) < H(L_{(2^\lambda)^+,\omega})$.

We first show $H(\boldsymbol{N}^{ss}_\lambda) \leq H(L_{(2^\lambda)^+,\omega})$ by constructing a map from $\boldsymbol{t} \in \boldsymbol{N}^{ss}_\lambda$ to $\psi_{\boldsymbol{t}} \in L_{(2^\lambda)^+,\omega}$; this construction depends heavily on the superstability hypothesis. Then we use some observations on Hanf numbers to show the inequality is strict: $H(\boldsymbol{N}^{ss}_\lambda) < H(L_{(2^\lambda)^+,\omega})$.

Lemma 1.15 *For each $\boldsymbol{t} = (T, T_1, p) \in \boldsymbol{N}^{ss}_\lambda$, there is a τ_2 extending τ_1 with $|\tau_1| = |\tau_2| = \lambda$ and a $\psi \in L_{(2^\lambda)^+,\omega}$ such that* $\mathrm{spec}(\boldsymbol{t}) = \mathrm{spec}(\psi)$.

Proof. In preparation consider a fixed saturated model M of cardinality $2^{|T|}$ of T.

To form τ_2, we add to τ_1 constants $\langle c_\alpha : \alpha < (2^\lambda)\rangle$ and as described below a unary predicate P and $2n+1$-ary functions H_n and function symbols $G_{n,m}$ indexing maps from N to N^m by $n+m$ tuples. The type p will be $\{P(x)\} \cup \{x \neq c_\alpha : \alpha < 2^\lambda\}$.

Notation 1.16

1. *We write $\mathbf{y}_1 \equiv_{\mathbf{x}_1,\mathbf{x}_2} \mathbf{y}_2$ to mean for every $\phi(\mathbf{v},\mathbf{w})$, $\phi(\mathbf{x}_1,\mathbf{y}_1) \leftrightarrow \phi(\mathbf{x}_2,\mathbf{y}_2)$.*
2. *$\mathbb{F}(\mathbf{x})$ is the collection (for $i < 2^\lambda$) of m-ary finite equivalence relations $E_i(\mathbf{x};\mathbf{y},\mathbf{z})$ over $\mathbf{x}$.*

We need these notions below.

Definition 1.17

1. *Recall that a model N is $F^a_{\kappa(T)}$-saturated (also called a-saturated and ϵ-saturated) if each strong type over a set of size less than $\kappa(T)$ is realized. For superstable theories $F^a_{\kappa(T)}$-saturated is just $F^a_{\aleph_0}$-saturated (each strong type over a finite set is realized).*
2. *A model N is strongly ω-homogeneous, if any two finite sequences that realize the same type over the empty set are automorphic in N.*

Fact 1.18 (III.3.10.2) *If a model M of a stable theory is $F^a_{\kappa(T)}$-saturated and for each set of infinite indiscernibles $\boldsymbol{I}$ in M there is an equivalent set of indiscernibles $\boldsymbol{I}'$ in M that has cardinality $|M|$, then M is saturated.*

Notation 1.19 *Now let $\psi_{\boldsymbol{t}} \in L_{(2^\lambda)^+,\omega}(\tau_1)$ assert of a model N:*

1. *The specified $p = p_{\boldsymbol{t}}$ is omitted.*
2. *P^M satisfies the complete $\tau \cup \{c_\alpha : \alpha < 2^\lambda\}$-diagram of M, the saturated model of cardinality $2^{|T|}$ specified at the beginning of the proof.*
3. *$N \restriction \tau$ is strongly ω-homogeneous. (Add $2n+1$-ary functions H_n satisfying if $\boldsymbol{a} \equiv \mathbf{b}$, $(\lambda z)H_n(\boldsymbol{a},\mathbf{b},z)$ is a τ-automorphism taking $\boldsymbol{a}$ to $\mathbf{b}$. This is expressible since having the same type over the empty set is expressible in $L_{(2^\lambda)^+,\omega}(\tau)$.)*

4. *For each $n < \omega, m < \omega$ there is an $(n+m+1)$-ary function G (into m-tuples) such that G witnesses that for any n-tuple $\boldsymbol{a}$ and m-tuple $\mathbf{b}$, if $\mathrm{stp}(\mathbf{b}/\boldsymbol{a})$ is realized infinitely often then it is realized $|N|$-times. Formally, N satisfies:*

$$(\forall \mathbf{x}\mathbf{z})[\bigwedge_{n<\omega}((\exists^{\geq n}\mathbf{y}) \bigwedge_{E_i(\mathbf{x};\mathbf{y},\mathbf{z})\in \mathbb{F}(\mathbf{x})} E_i(\mathbf{x},\mathbf{z},\mathbf{y})) \to (\forall w) \bigwedge_{E_i(\mathbf{x};\mathbf{y},\mathbf{z})\in \mathbb{F}(\mathbf{x})} E_i(\mathbf{x},\mathbf{z},G(\mathbf{x},\mathbf{z},w))]$$

where for every $\mathbf{x},\mathbf{z}$ $\lambda w G(\mathbf{x},\mathbf{z},w)$ is a 1-1 map from N into N^m.

Proof of $H(\boldsymbol{N}^{ss}_\lambda) \leq H(L_{(2^\lambda)^+,\omega})$: Suppose $\boldsymbol{t} \in \boldsymbol{N}^{ss}_\lambda$, $\psi_{\boldsymbol{t}}$ is constructed to satisfy Notation 1.19, and $N \models \psi_{\boldsymbol{t}}$. Since $|N| = |N{\restriction}\tau_1|$, it suffices to show $N{\restriction}\tau_1 \models \boldsymbol{t}$. Clearly N omits $p_{\boldsymbol{t}}$ and $(N{\restriction}P^N){\restriction}\tau$ is superstable; in particular it is an elementary extension of the $F^a_{\aleph_0}$-saturated model M. We must show $N{\restriction}\tau$ is saturated. But N is strongly ω-homogeneous by Notation 1.19.2. So each consistent strong type p over an n-element sequence $\boldsymbol{a} \in N$ is realized by $H_n^{-1}(\boldsymbol{a},\mathbf{b},\mathbf{c})$ where $\mathbf{b} \in M$ satisfies $\boldsymbol{a} \equiv \mathbf{b}$ and $\mathbf{c} \models H_n(\boldsymbol{a},\mathbf{b},q)$ (where the H_n transforms a strong type over $\boldsymbol{a}$ to one over $\mathbf{b}$ in the natural manner). Thus, N is $F^a_{\aleph_0}$-saturated so we may apply Fact 1.18. Every infinite indiscernible set J in $N{\restriction}\tau$ is based on a finite $\mathbf{d}$. That is, there is a strong type p_J over $\mathbf{d}$ such that J contains infinitely many realizations of p. Now the conditions on G of Notation 1.19.4 guarantee that p_J is realized $|N|$ times in N as required. $\square_{1.15}$

Now we strengthen the inequality $H(\boldsymbol{N}^{ss}_\lambda) \leq H(L_{(2^\lambda)^+,\omega})$ to a strict one.

Claim 1.20 $H(\boldsymbol{N}^{ss}_\lambda) < H(L_{(2^\lambda)^+,\omega})$.

Proof. VII.5.4 and VII.5.5.1 of [She91] shows for any μ, $\mathrm{cf}(H(L_{\mu^+,\omega})) \geq \mu^+$; in particular, $\mathrm{cf}(H(L_{(2^\lambda)^+,\omega}) \geq (2^\lambda)^+$. But there are at most 2^λ-classes in $\boldsymbol{N}^{ss}_\lambda$ and Lemma 1.15 implies that the supremum of the spec of each is less than $H(L_{(2^\lambda)^+,\omega})$. Thus, $H(\boldsymbol{N}^{ss}_\lambda) < H(L_{(2^\lambda)^+,\omega})$. $\square_{1.20}$

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