A COMMENT ON "$\mathfrak{p} < \mathfrak{t}$"

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ABSTRACT. Dealing with the cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$ of the continuum we prove that $\mathfrak{m} = \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{t} = \aleph_1$. In other words if $\mathrm{MA}_{\aleph_1}$ (or a weak version of this) then (of course $\aleph_2 \le \mathfrak{p} \le \mathfrak{t}$ and) $\mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{p} = \mathfrak{t}$. This is based on giving a consequence.

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§0

The cardinal invariants $\mathfrak{p}$ measure when a family of infinite subsets of ω, with any finitely many members, has a pseudo intersection (see below). A relative is $\mathfrak{t}$, which deals with towers, i.e. families well ordered by almost inclusion. They are classical cardinal invariants and closely related. In fact Rothenberg [Ro39], [Ro48] proved (stated in our terminology) that $\mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{p} = \mathfrak{t}$ and ask if $\mathfrak{p} = \mathfrak{t}$.

We prove (2.2) that $\mathfrak{m} = \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{p} = \mathfrak{t}$; considering that $\mathrm{MA}_{\aleph_0}$ is a theorem and $\mathfrak{m} = \aleph_2 \Rightarrow \mathrm{MA}_{\aleph_1}$. The parallelism with Rothenberg is clear. The reader may conclude that probably $\mathfrak{m} = \mathfrak{p} \Rightarrow \mathfrak{p} = \mathfrak{t}$; not unreasonable but it seemed that ([\Sh:769]: $\mathrm{CON}(\mathrm{MA}_{<\lambda} + \mathfrak{p} = \lambda + \mathfrak{t} = \lambda^+)$. The proof of 2.2 uses a characterization of $\mathfrak{p} < \mathfrak{t}$ from §1.

§1 A reduction

1.1 Hypothesis. $\lambda = \mathfrak{p} < \mathfrak{t}$.

Our aim is to prove 1.13, i.e., $\mathfrak{p} < \mathfrak{t}$ iff $({}^\omega\omega, <^*)$ has a peculiar cut. We give a self-contained proof (except using Bell theorem); this will give the background for a try to prove the consistency of CON($\mathfrak{p} < \mathfrak{t}$) in [Sh:F769]. The results up to 1.8 are well known and essentially covered by [BaJu95, §2.2] in particular $\mathfrak{t} \le \mathscr{M}$ is of Piotrowski and Szymanski.

Note also that Szymanski had proved that $\mathfrak{p}$ is regular (see, e.g., Fremlin [Fre], proposition 21K).

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1.2 Convention.: A, B denote members of $[\omega]^{\aleph_0}$.

1.3 Definition. 1) $\mathscr{B}$ exemplifies $\mathfrak{p}$ if:

- (a) $\mathscr{B} \subseteq [\omega]^{\aleph_0}$ has cardinality λ
- (b) $A, B \in \mathscr{B} \Rightarrow A \cap B \in \mathscr{B}$
- (c) for no $A \in [\omega]^{\aleph_0}$ do we have $B \in \mathscr{B} \Rightarrow A \subseteq^* B$.

1.4 Claim. *For any $\mathscr{B}$ exemplifying $\mathfrak{p}$ there are $\kappa = \mathrm{cf}(\kappa) < \lambda$ and $\subseteq^*$-decreasing $\langle A_i : i < \kappa \rangle$ such that $\bigwedge\limits_{i<\kappa} A \subseteq^* A_i \Rightarrow \bigvee\limits_{B \in \mathscr{B}} A \cap B \in [\omega]^{<\aleph_0}$, and of course, $\bigwedge\limits_{i<\kappa} \bigwedge\limits_{B \in \mathscr{B}} A_i \cap B$ is infinite.*

Proof. Let $\mathscr{B} = \{B_i : i < \lambda\}$ and try to choose $A_i \in [\omega]^{\aleph_0}$ by induction on $i < \lambda$ such that

- (α) $j < i \Rightarrow A_i \subseteq^* A_j$
- (β) $B \in \mathscr{B} \Rightarrow |B \cap A_i| = \aleph_0$
- (γ) $i = j + 1 \Rightarrow A_i \subseteq B_j$.

If we succeed then $\{A_i : i < \lambda\}$ has no pseudo-intersection so $\mathfrak{t} \le \lambda$, contradiction. So for some $i < \lambda$ we cannot choose A_i. Easily i is a limit ordinal and let $\kappa = \mathrm{cf}(i)$ so $\kappa \le i < \lambda$; choose $\langle j_\varepsilon : \varepsilon < \kappa \rangle$ which is increasing with limit i. So $\langle A_{j_\varepsilon} : \varepsilon < \kappa \rangle$ is as required.

1.5 Claim. $\mathfrak{b} \geq \mathfrak{t} > \lambda$.

Proof. Easy.

1.6 Claim. *If $\bar{A} = \langle A_i : i < \delta \rangle$ is a sequence of members of $[\omega]^{\aleph_0}, \delta < \lambda^+$ (or even $< \mathfrak{t}$) and $\bar{B} = \langle B_n : n < \omega \rangle$ is $\subseteq^*$-decreasing and $i < \delta$ & $n < \omega \Rightarrow A_i \cap B_n \in [\omega]^{\aleph_0}$ and $i < j < \delta \Rightarrow \bigvee_n [A_j \cap B_n \subseteq^* A_i \cap B_n]$ <u>then</u> for some $A \in [\omega]^{\aleph_0}$ we have $i < \delta \Rightarrow A \subseteq^* A_i$ and $n < \omega \Rightarrow A \subseteq^* B_n$.*

Proof. Without loss of generality $B_{n+1} \subseteq B_n$ and $\emptyset = \cap\{B_n : n < \omega\}$ (use $B'_n = \bigcap_{\ell \leq n} B_\ell \backslash \{0, \ldots, n\}$). For each $i < \delta$ let $f_i \in {}^\omega\omega$ be defined by $f_i(n) =$ $\text{Min}\{k+1 : k \in B_n \cap A_i$ and $k > f(m)$ for every $m < n\}$, so by 1.5 there is $f \in {}^\omega\omega$ such that $\bigwedge_{i<\kappa} f_i \leq^* f$ and $n < f(n) < f(n+1)$ for $n < \omega$. Let $B^* = \cup\{(B_{n+1} \backslash [0,n) \cap [0, f(n+1)] : n < \omega\}$ so $B^* \in [\omega]^{\aleph_0}$ as for n large enough, $\text{Min}[(B_{n+1} \backslash [0,n) \cap A_0] \in (B_{n+1} \backslash [0,n]) \cap [0, f(n+1))$ and $\exists^\infty n((B_{n+1} \backslash [0,n]) \cap A_0 \neq \emptyset)$. Clearly $n < \omega \Rightarrow B^* \backslash [0, f(n)] \subseteq B_n \Rightarrow B^* \subseteq^* B_n$ and also $i < \kappa \Rightarrow A_i \cap B^* \in [\omega]^{\aleph_0}$ (as above) so apply $\mathfrak{t} > \lambda$ to $\langle A_i \cap B^* : i < \delta \rangle$ getting A^*, it is as required. $\square_{1.6}$

1.7 Definition. 1) $\mathbf{S} = \{\bar{\eta} : \bar{\eta}$ is $\langle \eta_n : n \in A \rangle$ where $A \in [\omega]^{\aleph_0}, \eta_n \in {}^{[n,k)}2$ for some $k \in (n, \omega)\}$ and let $\text{Dom}(\bar{\eta}) = A$ and let $\text{set}(\bar{\eta}) = \cup\{\text{ set}(\eta_n) : n \in \text{ Dom}(\bar{\eta})\}$ where $\text{set}(\eta_n) = \{\ell : \eta_n(\ell) = 1\}$.
2) For $\bar{\eta}, \bar{\nu} \in \mathbf{S}$ let $\bar{\eta} \leq^* \bar{\nu}$ mean that for every n large enough, $n \in \text{ Dom}(\bar{\nu}) \Rightarrow n \in \text{ Dom}(\bar{\eta}) \wedge \eta_n \trianglelefteq \nu_n$.
3) For $\bar{\eta}, \bar{\nu} \in \mathbf{S}$ let $\bar{\eta} \leq^{**} \bar{\nu}$ mean that for every $n \in \text{ Dom}(\bar{\nu})$ large enough for some $m \in \text{ Dom}(\bar{\eta})$ we have $\eta_m \subseteq \nu_n$ (as functions).
4) For $\bar{\eta} \in \mathbf{S}$ let $C_{\bar{\eta}} =: \{\nu \in {}^\omega 2 : (\exists^\infty n)\eta_n \subseteq \nu\}$.

1.8 Claim. *1) The union of $\leq \lambda$ meagre subsets of ${}^\omega 2$ is meagre.*
2) Every $\leq^*$-increasing sequence members of $\mathbf{S}$ of length $\leq \lambda$ (and even $< \mathfrak{t}$) has an $\leq^*$-upper bound.
3) $(\mathbf{S}, \leq^*)$ is λ-directed.
4) Similarly for $\leq^{**}$.

Proof. 1) Can be translated to part (2) as by 1.9 below $\{C_{\bar{\eta}} : \bar{\eta} \in \mathbf{S}\}$ is dense among the co-meagre subsets of ${}^\omega 2$.

2) By 1.6. In full, let $\langle \bar\eta^\alpha : \alpha < \delta\rangle$ be as $\le^*$-increasing sequence and $\delta < \mathfrak{t}$. Let $A^*_\alpha =:$ Dom$(\bar\eta^\alpha)$ for $\alpha < \delta$, so $\langle A^*_i : \alpha < \delta\rangle$ is a $\subseteq^*$-decreasing sequence of members of $[\omega]^{\aleph_0}$. As $\delta < \mathfrak{t}$ there is $A^* \in [\omega]^{\aleph_0}$ such that $\alpha < \delta \Rightarrow A \subseteq^* A_\alpha$. Now for $n < \omega$ we define $B_n = \{\eta$: for some $m < k$ we have $m \in A^*, n \le m < k < \omega$ and $\eta \in {}^{[m,k)}2\}$, and for $\alpha < \delta$ we define $A_\alpha =: \{\eta$: for some $n \in$ Dom$(\bar\eta^\alpha)$ we have $\eta^\alpha_n \trianglelefteq \eta\}$. Now easily the assumptions of **?** holds (well, replacing ω by
—> scite{p.4} undefined
B_0!!) If A is as in the conclusion of **?** we let $A' = \{n$: for some $\eta \in A$ we have
—> scite{p.4} undefined
$\eta \in \cup\{{}^{[n,k)}2 : k \in (n,\omega)\}$, it is necessarily infinite and let $\bar\eta^* = \langle \eta_n : n \in A'\rangle$ where η_n is any member of $A \cap B_n \backslash B_{n+1}$.
3) Easy, too.
4) Similar to the proof of part (2). $\square_{1.8}$

1.9 Observation. 1) If $\bar\eta \le^* \bar\nu$ and then $\bar\eta \le^{**} \bar\nu$ which implies $C_{\bar\nu} \subseteq C_{\bar\eta}$.
2) For every $\bar\eta \in \mathbf{S}$ and meagre $B \subseteq {}^\omega 2$ there is $\bar\nu \in \mathbf{S}$ such that $\bar\eta \le^* \bar\nu$ and $C_{\bar\nu} \cap B = \emptyset$.

1.10 Definition. For $\mathfrak{A} \subseteq [\omega]^{\aleph_0}$ let $\mathbf{S}_{\mathfrak{A}} = \{\bar\eta \in \mathbf{S} : (\forall B \in \mathfrak{A})[\text{set}(\bar\eta) \subseteq^* B]$ and for every n, set$(\eta_n) \ne \emptyset\}$. If we write $\bar A' = \langle A'_i : i < \alpha\rangle$ instead $\mathfrak{A}$ we mean $\{A'_i : i < \alpha\}$.

1.11 Claim. *If $\mathfrak{A} \subseteq [\omega]^{\aleph_0}$ with f.i.p. and $|\mathfrak{A}| < \lambda$ then $\mathbf{S}_{\mathfrak{A}} \ne \emptyset$.*

Proof. Let $A \in [\omega]^{\aleph_0}$ be such that $B \in \mathfrak{A} \Rightarrow A \subseteq^* B$, exists as $|\mathfrak{A}| < \lambda$. Let $k_n =$ Min$\{k : k > n$ and $k \in A\}$, and let $\eta_n \in {}^{[n,k_n+1)}2$ be defined by $\eta_n(\ell)$ is 0 if $\ell \in [n, k_n)$ and is 1 if $\ell = k_n$. Now $\langle \eta_n : n < \omega\rangle$ is as required. $\square_{1.11}$

1.12 Claim. *Let $\bar A = \langle A_i : i < \kappa\rangle$ be $\subseteq^*$-decreasing, $\kappa < \lambda$ and $\mathscr{B} = \{B_\alpha : \alpha < \lambda\}$ exemplifying $\mathfrak{p}$ be as in 1.4 and let* pr:$\lambda \times \lambda \to \lambda$ *be one to one onto satisfying* pr$(\alpha_1, \alpha_2) \ge \alpha_1, \alpha_2$. *Then we can find $\langle \bar\eta^\alpha : \alpha \le \lambda\rangle$ such that*

(a) *$\bar\eta^\alpha \in \mathbf{S}_{\bar A}$ for $\alpha < \lambda$ and $\bar\eta^\lambda \in \mathbf{S}$ (sic!)*
(b) *$\langle \bar\eta^\alpha : \alpha \le \lambda\rangle$ is $\le^*$-increasing*
(c) *if $\alpha < \lambda$ and $n \in$ Dom$(\bar\eta^{\alpha+1})$ large enough then* set$(\eta^{\alpha+1}_n) \cap B_\alpha \ne \emptyset$ *(hence $(\forall^* n \in$ Dom$(\bar\eta^\beta))($set$(\eta^\beta_n) \cap B_\alpha \ne \emptyset)$ holds for every $\beta \in [\alpha+1, \lambda])$*
(d) *if $\alpha =$ pr(β,γ) then the truth value of* Min(set$(\eta^{\alpha+1}_n) \cap B_\beta) <$ Min(set$(\eta^{\alpha+1}_n) \cap B_\gamma$)) *and of set $(\eta^{\alpha+1}_n) \cap B_\beta \ne \emptyset$, and of set $(\eta^{\alpha+1}_n) \cap B_j \ne \emptyset$ are the same for all $n \in$ Dom$(\bar\eta^{\alpha+1})$*
(e) *in (d), if $\beta < \kappa$ we can replace B_β by A_β; similarly with γ and $\beta, \gamma < \kappa$ we can replace both.*

Proof. We choose $\bar\eta^\alpha$ by induction on α. For $\alpha = 0$ trivial. For α limit $< \lambda$ use 1.7(2) theorem (and $|\alpha| < \lambda$). For α successor first choose $B'_\alpha \in [\omega]^{\aleph_0}$ such that $B'_\alpha \subseteq B_\alpha$ and $i < \kappa \Rightarrow B'_\alpha \subseteq^* A_i$. Second, for $n \in \mathrm{Dom}(\bar\eta^{\alpha-1})$ choose η'_n such that $\eta^{\alpha-1}_n \triangleleft \eta'_n$ and $\{\ell : \eta'_n(\ell) = 1$ and $\ell g(\eta^{\alpha-1}_n) \le \ell < \ell g(\eta'_n)\}$ is not empty and is included in B'_α. Thirdly, let $\bar\eta^\alpha = \langle \eta'_n : n \in \mathrm{Dom}(\eta^{\alpha-1})\rangle$. By shrinking the domain of $\bar\eta^\alpha$ there is no problem to take care of clauses (d),(e).
For $\alpha = \lambda$, use 1.8. $\square_{1.12}$

1.13 Theorem. *In 1.12, we can find $\kappa = \mathrm{cf}(\kappa) < \lambda$ and $f_i, f^\alpha \in {}^\omega\omega$ for $i < \kappa, \alpha < \lambda$ satisfies*

- (α) $i < j < \kappa \Rightarrow f_j \le^* f_i$ *(and without loss of generality* $<^*$*)*
- (β) $\alpha < \beta < \lambda \Rightarrow f^\alpha \le^* f^\beta$ *(and without loss of generality* $<^*$*)*
- (γ) $i < \kappa \wedge \alpha < \lambda \Rightarrow f^\alpha \le^* f_i$ *(clearly* $<^*$ *holds)*
- (δ) *if* $f : A^* \to \omega$ *and* $\bigwedge\limits_{i<\kappa} f \le^* f_i$ *then* $\bigvee\limits_{\alpha<\lambda} f \le^* f^\alpha$
- (ε) *if* $f : A^* \to \omega$ *and* $\bigwedge\limits_{\alpha<\lambda} f^\alpha \le^* f$ *then* $\bigvee\limits_{i<\kappa} f_i \le^* f$.

Proof of 1.13. It is enough to find such $f_i (i < \kappa), f^\alpha (\alpha < \lambda)$ from ${}^{A^*}\omega$ for some infinite $A^* \subseteq \omega$ (so by renaming, it is ω) Let $\langle A_i : i < \kappa\rangle, \langle B_\alpha : \alpha < \lambda\rangle, \langle \bar\eta^\alpha : \alpha \le \lambda\rangle$ be as in 1.12. Let

$\circledast_0$ (a) $A^* = \mathrm{Dom}(\eta^\lambda)$

(b) for $i < \kappa$ let $f_i : A^* \to \omega$ be $f_i(n) = \mathrm{Min}\{\ell : \eta^\lambda_n(n+\ell) = 1, n+\ell \notin A_i$ or $\mathrm{Dom}(\eta^\lambda_n) = [n, n+\ell)\}$

(c) for $\alpha < \lambda$ let $f^\alpha : A^* \to \omega$ be $f^\alpha(n) =$ $\mathrm{Min}\{\ell + 1 : \eta^\lambda_n(n+\ell) = 1, n+\ell \in B_\alpha$ or $\mathrm{Dom}(\eta^\lambda_n) = [n, n+\ell)\}$.

Now $\kappa = \mathrm{cf}(\kappa)$ by 1.4 and $f_i, f^\alpha \in {}^{A^*}\omega$ by their definitions (remembering that $\eta^\lambda_n \in \cup\{{}^{[n,k)}2 : k \in (n, \omega)\}$ by the definition of $\mathbf{S}$).

Note that (by the choice of f_i, i.e., clause (b)):

$(*)_1$ $\cup\{[n, n+f_i(n)) \cap \mathrm{set}(\eta^\lambda_n) : n \in A^*\} \subseteq^* A_i$ for every $i < \kappa$.

$\circledast_1$ <u>Clause (α)</u> holds.

Why? Let $i < j < \kappa$ so (by 1.4) $A_j \subseteq^* A_i$ hence for some $n^*, A_j \backslash n^* \subseteq A_i$, hence for every $n \in A^* \backslash n^*$ in the definition of f_i, f_j in clause (b), if ℓ can serve as a candidate for $f_i(n)$ then it can serve for $f_j(n)$ so (as we use the minimum there) $f_j(n) \le f_i(n)$. Hence $f_j \le^* f_i$.

To have "without loss of generality $f_j <^* f_i$", it is enough to show that for every $i < \kappa$ for some $j \in (i, \kappa)$ we have $f_j <^* f_i$, so assume toward contradiction that for some $i(*) < \kappa$ we have $(\forall j)(i(*) < j < \kappa \to \neg(f_j <^* f_{i(*)}))$ hence for $j < \kappa$ let $B^*_j =: \{n \in A^* : f_j(n) \geq f_{i(*)}(n)\}$ so $B^*_j \in [A^*]^{\aleph_0}$ is $\subseteq^*$-decreasing, so there is a pseudo-intersection B^* of $\langle B^*_j : j < \kappa\rangle$; i.e., $B^* \in [\omega]^{\aleph_0}$ and $j < \kappa \Rightarrow B^* \subseteq^* B^*_j$. Now letting $A' = \cup\{\text{set}(\eta^\lambda_n) \cap [n, n + f_{i(*)}(n)) : n \in B^*\}$ it satisfies

- (i) it is an infinite subset of ω
  [Why? By recalling that by clause (a) of 1.12 we have $\bar\eta^0 \in \mathbf{S}_{\bar A}$ hence (see Definition 1.10), we have $n \in \text{Dom}(\bar\eta^0) \Rightarrow \text{set}(\eta_n) \neq \emptyset$ and $\text{set}(\bar\eta^0) \subseteq^* A_{i(*)}$. By clause (b) of $\circledast_0$ for every n large enough, $n \in \text{Dom}(\bar\eta^\lambda) \Rightarrow n \in \text{Dom}(\bar\eta^0)$ & $\eta^0_n \trianglelefteq \eta^\lambda_n$. Since for every n large enough $\text{set}(\bar\eta^0)\backslash\{0, \ldots, n-1\} \subseteq A_{i(*)}$ we know that for $n \in \text{Dom}(\bar\eta^\lambda)$ large enough $\eta^0_n \trianglelefteq \eta^\lambda_n \wedge \emptyset \neq \text{set}(\eta^0_n) \subseteq A_{i(*)}$ so $[n, f_{i(*)}(n)) \cap \text{set}(\eta^\lambda_n) \neq \emptyset$ so we are done.]
- (ii) $A' \subseteq^* A_j$ for $j \in (i(*), \kappa)$ (hence for $j < \kappa$)
  [as $f_j \restriction B^* =^* f_{i(*)} \restriction B^*$ for $j \in (i(*), \kappa)$]
- (iii) $A' \cap B_\alpha$ is infinite for $\alpha < \lambda$
  [Why? By clauses (c) + (a) of 1.12, we have: for every large enough $n \in \text{Dom}(\bar\eta^{\alpha+1})$ we have $\text{set}(\eta^{\alpha+1}_n) \cap B_\alpha \neq \emptyset$ and $\text{set}(\eta^{\alpha+1}_n) \subseteq A_{i(*)}$].

Together A' contradicts 1.4, hence $(\forall i < \kappa)(\exists j < \kappa)(f_j <^* f_i)$, so we are done proving $\circledast_1$.

$\circledast_2$ (i) the set (of function) $\{f_i : i < \kappa\} \cup \{f^\alpha : \alpha < \lambda\}$ is linearly ordered by $\leq^*$; moreover

(ii) in fact if f', f'' are in the family then $f' = f'' \mod J^{\text{bd}}_\omega$ or $f' < f'' \mod J^{\text{bd}}_\omega$ or $f'' < f' \mod J^{\text{bd}}_\omega$.

[Why? By clause (d) + (e) of 1.12.]
So we can choose $\langle \alpha(\varepsilon) : \varepsilon < \varepsilon^*\rangle$ such that:

$\circledast_3$ (i) $\alpha(\varepsilon)$ is the minimal $\alpha \in \lambda\backslash\{\alpha(\zeta) : \zeta < \varepsilon\}$ satisfying $i < \kappa \Rightarrow f^\alpha <^* f_i$ and $\zeta < \varepsilon \Rightarrow f^{\alpha(\zeta)} <^* f^\alpha$

(ii) we cannot choose $\alpha(\varepsilon^*)$.

We ignore (till $\circledast_7$) the question of the value of ε^*. Now

$\circledast_4$ $\langle f_i : i < \kappa\rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon(*)\rangle$ satisfies clauses $(\beta), (\gamma)$.

[Why? By clause (i) of $\circledast_3$.]

$\circledast_5$ $\langle f_i : i < \kappa\rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon(*)\rangle$ satisfies clause (δ).

[Why? Assume toward contradiction that $f : A^* \to \omega$ and $i < \kappa \Rightarrow f \le^* f_i$ but $\varepsilon < \varepsilon^* \Rightarrow \neg(f \le^* f^{\alpha(\varepsilon)})$. Clearly without loss of generality $n \in A^* \Rightarrow [n, n + f(n)) \subseteq \mathrm{Dom}(\eta^\lambda_n)$. Let $A' = \cup\{[n, n+f(n)) \cap \mathrm{set}(\eta^\lambda_n)$ in $\circledast_0(b) : n \in A^*\}$. Now for every $i < \kappa, A' \subseteq^* A_i$ because $f \le^* f_i$ and the definition of f_i.
Also, for every $\alpha < \lambda$, the set $A' \cap B_\alpha$ is infinite. Why? Because for some $\varepsilon < \varepsilon^*, f^\alpha \le^* f^{\alpha(\varepsilon)}$ (otherwise by $\circledast_2$ we have $\varepsilon < \varepsilon^* \Rightarrow f^{\alpha(\varepsilon)} <^* f^\alpha$, so α is a candidate to being $\alpha(\varepsilon^*)$ contradiction to clause (ii) of $\circledast_3$). Also we have $\neg(f \le^* f^{\alpha(\varepsilon)})$ (by the assumptions toward contradiction on f). Hence if $n \in A^*$ is large enough then $f^\alpha(n) \le f^{\alpha(\varepsilon)}(n)$ and for infinitely many $n \in A^*$ we have $f^\alpha(n) \le f^{\alpha(\varepsilon)}(n) < f(n) \le f_0(n) \le |\mathrm{dom}(\eta^\lambda_n)|$. For any such n let $\ell^*_n = \mathrm{Min}\{\ell : \eta^\lambda_n(n+\ell) = 1, n+\ell \in B_\alpha$ or $\mathrm{Dom}(\eta^\lambda_n) = [n, n+\ell)\}$, then $\ell^*_n + 1 \le f^\alpha(n) \le f^{\alpha(\varepsilon)}(n) < f(n) \le f_0(n) \le |\mathrm{Dom}(\eta^\lambda_n)|$ hence $n + \ell^*_n \in A' \cap B_\alpha$.
Together A' contradict the choice of $\langle A_i : i < \kappa\rangle, \langle B_\alpha : \alpha < \lambda\rangle$ from 1.4.]

$\circledast_6$ $\langle f_i : i < \kappa\rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon(*)\rangle$ satisfies clause (ε).

[Why? Assume toward contradiction that $f : A^* \to \omega$, and $\varepsilon < \varepsilon^* \Rightarrow f^{\alpha(\varepsilon)} \le^* f$ but $i < \kappa \Rightarrow \neg(f_i \le^* f)$. As $i < j < \kappa \Rightarrow f_j <^* f_i$ and we are assuming $i < \kappa \Rightarrow \neg(f_i \le^* f)$ there is an infinite $A^{**} \subseteq A^*$ such that $i < \kappa \Rightarrow (f \restriction A^{**}) <^* (f_i \restriction A^{**})$. Let[1] $A' = \cup\{[n, n+f(n)) \cap \mathrm{set}(\eta^\lambda_n) : n \in A^{**}\}$, so $A' \subseteq \omega$ and if $\alpha < \lambda$ then by $\circledast_2$ for some $\varepsilon, f^\alpha \le^* f^{\alpha(\varepsilon)}$ and by our assumption toward contradiction, $f^{\alpha(\varepsilon)} \le^* f$, so by the definition of $f^\alpha, f^{\alpha(\varepsilon)}$ we get $A' \cap B_\alpha$ is infinite. As $i < \kappa \Rightarrow (f \restriction A^{**}) <^* (f_i \restriction A^{**})$ we get $i < \kappa = A' \subseteq^* A_i$, so we have gotten a contradiction to the choice of $\langle A_i : i < \kappa\rangle, \langle B_\alpha : \alpha < \lambda\rangle$.]

$\circledast_7$ $\varepsilon^* \le \lambda$,

[Why? As we have chosen $\alpha(\varepsilon)$ "the minimal $\alpha < \lambda$ such that ...", the sequence $\langle \alpha(\varepsilon) : \varepsilon < \lambda\rangle$ is an increasing sequence of ordinals $< \lambda$, hence $\varepsilon(*) \le \lambda$.]

$\circledast_8$ $\varepsilon^* = \lambda$.

[Why? If $\varepsilon^* < \lambda$ by Bell theorem we get contradiction to $\circledast_4$ above.]

So $\langle f_i : i < \kappa\rangle, \langle f^\alpha : \alpha < \lambda\rangle$ are as required (in detail, clause (α) by $\circledast_1$, clause (β) by $\circledast_4$, clause (γ) also by $\circledast_4$, clause (δ) by $\circledast_5$, clause (ε) by $\circledast_6$). $\square_{1.13}$

An obvious compliment to 1.13 is

1.14 Claim. *Assume $\kappa \le \lambda$ are regular and $A^* \subseteq \omega$ is finite and $\langle f_i : i < \kappa\rangle, \langle f^\alpha < \lambda\rangle$ satisfies clauses $(\alpha) - (\delta)$ from Theorem 1.13.*

<u>Then</u>

[1] alternatively we could have replaced $\bar\eta^\lambda$ by $\bar\eta^\lambda \restriction A^{**}$ and use $\circledast_5$

(a) $\mathrm{MA}_{\kappa+\lambda}$*(σ-centered) fail*

(b) *for some σ-centered forcing notion* $\mathbb{Q}$ *of cardinality* κ *and sequence* $\langle \mathscr{I}_\alpha : \alpha < \lambda\rangle$ *of dense subsets of* $\mathbb{Q}$, *there is no directed* $G \subseteq \mathbb{Q}$ *such that* $\alpha < \lambda \Rightarrow G \cap \mathscr{I}_\alpha \neq \emptyset$.

Remark. In 1.13 by restricting the f_i, f^α's to A^* and renaming without loss of generality $A^* = \omega$.

Proof. We define the set of member of $\mathbb{Q}$ as the set of pairs $p = (\rho, u)$ where $\rho \in {}^{\omega>}\omega$ and $u \subseteq \kappa$ is finite.

The order is:

$(\rho_1, u) \leq_{\mathbb{Q}} (\rho_2, u)$ iff (both are in $\mathbb{Q}$ and)(a) $\rho_1 \triangleleft \rho_2$

(b) $u_1 \subseteq u_2$

(c) if $n \in [\ell g(\rho_1), \ell g(\rho_2))$ and $i \in u_1$ then $f_i(n) \leq \rho_2(n)$.

For $\alpha = 2j < \kappa$ let $\mathscr{I}_\alpha = \{(\rho, u) \in \mathbb{Q} : j \in u\}$.

For $\alpha = 2j \in [\kappa, \lambda)$ let $\mathscr{I}_\alpha = \mathscr{I}_0$.

For $\alpha = \omega\beta + 2n + 1 < \lambda$ let

$$\mathscr{I}_\alpha = \{(\rho, u) : \text{for some } m < \ell g(\rho) \text{ we have } m \geq n, m \in A^* \text{ and } \rho(m) < f^\beta(m)\}.$$

Clearly each $\mathscr{I}_\alpha$ is a dense open subset of $\mathbb{Q}$. Suppose toward contradiction there $G \subseteq \mathbb{Q}$ is directed non-disjoint to $\mathscr{I}_\alpha$ for every $\alpha < \lambda$. So $g = \cup\{\rho : (\rho, u) \in G\}$ is a function, its domain is ω (as $G \cap \mathscr{I}_{2n+1} \neq \emptyset$ for $n < \omega$) and $f_i \leq^* g$ (as $G \cap \mathscr{I}_{2i} \neq \emptyset$) and $\{n \in A^* : g(n) < f^\alpha(n)\}$ is infinite as $G \cap \mathscr{I}_{\omega\alpha+2n+1} \neq \emptyset$ for every n.

Lastly, trivially $\mathbb{Q}$ is σ-centered as for each $\rho \in {}^\omega\omega$, the subset $\{(\eta, u) \in \mathbb{Q} : \eta = \rho\}$ is directed. $\square_{\mathbf{1.14}}$

→ scite{y.17} ambiguous

§2

2.1 Claim. *In Theorem 1.13:*

(a) $\aleph_1 \le \kappa = \mathrm{cf}(\kappa) < \lambda = \mathfrak{p} = \mathrm{cf}(\lambda)$

(b) *if* $MA_{\aleph_1}$ *then* $\kappa = \aleph_1$ *is impossible.*

2.2 Conclusion. If $\mathrm{MA}_{\aleph_1}$ then $\mathfrak{p} = \aleph_2 \Leftrightarrow \mathfrak{t} = \aleph_0$. In other words $\mathfrak{m} = \mathfrak{p} = \aleph_1 \Rightarrow \mathfrak{p} = \aleph_2$.

Remark. The proof of (b) actually uses Hausdorff gaps on which much is known, see (xxxx).

Proof. Clause (a):

By 1.6 clearly $\kappa \ne \aleph_0$ and by its choice κ is regular $< \lambda = \mathfrak{p} = \mathrm{cf}(\lambda)$.

Clause (b):

Assume $\kappa = \aleph_1$. We define a forcing notion $\mathbb{Q}$ as follows:

$(*)_1$ $p \in \mathbb{Q}$ iff

- (a) $p = (u, \bar\rho) = (u_p, \bar\rho_p)$
- (b) $u \subseteq \omega_1$ is finite
- (c) $\bar\rho_0 = \langle \rho_\alpha : \alpha \in u \rangle$ and let $\rho_\alpha = \rho^p_\alpha$
- (d) for some $n = n_p$ we have $\alpha \in u \Rightarrow \rho_\alpha \in {}^n\omega$
- (e) $f_\alpha \restriction n_p \le \rho_\alpha$ for $\alpha \in u_p$, i.e. $n < n_p \Rightarrow f_\alpha(n) \le \rho_\alpha(n)$
- (f) $\langle f_\alpha \restriction [n_p, \omega) : \alpha \in u \rangle$ is increasing

$(*)_2$ $p \le_{\mathbb{Q}}$ iff

- (a) $u_p \subseteq u_q$
- (b) $\rho^p_\alpha \trianglelefteq \rho^q_\alpha$ for every $\alpha \in u_p$
- (c) if $\alpha < \beta$ are from u_p then $\rho^q_\alpha \restriction [n_p, n_q) < \rho^q_\beta \restriction [n_p, n_q)$
- (d) if $\alpha < \beta, \alpha \in u_q \backslash u_p$ and $\beta \in u_p$ then for some $n \in [n_p, n_q)$ we have $\rho^q_\beta(n) < \rho^q_\alpha(n)$.

$(*)_3$ $\mathbb{Q}$ is a quasi-order of cardinality $\aleph_1$.

[Why? Obvious.]

$(*)_4$ $\mathbb{Q}$ satisfies the c.c.c.

[Why? Let $p_\varepsilon \in \mathbb{Q}$ for $\varepsilon < \omega_1$. Without loss of generality $\langle p_\varepsilon : \varepsilon < \omega_1 \rangle$ is without repetition. We can find an unbounded $\mathscr{U} \subseteq \omega_1$ and $n(*) < \omega$

(a) if $\varepsilon \in \mathscr{U}$ then $|u_{p_\varepsilon}| = n(*) = \bar{\rho}_*$ so let $u_\varepsilon = \{\alpha_{\varepsilon,\ell} : \ell < n(*)\}$ and $\alpha_{\varepsilon,\ell}$ increases with ℓ.

By pigeon-hole pre? for some $m(*) \leq n(*)$

(b) $\alpha_{\varepsilon,\ell} = \alpha_\ell, \rho_{\varepsilon,\ell} = \rho^*_\ell$ for $\ell < m(*)$.

If $m(*) = n(*)$ then $p_\varepsilon = p_\zeta$ for $\varepsilon, \zeta \in \mathscr{U}$, contradiction any assumption, so $m(*) < n(*)$ so by the Δ-system lemma

(c) if $\varepsilon < \zeta$ are from $\mathscr{U}, \rho \in [m(*), n(*)), k \in [m(*), n(*))$ then $\alpha_{\varepsilon,\ell} < \alpha_{\zeta,k}$.

We can find $\delta(*) < \omega_1$ such that

(d) for every $\zeta \in \mathscr{U}$ and $k < \omega$ there is $\varepsilon \in \delta(*) \cap \mathscr{U}$ such that $\ell < n(*) \Rightarrow f_{\alpha_{\zeta,\ell}} \restriction k = f_{\alpha_{\varepsilon,\ell}} \restriction k$.

Now choose $\zeta_1 < \zeta_2$ from $\mathscr{U} \backslash \delta(*)$, hence we can find $k < \omega$ such that $\langle f_\alpha(k) : \alpha \in \{\alpha_{\varepsilon,\ell} : \ell < n(*)\} \cup \{\alpha_{\varepsilon,\ell} : \ell < n(*)\}\rangle$ is a strictly increasing sequence of natural numbers (in fact every k large enough is O.K. as $\langle f_\alpha : \alpha < \omega_1 \rangle$ is $<^*$-incrasing).

Apply clause (d) to $(\zeta_2, k+1)$ and get $\varepsilon \in \delta(*) \cap \mathscr{U}$. Now define $q = (u_1, \rho_q)$ as follows:

$$u_q = u_{p_\varepsilon} \cup u_{p_{\zeta_1}}$$

$$\rho^q_\alpha \text{ is } : \rho^{p_\varepsilon}_\alpha \cup (f_\alpha \restriction [n(*), k+1)) \text{ if } \alpha \in u_{p_\varepsilon}$$
$$\rho^{p_{\zeta_1}}_\alpha \cup (f_\alpha \restriction [n(*), k+1) \text{ if } \alpha \in u_{p_{\zeta_1}}.$$

It is well defined (as $\rho^{p_\varepsilon}_{\alpha_{\varepsilon,\ell}} \subset \rho^{p_\varepsilon}_{\alpha_{\zeta_1,\ell}}$ for $\ell < m(*)$).
Also $q \in \mathbb{Q}$.
Lastly, $p_\varepsilon \leq_{\mathbb{Q}} q, p_{\zeta_2} \leq q$, easily checked.

$(*)_5$ for each $\alpha < \omega_1$ and $n < \omega$ the set $\mathscr{I}_{\alpha,n}$ is a dense open subset of $\mathbb{Q}$ where $\mathscr{I}_{\alpha,n} = \{p : u_p \nsubseteq \alpha \text{ or for no } q, p \leq_{\mathbb{Q}} q \wedge u_q \nsubseteq \alpha\}$

$(*)_6$ for each α there is $p^*_\alpha \in \mathbb{Q}$ such that $u_{p_\alpha} = \{\alpha\}$

$(*)_7$ for some $\alpha(*), p^*_{\alpha(*)} \Vdash_{\mathbb{Q}}$ "$\{\beta : p_\beta \in \underset{\sim}{G}\}$ is unbounded in m.

[Why? By $(*)_4$.]

$(*)_8$ there is a directed $G \subseteq \mathbb{Q}$ such that $p^*_{\alpha(*)} \in G$ and $n < \omega \wedge \alpha < \omega_1 \Rightarrow \mathscr{I}_{\alpha,n} \cap G \neq 0$.

[Why? As $\mathrm{MA}_{\aleph_1}$ holds $+(*)_4$.]

$(*)_9$ $\mathscr{U} := \{u_p : p \in G\}$ is unbounded in ω_1.

[Why? As $p^*_{\alpha(*)} \in G$ and $G \cap \mathscr{I}_\alpha \neq \emptyset$ for $\alpha < \omega_1$.]

For $\alpha \in \mathscr{U}$ let $g_\alpha = \cup\{\rho^p_\alpha : p \in G\}$, clearly $g_\alpha \in {}^\omega\omega$ (as G is directed, $\mathscr{I}_{\alpha,n} \cap G \neq \emptyset$ for $\alpha < \omega_1, n < \omega$).

Also $f_\alpha \leq g_\alpha$ by clause (x) of the definition of $\mathbb{Q}$. Also $\langle g_\alpha : \alpha \in \mathscr{U}\rangle$ is $<_{J^{\mathrm{bd}}_\omega}$-decreasing by clause (y) of the definition of $\leq_{\mathbb{Q}}$. Hence for each $\alpha < \omega_1$ we have $\beta < \omega_1 \Rightarrow f_\beta <^* g_\alpha$ hence by clause (?) of Theorem 1.13 there is $\gamma(\alpha) < \lambda$ such that $f^{\gamma(\alpha)} <^* g_\alpha$. Let $\gamma(*) = \sup\{\gamma(\alpha) : \alpha < \omega_1\}$, so $\gamma(*) < \lambda$ (as $\lambda = \mathrm{cf}(\lambda) > \kappa = \aleph_1$) hence $\beta < \omega_1 \wedge \alpha < \omega_1 \Rightarrow f_\beta <^* f^{\gamma(*)} <^* g_\alpha$. Let n_α be minimal such that $n \in [n_\alpha, \omega) \Rightarrow f_\alpha(n) < f^{\gamma(*)} < g_\alpha(n)$. So for some n^* the set $\mathscr{U}_* = \{\alpha \in \mathscr{U} : n_\alpha = n_*\}$ is unbounded in ω_1.

Let $\alpha(*) \in \mathscr{U}_*$ be such that $\mathscr{U}_* \cap \alpha(*)$ is infinite and let $\langle q_n : n < \omega\rangle$ be an $\leq_{\mathbb{Q}}$-increasing sequence of members of G such that $\mathscr{U}_* \cap (\alpha(*)+1) \subseteq \cup\{u_{q_n} : n < \omega\}$. By clause (z) of the definition of the order $\leq_{\mathbb{Q}}$ we get a contradiction. $\square_{\mathbf{2.1}}$

$\longrightarrow$ scite{y.17} ambiguous

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