ALSO QUITE LARGE $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$ BEHAVE NICELY

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ABSTRACT. The present note is an answer to complains of E.Weitz on [Sh 371]. We present a corrected version of a part of chapter VIII of *Cardinal Arithmetic*.

Definition 1 ([Sh:g, VIII 3.1]). (1) $J_*[\mathfrak{a}] = \big\{\mathfrak{b} : \mathfrak{b} \subseteq \mathfrak{b}$ and for every inaccessible μ, we have $\mu > \sup(\mathfrak{b} \cap \mu)\big\}$.

(2) $\mathrm{pcf}_*(\mathfrak{a}) = \big\{\mathrm{tcf}(\prod \mathfrak{a}/D) : D$ is an ultrafilter on $\mathfrak{a}$, $D \cap J_*[\mathfrak{a}] \neq \emptyset\big\}$.

(3) If $|\mathfrak{a}| < \min(\mathfrak{a})$, for $\mu \in \mathrm{pcf}(\mathfrak{a})$ let $\mathfrak{b}^{\mathfrak{a}}_\mu = \mathfrak{b}_\mu[\mathfrak{a}]$ be a subset of $\mathfrak{a}$ such that $J_{\leq\mu}[\mathfrak{a}] = J_{<\mu}[\mathfrak{a}] + \mathfrak{b}_\mu[\mathfrak{a}]$.
(Note that $\mathfrak{b}^{\mathfrak{a}}_\mu$ exists by [Sh:g, VIII 2.6], also $\mathfrak{a}$ is a finite union of $\mathfrak{b}_\mu[\mathfrak{a}]$'s).

(4) If $|\mathfrak{a}| < \min(\mathfrak{a})$ let $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ be the ideal of subsets of $\mathrm{pcf}(\mathfrak{a})$ generated by $\{\mathrm{pcf}(\mathfrak{b}_\mu[\mathfrak{a}]) : \mu \in \lambda \cap \mathrm{pcf}(\mathfrak{a})\}$.
Let $J^{\mathrm{pcf}}_{\leq\lambda}[\mathfrak{a}] = J^{\mathrm{pcf}}_{<\lambda^+}[\mathfrak{a}]$.

Proposition 2 ([Sh:g, VIII 3.1A]). (1) *The ideal $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ depends on $\mathfrak{a}$ and λ only (and not on the choice of the $\mathfrak{b}_\mu[\mathfrak{a}]$'s).*

(2) *If $\mathfrak{b} \subseteq \mathfrak{a}$ then $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{b}] = \mathcal{P}(\mathfrak{b}) \cap J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ and $J_*[\mathfrak{b}] = \mathcal{P}(\mathfrak{b}) \cap J_*[\mathfrak{a}]$.*

Proof. (1) Let $\langle \mathfrak{b}'_\mu[\mathfrak{a}] : \mu \in \mathrm{pcf}(\mathfrak{a})\rangle$, $\langle \mathfrak{b}''_\mu[\mathfrak{a}] : \mu \in \mathrm{pcf}(\mathfrak{a})\rangle$ both be as in 1(3). So for each θ, $\mathfrak{b}'_\theta[\mathfrak{a}] \subseteq \mathfrak{b}''_\theta[\mathfrak{a}] \cup \bigcup_{\ell<n} \mathfrak{b}''_{\theta_\ell}[\mathfrak{a}]$ for some $n < \omega$, $\theta_0, \ldots, \theta_n - 1 < \theta$. Hence, if $\theta < \lambda$,

$$\mathrm{pcf}(\mathfrak{b}'_\theta[\mathfrak{a}]) \subseteq \mathrm{pcf}(\mathfrak{b}''_\theta[\mathfrak{a}]) \cup \bigcup_{\ell<n} \mathrm{pcf}(\mathfrak{b}''_{\theta_\ell}[\mathfrak{a}]),$$

and each is in $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ as defined by $\langle \mathfrak{b}''_\sigma[\mathfrak{a}] : \sigma \in \mathrm{pcf}(\mathfrak{a})\rangle$ (as $\theta_\ell < \theta < \lambda$). As this holds for every $\theta < \lambda$, all generators of $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ as defined by $\langle \mathfrak{b}'_\sigma[\mathfrak{a}] : \sigma \in \mathrm{pcf}(\mathfrak{a})\rangle$ are in $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ as defined by $\langle \mathfrak{b}''_\sigma[\mathfrak{a}] : \sigma \in \mathrm{pcf}(\mathfrak{a})\rangle$. As the situation is symmetric we finish.
(2) Similar proof. The first phrase follows from part (1), and check the second. $\square$

Lemma 3 ([Sh:g, VIII 3.2]). *Suppose $|\mathfrak{a}|^+ < \min(\mathfrak{a})$, $\mathfrak{a} \subseteq \mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, $\mathfrak{b} \notin J =: J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ and $\lambda = \max\mathrm{pcf}(\mathfrak{a})$.*
Then $\mathrm{tcf}(\prod \mathfrak{b}/J)$ is λ.

Proof. Remember that (by [Sh:g, VIII 2.6]) there is $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$, a generating sequence for $\mathfrak{a}$. Let for $\mu \in \mathrm{pcf}(\mathfrak{a})$, $\langle f^\mu_\alpha : \alpha < \mu\rangle$ exemplify $\mu = \mathrm{tcf}(\prod \mathfrak{b}_\mu[\mathfrak{a}], J_{<\mu}[\mathfrak{a}])$, $f^\mu_\alpha \in \prod \mathfrak{a}$; by [Sh 355, 3.1], without loss of generality

$(*)_0$ $\forall f \in \prod \mathfrak{a}[\bigvee_\alpha f \restriction \mathfrak{b}_\mu[\mathfrak{a}] \leq f^\mu_\alpha]$.

Without loss of generality for $\theta \in \mathfrak{a} : f^\theta_\alpha(\theta) = \alpha$ if $\alpha < \theta$, $f^\theta_\alpha(\theta^1) = 0$ if $\alpha < \theta < \theta^1 \in \mathfrak{a}$. We define $f^{\lambda,\mathfrak{b}}_\alpha \in \prod \mathfrak{b}$ by:

$$f^{\lambda,\mathfrak{b}}_\alpha \restriction \mathfrak{a} = f^\lambda_\alpha,$$

and for $\theta \in \mathfrak{b} \backslash \mathfrak{a}$:

$$f_\alpha^{\lambda,\mathfrak{b}}(\theta) = \min\left\{\beta : f_\alpha^\lambda \restriction \mathfrak{b}_\theta[\mathfrak{a}] \leq f_\beta^\theta \mod J_{<\theta}[\mathfrak{a}]\right\}.$$

Clearly

$(*)_1$ $f_\alpha^\lambda \leq f_\beta^\lambda \quad \Rightarrow \quad f_\alpha^{\lambda,\mathfrak{b}} \leq f_\beta^{\lambda,\mathfrak{b}}$.

Subfact 3.1 ([Sh:g, VIII 3.2A])**.**

$$\alpha < \beta < \lambda \quad \Rightarrow \quad f_\alpha^{\lambda,\mathfrak{b}} \leq f_\beta^{\lambda,\mathfrak{b}} \mod J.$$

Proof of the subfact. Let $\mathfrak{c} = \{\theta \in \mathfrak{a} : f_\alpha^\lambda(\theta) f_\beta^\lambda(\theta)\}$, so $\mathfrak{c} \in J_{<\lambda}[\mathfrak{a}]$ and hence for some $n < \omega$ and $\sigma_1 < \ldots < \sigma_n$ from $\lambda \cap \mathrm{pcf}(\mathfrak{a})$ (hence $< \lambda$), we have $\mathfrak{c} \subseteq \bigcup\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]$. So by the definition of the $f_\alpha^{\lambda,\mathfrak{b}}$'s we have:

$(*)_2$ if $\mu \in \mathfrak{b}$, and $\mathfrak{b}_\mu[\mathfrak{a}] \cap \bigcap\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}] \in J_{<\mu}[\mathfrak{a}]$, then $f_\alpha^{\lambda,\mathfrak{b}}(\mu) \leq f_\beta^{\lambda,\mathfrak{b}}(\mu)$.

However,

$(*)_3$ $\mathfrak{d} =: \left\{\mu \in \mathrm{pcf}(\mathfrak{a}) : \mathfrak{b}_\mu[\mathfrak{a}] \cap \bigcup\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}] \neq \emptyset \mod J_{<\mu}[\mathfrak{a}]\right\}$
(for our fixed $\sigma_1, \ldots, \sigma_n \in \lambda \cap \mathrm{pcf}(\mathfrak{a})$) belongs to J

[as $\mu \in \mathfrak{d}$ implies $\mu \in \bigcup\limits_{\ell=1}^{n} \mathrm{pcf}(\mathfrak{b}_{\sigma_\ell}[\mathfrak{a}])$ which is in J].

Together we get the subfact 3.1. □

Subfact 3.2 ([Sh:g, VIII 3.2B])**.** *For any $f \in \prod \mathfrak{b}$ for some α, $f \leq f_\alpha^{\lambda,\mathfrak{b}}$.*

Proof of the subfact. The family J_1 of sets $\mathfrak{c} \subseteq \mathfrak{b}$ for which this holds (i.e., for each $f \in \prod \mathfrak{c}$ there is $\alpha < \lambda$ such that $f \leq f_\alpha^{\lambda,\mathfrak{b}}$) satisfies:

(1) $\{\theta\} \in J_1$ for $\theta \in \lambda \cap \mathrm{pcf}(\mathfrak{a})$,
(2) J_1 is an ideal of subsets of $\mathrm{pcf}(\mathfrak{a})$,
(3) if $\mathfrak{c}_i$ (for $i < \kappa$) are in J_1, $\min(\mathfrak{c}_i) > \kappa^+$ for each i then $\bigcup\limits_{i<\kappa} \mathfrak{c}_i$ is in J_1.

We shall show their satisfaction below.

This suffices for 3.2 [as $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$; why? just prove that

$$\mathfrak{c} \subseteq \mathfrak{b} \ \&\ \mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})] \quad \Rightarrow \quad \mathfrak{c} \in J_1$$

by induction on $\sup\{\mu^+ : \mu \in \mathfrak{c}\}$. For successor use (1) + (2). For singular, let $\langle \mu_i : i < \kappa \rangle$ be such that μ_i is strictly increasing continuous with limit $\sup \mathfrak{c} = \sup\{\mu^+ : \mu \in \mathfrak{c}\}$, and $\kappa^+ < \mu_0$; by the induction hypothesis $\mathfrak{c} \cap \mu_0$, $\mathfrak{c} \cap [\mu_i, \mu_{i+1}]$ are in the ideal, by (3) we know that

$$\bigcup_{i<\kappa} (\mathfrak{c} \cap [\mu_i, \mu_{i+1})) = \mathfrak{c} \cap [\mu_0, \sup \mathfrak{c})$$

is in the ideal and by the induction hypothesis $\mathfrak{c} \cap \mu_0 \in J_1$ so by (2)

$$\mathfrak{c} = (\mathfrak{c} \cap \mu_0) \cup (\mathfrak{c} \cap [\mu_0, \sup \mathfrak{c}))$$

is J_1; note $\sup \mathfrak{c} \notin \mathfrak{c}$ as $\sup \mathfrak{c}$ is singular. As $\mathfrak{b} \in J_*[\mathfrak{a}]$, we have covered all cases].

Now why 1), 2), 3) holds? We shall use $(*)_1$ from above freely.

For (1): if $\theta \in \mathfrak{a}$ as $f_\alpha^{\lambda,\mathfrak{b}} \restriction \mathfrak{a} = f_\alpha^\lambda$ and $(*)_0$; if $\theta \in \mathfrak{b} \setminus \mathfrak{a} (\subseteq \mathrm{pcf}(\mathfrak{a}))$, $\alpha < \theta$ then for some $\beta < \lambda$, $f_{\alpha+1}^\theta \leq f_\beta^\lambda$, hence $f_\beta^{\lambda,\mathfrak{b}}(\theta) > \alpha$; this shows $\{\theta\} \in J_1$.

For (2): (trivially $\mathfrak{c} \subseteq \mathfrak{c}' \in J_1 \;\Rightarrow\; \mathfrak{c} \in J_1$;) if $\mathfrak{c}_1, \mathfrak{c}_2 \in J_1$, $\mathfrak{c} = \mathfrak{c}_1 \cup \mathfrak{c}_2$ and $f \in \prod \mathfrak{c}$, choose, for $\ell = 1, 2$, $\alpha_\ell < \lambda$ such that $f \restriction \mathfrak{c}_\ell \leq f^{\lambda,\mathfrak{b}}_{\alpha_\ell}$. Now let $f' \in \prod \mathfrak{a}$ be defined by $f'(\theta) = \max\left\{f^\lambda_{\alpha_1}(\theta), f^\lambda_{\alpha_2}(\theta)\right\}$, so by an assumption on $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ and $(*)_0$, for some α, $f' \leq f^\lambda_\alpha$, now $f^{\lambda,\mathfrak{b}}_\alpha$ is as required by $(*)_1$.

For (3): let $f \in \prod \mathfrak{c}$; by assumption for each $i < \kappa$ for some $\alpha(i) < \lambda$, $f \restriction \mathfrak{c}_i \leq f^{\lambda,\mathfrak{b}}_{\alpha(i)}$. Now $\left(\prod \mathfrak{a}, <_{J_{\leq\kappa}[\mathfrak{a}]}\right)$ is κ^+-directed, hence for some $f' \in \prod \mathfrak{a}$, $\bigwedge_{i<\kappa} f^\lambda_{\alpha(i)} <_{J_{\leq\kappa}[\mathfrak{a}]} f'$. By $(*)_0$ for some $\beta < \lambda$, $f' \leq f^\lambda_\beta$ and $f \restriction (\mathfrak{a} \cap \mathfrak{c}) \leq f^\lambda_\beta$ (necessarily $\bigwedge_{i<\kappa} \alpha(i) < \beta$). Now for each $\theta \in \bigcup\limits_{i<\kappa} \mathfrak{c}_i$; if $\theta \in \mathfrak{a}$ trivially $f(\theta) \leq f^\lambda_\beta(\theta)$, so assume $\theta \notin \mathfrak{a}$; now for some i, $\theta \in \mathfrak{c}_i$; so $\theta > \kappa$ and $f^\lambda_{\alpha(i)} <_{J_{\leq\kappa}[\mathfrak{a}]} f^\lambda_\beta$, hence $f^\lambda_{\alpha(i)} <_{J_{<\theta}[\mathfrak{a}]} f^\lambda_\beta$, hence by their definitions $f^{\lambda,\mathfrak{b}}_{\alpha(i)}(\theta) \leq f^{\lambda,\mathfrak{b}}_\beta(\theta)$.

So β is as required, i.e. we have proved subfact 3.2. $\square$

Now 3 follows from 3.1, 3.2 [using 3.2 for $f+1$ we can get there $f < f^{\lambda,\mathfrak{b}}_\alpha$, so (by 3.1) for some club C of λ,

$$\alpha < \beta \in C \quad\Rightarrow\quad f^{\lambda,\mathfrak{b}}_\alpha < f^{\lambda,\mathfrak{b}}_\beta \quad \bmod\ J.$$

Together $\langle f^{\lambda,\mathfrak{b}}_\alpha : \alpha \in C\rangle$ exemplify $\mathrm{tcf}(\prod \mathfrak{b}, <_J)$ is λ, as required]. $\square$

Theorem 4 ([Sh:g, VIII 3.3])**.** *Assume* $\min(\mathfrak{a}) > |\mathfrak{a}|$.

(1) *For an ultrafilter* D *on* $\mathrm{pcf}(\mathfrak{a})$ *not disjoint to* $J_*[\mathrm{pcf}(\mathfrak{a})]$,

$$\begin{aligned}\mathrm{tcf}(\textstyle\prod \mathrm{pcf}(\mathfrak{a})/D) &= \min\{\lambda \in \mathrm{pcf}(\mathfrak{a}) : \mathrm{pcf}(\mathfrak{b}_\lambda[\mathfrak{a}]) \in D\} \\ &= \min\left\{\lambda \in \mathrm{pcf}(\mathfrak{a}) : D \cap J^{\mathrm{pcf}}_{\leq\lambda}[\mathfrak{a}] \neq \emptyset\right\}.\end{aligned}$$

(2) *For* $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, $\mathrm{pcf}(\mathfrak{c})$ *is a subset of* $\mathrm{pcf}(\mathfrak{a})$ *and has a maximal element.*

(3) *For* $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, $\prod \mathfrak{b}/J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ *is* λ*-directed.*

(4) $\mathrm{pcf}_*(\mathfrak{a}) = \mathrm{pcf}(\mathfrak{a}) = \mathrm{pcf}_*(\mathrm{pcf}(\mathfrak{a}))$.

(5) *If* $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$ *and* $\mathfrak{c} \in J^{\mathrm{pcf}}_{\leq\lambda}[\mathfrak{a}]$ *then* $\prod \mathfrak{c}$ *has cofinality* $\leq \lambda$.

(6) *If* $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$ *and* $\mathfrak{c} \in J^{\mathrm{pcf}}_{\leq\lambda}[\mathfrak{a}] \backslash J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ *then* $\lambda = \mathrm{tcf}\left(\prod \mathfrak{c}, <_{J^{\mathrm{pcf}}_{<\lambda}}\right)$.

Proof. (1) Trivially the second and third terms are equal (see Definition 1(4)). Let λ be defined as in the second term, so $\mathrm{pcf}(\mathfrak{b}_\lambda[\mathfrak{a}]) \in D \cap J^{\mathrm{pcf}}_{\leq\lambda}[\mathfrak{a}]$. So by 2(2) without loss of generality $\mathfrak{a} = \mathfrak{b}_\lambda[\mathfrak{a}]$, so $\lambda = \max \mathrm{pcf}(\mathfrak{a})$. Using 3's notation, $\langle f^{\lambda,\mathfrak{b}}_\alpha : \alpha < \lambda\rangle$ exemplify $\lambda = \mathrm{tcf}(\prod \mathfrak{a}/D)$.

(2) By (1).

(3) This follows by the proof of 3, but as I was asked, we repeat the proof of 3 with the required changes. W.l.o.g. $\lambda \in \mathrm{pcf}(\mathfrak{a})$ [why? if $\lambda > \max \mathrm{pcf}(\mathfrak{a})$ then $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}] = \mathcal{P}(\mathrm{pcf}(\mathfrak{a}))$, so the conclusion is trivial, if not let $\lambda' = \min(\mathrm{pcf}(\mathfrak{a}) \setminus \lambda)$, so $\lambda' \in \mathrm{pcf}(\mathfrak{a})$ and $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}] = J^{\mathrm{pcf}}_{<\lambda'}[\mathfrak{a}]]$. We let $J =: J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$. Remember that (by [Sh:g, VIII 2.6]) there is $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$, a generating sequence for $\mathfrak{a}$. Let for $\mu \in \mathrm{pcf}(\mathfrak{a})$, $\langle f^\mu_\alpha : \alpha < \mu\rangle$ exemplify $\mu = \mathrm{tcf}(\prod \mathfrak{b}_\mu[\mathfrak{a}], J_{<\mu}[\mathfrak{a}])$, $f^\mu_\alpha \in \prod \mathfrak{a}$; by[Sh 355, 3.1] without loss of generality

$(*)_0$ $\forall f \in \prod \mathfrak{a}[\bigvee_\alpha f \restriction \mathfrak{b}_\mu[\mathfrak{a}] \leq f^\mu_\alpha]$.

Without loss of generality for $\theta \in \mathfrak{a} : f^\theta_\alpha(\theta) = \alpha$ if $\alpha < \theta$, $f^\theta_\alpha(\theta^1) = 0$ if $\alpha < \theta < \theta^1 \in \mathfrak{a}$. For any $f \in \prod \mathfrak{a}$ we define a function $f^\mathfrak{b} \in \prod \mathfrak{b}$ by:

$$f^\mathfrak{b} \restriction \mathfrak{a} = f$$

and for $\theta \in \mathfrak{b} \backslash \mathfrak{a}$:

$$f^{\mathfrak{b}}(\theta) = \min\left\{\beta : f \restriction \mathfrak{b}_\theta[\mathfrak{a}] \le f^\theta_\beta \mod J_{<\theta}[\mathfrak{a}]\right\}.$$

Let f vary on $\prod \mathfrak{a}$. Clearly

$(*)_1$ $f_1 \le f_2 \quad \Rightarrow \quad f^{\mathfrak{b}}_1 \le f^{\mathfrak{b}}_2$.

Subfact 4.1. *If $f_1 \le f_2 \mod J_{<\lambda}[\mathfrak{a}]$ (both in $\prod \mathfrak{a}$ of course) then $f^{\mathfrak{b}}_1 \le f^{\mathfrak{b}}_2 \mod J$.*

Proof of the subfact. Let $\mathfrak{c} = \{\theta \in \mathfrak{a} : f_1(\theta) \ge f_2(\theta)\}$, so $\mathfrak{c} \in J_{<\lambda}[\mathfrak{a}]$, hence for some $n < \omega$ and $\sigma_1 < \ldots < \sigma_n$ from $\lambda \cap \mathrm{pcf}(\mathfrak{a})$, (hence $< \lambda$) we have $\mathfrak{c} \subseteq \bigcup\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]$. So by the definition of the $f^{\mathfrak{b}}_\ell$'s we have:

$(*)_2$ if $\mu \in \mathfrak{b}$, and $\mathfrak{b}_\mu[\mathfrak{a}] \cap \bigcap\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}] \in J_{<\mu}[\mathfrak{a}]$, then $f^{\mathfrak{b}}_1(\mu) \le f^{\mathfrak{b}}_2(\mu)$.

However

$(*)_3$ $\mathfrak{d} =: \left\{\mu \in \mathrm{pcf}(\mathfrak{a}) : \mathfrak{b}_\mu[\mathfrak{a}] \cap \bigcup\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}] \ne \emptyset \mod J_{<\mu}[\mathfrak{a}]\right\}$
(for our fixed $\sigma_1, \ldots, \sigma_n \in \lambda \cap \mathrm{pcf}(\mathfrak{a})$) belongs to J

[as $\mu \in \mathfrak{d}$ implies $\mu \in \mathrm{pcf}(\bigcup\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]) = \bigcup\limits_{\ell=1}^{n} \mathrm{pcf}(\mathfrak{b}_{\sigma_\ell}[\mathfrak{a}])$ which is in J].

Together we get subfact 4.1. □

Subfact 4.2. *For any $g \in \prod(\mathfrak{b})$ for some $f \in \prod(\mathfrak{a})$ we have $g \le f^{\lambda,\mathfrak{b}}_\alpha$.*

Proof of the subfact. The family J_1 of sets $\mathfrak{c} \subseteq \mathfrak{b}$ for which this holds, i.e., for each $g \in \prod \mathfrak{c}$ there is $f \in \prod \mathfrak{a}$ such that $g \le f^{\mathfrak{b}}$, satisfies:

(1) $\{\theta\} \in J_1$ for $\theta \in \lambda \cap \mathrm{pcf}(\mathfrak{a})$,
(2) J_1 is an ideal of subsets of $\mathrm{pcf}(\mathfrak{a})$,
(3) if $\mathfrak{c}_i$ (for $i < \kappa$) are in J_1, $\min(\mathfrak{c}_i) > \kappa^+$ for each i *then* $\bigcup\limits_{i<\kappa} \mathfrak{c}_i$ is in J_1.

We shall show their satisfaction below.

Why (1)+(2)+(3) suffice for 4.2? As $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$; why? just prove that

$(*)_4$ $\mathfrak{c} \subseteq \mathfrak{b}$ & $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})] \quad \Rightarrow \quad \mathfrak{c} \in J_1$

by induction on $\sup\{\mu^+ : \mu \in \mathfrak{c}\}$. For successor use (1) + (2). For singular, let $\langle \mu_i : i < \kappa \rangle$ be such that μ_i is strictly increasing continuous with limit $\sup(\mathfrak{c}) = \sup\{\mu^+ : \mu \in \mathfrak{c}\}$, and $\kappa^+ < \mu_0$; by the induction hypothesis $\mathfrak{c} \cap \mu_0$, $\mathfrak{c} \cap [\mu_i, \mu_{i+1}]$ are in the ideal, by (3) we know that

$$\bigcup_{i<\kappa} (\mathfrak{c} \cap [\mu_i, \mu_{i+1})) = \mathfrak{c} \cap [\mu_0, \sup \mathfrak{c})$$

is in the ideal and, as said above, $\mathfrak{c} \cap \mu_0 \in J_1$ so by (2)

$$\mathfrak{c} = (\mathfrak{c} \cap \mu_0) \cup (\mathfrak{c} \cap [\mu_0, \sup \mathfrak{c}))$$

is J_1; note $\sup(\mathfrak{c}) \notin \mathfrak{c}$ as $\sup(\mathfrak{c})$ is singular. As $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$ implies $\mathfrak{c}$ has no inaccessible accumulation point, we have covered all cases in the induction, so $(*)_4$ holds. Now note that $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, so from $(*)_4$ we get $\mathfrak{b} \in J_1$ and by the definition of J_1 we are done.

Next, why 1), 2), 3) hold? We shall use $(*)_1$ from above freely.

For (1): let $g \in \prod \mathfrak{b}$; if $\theta \in \mathfrak{a}$ as $f^{\mathfrak{b}} \restriction \mathfrak{a} = f$ and $(*)_0$; if $\theta \in \mathfrak{b} \setminus \mathfrak{a}$ ($\subseteq \mathrm{pcf}(\mathfrak{a})$), then $g(\theta) < \theta$ let $f = f^{\theta}_{g(\theta)+1}$ hence

$$(\forall \gamma \leq g(0))(\neg f \leq f^{\theta}_{\gamma} \mod J_{<\theta}[\mathfrak{a}]).$$

Hence $g(\theta) < f^{\mathfrak{b}}(\theta)$; this shows $\{\theta\} \in J_1$.

For (2): (trivially $\mathfrak{c} \subseteq \mathfrak{c}' \in J_1 \Rightarrow \mathfrak{c} \in J_1$); if $\mathfrak{c}_1, \mathfrak{c}_2 \in J_1$, $\mathfrak{c} = \mathfrak{c}_1 \cup \mathfrak{c}_2$ and $g \in \prod \mathfrak{c}$, choose for $\ell = 1,2$ $f_\ell \in \prod \mathfrak{a}$ such that $g \restriction \mathfrak{c}_\ell \leq f^{\mathfrak{b}}_\ell$. Now let $f \in \prod \mathfrak{a}$ be defined by $f(\theta) = \max\{f_1(\theta), f_2(\theta)\}$, so $f \in \prod \mathfrak{a}$ and $g \restriction \mathfrak{c}_1 \leq f^{\mathfrak{b}}_1 \leq f^{\mathfrak{b}}$ and $g \restriction \mathfrak{c}_2 \leq f^{\mathfrak{b}}_2 \leq f^{\mathfrak{b}}$ hence $g \restriction (\mathfrak{c}_1 \cup \mathfrak{c}_2) \leq f^{\mathfrak{b}}$.

For (3): let $g \in \prod \mathfrak{c}$; by assumption for each $i < \kappa$ for some $f_i \in \prod \mathfrak{a}$, $g \restriction \mathfrak{c}_i \leq f^{\mathfrak{b}}_i$. Now $\left(\prod \mathfrak{a}, <_{J_{\leq\kappa}[\mathfrak{a}]}\right)$ is κ^+-directed, hence for some $f \in \prod \mathfrak{a}$, $\bigwedge_{i<\kappa} f_i <_{J_{\leq\kappa}[\mathfrak{a}]} f$. W.l.o.g. $g \restriction \mathfrak{a} \leq f \restriction \mathfrak{c}$. Now for each $\theta \in \bigcup\limits_{i<\kappa} \mathfrak{c}_i$; if $\theta \in \mathfrak{a}$ trivially $g(\theta) \leq f(\theta)$, so assume $\theta \notin \mathfrak{a}$. Now, for some i, $\theta \in \mathfrak{c}_i$; so $\theta > \kappa$ and $f_i <_{J_{\leq\kappa}[\mathfrak{a}]} f$, hence $f_i <_{J_{<\theta}[\mathfrak{a}]} f$, hence by their definitions $f^{\mathfrak{b}}_i(\theta) \leq f^{\mathfrak{b}}(\theta)$.

So (1), (2), (3) hold and hence $\mathfrak{b}$ is as required, i.e., we have proved subfact 4.2. □

We finish by

Subfact 4.3. *$\prod \mathfrak{b}/J$ is λ-directed.*

Proof of the subfact. Assume $g_i \in \prod \mathfrak{b}$ for $i < i^* < \lambda$, by 4.2 for each $i < i^*$ for some $f_i \in \prod \mathfrak{a}$ we have $g_i \leq f^{\mathfrak{b}}_i$. But $\prod \mathfrak{a}/J_{<\lambda}[\mathfrak{a}]$ is λ-directed hence for some $f \in \prod \mathfrak{a}$ we have

$$\bigwedge_{i<i^*} f_i < f \mod J_{<\lambda}[\mathfrak{a}].$$

By 4.1 we have

$$\bigwedge_{i<i^*} f^{\mathfrak{b}}_i \leq f^{\mathfrak{b}} \mod J,$$

hence by the previous sentence $i < i^* \Rightarrow g_i \leq f^{\mathfrak{b}}_i \leq_J f^{\mathfrak{b}}$, so $f^{\mathfrak{b}}+1$ is a $<_J$-upper bound of $\{g_i : i < i^*\}$, as required. □

□

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