On our paper 'Almost Free Splitter', a correction

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Abstract

Let R be a subring of $\mathbb{Q}$ and recall from [3] that an R-module G is a splitter if $\mathrm{Ext}_R(G,G)=0$. We correct the statement of Main Theorem 1.5 in [3]. Assuming CH any $\aleph_1$-free splitter of cardinality $\aleph_1$ is free over its nucleus as shown in [3]. Generally these modules are very close to being free as explained below. This change follows from [3] and is due to an incomplete proof (noticed thanks to Paul Eklof) in [3, first section on p. 207]. Assuming the negation of CH, in [6] it will be shown that under Martin's axiom these splitters are free indeed. However there are models of set theory having non-free $\aleph_1$-free splitter of cardinality $\aleph_1$.

1 Reductions

We refer to [3] for definitions and all details. Recall that an R-module G is a splitter if and only if $\mathrm{Ext}_R(G,G)=0$. We also assume may assume that splitters are torsion-free abelian group; see [3, p. 194]. Hence the nucleus R of a torsion-free abelian group $G\neq 0$ is defined to be the (now fixed) subring R of $\mathbb{Q}$ generated by all $\frac{1}{p}$ (p any prime) for which G is p-divisible, i.e. $pG=G$. Recall that G is an $\aleph_1$-free R-module if any countably generated R-submodule is free.

One of the main result in [3] should read as follows.

Theorem 1.1 *Assuming CH then any $\aleph_1$-free splitter of cardinality $\aleph_1$ is free over its nucleus.*

We must recall that G is of type I if there is an $\aleph_1$-filtration $G=\bigcup_{\alpha<\omega_1} G_\alpha$ of pure, free R-submodules such that $G_{\alpha+1}/G_\alpha$ are minimal non-free *for all* $\alpha>0$. Also

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recall that a non-free, torsion-free R-module of finite rank is minimal non-free, if all submodules of smaller rank are free.

Modules of type II and III are defined in [3] which is not needed here. It was shown in [3, Sections 3, 5, 6 and 7] that

(i) Any $\aleph_1$-free R-module G of cardinality $\aleph_1$ is either of type I, II or III.

(ii) Modules of type II or III are splitters if and only if they are free over the nucleus R (hence of type II).

(iii) Assuming ZFC + CH, then modules of type I are not splitters.

This shows Theorem 1.1, and in order to characterize $\aleph_1$-free splitters it remains to assume $\aleph_1 < 2^{\aleph_0}$ and to consider modules G of type I. In this case it is not needed to assume $\aleph_1$-freeness which essentially follows from Hausen [4], see [2].

We may assume that the splitter G has an $\aleph_1$-filtration $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ of pure and free R-submodules G_α such that $\operatorname{nuc} G_\alpha = R$ for all $\alpha \in \omega_1$ representing type I; see [3, p. 203]. Let G be such a fixed R-module which is not free.

If X is an R-submodule of G, then consider the set $\mathfrak{W} = \mathfrak{W}(X)$ of all finite sequences $\bar{a} = (a_0, a_1 \dots, a_n)$ such that
(i) $a_i \in G \ \ (i \le n)$
(ii) $\bigoplus_{i<n} (a_i + X)\, R$ is pure in G/X.
(iii) $\langle (a_i + X)R : i \le n\rangle_*$ is not a free R-module in G/X.

If $G_{\bar{a}} = \langle X,\ a_i R : i \le n\rangle_*$ of G, then $G_{\bar{a}}/X$ is a minimal non-free R-module of rank n. Hence there are natural numbers $p_{\bar{a}m}$ which are not units of R and elements $k_{\bar{a}im} \in R \ \ (i < n), \ \ g_{\bar{a}m} \in G_{\bar{a}}$ such that

$$y_{\bar{a}m+1}p_{\bar{a}m} = y_{\bar{a}m} + \sum_{i<n} a_i k_{\bar{a}im} + g_{\bar{a}m} \qquad (m \in \omega) \tag{1.1}$$

Choose a sequence $\bar{z} = (z_m : m \in \omega)$ of elements $z_m \in G$. The $\bar{z}$-inhomogeneous counter part of (1.1) is the system of equations

$$Y_{m+1}p_{\bar{a}m} \equiv Y_m + \sum_{i<n} X_i k_{\bar{a}im} + z_m \ \mathit{mod} \ \ X \qquad (m \in \omega). \tag{1.2}$$

Say that $\bar{a} \in \mathfrak{W}$ is contra-Whitehead if (1.2) has no solutions y_m $(m \in \omega)$ in G (hence in $G_{\bar{a}}$) for some $\bar{z}$ and $X_i = a_i$. Otherwise we say that $\bar{a}$ is pro-Whitehead. Using *this* definition the following was shown in [3, Proposition 4.4].

Proposition 1.2 *If* $G = \bigcup_{\alpha \in \omega_1} G_\alpha$ *and*

$$S = \{\alpha \in \omega_1 : \exists \bar{a} \in \mathfrak{W}(G_\alpha), \bar{a} \textit{ is contra-Whitehead } \}$$

is stationary in ω_1*, then* G *is not a splitter.*

By assumption on G follows that S is not stationary in ω_1 and we may assume that

- *all* G_α *are pro-Whitehead in* G*.*

This case is studied in the next result, which needs the extra assumption that $nuc(G/X) = R$.

Theorem 1.3 *Let* G *be a splitter of cardinality* $< 2^{\aleph_0}$ *with* $\operatorname{nuc} G = R$*. If* X *is a pure, countable* R*-submodule of* G *with* $nuc(G/X) = R$ *which is also pro-Whitehead in* G*, then* G/X *is an* $\aleph_1$*-free* R*-module.*

The proof is given in [3, p. 206 (first case)] applies.

Let $C = \{\alpha \in \omega_1 : nuc(G/G_\alpha) = R\}$. If $C = \emptyset$, then $G_{\alpha+1}/G_\alpha$ is free by the last Theorem 1.3 and the last assumption on G, hence G is a free R-module. This case was excluded. Otherwise $C \neq \emptyset$ and C is a final segment of ω_1, we get the following

However in general we get the following

Corollary 1.4 *Any non-free splitter of type I and cardinality at most* $\aleph_1 < 2^{\aleph_0}$ *has a countable* R*-submodule* X *such that* $nuc(G/X)$ *is strictly larger then* R*.*

If R is a local ring then by the corollary the module G is free-by-free, an extension of a countable free R-module by a divisible module - a free module over $\mathbb{Q}$.

It remains to consider splitters as in the corollary under $\aleph_1 < 2^{\aleph_0}$:

Assuming now in addition (to negation of CH) Martin's axiom, it follows that $\aleph_1$-free splitters of cardinality $\aleph_1$ are free, as shown by Shelah [6]. He ([6]) also constructs a model of set theory with non-free $\aleph_1$-free splitters of cardinality $\aleph_1$. Surely, in this model MA as well as CH cannot hold.

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