FPL MAY BE EQUIVALENT TO FO BUT NOT EQUIVALENT TO PFP

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Abstract. Given a class of finite models we would like to expand each model (allowing new elements but the old universe is a separate sort), making the expressive power of LFP (least fix point logic) and PFP (inductive logic) similar while not changing the expressive power of FO (first order logic). This continues in [GISh 525].

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Anotated Content

§1 The Construction

[We deal with the construction used, basically for coding $\bar{b} \in S_\ell[M_i]$ in M_i^2 we use an E_1-equivalent class, in it for each $a \in M_i$ we attach a set $B_{\langle \ell, \bar{b}, a \rangle}$, size determined by i and ℓ, and on it put a graph coding the answer if $a \notin$ Rang$(\bar{b})$ and coding b_j if $a = b_j$ (we can assume $\|M_i\| > 2\ell g(\bar{b}) > 100$). This is simpler than the previous random graph. Still inside we have the cliques of size $h(e)$ and the cycles. We show that F.O. says nothing new on M_i, while LFP logic expresses all the global relations attached. Also we essentially eliminate quantifiers showing what PFP in M_i^2 can say when restricted to M_i, the original model. So we can get: LFP, PFP expressing the same in $\{M_i^2 : i \in \mathbb{N}\}$ when restricted to $\{M_i : i \in \mathbb{N}\}$. But it is not shown: LFP, PFP are equivalent on $\{M_i^2 : i\}$ (i.e. closure under isomorphism).]

§2 Discussion

[Discuss some points.]

§3 Redoing

[Here we redo §1 somewhat differently to get the desired original question. For this $2^{h(i,\ell)} < h(i, \ell + 1)$ is really used. The main change is that the $S_\ell(M_i)$ is not a global $m(\ell)$-place relation on M_i, but a global $m(\ell)$-place function from M_i to $\mathbb{N} \restriction h_2(i, \ell + 3)$. Probably more natural is to ask $S_i(M_i)$ is a global $m(i)$ place function from M_i to $\mathbb{N} \restriction h_3(M, \ell)$ where $h_3(M, \ell) \leq h_2(i, \ell)$ but does not matter. We indicate the changes.]

§1 The Construction

⟨*1.1*⟩ *Hypothesis.*

(*a*) τ a finite vocabulary

(*b*) K a class of finite τ-models,

(*c*) $\langle S_i : i < \omega \rangle$ a list of global relations on K, or even on $K_1 = \{M :$ for some $i < \omega$ we have $M, M^1_i\}$ (see below), S_i has $m(S_i)$-places, without loss of generality $0 < m(S_i)$, for simplicity S_i is irreflexive and lastly $\langle M_j : j < \omega \rangle$ list K with no two isomorphic, i.e. this list a set of representatives

(*d*) $h : \omega \to \omega$ goes to infinite (slowly).

⟨1.2⟩ Choice: For $k, t, m < \omega$ let $G = G_{k,t,m}$ ($k \geq 10$ for simplicity so that Y below exists) be a graph such that we have $x_{\ell,j} \in G$ (for $\ell < k, j < m$, with no repetitions) and

(*a*) $\{x_{\ell,j} : \ell < k\}$ is a maximal clique

(*b*) if $j_1 = j_2 + 1$ mod m then $\langle x_{\ell,j_1} : \ell < k\rangle\hat{\ }\langle x_{\ell,j_2} : \ell < k\rangle$ has a special property (say for $\ell_1, \ell_2 < k$: $[\{x_{\ell_1,j_1}, x_{\ell_2,j}\}$ an edge $\Leftrightarrow (\ell_1, \ell_1 + k) \in Y]$ where $Y \subseteq \{0, \ldots, k-1\} \times \{k, k+1, \ldots, 2k-1\}$) is random enough: $\forall x < 2k \exists yz[(x,y) \in Y \wedge (x,z) \notin Y \wedge (z < k \equiv x \geq k)]$ and the bipartite graph $(\{0, \ldots, 2k-1\}, Y)$ is rigid

(*c*) if $i_1 \neq i_2, i_2 + 1$ mod m, then $\langle x_{\ell,i_1} : \ell < k\rangle\hat{\ }\langle x_{\ell,i_2} : \ell < k\rangle$ strongly fail the special property, say no $(x_{\ell_1,i_1}, x_{\ell_2,i_2})$ is an edge

(*d*) if $A, B \subseteq G$ are disjoint, $|A| + |B| < t$ and $[i < m \Rightarrow A$ does not contain $\{x_{\ell,i} : \ell < k\}]$, then for some $x \in G \backslash \{x_{\ell,j} : i,j\}$ we have $y \in A \Rightarrow yRx, y \in B \Rightarrow \neg yRx$.

⟨1.3⟩ Choice: We choose functions such that $h_0(i) < h(i), h_0(i)$ goes to infinity, $h_1 : \omega \to \omega, h_2(i,\ell)$ is defined for $\ell < h_0(i)$, the values are $< \log_2 \log_2 h(i)$, the functions $i \mapsto h_2(i,0)$ goes to infinity, $10 \leq h_1(\ell) < h_1(\ell+1) < h_2(i,\ell) < \log_2[h_2(i,\ell+1)]$.

⟨1.4⟩ Construction: We assume below no incidental equalities between elements occurs.

For $i < \omega, M_i \in K$ is given see ⟨1.1⟩(c), let

(*a*) $M^0_i = M_i$ and

(*b*) $M^1_i = M^0_i + \mathbb{N} \restriction h(i)$ so e.g. we have predicates $P_0, P_0^{M^1_i} = M^0_i$ and $P_1, P_1^{M^1_i} = (\mathbb{N} \restriction i) \cup M^0_i$, for R a predicate of τ let $R^{M^1_i} = R^{M^0_i}$ similarly for function symbols of τ and 0 individual constant, $+ \restriction h(i), \times \restriction h(i)$ and $(x+1) \restriction h(i)$ are, of course, $0, +, \times$ and $x+1$ restricted to $h(i)$ and $P_0, P_1 \notin \tau$

(c) M_i^2 is defined as follows:
<u>set of elements</u>: $|M_i^1| \cup \bigcup\limits_{\ell < h_0(i)} C_{i,\ell}$ where

$$B_{i,\ell} = \{\langle \ell, \bar b, a\rangle : \bar b \in {}^{m(S_\ell)}(|M_i^0|) \text{ with no repetition, i.e. } \bar b \text{ a sequence of members of } M_i^0 \text{ of length } m(S_\ell), s_1 < s_2 \Rightarrow b_{s_1} \neq b_{s_2} \text{ and } a \in M_i^0\}$$

for $u = \langle \ell, \bar b, a\rangle \in B_{i,\ell}$ we let $(k(u), t(u), m(u))$ be

$$k(u) = h_1(\ell)$$

$$m(u) = \begin{cases} m(S_i)h_2(i,\ell) & \underline{\text{if}}\ \bar b \in S_\ell[M_i] \\ m(S_\ell)h_2(i,\ell) + j + 2 & \underline{\text{if}}\ \bar b \notin S_\ell[M_1], a = b_j (j \text{ unique}) \\ m(S_\ell)h_2(i,\ell) + 1 & \underline{\text{if}}\ \bar b \notin S_\ell[M_1], a \notin \text{ Rang}(\bar b) \end{cases}$$

$$t(u) = 2k(u)m(u)$$

and let $u = \langle \ell(u), \bar b(u), a(u)\rangle$
and lastly

$$C_{i,\ell} = \{\langle u, c\rangle : c \in G_{k(u),t(u),m(u)} \text{ and } u \in B_{i,\ell}\}.$$

<u>Relations</u>:

- (α) those of M_i^1 (interpret same way)
- (β) F a partial one-place function, $\text{Dom}(F) = M_i^2 \backslash M_i^1, f((u,c)) = a(u) (\in M_i^1)$
- (γ) E_1 is an equivalence relation on M_i^2 such that $(u_1, c_1)E_1(u_2, c_2)$ iff $\ell(u_1) = \ell(u_2)$
- (δ) E_2 is an equivalence relation on M_i^2 refining E_1 : $(u_1, c_1)E_2(u_2, c_2)$ iff $\ell(u_1) = \ell(u_2)$ & $\bar b(u_1) = \bar b(u_2)$
- (ε) E_3 is the equivalence relation $(u_1, c_1), E_3(u_2, c_2)$ iff $u_1 = u_2$, so it refines E_2.

(d) Let $M_i^3 = M_i^2 \restriction (M_i^0 \cup \text{ Dom}(E^{M_i^2}))$ (i.e. we omit the arithmetic).
(e) Let for $t \leq h_0(i)$

$$M_{i,t}^4 = M_i^2 \restriction (M_i^0 \cup \text{ Dom}(E_0^{M_i^2}) \cup \mathbb{N} \restriction \tau)$$

So $M_{i,0}^4 = M_i^3, M_{i,h_0(i)}^4 = M^2$.

⟨1.5⟩ Definition. $K_\ell = \{M : M \cong M_i^\ell$ for some $i\}$ for $\ell = 0, 1, 2, 3$, so $K_0 = K$.

$$K_4 = \{M : M \cong_{M_i^2} M_{i,t}^\ell \text{ for some } t \le h_0(i) \text{ and } i < u\}.$$

⟨1.6⟩ Claim. *1) For $\ell_0 < \ell_1 \le 2$ and first order formula $\varphi(\bar x) \in \mathscr{L}_{\tau(K_{\ell_1})}$ there is a first order $\psi(\bar x) \in \mathscr{L}_{\tau(K_{\ell_0})}$ such that $M \in K_{\ell_1} \Rightarrow \varphi(\bar x)^M \restriction P_{\ell_0} = \psi(\bar x)^{(M \restriction P_{\ell_0} \restriction \tau(K_{\ell_0}))}$ equivalently $i < \omega \Rightarrow \varphi(\bar x)^{M_i^{\ell_1}} \restriction M_i^{\ell_0} = \psi(\bar x)^{M_i^{\ell_0}}$.*
2) Moreover, for every first order formula $\varphi(\bar x, \bar y) \in \mathscr{L}_{\tau(K_{\ell_0})}$ there is a first order formula $\psi(\bar x, \bar z) \in \mathscr{L}_{\tau(K_{\ell_0})}$ such that: if $M \in K_{\ell_1}, \bar c \in {}^{\ell g(\bar x)}M$, then for some $\bar d \in {}^{\ell g(\bar z)}M$ we have $\varphi(x, \bar c)^M \restriction P_{\ell_0} = \psi(\bar x, \bar d)^{M \restriction P_{\ell_0} \restriction \tau(K_{\ell_0})}$ equivalently if $i < \omega, \bar c \in {}^{\ell g(\bar y)}(M_i^{\ell_1})$ then for some $\bar d \in {}^{\ell g(\bar z)}(M_i^{\ell_0})$

$$\varphi(\bar x, \bar c)^{M_i^{\ell_1}} \restriction M_i^{\ell_0} = \psi(\bar x, \bar d)^{M_i^{\ell_0}}.$$

3) We can replace K_2 by K_4.

Proof. Enough to prove for $(\ell_0, \ell_1) = \{(0,1), (1,2)\}$.

$(\ell_0, \ell_1) = (0,1)$

By the addition theorem.

$(\ell_0, \ell_1) = (1,2)$

Let s be the quantifier depth of φ. We first look at $M_i' = M_i^2 \restriction \{x : x \in M_i^1$ or $x = (u, c), k(c) > s\}$, then all the graphs we add to M_i^1 (to get M_i') are s-random (as $G_{k,t,m}$ is Min$\{k-1, m\}$-random). Now find ψ directly or e.g. we can first replace all the graphs by copying one such graph, then add isomorphisms between them, then replace it by one graph and a representative of each E_2-equivalence class, then inspect.

The rest: going to full M_i^2. We can assume $h_2(i, 0) > s$, noting f.o. says little on circles.

Of course, we use: it is enough to find an equivalence formula for i large enough. $\square_{\langle 1.6\rangle}$

⟨1.7⟩ Claim. *1) For every ℓ, for some $\varphi(\bar x) \in LFP(\tau)$ we have $\varphi(\bar x)^{M_i^2} \restriction P_0 = \varphi(\bar x)^{M_i^2} \restriction M_i = S_\ell(M_i)$.*
2) Similarly using $M_{i,\tau}^4$.

Proof. Let $\varphi(\bar x)$ say:

there is $y \in$ Dom(E_1) such that:

- (a) in the graph $(y/E_3, R \restriction (y/E_3))$ there is a maximal clique of size $h_1(\ell)$
- (b) there are $y_0, y_1, \ldots, y_{m(S_i)-1}$ in y/E_2 pairwise not E_3-equivalent (remember "S_i is irreflexive") and for $j = 0, \ldots, m(S_i) - 1$, in the graph $(x_j/E_3, R \restriction (x_j/E_3))$ the set of maximal cliques form a cycle of length = mod $m(S_i)$ and $F(y_0) = x_0$ & $F(y_1) = x_1$ & $\ldots F(y_{\ell g(\bar x - 1)}) = x_{\ell g(\bar x - 1)}$.

This is easily expressible (well as alway we ignore the finitely many $\{M_i : h_1(\ell) < h_0(i)\}$ as it can be corrected). $\square_{\langle 1.7\rangle}$

$\langle$**1.8**$\rangle$ **Claim.** *1) If $\varphi(\bar x) \in PFP(\tau(K_2))$, then for some $\ell < \omega$ and $\psi(\bar x) \in PFP(\tau(K_1) \cup \{S_0, \ldots, S_{\ell-1}\})$ we have $M_2 \in K_2 \Rightarrow \varphi(\bar x)^{M_2} \restriction (P_1^{M_2}) = \psi(\bar x)^{M'}$ where $M_2 = (M_2 \restriction P_1 \restriction \tau(K_1), S_0^{M_2 \restriction P_0 \restriction \tau(K_0)}, \ldots, S_{\ell-1}^{M_2 \restriction P_0 \restriction (K_0)})$.*
2) For any $\varphi(\bar x, \bar y) \in PFP(\tau(K_2))$ for some $\varphi(\bar x, \bar z) \in LFP(\tau(K_1))$ we have: if $M_2 \in K_2, \bar c \in {}^{\ell g(\bar x)}(M_2)$, then for some $\bar d \in {}^{\ell g(\bar z)}(M_2)$ we have $\varphi(\bar x, \bar c)^{M_2} \restriction P_1^{M_2} = \psi(\bar x, \bar d)^{M_2 \restriction P_1 \restriction \tau(K_1)} \restriction P$.

$\langle$*1.8A*$\rangle$ *Remark.* If stuck in the middle, read $\langle 3.8\rangle$ which explicate more R_i's role.

Proof. 1), 2) We can consider a number n, a n-place predicate-variable Q and $\varphi = \varphi(Q, x_0, \ldots, x_{n-1})$ first order in the vocabulary of $\tau(K_2)$ and we would like to analyze for $M \in K_2, \langle Q^M_{\varphi,v} : v < v^M_\varphi\rangle$ where $Q^M_{\varphi,0} = \emptyset, Q^M_{\varphi,v+1} = \{\bar x : M_2 \models \varphi(Q^M_{\varphi,v}, \bar x)\}$; this will implicitly tell us how to choose φ (for (1) and for (2)). let $k(*)$ be: (the quantifier depth of $\varphi) + n + 2$. We look at $M \in K_2$. So let $M = M_i^2$. Let

$$A_i^3 = \{(u,c) \in M_i^2 \backslash M_i^1 : h_1(\ell(u)) \leq k(*)\}$$

$$A_i^4 = \{(u,c) \in M_i^2 \backslash M_i^1 : h_1(\ell(u)) \geq k(*)\}$$

if $d \in A_i^3$ we know $(d/E_2, R \restriction (x/E_2))$ is isomorphic to some G as in $\langle 1.2\rangle$ and call the cycle $\langle x^d_{j,\ell} : j < m^d, \ell < k^d\rangle$ so we demand $d' E_2 d'' \Rightarrow \bigwedge\limits_{j,\ell} x^{d'}_{j,\ell} = x^{d''}_{j,\ell}$ & $(m^{d'}, k^{d'}) = (m^{d''}, k^{d''})$.

Let us define a family $\mathscr{R}_i = \mathscr{R}(M_i^2)$ of n-place relations Q on M_i^2 such that: every function $f \in \mathscr{F}_i = \mathscr{F}[M_i^2]$ (defined below) preserve Q, i.e.

$$x_0, \ldots, x_{n-1} \in \text{ Dom}(f) \Rightarrow Q(x_0, \ldots, x_{n-1}) \equiv Q(f(x_0) \ldots f(x_{n-1})$$

where

$$\mathscr{F}_i = \Big\{ f : (a) \quad f \text{ is a partial permutation of } M_i^2,$$

(b) $f \restriction M_i^1$ is the identity

(c) $|\text{Dom } f| \leq k(*)$

(d) if $x \in (\text{Dom } f) \cap A_i^3$ then $x E_0^{M_i^2} f(x)$

(e) f is a partial isomorphism (i.e. preserve $F(-), \pm E_0, \pm E_1, \pm E_2$)

(f) we can find $\langle j_x^* : x \in (\text{Dom } f) \cap A_i^3 \rangle$ such that $x E_2 x' \Leftrightarrow j_x^* = j_{x'}^*$ $\Big\}$

and if $x \in (\text{Dom } f) \cap A_i^3$ let $x_0 = x, x_1 = f(x)$, <u>then</u> for $s \in \{0,1\}$ in x_s/E_3 letting the cycle (see ⟨1.2⟩) consist of $\langle x_{j,\ell}^s : \ell < k^s, j < m^s \rangle$ as $x_0 e_0^{M_i^2} f(x)$ necessarily $m^0 = m^1, k^0 = k^1 (\leq k(*))$ now the demand is $(\forall \ell < k^s)(\forall j < m^s)[x_0 = x_{j,\ell}^0 \leftrightarrow x_1 = s_{j+j_x,\ell}^1](\forall \ell < k^s)(\forall j < m^s)$ $[x_0 R x_{j,\ell}^0 \leftrightarrow x_1 R x_{j+j,\ell}^1]$.

Remember R is symmetric.

<u>⟨1.8B⟩ Fact</u>: $Q_{\varphi,v}^{M_i^2} \in \mathscr{R}_i$.

Proof. By induction on v.

Now each $Q \in \mathscr{R}_i$ we can code by a relation (with more than n places) on M_i^1, we need the $M_i' \restriction (P_j \backslash P_0) = \mathbb{N} \restriction h_0(i)$ to code the j_x^*'s.

⟨**1.8C**⟩ **Definition.** A coding of $Q \in \mathscr{R}_i$ is giving the following information:

$(*)_1$ for $n_1 \leq n$ and quantifier free complete type r in the variable $x_{n_1}, \dots, x_{n-1}$ and the predicate E_0, E_1, E_2, R, the n_1-place relation

$(*)_2$ $\{\bar{b}\char`^\bar{c} \in {}^{n_1}(M_i^1) : \ell g(\bar{b}) = n_1, \ell g(\bar{c}) = n - n_1$ for some $\bar{c}' \in {}^{n-n_1}(A_i^4)$, $\bar{c}$ realizes $\bar{r}, \bar{b}\char`^\bar{c} \in Q$ and $c_\ell = F(c'_\ell)\}$ (supercedes in margin)

$(*)_3$ for every division $\bar{w} = \langle w_0, w_1, w_2 \rangle$ of $\{0, \dots, n-1\}$ and quantifier free complete type $r = r(x_0, \dots, x_{n-1})$ is $\tau(K_2)$ such that

$$n' \in w_0 \Rightarrow P_1(x) \in r$$

$$n' \in w_1 \cup w_2 \Rightarrow \neg P_2(x) \in r$$

$$R^*_{\bar w,\bar r}(\bar Q)=\Big\{\bar b\hat{\ }\bar c : \ell g(\bar b)=n \text{ and for some } \bar b' \text{ we have}$$

(α) $\ell g(\bar b')=n$

(β) $m\in w_0\Rightarrow b'_m=b_m$

(γ) $m\in w_1\Rightarrow b'_m=F(b_m)\ \&\ b_m\in A^3_i$

(δ) $m\in w_2\Rightarrow b'_m=F(b_m)\ \&\ b_m\in A^4_i$

(ε) $\bar c=\langle g_m : m\in w_1\rangle$

(ζ) if $j\in w_1$, then $c_j\in P_1^{M_i^2}\backslash P_0^{M_i^2}(=M_i^1\backslash M_i^0)$ code the following two sequences (of length $k^{b_m}\times m^{b_m}$)

$\langle$truth value $(b_j=x^{b_j}_{k,m}) : k<k^{b_m}, m<m^{b_m}\rangle$

$\langle$truth value $(b_jRx^{b_j}_{k,m}) : k<k^{b_m}, m<m^{b_m}\rangle\Big\}$.

§2 Discussion

⟨2.1⟩ Question: What is the size of M_i^2?

Answer: If answer is $< \|M_i\|^{h_0(i)}$, now $h_0(i)$ could grow very slowly with i, so we are just barely above polynomial growth. But using Baire categoricity theorem and (finite) cardinality considerations, we cannot do better. Even if we like to code one global n-place relation on K, we may need to increase M_i to $\sim \|M_i\|^{n/m}$ if m is maximal arity of the new vocabulary.

⟨2.2⟩ Question: Can we use $h_2(i,\ell) = h_2(i,0)$?

Answer: I find in §1 no use of $h_2(i,\ell) < h_2(i,\ell+1)$ (or $h_2(i,\ell) \leq \log_2 h_2(i,\ell+1)$) though it had seemed to me natural, however, in §3 we shall use ?

⟨2.3⟩ Question: Can we not add $\mathbb{N} \restriction h_0(i)$?

Answer: If some S_i code a linear length $\geq h_0(i)$, then yes.
Also if some $\varphi(Q,\bar{x})$ is a in the proof of ⟨1.8⟩ and $\varphi(Q,\bar{x})$ is positive in Q, the induction on M_i, takes $\geq h_0(i)$ steps, we can easily do this.

§3 Redoing

We redo §1, so 3.x is a variant of 1.x and we say the changes
⟨**3.1**⟩ We replace clause (c) by:

(c) S_ℓ is a global functional, i.e. gives for every $M \in K$ a function S_ℓ^M from ${}^{m(S_\ell)}(M)$ to $\{0,\dots,h_2(M,\ell-1)\}, 0 < h_2(M < \ell-1) < \omega$ and we have $h_2(M_i,\ell-3) = h_2(i,\ell-1)$ (see later) and we let e.g. $h_2(M,-1) = 1 = h_2(M,-2) = h_2(M,-3)$ and

(∗) if f is an isomorphism from $M' \in K$ onto $M'' \in K$ then $(S_\ell(M'))(x_0,\dots,x_{m(S_\ell)-1}) = (S_\ell(M''))(f(x_0),\dots,f(x_{m(S_f)-1}$.

For notational simplicity $S_\ell(M)(\bar b)$ is defined only for $\bar b$ with no repetitions, $m(S_i) \geq 0$ and S_0,S_1,S_2,S_3 are trivial.

⟨3.2⟩ Minor Change: $x_{0,0}$ is connected to all $x_{\ell,i}$ for $(\ell,i) \neq (0,0)$ (make the choice of representatives clear).

⟨3.3⟩: We demand $2^{h_2(i,\ell)} \leq h_2(i,\ell+1)$ (when defined).

⟨3.4⟩: We define $M^2_{i,\ell(*)}$ and $M^2_i = M^2_{i,h_0(i)}$.
The universe of $M^2_{i,\ell}$ is

$$|M_i| \cup \bigcup \{C_{i,\ell} : \ell < \text{ Min}\{\ell(*), h_0(i)\}\}$$

where

$$B_{i,\ell} = \{\langle \ell,\bar b,a\rangle : \bar b \in {}^{m(S_\ell)}|M_i| \text{ and } a \in |M_i|\}$$

with no repetition for $u = \langle \ell,\bar b,a\rangle = \langle \ell(u),\bar b(u),a(u)\rangle \in B_{i,\ell}$ let

$$k(u) = h_1(\ell)(>0)$$

$$m[u] = \begin{cases} m(S_\ell)\times h_2(i,\ell) + (S_\ell(M_i))(\bar b) & \underline{\text{if}}\ a \notin \text{ Rang}(b[u]) \\ m(S_\ell)\times h_2(i,\ell) + j & \underline{\text{if}}\ a = b_j \end{cases}$$

(note j is unique as $\bar b$ with no repetition)

$$t(u) = 2m(u), k(u).$$

Lastly

$$C_{i,\ell} = \{(u,c) : u \in B_{i,\ell}, c \in G_{k(u),t(u),m(u)}\}.$$

Relations:

(a) those of M^0_i

(b) − (d) as before.

⟨**3.4A**⟩ **Definition.** We add

(*e*) $M^5_{i,\ell} = (M^2_i \restriction (M_i \cup \{(u,c) \in M^2_i \backslash M_i : \ell(u) \leq \ell\})$

$$M^6_{i,\ell} = M^5_{i,\ell} + \mathbb{N} \restriction h(i, \ell+3).$$

⟨**3.5**⟩ **Definition.** $K_2 = \cup\{M^2_i / \cong : i < \omega\}$

$$K^5_\ell = \cup\{M^5_{i,\ell} / \cong : i < \omega\}$$

$$K^0_\ell = \cup\{M^6_{i,\ell} / \cong : i < \omega\}.$$

⟨**3.6**⟩ **Definition.** For $(K_0, K_2), (K_0, K^5_\ell), (K^5_\ell, K^6_\ell)$ (continue as there).

—> MARTIN WARNS: Label 3.7 on next line is also used somewhere else (Perhaps should have used scite instead of stag?

⟨3.7⟩(1),(2): The meaning is transformed; see ⟨1.7⟩(3).

—> MARTIN WARNS: Label 3.7 on next line is also used somewhere else (Perhaps should have used scite instead of stag?

⟨**3.7**⟩**(3)Claim.** *For each ℓ there are a LFP formulas which interpret in M^2_i a copy of $M^0_{i,\ell}$ say by a mapping g_i, Dom$(g_i) = M^0_{i,\ell}, g_i \restriction M^5_{i,\ell} =$ the identity $g_i \restriction (\mathbb{N} \restriction h(i, \ell+3))$ has an image set of equivalence classes (using elements of $M^6_{i,\ell+3} \backslash M^6_{i,\ell+2}$. In particular for some formula $\theta_\ell(x)$ for every $i, (\forall b \in M^2_i)[d \in M^5_{i,\ell} \leftrightarrow M^2_i \models \theta_\ell[d]]$.*

Proof. Straight.

⟨**3.8A**⟩ **Claim.** *If $\varphi(\bar{x}) \in PFP(\tau(K_2))$, then for some $\ell_\varphi < \omega$ and some formula $\psi_s(\bar{x}) \in PFP(\tau(K^2_{\ell_\varphi}))(s < s^*)$ letting $\psi'_s(\bar{x}) = [\psi_s(\bar{x})]^{\theta_\ell(\omega)}$ (on $\theta_\ell(x)$ see above) (i.e. all free variables and quantifications are demanded to satisfy $\theta_\ell(-)$ for some Boolean combination $\psi(\bar{x})$ of $\psi'_s(\bar{x}), s < s^*$ and quantifier free formulas we have*

$$M \in K_2 \Rightarrow (\forall \bar{x})(\bar{\varphi}(\bar{x}) \equiv \psi(\bar{x})).$$

Proof. Like ⟨1.8⟩ just easier.

⟨**3.8B**⟩ **Claim.** *Letting n be fixed*

$$\mathscr{P}^n_i = \Big\{(Q,f) : Q \in \mathscr{R}_i \text{ a } n\text{-place relation on } M^5_{i,\ell}, f \text{ a function with } m(n)\text{-places (from the proof of } \langle 1.8\rangle\text{, implicit) from } M_i \text{ to } \mathbb{N} \restriction h_2(i,\ell+3) \text{ representing } Q\Big\}$$

give essential interpretations of $M^5_{i,\ell}$ in $M_i + \mathbb{N} \restriction h_2(i,\ell+3)$ in the sense that:

- $(*)_1$ *for every first order $\varphi(Q,\bar x)$ variable speaking on models in K^5_ℓ there is $\psi(f,\bar y,\bar z)$ first order on speaking on $\{M + \mathbb{N} \restriction h_2(i,\ell+3;M\}$*
- $(*)_2$ *if $(Q,f) \in \mathscr{P}^n_i, Q^+ = \{\bar x : \varphi(Q,\bar x)\}$ (relation) $f^+ = \{(\bar y,z) : \psi(Q,\bar y,z)\}$ (graph of the function so $\ell g(\bar y) = m_n$) <u>then</u> $(Q^+,f^+) \in \mathscr{P}^{\ell g(\bar y)}_i$*
- $(*)_3$ *for every $\psi(f,\bar y,z)$ above satisfying: if f is in $\{f : (\exists Q)(Qf) \in \mathscr{P}^n_i\}$ then so is the function defined by $\psi(f,\bar y,z)$ (i.e. it is syntactically clear not just semantically) then there is $\varphi(Q,\bar x)$ such that $(*)_2$ above holds.*

⟨**3.8**⟩ **Claim.** *Like ⟨1.8⟩.*

⟨**3.9**⟩ **Claim.** *Assume*

- $(*)$ *if $\ell(*) < \omega, \psi = \psi(\bar x,y)$ a PFP formula speaking on*

$$\{(M_i + \mathbb{N} \restriction h_2(i,\ell(*)+3), S_0(M_i),\dots,S_{\ell(*)}(M_i)) : i < \omega\}$$

giving an m-place function from M_i into $\mathbb{N} \restriction h_2(i,\ell(*)+3)$, <u>then</u> for some ℓ (generally $>> \ell(*)$), S_ℓ is ψ.

<u>Then</u> in K_2, LFP and PFP are equivalent.

Proof. Just put together all.

So we have completed the proof of the theorem originally desired.

REFERENCES.

[GISh 525] Yuri Gurevich, Neil Immerman, and Saharon Shelah. McColm Conjecture. In *Symposium on Logic in Computer Science*, pages 10–19. IEEE Computer Society Press, 1994. arxiv:math.LO/9411235.