

ON SOME PROBLEMS IN
GENERAL TOPOLOGY

SH-E3

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§0. INTRODUCTION

This work was done in 1977 and was widely quoted but not submitted.

In section 3 it is proved that Arhangel'skii's problem has a consistent positive answer: if $V \models CH$, then for some \aleph_1 -complete \aleph_2 -c.c. forcing notion P of cardinality \aleph_2 we have \Vdash_P " CH and there is a Lindelöf regular topological space of size \aleph_2 with clopen basis with every point of pseudo-character \aleph_0 (i.e. each singleton is the intersection of countably many open sets)".

Meanwhile this was continued in Hajnal and Juhasz [HJ], Stanley and Shelah [ShSt:167], I.Gorelic [Go] and Ch.Morgan.

In section 4 we prove the consistency of: " $CH + 2^{\aleph_1} > \aleph_2$ + there is no space as above with \aleph_2 points" (starting with a weakly compact cardinal).

Section 2 deals with $\beta(\mathbb{N})$, it is proved that the following is consistent with ZFC : $MA + 2^{\aleph_0} = \aleph_2 + (*)$ where

(*) if $A_i^0, A_i^1 \subseteq \omega$ (for $i < \omega_1$) and $A_i^0 \cap A_j^1$ is finite for $i, j < \omega_1$; and \mathfrak{D}_i^ℓ is a non-principal ultrafilter over ω such that $A_i^\ell \in \mathfrak{D}_i^\ell$ (for $i < \omega_1$ and $\ell \in \{0, 1\}$), then there is a $B \subseteq \omega$ such that $B \in \mathfrak{D}_i^\ell \Leftrightarrow \ell = 0$.

The scheme is as in Baumgartner [B].

Another problem on $\beta(\mathbb{N})$ which I remember was asked and published by E. van Douwen and G. Woods, is answered in §1: is there a discrete $D \subseteq \beta(\mathbb{N})$, of cardinality \aleph_1 and $A \subseteq D$ such that $cl(A) \cap cl(D \setminus A) \neq \emptyset$.

I thank U. Abraham for urging the publication (in this form) and for corrections, the referee for corrections and M. Džamonja for corrections. Compared with the old version we added details, explanations, the introduction, added 2.4 and stated 1.2, 4.2; one section was omitted.

§1. A PROBLEM ON $\beta(\mathbb{N})$

1.1 Question. Does there exist a discrete $D \subseteq \beta(\mathbb{N})$ of cardinality \aleph_1 , and an $A \subseteq D$ such that $cl(A) \cap cl(D \setminus A) \neq \emptyset$?

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1.2 Answer. Yes, moreover we can let D have any cardinality λ such that $\aleph_1 \leq \lambda = cf(\lambda) \leq 2^{\aleph_0}$ and have $\bigcap \{cl(D') : D' \subseteq D \text{ and } |D'| = |D|\}$ not empty.

1.3 Definition. Let B_i (for $i \leq \omega_1$) be the BA (Boolean algebra) freely generated by $\{x_\alpha : \alpha < i\}$, B_i^c is the completion of B_i , $B = B_{\omega_1}^c$.

1.4 Claim. *In the space of ultrafilters of $B_{\omega_1}^c$ we can find such a D .*

PROOF: We define by induction on $i \leq \omega_1$, an ultrafilter \mathfrak{D}_i of $B_{\omega_1}^c$ such that

- (i) $x_\alpha \in \mathfrak{D}_i \Leftrightarrow \alpha = i$,
- (ii) if $a \in B_i^c$, $a \in \mathfrak{D}_i$ then $a \in \mathfrak{D}_j$ for every $j > i$.

This is easy. Let $\mathfrak{D}_{\omega_1} = \bigcup_{i < \omega_1} (\mathfrak{D}_i \cap B_i^c)$. Now, $\{\mathfrak{D}_i : i < \omega_1\}$ is discrete by (i) and $\mathfrak{D}_{\omega_1} \in cl\{\mathfrak{D}_i : i \in S\}$ for any $S \subseteq \omega_1, |S| = \aleph_1$, because $B_{\omega_1}^c = \bigcup_{i < \omega_1} B_i^c$, as B_{ω_1} satisfies the countable chain condition.

1.5 Solution of the problem. Let $X_i \subseteq \omega$ ($i < \omega_1$) be independent, i.e. any non trivial Boolean combination of the X_i is not empty. Let $f : \mathcal{P}(\omega) \rightarrow B_{\omega_1}^c$ be any homomorphism such that $f(X_i) = x_i$ (exists as $B_{\omega_1}^c$ is complete and the X_i are independent). It is not hard to prove f is onto, and $\{f^{-1}(\mathfrak{D}_i) : i < \omega_1\}$ is as required (\mathfrak{D}_i from the claim), and

$$f^{-1}(\mathfrak{D}_{\omega_1}) \in cl(\{f^{-1}(\mathfrak{D}_{2i}) : i < \omega_1\}) \cap cl(\{f^{-1}(\mathfrak{D}_{2i+1}) : i < \omega_1\}).$$

□_{1.5}

§2 A QUESTION ON $\beta(\mathbb{N}) \setminus \mathbb{N}$

2.1 Claim. *Assuming the consistency of ZFC we prove the consistency of the following assertion with*

$ZFC + 2^{\aleph_0} = \aleph_2 + MA$:

⊗ if $A_i^0, A_i^1 \subseteq \omega$ (for $i < \omega_1$) and $A_i^0 \cap A_j^1$ is finite for $i, j < \omega_1$; \mathfrak{D}_i^ℓ is a non-principal ultrafilter over ω such that $A_i^\ell \in \mathfrak{D}_i^\ell$ (for $i < \omega_1$ and $\ell \in \{0, 1\}$), then there is a $B \subseteq \omega$ such that $B \in \mathfrak{D}_i^\ell \Leftrightarrow \ell = 0$.

PROOF: We assume $V \models 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2$. We repeat the proof of Solovay-Tennenbaum of $Con(ZFC + 2^{\aleph_0} = \aleph_2 + MA)$, similarly to Baumgartner [B]. That is, we define by induction on $\alpha < \omega_2$ a set of forcing conditions P_α , increasing (under \subseteq and even \triangleleft) and continuous with $\alpha, |P_\alpha| \leq \aleph_1$, and each P_α satisfies the countable chain condition. We start with $V \models 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2 + \diamond_{\{\delta < \omega_2 : cf(\delta) = \aleph_1\}}$.

Now, in addition, at some $\alpha < \omega_2$ we consider a system $\langle A_i^\ell : i < \omega_1, \ell = 0, 1 \rangle$, $\langle \mathfrak{D}_i^\ell : i < \omega_1, \ell = 0, 1 \rangle$, which belongs to V^{P_α} , $A_i^\ell \subseteq \omega$, $A_i^\ell \in \mathfrak{D}_i^\ell$, \mathfrak{D}_i^ℓ a family of subsets of ω , such that the filter it generates in V^{P_α} (which we denote by the same letter) is \aleph_1 -saturated (i.e. there are no $C_\alpha \neq \emptyset \text{ mod } \mathfrak{D}_i^\ell, C_\alpha \subseteq \omega$ for $\alpha < \omega_1$ such that $C_\alpha \cap C_\beta = \emptyset \text{ mod } \mathfrak{D}_i^\ell$ for $\alpha < \beta < \omega_1$). By the usual bookkeeping, every such system appears, and this is possible as $V \models 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$. Clearly $V^{P_\alpha} \models 2^{\aleph_0} = \aleph_1$. We define in V^{P_α} a set of forcing conditions Q satisfying the c.c.c., whose generic set gives a B such that $B \in \mathfrak{D}_i^0$ and $\omega \setminus B \in \mathfrak{D}_i^1$ (for

$i < \omega_1$), and define $P_{\alpha+1} = P_\alpha * Q$. This is sufficient, as if $p \in P_{\omega_2}$ forces that $\langle A_i^\ell : i < \omega_1 \rangle, \langle \mathfrak{D}_i^\ell : i < \omega_1, \ell = 0, 1 \rangle$ contradict \otimes , then for a club of E of ω_2 , for every $\alpha \in E$ of cofinality \aleph_1 , $\langle A_i^\ell : \ell < \omega_1 \rangle$ is a P_α -name, and $\langle \mathcal{P}(\omega)^{V^{P_\alpha}} \cap \mathfrak{D}_i^\ell : \ell, i \rangle$ is a P_α -name of an ultrafilter. Then, clearly for stationarily many $\alpha < \omega_2$ of cofinality \aleph_1 , in V^{P_α} the above holds and as $\diamond_{\{\delta < \omega_2 : cf(\delta) = \aleph_1\}}$ holds we can assume that we have considered the system in question at some α (of course in the bookkeeping we take care of MA too).

So it suffices to prove:

2.2 Claim. *If $V \models 2^{\aleph_0} = \aleph_1$, $A_i^\ell \subseteq \omega$ (for $\ell = 0, 1$ and $i < \omega_1$), $A_i^0 \cap A_j^1$ are finite for $i, j < \omega_1$, \mathfrak{D}_i^ℓ an \aleph_1 -saturated filter over ω , $A_i^\ell \in \mathfrak{D}_i^\ell$, then we can find a partial order Q of size \aleph_1 , satisfying c.c.c., and in V^Q there is an $X \subseteq \omega$ such that $X \in \mathfrak{D}_i^0$ and $\omega \setminus X \in \mathfrak{D}_i^1$ (for $i < \omega_1$; note: \mathfrak{D}_i^ℓ stands for the filter it generates).*

2.2A Remark. The sequence $\langle (B_i^0, B_i^1) : i < \omega_1 \rangle$ constructed below should be just generic enough (we do not use this presentation because some people do not like it¹). More specifically letting $f_{2i+l} \in {}^\omega 2$ be defined by

$$f_i^\ell(n) = 0 \Leftrightarrow [\text{the } n\text{th element of } A_i^\ell \text{ belongs to } B_i^\ell]$$

we demand: for every $n < \omega$ and open dense $J \subseteq {}^n({}^\omega 2)$, for some $\alpha_J < \omega_1$, for every $\alpha_0 < \dots < \alpha_{n-1}$ from (α_J, ω_1) we have $\langle f_{\alpha_0}, \dots, f_{\alpha_{n-1}} \rangle \in J$.

PROOF: We shall choose sets $B_i^\ell \subseteq A_i^\ell$, $B_j^\ell \not\subseteq \mathfrak{D}_i^\ell$, and let $Q = \{(f, g) : f, g \text{ are finite functions from } \omega_1 \text{ to } \omega, \text{ and } (A_i^0 \setminus B_i^0 \setminus f(i)) \cap (A_j^1 \setminus B_j^1 \setminus g(j)) = \emptyset \text{ when } i \in \text{Dom}(f) \text{ and } j \in \text{Dom}(g)\}$.

(let $Q_i = \{(f, g) \in Q : \text{Dom}(f) \cup \text{Dom}(g) \subseteq i\}$).

Q is ordered naturally: $(f_1, g_1) \leq (f_2, g_2)$ iff $f_1 \subseteq f_2$ & $g_1 \subseteq g_2$

For a generic $G \subseteq Q$ the set $X = \bigcup \{B_i^0 \setminus f(i) : \exists g ((f, g) \in G)\}$ is as required, and $|Q| = \aleph_1$. We should show only that we can choose B_i^ℓ such that Q satisfies the c.c.c. (the density condition is easy: for every $i, j < \omega_1$ by the almost disjoint condition, $A_i^0 \cap A_j^1$ is finite, hence for every $(f, g) \in Q$ and $i < \omega_1$ there is $n^* < \omega$ such that $A_i^0 \subseteq n^*$ is disjoint to A_j^1 for $j \in \text{Dom}(g) \cup \{i\}$ and $A_i^1 \setminus n^*$ is disjoint to A_j^0 for $j \in \text{Dom}(f) \cup \{i\}$. Let $f' = f \cup \{i, n^*\}$, $g' = g \cup \{i, n^*\}$. So, there is a $(f', g') \in Q$ such that $(f, g) \leq (f', g')$ and $i \in \text{Dom}(f') \cap \text{Dom}(g')$).

Suppose (f_i, g_i) (for $i < \omega_1$) exemplify a contradiction to c.c.c. Then, by the well known techniques, we can assume that there is a $(f, g) \leq (f_i, g_i)$, $\text{Dom}(f_i) = \text{Dom}(g_i) = w_i$, $\text{Dom}(f) = \text{Dom}(g) = w$, $w \subseteq w_i$ and the sets $w_i \setminus w$ are pairwise disjoint, $w_i \setminus w = \{\eta_i(0), \dots, \eta_i(t)\}$, $i < j \Rightarrow \eta_i(t) < \eta_j(0)$, $\eta_i(0) < \eta_i(1) < \dots < \eta_i(t)$, and for $m \leq t$ we have $f_i(\eta_i(m)) = k_m$ and $g_i(\eta_i(m)) = j_m$. For some δ we can get an “elementary submodel” of the whole system (so $\eta_i(t) < \delta$ for $i < \delta$). Let $t_\zeta, \delta_\zeta, f_i^\zeta, g_i^\zeta$ (for $i < \delta$), $\eta_i^\zeta(m), k_m^\zeta, j_m^\zeta$ (for $\zeta < \omega_1$ and $m \leq t$) enumerate all possible such systems. Now we shall choose B_α^ℓ by induction on α and then on ℓ with some restrictions: (say $\ell = 0$ for notational simplicity).

Let us try to explain the idea of the proof. If $\max\{\zeta, \delta_\zeta\} < \alpha < \omega_1$, then we think of $(\langle \alpha, k_0^\zeta \rangle, \emptyset) \in Q$ as a candidate to be a part of some (f_i, g_i) . Now,

¹at least they did not back in the seventies

either it is compatible with infinitely many (f_i^ζ, g_i^ζ) (for $i < \delta_\zeta$), or the condition on being an elementary submodel eliminates this possibility, and similarly if $Dom(f') = Dom(g') = \{\eta(0), \dots, \eta(m-1)\} \subseteq \alpha$ and (f, g) is compatible with infinitely many (f_i^ζ, g_i^ζ) , then either so is $(f' \cup \{\langle \alpha, k_m^\zeta \rangle\}, g')$, or the condition on elementary submodels is violated.

There are countably many such conditions and we can find 2^{\aleph_0} pairwise almost disjoint infinite $Y_\xi \subseteq A_i^\ell$ (for $\xi < 2^{\aleph_0}$), such that $A_\alpha^\ell \setminus Y_\xi$ are as required and all but countably many of the Y_ξ 's are $= \emptyset \text{ mod } \mathfrak{D}_\alpha^\ell$ (as it is \aleph_1 -saturated), so we have many candidates for B_α^ℓ .

Now we present the construction itself. Assume B_i^0, B_0^1 have been chosen for $i < \alpha$ and we shall define B_α^0, B_α^1 . As $\langle B_i^0, B_i^1 : i < \alpha \rangle$ is defined, so is Q_α .

Let

$$\begin{aligned} K_\alpha = \{(\zeta, m, n, i, f', g', \eta) : & \zeta < \alpha \wedge m \leq t_\zeta \wedge n < \omega \wedge i < \delta_\zeta \wedge (f', g') \in Q_\alpha \\ & \wedge |Dom(f')| = m \wedge \eta \in {}^m \alpha \wedge \eta \text{ strictly increasing} \\ & \wedge Dom(f') = Dom(g') = \{\eta(0), \dots, \eta(m-1)\} \\ & \wedge f'(\eta(s)) = k_s^\zeta \text{ (for } s < m) \\ & \wedge g'(\eta(s)) = j_s^\zeta \text{ (for } s < m)\}. \end{aligned}$$

Clearly, K_α is countable. Let $L_\alpha = \{(Y^0, Y^1) : Y^\ell \subseteq A_\alpha^\ell \text{ for } \ell = 0, 1\}$, and for $(Y^0, Y^1) \in L_\alpha$, let $Q_\alpha[Y^0, Y^1]$ be defined as $Q_{\alpha+1}$, had we chosen (B_α^0, B_α^1) to be $(A_\alpha^0 \setminus Y^0, A_\alpha^1 \setminus Y^1)$. We say that $(Y^0, Y^1) \in L_\alpha$ satisfies $(\zeta, m, n, i, f', g', \eta) \in K_\alpha$ if:

either

(α) for some $\beta < \delta_\zeta$, $(f_\beta^\zeta, g_\beta^\zeta)$ and $(f' \cup \{\langle \alpha, k_m^\zeta \rangle\}, g' \cup \{\langle \alpha, j_m^\zeta \rangle\}, h)$ are compatible conditions in $Q_\alpha[Y^0, Y^1]$ and $\beta e_{\zeta, n} i$ where $j e_{\zeta, n} i$ means:

$$\begin{aligned} (\forall \ell \leq t^\zeta) [A_{\eta_j^\zeta(\ell)}^0 \cap n = A_{\eta_i^\zeta(\ell)}^0 \cap n \wedge A_{\eta_j^\zeta(\ell)}^1 \cap n = A_{\eta_i^\zeta(\ell)}^1 \cap n \\ \wedge B_{\eta_j^\zeta(\ell)}^0 \cap n = B_{\eta_i^\zeta(\ell)}^0 \cap n \wedge B_{\eta_j^\zeta(\ell)}^1 \cap n = B_{\eta_i^\zeta(\ell)}^1 \cap n]. \end{aligned}$$

or

(β) for some natural number $u < \omega$, for every $(Z^0, Z^1) \in L_\alpha$ satisfying $Z^0 \cap u = Y^0 \cap u$ & $Z^1 \cap u = Y^1 \cap u$, clause (α) fails even for $Q_\alpha[Z^0, Z^1]$.

Now, L_α is a complete separable metric space (by the metric d defined as $d((Y^0, Y^1), (Z^0, Z^1)) = \min((Y^0 \Delta Z^0) \cup (Y^1 \Delta Z^1))$ where Δ is the symmetric difference i.e. $Y \Delta Z = (Y \setminus Z) \cup (Z \setminus Y)$.)

Clearly:

(*) for each $(\zeta, m, n, i, f', g', \eta) \in K_\alpha$ the set

$$L_{(\zeta, m, n, i, f', g', \eta)}^\alpha = \{(Y^0, Y^1) \in L_\alpha : (Y^0, Y^1)$$

satisfies $(\zeta, m, n, i, f', g', \eta)\}$ is an open dense set.

As K_α is countable, we can find a $\langle (Y_\xi^0, Y_\xi^1) : \xi < 2^{\aleph_0} \rangle$ such that:

- (a) $(Y_\xi^0, Y_\xi^1) \in \cap \{L_{(\zeta, m, n, i, f', g', \eta)}^\alpha : (\zeta, m, n, i, f', g', \eta) \in K_\alpha\}$
- (b) for $\zeta < \xi$ the set $Y_\zeta^0 \cap Y_\xi^0$ is finite.

(this is like building a perfect set of Cohen generic reals.)

So (by the \aleph_1 -saturation), for some $\xi < 2^{\aleph_0}$ we have $Y_\xi^0 = \emptyset \text{ mod } \mathfrak{D}_\alpha^0$ and $Y_\xi^1 = \emptyset \text{ mod } \mathfrak{D}_\alpha^1$.

We let $(B_\alpha^0, B_\alpha^1) = (A^0 \setminus Y^0, A_\alpha^1 \setminus Y^1)$.

So we have finished the inductive definition of the (B_α^0, B_α^1) 's.

Now we show that the construction guarantees that Q satisfies the c.c.c. Suppose $\{(f_i, g_i) : i < \omega_1\}$ is an uncountable antichain. As we explained above, we may assume there are $t_\zeta, \delta_\zeta, \dots$, and $\{(f_i, g_i) : i < \delta_\zeta\}$ is an "elementary subsystem". So in particular, if for some $u < \omega$ and $F^\ell : t_\zeta + 1 \rightarrow \mathcal{P}(u) \times \mathcal{P}(u)$ (where $\ell \in \{0, 1\}$) we have an $i < \omega_1$ such that $(A_i^\ell \cap u, B_{\eta_i^\zeta(m)}^\ell \cap u) = F^\ell(m)$ for each $\ell < 2$ and $m \leq t_\zeta$, then such i exists already below δ_ζ . So for each $u < \omega$ and $\gamma < \omega_1$ for some $i_{\gamma, u} < \delta_\zeta$ we have $\gamma e_{\zeta, u} i_{\gamma, u}$. But now consider $\gamma > \delta_\zeta$ such that $\eta_\gamma^\zeta(0) > \delta_\zeta$ and $\eta_\gamma^\zeta(0) > \zeta$, and let $m \leq t_\zeta$ be the least ordinal $n_\gamma^\zeta(m)$ for which condition (α) fails for some finite $u < \omega$ and $i = i_{\gamma, u}$ which is $< \delta_\zeta$ (note that if $\alpha = \eta_\gamma^\zeta(t_\zeta)$ and $u = 0$ then condition (α) cannot possibly hold). Let it fail for $u = u_m$. Then condition (β) holds. But if u witnesses (β) then for $\max\{u_m, u\}$ we have a failure for $m - 1$, contradiction.

The construction guarantees that Q satisfies the c.c.c, hence we have finished.

□_{2.2}

□_{2.1}

2.3 Remark. 1) We can act as in [Wi:C 2], and then use really \aleph_1 -saturation and all filters are in V .

2) Can we ignore CH and make 2^{\aleph_0} larger?

Assume $\lambda = \lambda^{\aleph_1} = cf(\lambda)$ and \diamond_S where $S = \{\delta < \lambda : cf(\delta) = \aleph_1\}$ and $(\forall \alpha < \lambda)[|\alpha|^{\aleph_0} < \lambda]$. We can find a forcing notion P , which is c.c.c. of cardinality λ , and \Vdash_P " $2^{\aleph_0} = \lambda$ and MA and $(*)$ of 2.1 holds."

Why? we use finite support iteration, if $\alpha \notin S, Q_\alpha$ is adding a Cohen real; if $\delta \in S$ and \diamond_S guesses $P, \langle A_i^\ell : i < \omega_1, \ell < 2 \rangle$, and $\langle \mathfrak{D}_\alpha^\ell \cap \mathcal{P}(\omega)^{V^{P_\alpha}} : \alpha < \omega_1, \ell < 2 \rangle$ as in the proof, we imitate the proof, but for $\langle (B_\alpha^0, B_\alpha^1) : \alpha < \omega_1 \rangle$ we use a sequence $\langle T_\alpha : \alpha < \omega_1 \rangle$, T_α a perfect set of members of L_α such that for every large enough $\alpha < \omega_1$, all branches of T_α are Cohen generic over V^{P_δ} .

To answer the question of the referee:

2.4 Claim. *The statement \otimes of 2.1 follows from PFA.*

PROOF Consider the forcing $P = Levy(\aleph_1, \aleph_2) * Q$ where Q is constructed as in the proof of 2.1 in the universe $V^{Levy(\aleph_1, \aleph_2)}$ (note that forcing with $Levy(\aleph_1, \aleph_2)$ adds no reals, so \mathfrak{D}_i is still an ultrafilter and CH holds) and let \underline{B} be the name of the desired set. So \Vdash_P "for $i < \omega_1, \ell < 2$, for some $A \in \mathfrak{D}_i^\ell$ we have $[\ell = 0 \Rightarrow \underline{B}^* \supseteq A]$ and $[\ell = 1 \Rightarrow \underline{B} \cap A =^* \emptyset]$ ". Apply PFA.

§3. CONCERNING ARHANGELSKII'S PROBLEM

3.1 Theorem. *The following is consistent with ZFC + GCH :*

$(*)$ *There is a regular space of cardinality \aleph_2 which is Lindelöf and has pseudo-character \aleph_0 .*

Remark. We had first said “a Hausdorff space...” but Kunen noted the proof actually yields a regular space.

PROOF: Assume V satisfies GCH . Let P be the set of tuples of the form $p = \langle A, f, E, T \rangle = \langle A^p, f^p, E^p, T^p \rangle$, where:

- (1) A is a countable subset of ω_2 ,
- (2) f is a two place function from A to $\omega + 1$, where we write $f_x(y)$ instead of $f(x, y)$, and $f_x(y) = \omega \Leftrightarrow y = x$ holds.
- (3) E is a three place relation on A , but we write $x E_\gamma y$, and demand that for each fixed γ , E_γ is an equivalence relation, and $\gamma \in A \wedge \gamma < \beta \wedge x E_\beta y \Rightarrow x E_\gamma y$, while $x < \gamma \wedge x E_\gamma y \Rightarrow x = y$. We stipulate E_{ω_2} as the equality on A .
- (4) T is a countable set. Each member B of T will be called a formal cover, $|B| = \aleph_0$, and B is of the form $\{\tau_n^B : n < \omega\}$, where each τ_n^B is the formal intersection of finitely many $U_x^n (x \in A, n \in \mathbb{Z})$ [the intended meaning is: $U_x^n = \{y : f_x(y) \geq n\}$ for $n \geq 0$, and $U_x^{-n} = \omega_2 \setminus U_x^n$ for $n > 0$; we say $p \models “y \in \tau”$ in the natural case (i.e. $y \in A, \tau = \bigcap_{i < k} U_{x(i)}^{\ell(i)}$, where $k < \omega, \ell(i) \in \mathbb{Z}, x(i) \in A$ and for each $i < k$ we have: $f_{x(i)}(y) \geq |\ell(i)|$ iff $\ell(i) \geq 0$]. We let, for a formal term τ , $dom(\tau) = \{x \in A : x \text{ is mentioned in } \tau\}$, $Dom(B) = \bigcup \{dom(\tau) : \tau \in B\}$.

The real restrictions are

- (5) if $z E_\gamma y$, and $x < \gamma$ then $f_x(y) = f_x(z)$.
- (6) (a) if $B \in T, x \in A$, then $p \models “x \in \tau_n^B”$ for some $\tau_n^B \in B$.
 (b) moreover, for each finite $A^1 \subseteq A$ and $h : A^1 \rightarrow \omega$, there is a $\tau \in B$ such that

$$[x \in A^1 \wedge U_x^n \text{ appears in } \tau] \Rightarrow [[0 \leq n \leq h(x)] \vee [n < -h(x)]]$$

- (c) moreover, for each $B \in T, x \in A$ and $\gamma \in \{\omega_2\} \cup Dom(B)$ satisfying $\gamma \leq x$, and a finite $A^1 \subseteq A$ and function $h : A^1 \rightarrow \omega$, there is a $\tau = \bigcap_{i < k} U_{x(i)}^{\ell(i)} \in B$ such that: $[x(i) < \gamma] \Rightarrow [x \in U_{x(i)}^{\ell(i)}]$ and $[x(i) \geq \gamma \wedge x(i) \in A^1] \Rightarrow [[0 \leq \ell(i) \leq h(y)] \vee [\ell(i) < -h(y)]]$. Note that for $\gamma = \min Dom(B)$, clause (c) reduces to clause (b) and for $\gamma = \omega_2$, clause (c) reduces to clause (a).

[Explanation: The set A approximates the set of points, the function f describes the U_x^n 's which will generate the topology as clopen sets, T is a set of “countable covers”, i.e. we think of a possible covering which is a counterexample to Lindelöfness and “promise” that a countable subfamily of such cover, will cover the entire space. In demand (6), clause (a) just says that each $B \in T$ really covers, clause (b) is necessary when we prove e.g. density of $\{p : x \in A^p\}$ for $x \in \omega_2$, (see the proof of Fact B). This is done by an increasing ω -sequence of descriptions of the important new values of f , so this clause tells us a finite information, so does not prevent us from preserving “ B is a cover”.

Still, why do we need clause (c) of demand (6)? We want that our forcing notion satisfies the \aleph_2 -c.c., so we use the Δ -system lemma, and during the construction (i.e. the proof of Fact E i.e. the construction of a common upper bound of p_1 ,

p_2) we have finitely many commitments on new values of f , we want to make $x \in \bigcup_{n < \omega} \tau_n^B$ for $B \in T^{p_j}$, $x \in A^{p_i} \setminus A^{p_j}$, so $f_y(x)$ is determined for all $y \in A^{p_i} \cap A^{p_j}$ and for finitely many $y \in A^{p_j} \setminus A^{p_i}$, and clause (6)(c) guarantees we can deal with this. We have above avoided “justifying” the use of the equivalence relation E^p , it is needed when in the proof of Fact D (Lindelöfness holds), to the union of a generic enough sequence $\langle p_n : n < \omega \rangle$ for the union q we add to the T^q a covering thus defeating a possible counterexample, we need E to verify condition (6)(c).]

If there are several p 's in consideration, we shall write A^p, f^p, \dots or $p^\ell = \langle A^\ell, f^\ell, E^\ell, T^\ell \rangle$. Now we define the order on $P : p \leq q$ iff $A^p \subseteq A^q, f^p = f^q \upharpoonright A^p, E^p = E^q \upharpoonright A^p, T^p \subseteq T^q$. In V^P we define the following topology on ω_2 . For $x \in \omega_2, n < \omega$, let $U_x^n = \{y : f_x^p(y) \geq n \text{ for some } p \text{ in the generic set}\}$, and if $n > 0$, we let $U_x^{-n} = \omega_2 \setminus U_x^n$. Now, $\{U_x^n : x \in \omega_2, n \in \mathbb{Z}\}$ will be closed and open, and the topology X is the minimal one which satisfies this. So the set of finite intersections of U_x^n 's forms a basis. By clause (2) in the definition of P , and the Fact B below, we know $\bigcap_{n < \omega} U_x^n = \{x\}$, so as each U_x^n is clopen, the space is Hausdorff and even regular, and has pseudo-character \aleph_0 .

Fact A. P is \aleph_1 -complete; in fact, any ascending ω -sequence has a naturally defined union.

In fact, we already use

Fact B. For every $p \in P$ and $z \in \omega_2$ there is a $q \in P$ such that $q \geq p$ and $z \in A^q$.

Moreover, if $z \notin A^p$, for any finite subset A^* of A^p and function h^* from A^* to \mathbb{Z} we can demand $f_x^q(z) = h^*(x)$ for $x \in A^*$.

PROOF OF B: The non-trivial part is to satisfy clause (6). We first define $f_x(z)$ for $x \in A^p$ to satisfy (6)(c) when z here stands for x there. So let $\{(B_k, A_k, h_k, \gamma_k) : k < \omega\}$ be a list of all tuples (B, A, h, γ) such that $\gamma \in \{\omega_2\} \cup \text{Dom}(B)$, $\gamma \leq z$, $B \in T^p$, $A \subseteq A^p$ is finite and $h : A \rightarrow \omega$. Now we define by induction on k a finite set $D_k \subseteq A^p$ and $f_x(z)$ for $x \in D_k$.

For $k = 0$, $D_0 = A^*$, $\bigwedge_{x \in D_0} f_x(z) = h^*(x)$. If we have defined D_k , let us define h'_k, A'_k as follows: $A'_k = A_k \cup D_k$ and $h'_k(x)$ is $h_k(x)$ if $x \in A_k$ and $f_x(z)$ if $x \in D_k \setminus A_k$; choose $\tau \in B_k$ as exists by (6)(c) (with A'_k in place of A^1 and h'_k in place of h), let $D_{k+1} = D_k \cup \text{Dom}(\tau)$, and define $f_x(z)$ for $k \in \text{Dom}(\tau) \setminus D_k$ as ℓ if U_z^ℓ appears in τ with $\ell \geq 0$ and as 0 otherwise.

We can at last complete the definition of $h_x(z)$ for $x \in A \setminus \bigcup_{k < \omega} D_k$. Lastly define $f_z(y)$ for $y \in A^p \cup \{z\}$ as ω if $y = z$ and 1 if $y \neq z$. If we let $q = \langle A \cup \{z\}, f^q \rangle$ is f expanded as described above, E^p (i.e. x is E_γ -equivalent only to itself), T^p then q is O.K. \square_B

Similarly, we can prove

Fact C.

- (1) For every $p \in P$, $z \in \omega_2 \setminus A^p$ and $\gamma \leq y$ in $A^p, \gamma \leq z$, there is a $q \in P$ such that $p \leq q$ and $q \models “zE_\gamma y”$. Moreover, for a given finite $A^1 \subseteq A^p \setminus \gamma$, and a function $h^1 : A^1 \rightarrow \omega$, we can demand $q \models “f_x(z) = h^1(x) \text{ for } x \in A^1”$.

(2) The following is a dense subset of P , closed under unions:

$$\mathcal{I} = \left\{ q \in P : \text{for every } \gamma \in A^q \cup \{\omega_2\} \text{ and finite } A \subseteq A^q \text{ and} \right. \\ \text{function } h : A \rightarrow \omega \text{ and } x \in A^q \text{ satisfying } \gamma \leq x \\ \text{there is } x' \in A^q \text{ such that } \gamma \leq x', x' E_\gamma x \text{ and} \\ \left. (\forall y \in A \setminus \gamma)[f_x(y) = h(y)] \right\}.$$

PROOF OF C: 1) like the proof of Fact B.

2) For any $p \in P$, choose $p_n \in P$, $p_0 = p$, $p_n \leq p_{n+1}$, each time use part (1) with a suitable bookkeeping and take the union by Fact A. \square_B

Fact D. The space is Lindelöf.

PROOF OF D: Let \mathcal{C} be a name of a cover and \mathcal{C}_x a member of it to which x belongs, and *w.l.o.g.* \mathcal{C}_x is a member of the basis (i.e. is a τ). Now, for each $p \in P$ and $x \in A$, there is a $q \geq p$, such that $q \Vdash \mathcal{C}_x = \tau$ for some specific τ . By Fact A for every $p \in P$ there is $q \in P$ such that

$$p \leq q \in \mathcal{I} \wedge \bigwedge_{x \in A^p} \bigvee \{q \Vdash \mathcal{C}_x = \tau : \tau \text{ as in demand (4)}\}.$$

So for every $p \in P$ we can find $\langle p_n : n < \omega \rangle$ such that $p \leq p_0$, $p_n \leq p_{n+1}$, $p_n \in \mathcal{I}$ and $x \in A^{p_n} \Rightarrow p_{n+1} \Vdash \mathcal{C}_x = \tau^x$. Let $q = \bigcup_{n < \omega} p_n$, now $q \in P$ is an upper bound of $\{p_n : n < \omega\}$ and $q \in \mathcal{I}$ by Fact C(2). Now

$$B^* = \{ \tau : q \Vdash \mathcal{C}_x = \tau \text{ for some } x \in A^q \text{ and } \tau \text{ as in (4) (for } q) \},$$

satisfies the requirements on B in clause (4). Now B^* also satisfies the requirements of part (6) in the definition: clause (a) holds as

$$x \in A^q \Rightarrow \bigvee_n x \in A^{p_n} \Rightarrow \bigvee_n (p_{n+1} \text{ forces a value to } \mathcal{C}_x).$$

Clause (c) holds by the above as $q \in \mathcal{I}$. So $q^* \stackrel{\text{df}}{=} \langle A^q, f^q, E^q, T^q \cup \{B^*\} \rangle \in P$ [why? check; the main point is (6)(c) which holds as $q \in \mathcal{I}$]. Also $q \leq q^*$, so q^* forces that B^* is (essentially) a countable subcover, as required. \square_D

Fact E. P satisfies the \aleph_2 -chain condition. (Hence in V^P , G has power $> 2^{\aleph_0}$).

PROOF: Let $p_i \in P (i < \omega_2)$. It is well known that we can assume that for some p and α_i (for $i < \omega_2$): α_i is increasing, $A^{p_i} \cap \alpha_i = A^p$, $A^{p_i} \subseteq \alpha_{i+1}$, $p \leq p_i$. Now, like in the proof of Fact B, we can prove p_0, p_1 can be extended to a condition $\langle A^{p_0} \cup A^{p_1}, f^{p_0} \cup f^{p_1}, E^{p_0} \cup E^{p_1}, T^{p_0} \cup T^{p_1} \rangle$. \square_E

$\square_{3.1}$

Remarks.

- (1) The proof works with \aleph_0 replaced by any λ such that $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$; countable is replaced by “of cardinality $\leq \lambda$ ” and τ are still finite formal intersections, $\text{Rang}(f) \leq \omega = 1$. We get λ -Lindelöf but still pseudo-character \aleph_0 .
- (2) It is well known that there is no Lindelöf space of pseudo-character \aleph_0 of power \geq “first measurable”.

§4. MORE ON ARHANGELSKII’S PROBLEM

We prove:

4.1 Theorem. *If (ZFC + GCH + \exists a weakly compact) is consistent, then so is the following: ZFC + CH + not(*) where*

- (*) *There is a regular space of cardinality \aleph_2 which is Lindelöf and has pseudo-character \aleph_0 .*

Question. (CH) Is there a Lindelöf space with pseudo-character \aleph_0 and with 2^{\aleph_1} points? More than \aleph_2 points?

PROOF: Let $V \models$ “GCH \wedge κ weakly compact.”

Let P_i ($i < \kappa^+$) be the forcing for adding a Cohen subset to ω_1 (so P_i is \aleph_1 -complete).

Q_2^κ is the Levy collapse of κ to \aleph_2 (i.e. every $\theta \in (\aleph_1, \kappa)$ is collapsed to \aleph_1 , and each condition is countable).

$Q_1 = \prod_{i < \kappa^+} P_i = \{p \in \prod_{i < \kappa^+} P_i : p \text{ has countable support (i.e. } p(i) = \emptyset \text{ for all except countably many } i)\}$.

$$Q = Q_2^\kappa \times Q_1.$$

In V^Q we know κ is \aleph_2 , CH holds, $2^{\aleph_1} = \kappa^+$ (and only the cardinals $\theta \in (\aleph_1, \kappa)$ were collapsed.)

We prove that in V^Q there is no Lindelöf space X with countable pseudo-character, such that $|X| = \kappa$.

If there is such an X , we can assume its set of points is ω_2 , and $x \in U_x^n, U_x^n$ open, $\bigcap_n U_x^n = \{x\}$ for all $x \in \omega_2$ (by the countable pseudo-character).

So the topology is the minimal one to which all U_x^n belong; this is O.K. as if we decrease family of open sets, the Lindelöfness is preserved.

We w.l.o.g. identify X with $\langle (x, n, U_x^n) : x < \omega_2, n < \omega \rangle$. So the topology is the minimal one to which all U_x^n belong; this is O.K. as if we decrease the family of open sets, the Lindelöfness is preserved. Note

\oplus if $X \in V_1 \subseteq V_2$ and $U \subseteq X$ with $U \in V_1$, then $V_1 \models$ “ U is open in X ” iff $V_2 \models$ “ U is open in X ”.

(because U is open iff $\forall x \in U \exists x_1, \dots, x_k \exists n_1, \dots, n_k (x \in \bigcap_{\ell=1}^k U_{x_\ell}^{n_\ell} \subseteq U)$)

Claim A. *In V^Q (but CH is used), for some closed unbounded $C \subseteq \kappa$, we have $\alpha \in C \wedge cf(\alpha) > \aleph_0 \Rightarrow (X \upharpoonright \alpha \text{ is not Lindelöf, moreover, there is a } g : \alpha \rightarrow \omega, \text{ so that } \{U_x^{g(x)} : x \in \alpha\} \text{ has no countable subcover).}$*

PROOF OF A: C will be the family of $\alpha < \omega_2$ such that:

if $Dom(h)$ is a countable bounded subset of α , $Range(h) \subseteq \omega, \omega_2 \neq \bigcup\{U_x^{h(x)} : x \in Dom(h)\}$ then there is a $\beta < \alpha$, $\beta \notin \bigcup\{U_x^{h(x)} : x \in Dom(h)\}$.

Clearly, C is closed and by CH it is unbounded.

If $cf(\alpha) > \aleph_0$, we can omit the “bounded”, as every countable subset of α is bounded.

For $\alpha \in C$ such that $cf(\alpha) > \aleph_0$, define $g : \alpha \rightarrow \omega$ as follows : $g(x)$ is the first $n < \omega$ such that $\alpha \notin U_x^n$ (exists as $\bigcap_{n < \omega} U_x^n = \{x\}$).

Clearly $\{U_x^{g(x)} : x < \alpha\}$ cover α . Suppose $\{U_x^{g(x)} : x < \alpha\}$ has a countable subcover $\{U_x^{g(x)} : x \in Y\}$, $|Y| \leq \aleph_0$. Let $h = g \upharpoonright Y$ and we get a contradiction to the definition of C (because α witnesses that the union is not ω_2). \square_A

Claim B. Suppose V satisfies CH and X and $U_x^n (x \in X)$ are as above. Suppose X is Lindelöf, P is an \aleph_1 -complete forcing, but in V^P , the space X is not Lindelöf.

Then also in V^{P_0} (remember that P_0 is adding one Cohen subset to ω_1), the space is not Lindelöf.

PROOF OF B: Suppose τ is a P -name for a cover contradicting Lindelöfness, wlog the cover consists of old sets. Let $p \in P$.

We define $p_\eta \in P$ for $\eta \in {}^{\omega_1} > \omega$ by induction on the length $\ell(\eta)$ of η , an old open set U_η (where $\ell(\eta)$ is a successor) such that:

- (1) $p_\eta \upharpoonright \alpha \leq p_\eta$ (\leq means “weaker than”)
- (2) $p_\eta \Vdash_P “U_\eta \in \tau”$ (for $\ell(\eta)$ a successor)
- (3) $X = \bigcup_{n < \omega} U_{\eta \hat{\ } \langle n \rangle}$.

For $\ell(\eta) = 0$, $\eta = \langle \rangle$, $p_\eta = p$, for limit only (1) applies and we use \aleph_1 -completeness. If $\eta \in ({}^\alpha \omega)$ and $p_{\eta \upharpoonright (\beta+1)}$ is defined for $\beta < \alpha$, let

$$F_\eta = \{U : U \text{ an open set of } X \text{ (in the universe } V)\}$$

and for some $q \geq p_\eta$, $q \Vdash “U \in \tau”$.

Clearly, F_η is a cover.

F_η is a cover, but X is Lindelöf, so for some countable $F' \subseteq F_\eta$ we have $X = \bigcup F'$. Let $F' = \{U_n^\eta : n < \omega\}$ (maybe with repetitions). Let $U_{\eta \hat{\ } \langle n \rangle} \stackrel{\text{def}}{=} U_n^\eta$, and let $p_{\eta \hat{\ } \langle n \rangle} \geq p_\eta$ be chosen so that $p_{\eta \hat{\ } \langle n \rangle} \Vdash_P “U_{\eta \hat{\ } \langle n \rangle} \in \tau”$.

Now we show that in V^{P_0} the space X is not Lindelöf. For a generic $g \in ({}^{\omega_1} 2)$ let σ be the family $\{U_{g \upharpoonright (\alpha+1)} : \alpha < \omega_1\}$. It is easily seen that σ is a cover of X . Suppose $X \subseteq \bigcup\{U_{g \upharpoonright (\alpha+1)} : \alpha < \beta\}$ for some $\beta < \omega_1$. Without loss of generality, β is a limit ordinal. Then $p_{g \upharpoonright \beta} \Vdash_P “\sigma \subseteq \tau”$, in contradiction with the choice of τ . \square_B

CONTINUATION OF THE PROOF OF 4.1: W.l.o.g. $y \in U_x^n$ is determined in $Q_3 = Q_2^\kappa \times \prod_{i < \kappa} P_i$. [Why? As Q satisfies the κ^+ -c.c. there are κ maximal antichains $\mathcal{I}_{y,n,x}$ of elements forcing a truth value to “ $y \in U_x^n$ ”. So $|\bigcup_{y,n,x} \mathcal{I}_{y,n,x}| \leq \kappa$, so for some α we have $\bigcup \mathcal{I}_{y,n,x} \subseteq Q_2^\alpha \times \prod_{i < \kappa} P_i$.] Also in V^{Q_3} , the space X (i.e. the space defined by letting a subset be open iff it is forced to be open, see \oplus above) is Lindelöf, noting that a cover in V^{Q_3} is also a cover in V^Q and those two universes

have the same ω -sequences of members of V^{Q_3} as no ω -sequences are added by $\Pi_{i \in [\kappa, \kappa^+)} P_i$. So, forcing by $P_0 = P_\kappa$ over V^{Q_3} does not contradict Lindelöfness, by Claim B.

As κ is weakly compact, for some stationary set S we have

$$S \subseteq \{\alpha < \kappa : \alpha \text{ strongly inaccessible}\},$$

and in addition, for each $\alpha \in S$, we can split the forcing Q to $Q'_\alpha \times Q''_\alpha$, both Q'_α and Q''_α are \aleph_1 -complete forcings, and $Q'_\alpha = Q_2^\alpha \times \Pi_{i < \alpha} P_i$ so that:

$$Q'_\alpha \text{ determines } "y \in U_x^n" \text{ for } y, x < \alpha, n < \omega,$$

and in $V^{Q'_\alpha}$, α becomes \aleph_2 . Also, the part of the space X that we get after forcing with Q'_α , X_α that is $X \upharpoonright \alpha$, is Lindelöf of pseudo-character \aleph_0 , as exemplified by the U_x^n , and adding a P_0 -generic does not contradict Lindelöfness. (Here we use the weak compactness of κ i.e. Π_1^1 -indescribability of κ .)

Now, by claim A, for some such α , in $V^{Q'_\alpha \times Q''_\alpha}$ the space X_α is no longer Lindelöf. Therefore, forcing by Q''_α abolishes Lindelöfness. Also Q'_α , Q''_α are \aleph_1 -complete, so Q''_α is \aleph_1 -complete in $V^{Q'_\alpha}$, hence (Claim B) P_0 forcing abolishes Lindelöfness, a contradiction. $\square_{4.1}$

4.2 Remark. Note that during the proof we did not use the regularity of X .

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