DETAILS ON SH:74

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Abstract. We give details on a claim from [Sh 74] (continuing [Sh 8]+ [Sh:E18]

Key words and phrases.

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Theorem -1.1. *3*

Let $\lambda_1 \geq \mu_1, \lambda \geq \mu$, then the following are equivalent

(A) $(\lambda_1, \mu_1) \rightarrow_{\aleph_0} (\lambda, \mu)$

(B) $(\lambda_1, \mu_1) \rightarrow_{\leq \mu} (\lambda, \mu)$

(C) *We can find functions $f_\ell : \lambda^\ell \rightarrow \mu$ for $\ell < \omega$ such that: if (n, E) is not an identity of (λ_1, μ_1) then $\langle f_\ell : \ell \leq n \rangle$ witness that it is not an identity of (λ, μ).*

Remark :

(1) If

$\boxdot$ the sets of (n, E) which are not identifies of (λ_1, μ_1) is recursive, we can add

(D) $(\lambda_1, \mu_1) \rightarrow_1 (\lambda, \mu)$.

(2) We can weaken $\boxdot$ to:

$\boxdot^+$ there is a recursive set of identities, including the one failing for (λ_1, μ_1) and included in the one holding for (λ, μ) (check).

Remark: Recall $(\lambda_1, \mu_1) \rightarrow_{\leq \kappa} (\lambda, \mu)$ mean that if T is a (first order theory of cardinality $\leq \kappa$, with the distinguish predicates P_1, P_2 and every finite $T' \subseteq T$ has a model M with $|P_1^M| = \lambda_1, |P_2^M| = \mu_1$ then T has a model N with $|P_1^N| = \lambda, |P_2^N| = \mu$

Proof. of theorem 3

The proof is by showing (for our given $\lambda_1, \mu_1, \lambda, \mu$)

$$(A) \Rightarrow (C) \Rightarrow (B) \Rightarrow (A)$$

$(B) \Rightarrow (A)$ trivially.

$(A) \Rightarrow (C)$: Let $\langle (n_i, E_i) : i < \omega \rangle$ list the identities. Let $m_i = \text{Max}\{i, n_0, \ldots, n_i\}$ For each i, choose if possible f^i_ℓ, an ℓ- place function from λ_1 to μ_1 for $\ell \leq m_i$ exemplifying $\lambda_1 \not\twoheadrightarrow (n_i, E_i)_{\mu_1}$; if impossible f^i_ℓ is choosen identically zero. We do it by induction on i and so without loss of generality

$\circledast_1$ $i < j < \omega \,\&\, \ell \leq m_i \Rightarrow f^j_\ell$ refine f^i_ℓ i.e. $f^j_\ell(\bar{x}) = f^j_\ell(\bar{y}) \rightarrow f^i_\ell(\bar{x}) = f^i_\ell(\bar{y})$ (recall $\mu^n = \mu$)

Let $M_1 = (\lambda_1, \mu_1 \ldots f^i_\ell, \ldots)$, so it has $i < \omega, \ell \leq n_i$ universe λ_1 and vocalulary $\{F^i_\ell : i < \omega, \ell \leq m_i\}$ where F^i_ℓ is an ℓ-place function.

This is not exaclly right so let M_2 be defind by

(a) M_2 has universe λ_1
relations:

(b) $P_1^{M_2} = \lambda_1$

(c) $P_2^{M_2} = \mu_1$

(d) F_ℓ an $(\ell{+}1)-$ place function such that for $i < \omega, \forall \bar{x}[F_\ell(\bar{x}, i) = f^i_\ell(\bar{x})]$ otherwise zero

(e) $P_3^{M_2} = \omega$

(f) $c_n^{M_2} = n$

(g) $<^{M_2}$- the usual order on λ_1

Let M_2^+ be the expansion of M_2 by Skolem functions.

Lastly let $T = \mathrm{Th}(M_2) \cup \{c_n < c \wedge P_3(c) : n < \omega\}$. Clearly.

$\circledast_2$ T is first order, countable and every finite subset has a (λ_1, μ_1)-model.

As we are assuming (1) there is a model N of T, with

$$|P_1^N| = \| N \| = \lambda,\ |P_2^N| = \mu;$$

so without loss of generality $P_1^N = \lambda, P_2^N = \mu$. Let $f^*_\ell : {}^\ell\lambda \to \lambda$ be

$$f^*_\ell(\bar{x}) = F^N_\ell(\bar{x}, c^N)$$

$\circledast_3$ if (n_i, E_i) is not an identity of (λ_1, μ_1) then $\langle f^*_\ell : \ell \leq n_i\rangle$ witness it is not an identity of (λ, μ)

[Why? because $M_2 \models$ "$\langle f^j_\ell : \ell \leq n_i\rangle$ witness (n_i, E_i) is not an identity of (λ_1, μ_1)" is known when $j = i$ by its choice, and if $j \leq i$ by $\circledast_1$, which means

$\boxtimes$ $M_2 \models (\forall y)(c_i \leq y \in P_3^{M_2} \to \langle F_\ell(-, y) : i \leq m_i\rangle$ is a witness to (n_i, E_i) being not an identity),

this ($\boxtimes$) is expressed by a first order sentence ψ_i which M_2 satisfied hence $\psi_i \in T$ hence $N \models \psi_i$.

In particular use $y = c$ in N recalling $N \models [c_i < c \,\&\, P_3(c)]$ so we have

$$\langle F^N_\ell(-, c) : \ell \leq n_i\rangle \quad \text{witness} \quad (P_1^N, P_2^N) = (\lambda, \mu)$$

fail the identity (n_i, E_i) which mean that $\langle f^*_\ell : \ell \leq n_i\rangle$ witness (λ, μ) fail the identity (n_i, E_i). So we have gotten (3).

$(C) \Rightarrow (B)$ Exactly as in [Sh 8]. Define an equivalence relation E on $\bigcup\limits_n {}^n\lambda$ as follows

$$\bar{b} E \bar{c} \quad \text{iff} \quad \bigvee_n [\bar{b}, \bar{c} \in {}^n\lambda \,\&\, f_n(\bar{b}) = f_n(\bar{c})]$$

By [Sh 8], Lemma 1 it suffice to show (*) of [[Sh 8] p. 194] which is stright.

Remark: in (3) we have $\langle f_\ell : \ell < \omega\rangle$ for all possible (n, E) not just for each $(n, E) \ldots$

$\square$

[References of the form `math.XX/···` refer to `arXiv.org`]

References

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