WHAT MAJORITY DECISIONS ARE POSSIBLE
E37

Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Mathematics Department
New Brunswick, NJ USA

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Annotated Content

§0 Introduction

Condorcet's "paradox" demonstrates that given three candidates A, B and C, the majority rule may result in the society preferring A to B , B to C and C to A. McGarvey [McG53] proved a far-reaching extension of Condorcet's paradox: For every asymmetric relation R on a finite set M of candidates there is a strict-preferences (linear orders, no ties) voter profile that has the relation R as its strict simple majority relation. In other words, for every assymetric relation (equivalently, a tournament) R on a set M of m elements there are n linear order relations on M, $R_1, R_2, \dots, R_n$ such that for every $a, b \in M$, aRb if and only if

$$|\{i : aR_ib\}| > n/2.$$

McGarvey's proof gave $n = m(m-1)$. Stearns [Ste59] found a construction with $n = m$ and noticed that a simple counting argument implies that n must be at least $m/\log m$. Erdős and Moser [ErMo64] were able to give a construction with $n = O(m/\log m)$. Alon [Alo02] showed that for some constant $c_1 > 0$ we can find $R_1, \dots, R_n$ with

$$|\{i : aR_ib\}| > (1/2 + c_1/\sqrt{n})n,$$

and that this is no longer the case if c_1 is replaced with another constant $c_2 > c_1$.

Gil Kalai asked to what extent the assertion of McGarvey's theorem holds if we replace the set of order relations by an arbitrary isomorphism class of choice functions on pairs of elements. Namely, the question is to characterize under which conditions clause (A) of 0.1 below holds (i.e., question 1.4).
Instead of choice functions we can speak on tournaments.

The main result is (follows by 2.1)

0.1 Theorem. *Let X be a finite set and $\mathfrak{D}$ be a non empty family of choice functions for $\binom{X}{2}$ closed under permutation of X. Then the following conditions are equivalent:*

(A) *for any choice function c on $\binom{X}{2}$ we can find a finite set J and $c_j \in \mathfrak{D}$ for $j \in J$ such that for any $x \neq y \in X$:*

$$c\{x,y\} = y \ \underline{iff}\ |J|/2 < \{j \in J : c_j\{x,y\} = y\}$$

(so equality never occurs)

(B) *for some $c \in \mathfrak{D}$ and some $x \in X$ we have $|\{y : c\{x,y\} = y\}| \neq (|X|-1)/2$.*

Gil Kalai further asks
0.2 Question: 1) In 0.1 can we bound $|J|$?
2) What is the result of demanding a "nontrivial majority"? (say 51%?)

Under 0.1 it seems reasonable to ask to characterize the two place relation $\mathbf{R}$ on the family of choice for $\binom{X}{2}$, see 1.5. We then give a complete solution: what is the closure of a set of choice functions by majority; in fact, there are just two possibilities (see 3.7).
I thank Gil for the simulating discussion and writing the historical background.
See later version [Sh 816].

0.3 Notation: Let n, m, k, ℓ, i, j denote natural numbers.
Let r, s, t, a, b denote real numbers.
Let x, y, z, u, v, w denote members of the finite set X.
Let $\binom{X}{k}$ be the family of subsets of X with exactly k members.
Let c, d denote choice functions on $\binom{X}{2}$.
Let $\text{conv}(A)$ be the convex hull of A, here for $A \subseteq \mathbb{R} \times \mathbb{R}$.
Let $\text{Per}(X)$ be the set of permutations of X.

§1 Basic definitions and facts

1.1 Hypothesis. Assume

(a) X is a (fixed) finite set with $\mathbf{n} \geq 3$ members, i.e. $\mathbf{n} = |X|$

(b) $\mathfrak{C} = \mathfrak{C}_X$ is the set of choice functions on $\binom{X}{2}$

(c) $\mathscr{C}, \mathscr{D}$ vary on non-empty subsets of $\mathfrak{C}, \mathfrak{D}$ vary on nonempty subsets of $\mathfrak{C}$ which are symmetric where

1.2 Definition. 1) $\mathscr{C} \subseteq \mathfrak{C}$ is symmetric if it is closed under permutations of X (i.e. for every $\pi \in \mathrm{Per}(X)$ the permutation $\hat{\pi}$ maps $\mathscr{C}$ onto itself where π induces $\hat{\pi}$, a permutation of $\mathfrak{C}$, that is we can write $c_1 = c_2^\pi$ or $c_1 = \hat{\pi} c_2$ where: $x_1 = \pi(x_2), y_1 = \pi(y_2)$ implies $c_1\{x_1, y_1\} = y_1 \Leftrightarrow c_2\{x_2, y_2\} = y_2$).

1.3 Definition. For $\mathscr{D} \subseteq \mathfrak{C}$ let maj-$cl(\mathscr{D})$ be the set of $d \in \mathfrak{C}$ such that for some real numbers $r_c \in [0,1]_{\mathbb{R}}$ for $c \in \mathscr{D}$ satisfying $\sum\limits_{c \in \mathscr{D}} r_c = 1$ we have[1]

$$d(\{x,y\}) = x \Leftrightarrow \frac{1}{2} < \Sigma\{r_c : c\{x,y\} = x \text{ and } c \in \mathscr{D}\}$$

(so the sum is never $\frac{1}{2}$).

So Kalai's question was

<u>1.4 Question</u>: For $|X|$ large enough for which symmeric $\mathfrak{D} \subseteq \mathfrak{C}$ do we have maj-cl$(\mathscr{D}^*) = \mathfrak{C}$?

1.5 Definition. 1) Let Dis $=$ Dis$(X) = \{\mu : \mu$ a distribution on $\mathfrak{C}_X\}$; of course, "μ a distribution on $\mathfrak{C}$" means μ is a function from $\mathfrak{C}$ into $[0,1]_{\mathbb{R}}$ such that $\Sigma\{\mu(c) : c \in \mathfrak{C}\} = 1$.
2) For $\mathscr{C} \subseteq \mathfrak{C}$ and $\mu \in \mathrm{Dis}(\mathfrak{C})$ let $\mu(\mathscr{C}) = \Sigma\{\mu(c) : c \in \mathscr{C}\}$ so $\mu(\{\mathscr{C}\}) \geq 0, \mu(\mathfrak{C}) = 1$.
3) For $\mathscr{D} \subseteq \mathfrak{C}$ let $\mathrm{Dis}_{\mathscr{D}} = \{\mu \in \mathrm{Dis} : \mu(\mathscr{D}) = 1\}$.
4) Let $\mathrm{pr}(\mathfrak{C}) = \{\bar{t} : \bar{t} = \langle t_{x,y} : x \neq y \in X\rangle$ such that $t_{x,y} \in [0,1]_{\mathbb{R}}$ and $t_{y,x} = 1 - t_{x,y}\}$, we may write $\bar{t}(x,y)$ instead of $t_{x,y}$; pr stands for probability.
5) For $T \subseteq \mathrm{pr}(\mathfrak{C})$ let pr-cl(T) be the convex hull of T.
6) For $d \in \mathfrak{C}$ let $\bar{t}[d] = \langle t_{x,y}[d] : x \neq y \in X\rangle$ be defined by $[t_{x,y}[d] = 1 \Leftrightarrow d\{x,y\} = y \Leftrightarrow t_{x,y}[d] \neq 0]$.
7) Let pr-cl$(\mathscr{D})$ for $\mathscr{D} \subseteq \mathfrak{C}$ be pr-cl$(\{\bar{t}[d] : d \in \mathscr{D}\})$, prd$(\mathscr{D}) = \{\bar{t}[c] : c \in \mathscr{D}\}$.
8) For $\mathscr{C} \subseteq \mathfrak{C}$ we let sym-$c\ell(\mathscr{C})$ be the minimal $\mathscr{D} \subseteq \mathfrak{C}$ which is symmetric and

[1] note that there is no reason to assume that $\mathscr{D}_2 = \mathrm{maj} - c\ell(\mathscr{D}_1)$ implies $\mathscr{D}_2 = \mathrm{maj} - c\ell(\mathscr{D}_2)$

includes $\mathscr{C}$.
9) For $T \subseteq \text{pr}(\mathfrak{C})$ let $\text{maj}(T) = \{c \in \mathfrak{C}$: for some $\bar{t} \in T$ for any $x \neq y$ from X we have $c\{x,y\} = y \Leftrightarrow t_{x,y} > \frac{1}{2} \Leftrightarrow t_{x,y} \geq \{\frac{1}{2}\}\}$ and for $\mathscr{D} \subseteq \mathfrak{C}$ let $\text{maj-cl}(\mathscr{D}) = \{\text{maj}(\bar{t}) : \bar{t} \in \text{pr-cl}(\mathscr{D})\}$ where for $\bar{t} \in \text{pr}(\mathfrak{C}), c = \text{maj}(\bar{t})$ if $c\{x,y\} = y \Leftrightarrow t_{x,y} > \frac{1}{2}$ (so we assume $x \neq y \Rightarrow t_{x,y} \neq \frac{1}{2}$.

1.6 Claim. *1) For $d \in \mathfrak{C}$ we have $\bar{t}[d] \in \text{pr}(\mathfrak{C})$.*
2) For $\mathscr{D} \subseteq \mathfrak{C}$ we have $\text{Dis}_{\mathscr{D}} \subseteq \text{Dis}$.
3) $\text{prd}(\mathfrak{C}) \subseteq \text{pr}(\mathfrak{C})$ and if $\mathscr{D} \subseteq \mathfrak{C}$ then $\text{prd}(\mathscr{D}) \subseteq \text{pr-cl}(\mathscr{D}) \subseteq \text{Dis}$.
4) If $\mathscr{C} \subseteq \mathfrak{C}$ then $\mathscr{C} \subseteq \text{sym} - c\ell(\mathscr{C}) \subseteq \mathfrak{C}$.
5) If $T \subseteq \text{pr}(\mathfrak{C})$ <u>then</u> $\text{maj}(T) \subseteq \mathfrak{C}$.
6) For $\mathscr{D} \subseteq \mathfrak{C}$ we have $\text{maj} - c\ell(\mathscr{D}) = \text{maj}(\text{pr-cl}(\mathscr{D}))$.

Proof. Obvious.

1.7 Claim. *(G. Kalai) If $\mathscr{C} \subseteq \mathfrak{C}$ and for every $c \in \mathscr{C}$ and $x \in X$, the in-valency and out-valency are equal, (i.e., $\text{val}_c(x) = (|X| - 1)/2$, see below) <u>then</u> every $d \in \text{maj-cl}(\mathscr{C})$ satisfies:*

- $(*)$ *if $\emptyset \neq Y \subsetneqq X$ then there are edges from Y to $X \backslash Y$ and from $X \backslash Y$ to Y, see 3.2(1),(2).*

1.8 Definition. 1) For $d \in \mathfrak{C}$ and $x \in X$ let $\text{val}_d(x)$, the valency of x for d, be $|\{y : y \in X, y \neq x \text{ and } d\{x,y\} = y\}|$ so $\text{val}_d(x) \in \{0, \ldots, |\mathbf{n}| - 1\}$.
We also call $\text{val}_d(x)$ the out-valency[2] of x in d and $\mathbf{n} - \text{val}_d(x) - 1$ the in-valency of x in d.
2) For $d \in \mathfrak{C}$ let $\text{Val}(d) = \{\text{val}_d(x) : x \in X\}$.
3) For $d \in \mathfrak{C}$ and $\ell \in \{0,1\}$ let $V_\ell(d) = \{(\text{val}_d(x),\text{val}_d(y)) : x \neq y \in X$ and $d\{x,y\} = y \Leftrightarrow \ell = 1\}$.
4) For $d \in \mathfrak{C}$ and $\ell \in \{0,1\}$ let $V^*_\ell(d) = \{\bar{k} - (\ell, 1 - \ell) : \bar{k} \in V_\ell(d)\}$ and let $V^*(d) = V^*_0(d) \cup V^*_1(d)$.
5) For $c \in \mathfrak{C}$ let $\text{dual}(c) \in \mathfrak{C}$ be $\text{dual}(c)\{x,y\} \in \{x,y\} \backslash \{c\{x,y\}\}$; similarly $\bar{t}' = \text{dual}(\bar{t})$ for $\bar{t} \in \text{pr}(\mathfrak{C})$ means that $t'_{x,y} = 1 - t_{x,y}$.

[2]natural under the tournament interpretation

1.9 Claim. *1)*

(α) $c_1 \in \text{sym-cl}\{c_2\}$ iff $\text{dual}(c_1) \in \text{sym-cl}\{\text{dual}\{c_2\}\}$

(β) $c_1 \in \text{maj-cl}(\text{sym-cl}\{c_2\})$ iff $\text{dual}(c_1) \in \text{maj-cl}(\text{sym-cl}\{\text{dual}(c_2)\})$.

2) $(k_0, k_1) \in V_0(d) \Rightarrow (k_1, k_0) \in V_1(d)$.
3) "$c_2 \in \text{sym-cl}\{c_2\}$" *is an equivalence relation on* $\mathfrak{C}$ *and it implies* $V_\ell(c_1) = V_\ell(c_2)$ *for* $\ell = 0, 1$.

Proof. Easy.

§2 When every majority choice is possible: a characterization

The following is the main part of the solution (probably $(c) \Leftrightarrow (g)$ should be stated as the main conclusion here).

2.1 Main Claim. *Assume that $\mathfrak{D} \subseteq \mathfrak{C}$ which is symmetric and not empty, (i.e., $\mathfrak{D}$ is a set of choice functions on $\binom{X}{2}$ closed under permutation on X, $\mathfrak{D} \neq \emptyset$) and for simplicity assuming that $\mathfrak{D} = \text{sym-cl}(d^*)$ for any $d^* \in \mathfrak{D}$, the following conditions on $\mathfrak{D}$ are equivalent, where x, y vary on distinct members of X:*

(a) $\text{maj-cl}(\mathfrak{D}) = \mathfrak{C}$

$(b)_{x,y}$ *there is* $\bar{t} \in \text{pr-cl}(\mathfrak{D}) \subseteq \text{pr}(\mathfrak{C})$ *such that*

- (i) $t_{x,y} > \frac{1}{2}$
- (ii) $\{x,y\} \neq \{u,v\} \in \binom{X}{r} \Rightarrow t_{u,v} = \frac{1}{2}$

(c) *for any $c \in \mathfrak{C}$ we can find a finite set J and sequence $\langle d_j : j \in J \rangle$ such that $d_j \in \mathfrak{D}$ and: if $u \neq v \in X$ then*
$c\{u,v\} = v \Leftrightarrow (\{j \in J : d_j\{u,v\} = j\}| > |J|/2$

(d) $(\frac{1}{2}, \frac{1}{2})$ *belongs to* $\text{Pr}_{>\frac{1}{2}}(\mathfrak{D})$, *see Definition 2.2 below*

(e) $(\frac{1}{2}, \frac{1}{2}) \in \text{Pr}_{\neq 1/2}(\mathfrak{D})$

(f) $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ *can be represented as* $r^*_0 \times \bar{s}_0 + r^*_1 \times \bar{s}_1$ *where*

- $(*)(i)$ $r^*_0, r^*_1 \in [0,1]_{\mathbb{R}} \backslash \{\frac{1}{2}\}$
- (ii) $1 = r^*_0 + r^*_1$
- (iii) *for $\ell = 0, 1$ the pair $\bar{s}_\ell$ belongs to the convex hull of $V^*_\ell(d^*)$ for some $d^* \in \mathfrak{D}$, see Definition 1.8(3), but recall that by a hypothesis of the claim, the choice of d^* is immaterial*

(g) *for some ($d^* \in \mathfrak{D}$ and) $x \in X$ we have* $\text{val}_{d^*}(x) \neq \frac{\mathbf{n}-1}{2}$.

Proof. $(b)_{x,y} \Leftrightarrow (b)_{x',y'}$:

(So $x, y, x', y' \in X$ and $x \neq y, x' \neq y'$). Trivial as $\mathfrak{D}$ is closed under permutations of X.

$(b)_{x,y} \Rightarrow (a)$:

Let $c \in \mathfrak{C}^*$.

Let $\{(u_i, v_i) : i < i(*)\}$ list the pairs (u,v) of distinct members of X such that

$c\{u,v\} = v$ so $i(*) = \binom{|X|}{2}$. For each $i < i(*)$ as $(b)_{x,y} \Rightarrow (b)_{u_i,v_i}$ clearly there is $\bar{t}^i \in$ pr-cl$(\mathfrak{D}^*)$ such that

$$t^i_{u_i,v_i} > \frac{1}{2} \text{ so } t^i_{v_i,u_i} = 1 - t_{u_i,v_i} < \frac{1}{2}$$

$$\{u_i, v_i\} \neq \{u,v\} \in \binom{X}{2} \Rightarrow t^i_{u,v} = \frac{1}{2}.$$

Let $\bar{t}^* = \langle t^*_{u,v} : u \neq v \in X\rangle$ be defined to

$$t^*_{u,v} = \Sigma\{t^i_{u,v} : i < i(*)\}/i(*).$$

As pr-cl$(\mathfrak{D})$ is convex and $i < i(*) \Rightarrow \bar{t}^i \in$ pr-cl$(\mathfrak{D})$ clearly $\bar{t} \in$ pr-cl$(\mathfrak{D})$. Now for each $j < i(*), t^i_{u_j,v_j}$ is $\frac{1}{2}$ if $i \neq j$ and is $> \frac{1}{2}$ if $i = j$. Hence $t^*_{u_j,v_j}$ being the average of $\langle t^i_{u_j,v_j} : i < i(*)\rangle$ is $> \frac{1}{2}$. Hence $t^*_{v_j,u_j} = 1 - t^*_{u_j,v_j} < \frac{1}{2}$. So $t^*_{u,v} > \frac{1}{2}$ iff $t^*_{u,v} \geq \frac{1}{2}$ iff $c\{u,v\} = v$ by the choice of $\langle (u_i,v_i) : i < i(*)\rangle$. So $\bar{t}^*$ witness $c \in$ maj-cl$(\mathfrak{D})$ as required in clause (a).

$(a) \Rightarrow (b)_{x,y}$:

By clause (a), for every $d \in \mathfrak{C}$ there is $\langle r_c : c \in \mathfrak{D}\rangle$ as in Definition 1.3, hence for some $\varepsilon_d > 0, d(\{x,y\}) = y \Rightarrow \frac{1}{2} + \varepsilon_d < \Sigma\{r_c : c \in \mathfrak{D}$ and $c\{x,y\} = y\}$. Hence $\varepsilon =$ Min$\{\varepsilon_d : d \in \mathfrak{D}\}$ is a real > 0.
Let $T = \{\bar{t} : \bar{t} \in$ pr-cl$(\mathfrak{D})$ and $t_{x,y} \geq \frac{1}{2} + \varepsilon$ and $u \neq v \in X \Rightarrow t_{u,v} \leq t_{x,y}\}$, so

$(*)_1$ $T \neq \emptyset$
[Why? As $\mathfrak{D}$ is symmetric clearly $\mathfrak{D}_{x,y} = \{c \in \mathfrak{D} : c\{x,y\} = y\}$ is non empty, and $\bar{t}[d] \in T$ for every $d \in \mathfrak{D}_{x,y}$]

$(*)_2$ T is convex and closed.

[Why? Trivial.]
For $\bar{t} \in T$ define

$\boxtimes$ err$(T) = \max\{|t_{u,v} - \frac{1}{2}| : u \neq v \in X$ and $\{u,v\} \neq \{x,y\}\}$

$(*)_3$ if $\bar{t} \in T$, err$(\bar{t}) > 0$ then we can find $\bar{t}' \in T$ such that err$(\bar{t}') \leq \frac{1}{2}$ err(t) and $t'_{x,y} \geq (t_{x,y} + \frac{1}{2} + \varepsilon)/2$.

Why? Choose $c \in \mathfrak{C}$ such that $c\{x,y\} = y$ and

$$u \neq v \in X \ \& \ \{u,v\} \neq \{x,y\} \ \& \ t_{u,v} > \frac{1}{2} \Rightarrow c\{u,v\} = u$$

(so if $t_{u,v} = t_{v,u} = \frac{1}{2}$ it does not matter).

So c is "a try to correct $\bar t$".

As we are assuming clause (a) and the choice of ε_d, we can find $\bar r^* = \langle r^*_d : d \in \mathfrak{D}\rangle$ with $r^*_d \in [0,1]_{\mathbb{R}}$ and $1 = \Sigma\{r^*_d : d \in \mathfrak{D}\}$ such that

$$\frac{1}{2} + \varepsilon_d < \Sigma\{r^*_d : d\{x,y\} = y\}$$

and if $\{u,v\} \neq \{x,y\}$ then

$$c\{u,v\} = v \Rightarrow \frac{1}{2} < \Sigma\{r^*_d : d\{u,v\} = v\}$$

hence

$$c\{u,v\} = u \Rightarrow \frac{1}{2} < \Sigma\{r^*_d : d\{u,v\} = u\}.$$

By the choice of ε, $\frac{1}{2} + \varepsilon < \Sigma\{r^*_d : d\{x,y\} = y\}$.
Let $\bar s = \langle s_{u,v} : u \neq v \in X\rangle$ be defined by $s_{u,v} = \Sigma\{r^*_d : d\{u,v\} = v\}$, so

- $\circledast(i)$ $\bar s \in$ pr-cl$(\mathfrak{D})$
- (ii) $s_{x,y} > \frac{1}{2} + \varepsilon$ (so $s_{y,x} < \frac{1}{2}$)
- (iii) if $t_{u,v} > \frac{1}{2}$ and $u \neq v \in X, \{u,v\} \neq \{x,y\}$ then $c\{u,v\} = u$ hence $s_{u,v} < \frac{1}{2}$
- (iv) if $t_{u,v} < \frac{1}{2}$ and $u \neq v \in X, \{u,v\} \neq \{x,y\}$ then $c\{u,v\} = v$ hence $s_{u,v} > \frac{1}{2}$.

Let $\bar t' = \frac{1}{2}\bar t + \frac{1}{2}\bar s$, i.e. $t'_{u,v} = \frac{1}{2}(t_{u,v} + s_{u,v})$ so clearly

- $\circledast(i)$ $\bar t' \in$ pr-cl$(\mathfrak{D})$
- (ii) $t'_{x,y} = (t_{x,y} + \frac{1}{2} + \varepsilon)/2$ (hence $\bar t' \in T$)
- (iii) if $u \neq v \in X, \{u,v\} \neq \{x,y\}$ then $|t'_{u,v} - \frac{1}{2}| \leq \frac{1}{2}|t_{u,v} - \frac{1}{2}|$.

So we are done proving $(*)_3$.

As T is closed (and is included in a $\{\bar t : \bar t = \langle t_{u,v} : u \neq v \in X\rangle, 0 \leq t_{u,v} \leq 1\}$), clearly there is $t \in T$ such that $u \neq v \in X$ & $\{u,v\} \neq \{x,y\} \Rightarrow t_{u,v} = \frac{1}{2}$ as required.

<u>(c) ⇒ (a)</u>:

Let $d \in \mathfrak{C}$ and let $\langle c_j : j \in J\rangle$ witness clause (c).
Let $r_c = |\{j \in J : c_j = c\}|/|J|$ now $\langle r_c : c \in \mathfrak{D}\rangle$ witness clause (a), i.e., witness that $d \in$ maj-cl$(\mathfrak{D})$.

<u>(a) ⇒ (c)</u>:

Let $d \in \mathfrak{C}$ and let $\langle r_c : c \in \mathfrak{D}\rangle$ be as guaranteed for d by clause (a). Let $n(*) > 0$ be large enough and for $c \in \mathfrak{D}$ let $k_c \in \{0,\dots,n(*)-1\}$ be such that $c \in \mathfrak{D} \Rightarrow k_c \leq$

$n(*) \times r_c < k_c + 1$; note that k_c exists as $r_c \in [0,1]_{\mathbb{R}}$. As $\sum\limits_c \frac{k_c}{n(*)} \le 1 \le \sum\limits_c \frac{k_c+1}{n(*)}$, we can choose $m_c \in \{k_c, k_c+1\}$ such that $r'_c = \frac{m_c}{n(*)}$ satisfies $\Sigma\{r'_c : c \in \mathfrak{D}^*\} = 1$. Let $J = \{(c,m) : c \in \mathfrak{D}$ and $m \in \{1,\dots,m_c\}\}$ and we let $\mathbf{c}_{(d,m)} = d$ for $(d,m) \in J$. Now the "majority" of $\langle \mathbf{c}_t : t \in J\rangle$, see Definition 1.3, choose d^* so clause (c) holds.

Before we deal with clauses (d),(e),(f) and (g) of 2.1, we define

2.2 Definition. 1) For $\mathscr{D} \subseteq \mathfrak{C}$ and $A \subseteq [0,1]_{\mathbb{R}}$ let $\mathrm{Pr}_A(\mathscr{D})$ be the set of pairs (s_0, s_1) of real numbers $\in [0,1]_{\mathbb{R}}$ such that for some $\bar t \in$ pr-cl$(\mathscr{D})$ and $x \ne y \in X$ and $a \in A$ we have $\bar t = \bar t\langle x,y,a,s_0,s_1\rangle$ where
2) $\bar t = \bar t\langle x,y,a,s_0,s_1\rangle$ where $x \ne y \in X, a \in [0,1]_{\mathbb{R}}$ and $s_0, s_1 \in [0,1]_{\mathbb{R}}$ and $\bar t = \langle t_{u,v} : u \ne y \in X\rangle \in$ pr$(\mathfrak{C})$ is defined by

- (α) $t_{x,y} = a$
- (β) if $z \in X\backslash\{x,y\}$ <u>then</u> $t_{y,z} = s_1$ (hence $t_{z,y} = 1 - s_1$)
- (γ) if $z \in X\backslash\{x,y\}$ <u>then</u> $t_{x,z} = s_0$ (hence $t_{z,x} = 1 - s_0$)
- (δ) if $z_1 \ne z_2 \in X\backslash\{x,y\}$ <u>then</u> $t_{z_1,z_2} = \frac{1}{2}$.

3) In $\mathrm{Pr}_A(\mathscr{D})$ we may replace A by $1, 0, \ne \frac{1}{2}, > \frac{1}{2}, < \frac{1}{2}$ if A is $\{1\}, \{0\}, [0,1]_{\mathbb{R}}\backslash\{\frac{1}{2}\}, (\frac{1}{2},1]_{\mathbb{R}}, [0,\frac{1}{2})_{\mathbb{R}}$ respectively.
4) For $\ell \in \{0,1\}$ and $\mathscr{D} \subseteq \mathfrak{C}^*$ let $\mathrm{Prd}_\ell(\mathscr{D})$ be the set of pairs $\bar s = (s_0, s_1)$ of real (actually rational) numbers $\in [0,1]_{\mathbb{R}}$ such that for some $c \in \mathscr{D}$ and $x \ne y \in X$ we have $\bar s = \bar s^{c,x,y} = (s_0^{c,x,y}, s_1^{c,x,y})$ where

- (i) $s_1^{c,x,y} = |\{z : z \in X\backslash\{x,y\}$ and $c\{y,z\} = z\}|/(\mathbf{n}-2)$
- (ii) $s_0^{c,x,y} = |\{z : z \in X\backslash\{x,y\}$ and $c\{x,z\} = z\}|/(\mathbf{n}-2)$
- (iii) $\ell = 1 \Leftrightarrow \ell \ne 0 \Leftrightarrow c\{x,y\} = y$.

5) $\mathrm{Prd}(\mathscr{D})$ is $\mathrm{Prd}_0(\mathscr{D}) \cup \mathrm{Prd}_1(\mathscr{D})$ and $\mathscr{D}_{x,y} = \{c \in \mathscr{D} : c\{x,y\} = y\}$.

2.3 Claim. *Let* $\mathscr{D} \subseteq \mathfrak{C}$

1. $\mathrm{Pr}_{A_1}(\mathscr{D}_1) \subseteq \mathrm{Pr}_{A_2}(\mathscr{D}_2)$ *if* $A_1 \subseteq A_2 \subseteq [0,1]_{\mathbb{R}}$ *and* $\mathscr{D}_1 \subseteq \mathscr{D}_2 \subseteq \mathfrak{C}$
2. $\mathrm{Pr}_A(\mathscr{D})$ *is a convex subset of* $[0,1]_{\mathbb{R}} \times [0,1]_{\mathbb{R}}$
3. $\mathrm{Prd}_\ell(\mathscr{D})$ *is finite and its convex hull is* $\subseteq \mathrm{Pr}_{\{\ell\}}(\mathscr{D})$, *increasing with* $\mathscr{D}(\subseteq \mathfrak{C})$ *for* $\ell = 0, 1$
4. *For* $x \ne y \in X$ *and* $c \in \mathfrak{C}$ *satisfying* $c\{x,y\} = y \Leftrightarrow \ell = 1 \Leftrightarrow \ell \ne 0$ *we have (see Claim 2.2(4))*

$$s_0^{c,x,y} = (\mathrm{val}_c(x) - \ell)/(\mathbf{n}-2)$$

$$s_1^{c,x,y} = (\mathrm{val}_c(y) - (1-\ell))/(\mathbf{n}-2).$$

(5) $\mathrm{Pr}_{A_1 \cup A_2}(\mathscr{D}) = \mathrm{Pr}_{A_1}(\mathscr{D}) \cup \mathrm{Pr}_{A_2}(\mathscr{D})$, *in fact* $\mathrm{Pr}_A(\mathscr{D}) = \cup\{\mathrm{Pr}_{\{a\}}(\mathscr{D}) : a \in A\}$.

Proof. Immediate.

2.4 Claim. *1) If $x \neq y \in X$ and $c \in \mathfrak{C}$ and $\ell \in \{0,1\}$ satisfies $(\ell = 1) \equiv (c(\{x,y\}) = y)$ __then__ $\bar t\langle x,y,\ell,s_0^{c,x,y},s_1^{c,x,y}\rangle = \frac{1}{|\Pi_{x,y}|}\Sigma\{\bar t[\hat\pi(c)] : \pi \in \Pi_{x,y}\}$ where $\Pi_{x,y} =: \{\pi \in \mathrm{Per}(X) : \pi(x) = x$ and $\pi(y) = y\}$ hence $|\Pi_{x,y}| = (\mathbf{n}-2)!$.*
2) If $\mathscr{D} \subseteq \mathfrak{C}$ is symmetric and $\bar t \in \mathrm{pr\text{-}cl}(\mathscr{D})$ and $\bar t^* = \Sigma\{\bar t^\pi : \pi \in \Pi_{x,y}\}/|\Pi_{x,y}|$ where $\bar t^\pi = \langle t^\pi_{u,v} : u \neq v \in X\rangle, t^\pi_{u,v} = t_{\pi^{-1}(u),\pi^{-1}(v)}$ __then__ $t^* \in \mathrm{pr\text{-}cl}(\mathscr{D})$ and $\bar t^* \in \mathrm{Pr}_{\{a\}}(s_0,s_1)$ where $a = t_{x,y}, s_0 = \Sigma\{t_{x,z} : z \in X\backslash\{x,y\}\}/(\mathbf{n}-2)!$ and $s_1 = \Sigma\{t_{y;z} : z \in X\backslash\{y,z\}/(\mathbf{n}-2)!$.

Proof. Easy.

2.5 Claim. *For any symmetric $\mathscr{D} \subseteq \mathfrak{C}$ (i.e., closed under permutations of X):*
1) For $\ell \in \{0,1\}$, the set $\mathrm{Pr}_\ell(\mathscr{D})$ is the convex hull of $\mathrm{Prd}_\ell(\mathscr{D})$ in $\mathbb{R} \times \mathbb{R}$.
2) $\mathrm{Pr}_{[0,1]}(\mathscr{D})$ is the convex hull of $\mathrm{Prd}(\mathscr{D})$.
3) Let $s_0^*, s_1^* \in [0,1]_{\mathbb{R}}$. __Then__ $\bar t^* = \bar t\langle x,y,a,s_0^*,s_1^*\rangle \in \mathrm{pr\text{-}cl}(\mathscr{D})$ __iff__ we can find $\langle r_{\bar s,\ell} : \ell \in \{0,1\}$ and $\bar s \in \mathrm{Prd}_\ell(\mathscr{D})\rangle$ such that $r_{\bar s,\ell} \in [0,1]_{\mathbb{R}}$ and $1 = \Sigma\{r_{\bar s,\ell} : \ell \in \{0,1\}, \bar s \in \mathrm{Prd}_\ell(\mathscr{D})\}$ and $\bar s = \Sigma\{r_{\bar s,\ell} \times \bar s : \ell \in \{0,1\}, \bar s \in \mathrm{Prd}_\ell(\mathscr{D})\}$ and $a = \Sigma\{r_{\bar s,1} : \bar s \in \mathrm{Prd}_1(\mathscr{D})\}$.

Proof. 1) By 2.3(3) we have one inclusion.

For the other direction assume $(s_0^*,s_1^*) \in \mathrm{Pr}_\ell(\mathscr{D})$ and we should prove that it belongs to the convex hull of $\mathrm{Prd}_\ell(\mathscr{D})$. Fix $x \neq y \in X$ and let $\bar t^* = \bar t\langle x,y,\ell,s_0^*,s_1^*\rangle$ so $\bar t^* = \langle t^*_{u,v} : u \neq v \in X\rangle$ is defined as follows $t^*_{x,y} = \ell, t^*_{u,v} = \frac{1}{2}$ if $u \neq v \in X\backslash\{x,y\}, t^*_{y,z} = s_1^*$ if $z \in X\backslash\{x,y\}, t_{x,z} = s_0^*$ if $z \in X\backslash\{x,y\}$. As $(s_0^*,s_1^*) \in \mathrm{Pr}_\ell(\mathscr{D})$ by Definition 2.2 we know that $\bar t^* \in \mathrm{pr\text{-}cl}(\mathscr{D})$ and let $\bar r = \langle r_c : c \in \mathscr{D}\rangle$ be such that

$\circledast_0$ $x,y,\bar r$ witness that $\bar t^* \in \mathrm{pr\text{-}cl}(\mathscr{D})$, so $r_c \geq 0$ and $1 = \Sigma\{r_c : c \in \mathscr{D}\}$ and

$$\bar t^* = \Sigma\{r_c \times \bar t[c] : c \in \mathscr{D}\}.$$

As $t^*_{x,y} = \ell$, necessarily

$\boxtimes_1$ $r_c \neq 0 \Rightarrow c \in \mathscr{D}_{x,y} =: \{c \in \mathscr{D} : (c\{x,y\} = y) \equiv (\ell = 1)\}$.

To make the rest of the proof also a proof of part (3) let $a = \ell$ (as the real number a maybe $\neq 0, 1$ we need $m \in \{0,1\}$ below).
Let $\Pi_{x,y} = \{\pi \in \text{Per}(X) : \pi(x) = x, \pi(y) = y\}$ and recall that for $\pi \in \text{Per}(X), \hat{\pi}$ is the permutation of $\mathfrak{C}$ which π induces so $\hat{\pi}$ maps $\mathscr{D}_{x,y}$ onto $\mathscr{D}_{x,y}$ recalling we assume $\mathscr{D}$ is symmetric. Clearly $|\Pi_{x,y}| = (\mathbf{n}-2)!$
For $(s_0, s_1) \in \text{Prd}_\ell(\mathscr{D}) \subseteq \{(\frac{m_1}{(\mathbf{n}-2)!}, \frac{m_2}{(\mathbf{n}-2)!}) : m_1, m_2 \in \{0, 1 \ldots, (\mathbf{n}-2)!\}\}$ let $r^*_{(s_0,s_1)} = \Sigma\{r_c : \bar{s}^{c,x,y} = \bar{s}\}$ and for $m \in \{0,1\}$ we let $\bar{r}^*_{(s_0,s_1),m} = \Sigma\{r_c : \bar{s}^{c,x,y} = \bar{s}$ and $c\{x,y\} = y \equiv (m=1)\}$.

Clearly $\pi \in \Pi_{x,y} \Rightarrow \langle t^*_{\pi(u),\pi(v)} : u \neq v \in X\rangle = \bar{t}^*$, just check the definition, hence recalling 2.3(4) we have

$$\boxtimes_2 \quad \bar{t}^* = \frac{1}{|\Pi_{x,y}|} \sum_{\pi \in \Pi_{x,y}} \langle t^*_{\pi(u),\pi(v)} : u \neq v \in X\rangle = \frac{1}{|\Pi_{x,y}|} \sum_{\pi \in \Pi_{x,y}} \sum_{c \in \mathscr{D}} r_c \times \bar{t}^*[\hat{\pi}(c)] =$$
$$\sum_{c \in \mathscr{D}} r_c (\frac{1}{|\Pi_{x,y}|} \sum_{\pi \in \Pi_{x,y}} \bar{t}^*[\hat{\pi}(c)]) = \sum_{c \in \mathscr{D}} r_c \times \bar{t}\langle x, y, a, s_0^{c,x,y}, s_1^{c,x,y}\rangle =$$
$$\sum_{m \in \{0,1\}} \sum_{(s_0,s_1) \in \text{ Pr}_m(\mathscr{D})} (\Sigma\{r_c : c \in \mathscr{D}, c\{x,y\} = y \equiv (k=1)$$
$$\text{and } \bar{s}^{c,x,y} = (s_0,s_1)\}) \times \bar{t}\langle x, y, m, \bar{s}_0, s_1) =$$
$$\sum_{m \in \{0,1\}} \sum_{(s_0,s_1) \in \text{ Pr}_m(\mathscr{D})} r^*_{(s_0,s_1),m} \bar{t}\langle x, y, m, s_0, s_1\rangle \text{ (we have used 2.4).}$$

Now concentrate again on the case $a = \ell \in \{0,1\}$, so $r^*_{\bar{s},1-\ell} = 0$ by $\boxtimes_1$ and $r^*_{\bar{s},\ell} = r^*_s$. So clearly

$\circledast_1$ $r^*_{\bar{s}} \geq 0$
[Why? As the sum of non negative integers]

$\circledast_2$ $1 = \Sigma\{r^*_{\bar{s}} : \bar{s} \in \text{ Prd}_\ell(\mathscr{D})\}$
[Why? As by Definition 2.2(2), $c \in \mathscr{D}$ & $r_c > 0 \Rightarrow c \in \mathscr{D}_{x,y} \Rightarrow \bar{s}^{c,x,y} \in \text{Prd}_\ell(\mathscr{D})$ and the definition of $r^*_{\bar{s}}$]

$\circledast_3$ we have

(α) $z \in X\backslash\{x,y\} \Rightarrow s^*_1 = t^*_{y,z} = \Sigma\{r^*_{(s_0,s_1)} \times s_1 : (s_0,s_1) \in \text{ Prd}_\ell(\mathscr{D})\}$,

(β) $z \in X\backslash\{y,z\} \Rightarrow s^*_0 = t^*_{x,z} = \Sigma\{r^*_{(s_0,s_1)} \times s_0 : (s_0,s_1) \in \text{ Prd}_\ell(\mathscr{D})\}$
[Why? By $\boxtimes_2$.]

So $\langle r^*_{\bar{s}} : \bar{s} \in \text{ Prd}_\ell(\mathscr{D})\rangle$ witness that $(s^*_0, s^*_1) \in$ convex hull of $\text{Prd}_\ell(\mathscr{D})$.
2) Similar proof (and not used).
3) One direction is as in 2.3(3). For the other, by the hypothesis, $\circledast_0$ in the proof of part (1) with ℓ replaced by a holds. So by the part of the proof of part (1) after $\boxtimes_1, r^*_{\bar{s},m}$ are defined and $\boxtimes_2$ holds. So

$\boxdot_1$ $r^*_{\bar{s},m} \geq 0$

$\boxdot_2$ $1 = \Sigma\{r^*_{\bar{s},m} : m \in \{0,1\}$ and $\bar{s} \in \mathrm{Prd}_m(\mathscr{D})\}$

$\boxdot_3$ $(s^*_0, s^*_1) = \Sigma\{r^*_{\bar{s},m} \times \bar{s} : m \in \{0,1\}, \bar{s} \in \mathrm{Prd}_m(\mathscr{D})\}$
[Why? By $\boxtimes_2$.]

$\boxdot_4$ $a = \Sigma\{r^*_{\bar{s},1} : \bar{s} \in \mathrm{Prd}_1(\mathscr{D})\}$.

So we are done. $\square_{2.5}$

Continuation of the proof of 2.1:

$(d) \Leftrightarrow (b)_{x,y}$:

Read the definition 2.2(1) (and the symmetry). $\square_{2.1}$

$(d) \Rightarrow (e)$:

By 2.3(1).

$(e) \Rightarrow (d)$:

Why? If clause (e) holds, for some $a \in [0,1]_{\mathbb{R}} \backslash \{\frac{1}{2}\}$ and $x \neq y \in X$ we have $\bar{t}^* =: \bar{t}\langle x, y, a, \frac{1}{2}, \frac{1}{2}\rangle \in$ pr-cl$(\mathfrak{D})$. If $a > \frac{1}{2}$ this witness $(\frac{1}{2}, \frac{1}{2}) \in \mathrm{Pr}_{>1/2}(\mathscr{D})$, so assume $a < \frac{1}{2}$. But trivially $\bar{t}\langle y, x, 1-a, \frac{1}{2}, \frac{1}{2}\rangle$ is equal to t^* hence (as in 1.9) is in pr-cl$(\mathfrak{D})$ and we are done.

$(e) \Leftrightarrow (f)$:

Clearly (e) means that

$(*)_0$ there are $r_c \in [0,1]_{\mathbb{R}}$ for $c \in \mathfrak{D}$ such that $1 = \Sigma\{r_c : c \in \mathfrak{D}\}$ and $a \in [0,1]_{\mathbb{R}} \backslash \{\frac{1}{2}\}$ such that $\bar{t}\langle x, y, a, \frac{1}{2}, \frac{1}{2}\rangle = \Sigma\{r_c \times \bar{t}[c] : c \in \mathfrak{D}\}$.

By 2.5(3) we know that $(*)_0$ is equivalent to

$(*)_1$ there are $r_{\bar{s},\ell} \in [0,1]_{\mathbb{R}}$ for $\bar{s} \in \mathrm{Prd}_\ell(\mathfrak{D}), \ell \in \{0,1\}$ such that
- (i) $1 = \Sigma\{r_{\bar{s},\ell} : \bar{s} \in \mathrm{Prd}_\ell(\mathfrak{D}), \ell \in \{0,1\}\}$
- (ii) $(\frac{1}{2}, \frac{1}{2}) = \Sigma\{r_{\bar{s},\ell} \times \bar{s} : \bar{s} \in \mathrm{Prd}_\ell(\mathfrak{D}), \ell \in \{0,1\}\}$
- (iii) $\frac{1}{2} \neq a = \Sigma\{r_{\bar{s},1} : \bar{s} \in \mathrm{Prd}_1(\mathfrak{D})\}$.

But by 2.3(4) and Definition 2.2(4) for $\ell \in \{0,1\}$:

$$\mathrm{Prd}_\ell(\mathfrak{D}) = \big\{\big(\frac{\mathrm{val}_c(x) - \ell}{\mathbf{n} - 2}, \frac{\mathrm{val}_c(y) - (1-\ell)}{\mathbf{n} - 2}\big) : c \in \mathfrak{D} \text{ and } x \neq y \text{ and } (c\{x,y\} = y) \equiv (\ell = 1)\big\}.$$

Let $d^* \in \mathfrak{D}$ recall that $\mathfrak{D} =$ sym-cl$(\{d^*\})$ by a hypothesis of 2.1 and $V_\ell(d^*) = \{(k_1, k_2) :$ for some $x_1 \neq x_2 \in X, k_1 = \text{val}_d(x_1), k_2 = \text{val}_d(x_2)$ and $d\{x_1, x_2\} = x_{\ell+1}\}$. So $(*)_1$ means (recalling the definition of $\text{Prd}_\ell(\mathfrak{D})$)

$(*)_2$ there is a sequence $\langle r_{\bar{k},\ell} : \bar{k} \in V_\ell(d^*)$ and $\ell \in \{0,1\}\rangle$ such that

- (i) $r_{\bar{k},\ell} \in [0,1]_{\mathbb{R}}$ and
- (ii) $1 = \Sigma\{r_{\bar{k},\ell} : \bar{k} \in V_\ell(d^*)\}$ and
- (iii) $(\frac{1}{2}, \frac{1}{2}) = \Sigma\{r_{(k_1,k_2),\ell} \times \left(\frac{k_1-\ell}{\mathbf{n}-2}, \frac{k_2-(1-\ell)}{\mathbf{n}-2}\right) : \ell \in \{0,1\}$ and $(k_1, k_2) \in V_\ell(d^*)\}$
- (iv) $\frac{1}{2} \neq \Sigma\{r_{\bar{k},1} : \bar{k} \in V_1(d)\}$.

Let us analyze $(*)_2$. Let $r^*_\ell = \Sigma\{r_{\bar{k},\ell} : \bar{k} \in V_\ell(d^*)\}$ for $\ell \in \{0,1\}$. So

$\circledast_1$ $r^*_\ell \in [0,1]_{\mathbb{R}}$ and $1 = r^*_0 + r^*_1$.

If $r^*_\ell = 0$ for some $\ell \in \{0,1\}$ we can finish easily. So we assume

$\circledast_2$ $r^*_0, r^*_1 \neq 0$.

Now clause (iii) of $(*)_2$ means $(iii)_1 + (iii)_2$ where

$(iii)_1$ $\frac{1}{2} = \frac{1}{\mathbf{n}-2}\left(\Sigma\{r_{\bar{k},\ell} \times k_1 : \bar{k} \in V_\ell(d^*) \text{ and } \ell \in \{0,1\}\}\right) - \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times \ell : \bar{k} \in V_\ell(d^*), \ell \in \{0,1\}\} = \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times k_1 : \bar{k} \in V_\ell(d^*) \text{ and } \ell \in \{0,1\}\} - \frac{r^*_1}{\mathbf{n}-2}$,

i.e.,

$(iii)'_1$ $\frac{\mathbf{n}}{2} - (1 - r^*_1) = \frac{\mathbf{n}-2}{2} + r^*_1 = \Sigma\{r_{\bar{k},\ell} \times k_1 : \bar{k} \in V_\ell(d^*) \text{ and } \ell \in \{0,1\}\}$

$(iii)_2$ $\frac{1}{2} = \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times k_2 : \bar{k} \in V_\ell(d^*) \text{ and } \ell \in \{0,1\}\} - \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times (1-\ell) : \bar{k} \in V_\ell(d^*) \text{ and } \ell \in \{0,1\}\} = \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times k_2 : \bar{k} \in V_\ell(d^*), \ell \in \{0,1\}\} - \frac{r^*_0}{\mathbf{n}-2}$,

i.e.,

$(iii)'_2$ $\frac{\mathbf{n}}{2} - (1 - r^*_0) = \frac{\mathbf{n}}{2} + r^*_1 = \Sigma\{r_{\bar{k},\ell} \times k_2 : \bar{k} \in V_\ell(d^*) \text{ and } \ell \in \{0,1\}\}$.

Together (iii) of $(*)_2$ is equivalent to

$(iii)^+$ $\left(\frac{\mathbf{n}}{2} - (1 - r^*_1), \frac{\mathbf{n}}{2} - (1 - r^*_0)\right) = \Sigma\{r_{\bar{k},\ell} \times \bar{k} : \bar{k} \in V_\ell(d^*) \text{ and } \ell \in \{0,1\}\}$.

Let $\bar{s}_\ell = \Sigma\{r_{\bar{k},\ell} \times \bar{k} : \bar{k} \in V_\ell(d^*)\}/r^*_\ell$, so $(*)_2$ is equivalent to ($V_\ell(d^*)$ is from Definition 1.8)

$(*)_3$ there are $\bar{s}_0, \bar{s}_1, r^*_0, r^*_1$ such that

(i) $\bar{s}_\ell \in$ convex$(V_\ell(d^*))$ for $\ell = 0, 1$

(ii) $r^*_0, r^*_1 \in [0,1]_{\mathbb{R}}$ and $r = r^*_0 + r^*_1$

(iii) $(\frac{\mathbf{n}}{2} - (1 - r^*_1), \frac{\mathbf{n}}{2} - (1 - r^*_0))$ is $r^*_0 \times \bar{s}_0 + r^*_1 \times \bar{s}_1$

(iv) $r^*_\ell \neq \frac{1}{2}$ (by clause (iv) in $(*)_2$ above).

Clearly $(*)_3(iii)$ is equivalent to

$(*)_4(iii)'$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ is $\big(r^*_0 \times (\bar{s}_0 - (0,1)), r^*_1 \times (\bar{s}_1 - (1,0)\big)$.

But this is clause (f).

$\underline{(g) \Rightarrow (f)}$: By 2.6, 2.8, 2.9 below (i.e., they show $(g) + \neg(f)$ lead to contradiction).

$\underline{(f) \Rightarrow (g)}$: As $\neg(g) \Rightarrow \neg(f)$ trivially.

We have proved $(b)_{x,y} \Leftrightarrow (b)_{x',y'}, (b)_{x,y} \Rightarrow (a) \Rightarrow (b)_{x,y}, (c) \Rightarrow (a) \Rightarrow (c), (d) \Leftrightarrow (b)_{x,y}, (d) \Rightarrow (e) \Rightarrow (d), (e) \Leftrightarrow (f), (g) \Rightarrow (f) \Rightarrow (g)$, so we are done proving 2.1. $\square_{2.1}$

2.6 Claim. *Assume that clause (f) of 2.1 fails, $d = d^* \in \mathfrak{D}$ but clause (g) of 2.1 holds (equivalently $\langle \text{val}_d(x) : x \in X \rangle$ is not constant). Then the following hold*

$\boxdot_1$ *there are no $\bar{s}^0 \in \text{conv}(V^*_0(d^*)), \bar{s}^1 \in \text{conv}(V^*_1(d^*))$ such that $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ lie on $\text{conv}\{\bar{s}^0, \bar{s}^1\}$ and for some $\ell \in \{0,1\}$ this set contains an interior point of $V^*_\ell(d^*)$*

$\boxdot_2$ *the lines $L^*_0 = \{(\frac{\mathbf{n}}{2} - 1, y) : y \in \mathbb{R}\}, L^*_1 = \{(x, \frac{\mathbf{n}}{2} - 1) : x \in \mathbb{R}\}$ divides the plane; and $\text{conv}(V^*(d^*))$ is*

(i) *included in one of the four closed half planes $\underline{\text{or}}$*

(ii) *is disjoint to at least one of the closed quarters minus $\{(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)\}$.*

2.7 Remark. 1) Recall $V^*(d^*) = V^*_0(d^*) \cup V^*_1(d^*)$ and $V^*_\ell(d^*) = \{\bar{s} - (\ell, 1 - \ell) : \bar{s} \in V_\ell(d^*)\}$.

2) So

(i) $(k_1, k_2) \in V^*_0(d^*) \Leftrightarrow (k_1, k_2) + (0,1) \in V_0(d^*) \Leftrightarrow (k_1, k_2 + 1) \in V_0(d^*)$
[Why? By Definition 1.8(4).]

(*ii*) $(k_2,k_1) \in V_1^*(d^*) \Leftrightarrow (k_2,k_1)+(1,0) \in V_1(d^*) \Leftrightarrow (k_2+1,k_1) \in V_1(d^*)$ (see 1.9(2)) hence

(*iii*) $(k_1,k_2) \in V_0^*(d^*) \Leftrightarrow (k_1,k_2-1) \in V_0(d^*) \Leftrightarrow (k_2-1,k_1) \in V_1(d^*) \Leftrightarrow (k_2,k_1) \in V_1^*(d^*)$.
[Why? By the above $(i)+(ii)$ and 1.9(2).]

Proof. Toward contradiction assume that $\boxdot_2$ or $\boxdot_1$ in the claim fails. So necessarily

$(*)_0$ $(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1) \notin V^*(d^*)$
[Why? If it belongs to $V_\ell^*(d^*)$ let $r_\ell^*=1, r_{1-\ell}^*=0$ and we get clause (f) of 2.1 which we are assuming fails]

$(*)_0'$ $(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1) \notin \text{conv}(V_\ell^*(d^*))$
[Why? As in the proof of $(*)_0$.]

$(*)_1$ $(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1)$ belongs to the convex hull of $V_0^*(d^*) \cup V_1^*(d^*)$ hence of $\text{conv}(V_0^*(d^*)) \cup \text{conv}(V_1^*(d^*))$
[Why? Otherwise $\boxdot_1$ trivially holds; also there is a line L through $(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1)$ such that $V^*(d^*)\backslash L$ lie in one half plane of L, so easily clause (ii) of $\boxdot_2$ holds so $\boxdot_2$ holds. But we are assuming toward contradiction that $\boxdot_1$ fails or $\boxdot_2$ fails.]

Let $E = \{(\bar{s}_0,\bar{s}_1) : \bar{s}_\ell \in \text{conv}(V_\ell^*(d^*))$ and $(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1)$ belongs to the convex hull of $\{\bar{s}_0,\bar{s}_1\}\}$

$(*)_2$ $E \neq \emptyset$
[Why? By $(*)_1$]

$(*)_3$ if $r_0,r_1 \in [0,1]_{\mathbb{R}}, 1 = r_0+r_1, (\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1) = r_0 \times \bar{s}_0 + r_1 \times \bar{s}_1$ and $\bar{s}_\ell \in \text{conv}(V_\ell^*(d^*))$ for $\ell=0,1$ <u>then</u> $r_0=r_1=\frac{1}{2}$
[Why? Otherwise clause (f) holds.]

$(*)_4$ if $(\bar{s}_0,\bar{s}_1) \in E$ then $(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1) = \frac{1}{2}(\bar{s}_0+\bar{s}_1)$
[Why? By $(*)_3$ and the definition of E]

$(*)_5$ if $(\bar{s}_0,\bar{s}_1) \in E, i \in \{0,1\}$, <u>then</u> $\bar{s}_\ell$ is the unique member of $\text{conv}(V_\ell^*(d^*))$ which lies on the line through $\{\bar{s}_0,\bar{s}_1\}$ for $\ell=0,1$
[Why? Otherwise let $\bar{s}_\ell'$ be a counterexample. If $(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1) \in \text{conv}\{\bar{s}_\ell,\bar{s}_\ell'\}$ then it belongs to $\text{conv}(V_\ell^*(d^*))$ hence clause (f) holds, contradiction. So letting $\bar{s}_{1-\ell}' = \bar{s}_{1-\ell}$ we know that $(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1) \in \text{conv}\{\bar{s}_{1-\ell}',\bar{s}_\ell'\}$ hence by the definition of E we get $(\bar{s}_0',\bar{s}_1') \in E$ hence $\frac{1}{2}(\bar{s}_0',\bar{s}_1') = (\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1) = \frac{1}{2}(\bar{s}_0+\bar{s}_1)$ hence subtracting the two equations, $\bar{s}_{1-\ell}$ is cancelled and we get $\bar{s}_\ell' = \bar{s}_\ell$, contradiction]

$(*)_6$ $\boxdot_1$ holds (so by the assumption towards contradiction $\boxdot_2$ fails).
[Why? Assume $\bar{s}_0, \bar{s}_1$ are as there hence (by the definition of E), $(\bar{s}_0, s_1) \in E$, now by $(*)_5$ the set (the line through $\bar{s}_0, \bar{s}_1$) $\cap$ conv$(V^*_\ell(d))$ is equal to $\{\bar{s}_\ell\}$. So the conclusion of $\boxdot_1$ for $\bar{s}_0, \bar{s}_1$ holds as $V^*_\ell(d^*)$ is convex for $\ell \in \{0,1\}$.]

Easily (by $(*)_4$)

$(*)_7$ $E_\ell = \{\bar{s}_\ell : (\bar{s}_0, \bar{s}_1) \in E\}$ is a convex subset of conv$(V^*_1(d^*)) \subseteq \mathbb{R}^2$.

Also

$(*)_8$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \notin E_\ell$
[Why? By $(*)_5 + (*)'_0$.]

Now we split the proof to three cases which trivially exhausts all the possibilities.
Case 1: E is not a singleton.
This implies by $(*)_4$ that $V^*_\ell(d^*)$ is not a singleton for $\ell = 0, 1$. As $|E| \geq 2$ by $(*)_4$ clearly $|E_1| \geq 2$. Also by $(*)_5$ if $\bar{s}_1 \in E_1$ so $\bar{s}_1 \neq (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ by $(*)_8$, then $\bar{s}_1$ is the unique member of conv$(V^*_1(d_1)) \cap$ (the line through $\bar{s}_1, (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$). Also E_1 is convex (by $(*)_7$) so necessarily E_1 lies on a line L_1 to which by $(*)_5$, the point $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ does not belong.
As $E_1 \subseteq V^*_1(d^*) \cap L_1$ is a convex set with ≥ 2 members and $(*)_5$ it follows that conv$(V^*_1(d^*))$ is included in this line L_1 and as $V^*_0(d^*) = \{(k_2, k_1) : (k_1, k_2) \in V_1(d^*)\}$ (by clause (iii) of 2.7(2) above) it follows that conv$(V^*_0(d^*))$ is included in the line $L_0 = \{(a_0, a_1) : (a_1, a_0) \in L_1\}$ to which $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ does not belong.

But $E_0 = \{\bar{s}_0 : (\bar{s}_0, \bar{s}_1) \in E\}$ is necessarily an interval of L_0 and by $(*)_4$ we have

$(*)_9$ $L_0 = \{(a_0, a_1) : 2(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) - (a_0, a_1) \in L_1\}$.

As L_1 is a line, for some reals r_0, r_1, r_2 we have

$$L_1 = \{(a_0, a_1) \in \mathbb{R}^2 : r_0 a_0 + r_1 a_1 + r_2 = 0\}$$

and

$$(r_0, r_1) \neq (0, 0).$$

Hence by the definition of L_0 we have

$$L_0 = \{(a_0, a_1) \in \mathbb{R}^2 : r_1 a_0 + r_0 a_1 + r_2 = 0\}$$

and by $(*)_4$ the line L_0 includes the interval $\{2(\frac{\mathbf{n}}{2}-1,\frac{\mathbf{n}}{2}-1)-\bar{s}_1 : \bar{s}_1 \in E_1\}$ so

$$L_0 = \{(a_0,a_1) : (-r_0)a_0 + (-r_1)a_1 + r_2' = 0\}$$

where $r_2' = 2r_0(\frac{\mathbf{n}}{2}-1) + 2r_1(\frac{\mathbf{n}}{2}-1) + r_2$.

So for some $s \in \mathbb{R}$ we have $r_0 = -sr_1, r_1 = -sr_0, r_2 = sr_2'$ but $(r_0,r_1) \neq (0,0)$ hence $s \in \{1,-1\}$ hence $r_0 \in \{r_1,-r_1\}$, so without loss of generality $r_0 = 1, r_1 \in \{1,-1\}$.

Subcase 1A: $r_1 = -1$.

So $d^*\{x,y\} = y \Rightarrow (\mathrm{val}_{d^*}(x),\mathrm{val}_{d^*}(y)) \in V_1(d^*) \Rightarrow (\mathrm{val}_{d^*}(x)-1,\ \mathrm{val}_{d^*}(y)) \in V_1(d^*) \Rightarrow (\mathrm{val}_{d^*}(*)-1, \mathrm{val}_{d^*}(y)) \in L_1 \Rightarrow\ \mathrm{val}_{d^*}(x) -\ \mathrm{val}_{d^*}(y) = -r_2+1$, i.e., is constant, is the same for any such pair (x,y). But the directed graph $(X,\{(u,v) : d^*\{u,v\} = v\})$ necessarily contains a cycle, so necessarily $-r_2+1 = 0$, so the $\mathrm{val}_{d^*}(x)$ is the same for all $x \in X$ hence is necessarily $(\frac{\mathbf{n}}{2}-1)$, which is an "allowable" case, in particular, contradict clause (g) of 2.1 which we are assuming.

Subcase 1B: $r_1 = 1$.

Clearly $x \neq y \in X\ \ \&\ \ d^*\{x,y\} = y \Rightarrow (\mathrm{val}_{d^*}(x),\mathrm{val}_{d^*}(y)) \in V_1(d^*) \Rightarrow (\mathrm{val}_{d^*}(x),\mathrm{val}_{d^*}(y)-1) \in V_1^*(d^*) \Rightarrow (\mathrm{val}_{d^*}(x),\mathrm{val}_{d^*}(y)-1) \in L_1 \Rightarrow\ \mathrm{val}_{d^*}(x) +\ \mathrm{val}_{d^*}(y) = -r_2-1$. As $\mathbf{n} \geq 3$ there are distinct $x_0,x_1,x_2 \in X$ so $\mathrm{val}_{d^*}(x_{\ell_1}) +\ \mathrm{val}_{d^*}(x_{\ell_2}) = -r_2+1$ for $\{\ell_1,\ell_2\} \in \{\{0,1\},\{0,2\},\{1,2\}\}$, the order is not important as $r_1 = r_2$ hence are equal and $-r_2+1$ is twice their value. So for $y \in X\backslash\{x_1\}$, we have $\mathrm{val}_{d^*}(x_1) +\ \mathrm{val}_{d^*}(y) = -r_2+1$ so $\mathrm{val}_{d^*}(y) =\ \mathrm{val}_{d^*}(x_2)$, so we are done as in case 1A.

Case 2: $E = \{(\bar{s}_0^*,\bar{s}_1^*)\}$ and $\bar{s}_0^* \neq \bar{s}_1^*$.

Let L be the line through $\{\bar{s}_0^*,\bar{s}_1^*\}$ so let the real r_0,r_1,r_2 be such that $L = \{(a_0,a_1) : r_0a_0 + r_1a_1 + r_2 = 0\}$ and $(r_0,r_1) \neq (0,0)$.

So by $(*)_5$ the set $\mathrm{conv}(V_\ell^*(d^*))$ intersect L in the singleton $\{\bar{s}_\ell^*\}$.

If one (closed) half plane for the line L contains $V_0^*(d^*) \cup V_1^*(d^*)$ then $\boxdot_2$ of 2.6 holds (if L is not parallel to the x-axis and the y-axis (i.e., $r_0,r_1 \neq 0$) then $\boxdot_2(ii)$ holds, otherwise $\boxdot_2(i)$ holds), so by $(*)_6$ we are done.

$(*)_0$ There are $\bar{k}_0 \in V_0^*(d^*)\backslash\{\bar{s}_0^*\}$ and $\bar{k}_1 \in V_1^*(d^*)\backslash\{\bar{s}_1^*\}$ and they are outside L in different sides.

[Why? First assume that there are $\ell \in \{0,1\}$ and $\bar{k}',\bar{k}'' \in V_\ell^*(d^*)\backslash L$ on different sides of L, also $V_{1-\ell}^*(d^*)$ has a member outside L (by clause (iii) of 2.7(2) and $(*)_5$); and call it $\bar{k}_{1-\ell}$ so the choice $\bar{k}_\ell = \bar{k}'$ or the choice $\bar{k}_\ell = \bar{k}''$ is as required.

Second, assume that there is no such $\ell \in \{0,1\}$, by the previous paragraph we can choose $\bar{b}_\ell \in V_\ell^*(d^*)\backslash L$ for $\ell = 0,1$ so $\bar{k}_0,\bar{k}_1$ are as required.]

As $(\frac{\mathbf{n}}{2}-1, \frac{\mathbf{n}}{2}-1)$ lie in the open interval spanned by $\bar s^*_0$ and $\bar s^*_1$, necessarily $(\frac{\mathbf{n}}{2}-1, \frac{\mathbf{n}}{2}-1)$ is an interior point of $\text{conv}\{\bar s^*_0, \bar k_0, \bar s^*_1, \bar k_1\}$, contradiction to the case assumption.

Case 3: $E = \{(\bar s^*_0, \bar s^*_1)\}$ and $\bar s^*_0 = \bar s^*_1$.

So $\bar s^*_\ell = (\frac{\mathbf{n}}{2}-1, \frac{\mathbf{n}}{2}-1)$, but this contradicts $(*)'_0$ above. $\square_{2.6}$

2.8 Claim. *In 2.6, clause (i) of $\boxdot_2$ is impossible.*

Proof. We know that $\langle \text{val}_{d^*}(x) : x \in X\rangle$ is not constant (as we assume clause (g) of 2.1 holds). As the average of $\text{val}_{d^*}(x), x \in X$ is $\frac{\mathbf{n}-1}{2} = \frac{\mathbf{n}}{2} - 1$ clearly

$(*)_1$ for some points $x \in X$ we have $\text{val}_{d^*}(x)$ is $< \frac{\mathbf{n}-1}{2} = \frac{\mathbf{n}}{2} - \frac{1}{2}$ and for some point $x \in X$ we have $\text{val}_{d^*}(x)$ is $> \frac{\mathbf{n}}{2} - \frac{1}{2}$.

The assumption leaves us four possibilities: which half plan and what side, so we have 4 cases. $\square_{2.8}$

Case 1: For no $(k_0, k_1) \in V^*(d^*)$ do we have $k_0 > \frac{\mathbf{n}}{2} - 1$.

It follows by 2.7 that

$$(k_0, k_1) \in V_0(d^*) \Rightarrow (k_0, k_1 - 1) \in V^*_0(d^*) \subseteq V^*(d^*) \Rightarrow k_0 \leq \frac{\mathbf{n}}{2} - 1.$$

So if $x \in X$ and for some $y \in X \backslash \{x\}$ we have $d^*\{x,y\} = x$ (this means just that, $\text{val}_{d^*}(x) < \mathbf{n} - 1$) then $\text{val}_{d^*}(x) \leq \frac{\mathbf{n}}{2} - 1$, so

$(*)_2$ if $x \in X$ and $\text{val}_{d^*}(x) < \mathbf{n} - 1$ then $\text{val}_{d^*}(x) \leq \frac{\mathbf{n}}{2} - 1$.

There can be at most one $x \in X$ with $\text{val}_{d^*}(x) = \mathbf{n} - 1$; if there is none then we have:

if $x \in X$ then $\text{val}_{d^*}(x) \leq \frac{\mathbf{n}}{2} - 1$.

But the average valency is $\frac{\mathbf{n}-1}{2}$ which is $> \frac{\mathbf{n}}{2} - 1$, contradiction. So there is $x^* \in X$ such that $\text{val}_{d^*}(x) = \mathbf{n} - 1$, of course, it is unique. Now $(X \backslash \{x^*\}, \{(y,z) : y \neq z \in X \backslash \{x^*\}, d^*\{y,z\} = z\})$ is a tournament with $\mathbf{n} - 1$ points each has out-valency $\leq \frac{\mathbf{n}}{2} - 1 = \frac{(\mathbf{n}-1)-1}{2}$, so necessarily $\mathbf{n}$ is even and every node in the tournament has out-valency exactly $\frac{\mathbf{n}}{2} - 1 = \frac{(\mathbf{n}-1)-1}{2}$. So $\mathbf{n}$ is even and as $\mathbf{n} \geq 3$ we can choose $y \neq z \in X \backslash \{x^*\}$ and without loss of generality $d^*(y,z) = z$. Now $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1), (\mathbf{n} - 1, \frac{\mathbf{n}}{2} - 1) \in V_1(d^*)$ as witnessed by the pairs $(y,z), (x^*, y)$ respectively, hence $(\frac{\mathbf{n}}{2} - 2, \frac{\mathbf{n}}{2} - 1), (\mathbf{n} - 2, \frac{\mathbf{n}}{2} - 1) \in V^*_1(d)$, again by Definition 1.8(4). Hence (as $\mathbf{n} \geq 3$ so $\mathbf{n} - 2 \geq \frac{\mathbf{n}}{2} - 1$) we have $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \in \text{conv}(V^*_1(d^*))$, contradiction to "clause (f) of 2.1 fails" assumed in 2.6.

Case 2: For no $(k_0, k_1) \in V^*(d^*)$ do we have $k_0 < \frac{\mathbf{n}}{2} - 1$.
By clause (i) of 2.7(2) it follows that

$$(k_0, k_1) \in V_0(d^*) \Rightarrow (k_0, k_1 - 1) \in V_0^*(d^*) \subseteq V^*(d^*) \Rightarrow k_0 \geq \frac{\mathbf{n}}{2} - 1.$$

So if $x \in X$ and for some $y \in X \backslash \{x\}$ we have $d^*\{x, y\} = x$ (equivalently $\text{val}_{d^*}(x) < \mathbf{n} - 1$) then $(\text{val}_{d^*}(x), \text{val}_{d^*}(y)) \in V_0(d^*) \Rightarrow (\text{val}_{d^*}(x), \text{val}_{d^*}(y) - 1) \in V_1^*(d^*)$ hence $\text{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - 1$, but $\mathbf{n} - 1 \geq \frac{\mathbf{n}}{2} - 1$ so in any case

$(*)_3$ if $x \in X$ then $\text{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - 1$.

So $x \in X \Rightarrow \text{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - 1$.

If $\mathbf{n}$ is odd we have $x \in X \Rightarrow \text{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - \frac{1}{2} = \frac{\mathbf{n}-1}{2}$, impossible by $(*)_1$ so $\mathbf{n}$ is even. Let $k = \frac{\mathbf{n}}{2} - 1$. The average $\text{val}_{d^*}(x)$ is necessarily $k + \frac{1}{2}$ hence $Y =: \{x \in X : \text{val}_{d^*}(x) \leq k \text{ (equivalently } = k)\}$ has at least $k + 1 = \frac{\mathbf{n}}{2}$ members. If $x \in X, \text{val}_{d^*}(x) = k+1$ then $x \notin Y, |\{y : d\{x, y\} = y\}| = k+1 = \frac{\mathbf{n}}{2} > |X \backslash Y \backslash \{x\}|$ so there is $y \in Y$ such that $d\{x, y\} = y$, hence $(k, k) = (\text{val}_{d^*}(x) - 1, \text{val}_{d^*}(y)) \in V_1^*(d^*)$ so clause (f) of 2.1 holds, contradiction. So

$(*)_4$ $x \in X \Rightarrow \text{val}_{d^*}(x) \neq k + 1$.

Now $|Y| = \mathbf{n}$ is impossible by $(*)_1$. Also if $|Y| = \mathbf{n} - 1$ let z be the unique element of X outside Y so in the tournament $(Y, \{(x, y) : x \neq y \text{ are from } Y \text{ and } d^*\{x, y\} = y\})$ each x has out-valency $\leq \text{val}_{d^*}(x) = k = \frac{(|Y|-1)}{2}$, but this is the average so equality holds and we get contradiction as in Subcase 1b. Hence

$(*)_5$ $|Y| \leq \mathbf{n} - 2$.

Clearly we can find $x_1 \in Y$ such that $|\{y \in Y : y \neq x_1, d\{x_1, y\} = y\}| \leq \frac{|Y|-1}{2}$ (as if we average thi snumber on the $x_1 \in Y$ we get $\frac{|Y|-1}{2}$) but $\frac{|Y|-1}{2} \leq \frac{\mathbf{n}}{2} - \frac{3}{2} = k - \frac{1}{2} < |\{y \in X : d\{x_1, y\} = y\}|$ hence there is $x_2 \in X \backslash Y$ such that $d^*\{x_1, x_2\} = x_2$. Now let $m = \text{val}_{d^*}(x_2)$ so $m > k$ as $x_2 \notin Y$ and $m \neq k+1$ by $(*)_4$ hence $m > k+1$ and (x_1, x_2) witness $(k-1, m) \in V_1^*(d^*)$. As $\mathbf{n} \geq 3, |Y| \geq \frac{\mathbf{n}}{2}$ obviously $|Y| \geq 2$ hence (as any pair of $y_1 \neq y_2$ from Y witness) also $(k-1, k) \in V_1^*(d^*)$ and as $|Y| \geq \frac{\mathbf{n}}{2} = k+1$, $\text{val}_d(x_2) > k + 1 \geq \mathbf{n} - |Y|$ easily there is $x_3 \in Y$ such that $d\{x_2, x_3\} = x_3$ also $(m-1, k) \in V_1^*(d^*)$ so $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) = (k, k) \in \text{conv}\{(k-1, k), (m-1, k)\})$ recalling $m > k+1$, contradiction to "not clause (f) of 2.1" with $r_0^* = 0$.

Case 3: For no $(k_0, k_1) \in V^*(d^*)$ do we have $k_1 > \frac{\mathbf{n}}{2} - 1$.
So

$$(k_0, k_1) \in V_1(d^*) \Rightarrow (k_0 - 1, k_1) \in V_1^*(d^*) \subseteq V^*(d^*) \Rightarrow k_1 \leq \frac{\mathbf{n}}{2} - 1.$$

So if $y \in X$ and for some $x \in X\backslash\{y\}$ we have $d\{x,y\} = y$ then $(\mathrm{val}_{d^*}(x), \mathrm{val}_{d^*}(y)) \in V_1(d^*)$ hence $\mathrm{val}_{d^*}(y) \le \frac{\mathbf{n}}{2} - 1$. But there is such x iff $\mathrm{val}_{d^*}(y) \ne \mathbf{n} - 1$, that is

$(*)_6$ if $y \in X$ and $\mathrm{val}_{d^*}(y) \ne \mathbf{n} - 1$ then $\mathrm{val}_{d^*}(y) \le \frac{\mathbf{n}}{2} - 1$.

We continue as in Case 1, (or dualize see 1.9(1)).

Case 4: For no $(k_0, k_1) \in V^*(d^*)$ do we have $k_1 < \frac{\mathbf{n}}{2} - 1$.

So $(k_0, k_1) \in V_1(d^*) \Rightarrow (k_0 - 1, k_1) \in V_1^*(d^*) \subseteq V^*(d^*) \Rightarrow k_1 \ge \frac{\mathbf{n}}{2} - 1$. So if $y \in X$ and for some $x \in X\backslash\{y\}$ we have $d\{x,y\} = y$ then $(\mathrm{val}_{d^*}(x), \mathrm{val}_{d^*}(y)) \in V_1(d^*) \Rightarrow \mathrm{val}_{d^*}(y) \ge \frac{\mathbf{n}}{2} - 1$. So if $y \in X$ and for some $x \in X\backslash\{y\}$ we have $d^*\{x,y\} = y$ then $\mathrm{val}_{d^*}(y) \ge \frac{\mathbf{n}}{2} - 1$ so if $\mathrm{val}_{d^*}(y) < \mathbf{n} - 1$ then $\mathrm{val}_{d^*}(y) \ge \frac{\mathbf{n}}{2} - 1$, but if $\mathrm{val}_{d^*}(y) \ge \mathbf{n} - 1$ we get the same conclusion, so

$(*)_7$ $\mathrm{val}_{d^*}(y) \ge \frac{\mathbf{n}}{2} - 1$

and we can continue as in case 2. $\square_{2.8}$

2.9 Claim. *In 2.6, clause (ii) of $\boxdot_2$ is impossible.*

Proof. Note that as we are assuming the failure of clause (f) of 2.1

$(*)_0$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \notin V^*(d^*)$.

Again we have four cases.

Case 1: If $a_0 \ge \frac{\mathbf{n}}{2} - 1, a_1 \ge \frac{\mathbf{n}}{2} - 1$ but $(a_0, a_1) \ne (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ then $(a_0, a_1) \notin \mathrm{conv}(V^*(d^*))$.
So

$(*)_1$ for at most one $x \in X$ we have $\mathrm{val}_{d^*}(x) \ge \frac{\mathbf{n}}{2}$.
[Why? If $x \ne y \in X$ and $\mathrm{val}_{d^*}(x) \ge \frac{\mathbf{n}}{2}$, $\mathrm{val}_{d^*}(y) \ge \frac{\mathbf{n}}{2}$ then $(\mathrm{val}_{d^*}(x) - 1, \mathrm{val}_{d^*}(y)) \in V_0^*(d^*) \subseteq V^*(d^*)$ or $(\mathrm{val}_{d^*}(x), \mathrm{val}_{d^*}(y) - 1) \in V_0^*(d^*) \subseteq V^*(d^*)$, a contradiction to the case assumption in both cases.]

If there is no $x \in X$ with $\mathrm{val}_{d^*}(x) \ge \frac{\mathbf{n}}{2}$ then $x \in X \Rightarrow \mathrm{val}_{d^*}(x) < \frac{\mathbf{n}}{2}$ and so $x \in X \Rightarrow \mathrm{val}_{d^*}(x) \le \frac{\mathbf{n}-1}{2}$ but this is the average valency, so always equality holds, contradicting an assumption of 2.6.
So assume

$(*)_2$ $x_0 \in X$, $\mathrm{val}_{d^*}(x_0) \ge \frac{\mathbf{n}}{2}$.

Now

$(*)_3$ $y \in X \backslash \{x_0\} \Rightarrow \ \mathrm{val}_{d^*}(y) < \frac{\mathbf{n}}{2} - 1$ (hence $\leq \frac{\mathbf{n}}{2} - \frac{3}{2}$)
[Why? If $d^*\{x_0, y\} = x_0$ then $(\mathrm{val}_{d^*}(x_0) - 1, \mathrm{val}_{d^*}(y)) \in V_1^*(d^*) \subseteq V^*(d^*)$, now $\mathrm{val}_{d^*}(x_0) - 1 \geq \frac{\mathbf{n}}{2} - 1$ hence by the case assumption $+(*)_0$ we have $\mathrm{val}_{d^*}(y) < \frac{\mathbf{n}}{2} - 1$. If $d^*(x_0, y) = y$ then $(\mathrm{val}_{d^*}(x_0) - 1, \mathrm{val}_{d^*}(y)) \in V_0^*(d^*) \subseteq V^*(d^*)$ and the proof if similar.]

So letting $Y = X \backslash \{x_0\}$ we have $(Y, \{\{y, z\}, y \neq z \in Y, d^*(y, z) = z\})$ is a tournament each node has out-valency $\leq \frac{\mathbf{n}-3}{2} < \frac{(\mathbf{n}-1)-1}{2} = \frac{|Y|-1}{2}$, contradiction.

Case 2: If $a_1 \leq \frac{\mathbf{n}}{2} - 1$ and $a_2 \leq \frac{\mathbf{n}}{2} - 1$ and $(a_1, a_2) \neq (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ then $(a_1, a_2) \notin \mathrm{conv}(V^*(d^*))$.
Clearly, as above in the proof of $(*)_1$

$(*)_1'$ there is at most one $x \in X$ with $\mathrm{val}_{d^*}(x) \leq \frac{\mathbf{n}}{2} - 1$.

If there is none then $x \in X \Rightarrow \ \mathrm{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - 1 + \frac{1}{2} = \frac{\mathbf{n}-1}{2}$, so considering the average of $\mathrm{val}_{d^*}(y)$ equality always holds so clause (g) of 2.1 fails contradicting an assumption of 2.6. So assume

$(*)_2'$ $x_0 \in X$, $\mathrm{val}_{d^*}(x_0) \leq \frac{\mathbf{n}}{2} - 1$

and we can show, as above that

$(*)_3'$ if $y \in X \backslash \{x_0\}$ then $\mathrm{val}_{d^*}(y) > (\frac{\mathbf{n}}{2} - 1) = \frac{\mathbf{n}-2}{2}$ so $\mathrm{val}_{d^*}(y) \geq \frac{\mathbf{n}-1}{2}$.

The directed graph $\mathbf{G}_c = (X \backslash \{x_0\}, \{(y, z\} : d\{y, z\} = z\})$ has $\mathbf{n} - 1$ nodes. Saharon????

Case 3: If $a_1 \geq \frac{\mathbf{n}}{2} - 1$ and $a_2 \leq \frac{\mathbf{n}}{2} - 1$ and $(a_1, a_2) \neq (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$, then $(a_1, a_2) \notin \mathrm{conv}(V^*(d^*))$.
So (as in the proof of $(*)_1$ using $(*)_0$)

$\bigodot_1$ there cannot be $x_0, x_1 \in X$ such that $\mathrm{val}_{d^*}(x_0) \geq \frac{\mathbf{n}}{2}$ and $\mathrm{val}_{d^*}(x_1) \leq \frac{\mathbf{n}}{2} - 1$ (the $x_0 \neq x_1$ follows)

so one of the following two subcases hold.

Subcase 3A: $x \in X \Rightarrow \ \mathrm{val}_{d^*}(x) < \frac{\mathbf{n}}{2}$.
So $x \in X \Rightarrow \ \mathrm{val}_{d^*}(x) \leq \frac{\mathbf{n}-1}{2}$ and (looking at average valency) equality holds, contradicting clause (g) of 2.1 which we are assuming.

Subcase 3B: $x \in X \Rightarrow \ \mathrm{val}_{d^*}(x) > \frac{\mathbf{n}}{2} - 1$.

So $x \in X \Rightarrow \text{val}_{d^*}(x) \geq \frac{\mathbf{n}-1}{2}$, and we finish as above.

Case 4: If $a_1 \leq \frac{\mathbf{n}}{2} - 1$ and $a_2 \geq \frac{\mathbf{n}}{2} - 1$ then $(a_1, a_2) \notin \text{conv}(V(d^*))$.
Similar (or dualize the situation by 1.9(1)). $\square_{2.9}$

§3 Balanced choice functions

3.1 Definition. 1) $c \in \mathfrak{C}$ is called balanced if $x \in X \Rightarrow \text{val}_c(x) = (\mathbf{n}-1)/2$, let $\mathfrak{C}^{\text{bl}} = \{c \in \mathfrak{C} : c \text{ is balanced}\}$.
2) $\bar{t} \in \text{pr}(\mathfrak{C})$ is called balanced if $x \in X \Rightarrow \Sigma\{t_{x,y} : y \in X \backslash \{x\}\} = (\mathbf{n}-1)/2$. Let $\text{pr}^{\text{bl}}(\mathfrak{C})$ be the set of balanced $\bar{t} \in \text{pr}(\mathfrak{C})$.
3) $c \in \mathfrak{C}$ is called pseudo-balanced if for some balanced $\bar{t} \in \text{pr}(\mathfrak{C})$ we have

$$c\{x,y\} = y \Rightarrow t_{x,y} > \frac{1}{2}.$$

4) We call $\mathscr{D} \subseteq \mathfrak{C}$ balanced if every $c \in \mathscr{D}$ is balanced, similarly $T \subseteq \text{pr}(\mathfrak{C})$ is called balanced if every $\bar{t} \in T$ is.
5) If $x, y, z \in X$ are distinct, let $\bar{t} = \bar{t}^{<x,y,z>}$ be defined by:
$t_{u,v}$
is 1 if $(u,v) \in \{(x,y),(y,z),(z,x)\}$.
is 0 if $(u,v) \in \{(y,x),(z,y),(x,z)\}$
is $\frac{1}{2}$ if otherwise.
6) For a sequence $\bar{x} = (x_0, \ldots, x_{k-1})$ with $x_\ell \in X$ and with no repetitions and $a \in [0,1]_\mathbb{R}$ let $\bar{t} = \bar{t}_{\bar{x},a} \in \text{pr}(\mathfrak{C})$ be defined by $t_{x_i,x_j} = a, t_{x_j,x_i} = 1-a$ if $j = i+1$ mod k and $t_{x,y} = \frac{1}{2}$ otherwise. If $a = 1$ we may omit it.

3.2 Fact 1) If $c \in \mathfrak{C}^{\text{bl}}$ then $\bar{t}[c]$ belongs to $\text{pr}^{\text{bl}}(\mathfrak{C})$; if $\bar{t} \in \text{pr}(\mathfrak{C})$ is balanced and $c \in \text{maj}\{\bar{t}\}$, then c is pseudo-balanced; (also if c is pseudo balanced then it satisfies the condition from 1.7; and it follows from (4a)).
2) If $\mathscr{D}$ is balanced, then pr-cl$(\mathscr{D})$ is balanced hence every member of maj-cl$(\mathscr{D})$ is pseudo-balanced.

Proof of 3.2. 1), 2) Check.

3.3 Claim. *If $\mathfrak{D} \subseteq \mathfrak{C}$ is non empty, symmetric and not balanced, then* maj-cl$(\mathfrak{D}) = \mathfrak{C}$.

Proof. Choose $d^* \in \mathfrak{D}$ which is not balanced, and let $\mathfrak{D}' =:$ sym-cl$(\{d^*\})$, so $\mathfrak{D}'$ is as in 2.1 and it satisfies clause (g) there hence it satisfies clause (a) there. This means that maj-cl$(\mathfrak{D}') = \mathfrak{C}$ but $\mathfrak{D}' \subseteq \mathfrak{D} \subseteq \mathfrak{C}$ hence $\mathfrak{C} =$ maj-cl$(\mathfrak{D}') \subseteq$ maj-cl$(\mathfrak{D}) \subseteq \mathfrak{C}$ so we are done. $\square_{3.3}$

3.4 Fact: 1) If $c \in \mathfrak{C}$ is pseudo-balanced, then every edge of $\mathbf{G}_c$ belongs to some directed cycle.
2) Assume that $\bar{t} \in \operatorname{pr}(\mathfrak{C})$ is balanced, then

(a) if $t_{u,v} > \frac{1}{2}$ then we can find $k \geq 3$ and $\langle x_0, \dots, x_{k-1} \rangle \in X$ with no repetitions such that $(x_0, x_1) = (u, v)$ and $j = i + 1 \mod k \Rightarrow t_{x_i, x_j} > \frac{1}{2}$

(b) we can find $m(*)$ and $\bar{x}_m, a_m$ as in 3.1(6) and $r_m \geq 0$ for $m < m(*)$ satisfying $\Sigma\{r_m : m < m(*)\} = 1$ such that $\bar{t}^* = \Sigma\{r_m \bar{t}_{\bar{x}_m, a_m} : m < m(*)\}$ is quite similar to $\bar{t}$: for some real $s \geq 1$ we have $u \neq v \in X \Rightarrow t_{u,v} - \frac{1}{2} = s(t^*_{u,v} - \frac{1}{2})$

(c) in (b), if all $t_{u,v}$ are rational then we can add $r_m = \frac{1}{m(*)}, s = m(*)$.

3) In fact (a), (b), (c) are equivalent.

Proof of 3.4. 1) By (a) of part (2).
2) Clause (a): by clause (b).

Clause (b): We define a directed graph $G = G(\bar{t})$ as follows: the set of nodes is X, the set of edge $E = E^G = \{(u, v) : u \neq v \in X, t_{u,v} > \frac{1}{2}\}$ and we define the function $\mathbf{w} = w^{\mathbf{t}} : E^{G(\bar{t})} \to \mathbb{R}^{>0}$ by $\mathbf{w}(u, v) = t_{u,v} - \frac{1}{2}$. Now $(G, \mathbf{w})$ or $(E^G, \mathbf{w})$ is balanced in the sense that

$(*)_{(E,\mathbf{w})}$ for every node the in-valency is equal to the out-valency, i.e., $\Sigma\{w(u) : (u, x) \in E^G\} = \Sigma\{w(x, v) : (x, v) \in E^G\}$.

For such pairs $(E, \mathbf{w})$ we shall prove that for some $(\bar{x}_m, a_m)$ as in 3.1(6) for $m = 0, \dots, m(*)-1$ we have $\mathbf{w}^{\mathbf{t}} = \Sigma\{r_m \mathbf{w}^{\bar{t}_{\bar{x}_m, a_m}} : m < m(*)\}$ for some $r_m \geq 0$ satisfying $1 = \Sigma\{r_m : m < m(*)\}$. We prove this by induction on $\{(u, v) \in E : \mathbf{w}(u, v) > 0\}$. For any such pair $(E, \mathbf{w})$ if $E \neq \emptyset$, by the equality $(*)_{(E,\mathbf{w})}$ for any node, the in-valency is > 0 iff the out-valency is > 0 hence there is a directed cycle $\bar{x} = \langle x_0, \dots, x_{k-1} \rangle$; so $a = \operatorname{Min}\{\mathbf{w}(x_0, x_1), \mathbf{w}(x_1, x_2), \dots, \mathbf{w}(x_{k-2}, x_{k-1}), \mathbf{w}(x_{k-1}, x_0)\}$ is > 0. Define $E' = E \backslash \{(x_i, x_j) : j = i + 1 \mod k$ and $\mathbf{w}(x_i, x_j) = a\}, \mathbf{w}'(u, v)$ is $\mathbf{w}(u, v) - a$ if $(u, v) = (x_i, x_j), j = i + 1 \mod k$ and $\mathbf{w}'(u, v) = \mathbf{w}(u, v)$ otherwise. Applying the induction hypothesis to $(E', \mathbf{w}')$ and adding $\bar{t}_{\bar{x},a}$ we are done.
3) Should be clear. $\square_{3.2}$

3.5 Claim. *Assume $|X| \geq 3, \mathfrak{D} \subseteq \mathfrak{C}$ is symmetric, non empty and balanced. Then, for any distinct $x, y, z \in Z$ we have $\bar{t}_{<x,y,z>} \in \operatorname{pr-cl}(\mathfrak{D})$.*

Proof. Let $d \in \mathfrak{D}$, now as $(X, \{(u, v) : d\{u, v\} = v\})$ is a directed graph even a tournament with equal out-valance and in-valance for every node, it has directed cycle and hence it has a triangle, i.e., $x, y, z \in X$ distinct such that

$(*)_1$ $d\{x,y\} = y, d\{y,z\} = z, d\{z,x\} = x.$

Let $\Pi_{x,y,z} = \{\pi \in \text{ Per}(X) : \pi \restriction \{x,y,z\}$ is the identity$\}$. Let $\bar{t} = \Sigma\{\bar{t}[c^\pi] : \pi \in \Pi_{x,y,z}\}/|\Pi_{x,y,z}|$.
Clearly $c^\pi \in \mathfrak{D}$ for $\pi \in \Pi_{x,y,z}$ hence $\bar{t} \in$ pr-cl$(\mathfrak{D})$. Also by $(*)_1$ and the definition of $\Pi_{x,y,z}$

$(*)_2$ $t_{x,y} = t_{y,z} = t_{z,y} = 1.$

Also

$$\begin{aligned}|\{w :& w \in X\backslash\{x,y,z\} \text{ and } d\{x,w\} = w\}| = \\ &|\{w : w \in X\backslash\{x\} \text{ and } d\{x,w\} = w\}| - \{w : w \in \{y,z\} \text{ and } d\{x,w\} = w\} = \\ &(|X|-1)/2 - 1 = (|X|-3)/2 = \\ &|\{w : w \in X\backslash\{x,y,z\} \text{ and } d\{x,w\} = x\}|\end{aligned}$$

hence

$(*)_3$ $t_{x,w} = 1/2$ for $w \in X\backslash\{x,y,z\}$.

Similarly

$(*)_4$ $t_{y,w} = 1/2$ for $w \in X\backslash\{x,y,z\}$
$(*)_5$ $t_{z,w} = 1/2$ for $w \in X\backslash\{x,y,z\}$

and even easier (and as in §2)

$(*)_6$ $t_{u,v} = 1/2$ if $u \neq v \in X\backslash\{x,y,z\}$.

So we are done. $\square_{3.5}$

3.6 Claim. *Assume $\mathfrak{D} \subseteq \mathfrak{C}$ is symmetric non empty and $c^* \in \mathfrak{C}$ is pseudo balanced <u>then</u> $c^* \in$ maj-cl$(\mathfrak{D})$.*

Proof. Without loss of generality $\mathfrak{D}$ is balanced (otherwise use 3.3). So by 3.5

⊛ if $x,y,z \in X$ are distinct then $\bar{t}_{<x,y,z>} \in$ pr-cl$(\mathfrak{D})$.

Let $\bar t^* = \bar t[c^*]$ and let $\langle \bar x^i : i < i(*)\rangle$ list the set cyc(d) of tuples $\bar x = \langle x_\ell : \ell \leq k\rangle$ such that:

$$k \geq 2, x_\ell \in X, \ell_1 < \ell_2 \leq k \Rightarrow x_{\ell_1} \neq x_{\ell_2}$$

$$c\{x_\ell, x_{\ell+1}\} = x_{\ell+1} \text{ for } \ell < k$$

$$c\{x_k, x_0\} = x_0$$

Note

$\otimes$ for every $\bar x \in$ Cyc(c^*) for some $\bar t = \bar t^{\bar x} \in$ pr-cl$(\mathfrak{D})$ we have
- (a) $t_{u,v} = \frac{1}{2} + \frac{1}{\ell g(\bar x) - 2}$ if $(u,v) \in \{(x_{\ell_1}, x_{\ell_2}) : \ell_1 < \ell g(\bar x) - 1 \ \& \ \ell_2 = \ell_1 + 1$ or $\ell_1 = \ell g(\bar x) - 1 \ \& \ \ell_2 = 0\}$
- (b) $t_{u,v} = \frac{1}{2} - \frac{1}{\ell g(\bar x) - 2}$ if (v,u) is as above
- (c) $t_{u,v} = \frac{1}{2}$ if otherwise.

[Why? If $\bar x = \langle x_\ell : \ell \leq k\rangle$, let $\bar t$ be the arithmetic average of $\langle t^{<x_0,x_1,x_2>}, \bar t^{<x_0,x_2,x_3>}, \ldots, \bar t^{<x_0,x_{k-1},x_k>}\rangle$.]

Now let

$$\bar t = \Sigma\{\frac{1}{i(*)} \bar t^{\bar x^i} : i < i(*)\}.$$

(In fact we just need that $c\{y_0, y_1\} = y_1 \Rightarrow (y_0, y_1)$ appears in at least one cycle $\bar x^i, i < i(*)$). Easily $c =$ maj$(\bar t)$. $\square_{3.6}$

So now we can give a complete answer.

3.7 Conclusion. Assume

- (a) $\mathfrak{D} \subseteq \mathfrak{C}$ is symmetric non empty
- (b) $c \in \mathfrak{D}$.

Then $c \in$ maj-cl$(\mathfrak{D})$ iff $\mathfrak{D}$ has a non balanced member or c^* is pseudo balanced.

Proof. If $\mathfrak{D}$ has no nonbalanced member and c is not pseudo-balanced, by 3.2(2) we know $c \notin$ maj-cl$(\mathfrak{D})$. For the other direction, if $d^* \in \mathfrak{D}$ is not balanced use 3.4 that is $(a) \Leftrightarrow (g)$ of claim 2.1 for sym-cl$\{d^*\}$. Otherwise $\mathfrak{D}$ is balanced, c is pseudo balanced and we use 3.6. $\square_{3.7}$

REFERENCES.

[Alo02] Noga Alon. Voting paradoxes and digraphs realizations. *Advances in Appl. Math.*, **29**:126–135, 2002.

[ErMo64] Paul Erdős and Leo Moser. On the representation of directed graphs as unions of orderings. *Magyar Tud. Akad. Mat. Kutats Int. Kvzl.*, **9**:125–132, 1964.

[McG53] David C. McGarvey. A theorem on the construction of voting paradoxes. *Econometrica*, **21**:608–610, 1953.

[Sh 816] Saharon Shelah. What majority decisions are possible. *Discrete Mathematics*, **309**:2349–2364, 2009. arxiv:math.CO/0405119.

[Ste59] Richard Stearns. The voting problem. *American Mathematical Monthly*, **66**:761–763, 1959.