

# Fully saturated extensions of standard universe

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It seems that it has been taken for granted that there is no distinguished, definable nonstandard model of the reals. (This means a countably saturated elementary extension of the reals.) Of course the axiom of constructibility implies the existence of such an extension (take the first one in the sense of the canonical well-ordering of  $\mathbf{L}$ ), but the existence provably in  $\mathbf{ZFC}$  was established quite recently in [3]. Note that without Choice the existence of *any* elementary extension of the reals, containing an infinitely large integer, is not provable.

The problem of the existence of a definable fully saturated (that is  $\kappa$ -saturated for any cardinal  $\kappa$ ) elementary extension of the whole set universe of  $\mathbf{ZFC}$  is even more challenging. In this case even the permission to use arbitrary sets as parameters of definition does not make things easier because any intended construction of an Ord-long chain of consecutive ultrapowers with increasing amount of saturation needs an Ord-long sequence of choices of suitable ultrafilters, which is hardly possible provably in  $\mathbf{ZFC}$ . Nevertheless we prove

**Theorem 1** *There exists, provably in  $\mathbf{ZFC}$ , a definable fully saturated elementary extension of the whole set universe of  $\mathbf{ZFC}$ .*

Such an extension can be viewed as an interpretation of bounded set theory  $\mathbf{BST}$  in  $\mathbf{ZFC}$ . The theory  $\mathbf{BST}$  is an improved, foundational-friendly version of Nelson's internal set theory  $\mathbf{IST}$ , in which every set belongs to a standard set.

**Corollary 2** *There exists, provably in  $\mathbf{ZFC}$ , an interpretation of  $\mathbf{BST}$  in  $\mathbf{ZFC}$  such that the standard core of the interpretation coincides with the  $\mathbf{ZFC}$  universe.*

The existence of such an interpretation of  $\mathbf{BST}$  was earlier known only on the base of Global Choice extensions of  $\mathbf{ZFC}$  [2]. Note that  $\mathbf{IST}$  itself does not admit such an interpretation, and essentially by the same reason discovered in [2] Theorem 1 cannot be strengthened to provide the saturation with respect to all  $\mathbf{V}$ -size families ( $\mathbf{V}$  being the  $\mathbf{ZFC}$  universe). For instance, if  $M$  is a minimal transitive model  $M$  of  $\mathbf{ZFC}$  then elementary extensions of  $M$  definable in  $M$  and saturated with respect to all definable  $M$ -size families do not exist [2], 2.8.

The proof of Theorem 1 is a combination of several arguments, most notably the iterated ultrapower construction (with finite support) in the sense of [1], 6.5. and the construction of Ord-long chains of consecutive ultrapower-like extensions.

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- [2] V. Kanovei and M. Reeken, Internal approach to external sets and universes, part 1, *Studia Logica*, 1995, 55, 2, pp. 229–257.
- [3] V. Kanovei and S. Shelah, A definable nonstandard model of the reals, *JSL* 2004, 69, 1, to appear.

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