AN $\aleph_2$-SOUSLIN TREE FROM A STRANGE HYPOTHESIS
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§1

Theorem. *Suppose CH holds and the filter $\mathscr{D}_{\omega_1}$ (see below) is $\aleph_2$-saturated. Then there is an $\aleph_2$-Souslin tree.*

Notation: $\mathscr{D}_{\omega_1}$ is the filter generated by the closed unbounded subset of ω_1. Let $S^\alpha_\beta = \{\delta < \aleph_\alpha : \mathrm{cf}(\delta) = \aleph_\beta\}$.

Proof. It is known that the assumption implies $2^{\aleph_1} = \aleph_2$. By Gregory [Gre] we know that if there is a stationary $S \subseteq S^2_0$ with no initial segment stationary, then there is an $\aleph_2$-Souslin tree. So assume there is no such S. By Gregory [Gre], $\diamondsuit(S^2_0)$ holds, and let $\langle A_\delta : \delta \in S^2_0\rangle$ exemplify this. For each $\alpha \in S^2_1$ define $\mathscr{P}_\alpha = \{B \subseteq \alpha : \{\delta < \alpha : B \cap \delta = A_\delta\}$ is a stationary subset of $\alpha\}$. If $|\mathscr{P}_\alpha| > \aleph_1$ let $B_i \in \mathscr{P}_\alpha (i < \aleph_2)$ be pairwise distinct and $\langle \gamma(\zeta) : \zeta < \omega_1\rangle$ be an increasing continuous sequence of ordinals such that $\gamma(\zeta) < \alpha$ and $\bigcup_\zeta \gamma(\zeta) = \alpha$; let $S_i =: \{\zeta < \omega_1 : B_i \cap \gamma(\zeta) = A_{\gamma(\zeta)}\}$. Now S_i is a stationary subset of ω_1 (as $B_i \in \mathscr{P}_\alpha$) and $S_i \cap S_j$ is bounded for $i \neq j$ (as for some $\zeta_0 < \omega_1, B_i \cap \gamma(\zeta_0) \neq B_j \cap \gamma(\zeta_0))$ hence $\langle S_i : i < \omega_2\rangle$ exemplifies that $\mathscr{D}_{\omega_1}$ is $\aleph_2$-saturated, contradiction, hence $|P_\alpha| \leq \aleph_1$. Also for every $A \subseteq \omega_2, |[\delta \in S^2_0 : A \cap \delta = A_\delta\}$ is stationary hence for some $\alpha \in S^2_1, \{\delta \in S^2_0 \cap \alpha : A \cap \delta = A_\delta\}$ is stationary below α. So $\langle \mathscr{P}_\alpha : \alpha \in S^2_1\rangle$ exemplify a variant of $\diamondsuit(S^2_1)$ which by Kunen implies $\diamondsuit(S^2_1)$; together with $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$ we finish.

Remark. We can replace $(\aleph_1, \mathscr{D}_{\omega_1})$ by $(\aleph_\alpha, D(\aleph_{\alpha+1}) + S^{\alpha+1}_\alpha)$ if $\aleph_\alpha$ is regular.
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REFERENCES.

[Gre] John Gregory. Higher Souslin trees and the generalized continuum hypothesis. *Journal of Symbolic Logic*, **41**(3):663–671, 1976.