Revised GCH

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Dedicated to Azriel Levy

Papers are available from Mathematics arXive
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Relevant papers in the author's list 829 (and [Sh:e], 460, 835, 840 and [GiSh:344], [DjSh:562], [DvSh:65], 775, E23)

A prelude

It is good to start with some ideological discussions: in this way we ensure that everybody will understand some part of the lecture, and can authoritatively agree or disagree with the lecturer.

Godel had suggested as a criterion for adopting additional axioms:

we can easily prove theorems which latter can be proved in ZFC.

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Under this criterion, $V = L$ fail miserably. In fact this axiom has been maltreated generally as a candidate for an axiom, Godel and Jensen who might have been expected to support it, has stated categorically it is not a true axiom. But it seem to me that ZFC essentially exhaust our intuition, however: there are many important semi axioms, my opinion appear at length among my dreams, [Sh: E23]. There are some obvious tests for the interest in such semi axiom:

- how strong is it (can it decide many questions)
- how reasonable it is (ZFC is inconsistent is low here)

See more there. An important criterion for me is

THESIS: Semi axiom is commendable if we can prove from it very strong theorems, which may fail under ZFC but

we can latter prove poor relatives of them, though probably, at least in many cases, we would not have thought on them otherwise.

We also expect that the proof under the semi axiom is easier and come more naturally to mind though this is not necessarily so.

Now a point of this lecture is that under this criterion, $V = L$ is successful.

We choose to concentrate on the GCH (which Godel prove in L) and on the diamonds $\diamondsuit_\lambda$, (discovered by Jensen in L),

Let us mention some cases fitting the thesis but we shall not speak much of them in this lecture

The ideal $I[\lambda]$

Guessing of clubs

Recall:

Definition: For a stationary subset S of a regular cardinal λ , let $\diamondsuit_S$ mean that we can find a sequenced $\langle A_\delta : \delta \in S \rangle$ with $A_\delta \subseteq \delta$ which guess everything; that is :

for every $A \subseteq \lambda$ for stationarily many $\delta \in S$ we have $A \cap \delta = A_\alpha$

Theorem (Jensen) If $V = L$ then for every stationary subset S of a regular cardinal λ, $\diamondsuit_S$ holds

Theorem (Gregory and Shelah) 1) If GCH then every successor cardinal except possibly $\aleph_1$ satisfies the diamond

2) Is $\kappa = cf(\kappa) \neq cf(\mu), \kappa \leq \mu$ and $\lambda = \mu^+ = 2^\mu$ then the set $S^\lambda_\kappa =^{def} \{\delta < \lambda : cf(\delta) = \kappa\}$ has diamond.

This says that diamond is not so far from GCH (in fact in a case by case way) but what about GCH itself?

The main thesis of [Sh:460] is that though GCH may fail miserably and persuasively "usually" does, if we represent it as a statement saying " λ has few subsets of cardinality κ, ie as a statement on two cardinals, it is conceivable that for most pairs it holds. Recall that, working hard

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Magidor prove that GCH may fail on any initial segment of the cardinals

Foreman and Woodin prove that it may fail on all of them.

The interpretation in [Sh:460] is

THESIS: Revised GCH for the pair (λ, κ) where $\kappa = cf(\kappa) < \lambda$ mean that there is a family P of $\leq \lambda$ subsets of λ each of cardinality κ such that any subset of λ of cardinality κ is equal/included in the union of $< \kappa$ members of the family

The difference between equal and included is not serious: for $\lambda \geq 2^\kappa$ they are equivalent, and otherwise only the "included" make sense.

The main result of [Sh:460] is that

Theorem: For any $\lambda \geq \beth_\omega$ for every large enough regular $\kappa < \beth_\omega$ the revised GCH holds for the pair (λ, κ)

By Gitik Shelah [GiSh:344], we cannot get better than "except finitely many"

Conclusion: If $\lambda = \mu^+ = 2^\mu \geq \beth_\omega$ then for every large enough regular $\kappa < \beth_\omega$ the stationary set S^λ_κ holds.

Seemingly the most striking result of [Sh:829] is the following improvement of the conclusion above:

Conclusion: If $\lambda = \mu^+ = 2^\mu \geq \beth_\omega$ then for all but finitely many regular $\kappa < \beth_\omega$ for the stationary set S^λ_κ the diamond holds.

Still we start here from GCH

Let us go back

Theorem: (Devlin and Shelah) 1) If $2^{\aleph_1} > 2^{\aleph_0}$ then the weak diamond holds for $\aleph_1$ (see below)

2) if $\theta < \lambda = cf(\lambda), 2^\theta = 2^{<\lambda} < 2^\lambda$ then λ satisfies the weak diamond

The first part is not a ZFC result, the second is, there is a class of such cardinals. The weak diamond differ from the diamond as follows: instead of guessing what is $A \cap \delta$ we just guess whether it is black or white.

Definition: We say that a stationary subset S of a regular uncountable cardinal λ has the weak diamond when for every colouring $\mathbf{c}$ of $\cup\{{}^\delta 2 : \delta \in S\}$ by the two colours $\{0, 1\}$ there is a guessing sequence $g \in {}^\lambda 2$ which mean that for every $f \in {}^\lambda 2$ for stationarily many $\delta \in S$ we have $\mathbf{c}(f \restriction \delta) = g(\delta)$.

Part (1) of this theorem is not in ZFC, but part (2) is, there is a class of cardinals for which this holds in ZFC but our control is very limited: we do no know on the cofinality of S, and it may occurs quite rarely (e.g. if 2^λ is the first weakly inaccessible cardinal above it).

Consider another statement which sit in the middle between the weak diamond and the true diamond, hence it has the not so surprising name "middle diamond".

Definition: ([Sh:775]) Assume that λ is regular uncountable $\kappa = cf(\kappa) < \lambda$ and $S \subseteq S^{\lambda}_{\kappa}$ is stationary, we say that S has the middle diamond (and then say that (λ, κ) has it) when:

there are a sequence $\langle C_\delta : \delta \in S \rangle$ and f_δ a function from C_δ into $\{0, 1\}$ for $\delta \in S$ such that

a. C_δ is a club of order type $cf(\delta)$

b. for every $f \in {}^{\lambda}2$ for stationarily many $\delta \in S$ we have $f \restriction C_\delta = f_\delta$

Theorem: for every regular not strongly inaccessible cardinal $\lambda > \beth_{\omega_1}$ for all but finitely many regular $\kappa < \beth_{\omega_1}$ the pair (λ, κ) has the middle diamond

(We can demand more on the C_δ-s)

Theorem: Assume

1. $\aleph_0 < \sigma = cf(\sigma) \leq \kappa \leq \theta$
2. J is a κ-complete ideal on κ
3. $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$
4. $T_J(\bar{\lambda}) = \lambda$
5. $\lambda_i^{[\sigma,\theta]} = \lambda_i$ for $i < \kappa$
6. if $\sigma_i < \sigma$ for $i < \kappa$ then $\prod_{i<\kappa} \sigma_i < \sigma$
7. $\theta = \theta^\kappa$ and $2^\theta \leq \lambda$.

Then $\lambda^{[\sigma,\theta]} = \lambda$.