Good Frames

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A prelude

I have heard about a legendary country of very honest people. When war broke, no one pointed a blaming finger against another, but each searched to the bottom of his soul and find himself guilty;

the army logistics man for not buying enough armament,

the pacifist for not lying down in the tanks way,

the generals for not urging a preemptive strike,

and the diplomats for not talking enough.

Certainly legendary, but reflecting I have discovered that I am in such a land: it seem mathematical logic is quite closed to this ideal:

- Woodin wonder how could he believed merely in $AD_{L[\mathbb{R}]}$, this is not going far enough, we should know upon deep reflection that $(*)$ is self evident

- Macintyre decided it is time that model theory should be purged from any remnants of ensemblist (= set theoretical) contamination

- Probably Jensen thinks he has not paid enough attention to fine structure of additional inner models, toward an ultimate one.

- Friedman wonder how could he had thought that his finitary principles, equiconsistent with large cardinals, are to be used merely for settling existing mathematical problems, surely this is below their dignity, of course they will be essential for the central problems of the mainstream mathematical research to be developed.

For my part I feel that the first order classification theory is not general enough, we should not work just in this restricted first order framework. Even the context of a monster model which is sequence homogeneous or a.e.c. with no maximal model are too restricted. We shall go after Abstract Elementary Classes (a.e.c., see below).

Hopefully in twenty years, like Faust of Goethe, I will find that some gold of ”suspiciously set theoretic minded, Tarski school tainted” (see a late BJSL article) has trickled through Gretchen, blessed and purified by the holy water of the high priest of the church of the Robinson school.

What are you looking for?

- To find the right context for having the parallel of the theory of superstable classes.

Why "superstable? as at present, it seem to me that maximize the "area",, i.e. (size of family of classes covered) times (how much we can say about the superstable ones (and also complement the complementary non-structure)) Now it is nice to axiomatize what we have done, but I prefer that (a)we have "hard" , "objective" criterions to check has we grind water or have gotten something substantial (b) have a parallel role in the family of our classes Minimal requirement are that

- It should shed some light on the categoricity spectrum of our class of models.
- include any $(MOD(\varphi), \prec_{\mathbb{L}_{\lambda^+,\omega}})$
- Work in ZFC or with soft additional assumptions; say by cardinal arithmetic

Review

Let us review relevant contexts; in all those discussed here the main "loss" compared to first order logic is compactness. The oldest (in this context) is the sequence homogeneous case, for a finite diagram D; here, types have the local property, i.e., a type is determined by its restriction to finite subsets of the domain, see Keisler and Morley [KM67], Shelah [Sh:3], [Sh:54], Hyttinen [Hy98], [Hy98a], Hyttinen and Shelah [HySh:629], [HySh:632], [HySh:676], Grossberg and Lessmann [GrLe0x], [GrLe0y], [GrLe0z], Lessmann [Le0x], [Le0y].

We may like to be closer to the first order case. Such cases are the models (or existentially closed models) of a universal theory or one with the amalgamation property, classes like Banach spaces and more; see [Sh:54], Iovino [Iov], Pillay [Pi0x], Ben Yaacov [BY0x]. Shelah and Usvyatsov [ShUs]

See on related research: [Sh:87a], [Sh:87b] (from $\mathbb{L}_{\omega_1,\omega}$ to excellent classes), [Sh:88], (a.e.c., mostly around $\aleph_1$) and [Sh:300], [Sh:h] (both on universal classes),

Makkai and Shelah [MaSh:285], $\mathbb{L}_{\kappa^+,\omega}, \kappa$ compact)

Grossberg and Hart [GrHa89], (excellent classes) Hart and Shelah [HaSh:323] (a counterexample) Kolman and Shelah [KlSh:362], [Sh:472], ($\mathbb{L}_{\kappa,\omega}, \kappa$ measurable), [Sh:394], (a.e.c. with amalgamation), [Sh:576], [Sh:600] (from a.e.c. to good frames), Zilber Zi0xa], [Zi0xb], Villaveces and Shelah [ShVi:635], and lately

van Dieren [Va02] (a.e.c. with no maximal models).

See also the closely related Grossberg and Shelah [GrSh:222], [GrSh:238], [GrSh:259], Grossberg [Gr91] and Baldwin and Shelah [BlSh:330], [BlSh:360], [BlSh:393].

Why not more?

Why the sudden modesty of dealing only with $\varphi \in \mathbb{L}_{\lambda^+,\omega}$ and not $\mathbb{L}_{\lambda^+,\aleph_1}$?

Because the answer to such questions will be of totally different character; (a)may well be like the " the class of cardinals in which a first order sentence has a rigid model (that is without automorphism), is essentially any Σ^1_2 class of cardinals" and (b) certainly is far from what we seek.

WHY you do not generalize the main gap theorem?

Well, probably a true gentleman should say "we should leave something for the future generation", but I have to admit that it is far beyond me at present.

WHY ZFC?

Well, first of all because those are the true axioms; and also investigation assuming $\mathbf{V} = \mathbf{L}$ or assumptions with descriptive set theoretical character will give very different information, not what we are looking for here.

Abstract Elementary Classes, a.e.c.

In short they axiomatize the obvious properties So

I. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$, K is a class of models , of a fixed vocabulary $\tau(\mathfrak{K})$ $\leq_{\mathfrak{K}}$ a partial order on it , both closed under isomorphism such that:

the two halves of Tarski-Vaught theorem

III. if $\langle M_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing then $\alpha < \delta \rightarrow M_\alpha \leq_{\mathfrak{K}} M_\delta$ where we let $M_\delta =^{\mathrm{df}} \cup\{M_\alpha : \alpha < \delta\}$

IV. above if $\alpha < \delta \Rightarrow M_\alpha \leq_{\mathfrak{k}} N$ then $M_\delta \leq_{\mathfrak{K}} N$

V If $M_i \leq \mathfrak{K} N$ for $i = 0, 1$ and $M_0 \subseteq M_1$ then $M_0 \leq \mathfrak{K} M_1$

VI. LS $(\mathfrak{K})$ exists

(Jonsson introducing mode-homogeneous universal models use such system but with **amalgamation** . , [Sh :88] suggest it as a frame to do model theory, prove that it can be represented as PC class with omitting types to has low Hanf number, Definition: We say $\mathfrak{s} = (\mathfrak{K}, NF_\lambda, \mathscr{S}_\lambda^{\mathfrak{s}}) = (\mathfrak{K}^{\mathfrak{s}}, NF_{\mathfrak{s}}, \mathscr{S}_{\mathfrak{s}}^{bs})$ is a good frame in λ or a good λ-frame (λ omitted when clear, note, that $\lambda = \lambda^{\mathfrak{s}} = \lambda(\mathfrak{s})$ is determined by $\mathfrak{s}$ and we may write $\mathscr{S}_{\mathfrak{s}}^{bs}(M)$ instead of $\mathscr{S}_{\mathfrak{K}_{\mathfrak{s}}}^{bs}(M)$) if the following conditions hold:

- (A) $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an abstract elementary class with LS($\mathfrak{K}$), the Löwenheim Skolem number of $\mathfrak{K}$, being $\leq \lambda$; there is no harm in assuming $M \in K \Rightarrow \|M\| \geq \lambda$; let $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda^{\mathfrak{s}}$ i.e. restrict ourselves to models of this cardinals only and $\leq_{\mathfrak{s}} = \leq_{\mathfrak{K}} \restriction K_\lambda$, so $K_{\mathfrak{s}} = (K_\lambda, \leq_{\mathfrak{s}})$
- (B) $\mathfrak{K}$ has a superlimit model (see below) in λ which is not $<_{\mathfrak{K}}$-maximal; e.g. is categorical in λ , which in this frame is not so strong assumption.
- (C) $\mathfrak{K}_\lambda$ has the amalgamation property, the JEP (joint embedding property), and has no $\leq_{\mathfrak{K}}$-maximal member.

- (D)

  (a) $\mathscr{S}^{bs}_{\mathfrak{s}} = \mathscr{S}^{bs}_{\lambda}$ (the class of basic types for $\mathfrak{K}_\lambda$) is included in i.e. set of 1-types (defined by chasing arrows) $\bigcup\{\mathscr{S}^{bs}_{\mathfrak{s}}(M) : M \in K_\lambda\}$ and is closed under isomorphisms including automorphisms; for $M \in K_\lambda$ let $\mathscr{S}^{bs}_{\lambda}(M) = \mathscr{S}^{bs}_{\mathfrak{s}} \cap \mathscr{S}(M)$; no harm in allowing types of finite sequences.

  (b) if $p \in \mathscr{S}^{bs}_{\mathfrak{s}}(M)$, then p is non-algebraic (i.e. not realized by any $\bar{a} \in M$).

  (c) (density)
  if $M \leq_{\mathfrak{K}} N$ are from K_λ and $M \neq N$, then for some $a \in N \backslash M$ we have
  $tp(a, M, N) \in \mathscr{S}^{bs}_{\mathfrak{s}}(M)$

  [intention: examples are: minimal types in [Sh:576], regular types for superstable theories].

  (d) $\mathfrak{s}$-stability
  $\mathscr{S}^{bs}_{\mathfrak{s}}(M)$ has cardinality $\leq \lambda$ for $M \in K_\lambda$.

(E)

- (a) NF codes, for $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_3$ from $K_\lambda, a \in M_3 \setminus M_1$ when $tp(a, M_0, M_3) \in \mathscr{S}^{bs}_{\mathfrak{s}}(M_0)$ and $tp(a, M_1, M_3)$ do not fork over M_0
- (b) (monotonicity), left to the reader:
- (c) (local character):
  if $\langle M_i : i \leq \delta + 1 \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous in $K_\lambda, a \in M_{\delta+1}$ and $\mathrm{tp}(a, M_\delta, M_{\delta+1}) \in \mathscr{S}^{bs}_{\mathfrak{s}}(M_\delta)$ then for every $i < \delta$ large enough $\mathrm{tp}(a, M_\delta, M_{\delta+1})$ does not fork over M_i.

  [Explanation: This is a replacement for superstability which says that: if $p \in \mathscr{S}(A)$ then there is a finite $B \subseteq A$ such that p does not fork over A.]
- (d) (transitivity):
  if $M_0 \leq_{\mathfrak{s}} M'_0 \leq_{\mathfrak{s}} M''_0 \leq_{\mathfrak{s}} M_3$ and $a \in M_3$ and $\mathrm{tp}(a, M''_0, M_3)$ does not fork over M'_0 and $\mathrm{tp}(a, M'_0, M_3)$ does not fork over M_0 then $\mathrm{tp}(a, M''_0, M_3)$ does not fork over M_0

- (e) uniqueness: if $p, q \in \mathscr{S}^{bs}_{\mathfrak{s}}(M_1)$ do not fork over $M_0 \leq_{\mathfrak{K}} M_1$ (all in K_λ) and $p \restriction M_0 = q \restriction M_0$ then $p = q$

- (f) symmetry: if $M_0 \leq_{\mathfrak{K}} M_3$ are in $\mathfrak{K}_\lambda$ and for $\ell = 1, 2$ we have $a_\ell \in M_3$ and $tp(a_\ell, M_0, M_3) \in \mathscr{S}^{\mathfrak{s}}(M_0)$, then the following are equivalent:

  (α) there are M_1, M_3' in K_λ such that $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_3'$, $a_1 \in M_1, M_3 \leq_{\mathfrak{K}} M_3'$ and $tp(a_2, M_1, M_3')$ does not fork over M_0

  (β) there are M_2, M_3' in K_λ such that $M_0 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3'$, $a_2 \in M_2, M_3 \leq_{\mathfrak{K}} M_3'$ and $tp(a_1, M_2, M_3')$ does not fork over M_0.

- (g) extension existence: if $M \leq_{\mathfrak{K}} N$ are from K_λ and $p \in \mathscr{S}^{bs}_{\mathfrak{s}}(M)$ then some $q \in \mathscr{S}^{bs}_{\mathfrak{s}}(N)$ does not fork over M and extends p

The Dull Canonical Misleading Example

Where can we find such frames? can we find them in "nature" or are they just a convenient choice?

Recall that if T is a first order superstable, then the class of $2^{|T|}$-saturated models of T is an a.e.c., we can choose as basic types the non algebraic complete types over a model or the regular ones. This is "cheating", but very instructive. The class of models of a superstable first order complete theory, not only is a an a.e.c. but if M is a saturated model of T of cardinality λ and we restrict ourselves to models isomorphic to it, we get an a.e.c. (ignoring models of higher cardinalities. A model with this property (in an a.e.c.) is called **superlimit** . So the main point is that (elementary) increasing chains of models isomorphic to it , if of the same cardinality, is isomorphic to it. This does not holds for saturated models only for saturated models of superstable theories (and some countable models).

This give an exaggerated picture. Though we can find good frames for first order theories iff they are superstable, most good frames do not resemble superstable at all., see later. Still, very little basic theory of stable first order classes can be carried in this context.

They are found in nature in the sense that under categoricity assumptions they are found; First we review the older results. Assuming cases of weak GCH, (we ignore the case of reasonable classes which has few models in $\aleph_1$ and concentrate on):

Theorem : ($2^{\lambda} < 2^{\lambda^+} < 2^{\lambda^{++}}$) If an a.e.c. $\mathfrak{K}$ is categorical in λ, λ^+ and has intermediate number of models in λ^{++} then (it has models in λ^{+3} and) we can find a good frame $\mathfrak{s}$ with $\mathfrak{K}_{\mathfrak{s}}$ being $\mathfrak{K}_{\lambda^+}$ (Sh:576[,[Sh:600])

Theorem : (1) (λ as above) If $\mathfrak{s}$ is a good λ-frame and there are not many models in λ^{++} [a] then there is a good λ^+-frame $\mathfrak{s}^+$ such that $\mathfrak{K}_{\mathfrak{s}^+}$ is the restriction of $\mathfrak{K}^{\mathfrak{s}}{}_{\lambda^+}$ so we can repeat this ω times ([Sh:600]), ; if we succeed we call $\mathfrak{s}$ successful (but $\mathfrak{K}_{\mathfrak{s}}$ is in general smaller than $\mathfrak{K}^{\mathfrak{s}}{}_{\lambda^+}$

(2) If we succeed we can go on, for this we redevelop superstability theory as in [Sh:a,c] in this context including regular types... ([Sh:705]) in particular categoricity is carried up (as in excellent classes; Moreover we can after adaptation get even canonical basis (weight local weight ...[Sh:F569])

because

Thesis : It is sometime good to be poor as then you have little to lose. (e.g. more freedom in choosing e.g. canonical base)

Thesis It is important that during proofs we can change our frame (as done in [Sh:600])

[a] at least one always exists

Concerning this, note

Phenomena : In this context, we may have nice superstability in one cardinal though the class is far from stable few cardinals above

Phenomena : Here as in previous cases we first analyze models which are the parallel of saturated, even more, above we end up analyzing $\lambda^{+\omega}$-saturated models

Phenomena : Looking at superstability theory as a tower, almost all tools we have disappear, formulas , compactness, (in the case discussed above also the definability of well-ordered). Some, like amalgamation, instead of being pregiven (i.e proved by previous generation of model theorists) it is parcel out according to cardinality and become a dividing line

Phenomena We have for long to keep working on new family of dividing lines before arriving to the promised land of something like superstable-like classes. E.g.

$(*)_0$ amalgamation in $\mathfrak{K}_\lambda$, i.e. fixing λ ; and

$(*)_1$ density of triples $(M, N, a) \in K^3_\lambda$ with the uniqueness property, which mean: for any given M_1 extending M ; if we like to amalgamate M_1, N over M say inside N_1 such that $\mathrm{tp}(a, M_1, N_1)$ does not fork over M then , (up to chasing arrows) this is unique

$(*)_2$ not for now

Phenomena : So though the beginning is different, when we arrive to the "higher floors" eventually the situation is similar

Phenomena : Upward categoricity [which was hard previously] follows (well, under mild set theoretic assumptions) to be eliminated.

Note that earlier the downward categoricity seem easier, hence this implies solution there (well, assuming we have suitable λ such that 2^{λ^n} is strictly increasing , will be repaired

HOPE: We can use such frames to investigate first order classes

From high categoricity to good frames

Now we begin our true subject of this lecture , [Sh:734], mostly several years old.

AIM : We would like to prove that the categoricity spectrum is large when we are high enough ; so we try only at (minimum for passing reasonable behavior)

(*) In ZFC; $\mathfrak{K}$ is categorical in every large enough cardinal or is not categorical in every large enough cardinal

[Large cardinals? not in the set theoretic sense, rather like Ackerman function in number theory. Let us define $\beth_{1,\alpha}(\lambda)$ by induction on α , $\beth_{1,0}(\lambda) = \lambda, \beth_{1,\alpha+1}(\lambda) = \beth_\gamma$ where $\gamma = (2^{\beth_{1,\alpha}})^+$

HYPO: $\mathfrak{K}$ is an a.e.c. with LS number $LS(\mathfrak{K})$ or just the class of models of $\varphi \in \mathbb{L}_{\omega_7,\omega}$ with appropriate notion of sub-model

We assume $\mathfrak{K}$ is categorical in some large enough μ and we shall try to see what occurs in λ when $LS(\mathfrak{K}) < \lambda = \beth_\lambda < \mu$ (further restrictions later)

Like many times, in the beginning the problems look like:

I understand very well how he become a billioner, just I do not understand how he has the first million".

How can we get anything at all in this context?

Well we use E.M. models and the omitting types theorem and $\mathbb{L}_{\infty,\kappa}$-complete types

[it is fortunate that the climate of Finland seem to favour them]

Specifically, the model from $\mathfrak{K}_\mu$ is existentially closed for $\mathbb{L}_{\infty,\theta}$. If $\lambda = \lambda^{<\theta}$ this is easy; in general a priory bound on the number of exceptional μ (i.e. those for which we have categoricity but no model completion).

Now, our written accomplishment is

Theorem : For $\mathfrak{K}, \mu\lambda$ as above (except for few μ of small cofinality) ; **if** $\mathrm{cf}(\lambda) = \aleph_0$:

a. there is a universal model in $\mathfrak{K}_\lambda$ hence such model has extension in all higher cardinalities

b. there is a superlimit model in $\mathfrak{K}_\lambda$

c. there is a good λ-frame $\mathfrak{s}$, such that $\mathfrak{K}_\lambda$ is categorical in λ and such that $\mathfrak{K}_\sigma$ is equal to $\mathfrak{K}_\lambda$ restricted

This support the thesis the we can find good frames in nature; in particular, not by prearrangement.

It also fit well with the program to prove categoricity in all large enough cardinals.

on the proof:

- EM does not have the order property for κ sequences, for any formulas, i.e. ask on isomorphism types
- We fix a blueprint Φ with as mush homogeneity properties as we can get
- increasing the skeleton gives us elementary extensions for those strong logics
- Def : Let ${\mathfrak{K}^*}_\theta$ be the class of EM-models for our Φ of cardinality θ with the induced order frames $\mathfrak{K}$
- $\mathfrak{K}^*_\theta$ is categorical in θ , but not a priory an a.e.c.

  Next we investigate member of ${\mathfrak{K}^*}_{\geq\theta}$ how the elementary sub-model relation behave on them when not with fitting representation, using the omitting type theorem
- under reasonable cardinality assumptions we get their being strong sub-model

- We define for those logic non-splitting
- prove the existence of sequences converging to types in those logics the sequences in a model M , the type realized in some extension

  BE AWARE: there are different interpretation of the notion of type around; we know what to expect but this does not mean it has been proved

Now come the punch line: for the cardinal $\lambda = \beth_\lambda$ trying to understand the type of a sequence of length $\kappa \in LS(\mathfrak{K})$ we need logic $\mathbb{L}_{\infty,\theta}, \theta \geq \beth_{1,1}(\kappa)$ which belong to the same interval.

Hence given any model from $\mathfrak{K}_\lambda$ we can approximate an EM-model extending it by approximations of size λ_n ending in

(*) every $M \in \mathfrak{K}_\lambda$ has elementary extension in any higher cardinal

(*) $\mathfrak{K}^*_\lambda$ has amalgamation

(*) at least if λ is the ω limit of such cardinals then $\mathfrak{K^*}_\lambda$ is closed under increasing union. In fact the two sides of Tarski-Vaught theorem; that is:

- for an increasing chain (in the sense of our a.e.c.) if the union is of the same cardinality λ, then it belong to this class ($\mathfrak{K}^*_\lambda$)

  the other side is obvious, it is

- if each member of the chain is a sub-model of N then so is the union

  So we have end the description of the proof of the main theorem of [Sh:734].

To prove that we have a good frame we use the definability of types (e.g by convergent sequences

This finish surveying the proof of the theorem above

Cheating:: A little cheating: in fact, intermediate number of models in λ^{++} does not mean $2^{\lambda^{++}}$ but a number which is quite to it and if $\lambda \geq \beth_\omega$ it is eqaul to it. But for this we sometimes have to use semi-good frames, very exciting to me but not for now.

Many years ago, when as a green Ph.D., I have visited Morley, he told me model theory of first order classes is finished.

I thought that there is still much to be done first order (though I have not been a good lawyer of this cause).

I still think so, though model theory of first order classes not not seem to be of great need of such a defender). In fact, I know that the following is self evident: [a]

[a] In practice, this usually precede highly disputed statements, of course

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Thesis a. There is not ”conflict of interest” between model theory of first order classes and more general ones; in fact they shed light one on the other

b. this is true in particular for classification theory; including stability theory

c. similarly , the structure part of classification theory (from forking to local weight) is the other face of Janus, the other one being the non structure part which naturally use stationary sets, many non isomorphic model or other criterion’s of complicatedness