CHAIN CONDITIONS IN DEPENDENT GROUPS

ITAY KAPLAN AND SAHARON SHELAH

Abstract. In this note we prove and disprove some chain conditions in type definable and definable groups in dependent, strongly dependent and strongly^2 dependent theories.

1. Introduction

This note is about chain conditions in dependent, strongly dependent and strongly^2 dependent theories.

Throughout, all formulas will be first order, T will denote a complete first order theory, and $\mathfrak{C}$ will be the monster model of T — a very big saturated model that contains all small models. We do not differentiate between finite tuples and singletons unless we state it explicitly.

Definition 1.1. A formula $\varphi(x, y)$ has the independence property in some model if for every $n < \omega$ there are $\langle a_i, b_s \,|i < n, s \subseteq n \rangle$ such that $\varphi(a_i, b_s)$ holds iff $i \in s$.

A (first order) theory T is dependent (sometimes also NIP) if it does not have the independence property: there is no formula $\varphi(x, y)$ that has the independence property in any model of T. A model M is dependent if $Th(M)$ is.

A good introduction to dependent theories appears in [Adl08], but we shall give an exact reference to any fact we use, so no prior knowledge is assumed.

What do we mean by a chain condition? Rather than giving an exact definition, we give an example of such a condition — the first one. It is the Baldwin-Saxl Lemma, which we shall present with the (very easy and short) proof.

Definition 1.2. Suppose $\varphi(x, y)$ is a formula. Then if G is a definable group in some model, and for all $c \in C$, $\varphi(x, c)$ defines a subgroup, then $\{\varphi(\mathfrak{C}, c) \,|\, c \in C\}$ is a family of *uniformly definable subgroups*.

Lemma 1.3. [BS76] *Let* G *be a group definable in a dependent theory. Suppose* $\varphi(x, y)$ *is a formula and that* $\{\varphi(x, c) \,|\, c \in C\}$ *defines a family of subgroups of* G*. Then there is a number* $n < \omega$ *such that any finite intersection of groups from this family is already an intersection of* n *of them.*

The first author's research was partially supported by the SFB 878 grant.

The second author would like to thank the Israel Science Foundation for partial support of this research (Grants no. 710/07 and 1053/11). No. 993 on the author's list of publications.

Proof. Suppose not, then for every $n < \omega$ there are $c_0, \ldots, c_{n-1} \in C$ and $g_0, \ldots, g_{n-1} \in G$ (in some model) such that $\varphi(g_i, c_j)$ holds iff $i \neq j$. For $s \subseteq n$, let $g_s = \prod_{i \in s} g_i$ (the order does not matter), then $\varphi(g_s, c_j)$ iff $j \notin s$ — this is a contradiction. □

In stable theories (which we shall not define here), the Baldwin-Saxl lemma is even stronger: every intersection of such a family is really a finite one (see [Poi01, Proposition 1.4]).

The focus of this note is type definable groups in dependent theories, where such a proof does not work.

Definition 1.4. A *type definable group* for a theory T is a type — a collection $\Sigma(x)$ of formulas (maybe over parameters), and a formula $\nu(x, y, z)$, such that in the monster model $\mathfrak{C}$ of T, $\langle \Sigma(\mathfrak{C}), \nu \rangle$ is a group with ν defining the group operation (without loss of generality, $T \models \forall xy \exists^{\leq 1} z (\nu(x, y, z))$). We shall denote this operation by $\cdot$.

In stable theories, their analysis becomes easier as each type definable group is an intersection of definable ones (see [Poi01]).

Remark 1.5. In this note we assume that G is a finitary type definable group, i.e. x above is a finite tuple.

Definition 1.6. Suppose $G \geq H$ are two type definable groups (H is a subgroup of G). We say that the index $[G : H]$ is *unbounded*, or ∞, if for any cardinality κ, there exists a model $M \models T$, such that $\left[G^M : H^M\right] \geq \kappa$. Equivalently (by the Erdős-Rado coloring theorem), this means that there exists (in $\mathfrak{C}$) a sequence of indiscernibles $\langle a_i \,|\, i < \omega \rangle$ (over the parameters defining G and H) such that $a_i \in G$ for all i, and $i < j \Rightarrow a_i \cdot a_j^{-1} \notin H$. In $\mathfrak{C}$, this means that $\left[G^{\mathfrak{C}} : H^{\mathfrak{C}}\right] = |\mathfrak{C}|$. When G and H are definable, then by compactness this is equivalent to the index $[G : H]$ being infinite.

So $[G : H]$ is *bounded* if it is not unbounded.

This leads to the following definition

Definition 1.7. Let G be a type definable group.

(1) For a set A, G^{00}_A is the minimal A-type definable subgroup of G of bounded index.
(2) We say that G^{00} exists if $G^{00}_A = G^{00}_\emptyset$ for all A.

Shelah proved:

Theorem 1.8. [She08] If G is a type definable group in a dependent theory, then G^{00} exists.

Even though fields are not the main concern of this note, the following question is in the basis of its motivation. Recall

Theorem 1.9. [Lan02, Theorem VI.6.4] *(Artin-Schreier) Let* k *be a field of characteristic* p*. Let* ρ *be the polynomial* $X^p - X$*.*

(1) *Given* $a \in k$*, either the polynomial* $\rho - a$ *has a root in* k*, in which case all its root are in* k*, or it is irreducible. In the latter case, if* α *is a root then* $k(\alpha)$ *is cyclic of degree* p *over* k*.*
(2) *Conversely, let* K *be a cyclic extension of* k *of degree* p*. Then there exists* $\alpha \in K$ *such that* $K = k(\alpha)$ *and for some* $a \in k$*,* $\rho(\alpha) = a$*.*

Such extensions are called Artin-Schreier extensions.

The first author, in a joint paper with Thomas Scanlon and Frank Wagner, proved:

Theorem 1.10. [KSW11] *Let* K *be an infinite dependent field of characteristic* $p > 0$*. Then* K *is Artin-Schreier closed — i.e.* ρ *is onto.*

What about the type definable case? What if K is an infinite type definable field?

In simple theories (which we shall not define), we have:

Theorem 1.11. [KSW11] *Let* K *be a type definable field in a simple theory. Then* K *has boundedly many AS extensions.*

But for the dependent case we only proved:

Theorem 1.12. [KSW11] *For an infinite type definable field* K *in a dependent theory there are either unboundedly many Artin-Schreier extensions, or none.*

From these two we conclude:

Corollary 1.13. *If* T *is stable (so it is both simple and dependent), then type definable fields are AS closed.*

The following, then, is still open:

Question 1.14. *What about the dependent case? In other words, is it true that infinite type definable fields in dependent theories are AS-closed?*

Observing the proof of Theorem 1.10, we see that it is enough to find a number n, and $n+1$ algebraically independent elements, $\langle a_i \mid i \leq n \rangle$ in $k := K^{p^\infty}$, such that $\bigcap_{i<n} a_i \rho(K) = \bigcap_{i \leq n} a_i \rho(K)$. So the Baldwin-Saxl applies in the case where the field K is definable. If K is type definable, we may want something similar. But what can we prove?

A conjecture of Frank Wagner is the main motivation question

Conjecture 1.15. *Suppose* T *is dependent, then the following holds*

☺ *Suppose* G *is a type definable group. Suppose* $p(x,y)$ *is a type and* $\langle a_i \mid i<\omega \rangle$ *is an indiscernible sequence such that* $G_i = p(x,a_i) \leq G$*. Then there is some* n*, such that for all finite sets,* $v \subseteq \omega$*, the intersection* $\bigcap_{i\in v} G_i$ *is equal to a sub-intersection of size* n*.*

Let us refer to ☺ as *Property A* (of a theory T) for the rest of the paper. So we have

Fact 1.16. *If Property A is true for a theory* T*, then type definable fields are Artin-Schreier closed.*

In Section 2, we deal with strongly^2 dependent theories (this is a much stronger condition than merely dependence), and among other things, prove that Property A is true for them.

In Section 3, we give some generalizations and variants of Baldwin-Saxl for type definable groups in dependent and strongly dependent theories (which we define below). One of them is joint work with Frank Wagner. We prove that Property A holds for theories with bounded dp-rank.

In Section 4, we provide a counterexample that shows that property A does not hold in stable theories, so Conjecture 1.15 as it is stated is false.

Question 1.17. *Does Property A hold for strongly dependent theories?*

We would like to thank the referee for his careful reading.

2. Strongly^2 dependent theories

Notation 2.1. We call an array of elements (or tuples) $\langle a_{i,j} \mid i,j<\omega \rangle$ an *indiscernible array* over A if for $i_0<\omega$, the i_0-row $\langle a_{i_0,j} \mid j<\omega \rangle$ is indiscernible over the rest of the sequence ($\{a_{i,j} \mid i\neq i_0, i,j<\omega\}$) and A, i.e. when the rows are mutually indiscernible.

Definition 2.2. A theory T is said to be not *strongly^2 dependent* if there exists a sequence of formulas $\langle \varphi_i(x,y_i,z_i) \mid i<\omega \rangle$, an array $\langle a_{i,j} \mid i,j<\omega \rangle$ and $b_k \in \{a_{i,j} \mid i<k, j<\omega\}$ such that

- The array $\langle a_{i,j} \mid i,j<\omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x,a_{i,0},b_i) \wedge \neg\varphi_i(x,a_{i,1},b_i) \mid i<\omega\}$ is consistent.

So T is *strongly^2 dependent* when this configuration does not exist.

Note that the roles of i and j are not symmetric.

(In the definition above, x, z_i, y_i can be tuples, the length of z_i and y_i may depend on i).

This definition was introduced and discussed in [She12] and [She09].

Remark 2.3. By [She12, Claim 2.8], we may assume in the definition above that x is a singleton.

Fact 2.4. [She12, Claim 2.9] *An equivalent definition is* T *is not strongly^2 dependent if there exists an array* $\langle a_{i,j} \mid i,j<\omega \rangle$*, a set* A *and some finite tuple* c *such that*

- *The array* $\langle a_{i,j} \mid i,j<\omega \rangle$ *is an indiscernible array over* A*.*
- *For* $i_0<\omega$*, the row* $\bar{a}_{i_0} := \langle a_{i_0,j} \mid j<\omega \rangle$ *is not indiscernible over* $\bigcup_{i<i_0} \bar{a}_i \cup c$*.*

Proposition 2.5. *Suppose* T *is strongly*[2] *dependent, then it is impossible to have a sequence of type definable groups* $\langle G_i \mid i < \omega \rangle$ *such that* $G_{i+1} \leq G_i$ *and* $[G_i : G_{i+1}] = \infty$ *(see Definition 1.6).*

Proof. Without loss of generality, we shall assume that all groups are type definable over $\emptyset$. Suppose there is such a sequence $\langle G_i \mid i < \omega \rangle$. Let $\langle a_{i,j} \mid i, j < \omega \rangle$ be an indiscernible array such that for each $i < \omega$, the sequence $\langle a_{i,j} \mid j < \omega \rangle$ is a sequence from G_i (in $\mathfrak{C}$) such that $a_{i,j'}^{-1} \cdot a_{i,j} \notin G_{i+1}$ for all $j < j' < \omega$. We can find such an array because of our assumption and Ramsey (for more details, see the proof of Corollary 2.9 below).

For each $i < \omega$, let $\psi_i(x)$ be in the type defining G_{i+1} such that $\neg\psi_i\left(a_{i,j'}^{-1} \cdot a_{i,j}\right)$. By compactness, there is a formula $\xi_i(x)$ in the type defining G_{i+1} such that for all $a, b \in \mathfrak{C}$, if $\xi_i(a) \wedge \xi_i(b)$ then $\psi_i\left(a \cdot b^{-1}\right)$ holds. Let $\varphi_i(x, y, z) = \xi_i\left(y^{-1} \cdot z^{-1} \cdot x\right)$. For $i < \omega$, let $b_i = a_{0,0} \cdot \ldots \cdot a_{i-1,0}$ (so $b_0 = 1$).

Let us check that the set $\{\varphi_i(x, a_{i,0}, b_i) \wedge \neg\varphi_i(x, a_{i,1}, b_i) \mid i < \omega\}$ is consistent. Let $i_0 < \omega$, and let $c = b_{i_0}$. Then for $i < i_0$, $\varphi_i(c, a_{i,0}, b_i)$ holds iff $\xi_i(a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0})$ holds, but the product $a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0}$ is an element of G_{i+1} and ξ_i is in the type defining G_{i+1}, so $\varphi_i(c, a_{i,0}, b_i)$ holds. Now, $\varphi_i(c, a_{i,1}, b_i)$ holds iff $\xi_i\left(a_{i,1}^{-1} a_{i,0} \cdot \ldots \cdot a_{i_0-1,0}\right)$ holds. So if $\varphi_i(c, a_{i,1}, b_i)$ holds then, since $\xi_i(a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0})$ holds, by the choice of ξ_i we get

$$\psi_i\left(\left[a_{i,1}^{-1} a_{i,0} \cdot \ldots \cdot a_{i_0-1,0}\right] \cdot \left[a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0}\right]^{-1}\right),$$

i.e. $\psi_i\left(a_{i,1}^{-1} \cdot a_{i,0}\right)$ holds — a contradiction. □

Remark 2.6. It is well known (see [Poi01]) that in superstable theories the same proposition hold.

The next corollary already appeared in [She12, Claim 0.1] with definable groups instead of type definable (with proof already in [She09, Claim 3.10]).

Corollary 2.7. *Assume* T *is strongly*[2] *dependent. If* G *is a type definable group and* h *is a definable homomorphism* $h : G \to G$ *with finite kernel then* h *is almost onto* G*, i.e., the index* $[G : h(G)]$ *is bounded (i.e.* $< \infty$*). If* G *is definable, then the index must be finite.*

Proof. Consider the sequence of groups $\left\langle h^{(i)}(G) \mid i < \omega \right\rangle$ (i.e. G, $h(G)$, $h(h(G))$, etc.). By Proposition 2.5, for some $i < \omega$, $\left[h^{(i)}(G) : h^{(i+1)}(G)\right] < \infty$. Now the Corollary easily follows from:

Claim. If G is a group, $h : G \to G$ a homomorphism with finite kernel, then $[G : h(G)] + \aleph_0 = [h(G) : h(h(G))] + \aleph_0$.

Proof. (of claim) Let $H = h(G)$. Easily, one has $[H : h(H)] \leq [G : H]$.

We may assume that $[G : H]$ is infinite. Let $\ker(h) = \{g_0, \ldots, g_{k-1}\}$. Suppose that $[G : H] = \kappa$ but $[H : h(H)] < \kappa$. So let $\{a_i \mid i < \kappa\} \subseteq G$ are such that $a_i^{-1} \cdot a_j \notin H$ for $i \neq j$. So there must be some coset $a \cdot h(H)$ in H such that for infinitely many $i < \kappa$, $h(a_i) \in a \cdot h(H)$. Let us enumerate them as $\langle a_i \mid i < \omega \rangle$. So for $i < j < \omega$, let $C(a_i, a_j)$ be the least number $l < k$ such that there is some $y \in h(G)$ with $y^{-1} a_i^{-1} a_j = g_l$. By Ramsey, we may assume that $C(a_i, a_j)$ is constant. Now pick $i_1 < i_2 < j < \omega$. So we have $y^{-1} a_{i_1}^{-1} a_j = (y')^{-1} a_{i_2}^{-1} a_j$ for some $y, y' \in H$, so $y^{-1} a_{i_1}^{-1} = (y')^{-1} a_{i_2}^{-1}$ and hence $a_{i_1}^{-1} a_{i_2} = y (y')^{-1} \in H$ — a contradiction. □

Corollary 2.8. *If* K *is a strongly*^{2} *dependent field, (or even a type definable field in a strongly*^{2} *dependent theory) then for all* $n < \omega$, $[K^\times : (K^\times)^n] < \infty$.

Corollary 2.9. *Let* G *be a type definable group in a strongly*^{2} *dependent theory* T.

(1) Given a family of uniformly type definable subgroups $\{p(x, a_i) \mid i < \omega\}$ such that $\langle a_i \mid i < \omega \rangle$ is an indiscernible sequence, there is some $n < \omega$ such that $\bigcap_{j<\omega} p(\mathfrak{C}, a_j) = \bigcap_{j<n} p(\mathfrak{C}, a_j)$. In particular, T has Property A.

(2) Given a family of uniformly definable subgroups $\{\varphi(x, c) \mid c \in C\}$, the intersection

$$\bigcap_{c \in C} \varphi(\mathfrak{C}, c)$$

is already a finite one.

Proof. (1) Assume without loss of generality that G is defined over $\emptyset$. Let $G_i = p(\mathfrak{C}, a_i)$, and let $H_i = \bigcap_{j<i} G_i$. By Proposition 2.5, for some $i_0 < \omega$, $[H_{i_0} : H_{i_0+1}] < \infty$. For $r \geq i_0$, let $H_{i_0,r} = \bigcap_{j<i_0} G_j \cap G_r$ (so $H_{i_0+1} = H_{i_0,i_0}$). By indiscerniblity, $[H_{i_0} : H_{i_0,r}] < \infty$. This means (by definition of $H_{i_0}^{00}$) that $H_{i_0}^{00} \leq H_{i_0,r}$ for all $r > i_0$. However, if $H_{i_0,i_0} \neq H_{i_0,r}$ for some $i_0 < r < \omega$, then by indiscerniblity $H_{i_0,r} \neq H_{i_0,r'}$ for all $i_0 \leq r < r'$, and by compactness and indiscerniblity we may increase the length ω of the sequence to any cardinality κ, so that the size of $H_{i_0}/H_{i_0}^{00}$ is unbounded — a contradiction. This means that $H_{i_0+1} \subseteq G_r$ for all $r > i_0$, and so $\bigcap_{i<\omega} G_i = \bigcap_{i<i_0+1} G_i$.

(2) Assume not. Then we can find a sequence $\langle c_i \mid i < \omega \rangle$ of elements of C such that

$$\bigcap_{j<i} \varphi(\mathfrak{C}, c_j) \neq \bigcap_{j<i+1} \varphi(\mathfrak{C}, c_j).$$

By Ramsey and compactness (see e.g. [TZ12, Lemma 5.1.3]), there is an indiscernible sequence $\langle a_i \mid i < \omega \rangle$ such that for any n, and any formula $\psi(x_0, \ldots, x_{n-1})$, if $\psi(a_0, \ldots, a_{n-1})$ holds then there are $i_0 < \ldots < i_{n-1}$ such that $\psi(c_{i_0}, \ldots, c_{i_{n-1}})$ holds. In particular, $\varphi(\mathfrak{C}, a_i)$ defines a subgroup of G and $\bigcap_{j<i} \varphi(\mathfrak{C}, a_j) \neq \bigcap_{j<i+1} \varphi(\mathfrak{C}, a_j)$. But this contradicts (1). □

As further applications, we show that some theories are not strongly^2 dependent.

Example 2.10. Suppose $\langle G, +, < \rangle$ is an ordered abelian group. Then its theory $Th(G, +, 0, <)$ is not strongly^2 dependent.

Proof. We work in the monster model $\mathfrak{C}$. Let $G_d = \{x \in \mathfrak{C} \mid \forall n < \omega\,(n \mid x)\}$, so it is a type definable divisible ordered subgroup of G. Note that since G is ordered, it is torsion free, so G_d is a $\mathbb{Q}$-vector space. We shall define a descending sequence of infinite type definable groups $G_d^i \leq G_d$ for $i < \omega$ such that $\left[G_d^i : G_d^{i+1}\right] = \infty$, which contradicts Proposition 2.5. Let $G_d^0 = G_d$, and suppose we have chosen G_d^i. Let $a_i \in G_d^i$ be positive. Let $G_d^{i+1} = G_d^i \cap \bigcap_{n<\omega} (-a_i/n, a_i/n)$. This is a type definable subgroup of G_d^i. The sequence $\langle k \cdot a_i \mid k < \omega \rangle$ satisfies $(k-l) \cdot a_i \notin (-a_i/2, a_i/2)$ for any $k \neq l$, and by Ramsey (as in the proof of Corollary 2.9 (2)) we get $\left[G_d^i : G_d^{i+1}\right] = \infty$. □

Example 2.11. The theory $Th(\mathbb{R}, +, \cdot, 0, 1)$ is strongly dependent (it is even o-minimal, so dp-minimal — see Definitions 3.8 and 3.5 below). However it is not strongly^2 dependent.

Example 2.12. The theory $Th(\mathbb{Q}_p, +, \cdot, 0, 1)$ of the p-adics is strongly dependent (it is also dp-minimal), but not strongly^2 dependent: The valuation group $(\mathbb{Z}, +, 0, <)$ is interpretable.

Adding some structure to an algebraically closed field, we can easily get a strongly^2 dependent theory which is not stable.

Example 2.13. Let $L = L_{rings} \cup \{P, <\}$ where L_{rings} is the language of rings $\{+, \cdot, 0, 1\}$, P is a unary predicate and $<$ is a binary relation symbol. Let K be $\mathbb{C}$ (so it is an algebraically closed field), and let $P \subseteq K$ be a countable set of algebraically independent elements, enumerated as $\{a_i \mid i \in \mathbb{Q}\}$. Let $M = \langle K, P, < \rangle$ where $a <^M b$ iff $a, b \in P$ and $a = a_i$, $b = a_j$ where $i < j$. Let $T = Th(M)$.

Claim 2.14. T is strongly^2 dependent.

Proof. Note that T is axiomatizable by saying that the universe is an algebraically closed field, P is a subset of algebraically independent elements and $<$ is a dense linear order on P (to see this, take two saturated models of the same size and show that they are isomorphic).

Let us fix some terminology:

- When we write acl, we mean the algebraic closure in the field sense. When we say basis, we mean a transcendental basis.
- When we say that a set is independent / dependent over A for some set A, we mean that it is dependent / independent in the pregeometry induced by $\mathrm{cl}(X) = \mathrm{acl}(AX)$.
- $\mathrm{dcl}(X)$ stands for the definable closure of X.

We work in a saturated model $\mathfrak{C}$ of T.

Suppose X is some set. Let X_0 be some basis for X over P, and let $\mathrm{dcl}^P(X)$ be the set of $p \in P$ such that there exists some minimal finite $P_0 \subseteq P$ with $p \in P_0$ and some $x \in X$ such that $x \in \mathrm{acl}(P_0 X_0)$. Note that this set is contained in $\mathrm{dcl}(X)$ (since P is linearly ordered) and that it does not depend on the choice of X_0.

For a set (or a tuple) A, let $A^P = \mathrm{dcl}^P(A)$.

Subclaim. Suppose $M_1 = (K_1, P_1, <_1)$ and $M_2 = (K_2, P_2, <_2)$ are two saturated models of T and $A \subseteq K_1$ is a small set. Suppose that $K_1 = K_2$ and $\left(A^{P_1}, <^{P_1}\right) = \left(A^{P_2}, <^{P_2}\right)$. Then there is an isomorphism $f : M_1 \to M_2$ fixing $A \cup A^{P_1}$.

Proof. Let $\tau : P_1 \to P_2$ be any isomorphism fixing A^{P_1}. Since both $P_1 \backslash A^{P_1}$ and $P_2 \backslash A^{P_1}$ are algebraically independent over A, $\tau \cup (\mathrm{id} \restriction A)$ is an elementary map in the field language. This map can be extended to an automorphism f of K_1, which is the desired isomorphism. $\square$

Let $\mathrm{tp}_K(a/A)$ be the type of $a \frown (Aa)^P$ (considered as a tuple, ordered by $<^{\mathfrak{C}}$) over $A \cup A^P$ in the field language, and $\mathrm{tp}_P(a/A)$ the type of the tuple $(Aa)^P$ over A^P in the order language.

Subclaim. For finite tuples a, b and a set A, $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ iff $\mathrm{tp}_P(a/A) = \mathrm{tp}_P(b/A)$ and $\mathrm{tp}_K(a/A) = \mathrm{tp}_K(b/A)$.

Proof. Denote by K the field structure of $\mathfrak{C}$. There is an automorphism σ of K that maps $a \frown (Aa)^P$ to $b \frown (Ab)^P$ and fixes $A \cup A^P$ pointwise. Since $\mathrm{tp}_P(a/A) = \mathrm{tp}_P(b/A)$, the restriction $\sigma \restriction A^P \cup (Aa)^P$ is order preserving. Let $\mathfrak{C}' = (K, \sigma(P), \sigma(<))$. By the first subclaim, there is an isomorphism $\tau : \mathfrak{C}' \to \mathfrak{C}$ fixing $Ab \cup (Ab)^P$. Now, $\tau \circ \sigma$ is an automorphism of $\mathfrak{C}$ that takes a to b and fixes A. $\square$

Suppose that $\langle a_{i,j} \mid i, j < \omega \rangle$ is an indiscernible array over a parameter set A as in Definition 2.2 and that c is a singleton such that:

- The sequence $I_0 := \langle a_{0,j} \mid j < \omega \rangle$ is not indiscernible over Ac, and moreover $\mathrm{tp}(a_{0,0}/Ac) \neq \mathrm{tp}(a_{0,1}/Ac)$.
- For $i > 0$, the sequence $I_i := \langle a_{i,j} \mid j < \omega \rangle$ is not indiscernible over $c \cup \bigcup_{k<i} I_k \cup A$.

Suppose that $c \notin \mathrm{acl}(APa_{0,0}a_{0,1})$. Then, by the second subclaim, $\mathrm{tp}(ca_{0,0}/A) = \mathrm{tp}(ca_{0,1}/A)$ — a contradiction. So $c \in \mathrm{acl}(APa_{0,0}a_{0,1})$. Increase the parameter set A by adding the first row $\langle a_{0,j} \mid j < \omega \rangle$. So we may assume that $c \in \mathrm{acl}(AP)$. Since $c \in \mathrm{acl}\left(A(Ac)^P\right)$, we may replace c by a finite tuple contained in $(Ac)^P$ and assume that c is a finite tuple of elements in P (here we use the fact that in general, if I is indiscernible over C then it is also indiscernible over $\mathrm{acl}(C)$).

Expand all the sequences to order type $\omega^* + \omega + \omega$. Let $B = \bigcup\{a_{i,j} \mid i < \omega, j < 0 \vee \omega \leq j\} \cup A$. For each $i < \omega$ and $0 \leq j < \omega$, let $a^P_{i,j}$ be $\mathrm{dcl}^P(a_{i,j}B)$ considered as a tuple ordered by $<^{\mathfrak{C}}$, and let $B^P = \mathrm{dcl}^P(B)$. Then $\left\langle a^P_{i,j} \mid 0 \leq i, j < \omega \right\rangle$ is an indiscernible array over B^P and $\left\langle a_{i,j} \frown a^P_{i,j} \mid 0 \leq i, j < \omega \right\rangle$ is an indiscernible array over $B \cup B^P$.

As both the theories of dense linear orders and algebraically closed fields are strongly^2 dependent (this is easy to check), by Fact 2.4 there is some i_0 such that $\left\langle a^P_{i_0,j} \mid 0 \leq j < \omega \right\rangle$ is indiscernible over $cB^P \cup \left\{a^P_{i,j} \mid i < i_0, 0 \leq j < \omega\right\}$ in the order language and $\left\langle a_{i_0,j} \frown a^P_{i_0,j} \mid 0 \leq j < \omega \right\rangle$ is indiscernible over $cB \cup B^P \cup \left\{a_{i,j} \frown a^P_{i,j} \mid i < i_0, 0 \leq j < \omega\right\}$ in the field language.

Let $C = \bigcup\{a_{i,j} \mid i < i_0, 0 \leq j < \omega\}$. We must check that $\langle a_{i_0,j} \mid 0 \leq j < \omega\rangle$ is indiscernible over BCc. Let us show, for instance, that $\mathrm{tp}(a_{i_0,0}/BCc) = \mathrm{tp}(a_{i_0,1}/BCc)$. For this we apply the second subclaim. For each $0 \leq i, j < \omega$, let $a'_{i,j}$ be a basis for $a_{i,j}$ over BP. Then, by indiscernibility, $\{a'_{i,j} \mid i < i_0, 0 \leq j < \omega\}$ is a basis for C over BP (this is why we expanded the sequences). Now it follows that $\mathrm{dcl}^P(BCc) = \bigcup\{a^P_{i,j} \mid i < i_0, 0 \leq j < \omega\} \cup B^P \cup c$. Similarly, for $j \geq 0$, $\mathrm{dcl}^P(a_{i_0,j}BCc) = a^P_{i_0,j} \cup \mathrm{dcl}^P(BC) \cup c$. By the second subclaim above, we are done. □

Remark 2.15. With the same proof, one can show that if T is strongly minimal, and $P = \{a_i \mid i < \omega\}$ is an infinite indiscernible set in $M \models T$ of cardinality $\aleph_1$, the theory of the structure $\langle M, P, <\rangle$ where $<$ is some dense linear order with no end points on P, is strongly^2 dependent.

We finish this section with the following conjecture:

Conjecture 2.16. *All strongly^2 dependent groups are stable, i.e. if G is a group such that $\mathrm{Th}(G, \cdot)$ is strongly^2 dependent, then it is stable.*

Example 2.10 and Corollary 2.9 show that this might be reasonable. This is related to the conjecture of Shelah in [She12] that all strongly^2 dependent infinite fields are algebraically closed.

3. Baldwin-Saxl type lemmas

The next lemma is the type definable version of the Baldwin-Saxl Lemma (see Lemma 1.3). But first,

Notation 3.1. If $p(x, y)$ is a partial type, then $|p|$ is the size of the set of formulas $\varphi(x, z_1, \ldots, z_n)$ (where z_i is a singleton) such that for some finite tuple $y_1, \ldots, y_n \in y$, $\varphi(x, y_1, \ldots, y_n) \in p$. In this sense, the size of any partial type over $\emptyset$ is bounded by $|T|$.

Lemma 3.2. *Suppose G is a type definable group in a dependent theory T.*

1. *If $p_i(x, y_i)$ is a type for $i < \kappa$ (y_i may be an infinite tuple), $|\bigcup p_i| < \kappa$, and $\langle c_i \mid i < \kappa\rangle$ is a sequence of tuples such that $p_i(\mathfrak{C}, c_i)$ is a subgroup of G, then for some $i_0 < \kappa$, $\bigcap_{i<\kappa} p_i(\mathfrak{C}, c_i) = \bigcap_{i<\kappa, i\neq i_0} p_i(\mathfrak{C}, c_i)$.*
2. *In particular, given a family of uniformly type definable subgroups, defined by $p(x, y)$, and C of size $|p|^+$, there is some $c_0 \in C$ such that $\bigcap_{c\neq c_0} p(\mathfrak{C}, c) = \bigcap_{c\in C} p(\mathfrak{C}, c)$.*
3. *In particular, if $\{G_i \mid i < |T|^+\}$ is a family of type definable subgroups (defined with parameters), then there is some $i_0 < |T|^+$ such that $\bigcap G_i = \bigcap_{i\neq i_0} G_i$.*

Proof. (1) Without loss of generality $p_i(x, y_i)$ are closed under finite conjunctions. Let $H_i = p_i(\mathfrak{C}, c_i)$. Suppose not, i.e. for all $i < \kappa$, there is some g_i such that $g_i \in H_j$ iff $i \neq j$. If $d_1, d_2 \in H_i$ then $d_1 \cdot g_i \cdot d_2 \notin H_i$. Hence by compactness there is some formula $\varphi_i(x, c_i) \in p_i(x, c_i)$ such that for all such $d_1, d_2 \in H_i$, $\neg\varphi_i(d_1 g_i d_2, c_i)$ holds. Since $|\bigcup p_i| < \kappa$, we may assume that for $i < \omega$,

φ_i is constant and equals $\varphi(x,y)$. Now for any finite subset $s \subseteq \omega$, let $g_s = \prod_{i \in s} g_i$ (the order does not matter). So we have $\varphi(g_s, c_i)$ iff $i \notin s$ — a contradiction.

(2) and (3) now follow easily from (1). □

In (2) of Lemma 3.2, if C is an indiscernible sequence, then the situation is simpler:

Corollary 3.3. *Suppose* G *is a type definable group in a dependent theory* T. *Given a family of uniformly type definable subgroups, defined by* $p(x,y)$, *and an indiscernible sequence* $C = \langle a_i \mid i \in \mathbb{Z} \rangle$, $\bigcap_{i \neq 0} p(\mathfrak{C}, a_i) = \bigcap_{i \in \mathbb{Z}} p(\mathfrak{C}, a_i)$.

Proof. Assume not. By indiscernibility, we get that for all $i \in \mathbb{Z}$, $\bigcap_{j \neq i} p(\mathfrak{C}, a_j) \nsubseteq p(\mathfrak{C}, a_i)$. Let I be an indiscernible sequence which extends C to length $|p|^+$. Then by indiscernibility and compactness the same is true for this sequence. This contradicts Lemma 3.2. □

Remark 3.4. The above corollary is in the kernel of the proof that G^{00} exists in dependent theories.

If T is strongly dependent, and C is indiscernible, we can even assume that the order type is ω. Let us recall,

Definition 3.5. A theory T is said to be *not strongly dependent* if there exists a sequence of formulas $\langle \varphi_i(x, y_i) \mid i < \omega \rangle$ and an array $\langle a_{i,j} \mid i,j < \omega \rangle$ such that

- The array $\langle a_{i,j} \mid i,j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x, a_{i,0}) \wedge \neg\varphi_i(x, a_{i,1}) \mid i < \omega\}$ is consistent.

So T is *strongly dependent* when this configuration does not exist.

Remark 3.6. This definition is not exactly the original definition given in [She12, Definition 1.2], but it is equivalent to it by [She12, Definition 1.2]

Lemma 3.7. *Suppose* G *is a type definable group in a strongly dependent theory* T. *Given a family of type definable subgroups* $\{p_i(x, a_i) \mid i < \omega\}$ *such that* $\langle a_i \mid i < \omega \rangle$ *is an indiscernible sequence and* $p_{2i} = p_{2i+1}$ *for all* $i < \omega$, *there is some* $i < \omega$ *such that* $\bigcap_{j \neq i} p_j(\mathfrak{C}, a_j) = \bigcap_{j < \omega} p_j(\mathfrak{C}, a_j)$.

In particular, this is true when p *is constant.*

Proof. Without loss of generality $p_i(x, y_i)$ are closed under finite conjunctions. Let $H_i = p_i(\mathfrak{C}, a_i)$. Assume not, i.e. for all $i < \omega$, there exists some $g_i \in G$ such that $g_i \in H_j$ iff $i \neq j$. For each even $i < \omega$ we find a formula $\varphi_i(x,y) \in p_i(x,y)$ such that for all $d_1, d_2 \in H_i$, $\neg\varphi_i(d_1 g_i d_2, a_i)$. Let $n < \omega$, and consider the product $g_n = \prod_{i<n,\, 2|i} g_i$ (the order does not matter). Then for odd $i < n$, $\varphi_{i-1}(g_n, a_i)$ holds (because $\varphi_{i-1} \in p_{i-1} = p_i$ by assumption), and for even $i < n$, $\neg\varphi_i(g_n, a_i)$ holds. By compactness, we can find $g \in G$ such that $\varphi_{i-1}(g, a_i)$ holds for all odd $i < \omega$ and $\neg\varphi_i(g, a_i)$ for all even $i < \omega$. Now expand the sequence by adding a sequence $\langle b_{i,j} \mid j < \omega \rangle$ after each pair a_{2i}, a_{2i+1}. Then the array defined by $a_{i,0} = a_{2i}$, $a_{i,1} = a_{2i+1}$ and $a_{i,j} = b_{i,j-2}$ for $j \geq 2$ will show that the theory is not strongly dependent. □

If the theory is of bounded dp-rank, then we can say even more.

Definition 3.8. A theory T is said to have *bounded dp-rank*, if there is some $n < \omega$ such that the following configuration does not exist: a sequence of formulas $\langle \varphi_i(x, y_i) \mid i < n \rangle$ where x is a singleton and an array $\langle a_{i,j} \mid i < n, j < \omega \rangle$ such that

- The array $\langle a_{i,j} \mid i < n, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x, a_{i,0}) \wedge \neg\varphi_i(x, a_{i,1}) \mid i < n\}$ is consistent.

T is *dp-minimal* if $n = 2$.

Note that if T has bounded dp-rank, then it is strongly dependent.

Remark 3.9. All dp-minimal theories are of bounded dp-rank. This includes all o-minimal theories and the p-adics.

The name is justified by the following fact:

Fact 3.10. [UOK11] *If T has bounded dp-rank, then for any $m < \omega$, there is some $n_m < \omega$ such that a configuration as in Definition 3.8 with n_m replacing n is impossible for a tuple x of length m (in fact $n_m \leq m \cdot n_1$).*

Lemma 3.11. *Let G be type definable group in a bounded dp-rank theory T.*

Given a family of type definable subgroups $\{p_i(x, a_i) \mid i < \omega\}$ such that $\langle a_i \mid i < \omega \rangle$ is an indiscernible sequence and $p_{2i} = p_{2i+1}$ for all $i < \omega$, there is some $n < \omega$ and $i < n$ such that $\bigcap_{j \neq i, j < n} p_j(\mathfrak{C}, a_j) = \bigcap_{j<n} p_j(\mathfrak{C}, a_j)$.

In particular, if p_i is constant (say p) and $\langle a_i \mid i < \omega \rangle$ is an indiscernible set, then $\bigcap_{i<\omega} p(\mathfrak{C}, a_i) = \bigcap_{i<n} p(\mathfrak{C}, a_i)$.

In particular, T has Property A.

Proof. The proof is exactly the same as the proof of Lemma 3.7, but we only need to construct g_n for n large enough. ☐

Another similar proposition:

Proposition 3.12. *Assume T is strongly dependent, G a type definable group and $G_i \leq G$ are type definable normal subgroups for $i < \omega$. Then there is some i_0 such that $\left[\bigcap_{i \neq i_0} G_i : \bigcap_{i<\omega} G_i\right] < \infty$.*

Proof. Assume not. Then, for each $i < \omega$, we have an indiscernible sequence $\langle a_{i,j} \mid j < \omega \rangle$ (over the parameters defining all the groups) such that $a_{i,j} \in \bigcap_{k \neq i} G_k$ and for $j_1 < j_2 < \omega$, $a_{i,j_1}^{-1} \cdot a_{i,j_2} \notin G_i$. Note that if $d_1, d_2, d_3 \in G_i$, then $d_1 \cdot a_{i,j_1}^{-1} \cdot d_2 \cdot a_{i,j_2} \cdot d_3 \notin G_i$, since G_i is normal. By compactness there is a formula $\psi_i(x)$ in the type defining G_i such that for all $d_1, d_2, d_3 \in G_i$, $\neg\psi_i\left(d_1 \cdot a_{i,j_1}^{-1} \cdot d_2 \cdot a_{i,j_2} \cdot d_3\right)$ holds (by indiscernibility it is the same for all $j_1 < j_2$). We may assume, applying Ramsey, that the array $\langle a_{i,j} \mid i,j<\omega\rangle$ is indiscernible (i.e. the sequences are mutually indiscernible). Let $\varphi_i(x,y)=\psi_i(x^{-1}\cdot y)$.

Now we check that the set $\{\varphi_i(x,a_{i,0})\wedge\neg\varphi_i(x,a_{i,1}) \mid i<n\}$ is consistent for each $n<\omega$. Let $c=a_{0,0}\cdot\ldots\cdot a_{n-1,0}$ (the order does not really matter, but for the proof it is easier to fix one). So $\varphi_i(c,a_{i,0})$ holds iff $\psi_i\left(a_{n-1,0}^{-1}\cdot\ldots\cdot a_{i,0}^{-1}\cdot\ldots\cdot a_{0,0}^{-1}\cdot a_{i,0}\right)$ holds. But since G_i is normal, $a_{i,0}^{-1}\cdot\ldots\cdot a_{0,0}^{-1}\cdot a_{i,0}\in G_i$, so the entire product is in G_i, so $\varphi_i(c,a_{i,0})$ holds. On the other hand, $\psi_i\left(a_{n-1,0}^{-1}\cdot\ldots\cdot a_{i,0}^{-1}\cdot\ldots\cdot a_{0,0}^{-1}\cdot a_{i,1}\right)$ does not hold by the choice of ψ_i. □

The following Corollary is a weaker version of Corollary 2.8:

Corollary 3.13. *If* G *is an abelian definable group in a strongly dependent theory and* $S\subseteq\omega$ *is an infinite set of pairwise co-prime numbers, then for almost all (i.e. for all but finitely many)* $n\in S$, $[G:G^n]<\infty$. *In particular, if* K *is a definable field in a strongly dependent theory, then for almost all primes* p, $\left[K^\times:(K^\times)^p\right]<\infty$.

Proof. Let $K\subseteq S$ be the set of $n\in S$ such that $[G:G^n]<\infty$. If $S\backslash K$ is infinite, replace S with $S\backslash K$.

For $i\in S$, let $G_i=G^i$ (so it is definable). By Proposition 3.12, there is some $n\in S$ such that $\left[\bigcap_{i\neq n}G_i:\bigcap_{i\in S}G_i\right]<\infty$. If $[G:G_n]=\infty$, then there is an indiscernible sequence $\langle a_i \mid i<\omega\rangle$ of elements of G, such that $a_i^{-1}\cdot a_j\notin G_n$. Suppose $S_0\subseteq S\backslash\{n\}$ is a finite subset and let $r=\prod S_0$. Then $\langle a_i^r \mid i<\omega\rangle$ is an indiscernible sequence in $G^r\subseteq\bigcap_{i\in S_0}G_i$ such that $a_i^{-r}\cdot a_j^r\notin G_n$. So by compactness, we can find such a sequence in $\bigcap_{i\neq n}G_i$ — a contradiction. □

Remark 3.14. The above Proposition and Corollary can be generalized (with almost the same proofs) to the case where the theory is only *strong*. For the definition, see [Adl].

Remark 3.15. This Corollary generalizes in some sense [KP11, Proposition 2.1] (as they only assumed finite weight of the generic type). And so, as in [KP11, Corollary 2.2], we can conclude that if K is a field definable in a strongly stable theory (i.e. the theory is strongly dependent and stable), then $K^p=K$ for almost all primes p.

Problem 3.16. Is Proposition 3.12 is still true without the assumption that the groups are normal?

Note that in strongly dependent[2] theories, this assumption is not needed: Let $H_i=\bigcap_{j<i}G_i$. Then $[H_i:H_{i+1}]<\infty$ for all i big enough by Proposition 2.5. But this implies $\left[\bigcap_{j\neq i}G_j:\bigcap_j G_j\right]<\infty$.

κ-intersection.

This part is joint work with Frank Wagner.

Definition 3.17. For a cardinal κ and a family $\mathfrak{F}$ of subgroups of a group G, the κ-intersection $\bigcap_\kappa \mathfrak{F}$ is $\{g \in G \mid |\{F \in \mathfrak{F} \mid g \notin F\}| < \kappa\}$.

The following proposition shows that in some sense, the intersection of a family of uniformly type definable subgroups can be understood via its κ-intersection and a small intersection.

Proposition 3.18. *Let G be a type definable group in a dependent theory. Suppose*

- *$\mathfrak{F}$ is a family of uniformly type definable subgroups defined by $p(x, y)$.*

Then for any infinite regular cardinal $\kappa > |p|$ (in the sense of Notation 3.1), and any subfamily $\mathfrak{G} \subseteq \mathfrak{F}$, there is some $\mathfrak{G}' \subseteq \mathfrak{G}$ such that

$\star$ $|\mathfrak{G}'| < \kappa$ *and* $\bigcap \mathfrak{G}$ *is* $\bigcap \mathfrak{G}' \cap \bigcap_\kappa \mathfrak{G}$.

Remark 3.19. In the context of the proposition, this means that $\mathfrak{G}'$ has the property that for every subset $\mathfrak{G}'' \subseteq \mathfrak{G}$ such that $|\mathfrak{G} \backslash \mathfrak{G}''| < \kappa$, $\bigcap \mathfrak{G} = \bigcap \mathfrak{G}' \cap \bigcap \mathfrak{G}''$.

Proof. (of proposition) Let κ be such a cardinal. Assume that there is some family $\mathfrak{G} = \{H_i \mid i < \varkappa\}$, which is a counterexample of the proposition. For $g \in G$, let $J_g = \{i < \varkappa \mid g \in H_i\}$. So $g \in \bigcap_\kappa \mathfrak{G}$ iff $|\varkappa \backslash J_g| < \kappa$.

For $i < \kappa$ we define by induction $g_i \in \bigcap_\kappa \mathfrak{G}$, $I_i \subseteq \varkappa$, $R_i \subseteq \varkappa$ and $\alpha_i < \varkappa$ such that:

1. $R_0 = [0, \alpha_0)$ and for $0 < i$, $R_i = \bigcup_{j<i} R_j \cup \left[[\sup_{j<i} \alpha_j, \alpha_i) \cap \bigcap_{j<i} I_j \right]$ (so $R_i \subseteq \alpha_i$).
2. $\bigcap_{j \le i} J_{g_j} \subseteq R_i \cup I_i$ (so by the definition of $\bigcap_\kappa$, and by the regularity of κ,$|\varkappa \backslash (R_i \cup I_i)| < \kappa$).
3. $\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\alpha \in R_i} H_\alpha$.
4. $I_i \cap [0, \alpha_i] = \emptyset$.
5. I_i is $\subseteq$-decreasing.
6. α_i is $<$-increasing.
7. $I_i \subseteq J_{g_i}$.
8. For $j < i$, $g_i \in H_{\alpha_j}$, $g_j \in H_{\alpha_i}$ and $g_i \notin H_{\alpha_i}$.

Let $\alpha_0 < \varkappa$ be minimal such that there is some $g_0 \in \bigcap_\kappa \mathfrak{G} \backslash H_{\alpha_0}$ (it must exist, otherwise $\bigcap_\kappa \mathfrak{G} = \bigcap \mathfrak{G}$). Let $I_0 = \{j > \alpha_0 \mid g_{\alpha_0} \in H_j\}$.

For α_0, (2), (3), (4), (7) and (8) are true, by the definition of $\bigcap_\kappa$ and the choice of α_0.

Suppose we have chosen g_j, I_j and α_j (so R_j is already defined by (1)) for $j < i$.

Let $J = \bigcap_{j<i} I_j$. Choose $g_i \in \left(\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \right) \backslash H_{\alpha_i}$ where $\alpha_i \in J$ is the smallest possible such that this set is nonempty. Suppose for contradiction that we cannot find such α_i, then $\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\alpha \in J} H_\alpha$, so

$$\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \cap \bigcap_{j \in \varkappa \backslash J} H_j = \bigcap \mathfrak{G}.$$

Let $J' = J \cup \bigcup_{j<i} R_j$, then by (3), $\bigcap \mathfrak{G}$ equals

$$\bigcap_{\kappa} \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \cap \bigcap_{j \in \varkappa \setminus J'} H_j.$$

Note that $\bigcap_{j<i} (R_j \cup I_j) \subseteq J'$, so by the regularity of κ, and by (2), $|\varkappa \setminus J'| < \kappa$, so we get a contradiction.

Let $I_i = \{\alpha_i < j \in J \,|\, g_i \in H_j\}$, and let us check the conditions above.

Conditions (4) – (7) are easy.

Condition (2): By induction we have

$$\bigcap_{j \leq i} J_{g_j} = \bigcap_{j<i} J_{g_j} \cap J_{g_i} \subseteq J' \cap J_{g_i} \subseteq R_i \cup (J \cap J_{g_i}).$$

But by (4) and the definition of R_i, letting $\alpha = \sup_{j<i} \alpha_j$, we have

$$J \cap J_{g_i} \subseteq \left[[\alpha, \alpha_i) \cap \bigcap_{j<i} I_j \right] \cup I_i \subseteq R_i \cup I_i.$$

Condition (3) is true by the minimality of α_i: $\bigcap_{\kappa} \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\beta \in J \cap [\alpha, \alpha_i)} H_\beta$, so by the induction hypothesis, we are done.

Condition (8): We show that $g_j \in H_{\alpha_i}$ for $j < i$. We have that $\alpha_i \in J$ so also in I_j which, by (7), is a subset of J_{g_j}, so $g_j \in H_{\alpha_i}$.

Finally, we have that for each $i, j < \kappa$, $g_i \in H_{\alpha_j}$ iff $i \neq j$. But by Lemma 3.2, there is some $i_0 < |p|^+$ such that $\bigcap_{i \neq i_0} H_{\alpha_i} = \bigcap_{i < |p|^+} H_{\alpha_i}$ — a contradiction. □

Remark 3.20. So far we have not found applications for this proposition, but it seems like a very nice proposition in its own right, and it might turn out to be useful.

4. A counterexample

In this section we shall present an example that shows that Property A does not hold in general dependent (or even stable) theories.

Let $S = \{u \subseteq \omega \,|\, |u| < \omega\}$, and $V = \{f : S \to 2 \,|\, |\text{supp}(f)| < \infty\}$ where $\text{supp}(f) = \{x \in S \,|\, f(x) \neq 0\}$. This has a natural group structure as a vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

For $n, m < \omega$, define the following groups:

- $G_n = \{f \in V \,|\, u \in \text{supp}(f) \Rightarrow |u| = n\}$
- $G_\omega = \prod_n G_n$
- $G_{n,m} = \{f \in V \,|\, u \in \text{supp}(f) \Rightarrow |u| = n \,\&\, m \in u\}$ (so $G_{0,m} = 0$)
- $H_{n,m} = \{\eta \in G_\omega \,|\, \eta(n) \in G_{n,m}\}$

Now we construct the model:

Let L be the language (vocabulary) $\{P, Q\} \cup \{R_n \,|\, n < \omega\} \cup L_{AG}$ where L_{AG} is the language of abelian groups, $\{0, +\}$; P and Q are unary predicates; and R_n is binary. Let M be the following

L-structure: its universe is $G_\omega \coprod \omega$, $P^M = G_\omega$ (with the group structure), $Q^M = \omega$ and $R_n = \{(\eta, m) \mid \eta \in H_{n,m}\}$. Let $T = Th(M)$.

Let $p(x, y)$ be the type $\{R_n(x, y) \mid n < \omega\}$. Note that since $H_{n,m}$ is a subgroup of G_ω, for each $m < \omega$, $p(M, m)$ is a subgroup of G_ω (and this remains true in elementary extensions).

Claim 4.1. Let $N \models T$ be $\aleph_1$-saturated. For any $m < \omega$, and any distinct $\alpha_0, \ldots, \alpha_m \in Q^N$, $\bigcap_{i \leq m} p(N, \alpha_i)$ is different than any sub-intersection of size m.

Proof. We show that $\bigcap_{i\leq m} p(N, \alpha_i) \subsetneq \bigcap_{i<m} p(N, \alpha_i)$ (the general case is similar). More specifically, we show that

$$\bigcap_{i<m} p(N, \alpha_i) \setminus \bigcap_{i\leq m} R_m(N, \alpha_i) \neq \emptyset.$$

By saturation, it is enough to show that this is the case in M, so we assume $M = N$. Note that if $\eta \in \bigcap_{i\leq m} R_m(M, \alpha_i)$, then $\eta \in H_{m,\alpha_i}$ for all $i \leq m$. So for all $i \leq m$, $u \in \operatorname{supp}(\eta(m)) \Rightarrow |u| = m \,\&\, \alpha_i \in u$. This implies that $\operatorname{supp}(\eta(m)) = \emptyset$, i.e. $\eta(m) = 0$. But we can find $\eta \in \bigcap_{i<m} p(M, \alpha_i)$ such that $\eta(m) \neq 0$. For instance let $\eta(n) = 0$ for all $n \neq m$ while $|\operatorname{supp}(\eta(m))| = 1$ and $\eta(m)(\{\alpha_0, \ldots, \alpha_{m-1}\}) = 1$. □

Next we shall show that T is stable. For this we will use κ-resplendent models. This is a very useful (though not a very well known) tool for proving that theories are stable, and we take the opportunity to promote it.

Definition 4.2. Let κ be a cardinal. A model M is called κ-resplendent if whenever

- $M \prec N$; N' is an expansion of N by less than κ many symbols; $\bar{c}$ is a tuple of elements from M and $\lg(\bar{c}) < \kappa$

There exists an expansion M' of M to the language of N' such that $\langle M', \bar{c}\rangle \equiv \langle N', \bar{c}\rangle$.

The following remarks are not crucial for the rest of the proof.

Remark 4.3. [She]

(1) If κ is regular and $\kappa > |T|$, and $\lambda = \lambda^{<\kappa}$, then T has a κ-resplendent model of size λ.
(2) A κ-resplendent model is also κ-saturated.
(3) If M is κ-resplendent then M^{eq} is also such.

The following is a useful observation:

Claim 4.4. If M is κ-resplendent for some κ, and $A \subseteq M$ is definable and infinite, then $|A| = |M|$.

Proof. Enrich the language with a function symbol f. Let $T' = T \cup \{f : M \to A \text{ is injective}\}$. Then T' is consistent with an elementary extension of M (for example, take an extension N of M where $|A^N| = |M|$, and then take an elementary substructure $N' \prec N$ of size $|M|$ containing M and A^N). Hence we can expand M to a model of T'. □

The main fact is

Theorem 4.5. [She, Main Lemma 1.9] *Assume* κ *is regular and* $\lambda = \lambda^{\kappa} + 2^{|T|}$. *Then, if* T *is unstable then* T *has* $> \lambda$ *pairwise nonisomorphic* κ*-resplendent models of size* λ[1]. *On the other hand, if* T *is stable and* $\kappa \geq \kappa(T) + \aleph_1$ *then every* κ*-resplendent model is saturated.*

Proposition 4.6. T *is stable.*

Proof. We may restrict T to a finite sub-language, $L_n = \{P, Q, \} \cup \{R_i \,|\, i < n\} \cup L_{AG}$.

Our strategy is to prove that our theory has a unique model in size λ which is κ-resplendent where $\kappa = \aleph_0$, $\lambda = 2^{\aleph_0}$. Let N_0, N_1 be two κ-resplendent models of size λ.

By Claim 4.4, $|Q^{N_0}| = |Q^{N_1}| = \lambda$ and we may assume that $Q^{N_0} = Q^{N_1} = \lambda$.

Let $G_0 = P^{N_0}$ and $G_1 = P^{N_1}$ with the group structure. For $i < n$, $j < 2$ and $\alpha < \lambda$, let $H^j_{i,\alpha} = \left\{x \in G_j \,\middle|\, R_i^{N_j}(x, \alpha)\right\}$. This is a definable subgroup of G_j. For $k \leq n$, let $G^k_j = \bigcap_{\alpha<\lambda,\, i\neq k,\, i<n} H^j_{i,\alpha}$. In our original model M, this group is $\{\eta \in G_\omega \,|\, \forall i \neq k,\, i < n\,(\eta(i) = 0)\}$. Note that $G_j = \sum_{k<n} G^k_j$, and that $G^{k_0}_j \cap \sum_{k<n,k\neq k_0} G^k_j = G^n_j$ (this is true in our original model M, so it is part of the theory). We give each G^k_j the induced L-structure $N^k_j = \left\langle G^k_j, \lambda\right\rangle$, i.e. we interpret $R_i^{N^k_j} = R_i \cap \left(G^k_j \times \lambda\right)$.

Since these groups are definable and infinite, their cardinality is λ, and hence their dimension (over $\mathbb{F}_2$) is λ. In particular there is a group isomorphism $f_n : G^n_0 \to G^n_1$. Note that f_n is an isomorphism of the induced structure on $N^n_j = \left\langle G^n_j, \lambda\right\rangle$ (because it is trivial).

Subclaim. For $k < n$, there is an isomorphism $f_k : G^k_0 \to G^k_1$ which is an isomorphism of the induced structure $N^k_j = \left\langle G^k_j, \lambda\right\rangle$ and extends f_n.

Assuming this claim, we shall finish the proof. Define $f : G_0 \to G_1$ by: given $x \in G_0$, write it as a sum $\sum_{k<n} x_k$ where $x_k \in G^k_0$, and define $f(x) = \sum_{k<n} f_k(x_k)$. This is well defined because if $\sum_{k<n} x_k = \sum_{k<n} x'_k$ then $\sum_{k<n}(x_k - x'_k) = 0$ so for all $k < n$, $x_k - x'_k \in G^n_0$ and

$$\begin{aligned}\sum_{k<n}\left(f_k(x_k) - f_k(x'_k)\right) &= \sum_{k<n}\left(f_k(x_k - x'_k)\right) = \sum_{k<n}\left(f_n(x_k - x'_k)\right) = \\ &= f_n\left(\sum_{k<n} x_k - x'_k\right) = f_n(0) = 0.\end{aligned}$$

It follows similarly that f is a group isomorphism. Also, f is an L_n-isomorphism because if $R_i^{N_0}(a, \alpha)$ holds for some $i < n$, $\alpha < \lambda$ and $a \in G_0$, then write $a = \sum_{k<n} a_k$ where $a_k \in G^k_0$. Since $R_i^{N_0}(a, \alpha)$ holds and $R_i^{N_0}(a_k, \alpha)$ holds for all $k \neq i$, it follows that $R_i^{N_0}(a_i, \alpha)$ holds, so $R_i^{N_1}(f_k(a_k), \alpha)$ holds for all $k < n$, and so $R_i^{N_1}(f(a), \alpha)$ holds. The other direction is similar.

[1]In fact, by [She, Claim 3.1], if T is unstable there are 2^λ such models.

Proof. (of subclaim) For a finite set b of elements of λ, let $L_b^j = G_j^k \cap \bigcap_{\alpha \in b} H_{k,\alpha}^j$. For $m \leq k+1$, let $K_m^j = \sum_{|b|=m} L_b^j$ (as a subspace of G_j^k), so K_m^j is not necessarily definable (however $K_0^j = G_j^k$ and $K_{k+1}^j = G_j^n$ are). This is a decreasing sequence of subgroups (so subspaces), $G_j^k = K_0^j \geq \ldots \geq K_{k+1}^j = G_j^n$. Now it is enough to show that:

Subclaim. For $m \leq k+1$, there is an isomorphism $f_m : K_m^0 \to K_m^1$ which is an isomorphism of the induced structure $\left\langle K_m^j, \lambda \right\rangle$ which extends f_n.

Proof. (of subclaim) The proof is by reverse induction. For $m = k+1$ we already have this. Suppose we have f_{m+1} and we want to construct f_m. Let $b \subseteq \lambda$ be of size m. If $m = k$, then it is easy to see that $\left| L_b^j / \left(K_{m+1}^j \cap L_b^j \right) \right| = 2$ (this is true in M), so there is an isomorphism $g_b : L_b^0 / \left(K_{m+1}^0 \cap L_b^0 \right) \to L_b^1 / \left(K_{m+1}^1 \cap L_b^1 \right)$.

Assume $|b| < k$. In our original model M, $L_b \subseteq K_k$, but here one can find infinitely many pairwise distinct cosets in $L_b^j / \left(K_{m+1}^j \cap L_b^j \right)$. Indeed, we can find a type in λ infinitely many variables $\{x_i \mid i < \lambda\}$ over b saying that $x_i \in L_b$ and $x_i - x_j \notin K_{m+1}$ for $i \neq j$ — for all $r < \omega$, it will contain a formula of the form

$$\forall (z_0, \ldots, z_{r-1}) \forall_{t<r} (\bar{y}_t) \left(\left[\bigwedge_{t<r} (z_t \in L_{\bar{y}_t} \wedge |\bar{y}_t| = m+1) \right] \to x_i - x_j \neq \sum_{t=0}^{r-1} z_t \right).$$

To show that this type is consistent, we may assume that $b \subseteq Q^M$ so we work in our original model M. For such r and b, choose distinct $\eta_0, \ldots \eta_{l-1} \in G_\omega$ such that for $s, s' < l$

- $\eta_s(i) = 0$ for $i \neq k$.
- $|\mathrm{supp}(\eta_s(k))| = r+1$.
- $u_1 \in \mathrm{supp}(\eta_s(k))\ \&\ u_2 \in \mathrm{supp}(\eta_{s'}(k)) \Rightarrow u_1 \cap u_2 = b$ (s might be equal to s' but $u_1 \neq u_2$).

Then $\{\eta_s \mid s < l\}$ is such that η_{s_1}, η_{s_2} satisfies the formula above for all $s_1 \neq s_2 < l$: if not, there are $z_0 \in L_{c_0}, \ldots, z_{r-1} \in L_{c_r}$ where $|c_t| = m+1$ such that $\sum_{t<r} z_t = \eta_{s_1} - \eta_{s_2}$. We may assume that

$$\bigcup_{t<r} \mathrm{supp}(z_t(k)) = \mathrm{supp}(\eta_{s_1}(k) - \eta_{s_2}(k)) = \mathrm{supp}(\eta_{s_1}(k)) \cup \mathrm{supp}(\eta_{s_2}(k)),$$

but then for $t < r$, $|\mathrm{supp}(z_t(k))| \leq 1$ by our choice of η_s and this is a contradiction.

Now, let N_j' be an elementary extension of N_j with realizations $D = \{c_i \mid i < \lambda\}$ of this type, and we may assume $\left|N_j'\right| = \lambda$. Then, add a predicate for the set D, and an injective function from N_j' to D. Finally, by resplendence of N_j, $\left| L_b^j / \left(K_{m+1}^j \cap L_b^j \right) \right| = \lambda$.

Hence it has a basis of size λ, and let $g_b : L_b^0 / \left(K_{m+1}^0 \cap L_b^0 \right) \to L_b^1 / \left(K_{m+1}^1 \cap L_b^1 \right)$ be an isomorphism of $\mathbb{F}_2$-vector spaces.

Note that $f_{m+1} \restriction K_{m+1}^0 \cap L_b^0$ is onto $K_{m+1}^1 \cap L_b^1$ (this is because f_{m+1} is an isomorphism of the induced structure). We can write $L_b^j = \left(K_{m+1}^j \cap L_b^j \right) \oplus W^j$ where $W^j \cong L_b^j / \left(K_{m+1}^j \cap L_b^j \right)$, so g_b

induces an isomorphism from W^0 to W^1. Now extend $f_{m+1} \restriction K^0_{m+1} \cap L^0_b$ to $f^b_m : L^0_b \to L^1_b$ using g_b.

Next, note that $\left\{ L^j_b \,|\, b \subseteq \lambda, |b| = m \right\}$ is independent over K^j_{m+1}, i.e. for distinct $b_0, \ldots, b_r$, $L^j_{b_r} \cap \sum_{t<r} L^j_{b_t} \subseteq K^j_{m+1}$. Indeed, in our original model M, the intersection $L_{b_r} \cap \sum_{t<r} L_{b_t}$ is equal to $\sum_{t<r} L_{b_r \cup b_t}$, so this is true also in N_j (in fact, this is true for every choice of finite sets b_t — regardless of their size).

Define f_m as follows: given $a \in K^0_m$, we can write $a = \sum_{b \in B} a_b$ where $a_b \in L_b$ for a finite $B \subseteq \{b \subseteq \lambda \,|\, |b| = m\}$, and define $f_m(a) = \sum f^b_m(a_b)$. It is well defined: if $\sum_{b \in B} x_b = \sum_{b' \in B'} y_{b'}$, then for $b_1 \in B \cap B'$, $b_2 \in B \backslash B'$ and $b_3 \in B' \backslash B$, $(x_{b_1} - y_{b_1}), x_{b_2}, y_{b_3} \in K^0_{m+1}$, so

$$\sum_{b \in B} f^b_m(x_b) - \sum_{b' \in B'} f^{b'}_m(y_{b'}) =$$

$$\sum_{b \in B \cap B'} f_{m+1}(x_b - y_b) + \sum_{b \in B \backslash B'} f_{m+1}(x_b) - \sum_{b \in B' \backslash B} f_{m+1}(y_b) = 0.$$

It follows similarly that f_m is a group isomorphism.

We check that f_m is an isomorphism of the induced structure. So suppose $a \in K^0_m$, $\alpha < \lambda$ and $i < \omega$. If $i \neq k$, then since $K^j_m \subseteq G^k_j$ for $j < 2$, both $R^{N_0}_i(a, \alpha)$ and $R^{N_1}_i(f(a), \alpha)$ hold. Suppose $R^{N_0}_k(a, \alpha)$ holds. Write $a = \sum_{b \in B} a_b$ as above. Then, as $a \in L_{\{\alpha\}} \cap \sum_{b \in B} L_b = \sum_{b \in B} L_{b \cup \{\alpha\}}$, we may assume that $b \in B \Rightarrow \alpha \in b$. So by definition of f_m, $R^{N_1}_k(f_m(a), \alpha)$ holds. The other direction holds similarly and we are done. □

Note 4.7. This example is not strongly dependent, because the sequence of formulas $R_n(x, y)$ is a witness of that the theory is not strongly dependent. So as we said in the introduction, it is still open whether Property A holds for strongly dependent theories.

References

[Adl] Hans Adler. Strong theories, burden, and weight. Preprint.

[Adl08] Hans Adler. An introduction to theories without the independence property. *Archive for Mathematical Logic*, 2008. accepted.

[BS76] J. T. Baldwin and Jan Saxl. Logical stability in group theory. *J. Austral. Math. Soc. Ser. A*, 21(3):267–276, 1976.

[KP11] Krzysztof Krupiński and Anand Pillay. On stable fields and weight. *J. Inst. Math. Jussieu*, 10(2):349–358, 2011.

[KSW11] Itay Kaplan, Thomas Scanlon, and Frank O. Wagner. Artin-Schreier extensions in nip and simple fields. *Israel J. Math.*, 185:141–153, 2011.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[Poi01] Bruno Poizat. *Stable groups*, volume 87 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the 1987 French original by Moses Gabriel Klein.

[She] Saharon Shelah. On spectrum of κ-resplendent models. *preprint, 363 in Shelah archive*. arXiv:1105.3774.

[She08] Saharon Shelah. Minimal bounded index subgroup for dependent theories. *Proc. Amer. Math. Soc.*, 136(3):1087–1091 (electronic), 2008.

[She09] Saharon Shelah. Dependent first order theories, continued. *Israel J. Math.*, 173:1–60, 2009.

[She12] Saharon Shelah. Strongly dependent theories. *Israel Journal of Mathematics*, 2012. accepted.

[TZ12] Katrin Tent and Martin Ziegler. *A Course in Model Theory (Lecture Notes in Logic)*. Cambridge University Press, 2012.

[UOK11] Alexander Usvyatsov, Alf Onshuus, and Itay Kaplan. Additivity of the dp-rank. *Transactions of the AMS*, 2011. accepted, arXiv:1109.1601.