

AXIOM OF CHOICE AND CHROMATIC NUMBER OF THE PLANE

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Question.

Define a graph U^2 on the set of all points of the plane R^2 as its vertex set, with two points adjacent iff they are distance 1 apart. The graph U^2 ought to be called *Unit Distance Plane*, and its chromatic number χ is called *chromatic number of the plane*¹.

1950 the 18-year old Edward Nelson posed the problem of finding χ (see its history in [S]). A number of relevant results were obtained under additional restrictions on monochromatic sets. K. Falconer, for example, showed [F] that χ is at least 5 if monochromatic sets are Lebesgue measurable. Amazingly though, the problem has withstood all assaults in general case, leaving us with embarrassingly wide range for χ being 4, 5, 6 or 7.

In their fundamental 1951

naturally channeled much of research in the direction of finite unit-distance graphs. One s-de-Bruijn result, however, has remained a low key: they used quite essentially the axiom of choice. So, it is natural to ask, what if we have no choice? Absence of choice in mathematics as in life may affect outcome.

We will present here an example of a distance graph on the line R , whose chromatic number depends a great deal upon the axiom of choice. While the setting of our example differs from that of chromatic number of the plane problem, the example illuminates how the value of chromatic number may be affected by inclusion or exclusion of the axiom of choice in the system of axioms for sets.

Preliminaries.

¹ Chromatic number $\chi(G)$ of a graph G is the smallest number of colors required for coloring the vertices, so that no two vertices of the same color are connected by an edge.

Let us recall basic set theoretic definitions and notations.

Axiom of Choice (AC). Every family Φ of nonempty sets has a choice function, i.e., there is a function f such that $f(S) \in S$ for every S from Φ .

Many results really need just a countable version of choice:

Countable Axiom of Choice (AC $_{\aleph_0}$). Every countable family of nonempty sets has a choice function.

In 1942 P. Bernays introduced the following axiom [B]:

Principle of Dependent Choices (DC). If E is a binary relation on a nonempty set A , and for every $a \in A$ there exists $b \in A$ with aEb , then there is a sequence $a_1, a_2, \dots, a_n, \dots$ such that $a_n E a_{n+1}$ for every $n < \omega$.

AC implies DC (see theorem 8.2 in [J], for example), but not conversely. In turn, DC implies AC $_{\aleph_0}$, but not conversely. DC is a weak form of AC and is sufficient for classical theory of Lebesgue measure. We observe that, in particular, DC is sufficient for the

As always, ZF stands for Zermelo-Fraenkel system of axioms for sets, and ZFC for Zermelo-Fraenkel with the addition of the axiom of choice. We will need the following theorem (see, for example, theorem 10.10 in [J]).

Theorem.

is consistent.

Finally, we say a set $X \subseteq R$ has the Baire property if there is an open set U such that $X \triangle U$ (symmetric difference) is meager, or of first category, i.e., a countable union of nowhere dense sets.

Example.

We define a graph G as follows: the set R of real numbers serves as the vertex set, and the set of edges is $\{(s, t) : s - t - \sqrt{2} \in Q\}$.

Claim 1: In ZFC the chromatic number of G is equal to 2.

Proof. Let $S = \{q + n\sqrt{2} : q \in Q, n \in Z\}$. We define an equivalence relation E on R as follows: $sEt \Leftrightarrow s - t \in S$.

Let Y be a set of representatives for E . For $t \in R$ let $y(t) \in Y$ be such that $tEy(t)$. We define a 2-coloring $c(t)$ as follows: $c(t) = l$, $l = 0, 1$ iff there is $n \in Z$ such that $t - y(t) - 2n\sqrt{2} - l\sqrt{2} \in Q$.

Without AC the chromatic situation changes dramatically:

Claim 2: In $ZF + AC_{\aleph_0} +$

chromatic number of the graph G cannot be equal to any positive integer n nor even to \aleph_0 .

The proof of Claim 2 immediately follows from the first of the following two statements:

1. If A_1, \dots, A_n, \dots are measurable subsets of R and $\bigcup_{n < \omega} A_n \supseteq [0, 1)$, then at least one set A_n contains two adjacent vertices of the graph G .
2. If $A \subseteq [0, 1)$ and A contains no pair of adjacent vertices of G , then A is null (Lebesgue measure zero).

Proof. We start with the proof of statement 2. Assume to the contrary that A contains no pair of adjacent vertices of G yet A has positive measure. Then there is an interval I such that

$$\frac{\mu(A \cap I)}{\mu(I)} > \frac{9}{10} \quad (1.1)$$

Choose $q \in Q$ such that $\sqrt{2} < q < \sqrt{2} + \frac{1}{10}$.

Let $B = A - (q - \sqrt{2}) = \{x - q + \sqrt{2} : x \in A\}$. Then

$$\frac{\mu(B \cap I)}{\mu(I)} > \frac{8}{10} \quad (1.2)$$

Inequalities (1.1) and (1.2) imply that there is $x \in I \cap A \cap B$. As $x \in B$, we have $y = x + (q - \sqrt{2}) \in A$. So, we have $x, y \in A$ and $x - y - \sqrt{2} = -q \in Q$. Thus, $\{x, y\}$ is an edge of the graph G with both endpoints in A , which is the desired contradiction.

The proof of the statement 1 is now obvious. Since $\bigcup_{n < \omega} A_n \supseteq [0, 1)$ and Lebesgue measure is an countably-additive function in AC_{\aleph_0} (see section 2.4.5 in [J]), there is a positive

integer n such that A_n is a non-null set of reals. By statement 2, A_n contains a pair of adjacent vertices of G as required.

Remark. We can replace ZF

Epilogue.

Is AC relevant to the problem of chromatic number χ of the plane? The answer depends upon the value of χ which we, of course, do not know yet. However, the presented here example points out circumstances in which AC would be quite relevant:

Assume that any graph on finite set of points in the plane as its vertex set, with two points adjacent iff they are distance 1 apart, has chromatic number not exceeding 4. Then:

*) due to [EB], in ZFC the chromatic number of the plane is 4;

***) due to [F], in

chromatic number of the plane is at least 5.

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