ON MODEL COMPLETION OF T_{aut}
E34

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Annotated Content

§0 Introduction

On the subject, history and background see [BlSh 759]. For a complete first order T they dealt with the existence of the model completion $T_{\rm aut}$ of $T \cup \{\sigma$ is an automorphism (for τ_T)$\}$.
We may ask:
<u>0.1 Question</u>: If T is stable and $T_{\rm aut}$ has model completion $T^{\rm mc}_{\rm aut}$, when is (every) completion of $T^{\rm mc}_{\rm aut}$ stable?

We answer in 1.6 (observation 1.7 deals with some obvious things).
Section 1 raises some question which we discuss below (assuming T stable, $T^{\rm mc}_{\rm aut}$ exists) some of which are answered below.
<u>0.2 Question</u>: 1) Can we in Claim 1.6 below replace "every completion of $T^{\rm mc}_{\rm aut}$ is stable" by "some completion of $T^{\rm mc}_{\rm aut}$ is stable"?
2) The "unstable" in 1.6 clause (a) can be replaced by "having the independence property"; but can $T^{\rm mc}_{\rm aut}$ be completed to a theory with the strict order property? The SOP$_n$'s?
3) What occurs if $T^{\rm mc}_{\rm aut}$ does not exist, can we still say something?
4) Point out that $(a)(\equiv (b))$ of 1.6 holds (for some stable T for which $T^{\rm mc}_{\rm aut}$ exists) and fails for others.
5) Show for stable T with $T^{\rm mc}_{\rm aut}$, that no completion T_* of $T^{\rm dc}_{\rm ut}$ has the explicit ncp (which means that for some first order $E(\bar x, \bar y, \bar z)$, for every n for some $\bar c \subseteq \mathfrak{C}, E(\bar x, \bar y, \bar c)$ is an equivalent relation which has $\ge n, < \aleph_0$ equivalence classes);
a stronger version is
6) For such T, T_* can T_* have obstructions (see §4)?
7) What if we use σ_1, σ_2? What about $\sigma_1, \ldots, \sigma_n$? What about pairwise commuting $\sigma_1, \ldots, \sigma_n$? This is like $(T_{\rm aut})_{\rm aut}$ for $n = 2$.
8) Is there unstable T such that $T_{\rm aut}$ has model completion? (A conjecture stating that had been the starting point of Kikyo Shelah [KkSh 748]).

<u>0.3 Discussion</u>: We prove that:

- (A) on 0.2(1), for some T (stable with $T^{\rm mc}_{\rm aut}$ existing), some completion of $T^{\rm mc}_{\rm aut}$ are stable and some are not (still we may wonder on a general characerization, see 2.7 below)
- (B) we shall show that for no such T is any completion of $T^{\rm mc}_{\rm aut}$ with the strict order property and even have NSOP$_3$, see 3.1
- (C) we can look at the class of existentially closed models of $T_{\rm aut}$ (see [ShUs 789] and references there); the results are similar.

Note

0.4 Observation. [Here?]

- (α) for $T =$ theory of equality, $T_{\rm aut}$ has a model completion and all completion of $T^{\rm mc}_{\rm aut}$ are stable
- (β) for T from 2.1, some completions of $T^{\rm mc}_{\rm aut}$ are stable and some are not
- (γ) for $T = \mathrm{Th}(M \restriction \{E, F_1, F_2, Q\}), M$ from 2.1, we get that all the completions of $T^{\rm mc}_{\rm aut}$ are unstable.

I think
0.5 Quesiton: What about getting (in §3) that

- (a) $T^{\rm mc}_{\rm aut}$ is simple in §3?
- (b) even if T is just simple, $T^{\rm mc}_{\rm aut} \models \mathrm{NSOP}_3$
- (c) non elementary class (true).

See below.

§1 On the stability of model completion for T_{aut} ($= T + \sigma$ an automorphism)

1.1 Hypothesis. 1) T is first order complete and for notational simplicity every formula is equivalent to a relation and τ_T having only predicates.
2) $\mathfrak{C}$ is the monster model of T.

1.2 Definition. 1) T_{aut} is $T \cup \{\sigma$ is an automorphism (for τ_T)$\}$, so σ is a new unary function symbol that is $T_{\mathrm{aut}} = T \cup \{(\forall x_0, \ldots, x_{n-1})[R(x_0, \ldots, x_{n-1}) \equiv R(\sigma(x_0), \ldots, \sigma(x_{n-1}))] : R$ an n-place predicate of $\tau_T\}$.
2) $T^{\mathrm{mc}}_{\mathrm{aut}}$ is the model completion, if it exists.
3) Let T_* denote any completion of $T^{\mathrm{mc}}_{\mathrm{aut}}$ and σ_* or σ^{N^+} is an automorphism.
4) A completion T_* of $T^{\mathrm{mc}}_{\mathrm{aut}}$ is cute <u>if</u> it has a model N^+ such that for some $M^+ \subseteq N^+$ we have $\sigma^{N^+} = \mathrm{id}_{N^+}$.

1.3 Definition. For T as in 0.2 let:
1) $K_{\mathrm{aut}}(T) =$ the class of models of T_{aut}.
2) $K^{\mathrm{ec}}_{\mathrm{aut}}(T) =$ the class of e.c. models of T_{aut}.
3) $K_*(T)$ is a subclass of $K^{\mathrm{ec}}_{\mathrm{aut}}(T)$ such that $M \cong N \in K_* \Rightarrow M \in K_*$ and if $M \subseteq N$ are from $K^{\mathrm{ec}}_{\mathrm{aut}}$ then $M \in K_* \Leftrightarrow N \in K_*$; there are $\leq 2^{|T|}$ such classes.
4) K_* is cute, etc.
5) $\mathfrak{C}_{\mathrm{aut}}$ is a monster model for $K^{\mathrm{ec}}_{\mathrm{aut}}$, i.e., a member of $K^{\mathrm{ec}}_{\mathrm{aut}}$ which is $\bar\kappa$-saturated of cardinality $\bar\kappa$; it is unique if $K_{\mathrm{aut}}(T)$ has the JEP.
6) A class K_* is stable[1] <u>if</u> for some $\lambda < \bar\kappa$ there is no model $M \in K_*, m < \omega, \bar a_i \in {}^m M, i < \lambda$ and q.f. formula $\varphi(\bar x, \bar y)$ which order $\{\bar a_i : i < \lambda\}$.
7) K_* is simple <u>if</u> there is a q.f. formula $\varphi(\bar x, \bar y)$ and m such that for every λ, κ we can find $M \in K_*, \bar a_\eta \subset {}^{\ell g(\bar y)}M$ for $\eta \in {}^{\kappa >}\lambda$ and $\bar b_\nu \in {}^{\ell g(\bar x)}M$ for $\nu \in {}^\kappa\lambda$ such that:

- (i) $M \models \varphi(\bar b_\eta, \bar a_{\eta \restriction \alpha})$ for $\alpha < \kappa, \eta \in {}^\kappa\lambda$
- (ii) no sequence in m realizes $\geq m$ of the formulas $\langle \varphi(\bar x, \bar a)_{\eta\hat{}<1>} : i < \lambda\}$.

On such models see [Sh 54], [xx], [xx].
<u>1.4 Fact</u>: If $T^{\mathrm{mc}}_{\mathrm{aut}}$ exists then $K^{\mathrm{ec}}_{\mathrm{aut}}(T)$ is the class of its models.

1.5 Claim. *In the claims below we can replace "T has model completion" by dealing with the class $K^{\mathrm{ec}}_{\mathrm{aut}}(T)$, and replace T^* is a model completion by dealing with K_*.*

[1] this is for classes as above, for general non first order classes this does not fit

1.6 Claim. *Let T be stable, $T^{\rm mc}_{\rm aut}$ exists. The $(a) \Leftrightarrow (b)$ where*

(a) *$T^{\rm mc}_{\rm aut}$ is stable (i.e., every completion is stable)*

(b) *if $M_0 \prec M_\ell \prec \mathfrak{C}$ for $\ell = 1,2$ and $M_1 \underset{M_0}{\bigcup} M_2$ then in $\mathfrak{C}^{\rm eq}$, $\mathrm{acl}_{\mathfrak{C}^{\rm eq}}(M_1 \cup M_2) = \mathrm{dcl}_{\mathfrak{C}^{\rm eq}}(M_1 \cup M_2)$*

(c) *$T^{\rm mc}_{\rm aut}$ is dependent (i.e., every completion does not have the independence property).*

Proof. (b) $\Rightarrow$ (a)

We work in $\mathfrak{C}^{\rm eq}$ and use observation 1.7 below. Suppose $\mathfrak{C}_* = (\mathfrak{C}, \sigma_*)$ is an expansion of $\mathfrak{C}^{\rm eq}$ to a model of $T^{\rm mc}_{\rm aut}$ and let $\sigma^{\rm eq}_*$ be the unique extension of σ_* to an automorphism of $\mathfrak{C}^{\rm eq}$. Let $\lambda = \lambda^{|T|}, M^+ \prec (\mathfrak{C}^{\rm eq}, \sigma^{\rm eq}_*), |M^+| = \lambda$ (note $|T| \geq \aleph_0$ here (by 1.1(1))).

Now for every $p \in \mathscr{S}(M^+, \mathfrak{C}^{\rm eq}_*)$ let $a_p \in \mathfrak{C}$ realize p in $(\mathfrak{C}, \sigma_*)$ and let M^+_p, N^+_p be such that

$$M^+_p \prec M^+, \|M^+_p\| = |T| + \aleph_0$$

$$M^+_p \prec N^+_p \prec \mathfrak{C}_\sigma, \|N^+_p\| = |T|$$

$$a_p \in N^+_p$$

$$N^+_p \restriction \tau_T \underset{M^+_p \restriction \tau_T}{\overset{\mathfrak{C}}{\bigcup}} M^+ \restriction \tau_T.$$

Let $A_p =$ $\mathrm{acl}_{\mathfrak{C}^{\rm eq}}(|N^+_p| \cup |M^+_p|)$. We define a two-place relation E on $\mathscr{S}(M^+, \mathfrak{C}_\sigma)$ as follows:

$\circledast$ pEq <u>iff</u> $M^+_p = M^+_q$ and there is an isomorphism f from N^+_p onto N^+_q which is the identity on M^+_p and satisfying $f_p(a_p) = a_q$.

Clearly

$\circledast_0$ E is an equivalence relation on $\mathscr{S}(M^+, \mathfrak{C}_*)$

$\circledast_1$ $|\mathscr{S}(M^+, \mathfrak{C}_*)/E| \leq \lambda^{|T|}$.

Hence it is enough to prove that

$\circledast_2$ $pEq \Rightarrow p = q$.

Proof of $\circledast_2$. Let f witness pEq,

Let $f^+ : A_p = \mathrm{dcl}_{\mathfrak{C}^{\rm eq}}(|N_p^+| \cup |M^+|) \rightarrow A_q$ extends $f \cup \mathrm{id}_M$ and be an elementary mapping (in $\mathfrak{C}^{\rm eq}$); by non forking calculus it exists and is unique. Obviously it commutes with σ_*. Also A_p (and A_q) are algebraically closed sets in $\mathfrak{C}^{\rm eq}$ by our hypothesis (that is, clause (b)) applied to $|M_p^+|, |N_p^+|, |M^+|$ hence by 1.7(4), 1.8(4) below, f^+ can be extended to an automorphism of $\mathfrak{C}^{\rm eq}$. So by properties of model completion (and the obvious 1.8(1) below) we are done.

$\neg(b) \Rightarrow \neg(a)$:

Let M_0, M_1, M_2 form a counterexample to (b). So let $e \in \mathrm{acl}_{\mathfrak{C}^{\rm eq}}(M_1 \cup M_2) \setminus \mathrm{dcl}_{\mathfrak{C}^{\rm eq}}(M_1 \cup M_2)$ hence we can find $\bar{a} \in {}^{\omega>}(M_1), \bar{b} \in {}^{\omega>}(M_2)$ and $n < \omega, \varphi(x, \bar{b}, \bar{a})$ such that

$\circledast(i)$ $\mathfrak{C}^{\rm eq} \models \varphi[e, \bar{b}, \bar{a}]$

(ii) $\models (\exists^{!n} x)\varphi(x, \bar{b}, \bar{a})$

(iii) n minimal under $(i) + (ii)$.

We know $\varphi(x, \bar{b}, \bar{a}) \vdash \mathrm{tp}(e, M_1 \cup M_2)$ and let $\{e_0, \ldots, e_{n-1}\}$ list $\varphi(\mathfrak{C}^{\rm eq}, \bar{b}, \bar{a})$.

Let $\bar{e} = \langle e_0, \ldots, e_{n-1} \rangle$. Possibly increasing $\bar{a}, \bar{b}$ for some formula $\psi = \psi(\bar{x}, \bar{b}, \bar{a})$ with $\bar{x} = \langle x_\ell : \ell < n \rangle$ we have $\models \psi(\bar{e}, \bar{b}, \bar{a})$ and $\psi(\bar{x}, \bar{b}, \bar{a}) \vdash \mathrm{tp}(\bar{e}, M_1 \cup M_2)$.
So we can find f such that

$\circledast$ f is an elementary mapping in $\mathfrak{C}$
$\mathrm{Dom}(f) = M_1 \cup M_2 \cup \bar{e}$
$f \restriction (M_1 \cup M_2)$ is the identity
$f(e_0) \neq e_0$ (but of course f permutes $\{e_\ell : \ell < n - 1\}$).

Let $f(\bar{e}) = \bar{e}'$. Let $\bar{e}_0 = \bar{e}, \bar{e}_1 = f(\bar{e})$.
We can find a sequence of $\mathfrak{C}^{\rm eq}$-elementary mapping $\langle g_i : i < |T|^+ \rangle$ such that

$$\mathrm{Dom}(g_i) = \mathrm{acl}_{\mathfrak{C}^{\rm eq}}(M_1 \cup M_2)$$

$$g_i \restriction M_2^{\rm eq} = \mathrm{id}$$

$$\bigcup_{M_2^{\rm eq}} \{\mathrm{Rang}(g_i) : i < |T|^+\}.$$

Now

$\circledast$ if $k < \omega, i_0 < \ldots < i_{k-1} < \omega$ and $\eta \in {}^n 2$ then the type $p_\eta = \operatorname{tp}(g_{i_0}(\bar{e}_{\eta(0)})\hat{\ } g_{i_1}(\bar{e}_{\eta(1)}\hat{\ } \ldots g_{i_{k-1}}(\bar{e}_{\eta(k)}), \bigcup_{i<|T|} \operatorname{Rang}(g_i))$ does not depend on η.

[Why? By induction on k, hence by transitivity of equality it is enough to prove $p_\eta = p_\nu$ when $1 = |\{\ell : \eta(\ell) \neq \nu(\ell)\}|$.

By an indiscernible sequence = indiscernible set (= symmetry of nonforking, etc.) without loss of generality $\eta(0) \neq \nu(0)$. As $\operatorname{Rang}(\bar{e}_0) = \operatorname{Rang}(\bar{e})$, without loss of generality $\bigwedge_{\ell<k-1} \eta(1+\ell) = 0 = \nu(1+\ell)$. Lastly, $\operatorname{tp}(\bigcup_{i>0} \operatorname{Rang}(g_i), \operatorname{Rang}(g_0))$ is finitely satisfiable in M_2 so by the choice of ψ we are done.]

Now for any $\eta \in {}^{(|T|^+)}2$ we define the function h_η:

$$\operatorname{Dom}(h_\eta) = M_2^{\text{eq}} \cup \cup \{g_i''(M_1^{\text{eq}}) : i < |T|^+\} \cup \{g_i(\bar{e}) : i < |T|^+\}$$

$$h_\eta \restriction M_2^{\text{eq}} = \text{ identity}$$

$$h_\eta \restriction g_i''(M_1^{\text{eq}}) = \text{ identity}$$

$$h_\eta(g_i(\bar{e})) = \begin{cases} g_i(\bar{e}) = g_i(\bar{e}_0) & \underline{\text{if}}\ \eta(i) = 0 \\ g_i(\bar{e}_1) & \underline{\text{if}}\ \eta(i) = 1 \end{cases}$$

We can find M_3, M_4, σ such that

$$\cup\{g_i(M_1) : i < |T|^+\} \subseteq M_3 \prec M_4 \prec \mathfrak{C}$$

$$M_2 \underset{M_0}{\bigcup\!\!\!\!\bigcup} M_4$$

$$M_4 \text{ is saturated of cardinality } > \|M_3\|$$

$$\sigma \in \operatorname{Aut}(M_4), \sigma \restriction M_3 = \text{ identity}$$

$$(M_4, \sigma) \text{ is a model of } T_{\text{aut}}^{\text{mc}}.$$

Now for every $\eta \in {}^{(|T|^+)}2$ we can find $(M_\eta^5, \sigma) \models T_{\text{aut}}$ such that $(M_4, \sigma) \subseteq (M_5, \sigma)$ and $\bar{b}_\eta$ realizing $\operatorname{tp}_{\mathfrak{C}^{\text{eq}}}(\bar{b}, M_0, \mathfrak{C})$ such that

$$\eta(i) = 0 \Leftrightarrow (\exists \bar x)(\psi(\bar x, \bar b_\eta, g_i(\bar a)) \wedge \sigma(\bar x) = x).$$

So $\{(\exists \bar x)(\psi(\bar x, \bar y, g_i(\bar a)) : i < |T|^+\}$ is an independent set of formulas in (M_4, σ) hence $T^{\rm mc}_{\rm aut}$ is unstable.

<u>$(a) \Rightarrow (d)$</u>:

Trivial.

<u>$\neg(b) \Rightarrow \neg(c)$</u>:

Included in the proof of $\neg(b) \Rightarrow \neg(a)$. $\square_{1.6}$

1.7 Observation. Assume $T^{\rm mc}_{\rm aut}$ exists, T_* any completion of it.
1) If $\mathfrak{C}$ is a saturated model of T of cardinality $\bar\kappa = \bar\kappa^{<\bar\kappa}$, can be expanded to a model $\mathfrak{C}_*$ of T_*.
2) If $M \models T, \sigma \in$ Aut(M), let $\sigma^{\rm eq}$ be the natural extension of σ to an automorphism of $M^{\rm eq}$, then (it exists and is unique) $(M^{\rm eq}, \sigma^{\rm eq}) \models (T^{\rm eq})_{\rm aut}$.
3) $(T^{\rm eq})_{\rm aut}$ has a model completion T and there is a natural one to one correspondence between the completions of the model completions of $(T^{\rm eq})_{\rm aut}$ and $\{T_{**} : T_{**}$ a model completion of $T^{\rm mc}_{\rm aut}\}$ any one of the former is essentially bi-interpretable with the corresponding one of the latter (but we have the elements not in any $P_{E(\bar x, \bar y)}$.
4) Let $\mathfrak{C}_* = (\mathfrak{C}, \sigma_*)$ be a $\bar\kappa$-saturated model of T_* expanding $\mathfrak{C}$. If $A_\ell \subseteq \mathfrak{C}^{\rm eq}, A_\ell =$ acl$_{\mathfrak{C}^{\rm eq}}(A_\ell), A_\ell$ closed under σ_*, f is an $\mathfrak{C}^{\rm eq}$-elementary mapping from A_1 onto A_2 commuting with σ <u>then</u> f can be extended to an automorphism of $(\mathfrak{C}^{\rm eq})_{\rm aut}$ (it is $\mathfrak{C}^{\rm eq}$ expanded by σ naturally extended to σ^+.

1.8 Observation. 1) M is a model of $T, \sigma_* \in$ Aut(M) <u>iff</u> (M, σ_*) is a model of $T_{\rm aut}$.
2) If $M \prec \mathfrak{C}$ and (M, σ_*) as a model of $T_{\rm aut}$ then for one and only one $\sigma^{\rm eq}_* \in$ Aut$(M^{\rm eq})$ extend σ_*.
3) If $M \prec \mathfrak{C}, \sigma^{\rm eq}_* \in$ Aut$(M^{\rm eq})$ then $\sigma^{\rm eq}_* \restriction M \in$ Aut(M).
4) If $A_\ell \subseteq \mathfrak{C}^{\rm eq}$ and $A_0 =$ acl$_{\mathfrak{C}^{\rm eq}}(A_0)$ and f_ℓ is an $\mathfrak{C}^{\rm eq}$-elementary mapping from A_ℓ onto A_ℓ for $\ell = 0, 1, 2$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ <u>then</u> for some automorphism F of $\mathfrak{C}^{\rm eq}, F \restriction A_0 =$ id$_{A_0}$ and $f_2 \cup (F f_1 F^{-1})$ is an elementary mapping in $\mathfrak{C}^{\rm eq}$ (hence can be extended to an automorphism of $\mathfrak{C}^{\rm eq}$; if $A_1 \bigcup\limits_{A_0} A_2$ then without loss of generality $F \restriction (A_1 \cup A_2) =$ id$_{A_1 \cup A_2}$).

§2

2.1 Example: There is T such that:

(a) T is as in 1.1, stable $T^{\rm mc}_{\rm aut}$ exists. Moreover T is superstable, countable $I(\aleph_\alpha, T) \le 2^{|\alpha|}$ for $\alpha \ge 2^{\aleph_0}$ (hence NDOP, NOTOP, shallow with small depths, with $\le 2^{\aleph_0}$ dimensions)

(b) $T^{\rm mc}_{\rm aut}$ exist

(c) some completions of $T^{\rm mc}_{\rm aut}$ are stable and some are not.

Proof. Let us define M, I

$|M|$ is $\{(\eta, k, n, \ell) : k, n < \omega, \ell < 2$ and $\eta \in {}^\omega 2\}$ and $k = n \Rightarrow \ell = 0$

E^M_n, a two-place relation is $\{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2)) \in |M| \times |M| : \eta_1 \restriction n = \eta_2 \restriction n\}$

E^M, a two-place relation is $\{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 = \eta_2\}$

Q^M, a one-place relation is $\{(\eta, k, n, \ell) \in |M| : k = n\}$

F^M_1, a one-place relation is: $F^M_1((\eta, k, n, \ell)) = (\eta, k, k, 0)$

F^M_2, a one-place relation is: $F^M_2(\eta, k, n, \ell)) = (\eta, n, n, 0)$

Let $T =$ Th(M). Clearly it satisfies (a):

$\circledast_1$ $T^{\rm mc}_{\rm aut}$ exists.
[Why? Check that there are no obstructions.]

$\circledast_2$ $T^{\rm mc}_{\rm aut}$ has an unstable completion.
[Why? By 1.6, or more specifically, see below.]

We shall now prove

$\circledast^+_2$ for T_* a completion of $T^{\rm mc}_{\rm aut}, T_*$ is unstable <u>if</u>:
for some $M^+ \models T_*$, for some $a \in M^+$ we have $\bigwedge\limits_n a E_n (\sigma^{M^+}(a))$ or just $(\exists m) \bigwedge\limits_{n<\omega} a E_n ((\sigma^{M^+})^m(a))$, i.e., for some $m^* \in [1, \omega)$ we have $\bigwedge\limits_n a E_n a_{M^+}$ where $a_0 = a, a_{\ell+1} = \sigma^{M^+}(a_\ell)$ for $\ell < \omega$.

Let $m^*, a, \langle a_\ell : \ell < \omega\rangle$ be as above. We define N a model of T: let $|N|$, the universe of N be

$$|M^+| \cup \{(m, k, n, \ell) : m < m^*, k, n < \omega, \ell < 2, k = n \Rightarrow \ell = 0\}$$

we assume no incidental identification.

$$E_n^N : \begin{cases} E_n^N \text{ is an equivalence relation} \\ E_n^N \restriction |M^+| = E_n^{M^+} \\ \text{every } (m,k,n,\ell) \in |N| \backslash |M^+| \text{ is } E_n\text{-equivalent to } a_m \end{cases}$$

$$E^N : \begin{cases} E^N \text{ is an equivalence relation} \\ E^N \restriction |M^+| = E^N \\ \{(m,k,n,\ell) \in |N| \backslash |M^+| : k,n < \omega, \ell < 2, k = n \Rightarrow \ell = 0\} \\ \quad \text{is an } E^N\text{-equivalence class (for each } m < m^*) \end{cases}$$

$$Q^N = Q^N \cup \{(m,k,k,0) : k < \omega\}$$

$$F_1^N \text{ extends } F_1^{M^+}, F_1^N((m,k,n,\ell)) = (m,k,k,0)$$

$$F_2^N \text{ extends } F^{M^+}, F_2^N((m,k,n,\ell)) = (m,n,n,0).$$

Easily

$\boxdot_1$ $M^+ \restriction \tau_T \prec N$.

Now we define an automorphism σ^+ of N:

$\boxdot_2$ $\sigma^+ \restriction |M^+| = \sigma^{M^+}$

$\boxdot_3$ if $m_1, m_2 < m^*, m_2 = m_1 + 1 \bmod m^*$ then
$\sigma(m_1, n, k, \ell)$ is:
$(m_2, n, k, 1 - \ell)$ if $m_1 = m^* - 1$ & $n < k$
(m_2, n, k, ℓ) otherwise.

Easy to check that $\sigma^+ \in$ Aut(N), so $(N, \sigma) \supseteq M^+$ is a model of $T_{\rm aut}$. As $T_{\rm aut}^{\rm mc}$ exists and $M^+ \models T_{\rm aut}^{\rm mc}$ there is a model $N^+ \models T_{\rm aut}^{\rm mc}$ such that $M^+ \prec M^+, (N, \sigma) \subseteq N^+$.

Let

$$\varphi(x,y) = Q(x) \ \& \ Q(y) \ \& \ xEy \ \& \ (\exists z)(F_1(z) \ \& \ F_2(z) = y \ \& \ (\sigma^{m^*}(z) \neq z)]$$

This is a first order formula in $\mathbb{L}(\tau_{{\rm Th}(M^+)}) = \mathbb{L}(\tau_{T_{\rm aut}})$ and $N^+ \models \varphi[b_n, b_k]$ iff $n < \omega$ where $b_n = (0, n, n, 0) \in N \subseteq N^+$, so this formula has the order property in Th$(N^+) =$ Th(M^+). So Th(M^+) is unstable as required in $\circledast_2^+$

⊛$_3$ if T_* is a completion of $T^{\rm mc}_{\rm aut}$ not satisfying the demand in ⊛$^+_2$ then T_* is stable.
[Why? As any model M^+ of T_*, σ^{M^+} acts as a permutation of $|M^+|/E^{M^+}$ which has no fix point and even no finite cycle. Now reflect.]

⊛$_4$ there is a completion T_* of $T^{\rm mc}_{\rm aut}$ which is stable.
Why? Let f be a permutation of ${}^\omega 2$ such that

- (α) $\eta, \nu \in {}^\omega 2 \wedge \eta \restriction n = \nu \restriction n \Rightarrow f(\eta) \restriction n = f(\nu) \restriction n$
- (β) for every $m < \omega$ (≥ 2) for some $n < \omega$ we have if $\eta \in {}^\omega 2$, then $\eta, f^m(\eta)$ are not E_n-equivalent.

Easy to construct (or use $\prod\limits_{n<\omega}(n+1)$ instead ${}^\omega 2$) and define M^+, a $\tau_{T_{\rm aut}}$-expansion of M by defining

$$\sigma^{M^+}((\eta, k, n, \ell)) = (f(\eta), k, n, \ell)).$$

So if $M^+ \subseteq N^+ \models T^{\rm mc}_{\rm aut}$ then $T_* = {\rm Th}(N^+)$ fail the demand in ⊛$^+_2$ hence by ⊛$_3$ it is stable as required (and it is uniquely determined by M^+, really just the action on ${\rm acl}_{\mathfrak{C}^{eq}}(\emptyset)$, suffice. So ⊛$_4$ holds. $\square_{2.1}$

2.2 Discussion: It seems reasonable that we can characterize when this occurs thus answering fully 0.1; see below.

A closely related example is

2.3 Claim. *There is T such that:*

- (a) *T is stable (complete countable first order theory) and has elimination of quantifiers for simplicity*
- (b) *T is superstable and small, i.e., with countable $D(T)$*
- (c) *$T_{\rm aut}$ has no model completion*
- (d) *some $T_{\rm aut}(M^+)$ has a model completion where*

2.4 Definition. 1) For a model $M^+ = (M, \sigma^{M^+})$ of $T_{\rm Aut}$ let $T_{\rm aut}(M^+) = T_{\rm aut} \cup {\rm Th}(M, c)_{c \in M} \cup \{\sigma(c_1) = c_2 : \sigma^{M^+}(c_1) = c_1\}$.

2.5 Remark. Actually we can use any completion of $T_{\rm aut} \cup$(the action of σ on ${\rm acl}_{\mathfrak{C}^{\rm eq}}(\emptyset, \mathfrak{C}_T)$ (i.e., on the E-equivalence classes for each n).

Proof. Define M

(a) $\tau_M = \{E_n, P_n : n < \omega\} \cup \{E, E_*\}$
(b) $|M| = \{(\eta, k, n, \ell) : \eta \in {}^\omega 2, k < \omega, n < \omega, \ell < 2\}$
(c) $E^M_n = \{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2)) \in |M| \times |M| : \eta_1 \restriction n = \eta_2 \restriction n\}$
(d) $E^M = \{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2)) \in |M| \times |M| : \eta_1 = \eta_2$ and $k_1 = k_2\}$
(e) $E^M_* = \{((\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, \eta_2, \ell_2)) \in |M| \times |M| : \eta_1 = \eta_2, k_1 = k_2, n_1 = n_2\}$
(f) $P^M_n = \{(\eta, k, n, \ell) \in |M| : n = m\}$.

We choose σ^M such that $\sigma(\eta, k, n, \ell) = (\eta', k, n, \ell)$ and (η, η') are as in the proof of 2.1.

Remark. If we let $(d)'$ be as in 2.8 below we add $\sigma =$ the identity then $(a)+(c)+(d)'$ is impossibly by [BlSh 759].

Actually the case σ is the identity on some M is the real one because

2.6 Claim. *For any first order complete T_1 (with τ_{T_1} a set of predicates for simplicity) there is T such that:*

(a) *T is first order complete*
(b) *if $a \in M, M \models T$ then we can interpret T_1 in (M, a)*
(c) *$\tau_T \backslash \tau_{T_1}$ countable*
(d) *some $T_{\rm aut}(M^+)$ has a model completion.*

Proof. As in 2.3 without $E_*, P_n(n < \omega)$ in any E^M-equivalence class we "plant" a model of T_1.

2.7 Claim. *Let T_* be a completion of $T^{\rm mc}_{\rm aut}$.*
The following are equivalent:
Condition (a): *T_* is stable.*
Condition (b): *If T is stable and $(\alpha) + (\beta) + (\gamma)$ below holds, then $(*)$ below holds where*

(α) $M^+_0 \prec M^+_\ell < M^+_3$ *for* $\ell = 1, 2, M_0 \models T_*, M_\ell \models T_{\rm aut}$ *for* $\ell = 1, 2, 3$ *and*
(β) $M_\ell = M_\ell \restriction \tau_T$ *and* $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} M_2$ *without loss of generality* $M_3 \prec \mathfrak{C} = \mathfrak{C}_T$

(γ) *if* f *is an elementary mapping from* $\mathrm{acl}_{\mathfrak{C}^{\mathrm{eq}}}(M_1 \cup M_2)$ *onto itself extending* $\sigma^{M_1^+} \cup \sigma^{M_2^+}$

(∗) *there is an elementary mapping* h *from* $\mathrm{acl}_{\mathfrak{C}}, (M_1 \cup M_2)$ *onto itself such that* $h \restriction (M_1 \cup M_2) = \mathrm{identity}_{M_1 \cup M_2}$ *and* $hfh^{-1} = \sigma^{M_3^+} \restriction \mathrm{acl}_{\mathfrak{C}^{eq}}(M_1 \cup M_2)$.

Proof. $\underline{(b) \Rightarrow (a)}$:

As in the proof of 1.6.

$\underline{\neg(b) \Rightarrow \neg(a)}$:

We can use compactness to replace $\neg(b)$ by a finite failure, and continue as in the proof of 1.6.

2.8 Remark. We can make $\neg(b)$ more explicit as in the proof of 2.7.

§3 NSOP$_3$

As by [KkSh 748], if $T^{\rm mc}_{\rm aut}$ exists, then T fails the strict order property. It seems reasonable to ask if any $T^{\rm mc}_{\rm aut}$, which exists, can have the strict order property. As we understand the stable case, it seems reasonable to deal with it. In fact, more turn out to hold.

3.1 Claim. *[T as in 1.1.] If T is stable, any completion T_* of $T^{\rm mc}_{\rm aut}$ satisfies NSOP$_3$ (see [Sh 500, §2] and [ShUs 789]).*

Proof. 1) Clause (a):

Let T_* be completion of $T^{\rm mc}_{\rm aut}$ and $\varphi(\bar x,\bar y)(\ell g(\bar x) = \ell g(\bar y) = n^* < \omega)$ a first order formula in $\mathbb{L}(\tau_{T_*})$ and for some $M \models T_*$ we have $M \models \varphi(\bar a_n, \bar a_m)^{{\rm if}(n<m)}$. Hence we can find an E.M.-template Φ such that $\tau_\Phi \supseteq \tau_{T_*} = \tau_T \cup \{\sigma\}$ and for linear orders $I \subseteq J$, ${\rm EM}(I,\Phi) \prec {\rm EM}(J,\Phi) \neq T_*$, with skeleton $\langle \bar a_t : t \in J\rangle$ such that ${\rm EM}(J,\Phi) \models \varphi[\bar a_s, \bar a_t]^{{\rm if}(s<_J t)}$ for $s,t \in J$ (so $\bar a_t \in {\rm EM}(\{t\},\Phi)$ (see, e.g., [Sh:c, VII] or [Sh:e, III]). Now (recalling that ${\rm EM}_\tau(I,\Phi) = {\rm EM}(I,\Phi) \restriction \tau$)) without loss of generality

$\circledast_1$ if $I_1, I_2 \subseteq J, I_0 = I_1 \cap I_2$ and if $t \in I_1 \backslash I_0$ then there is $s \in I_0$ such that $s < t$ & $]s,t]_J \cap I_2 \subseteq I_0$ or $t < s$ & $[t,s]_J \cap I_2 \subseteq I_0$ then ${\rm tp}_{L(\tau_{T^*})}({\rm EM}_{\tau_{T_*}}(I_1,\Phi), {\rm EM}_{\tau_{T_*}}(I_2,\Phi))$ is f.s. (finitely satisfiable) in ${\rm EM}_{\tau_{T^*}}(I_0,\Phi)$
[Why? Let $I \times \mathbb{Z}$ be ordered lexicographically, choose Φ' such that ${\rm EM}(I,\Phi') = {\rm EM}(I \times \mathbb{Z},\Phi)$, with skeleton $\bar a'_t = \bar a_{(t,0)}$; can look at [Sh 394].]

For $u \subseteq \{0,1,2\}$ let $M^2_u = {\rm EM}(u,\Phi)$ and if $|u| = |v|$ both subsets of $\{0,1,2\}$ let $f_{v,u}$ be the canonical isomorphism from M_u onto M_v. Let $M^1_u = M^2_u \restriction \tau_{T_*}, M^0_u = M^2_u \restriction \tau_T$. Let N be such that $M^0_{\{0,1,2,\}} \prec N, N$ is $\|M^0_{\{0,1,2,\}}\|^+$-saturated

$\circledast_2$ in N, $\bigcup\limits_{M^0_\emptyset} \{M^0_{\{0\}}, M^0_{\{1\}}, M^0_{\{2\}}\}$
[Why? By $\circledast_1$ and nonforking calculus.]

Let $g_0 =: f_{\{0\},\{2\}} \cup f_{\{2\},\{0\}}$

$\circledast_3$ g_0 is an elementary mapping (inside N)
[Why? Nonforking calculus.]

Let g_1 be an elementary mapping inside N extending g_0 with domain $M^0_{\{0,2\}}$.

Let $M^{0,*}_{\{0,2\}} = g(M^0_{\{0,1\}})$.

Let $M^{1,*}_{\{0,2\}}$ be an expansion of $M^{0,*}_{\{0,2\}}$ by an automorphism $\sigma^{M^{1,*}_{\{0,2\}}}$ such that g_1 is an isomorphism from $M^1_{\{0,2\}}$ onto $M^{1,*}_{\{0,2\}}$, clearly exists.

As N is a model of the stable theory T without loss of generality $\text{tp}_{L^*(\tau_T)}(|M^{1,*}_{\{0,2\}}|, |M^0_{\{0,1,2\}}|)$ does not fork over $|M^0_{\{0\}}| \cup |M^0_{\{2\}}|$.

Now the point is that

$\odot$ $h = \sigma^{M^1_{\{0,1\}}} \cup \sigma^{M^{1,*}_{\{0,2\}}} \cup \sigma^{M^1_{\{1,2\}}}$ is a permutation of $|M^{1,*}_{\{0,1\}}| \cup |M^1_{\{0,1\}}| \cup |M^1_{\{1,2\}}|$ and is an elementary mapping.
[Why? Let $B_0 = |M^0_{\{0\}}| \cup |M^0_{\{2\}}|, B_1 = |M^0_{\{0,1\}}| \cup |M^0_{\{2,2\}}|$.
By [Sh:c, XII], the pair (B_0, B_1) satisfies the T.V. condition inside N (i.e., if $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T), N \models \varphi[\bar{a}, \bar{b}], \bar{a} \subseteq B_1, \bar{b} \subseteq B_0$ then for some $\bar{a}' \subseteq B_0, N \models \varphi[\bar{a}', \bar{b}]$. Moreover, we can allow $\bar{b} \subseteq |M^{0,*}_{\{0,2\}}|$ then this follows.]

So for some $N', N \prec N' \models T$ and there is an automorphism h' of N' extending h and we can extend (N', h') to a model (N'', h'') of T_*. By this model clearly

$$(N'', h'') \models \varphi[\bar{a}_0, a_1] \text{ using } M^1_{\{0,1\}}$$

$$(N'', h'') \models \varphi[\bar{a}_1, \bar{a}_2] \text{ using } M^1_{\{1,2\}}$$

$$(N'', h'') \models \varphi[\bar{a}_2, \bar{a}_0] \text{ using } M^{1,*}_{\{0,2\}} \text{ and}$$
$$g_1 \text{ being an isomorphism from } M^1_{\{0,2\}} \text{ onto } M^{1,*}_{\{0,2\}}.$$

This is enough to show $T_* \models$ NDOP$_3$.

3.2 Claim. *T is stable or just simple <u>then</u> any T_* (assuming it exists, K_* in general) is simple.*

Proof. We write it for K_*. Choose $\kappa = \text{cf}(\kappa) > |T|$ and μ a strong limit singular cardinal of cofinality κ. Let $\langle \lambda_i : i < \kappa \rangle$ be increasing with limit $\mu, \lambda_0 > \kappa, \lambda_\kappa = \mu, \langle {}_*M^+_i : i < \kappa \rangle$ be an increasing sequence of elementary submodels of $\mathfrak{C}_{K_*}$ (check notation), $\|{}_*M^+_i\| = 2^{\lambda_i}, {}_*M^+_i$ is λ^+_i-homo universal (in $K^{\text{ec}}_{\text{aut}}(T)$), $M^+ = \cup\{{}_*M^+_i : i < \kappa\}$. Let $\langle p^+_i : i < \mu^+ \rangle$ be a sequence of existential types in $\mathbb{L}(\tau \cup \{\sigma\})$ each of cardinality $\leq \kappa$ with domain $\subseteq M$, and we shall prove that for some $\alpha < \beta < \mu^+, p^+_\alpha \cup p^+_\beta$ is realized in $\mathfrak{C}_{K_*}$, this suffices.

For each $\alpha < \mu^+$, we can find $a_\alpha \in \mathfrak{C}_{K_*}$ realizing p_i and $N^*_{3,\alpha} \prec \mathfrak{C}_{K_*}$ of cardinality κ to which a_i belongs and $N^+_{2,\alpha} = N^+_{3,\alpha} \cap M^+ \prec M^+$ and $\mathrm{tp}_{\mathfrak{C}}(|N^+_{3,\alpha}|, |M^+|)$ does not fork over $|N^+_{2,\alpha}|$. Let $N^+_{1,\alpha} \prec N^+_{3,\alpha}$ be of cardinality $|T|$ such that $a_i \in N^+_{1,\alpha}$, $\mathrm{tp}_{\mathfrak{C}}(|N^+_{1,\alpha}|, |M^+|)$ does not fork over $|N^+_{0,i}$ where $N^+_{0,i} = N^+_{1,i} \restriction M^+ \prec M^+$. Without loss of generality $\alpha < \mu^+ \Rightarrow N^+_{0,\alpha} = N^+_0$ and for every $\alpha, \beta < \mu^+$ there is an isomorphism $h_{\beta,\alpha}$ from $N^+_{3,\alpha}$ onto $N^+_{3,\beta}$ mapping $a_\alpha, N^+_{1,\alpha}, N^+_{2,\alpha}$ to $a_\beta, N^+_{1,\beta}, N^+_{2,\beta}$ respectively and $h_{\beta,\alpha} \restriction N^+_0 = \mathrm{id}_{N^+_0}$. Moreover, without loss of generality for some well ordering $<^*$ all $h_{\beta,\alpha}$ are order preserving.
Let $\kappa > \bar{\kappa}, \mathfrak{B}$ be an elementary submodel of $(\mathscr{H}(\chi), \in)$ of cardinality 2^κ such that $T, \kappa, \mu, \mathfrak{C}, \mathfrak{C}_{K_*}, M^+, \langle N^+_i : i < \mu^+ \rangle$ belongs and such that $[\mathfrak{B}] \leq \kappa \subseteq \mathfrak{B}$. Now choose $\alpha(2) \in \mu^+ \backslash \mathfrak{B}$, and let $M^+_0 = N^+_{1,\alpha} \restriction \mathfrak{B}$. Clearly $M^+_0 \prec M^+$ and there is $\alpha(1) \in \mu^+ \cap \mathfrak{B}$ such that $h_{\alpha(1),\alpha(2)}$ is the identity on M^+_0.
[FILL?]

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